Superradiance and instability of small rotating AdS black holes in all spacetime dimensions

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Outline

- **A Brief Review of Superradiance**
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- **Low-Frequency Solutions and Matching Procedure**
- **Damping Parameter and Superradiant Instability**

Superradiance

- As is known, classically the stationary black holes are "dead" objects, it is of crucial importance to explore their characteristic responses to external perturbations of different sorts.

- Superradiance is one of such responses.

- Superradiance is a property of rotating black holes by which waves of certain frequencies are amplified when scattering by the black holes.

- The quantum aspects of this phenomenon traces back to the so-called Klein paradox (1929) whose subsequent resolution revealed the existence of superradiant boson (not fermion!) states in the presence of strong electromagnetic fields.

Superradiance

- **The superradiant effect also arises in many classical systems moving through a medium with the linear velocity that exceeds the phase velocity of waves under consideration.**

- **As early as 1934 it was known that the reflection of sound waves from the boundary of a medium, which moves with supersonic velocity, occurs with amplification.**

N. N. Andreev and I. G. Rusakov, *Acoustics of a Moving Medium*, **GTTI, Moscow 1934.**

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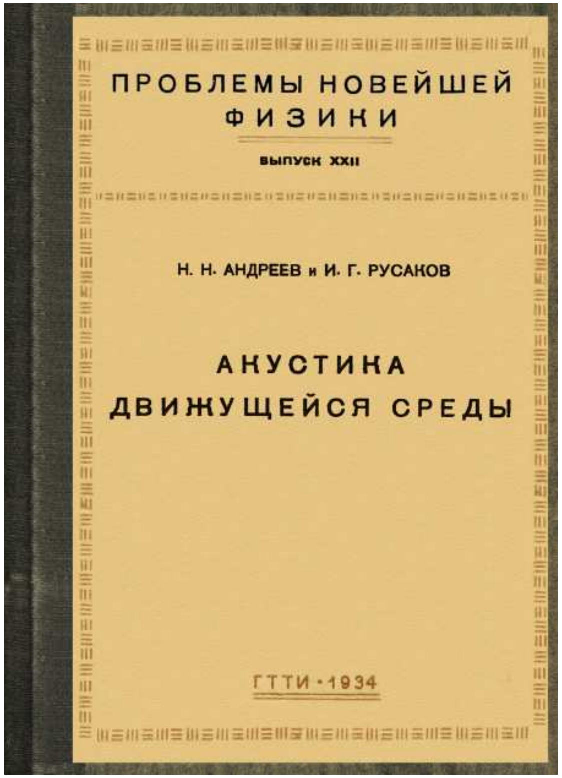

“the reflection of sound waves from the boundary of a medium, which moves with supersonic velocity, occurs with amplification...”

Superradiance and Black Holes

- The superradiant condition can be fulfilled in a rotational case as well. For a wave of frequency ω and angular momentum m, the angular velocity of a body Ω can exceed the angular phase velocity ω/m of the wave, ($\Omega > \omega/m$). For this case, Zeldovich (1971) demonstrated the amplification of waves reflected from a rotating and conducting cylinder.

- Zel'dovich put forward the idea that a semitransparent mirror surrounding the cylinder could provide exponential amplification of waves. He also anticipated that the phenomenon of superradiance and the process of exponential amplification of waves would occur in the field of a Kerr black hole.

- The quantitative theory of superradiance for scalar, electromagnetic and gravitational waves in the Kerr metric was developed in classic papers by Starobinsky (1973) and Starobinsky and Churilov (1974).

Black Hole Bomb

- The black hole superradiance was independently predicted by Misner (1972), who pointed out that certain modes of scalar waves scattered off the Kerr black hole undergo amplification.
- The black hole superradiance on its own has only a conceptual significance.
- The possible applications of the superradiant mechanism were explored by Press and Teukolsky (1972). In particular, by locating a spherical mirror around a rotating black hole, they pointed out that such a system would eventually develop a strong instability against exponentially growing modes in the superradiant regime, thus creating a black hole bomb.

