Topological partition relations for ω^2

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Workshop on Set Theory and its applications in Topology

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Partition relations for ordinal spaces

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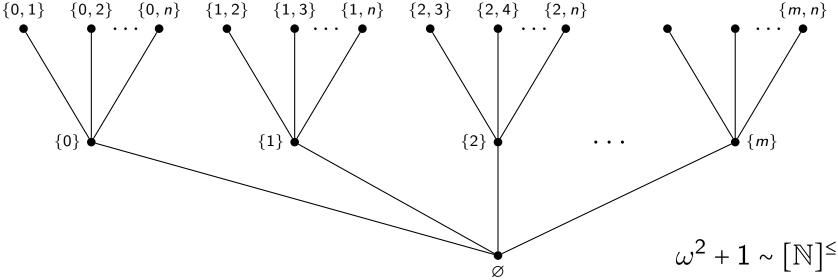
In order to get $\mathcal{H} \sim \omega^2 + 1$ we chose $\mathcal{H}$ which *behaves* as $[\mathbb{N}]^{\leq 2}$.

Copies of $\omega^2 + 1$



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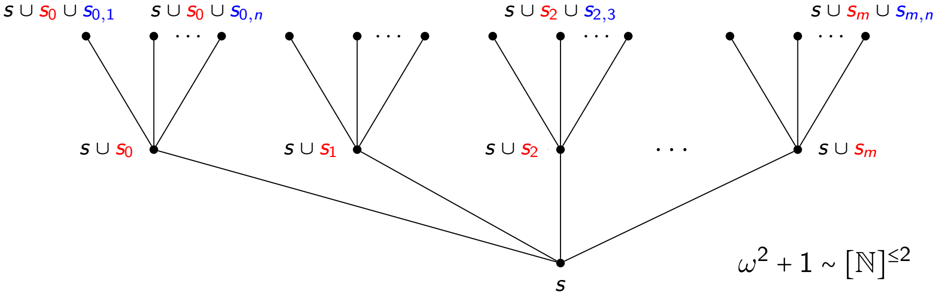

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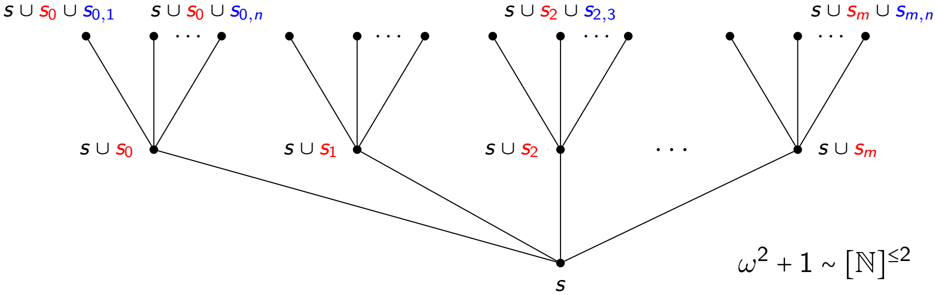

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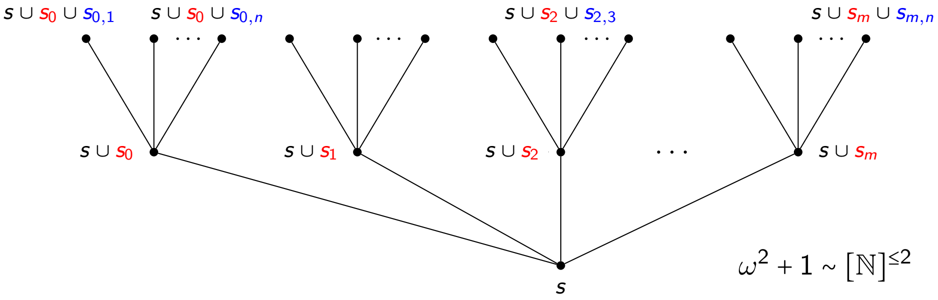

