Higher dimensional spacetimes with a separable Klein–Gordon equation

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May 17, 2016

Contents

I. Kolář and P. Krtouš, Phys. Rev. D **93**, 024053 (2016), eprint: `1509.01667[gr-qc]`.

I. Kolář and P. Krtouš, Phys. Rev. D **91**, 124045 (2015), eprint: `1504.00524[gr-qc]`.

$$\boldsymbol{g} = \frac{P_1Z_2 - P_2Z_1}{X_1}\big(\mathbf{d}x_1\big)^{\mathbf{2}} + \frac{P_2Z_1 - P_1Z_2}{X_2}\big(\mathbf{d}x_2\big)^{\mathbf{2}} + \frac{X_1}{P_1Z_2 - P_2Z_1}\big(P_2\mathbf{d}\psi_0 + Z_2\mathbf{d}\psi_1\big)^{\mathbf{2}} + \frac{X_2}{P_2Z_1 - P_1Z_2}\big(P_1\mathbf{d}\psi_0 + Z_1\mathbf{d}\psi_1\big)^{\mathbf{2}}$$

Separability condition:

$$P_1Z_2 - P_2Z_1 = W_1 + W_2$$

where $P_\mu = P_\mu(x_\mu),\ Z_\mu = Z_\mu(x_\mu),\ W_\mu = W_\mu(x_\mu),\ X_\mu = X_\mu(x_\mu)$

$\rightarrow$ Carter's classification: [A], [B$\pm$], [C$\pm$], [D]

Important case:

P_μ are nonzero constants

coordinate transformation $\rightarrow P_\mu = 1$

B. Carter, Commun. Math. Phys. **10**, 280 (1968).

B. Carter, General Relativity and Gravitation **41**, 2873 (2009).

Kerr–NUT–(A)dS metric ($D = 2N$):

$$\boldsymbol{g} = \sum_{\mu} \left[\frac{U_\mu}{X_\mu} (\mathbf{d}x_\mu)^{\mathbf{2}} + \frac{X_\mu}{U_\mu} \Big(\sum_j A_\mu^{(j)} \mathbf{d}\psi_j \Big)^{\mathbf{2}} \right]$$

where

$$U_\mu = \prod_{\substack{\nu \\ \nu \neq \mu}} (x_\nu^2 - x_\mu^2) \qquad A_\mu^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1 < \dots < \nu_k \\ \nu_i \neq \mu}} x_{\nu_1}^2 \dots x_{\nu_k}^2$$

Coordinates:

x_μ	$\mu = 1, \dots, n$	radial and latitudinal directions
ψ_j	$j = 0, \dots, n-1$	temporal and longitudinal directions

Metric functions:

$X_\mu = X_\mu(x_\mu)$	off-shell
$X_\mu = \sum_{k=0}^{n} c_k \big(-x_\mu^2\big)^k + b_\mu x_\mu$	on-shell

$c_k,\ b_\mu$: cosmological constant, angular momenta, mass, and NUT charges

W. Chen et al., Class. Quantum Grav. **23**, 5323 (2006), eprint: `hep-th/0604125`.

Properties:

- integrability of geodesic motion[1]
- separability of the Hamilton–Jacobi, Dirac, and Klein–Gordon equations[2]
- tower of Killing vectors and Killing tensors[3]
- uniquely determined by the existence of a principal conformal Killing–Yano tensor[4]

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[1] D. N. Page et al., Phys. Rev. Lett. **98**, 061102 (2007), eprint: `hep-th/0611083`.
[2] V. P. Frolov et al., J. High Energy Phys. **0702**, 005 (2007), eprint: `hep-th/0611245`.
A. Sergyeyev and P. Krtouš, Phys. Rev. D **77**, 044033 (2008), eprint: `0711.4623[hep-th]`.
M. Cariglia et al., Phys. Rev. D **84**, 024004 (2011), eprint: `1102.4501[hep-th]`.
M. Cariglia et al., Phys. Rev. D **84**, 024008 (2011), eprint: `1104.4123[hep-th]`.
[3] P. Krtouš et al., J. High Energy Phys. **0702**, 004 (2007), eprint: `hep-th/0612029`.
P. Krtouš et al., Phys. Rev. D **76**, 084034 (2007), eprint: `0707.0001[hep-th]`.
[4] T. Houri et al., Phys. Lett. **B656**, 214 (2007), eprint: `0708.1368[hep-th]`.
P. Krtouš et al., Phys. Rev. D **78**, 064022 (2008), eprint: `0804.4705[hep-th]`.
T. Houri et al., Class. Quant. Grav. **26**, 045015 (2009), eprint: `0805.3877[hep-th]`.