The Cosmological Constant

- **Another realization of the black hole bomb effect occurs in pure geometrical settings namely, in anti-de Sitter (AdS) spacetimes due to their causal structure.**

- **The causal structure of the AdS spacetime shows that spatial infinity in it corresponds to a finite region with a timelike boundary; The spacetime exhibits a "box-like" behavior, ensuring the repetitive reflections of massless bosonic waves between spatial infinity and a Kerr-AdS black hole.**

- **The small Kerr-AdS black holes may become unstable against external perturbations (S. W. Hawking et al, 1999).**

The Hawking idea was further developed by many investigators; (Cardoso, 2004; Kodama, 2008 etc.

Our Purpose

- Continuing this line of investigation, we wish to give a new insight into the Black Hole Bomb Effect.
- To develop a quantitative theory of the black hole superradiant instability to low-frequency scalar perturbations in a vein which consists of a higher-dimensional rotating (charged) AdS black hole with a single angular momentum.
- To give a universal analytic expression for the superradiant instability for slowly rotating charged AdS black holes in all spacetime dimensions.
- To figure out how do the instability properties depend on the number of spacetime dimensions?

The Spacetime Metric

$$
\begin{aligned}
ds^2 \;&=\; -\frac{\Delta_r}{\Sigma}\left(dt - \frac{a\sin^2\theta}{\Xi}\,d\phi\right)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_\theta}\,d\theta^{\,2} \\
&\quad + \frac{\Delta_\theta \sin^2\theta}{\Sigma}\left(a\,dt - \frac{r^2+a^2}{\Xi}\,d\phi\right)^2 + r^2\cos^2\theta\, d\Omega^2_{N-3}\,, \qquad (1)
\end{aligned}
$$

where N is the number of spatial dimensions ($N \geq 3$) and

$$
d\Omega^2_{N-3} = d{\chi_1}^2 + \sin^2\chi_1\left(d{\chi_2}^2 + \sin^2\chi_2\left(...d{\chi_{N-3}}^2...\right)\right), \qquad (2)
$$

is the metric on a unit $(N-3)$-sphere. The metric functions are given by $\Delta_r = \left(r^2+a^2\right)\left(1+\frac{r^2}{l^2}\right) - m\,r^{4-N}$, $\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta$, $\Sigma = r^2 + a^2\cos^2\theta$, $\Xi = 1 - \frac{a^2}{l^2}$, $\Lambda = -l^{-2}N(N-1)/2$.

Properties 1

The stationary and azimuthal isometries: $\xi_{(t)} = \frac{\partial}{\partial t}\,, \quad \xi_{(\phi)} = \frac{\partial}{\partial \phi}$

$\Omega_H = \frac{a\,\Xi}{r_+^2 + a^2}\,, \qquad \Leftrightarrow \qquad \chi = \xi_{(t)} + \Omega_H\, \xi_{(\phi)}$

The potential one-form for a small electric charge of the black hole is determined by the difference between the timelike Killing isometries of the metric and those of its reference (the vanishing mass parameter, $m = 0$) background

$$A = -\frac{Q\, r^{4-N}}{(N-2)\,\Sigma} \left(dt - \frac{a \sin^2\theta}{\Xi}\, d\phi \right) \tag{3}$$

The electrostatic potential of the horizon, defined as $\Phi_H = -A \cdot \chi$, is given by

$$\Phi_H = \frac{Q}{(N-2)}\, \frac{r_+^{4-N}}{r_+^2 + a^2}\,. \tag{4}$$

Properties 2

By a rescaling of the mass parameter in the spacetime metric (1),

$$m \to m - q^2/r^{N-2}\,, \tag{5}$$

one can introduce a generic electric charge into the black hole spacetime. However, in higher dimensions the system of Einsten-Maxwell equations becomes consistent only in the limit of slow rotation (Aliev, 2006). The horizon of such a black hole is governed by the equation (dropping the a^2 term)