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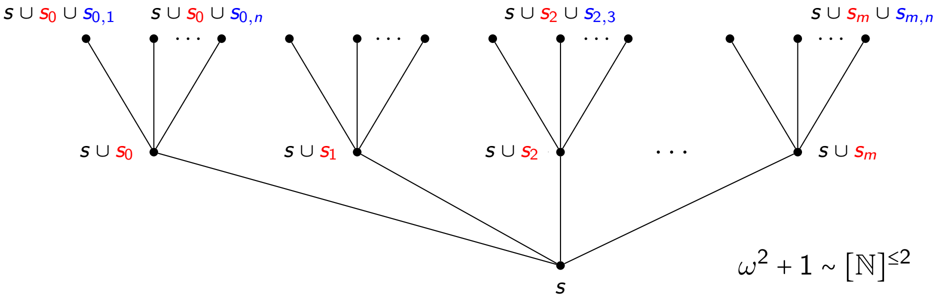

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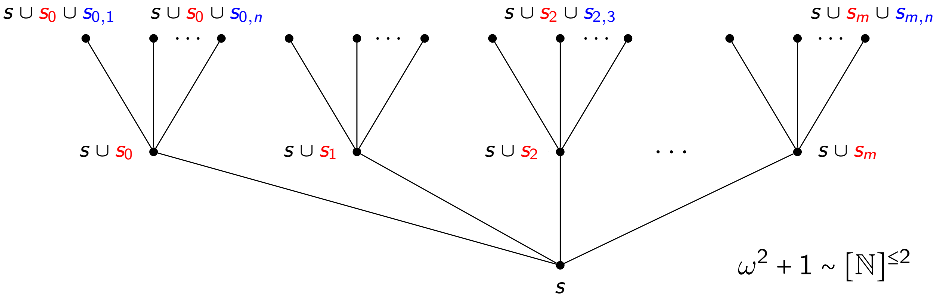

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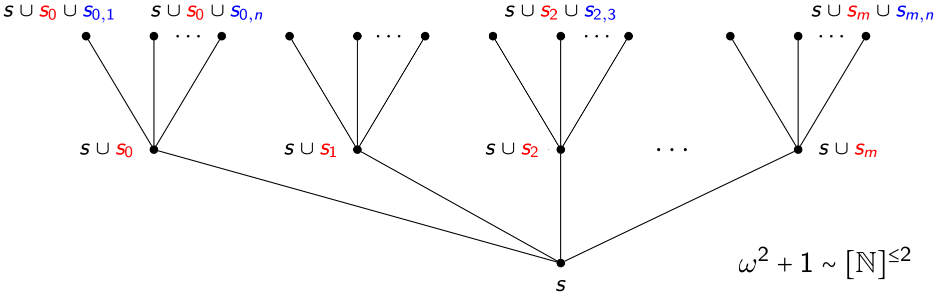

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- any two elements intersect just in a root.

For every $\omega^2 + 1 \sim \mathcal{A} \subseteq [\mathbb{N}]^{<\infty}$ there is $\mathcal{H} \subseteq \mathcal{A}$ as before.

Configurations of the pairs in $\mathcal{H}$

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Theorem (Todorcevic)

There is a coloring $\mathrm{osc} : \left[[\mathbb{N}]^{<\infty} \right]^2 \longrightarrow \mathbb{N}$ *such that given* $\mathcal{A} \subseteq [\mathbb{N}]^{<\infty}$ *and* $n < \omega$*, if* $\mathcal{A}$ *is homeomorphic to* $\omega^n + 1$ *then* $\{1, 2 \ldots, 2n\} \subseteq \mathrm{osc}''[\mathcal{A}]^2$*.*

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$\operatorname{osc}''[\mathcal{H}]^2 = \{1, 2, 3, 4\}$ for $\mathcal{H}$ as before.

