Kinetic and Related Equations

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1 First Comments

This meeting was only the third to be held at CMO Oaxaca, and as the buildings for CMO are still in the planning stage, we lived, ate and worked in the beautiful Hacienda Las Laureles, which offers pleasant and large rooms, spectacular gardens (in which all meals were served) and very suitable conference and discussion rooms. The technical support provided by CMO was flawless, and the whole meeting went by without any glitches.

Mexico is, as was mentioned during the opening, a happy place, and that happiness invaded the meeting. It was easy to smile at breakfast, although a full day of work lay ahead. We prepared a full schedule of talks for the first two days. On Wednesday morning, the entire conference visited the spectacular ruins of Monte Alban, a site which grew to prominence at the same time as the city of Rome. Its dramatic mountaintop location combines an amazing historical dimension with fantastic views.

We were back in the lecture hall by 2:30, and managed to work through the heavy late afternoon downpour with thunder and lightning in safe and dry conditions. A couple of speakers had to compete with thunder and heavy rain on the roof. Thursday brought another full day of talks, and on Friday the remaining participants (about half; many had to leave early to reach their planes) spent the morning discussing open problems, loose ends and applications.

The formula for these meetings is a recipe for total success.

2 Overview of Kinetic and Related Equations

Kinetic Equations are, by now, a rather classical subject, starting with the Boltzmann equation in 1872. Other kinetic equations, such as the Vlasov-Poisson and Vlasov-Maxwell equations arising in plasma physics, or the neutron transport equation in reactor physics, have also been studied for many decades now, and continue to resist complete understanding and analysis. The importance of these equations is clear from the many applications in which they arise (rarefied gas dynamics in space applications and micro-machines, plasma simulations for tokamak simulations, and many other matters), and decades of work have gone into efforts towards rigorous derivations, analysis, and effective numerical schemes.

All these subjects were represented at the meeting. In addition, there were many talks about more recent kinetic models, for examples, equations which model the flocking or swarming of birds, fish, locusts or other "agents". These new developments even encompass the social and marketing behaviour of human beings, thus extending the applicability of kinetic theory to the social sciences.

3 Recent Developments and Open Problems

The recent developments alluded to in the previous section include flocking and swarming models (such as the well-known Cucker-Smale model), kinetic equations to model the purchase of and competition for microchips in the gaming industry (Intel vs. AMD), or even kinetic equations derived specifically to provide efficient numerical tools for a search algorithm [9]. The links to macroscopic equations (for example, fractional diffusion equations [3] in cloud dynamics) play an ever increasing part in a growing number of applications.

Mathematical biology applications of kinetic theory are here to stay. Many different models were discussed during the presentations. Some open problems regard the different behaviour at the levels of particle and continuum descriptions. The example of the follow the leader model in which propagation of chaos fails was discussed in detail. These open problems related to mean-field limits and its connections to agent based modelling will generate future work in the community. Other interesting models include mutation-selection balance used in qualitative genetics: given the phenotypical trait $x \in \mathbb{R}$, β fertility function, μ the mortality rate, consider

$$\partial_t f(t,x) = \int K(x-x')\beta(x')f(t,x')\,dx' - \mu f(t,x)$$

in a changing environment: $\mu = \mu(x - st)$. What is the lag in approaching a sort of periodic behaviour, if any?

A general open problem in the interplay between kinetic equations and macroscopic descriptions is how to treat analytically the break of symmetry for patterns/minimizers/steady states. There was an excellent contribution by J. Dolbeault showing the case of the Caffarelli-Kohn-Nirenberg inequality, for which a sharp result separating radial optimizers to non-radial ones has recently been achieved using nonlinear flows. These ideas should give strategies to attack similar problems for the flocking patterns of many models that coincide with the minimizers of the interaction energy

$$\iint W(x-y)\rho(x)\rho(y)\,dx\,dy\,.$$

Depending on the potential, one may have numerical solutions that seem to loose radial symmetry. An understanding of this problem using variational techniques would mean a big advance in the theory of the macroscopic model

$$\rho_t = \nabla\cdot[(\nabla W * \rho)\rho]$$

and the Vlasov-like equations for collective behaviour that show flocking patterns.