$$r^{2(N-2)}\left(1+\frac{r^2}{l^2}\right) - mr^{N-2} + q^2 = 0\,, \tag{6}$$

where the parameter q is related to the electric charge of the black hole by the relation

$$q^2 \;=\; \frac{8\pi G Q^2}{(N-2)\,(N-1)}\,. \tag{7}$$

Scalar Field

It is straightforward to show that the Klein-Gordon equation $D^\mu D_\mu \Phi = 0$, $\quad D_\mu = \nabla_\mu - ieA_\mu$, can be written out in the form

$$\frac{1}{\Sigma\, r^{N-3}} \frac{\partial}{\partial r} \left(\Delta_r\, r^{N-3}\, \frac{\partial \Phi}{\partial r} \right) + \frac{1}{\Sigma \sin\theta \cos^{N-3}\theta} \frac{\partial}{\partial \theta} \left(\Delta_\theta \sin\theta \cos^{N-3}\theta\, \frac{\partial \Phi}{\partial \theta} \right)$$

$$+ g^{ab} \frac{\partial^2 \Phi}{\partial x^a \partial x^a} - 2ieA^a \frac{\partial \Phi}{\partial x^a} - e^2 A_a A^a \Phi + \frac{1}{r^2 \cos^2\theta} \triangle_{(N-3)} \Phi = 0\,. \qquad (8)$$

Here we have introduced the Laplace-Beltrami operator $\triangle_{(N-3)}$ on a unit $(N-3)$-sphere,

$$\triangle_{(N-3)} \Phi \;=\; \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\gamma}\, \gamma^{\alpha\beta}\, \frac{\partial \Phi}{\partial x^\beta} \right), \qquad (9)$$

decomposing the indices as $\mu = \{r, \theta,\, a,\, \alpha\}$, where $a = 0, 3 \equiv t,\, \phi$ and $\alpha = 1, ..., N-3$.

Separability

Next, to separate variables in equation (8) we assume the ansatz in the form

$$\Phi = e^{-i\omega t + im\phi}\, R(r)\, S(\theta)\, Y_j(\Omega)\,, \tag{10}$$

where m is the "magnetic" quantum number, $(\omega > 0, m > 0)$.

The hyperspherical harmonics $Y_j(\Omega)$ are eigenfunctions of the Laplace-Beltrami operator. The corresponding eigenvalues are given by

$$\triangle_{(N-3)} Y_j(\Omega) \;=\; -j(j+N-4) Y_j(\Omega). \tag{11}$$

With this in mind, it is not difficult to show that the separation ansatz yields two decoupled ordinary differential equations; the angular equation

Angular Equation

$$
\begin{aligned}
\frac{1}{\sin\theta\cos^{N-3}\theta}\frac{d}{d\theta}\left(\Delta_\theta\sin\theta\cos^{N-3}\theta\frac{dS}{d\theta}\right)+\Bigg[\lambda-\frac{1}{\Delta_\theta}\left(\frac{m\Xi}{\sin\theta}-a\omega\sin\theta\right)^2 & \\
-\frac{j(j+N-4)}{\cos^2\theta}\Bigg]S &= 0.
\end{aligned}
$$

With the regular boundary conditions at $\theta = 0$ and $\theta = \pi/2$, this equation yields a well-defined eigenvalue problem for the separation constant $\lambda = \lambda_\ell(\omega)$. The associated eigenfunctions are spheroidal harmonics $S(\theta) = S_{\ell\, mj}(\theta|a\omega)$. **Assuming that $a\omega \ll 1$ and $a/l \ll 1$ a one can show that**

$$
\lambda = \ell(\ell+N-2)+\mathcal{O}\left(a^2\omega^2, a^2/l^2\right), \tag{12}
$$

where ℓ is constrained by the condition $\ell \geq m + j$ (see e.g. Berti et al, 2006).