Could Carter's ansatz be generalized to higher dimensions?

Yes, by introducing a functional freedom in each coordinate x_μ

$$x_\mu^2 \to Z_\mu(x_\mu)$$

- separability of the Klein–Gordon equation?
- Killing vectors, Killing tensors, and CKY?
- solutions of the Einstein equations?

Klein–Gordon simple separable metric ($D = 2N$):

$$\boldsymbol{g} = \sum_{\mu} \left[\frac{U_\mu}{X_\mu} \left(\mathbf{d}x_\mu\right)^{\mathbf{2}} + \frac{X_\mu}{U_\mu} \Big(\sum_j A_\mu^{(j)} \mathbf{d}\psi_j \Big)^{\mathbf{2}} \right]$$

where $X_\mu = X_\mu(x_\mu)$

$$U_\mu = \prod_{\substack{\nu \\ \nu \neq \mu}} \left(Z_\nu - Z_\mu\right) \qquad A_\mu^{(j)} = \sum_{\substack{\nu_1, \dots, \nu_j \\ \nu_1 < \dots < \nu_j \\ \nu_k \neq \mu}} Z_{\nu_1} \dots Z_{\nu_j}$$

and

$$Z_\mu = Z_\mu(x_\mu) \qquad Z'_\mu \neq 0$$

Alternative form:

$$\boldsymbol{g} = \sum_{\mu} \left[\frac{U_\mu}{Y_\mu} \left(\mathbf{d}x_\mu\right)^{\mathbf{2}} + \frac{X_\mu}{U_\mu} \Big(\sum_j A_\mu^{(j)} \mathbf{d}\psi_j \Big)^{\mathbf{2}} \right]$$

where $X_\mu = X_\mu(x_\mu),\ Y_\mu = Y_\mu(x_\mu),\ Z_\mu = x_\mu^2$

Orthonormal frame $\boldsymbol{e}_\mu$, $\hat{\boldsymbol{e}}_\mu$ and coframe $\boldsymbol{e}^\mu$, $\hat{\boldsymbol{e}}^\mu$:

$$\boldsymbol{e}_\mu = \sqrt{\frac{X_\mu}{U_\mu}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x_\mu} \qquad \hat{\boldsymbol{e}}_\mu = \sqrt{\frac{U_\mu}{X_\mu}}\sum_k \frac{\left(-Z_\mu\right)^{N-1-k}}{U_\mu}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\psi_k}$$

$$\boldsymbol{e}^\mu = \sqrt{\frac{U_\mu}{X_\mu}}\mathbf{d}x_\mu \qquad \hat{\boldsymbol{e}}^\mu = \sqrt{\frac{X_\mu}{U_\mu}}\sum_k A_\mu^{(k)}\mathbf{d}\psi_k$$

$$\rightarrow \boldsymbol{g} = \textstyle\sum_\mu \left(\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \hat{\boldsymbol{e}}^\mu \hat{\boldsymbol{e}}^\mu\right)$$

Ricci tensor:

$$\mathbf{Ric} = \sum_\mu T_\mu \boldsymbol{e}^\mu\boldsymbol{e}^\mu + \sum_\mu \left[T_\mu + \frac{1}{2}\sum_{\substack{\nu\\ \nu\neq\mu}}\frac{S_{\nu,\mu}}{Z'_\mu}\frac{X_\mu}{U_\mu}\right]\hat{\boldsymbol{e}}^\mu\hat{\boldsymbol{e}}^\mu + \sum_{\substack{\mu,\nu\\ \nu\neq\mu}}\frac{1}{2}\frac{S_\mu - S_\nu}{Z_\mu - Z_\nu}\sqrt{\frac{X_\mu}{U_\mu}\frac{X_\nu}{U_\nu}}\hat{\boldsymbol{e}}^\mu\hat{\boldsymbol{e}}^\nu$$

where

$$S_\mu = Z''_\mu + \sum_{\substack{\kappa\\ \kappa\neq\mu}}\frac{{Z'_\mu}^2 - {Z'_\kappa}^2}{Z_\mu - Z_\kappa} \qquad T_\mu = -\frac{1}{2}\frac{X''_\mu}{U_\mu} - \frac{1}{2}\sum_{\substack{\nu\\ \nu\neq\mu}}\frac{S_{\mu,\nu}}{Z'_\nu}\frac{X_\nu}{U_\nu} + \frac{1}{2}\sum_{\substack{\nu\\ \nu\neq\mu}}\frac{1}{U_\mu}\left[\frac{Z'_\mu X'_\mu - Z''_\mu X_\mu}{Z_\mu - Z_\nu} + \frac{Z'_\nu X'_\nu - Z''_\nu X_\nu}{Z_\nu - Z_\mu}\right]$$