Another general topic of interest in the mathematical analysis in kinetic theory is how to analyse phase transitions at the kinetic level. There are cases in which this phase transition has been characterized in homogeneous situations. However, more general properties of stability of these situations are still lacking. Studying these problem in inhomogeneous settings is still out of reach and only numerical studies have been done.

A list of applied topics in which kinetic theory is likely to have an impact in the next years to come was discussed in the open meeting on Friday. Some of the discussed fields include: climate change questions such as cloud formation, erosion, plant distribution...; the classical field of fusion reactors modelling has been resurrected by the ITER project, new questions concerning Vlasov-Maxwell approximations are likely to show up; the recent surge of applications of kinetic modelling in mathematical biology and social sciences is likely to be a great source of open problems. Big data applications are probably the next field in which kinetic theory can contribute finding a way to mix clustering, mean-field limits and continuum approaches.

4 Presentation Highlights

The talks were loosely grouped according to their emphasis:

- Application-oriented talks
- Numerical methodology (typically related to applications)
- Modeling and Derivations

- Analysis
- Other related topics

The vast majority of the talks was highly informative, well delivered and perfectly within the scope of the meeting. From that point of view almost every talk was a "presentation highlight." We describe a few of these highlights here but emphasize that our selection is subjective and in no way dismissive of other presentation. Our selection follows the order in which these talks appear in the program.

Among the talks on the Monday morning, D. Armbruster's showed a highly unusual application of kinetic modelling to the social sciences, namely the purchase of competitive multi-generational products in a market where availability of funds and eagerness to purchase the next generation are the independent variables. The motivating example for this is the gaming industry, and computer chips by Intel and AMD are the the relevant products in this case [1]. The proper setting of initial and boundary conditions is of crucial importance. This work has the potential to be of use in general marketing.

Martin Frank discussed a non-classical transport model which applies, for example, to droplet transport in clouds. The model contains feedback mechanisms leading to unconventional nonlinearities, and scaling arguments in the limit of vanishing mean free path produce a fractional diffusion equation in the diffusive limit. This is of potentially high significance in global climate models [2].

Monday afternoon, among several other presentations, Luc Mieussens showed a state-of-the art numerical simulation of the "toy" known as the Crookes radiometer, a famous device which demonstrates rarefied gas effects: It is really a "windmill" inside a near vacuum tube, and when it is illuminated the temperature gradient between the dark and white sides of the panels induces a thermal creep flow that flows from cold to hot (rather in violation of common intuition) and leads to a rotation of the windmill in the "wrong" direction. This problem contains a multitude of difficulties: Choosing a manageable kinetic model which will produce thermal creep; dealing with moving boundaries; adaptive mesh refinement; and others. Mieussens' simulations are among the first to reproduce the radiometer's behaviour satisfactorily. [4].

The last two talks on Monday were "tandem talks" by M.Agueh and G. Carlier on a granular flow model originally suggested by Benedetto, Caglioti and Pulvirento [5, 6]. The highly singular kinetic equation of interest is

$$\partial_t f + v \cdot \nabla_x f = \lambda \,\mathrm{div}_v \Big((\nabla W *_v f) f \Big), \; f|_{t=0} = f_0, \tag{1}$$

where the convolution $\nabla W *_v f$ is with respect to the velocity variable only i.e. $(\nabla W *_v f)(x,v) = \int_{\mathbf{R}^d} \nabla W(v-u) f(x,u) \mathrm{d}u$. Agueh discussed local existence and uniqueness for this problem with an interesting proof based on a splitting scheme and employing the Wasserstein metric in the collision step. Carlier showed a number of recent estimates based on "pseudo-conformal conservation laws," classic invariants and an adaptation of Bony's "potential for interaction" (which restricts the discussion to one space dimension). These tools provide a great deal of information about solutions to (1), though they fall short of providing general global existence. [7].

