# Affine Geometric Analysis

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### Isoperimetric inequalities in hermitian vector spaces

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In this talk I will present recent generalizations of the classical isoperimetric inequalities for quermassintegrals in hermitian vector spaces.
Consider the Grassmannian of real $k$-planes in $\mathbb{C}^n$. For $k = 2, 3$ and $n \geq k$ the action of the unitary group decomposes the Grassmannian into infinitely many orbits parametrized by a single real parameter, known as the Kähler angle $\theta \in [0, \pi/2]$. Each orbit gives rise to a quermassintegral by averaging the volume of projections onto the planes of the orbit.
In the first part of the talk I will present an Aleksandrov-Fenchel type inequality satisfied by hermitian quermassintegrals of degree 2 and 3 which allows to deduce an isoperimetric type inequality for some hermitian quermassintegrals described before. For instance, for orbits of isotropic or Lagrangian planes, the corresponding quermassintegrals satisfy isoperimetric inequalities similar to the ones discovered by Minkowski, Aleksandrov, and Fenchel.
In a second part, I will give an isoperimetric inequality relating the volume with positive unitary valuations of codegree 2.

This is a joint project with T. Wannerer.

### Rotations of shadows of convex bodies: Positive Results and Counterexamples

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We construct examples of two convex bodies $K, L$ in $\mathbb{R}^n$, such that every projection of $K$ onto a $(n-1)$-dimensional subspace can be rotated to be contained in the corresponding projection of $L$, but $K$ itself cannot be rotated to be contained in $L$. We also find necessary conditions on $K$ and $L$ to ensure that $K$ can be rotated to be contained in $L$ if all the $(n-1)$-dimensional projections have this property.

**The hyperbolic floating body**

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The notion of convex floating body of a Euclidean convex body has recently been extended to the Euclidean unit sphere. We will present an extension to the hyperbolic setting. Thus a complete description of floating bodies in real space forms, that is, manifolds with constant curvature, is obtained. Differentiation of the volume of the floating body gives rise to the floating area. In the Euclidean setting the floating area is better known as affine surface area. It is a classical and powerful tool in the (equi-)affine geometry of convex bodies.

**Symmetrizations in geometry**

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I will present an ongoing research in collaboration with Richard Gardner and Paolo Gronchi. We introduce the concept of i-symmetrization, which provides a convenient framework for most of the familiar symmetrization processes on convex sets. We characterize Steiner and Minkowski symmetrizations in terms of some of the natural properties that they enjoy. We introduce several new symmetrizations, consider various properties of $i$-symmetrizations and discuss the relations between them.

**Random reflections, symmetrizations, and foldings on the sphere**

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Two-point symmetrizations are simple rearrangements that have been used to prove isoperimetric inequalities on the sphere. For each unit vector u, there is a two-point symmetrization that pushes mass towards u across the normal hyperplane.

In this talk, I will talk about recovering full rotational symmetry from partial information. How many two-point symmetrizations are needed to verify that only radial functions are invariant under these symmetrizations? I will show that d+1 suffice, and will discuss the ergodicity of the random walk generated by the corresponding folding maps on the sphere.

(Joint work with G.R. Chambers and A. Dranovski)

**Sobolev trace inequalities for functions of bounded variation and related geometric inequalities**

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The sharp constant in Poincaré type inequalities for the trace of functions of bounded variation on the boundary of their domain is characterized in terms of geometric inequalities involving subsets of the relevant domain. This characterization is exploited to show that balls have the smallest optimal constant, among all admissible Euclidean domains, in Poincaré trace inequalities for functions with vanishing median or mean value. This is a joint work with V. Ferone, C. Nitsch and C. Trombetti.