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$$\mathrm{osc}''[\mathcal{H}]^2 = \{1, 2, 3, 4\} \text{ for } \mathcal{H} \text{ as before.}$$

Fact: Given $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ of topological type $\alpha > \omega^2$, any partition $[\mathcal{F}]^2 = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{\ell-1}$ and $\mathcal{H} \subseteq \mathcal{F}$ as before, then for every $i \in \{1, 2, 3, 4\}$ there is $k_i < \ell$ (hopefully unique) satisfying

$$u, v \in \mathcal{H} \left(\mathrm{osc}(\{u, v\}) = i \longrightarrow \{u, v\} \in \mathcal{F}_{k_i} \right).$$

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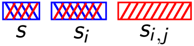

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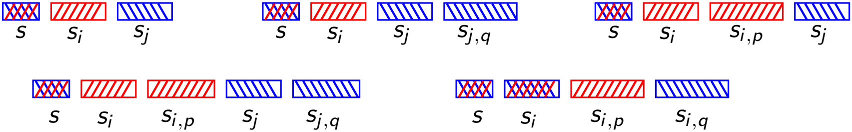

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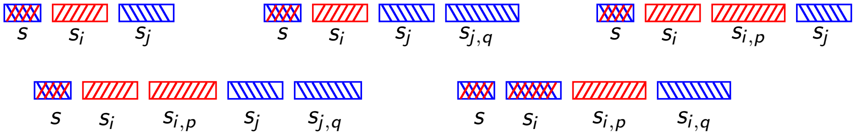

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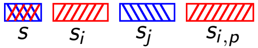

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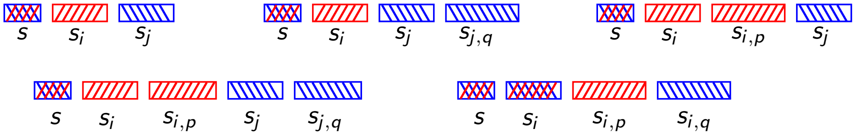

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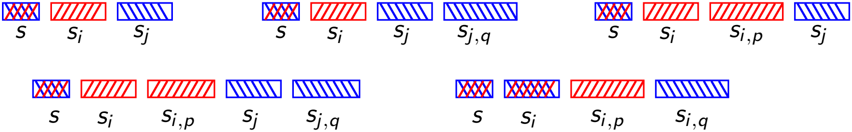

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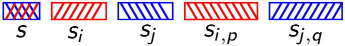

Some optimal partition relations

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Theorem 1

$\omega^\omega + 1 \to (\text{top } \omega^2 + 1)^2_{\ell,6}$ *for every* $\ell > 1$.

Idea of the proof

Fix $\ell > 1$ and $\left[\,\overline{\mathcal{S}}\,\right]^2 = \mathcal{A}_0 \cup \cdots \cup \mathcal{A}_{\ell-1}$, where $\mathcal{S} = \{s \in \mathrm{FIN} : |s| = \min(s) + 1\}$.

Idea of the proof

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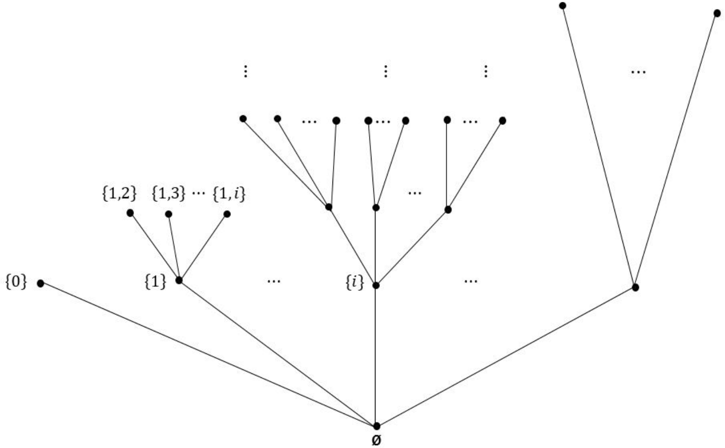
$$\overline{\mathcal{S}} = \{s \in \mathrm{FIN} : |s| \leq \min(s) + 1\} \sim \omega^\omega + 1.$$