Radial Equation

$$\frac{\Delta_r}{r^{N-3}}\frac{d}{dr}\left(\Delta_r r^{N-3}\frac{dR}{dr}\right)+U(r)\,R=0\,, \qquad (13)$$

$$U(r)=-\Delta_r\left[\lambda+\frac{j(j+N-4)a^2}{r^2}\right]+(r^2+a^2)^2\left(\omega-\frac{am\Xi}{r^2+a^2}-\frac{eQ}{N-2}\frac{r^{4-N}}{r^2+a^2}\right)^2.$$

The radial equation can be put in the Schrödinger form

$$\frac{d^2Y}{dr_*^2}+V(r)Y=0\,,\quad Y=\left[r^{N-3}(r^2+a^2)\right]^{1/2}R\,,\qquad \frac{dr_*}{dr}=\frac{r^2+a^2}{\Delta_r} \qquad (15)$$

where the effective potential is given by

$$V(r)=\frac{U(r)}{(r^2+a^2)^2}-\frac{1}{f}\frac{d^2f}{dr_*^2}\,,\quad f=\left[r^{N-3}(r^2+a^2)\right]^{1/2}. \qquad (16)$$

Boundary Conditions 1

Recalling that the AdS spacetime yields a natural reflective boundary at spatial infinity due to its confining-box behavior, it is tempting to impose the vanishing field boundary condition,

$$\Phi \to 0\,, \qquad r \to \infty\,. \tag{17}$$

Meanwhile, it is clear that at the horizon one must impose an ingoing wave boundary condition. At the horizon, the effective potential reduces to

$$V(r_+) \;=\; (\omega - m\Omega_H - e\Phi_H)^2\,. \tag{18}$$

Boundary Conditions 2

This in turn yields the asymptotic solution that represents a purely ingoing wave at the horizon,

$$\Phi = e^{-i\omega t + im\phi}\, e^{-i(\omega - \omega_p)r_*} S(\theta) Y_j(\Omega)\,, \tag{19}$$

where ω_p is the threshold frequency, given by

$$\omega_p = m\Omega_H + e\Phi_H\,. \tag{20}$$

It follows that for the frequency range

$$0 < \omega < \omega_p\,, \tag{21}$$

the phase velocity of the wave, $v_{ph} = \omega/(\omega_p - \omega)$, is in the opposite direction with respect to the group velocity, $v_{gr} = -1$. This fact signifies the appearance of superradiance, resulting in the energy outflow from the black hole.

Quasinormal Modes

Thus, requiring a purely ingoing wave at the horizon and a purely damping wave at infinity, we arrive at a characteristic-value problem for complex frequencies of quasinormal (ringing) modes of the massless scalar field. $(\omega = \omega_n + i\delta)$

As follows from the decomposition

$$\Phi = e^{-i\omega t + im\phi} \, R(r) \, S(\theta) \, Y_j(\Omega) \, , \tag{22}$$

the imaginary part of these frequencies describes the damping of the modes. A characteristic mode is stable if the imaginary part of its complex frequency is negative (*the positive damping*), while for the positive imaginary part, the mode undergoes exponential growth (*the negative damping*). In the latter case, the system will develop instability

Solutions

- We find low-frequency solutions, $\omega \ll 1/r_+$, to some approximated versions of the radial equation, which are applicable in various regions of the spacetime
- We divide the spacetime into the near horizon,$r - r_+ \ll 1/\omega$, and far-horizon, $r - r_+ \gg r_+$, regions and approximate the radial equation for each of these regions.
- The solution of in the far-region which is valid only for large r, might also correspond to small $\omega(r - r_+)$ for sufficiently small frequencies, i.e. provided that$\omega r_+ \ll \omega(r - r_+) \ll 1$.
- Meanwhile, for $\omega \to 0$, the near-horizon region solution tends to cover whole spacetime.
- Altogether, one can conclude that for sufficiently small frequencies there must exist a region, given by $r_+ \ll r - r_+ \ll 1/\omega$, where the near-horizon solution overlaps the far-region solution.