Killing vectors:

$$ {}^{j}\boldsymbol{l} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\psi_j} $$

Rank-two Killing tensors:

$$ {}^{j}\boldsymbol{k} = \sum_{\mu} A_{\mu}^{(j)} \left(\boldsymbol{e}_{\mu}\boldsymbol{e}_{\mu} + \hat{\boldsymbol{e}}_{\mu}\hat{\boldsymbol{e}}_{\mu} \right) $$

GCCKY 2-form with torsion:[5]

$$ \boldsymbol{h} = \sum_{\mu} \sqrt{Z_\mu}\, \boldsymbol{e}^{\mu} \wedge \hat{\boldsymbol{e}}^{\mu} $$

[5] T. Houri et al., Class. Quantum Grav. **29**, 165001 (2012), eprint: `1203.0393[hep-th]`.

Operators:

$$\mathsf{K}_j = -\boldsymbol{\nabla}_a {}^j\boldsymbol{k}^{ab}\boldsymbol{\nabla}_b \qquad \mathsf{L}_j = -i\,{}^j\boldsymbol{l}^a\boldsymbol{\nabla}_a$$

here $\mathsf{K}_0 = \Box$

Operators in coordinates x_μ, ψ_j:

$$\mathsf{K}_j = \sum_\mu \frac{A_\mu^{(j)}}{U_\mu}\left[-\frac{\partial}{\partial x_\mu}X_\mu\frac{\partial}{\partial x_\mu} + \frac{1}{X_\mu}\left[\sum_k \left(-Z_\mu\right)^{N-1-k}\mathsf{L}_k\right]^2\right] \qquad \mathsf{L}_k = -i\frac{\partial}{\partial\psi_k}$$

Mutual commutation:

$$[\mathsf{L}_j\,,\mathsf{L}_k] = 0 \qquad [\mathsf{L}_j\,,\mathsf{K}_k] = 0 \qquad [\mathsf{K}_j\,,\mathsf{K}_k] = 0$$

$$\Downarrow \quad \text{eigenfunctions } \phi$$

$$\mathsf{K}_j\phi = \Xi_j\phi$$
$$\mathsf{L}_j\phi = \Psi_j\phi$$

where Ξ_j and Ψ_j are eigenvalues

Separability ansatz:

$$\phi = \prod_\mu R_\mu \prod_k \exp\big(i\Psi_k \psi_k\big)$$

$\to R_\mu = R_\mu(x_\mu)$ satisfy ordinary differential equations

$$\Big(X_\mu R_\mu'\Big)' + \left(\breve{\Xi}_\mu - \frac{\breve{\Psi}_\mu^2}{X_\mu}\right) R_\mu = 0$$

where

$$\breve{\Psi}_\mu = \sum_k \Psi_k \big(-Z_\mu\big)^{N-1-k} \qquad \breve{\Xi}_\mu = \sum_k \Xi_k \big(-Z_\mu\big)^{N-1-k}$$

Vacuum Einstein equations:

$$\mathbf{Ric} = \frac{\Lambda}{N-1}\boldsymbol{g}$$

Solution (off-diagonal part $\rightarrow$ trace $\rightarrow$ diagonal part):

$$Z_\mu = p x_\mu^2 + q_\mu x_\mu + r_\mu \qquad X_\mu = \alpha_\mu x_\mu + \beta_\mu + \sum_{k=1}^{N+1} d_k \big(-Z_\mu\big)^k$$

$$q_\mu^2 - 4 p r_\mu = r \qquad \alpha_\mu q_\mu - 2\beta_\mu p = s$$

$$d_{N+1} = \begin{cases} 0 & p \neq 0 \\ -\frac{2}{N^2-1}\frac{\Lambda}{r} & p = 0, r \neq 0 \end{cases}$$