### Bounding marginal densities via affine isoperimetry

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Let $\mu$ be a probability measure on $\mathbb{R}^n$ with a bounded density $f$. We prove that the marginals of $f$ on most subspaces are well-bounded. For product measures, studied recently by Rudelson and Vershynin, our results show there is a trade-off between the strength of such bounds and the probability with which they hold. Our proof rests on new affinely-invariant extremal inequalities for certain averages of $f$ on the Grassmannian and affine Grassmannian. These are motivated by Lutwak's dual affine quermassintegrals for convex sets. We show that key invariance properties of the latter, due to Grinberg, extend to families of functions. The inequalities we obtain can be viewed as functional analogues of results due to Busemann–Straus, Grinberg and Schneider. As an application, we show that without any additional assumptions on $\mu$, any marginal $\pi_E(\mu)$, or a small perturbation thereof, satisfies a nearly optimal small-ball probability.
This is joint work with Grigoris Paouris and Peter Pivovarov.

### On the rate of convergence of random two-point symmetrizations

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It is known that i.i.d. random sequences of two-point symmetrizations almost surely transform every subset of $S^d$ into a spherical cap. In fact, the expected distance decreases at least like a power law in the number of iterations. In this talk, I will show that the rate of convergence on $S^1$ exactly obeys the power law. The key to the proof is an analogue of the Riesz rearrangement inequality on $S^1$ which we conjecture to extend to higher dimensions.
(Joint work with A. Burchard)

## The class of $j$-projection bodies

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Dual to Koldobsky's notion of j-intersection bodies, the class of $j$-projection bodies is introduced, generalizing Minkowski's notion of projection bodies of convex bodies. A fundamental Fourier-analytic characterization of $j$-intersection bodies due to Koldobsky initiated further investigations of this class. Here a dual version of this theorem for $j$-projection bodies will be discussed. It turns out that this characterization is closely related to another - valuation-theoretic - characterization involving the Alesker-Fourier transform.
(Joint work with Franz Schuster)

## The reverse Prekopa-Leindler inequality and an application to the Godbersen conjecture

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We prove a reverse form of the Prekopa-Leindler inequality for convex functions. As an application we provide the best known bound for Godbersen's conjecture. We also provide a natural generalization of a geometric conjecture of Fary and Redei regarding the volume of the convex hull of K and K, and show that it implies the precise form of Godbersen's conjecture.

**The interplay between curvature flows and the stability of inequalities**

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In this talk, first, I will describe a method for obtaining the stability of geometric inequalities such as the Busemann-Petty centroid inequality and the affine isoperimetric inequality in the plane by using the centro-affine normal flow and the affine normal flow. Second, I will present an application of the stability of Firey's functional to the asymptotic roundness of the normalized solutions of the Gauss curvature flow provided Firey's functional is initially small enough.

**Sharp affine Sobolev type inequalities via the $L_p$ Busemann-Petty centroid inequality**

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In this talk we show how to recover some sharp affine functional inequalities, like log-Sobolev, Sobolev and Gagliardo-Nirenberg, using the $L_p$ Busemann-Petty centroid inequality along with some classical results for general norms. This approach allows us to recover as well the equality cases with optimal constants. Joint work with J. Haddad and M. Montenegro.

**Group actions on hyperspaces of compact convex subsets of $\mathbb{R}^n$**

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Let $n \geq 2$. We will denote by $\mathrm{Aff}(n)$ the group of all affine transformations of $\mathbb{R}^n$ while $cc(\mathbb{R}^n)$ will be the hyperspace of all compact convex subsets of $\mathbb{R}^n$ equipped with the Hausdorff distance topology. A very well-known result of Nadler, Quinn and Stavrakas states that $cc(\mathbb{R}^n)$ is homeomorpic to the punctured Hilbert cube. In this talk we are interested in showing some results about the topology of certain subspaces of $cc(\mathbb{R}^n)$ obtained by the study of the natural action of $\mathrm{Aff}(n)$ on $cc(\mathbb{R}^n)$.

We will see how the topological structure of certain subspaces of $cc(\mathbb{R}^n)$ is directly related with the geometry of the action of a specific subgroup of $\mathrm{Aff}(n)$. Understanding the dynamic of such action, allows us to give a concrete description of the subspace's topology. Using these techniques we prove that the hyperspace $cb(\mathbb{R}^n)$ of all compact convex bodies of $\mathbb{R}^n$ is homeomorphic to the topological product $Q \times \mathbb{R}^{\frac{n(n+3)}{2}}$, where $Q$ denotes de Hilbert cube. Similarly, the hyperspace $cc_1(\mathbb{R}^n)$ of all compact convex subsets of $\mathbb{R}^n$ of dimension at least 1 is homeomorphic to $Q \times \mathbb{R}^{n+1}$.