Tuesday brought more highlights, in particular the first lecture by L. Pareschi, who showed how kinetic constrained alignment models manage to simulate consensus building (in sociological applications). Pareschi managed to link his abstract kinetic model with videos of real experiments in which peer pressure led to consensus, in some cases "nonsensical" consensus. Another application-oriented talk was offered by Maria Carvalho, who presented a rigorous and detailed analysis of a Boltzmann model related to the Vicsek alignment interaction, and which is also very similar to a model for rod alignment of Ben-Naim and Krapivsky. The basic interaction is to make pairs of rods align, but noise in the model redistributes the directions. A uniform distribution of directions is always an equilibrium, but when the noise is small, there is a non-uniform equilibrium as well, whose appearance can be described by a pitchfork bifurcation.

In the afternoon Jean Dolbeault presented a survey, and some striking recent results [8], on functional inequalities (like the Cafarelli-Kohn-Nirenberg inequalities) and how such inequalities arise in the context of nonlinear diffusion via entropy methods. In the Friday session on open problems Dolbeault presented several loose ends to be pursued in this context, as mentioned above in the section on open problems.

There were several other rather technical and highly informative talks on Tuesday, on traveling waves for the Vlasov-Poisson-Boltzmann system, the Fokker-Planck equation with boundaries, and new progress for Cercignani's conjecture on the spatially homogeneous Boltzmann equation.

Renjun Duan's work studies the convergence of solutions to the one-dimensional Vlasov-Poisson-Boltzmann system towards the profile of rarefaction waves constructed through the quasi-neutral Euler system. Here, the large time profile of the electric potential can take distinct constant states at both far-fields. The key point in the proof is motivated by the recent study of the same problem for the viscous compressible fluid with the self-consistent electric field.

Jaewoo Jung discussed the kinetic Fokker-Planck equation in general multidimensional bounded domains with absorbing boundary conditions. Gllobal existence of the solutions, hypoellipticity in the interior and the boundary away from the singular set, and local Hölder continuity up to the singular set were obtained. In addition, if the principal curvatures of the boundary surface do not vanish, the Hölder continuity is global.

Thursday saw several talks on numerical methods. F. Filbet showed higher-order implicit numerical methods for reduced Vlasov-Maxwell equations as they appear in applications to plasma dynamics, in particular tokamak simulations as needed for the ITER project.

F. Charles discussed an interesting variation of particle methods for fluid simulations; S. Martin showed how kinetic particle schemes can be used for complex optimization problems, for example, finding the global minimizer of a complicated function on a high-dimensional domain; J. Morales used Galerkin approximations to compute electron transport in semiconductors. Shen showed a generalizaton of the Vlasov-Poisson system to curved geometries, research which has the potential to assist in fundamental cosmology questions such as what the curvature of spacetime is really like.

5 Scientific Progress Made, and Outcome of the Meeting

The primary purpose of a meeting such as this is an open exchange of methods, ideas and results. These objectives were certainly met; in particular, as this meeting was very well attended by the next generation (which was certainly intended by the organizers, though not automatically guaranteed), many new contacts and potential collaborations were created. Among the ideas which remain in the mind of this reporter, here are a few that are likely to produce results in the future:

- The applications of refined numerical techniques, in particular implicit schemes, to central benchmark problems in applied physics (talks by Filbet and Mieussens)
- Using kinetic models and associated particle schemes to solve optimization problems arising elsewhere in engineering (Frank)
- Development and analysis of kinetic models on manifolds (with applications in cosmology; Shen)
- Crossover methodology between kinetic equations and PDE theory (functional inequalities, entropy methods; talk by Dolbeault)
- Flocking models, their refinement and analysis

References

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