Combinatorial Properties of MAD families

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Introduction

An infinite family $\mathcal{A} \subseteq [\omega]^{\omega}$ is *almost disjoint* (*AD*) if the intersection of any two different of its elements is finite and a *MAD* family is a maximal AD family. Almost disjoint families and MAD families have many applications in forcing theory (almost disjoint coding), in topology (Mrówka-Isbell spaces) and in functional analysis (in the study of masas in the Calkin algebra).

It is easy to prove that the Axiom of Choice implies the existence of MAD families, however constructing MAD families with special combinatorial or topological properties has become a very difficult task without the aid of additional hypothesis beyond ZFC. On the other hand, constructing models of set theory where there are no certain kinds of MAD families has also been very difficult. We would like to mention some important examples regarding the existence or non existence of special MAD families:

1. (Simon) There is a MAD family which can be partitioned into two nowhere MAD families.
2. (Mrówka) There is a MAD family for which its Ψ-space has a unique compactification.
3. (Raghavan) There is a van Douwen MAD family.
4. (Raghavan) There is a model with no strongly separable almost disjoint families.
5. (Shelah, $\mathsf{ZFC} + \varepsilon$) There is a completely separable MAD family.
6. (Mildenberger, Raghavan, Steprāns, $\mathsf{ZFC} + \delta$) There is a weakly tight MAD family.

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- A recursive construction of lenght $\mathfrak{c}$. This method often requires certain cardinal invariant to be big.
- A short recursion of lenght ω_1. This is usually done with a guessing axiom (like a parametrized diamond).
- "The *SANE player* method". This is a very clever method to build a MAD families by placing its elements wisely on a binary tree. This method has been used to construct (under certain hypothesis) a completely separable MAD family and a weakly tight family.

We will need the following definitions:

- If $\mathcal{I}$ is an ideal, we denote by $\mathcal{I}^+$ the subsets of ω that are not in $\mathcal{I}$.
- An ideal $\mathcal{I}$ is *tall* if for every $X \in [\omega]^\omega$ there is $A \in \mathcal{I}$ such that $A \cap X$ is infinite.
- If $\mathcal{A}$ is a an AD family then $\mathcal{I}(\mathcal{A})$ will denote the ideal generated by $\mathcal{A}$ and the finite sets.
- If $\mathcal{A}$ is MAD then $X \in \mathcal{I}(\mathcal{A})^+$ if and only if the set $\{A \in \mathcal{A} \mid |A \cap X| = \omega\}$ is infinite.

Definition

Let $\mathcal{I}$ be an ideal.

1. A tree $T \subseteq \omega^{<\omega}$ is called an $\mathcal{I}^{+}$-*branching tree* if for every $s \in T$ we have that $suc_T(s) = \{n \mid s^\frown \{n\} \in T\} \in \mathcal{I}^{+}$.
2. $\mathcal{I}$ is $+$-*Ramsey* if every $\mathcal{I}^{+}$-branching tree has a branch in $\mathcal{I}^{+}$ (i.e there is branch f such that $im(f) \in \mathcal{I}^{+}$).

Theorem (Hrušák)

1. *Every ideal generated by less than $cov(\mathcal{M})$ sets is $+$-Ramsey.*
2. *There is an ideal generated by $cov(\mathcal{M})$ that is not $+$-Ramsey.*
3. *There is an ideal generated by ω_1 sets that is $+$-Ramsey.*
4. *There is a MAD family that is not $+$-Ramsey.*

He then asked the following:

Problem (Hrušák)

Is there a $+$-Ramsey MAD family?

It was known the answer is positive under $\mathfrak{a} < cov(\mathcal{M})$, $\mathfrak{b} = \mathfrak{c}$ or $\diamondsuit(\mathfrak{b})$. However, such families can be constructed without any additional axiom:

Theorem (G.)

There is a $+$-Ramsey MAD family.

The proof is divided into two cases depending on the relationship between $\mathfrak{a}$ and $\mathfrak{s}$.

Definition

Let $\mathcal{A}$ be a MAD family and $\mathbb{P}$ a partial order. Then $\mathcal{A}$ is $\mathbb{P}$-*indestructible* if $\mathcal{A}$ is still a MAD family after forcing with $\mathbb{P}$.

Destroying a MAD family means destroying its maximality. The indestructibility of MAD families (and ideals in general) is a very interesting topic with many important questions still unsolved.

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- Any MAD family containing a perfect set is Sacks destructible, hence there are Sacks destructible MAD families.

Theorem

Every Miller indestructible MAD family is $+$-Ramsey.

The previous result improves a result of Hrušák. It is known (Hrušák? Brendle and Yatabe? Folklore?) that every MAD family of size less than $\mathfrak{d}$ is Miller indestructible, hence we conclude the following:

Corollary

Every MAD family of size less than $\mathfrak{d}$ is $+$-Ramsey. In particular, if $\mathfrak{a} < \mathfrak{s}$ then there is a $+$-Ramsey MAD family.

A family $\mathcal{S} \subseteq [\omega]^\omega$ is called a (ω, ω)-*splitting family* if for every $\{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$ there is $S \in \mathcal{S}$ such that there are infinitely many $n \in \omega$ for which $X_n \cap S$ is infinite and there are infinitely many $m \in \omega$ for which $X_m \cap (\omega \setminus S)$ is infinite.