On the other hand, by studying the topology of the orbit spaces generated by the action of some subgroups of $\mathrm{Aff}(n)$ on certain subspaces of $cc(\mathbb{R}^n)$ we get some interesting results. In this line, we show that the orbit spaces $cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ and $cc_1(\mathbb{R}^n)/\mathrm{Sim}(n)$ (where $\mathrm{Sim}(n)$ stands for the group of all similarities of $\mathbb{R}^n$) are both homeomorphic to the Banach-Mazur compactum $BM(n)$. Furthermore, if $E(n)$ denotes de Euclidean group, the orbit space $cc(\mathbb{R}^n)/E(n)$ (which corresponds with the Gromov-Hausdorff hyperspace of all compact convex subsets of $\mathbb{R}^n$) is homeomorphic to the open cone over $BM(n)$.

These results were obtained in collaboration with Sergey Antonyan and Bernardo González Merino.

**Proof of a conjecture of B. Grünbaum about affine invariant points**

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An affine invariant point $p$ is a continous map from the set of d-dimensional convex bodies $\mathcal{K}_d$ equipped with the Hausdorff-distance to $\mathbb{R}^d$ such that for every $T$ affine linear and invertible map on $\mathbb{R}$ and every $C \in \mathcal{K}^d$ we have

$$T(p(C)) = p(T(C))$$

Famous examples are the centroid, the Santalo-, Löwner- and John-point of a convex body. Grünbaum conjectured that for every convex body $K$ we have $\mathcal{P}_d(K) := \{p(K) : p \text{ affine invariant point}\}$ equals to $\mathcal{F}_d(K) = \{x : Tx = x \text{ for every } T \text{ affine linear with } T(K) = K\}$. In this talk we want to present a proof of this conjecture.

## Algebraically inspired constructions in convex geometry

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We will show how algebraic identities, inequalities and constructions, which hold for numbers or matrices, often have analogs in the geometric classes of convex bodies or convex functions. By letting the polar body $K^\circ$ or the dual function $\varphi^*$ play the role the inversion we will be able to prove some new results, and cast some old results in a new light.
We will begin by explaining this idea and giving a few examples of old and new identities. Next we will explain how Molchanov used such ideas to construct continued fractions of convex bodies, and discuss some amusing corollaries.
In the main part of the talk we will construct a new, natural definition for the geometric mean of convex bodies or convex functions. We will prove some basic properties of this construction, such as affine equivariance, and some less basic properties such as concavity. We will also discuss the relationship between our construction and the log-Brunn-Minkowski conjecture of Böröczky, Lutwak, Yang and Zhang.
Based on joint work with Vitali Milman.

## On bodies with congruent projections in $\mathbb{R}^4$

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Let $K$ and $L$ be two convex bodies in $\mathbb{R}^4$ and let $\xi^\perp$ be a three-dimensional subspace orthogonal to the unit vector $\xi$. Assume that for every $\xi$, the projections $K|\xi^\perp$, $L|\xi^\perp$ are directly congruent. Does it follow that $K$ and $L$ coincide up to translation and reflection in the origin? We show that if the set of diameters of bodies satisfy an additional condition and certain projections do not have $\pi$-symmetries, then the answer is affirmative.

### Affine invariant points and their duals

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We consider which affine invariant points have dual points.

### Functional inequalities involving the geometric inf-convolution

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We present a natural geometric analogue of the Prekopa-Leindler and Borell-Brascamp-Lieb inequalities. We also generalize a classical theorem of Busemann and relate it to the above mentioned inequalities. Finally, we present an implication between Busemann's theorem and a particular case of the Borell-Brascamp-Lieb inequality.

### Remarks on Dvoretzky's theorem

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We are going to discuss the dependence on $\varepsilon$ in the random version of Dvoretzky's theorem for the classical spaces. This is joint work with G. Paouris and J. Zinn (Texas A&M University).