Near Region Solution 1

In the region near the horizon, $r - r_+ \ll 1/\omega$, and for low frequency perturbations $r_+ \ll 1/\omega$, the radial equation reduces to the hypergeometric type equation

$$z(1-z)\frac{d^2R}{dz^2}+(1-z)\frac{dR}{dz}+\left[\frac{1-z}{z}\,\Omega^2 - \frac{\ell}{N-2}\left(1+\frac{\ell}{N-2}\right)\frac{1}{1-z}\right]R = 0\,, \tag{23}$$

where

$$\Omega = \frac{x_+^{\frac{N-1}{N-2}}}{N-2}\,\frac{\omega-\omega_p}{x_+-x_-}\,,\quad z = \frac{x-x_+}{x-x_-}\,,\quad x = r^{N-2}\,. \tag{24}$$

This equation can be solved in a standard way by the ansatz

$$R(z) = z^{i\Omega}\,(1-z)^{1+l/2}\,F(z)\,, \tag{25}$$

where $F(z) = F(\alpha\,,\beta\,,\gamma,z)$ is the hypergeometric function.

Near Region Solution 2

The desired solution is given by

$$R(z) = A^{in}_{(+)}\, z^{-i\Omega}\, (1-z)^{1+\frac{\ell}{N-2}}\, F\left(1 + \frac{\ell}{N-2}\,,\ 1 + \frac{\ell}{N-2} - 2i\Omega\,,\ 1 - 2i\Omega\,,\ z\right), \tag{26}$$

where $A^{in}_{(+)}$ is a constant.

Near Region Solution 3

The large $r\ (z \to 1)$ limit of the near-horizon solution is given by

$$R \simeq A^{in}_{(+)}\Gamma(1-2i\Omega)\left[\frac{\Gamma\left(-1-\frac{2\ell}{N-2}\right)\ (x_+ - x_-)^{1+\frac{\ell}{N-2}}}{\Gamma\left(-\frac{\ell}{N-2}\right)\ \Gamma\left(-\frac{\ell}{N-2}-2i\Omega\right)}\ r^{2-N-\ell} + \frac{\Gamma\left(1+\frac{2\ell}{N-2}\right)\ (x_+ - x_-)^{-\frac{\ell}{N-2}}}{\Gamma\left(1+\frac{\ell}{N-2}\right)\ \Gamma\left(1+\frac{\ell}{N-2}-2i\Omega\right)}\ r^{\ell}\right] \quad (27)$$

It turns out that the quotient of gamma functions $\Gamma\left(-1-\frac{2\ell}{N-2}\right)/\Gamma\left(-\frac{\ell}{N-2}\right)$ appearing in the first line requires a special care for $N \geq 4$, yielding in some cases divergent results for an integer ℓ.

The matching procedure fails for some values of ℓ (in odd spacetime dimensions !)

Matching Procedure

The key idea to avoid this failure:

Assume that ℓ is nearly integer, approaching its exact value only in the limit.

This can always be thought of as a pure mathematical trick by introducing a small deviation from its exact value. In the rotating case, such an assumption could be argued physically as well, if one takes into account the correction terms in the eigenvalue equation.

However, we are dealing with the regime of slow rotation (ignoring a^2 and higher-order terms).

Therefore, we will henceforth employ a formal mathematical trick, assuming that ℓ is an approximate integer.

See also an old paper by D. Page (1976).