$$d_N = \begin{cases} \frac{1}{(N-1)(2N-1)}\frac{\Lambda}{p} & p \neq 0 \\ \text{const.} \in \mathbb{R} & p = 0, r \neq 0 \end{cases}$$

$$d_{N-1} = \text{const.} \in \mathbb{R}$$

$$\vdots$$

$$d_1 = \text{const.} \in \mathbb{R}$$

$$\boxed{Z_\mu = p x_\mu^2 + q_\mu x_\mu + r_\mu \qquad p \neq 0}$$

New coordinates $\check{x}_\mu$, $\check{\psi}_j$ (linear transformation):

$$\boldsymbol{g} = \sum_\mu \left[\frac{\check{U}_\mu}{\check{X}_\mu} \big(\mathbf{d}\check{x}_\mu\big)^{\mathbf{2}} + \frac{\check{X}_\mu}{\check{U}_\mu} \Big(\sum_j \check{A}_\mu^{(j)} \mathbf{d}\check{\psi}_j \Big)^{\mathbf{2}} \right]$$

where

$$\check{Z}_\mu = \check{x}_\mu^2 \qquad\qquad \check{X}_\mu = \sum_{k=0}^{N} \check{c}_k \big(-\check{x}_\mu^2\big)^k + \check{b}_\mu \check{x}_\mu$$

$$\check{c}_N = \frac{\Lambda}{(N-1)(2N-1)}$$

$\rightarrow$ Kerr–NUT–(A)dS spacetime

$$\boxed{Z_\mu = q x_\mu + r_\mu \qquad q \neq 0}$$

New coordinates $\check{x}_\mu,\ \check{\psi}_j$ (linear transformation):

$$\boldsymbol{g} = \sum_\mu \left[\frac{\check{U}_\mu}{\check{X}_\mu} \big(\mathbf{d}\check{x}_\mu\big)^{\mathbf{2}} + \frac{\check{X}_\mu}{\check{U}_\mu} \Big(\sum_j \check{A}^{(j)}_\mu \mathbf{d}\check{\psi}_j \Big)^{\mathbf{2}} \right]$$

where

$$\check{Z}_\mu = \check{x}_\mu \qquad \check{X}_\mu = \sum_{k=1}^{N+1} \check{c}_k \big(-\check{x}_\mu\big)^k + \check{b}_\mu$$

$$\check{c}_{N+1} = -\frac{2\Lambda}{N^2 - 1}$$

- scaling limit of the Kerr–NUT–(A)dS spacetime[6]
- Kähler 2-form $\boldsymbol{\Omega} = \sum_\mu \boldsymbol{e}^\mu \wedge \hat{\boldsymbol{e}}^\mu$
- Euclidean signature

[6] D. Kubizňák, Phys. Lett. **B675**, 110 (2009), eprint: `0902.1999[hep-th]`.

Manifold ($D = \tilde{D} + \bar{D}$):

$$M = \tilde{M} \times \bar{M}$$

Metric:

$$\boldsymbol{g} = \tilde{\boldsymbol{g}} + \tilde{w}^2 \bar{\boldsymbol{g}}$$

Ricci tensor:

$$\mathbf{Ric} = \tilde{\mathbf{Ric}} - \frac{\bar{D}}{\tilde{w}} \tilde{\boldsymbol{H}} + \bar{\mathbf{Ric}} - \tilde{w}^2 \left(\frac{\tilde{\mathcal{H}}}{\tilde{w}} + (\bar{D} - 1) \tilde{\lambda}^2 \right) \bar{\boldsymbol{g}}$$

where

$$\tilde{\boldsymbol{H}} = \tilde{\boldsymbol{\nabla}} \tilde{\boldsymbol{\nabla}} \tilde{w} \qquad \tilde{\mathcal{H}} = \tilde{\boldsymbol{g}}^{ab} \tilde{\boldsymbol{H}}_{ab} \qquad \tilde{\boldsymbol{\lambda}} = \mathbf{d} \ln \tilde{w} \qquad \tilde{\lambda}^2 = \tilde{\boldsymbol{g}}^{ab} \tilde{\boldsymbol{\lambda}}_a \tilde{\boldsymbol{\lambda}}_b$$

Rank-two Killing tensors:[7]

$$\tilde{\boldsymbol{k}} + \frac{\tilde{A}}{\tilde{w}^2} \bar{\boldsymbol{g}}^{-1} \qquad \bar{\boldsymbol{k}}$$

if $\tilde{\boldsymbol{k}}$, $\bar{\boldsymbol{k}}$ are Killing tensors and $\tilde{\boldsymbol{\nabla}}^a \tilde{A} = 2\big(\tilde{A}\, \tilde{\boldsymbol{g}}^{ab} - \tilde{\boldsymbol{k}}^{ab}\big) \tilde{\boldsymbol{\lambda}}_b$

[7] P. Krtouš et al., Phys. Rev. D **93**, 024057 (2015), eprint: `1508.02642[gr-qc]`.