Theorem (Mildenberger, Raghavan, Steprāns)

The minimum size of a (ω, ω)-splitting family is $\mathfrak{s}$.

The key feature of (ω, ω)-splitting families is that they are able to split any $\mathcal{I}(\mathcal{A})$ positive set into two $\mathcal{I}(\mathcal{A})$ positive sets (for any AD family $\mathcal{A}$):

Lemma (Raghavan, Steprāns)

If $\mathcal{S}$ is a (ω, ω)-splitting family then for every AD family $\mathcal{A}$ and every $X \in \mathcal{I}(\mathcal{A})^+$ there is $S \in \mathcal{S}$ such that both $X \cap S$ and $X \cap (\omega - S)$ are elements of $\mathcal{I}(\mathcal{A})^+$.

A very brief (and most likely incomprehensible) explanation of Shelah's construction of a completely separable MAD family under $\mathfrak{s} \leq \mathfrak{a}$:

Fix $\mathcal{S} = \{S_\alpha \mid \alpha < \mathfrak{s}\}$ a (ω, ω)-splitting family and $[\omega]^\omega = \{X_\alpha \mid \alpha < \mathfrak{c}\}$. We will recursively build $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha \mid \alpha < \mathfrak{c}\}$ with the following properties:

1. $\mathcal{A}$ is almost disjoint.
2. $\{\sigma_\alpha \mid \alpha < \mathfrak{c}\} \subseteq 2^{<\mathfrak{s}}$.
3. If $\alpha \neq \beta$ then $\sigma_\alpha \neq \sigma_\beta$.
4. If $\xi < dom(\sigma_\alpha)$ then $A_\alpha \subseteq^* S_\xi^{\sigma_\alpha(\xi)}$.
5. If $X_\alpha \in \mathcal{I}(\mathcal{A}_{<\alpha})^+$ then $A_\alpha \subseteq X_\alpha$.

A key point of the construction is that while at the end we will end with a MAD family, every $\mathcal{A}_{<\alpha}$ is nowhere MAD (even if $\mathfrak{a} < \mathfrak{c}$).

For the construction of the $+$-Ramsey MAD family we need a notion for "splitting Miller trees":

Let p be a Miller tree. Given $f \in [p]$ we define
$Sp(p, f) = \{f(n) \mid f \upharpoonright n \in Split(p)\}$ and $[p]_{split} = \{Sp(p, f) \mid f \in [p]\}$.

Definition

Let p be a Miller tree and $S \in [\omega]^{\omega}$. We say S *tree-splits* p if there are Miller trees $q_0, q_1 \leq p$ such that $[q_0]_{split} \subseteq [S]^{\omega}$ and $[q_1]_{split} \subseteq [\omega - S]^{\omega}$.
$\mathcal{S}$ is a *Miller tree-splitting family* if every Miller tree is tree-splitted by some element of $\mathcal{S}$.

It is easy to see that every Miller-tree splitting family is a splitting family and it is also easy to see that every ω-splitting family is a Miller-tree splitting family.

Theorem

There is a Miller-tree splitting family of size $\mathfrak{s}$.

Constructing a $+$-Ramsey MAD family under $\mathfrak{s} \leq \mathfrak{a}$:
Fix $\mathcal{S} = \{S_\alpha \mid \alpha < \mathfrak{s}\}$ an (ω, ω)-splitting family that is also a Miller-tree splitting family. Let $\{L, R\}$ be a partition of the limit ordinals smaller than $\mathfrak{c}$ such that both L and R have size continuum. Enumerate $\{X_\alpha \mid \alpha \in L\}$ all infinite subsets of ω and $\{T_\alpha \mid \alpha \in R\}$ all subtrees of $\omega^{<\omega}$. We will recursively construct $\mathcal{A} = \{A_\xi \mid \xi < \mathfrak{c}\}$ and $\{\sigma_\xi \mid \xi < \mathfrak{c}\}$ as follows:

1. $\mathcal{A}$ is an AD family and $\sigma_\alpha \in 2^{<\mathfrak{s}}$ for every $\alpha < \mathfrak{c}$.
2. If $\sigma_\alpha \in 2^\beta$ and $\xi < \beta$ then $A_\alpha \subseteq^* S_\xi^{\sigma_\alpha(\xi)}$.
3. If $\alpha \neq \beta$ then $\sigma_\alpha \neq \sigma_\beta$.
4. If $\delta \in L$ and $X_\delta \in \mathcal{I}(\mathcal{A}_\delta)^+$ then $A_{\delta+n} \subseteq X_\delta$ for every $n \in \omega$.
5. If $\delta \in R$ and T_δ is an $\mathcal{I}(\mathcal{A}_\delta)^+$-branching tree then there is $f \in [T_\delta]$ such that $A_{\delta+n} \subseteq im(f)$ for every $n \in \omega$.

The important point is that we will always be able to find a positive branch for an $\mathcal{I}(\mathcal{A}_\delta)^+$-branching tree and then we seal it by adding countably many of its subsets to our almost disjoint family.

Thank you very much!