Far Region Solution 1

In the far-horizon region, $r - r_+ \gg r_+$, one can approximate the radial equation by its purely AdS limit. Finally, we have the equation

$$y(1-y)\frac{d^2R}{dy^2} + \left[1 - \left(1 + \frac{N}{2}\right) y\right] \frac{dR}{dy} - \frac{1}{4}\left[\frac{\omega^2 l^2}{y} - \frac{\ell(\ell + N - 2)}{y-1}\right] R = 0\,. \tag{28}$$

The solution to this equation which is regular at $r \to 0$ and satisfies the vanishing field condition at $r \to \infty$ is given by

$$R(y) = A_\infty\, y^{-\frac{\ell}{2}-\frac{N}{2}}\,(1-y)^{\frac{\ell}{2}}\, F\left(\frac{N}{2} + \frac{\ell}{2} + \frac{\omega l}{2},\ \frac{N}{2} + \frac{\ell}{2} - \frac{\omega l}{2},\ 1 + \frac{N}{2},\ 1/y\right)\,, \tag{29}$$

where A_∞ is a constant and

$$y = \left(1 + \frac{r^2}{l^2}\right)\,. \tag{30}$$

Far Region Solution 2

The small r behavior of this solution, $y \to 1$, is given by

$$R(r) \simeq A_\infty \, (-1)^{\ell/2} \, \Gamma\left(1+\frac{N}{2}\right) \left[\frac{\Gamma\left(\frac{N}{2}+\ell-1\right) \, l^{N+\ell-2} \;\, r^{2-N-\ell}}{\Gamma\left(\frac{N}{2}+\frac{\ell}{2}+\frac{\omega l}{2}\right) \Gamma\left(\frac{N}{2}+\frac{\ell}{2}-\frac{\omega l}{2}\right)} \right.$$

$$\left. + \; \frac{\Gamma\left(1-\ell-\frac{N}{2}\right) \, l^{-\ell} \;\, r^{\ell}}{\Gamma\left(1-\frac{\ell}{2}+\frac{\omega l}{2}\right) \Gamma\left(1-\frac{\ell}{2}-\frac{\omega l}{2}\right)} \right] , \qquad (31)$$

In order that this solution be finite at the origin of the AdS space, $r = 0$, we must set the "quantization" condition $\frac{N}{2} + \frac{\ell}{2} - \frac{\omega l}{2} = -n,$ which in turn governs the discrete frequency spectrum for scalar perturbations by the remarkably simple formula

$$\omega_n = \frac{2n+\ell+N}{l} \, , \qquad \omega = \omega_n + i\delta \, , \qquad (32)$$

Damping Parameter 1

Performing the matching of these solutions in the overlapping region $r_+ \ll r - r_+ \ll 1/\omega$, allows us to find the damping parameter by iteration and, to first order, it is given by

$$
\begin{aligned}
\delta \quad = \quad & 2i\,\frac{(-1)^n}{n!}\;\frac{(x_+ - x_-)^{1+\frac{2\ell}{N-2}}}{l^{N+2\ell-1}}\;\frac{\Gamma\left(1+\frac{\ell}{N-2}\right)}{\Gamma\left(1+\frac{2\ell}{N-2}\right)}\;\frac{\Gamma\left(N+\ell+n\right)}{\Gamma\left(\frac{N}{2}+n+1\right)\Gamma\left(\frac{N}{2}+\ell-1\right)}\times \\
& \frac{\Gamma\left(-1-\frac{2\ell}{N-2}\right)}{\Gamma\left(-\frac{\ell}{N-2}\right)}\;\frac{\Gamma\left(1+\frac{\ell}{N-2}-2i\Omega\right)}{\Gamma\left(-\frac{\ell}{N-2}-2i\Omega\right)}\;\frac{\Gamma\left(1-\ell-\frac{N}{2}\right)}{\Gamma\left(1-\ell-\frac{N}{2}-n\right)}\,. \qquad (33)
\end{aligned}
$$

To proceed further with this expression, we need to establish its sign for the cases of interest.

We begin by simplifying the quotients of gamma functions, appearing in the second line of this expression.