Warped Klein–Gordon separable metric:

$$\boxed{\boldsymbol{g} = \tilde{\boldsymbol{g}} + \tilde{w}^2 \bar{\boldsymbol{g}}}$$

where

$$\boxed{\tilde{w}^2 = \tilde{Z}_1 \dots \tilde{Z}_{\tilde{N}}}$$

and

$\tilde{\boldsymbol{g}}$, $\bar{\boldsymbol{g}}$ are Klein–Gordon simple separable metrics

Killing vectors:

$$^{\tilde{j}}\boldsymbol{l} = {}^{\tilde{j}}\tilde{\boldsymbol{l}} \qquad ^{\tilde{N}+\bar{j}}\boldsymbol{l} = {}^{\bar{j}}\bar{\boldsymbol{l}}$$

Rank-two Killing tensors:

$$^{\tilde{j}}\boldsymbol{k} = {}^{\tilde{j}}\tilde{\boldsymbol{k}} + \frac{\tilde{A}^{(\tilde{j})}}{\tilde{w}^2}\bar{\boldsymbol{g}}^{-1} \qquad ^{\tilde{N}+\bar{j}}\boldsymbol{k} = {}^{\bar{j}}\bar{\boldsymbol{k}}$$

where

$$A^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j \\ \nu_1<\dots<\nu_j}} Z_{\nu_1} \dots Z_{\nu_j}$$

Operators:

$$\mathsf{K}_j = -\boldsymbol{\nabla}_a {}^j\boldsymbol{k}^{ab} \boldsymbol{\nabla}_b \qquad \mathsf{L}_j = -i \, {}^j\boldsymbol{l}^a \boldsymbol{\nabla}_a$$

Operators in coordinates $\tilde{x}_{\tilde{\mu}},\, \tilde{\psi}_{\tilde{j}},\, \bar{x}_{\bar{\mu}},\, \bar{\psi}_{\bar{j}}$:

$$\mathsf{K}_{\tilde{j}} = \sum_{\tilde{\mu}} \frac{\tilde{A}^{(\tilde{j})}_{\tilde{\mu}}}{\tilde{U}_{\tilde{\mu}}} \left[-\frac{1}{\tilde{Z}_{\tilde{\mu}}^{\frac{\tilde{N}}{2}}} \frac{\partial}{\partial \tilde{x}_{\tilde{\mu}}} \tilde{Z}_{\tilde{\mu}}^{\frac{\tilde{N}}{2}} \tilde{X}_{\tilde{\mu}} \frac{\partial}{\partial \tilde{x}_{\tilde{\mu}}} + \frac{1}{\tilde{X}_{\tilde{\mu}}} \left[\sum_{\tilde{k}} \left(-\tilde{Z}_{\tilde{\mu}}\right)^{\tilde{N}-1-\tilde{k}} \mathsf{L}_{\tilde{k}} \right]^2 \right] + \frac{\tilde{A}^{(\tilde{j})}}{\tilde{A}^{(\tilde{N})}} \mathsf{K}_{\tilde{N}} \qquad \mathsf{L}_{\tilde{k}} = -i \frac{\partial}{\partial \tilde{\psi}_{\tilde{k}}}$$

$$\mathsf{K}_{\tilde{N}+\bar{j}} = \sum_{\bar{\mu}} \frac{\bar{A}^{(\bar{j})}_{\bar{\mu}}}{\bar{U}_{\bar{\mu}}} \left[-\frac{\partial}{\partial \bar{x}_{\bar{\mu}}} \bar{X}_{\bar{\mu}} \frac{\partial}{\partial \bar{x}_{\bar{\mu}}} + \frac{1}{\bar{X}_{\bar{\mu}}} \left[\sum_{\bar{k}} \left(-\bar{Z}_{\bar{\mu}}\right)^{\bar{N}-1-\bar{k}} \mathsf{L}_{\tilde{N}+\bar{k}} \right]^2 \right] \qquad \mathsf{L}_{\tilde{N}+\bar{k}} = -i \frac{\partial}{\partial \bar{\psi}_{\bar{k}}}$$