**Affine variational capacity**

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This talk will address the so-called affine variational $1 < p < n$ capacity in the Euclidean n-space and its convex geometric nature.

**Affine isoperimetric inequalities for geominimal surface areas**

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Affine isoperimetric inequalities are central to (affine) convex geometry with many applications, such as, in quantum information theory. These inequalities aim to find the best possible upper and/or lower bounds, in terms of volume, for affine invariant functionals on convex bodies (i.e., functionals defined on convex bodies remain the same under all linear transforms with determinant to be 1 or -1).

In this talk, I will talk about recent progress on affine isoperimetric inequalities for geominimal surface areas. In particular, I will explain how to define the $L_p$ geominimal surface areas for all $-n \neq p < 1$ and its Orlicz extension, which generalize the $L_p$ geominimal surface areas for $p \geq 1$ (defined by Petty for $p = 1$ and by Lutwak for $p > 1$). Related properties and affine isoperimetric inequalities will be provided as well.

**On a linear refinement of the Prékopa-Leindler inequality under projection assumptions**

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The Prékopa-Leindler inequality is a powerful result closely related to a number of classical integral inequalities, such as Hölder's inequality or the reverse Young's inequality, as well as to some geometric ones like the well-known Brunn-Minkowski inequality.
More precisely, it states that, given $\lambda \in (0,1)$ and non-negative measurable functions $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ such that, for any $x, y \in \mathbb{R}^n$,

$$h\big((1-\lambda)x + \lambda y\big) \geq f(x)^{1-\lambda} g(y)^{\lambda},$$

then

$$\int_{\mathbb{R}^n} h \, \mathrm{d}x \geq \left(\int_{\mathbb{R}^n} f \, \mathrm{d}x\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \, \mathrm{d}x\right)^{\lambda}.$$

In this talk we will show that under the sole assumption that $f$ and $g$ have a common *projection* onto a hyperplane (which is the analytic counterpart of the projection of a set onto a hyperplane), the Prékopa-Leindler inequality admits a linear refinement. That is, under such an assumption for the functions $f$ and $g$, the right-hand side in the above integral inequality may be exchanged by the convex combination of the integrals, which yields a stronger inequality. Moreover, the same inequality can be obtained when assuming that both projections (not necessarily equal as functions) have the same integral. We will explore the main idea of the proof of the latter result and we will discuss some questions in that sense.
This is joint work with A. Colesanti and E. Saorín Gómez.

**On $L_p$-affine surface area and curvature measures**

Yiming Zhao
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Affine surface area is a fundamental concept in affine convex geometry with applications ranging from PDE to affine analytical isoperimetric inequalities. In this talk, the three existing representations of affine surface area will be discussed and a new representation of affine surface area based on curvature measures of the convex body will be introduced. It will be explained how the new representation is a *missing* piece from the three already known definitions. The new representation will be *polar* to that of Lutwaks and *dual* to that of Schütt & Werner. It will also be discussed briefly on how one can prove that the new representation is equivalent to the existing ones.

**Bezout inequality for mixed volumes**

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In this talk we will discuss the following analog of Bezout inequality for mixed volumes:

$$V(P_1, \ldots, P_r, \Delta^{n-r}) V_n(\Delta)^{r-1} \leq \prod_{i=1}^{r} V(P_i, \Delta^{n-1}) \quad \text{for } 2 \leq r \leq n.$$

We will show that the above inequality is true when $\Delta$ is an $n$-dimensional simplex and $P_1, \ldots, P_r$ are convex bodies in $\mathbb{R}^n$. We will also discuss the conjecture that if the above inequality is true for all convex bodies $P_1, \ldots, P_r$, then $\Delta$ must be an $n$-dimensional simplex. We prove that if the above inequality is true for all convex bodies $P_1, \ldots, P_r$, then $\Delta$ must be indecomposable (i.e. cannot be written as the Minkowski sum of two convex bodies which are not homothetic to $\Delta$), which confirms the conjecture when $\Delta$ is a simple polytope and in the 2-dimensional case. Finally, we connect the inequality to an inequality on the volume of orthogonal projections of convex bodies as well as prove an isomorphic version of the inequality. This is a joint work with Ivan Soprunov.