Damping Parameter 2

$$\delta = \frac{(x_+ - x_-)^{1+\frac{2\ell}{N-2}}}{n!\, l^{N+2\ell-1}} \frac{\Gamma^2\left(1+\frac{\ell}{N-2}\right)}{\Gamma\left(1+\frac{2\ell}{N-2}\right)\Gamma\left(2+\frac{2\ell}{N-2}\right)} \frac{\left|\Gamma\left(1+\frac{\ell}{N-2}-2i\Omega\right)\right|^2}{\pi\cos\left(\frac{\pi\ell}{N-2}\right)} \times$$
$$\left[i\sin\left(\frac{\pi\ell}{N-2}\right)\cosh(2\pi\Omega) - \cos\left(\frac{\pi\ell}{N-2}\right)\sinh(2\pi\Omega)\right] \times$$
$$\frac{\Gamma(N+\ell+n)}{\Gamma\left(\frac{N}{2}+n+1\right)\Gamma\left(\frac{N}{2}+\ell-1\right)} \prod_{k=1}^{n}\left(\frac{N}{2}+\ell+k-1\right). \tag{34}$$

The real part of this quantity describes the damping of modes and is positive for $\Omega < 0$, i.e. in the superradiant regime. Meanwhile, the imaginary part gives the frequency-shift of modes with respect to the AdS spectrum and behaves as being not sensitive to the superradiance.

Exhaustive Cases: (i)

$$\delta = \frac{(x_+ - x_-)^{1+\frac{2\ell}{N-2}}}{n!\, l^{N+2\ell-1}} \frac{\Gamma^2\left(1+\frac{\ell}{N-2}\right)}{\Gamma\left(1+\frac{2\ell}{N-2}\right)\Gamma\left(2+\frac{2\ell}{N-2}\right)} \frac{\left|\Gamma\left(1+\frac{\ell}{N-2}-2i\Omega\right)\right|^2}{\pi\cos\left(\frac{\pi\ell}{N-2}\right)} \times$$

$$\left[i\sin\left(\frac{\pi\ell}{N-2}\right)\cosh(2\pi\Omega) - \cos\left(\frac{\pi\ell}{N-2}\right)\sinh(2\pi\Omega)\right] \times$$

$$\frac{\Gamma\left(N+\ell+n\right)}{\Gamma\left(\frac{N}{2}+n+1\right)\Gamma\left(\frac{N}{2}+\ell-1\right)} \prod_{k=1}^{n}\left(\frac{N}{2}+\ell+k-1\right). \qquad (35)$$

(i) $\frac{\ell}{N-2} = p + \epsilon$, where p is a non-negative integer, the imaginary part of (36) vanishes and the real part correctly describes the instability of the associated modes in the superradiant regime, $\Omega < 0$. This choice also encompasses the case of instability for rotating AdS black holes in 4D spacetime $(N = 3)$, Cardoso et al (2004).

Exhaustive Cases: (ii)

$$\delta = \frac{(x_+ - x_-)^{1+\frac{2\ell}{N-2}}}{n!\, l^{N+2\ell-1}} \frac{\Gamma^2\left(1+\frac{\ell}{N-2}\right)}{\Gamma\left(1+\frac{2\ell}{N-2}\right)\Gamma\left(2+\frac{2\ell}{N-2}\right)} \frac{\left|\Gamma\left(1+\frac{\ell}{N-2}-2i\Omega\right)\right|^2}{\pi\cos\left(\frac{\pi\ell}{N-2}\right)} \times$$
$$\left[i\sin\left(\frac{\pi\ell}{N-2}\right)\cosh(2\pi\Omega) - \cos\left(\frac{\pi\ell}{N-2}\right)\sinh(2\pi\Omega)\right] \times$$
$$\frac{\Gamma(N+\ell+n)}{\Gamma\left(\frac{N}{2}+n+1\right)\Gamma\left(\frac{N}{2}+\ell-1\right)} \prod_{k=1}^{n}\left(\frac{N}{2}+\ell+k-1\right). \qquad (36)$$

(ii) $\frac{\ell}{N-2} \neq (p+1/2)+\epsilon$**. In this case, the damping parameter in (36) remains complex, describing both the frequency-shift and superradiant instability of the associated modes, by its imaginary and real parts, respectively;**

Exhaustive Cases: (iii)