Mutual commutation:

$$[\mathsf{L}_j \,, \mathsf{L}_k] = 0 \qquad [\mathsf{L}_j \,, \mathsf{K}_k] = 0 \qquad [\mathsf{K}_j \,, \mathsf{K}_k] = 0$$

$$\Downarrow \quad \text{eigenfunctions } \phi$$

$$\mathsf{K}_j \phi = \Xi_j \phi$$
$$\mathsf{L}_j \phi = \Psi_j \phi$$

where Ξ_j and Ψ_j are eigenvalues

Warped Klein–Gordon separable spacetimes
Separability of the Klein–Gordon equation

Separability ansatz:

$$\phi = \prod_{\tilde\mu} \tilde R_{\tilde\mu} \prod_{\bar\mu} \bar R_{\bar\mu} \prod_{\tilde k} \exp\left(i\Psi_{\tilde k}\tilde\psi_{\tilde k}\right) \prod_{\bar k} \exp\left(i\Psi_{\tilde N+\bar k}\bar\psi_{\bar k}\right)$$

$\rightarrow \tilde R_{\tilde\mu} = \tilde R_{\tilde\mu}(\tilde x_{\tilde\mu}),\ \tilde R_{\tilde\mu} = \bar R_{\bar\mu}(\bar x_{\bar\mu})$ satisfy ordinary differential equations

$$\left(\tilde Z_{\tilde\mu}^{\frac{\bar N}{2}} \tilde X_{\tilde\mu} \tilde R'_{\tilde\mu}\right)' + \left(\breve{\tilde\Xi}_{\tilde\mu} - \frac{\breve{\tilde\Psi}^2_{\tilde\mu}}{\tilde X_{\tilde\mu}} + \frac{\Xi_{\tilde N}}{\tilde Z_{\tilde\mu}}\right) \tilde Z_{\tilde\mu}^{\frac{\bar N}{2}} \tilde R_{\tilde\mu} = 0$$

$$\left(\bar X_{\bar\mu} \bar R'_{\bar\mu}\right)' + \left(\breve{\bar\Xi}_{\bar\mu} - \frac{\breve{\bar\Psi}^2_{\bar\mu}}{\bar X_{\bar\mu}}\right) \bar R_{\bar\mu} = 0$$

where

$$\breve{\tilde\Xi}_{\tilde\mu} = \sum_{\tilde k} \Xi_{\tilde k}\left(-\tilde Z_{\tilde\mu}\right)^{\tilde N-1-\tilde k} \qquad \breve{\bar\Xi}_{\bar\mu} = \sum_{\bar k} \Xi_{\tilde N+\bar k}\left(-\bar Z_{\bar\mu}\right)^{\bar N-1-\bar k}$$

$$\breve{\tilde\Psi}_{\tilde\mu} = \sum_{\tilde k} \Psi_{\tilde k}\left(-\tilde Z_{\tilde\mu}\right)^{\tilde N-1-\tilde k} \qquad \breve{\bar\Psi}_{\bar\mu} = \sum_{\bar k} \Psi_{\tilde N+\bar k}\left(-\bar Z_{\bar\mu}\right)^{\bar N-1-\bar k}$$

Vacuum Einstein equations:

$$\tilde{\mathbf{Ric}} = \frac{\bar{D}}{\tilde{w}}\tilde{\boldsymbol{H}} + \frac{2\Lambda}{D-2}\tilde{\boldsymbol{g}} \qquad \bar{\mathbf{Ric}} = \underbrace{\tilde{w}^2\left(\frac{\tilde{\mathcal{H}}}{\tilde{w}} + (\bar{D}-1)\tilde{\lambda}^2 + \frac{2\Lambda}{D-2}\right)}_{\Upsilon=\text{const.}}\bar{\boldsymbol{g}}$$

Solution of tilded equations:

$$\tilde{X}_{\tilde{\mu}} = \sum_{\tilde{k}=0}^{\tilde{N}} \tilde{c}_{\tilde{k}}\left(-\tilde{x}_{\mu}^2\right)^{\tilde{k}} + \frac{\tilde{b}_{\tilde{\mu}}}{\tilde{x}_{\tilde{\mu}}^{2\bar{N}-1}} \qquad \tilde{c}_{\tilde{N}} = \frac{\Lambda}{(2N-1)(N-1)} \qquad \tilde{c}_0 = \frac{\Upsilon}{2\bar{N}-1}$$

Solutions of barred equation:

$$\bar{X}_{\bar{\mu}} = \sum_{\bar{k}=0}^{\bar{N}} \bar{c}_{\bar{k}}\left(-\bar{x}_{\bar{\mu}}^2\right)^{\bar{k}} + \bar{b}_{\bar{\mu}}\bar{x}_{\bar{\mu}} \qquad \bar{c}_{\bar{N}} = \frac{\Upsilon}{2\bar{N}-1}$$

$\rightarrow$ limit of vanishing rotations of the Kerr–NUT–(A)dS spacetime[8]

$$\bar{X}_{\bar{\mu}} = \sum_{\bar{k}=1}^{\bar{N}+1} \bar{c}_{\bar{k}}\left(-\bar{x}_{\bar{\mu}}\right)^{\bar{k}} + \bar{b}_{\bar{\mu}} \qquad \bar{c}_{\bar{N}+1} = -\frac{2\Upsilon}{\bar{N}+1}$$

[8] P. Krtouš et al., Class. Quantum Grav. **33**, 115016 (2016), eprint: `1511.02536[hep-th]`.

'Charged' classical observables:

$$
\begin{aligned}
{}^q K_j &= \big(\boldsymbol{p}_a - q\boldsymbol{A}_a\big)\, {}^j\boldsymbol{k}^{ab} \big(\boldsymbol{p}_b - q\boldsymbol{A}_b\big) \\
{}^q L_j &= {}^j\boldsymbol{l}^a \boldsymbol{p}_a
\end{aligned}
$$

here $\boldsymbol{A}$ is a vector potential, q is a charge, ${}^q K_0 = H$

$\forall\ {}^q K_j,\ {}^q L_k$ mutually Poisson-commute $\iff$
$$
\begin{aligned}
\pounds_{{}^j\boldsymbol{l}}\, \boldsymbol{A} &= 0 \\
{}^j\boldsymbol{k}^{c(a}\, \boldsymbol{F}_{cd}\, {}^k\boldsymbol{k}^{b)d} &= 0
\end{aligned}
$$

Solution:

$$
\boxed{\boldsymbol{A} = \sum_\mu \frac{f_\mu}{\sqrt{U_\mu X_\mu}}\, \hat{\boldsymbol{e}}^\mu}
$$

where $f_\mu = f_\mu(x_\mu)$

Maxwell tensor $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$:

$$
\boldsymbol{F} = \sum_\nu \left(\frac{f'_\nu}{U_\nu} + 2x_\nu \sum_{\substack{\mu \\ \mu\neq\nu}} \frac{1}{U_\mu} \frac{f_\mu - f_\nu}{x_\mu^2 - x_\nu^2} \right) \boldsymbol{e}^\nu \wedge \hat{\boldsymbol{e}}^\nu
$$

Special cases:

- aligned with the primary Killing vector[9] $\boldsymbol{\xi} = \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\psi_0}\right)^\flat$

$$\boldsymbol{A} = e\,\boldsymbol{\xi} \qquad f_\mu = e\,X_\mu$$

  where e is a constant parameter

- solution of source-free Maxwell's equations[10]

$$\boldsymbol{A} = \sum_\mu \frac{e_\mu x_\mu}{\sqrt{U_\mu X_\mu}}\,\hat{\boldsymbol{e}}^\mu \qquad f_\mu = e_\mu x_\mu$$

  where e_μ are constant parameters

[9] V. P. Frolov and P. Krtouš, Phys. Rev. D **83**, 024016 (2011), eprint: `1010.2266[hep-th]`.
M. Cariglia et al., Phys. Rev. D **87**, 064003 (2013), eprint: `1211.4631[gr-qc]`.
[10] P. Krtouš, Phys. Rev. D **76**, 084035 (2007), eprint: `0707.0002[hep-th]`.