(iii) $\frac{\ell}{N-2} = (p+1/2) + \epsilon$, $\quad \epsilon \to 0$.

$$\delta = -\frac{(x_+ - x_-)^{1+\frac{2\ell}{N-2}}}{\pi\, n!\, l^{N+2\ell-1}} \frac{\Gamma^2\left(1+\frac{\ell}{N-2}\right)\left|\Gamma\left(1+\frac{\ell}{N-2}-2i\Omega\right)\right|^2}{\Gamma\left(1+\frac{2\ell}{N-2}\right)\Gamma\left(2+\frac{2\ell}{N-2}\right)}$$

$$\left[\sinh(2\pi\Omega) + \frac{i}{\pi\epsilon}\cosh(2\pi\Omega)\right] \times$$

$$\frac{\Gamma\left(N+\ell+n\right)}{\Gamma\left(\frac{N}{2}+n+1\right)\Gamma\left(\frac{N}{2}+\ell-1\right)} \prod_{k=1}^{n}\left(\frac{N}{2}+\ell+k-1\right). \tag{37}$$

Thus the imaginary part contains $1/\epsilon$ type divergence in the limit $\epsilon \to 0$.

Since the horizon radius r_+ is small enough, $\frac{(x_+ - x_-)^{1+\frac{2\ell}{N-2}}}{\epsilon}$ to high accuracy, can be driven to a small finite quantity even for the lowest mode.

Exhaustive Cases: (iii) Numeric Analysis

(iii) $\frac{\ell}{N-2} = (p+1/2) + \epsilon$, $\quad \epsilon \to 0$.

$$\delta \;=\; -\frac{(x_+ - x_-)^{1+\frac{2\ell}{N-2}}}{\pi\, n!\, l^{N+2\ell-1}} \; \frac{\Gamma^2\left(1+\frac{\ell}{N-2}\right)\left|\Gamma\left(1+\frac{\ell}{N-2}-2i\Omega\right)\right|^2}{\Gamma\left(1+\frac{2\ell}{N-2}\right)\Gamma\left(2+\frac{2\ell}{N-2}\right)}$$

$$\left[\sinh(2\pi\Omega) + \frac{i}{\pi\epsilon}\cosh(2\pi\Omega)\right] \times$$

$$\frac{\Gamma\left(N+\ell+n\right)}{\Gamma\left(\frac{N}{2}+n+1\right)\Gamma\left(\frac{N}{2}+\ell-1\right)} \prod_{k=1}^{n}\left(\frac{N}{2}+\ell+k-1\right). \qquad (38)$$

Numeric analysis shows that the imaginary part of this parameter can be thought of as representing a small frequency-shift in the spectrum by choosing $\epsilon \to 10^{-8}$ in 5 dimensions and $\epsilon \to 10^{-15}$ in 7 dimensions.

Concluding Remarks 1

- In four dimensions, the superradiant instability of small Kerr-AdS black holes to low-frequency scalar perturbations appears to be amenable to a complete quantitative description.
- In higher dimensions, it appears that there exist some subtleties with the matching procedure, where it fails to be valid for certain modes of scalar perturbations.
- This makes the use of numerical integration inevitable, thereby creating a gap in the complete analytic description of the superradiant instability in all dimensions.
- We have filled this gap, extending the complete analytic description of the black hole superradiant instability to all higher dimensions and focusing on a small rotating charged AdS black hole, in the regime of slow rotation and with a single angular momentum.

Concluding Remarks 2

- We have employed the idea that the orbital quantum number ℓ can be thought of as an approximate integer and managed to perform the matching procedure, resulting in the complete low-frequency solution.
- Finally, we have given a remarkably instructive expression for the damping parameter, which appears to be a complex quantity in general.
- We have shown that the real part of the damping parameter can be used to give a *universal* analytic description of the superradiant instability for slowly rotating charged AdS black holes in all spacetime dimensions.
- The instability time scale, $\tau = 1/\delta$, significantly grows as the number of dimensions increases.

Thank You !