Conserved quantities:

$$ {}^{q}K_j = \Xi_j \qquad {}^{q}L_j = \Psi_j $$

$$ \Big\Downarrow \; \boldsymbol{p} \to \boldsymbol{\nabla} S $$

$$ \boxed{\begin{gathered} \big(\boldsymbol{\nabla}_a S - q\boldsymbol{A}_a\big)\, {}^{j}\boldsymbol{k}^{ab} \big(\boldsymbol{\nabla}_b S - q\boldsymbol{A}_b\big) = \Xi_j \\ {}^{j}\boldsymbol{l}^{a}\,\boldsymbol{\nabla}_a S = \Psi_j \end{gathered}} $$

Separability ansatz:

$$ S = \sum_\mu S_\mu + \sum_k \Psi_k \psi_k $$

$\to S_\mu = S_\mu(x_\mu)$ satisfy ordinary differential equations

$$ \big(S'_\mu\big)^2 = \frac{\breve{\Xi}_\mu}{X_\mu} - \frac{1}{X_\mu^2}\big(\breve{\Psi}_\mu - q f_\mu\big)^2 $$

where

$$ \breve{\Psi}_\mu = \sum_k \Psi_k \big(-x_\mu^2\big)^{N-1-k} \qquad \breve{\Xi}_\mu = \sum_k \Xi_k \big(-x_\mu^2\big)^{N-1-k} $$

'Charged' field operators ($\boldsymbol{p} \to -i\boldsymbol{\nabla}$):

$$
\begin{aligned}
{}^{q}\mathsf{K}_j &= -\big[\boldsymbol{\nabla}_a - iq\boldsymbol{A}_a\big]\,{}^{j}\boldsymbol{k}^{ab}\big[\boldsymbol{\nabla}_b - iq\boldsymbol{A}_b\big] \\
{}^{q}\mathsf{L}_j &= -i\,{}^{j}\boldsymbol{l}^{a}\boldsymbol{\nabla}_a
\end{aligned}
$$

$\forall\ {}^{q}\mathsf{K}_j,\ {}^{q}\mathsf{L}_k$ mutually commute $\iff$

$$
\begin{aligned}
\pounds_{{}^{j}\boldsymbol{l}}\,\boldsymbol{A} &= 0 \\
{}^{j}\boldsymbol{k}^{c(a}\boldsymbol{F}_{cd}\,{}^{k}\boldsymbol{k}^{b)d} &= 0 \\
\boldsymbol{\nabla}_a\Big({}^{j}\boldsymbol{k}^{ab}\boldsymbol{\nabla}_b\big(\boldsymbol{\nabla}_c({}^{k}\boldsymbol{k}^{cd}\boldsymbol{A}_d)\big) - (j \leftrightarrow k)\Big) &= 0
\end{aligned}
$$

Solution:

$$
\boxed{\boldsymbol{A} = \sum_{\mu} \frac{f_\mu}{\sqrt{U_\mu X_\mu}}\,\hat{\boldsymbol{e}}^{\mu}}
$$

anomalous condition is automatically satisfied

Eigenfunctions ϕ:

$$\boxed{\begin{aligned} {}^q\mathsf{K}_j\phi &= \Xi_j\phi \\ {}^q\mathsf{L}_j\phi &= \Psi_j\phi \end{aligned}}$$

Separability ansatz:

$$\phi = \prod_\mu R_\mu \prod_k \exp\big(i\Psi_k\psi_k\big)$$

$\to R_\mu = R_\mu(x_\mu)$ satisfy ordinary differential equations

$$\big(X_\mu R_\mu'\big)' + \left(\breve{\Xi}_\mu - \frac{1}{X_\mu}\big(\breve{\Psi}_\mu - q f_\mu\big)^2\right) R_\mu = 0$$

where

$$\breve{\Psi}_\mu = \sum_k \Psi_k\big(-x_\mu^2\big)^{N-1-k} \qquad \breve{\Xi}_\mu = \sum_k \Xi_k\big(-x_\mu^2\big)^{N-1-k}$$

Conclusions

Klein–Gordon simple separable spacetimes:

- generalization of Carter's ansatz to higher dimensions
- separability of the Klein–Gordon equation
- solutions of the Einstein equations (Kerr–NUT–(A)dS, scaling limit)

Warped Klein–Gordon separable spacetimes:

- separability of the Klein–Gordon equation
- solutions of the Einstein equations (limit of vanishing rotations, ?)

Weak electromagnetic field in the Kerr–NUT–(A)dS:

- charged Hamilton–Jacobi equation
- charged Klein–Gordon equation

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