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$$\mathcal{H} = \{\emptyset\} \cup \{s_i \cap (x_i + 1) : i < \omega\} \cup \{s_{i,j} \cap (y_{i,j} + 1) : i < j < \omega\}$$

with the help of an infinite set $M \in [\mathbb{N}]^\infty$ and subsets $\varphi(s) \subseteq s$ for every $s \in \mathcal{S} \upharpoonright M = \{s \in \mathcal{S} : s \subseteq M\}$.

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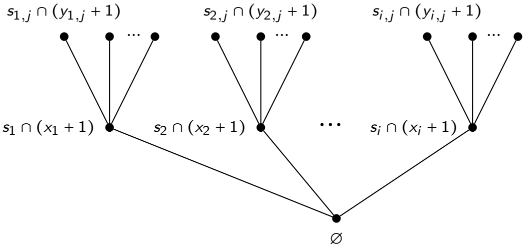

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- $x_i \in \varphi(s_i) \subseteq s_i$,
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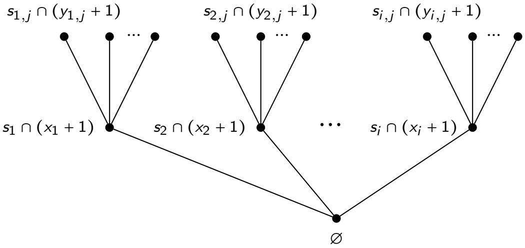

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We will control de colors in $[\mathcal{H}]^2$ by carefully choosing M and φ.

Ramsey and Nash-Williams

$$\left[\,\overline{\mathcal{S}}\,\right]^2 = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{\ell-1}$$

For pairs with oscillation 1**:** We color each $s \in \mathcal{S}$ into ℓ colors by $x \mapsto i$ iff $\{\varnothing, s \cap (x+1)\} \in \mathcal{A}_i$. Then, we get $\varphi_1(s) \subseteq s$ for each $s \in \mathcal{S}$, $M_1 \in [\mathbb{N}]^\infty$ and $i_1 < \ell$ such that:

$$\{\varnothing, s \cap (x+1)\} \in \mathcal{A}_{i_1} \quad \forall\, x \in \varphi_1(s)\ \forall\, s \in \mathcal{S} \restriction M_1.$$

Analogously, by colorings each $[\varphi_1(s)]^2$ into ℓ by $\{x, y\} \mapsto i$ iff $\{s \cap (x+1), s \cap (y+1)\} \in \mathcal{A}_i$, we get $\varphi_2(s) \subseteq \varphi_1(s)$ for each $s \in \mathcal{S}$, an infinite set $M_2 \subseteq M_1$ and $i_2 < \ell$ such that:

$$\{s \cap (x+1), s \cap (y+1)\} \in \mathcal{A}_{i_2} \quad \forall\, x, y \in \varphi_2(s)\ \forall\, s \in \mathcal{S} \restriction M_2.$$

For pairs with oscillation $2, 3$ **and** 4**:** We use moreover the infinite Ramsey theorem and diagonalization processes.

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For pairs with oscillation 1**:** We color each $s \in \mathcal{S}$ into ℓ colors by $x \mapsto i$ iff $\{\varnothing, s \cap (x+1)\} \in \mathcal{A}_i$. Then, we get $\varphi_1(s) \subseteq s$ for each $s \in \mathcal{S}$, $M_1 \in [\mathbb{N}]^\infty$ and $i_1 < \ell$ such that:

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Analogously, by colorings each $[\varphi_1(s)]^2$ into ℓ by $\{x, y\} \mapsto i$ iff $\{s \cap (x+1), s \cap (y+1)\} \in \mathcal{A}_i$, we get $\varphi_2(s) \subseteq \varphi_1(s)$ for each $s \in \mathcal{S}$, an infinite set $M_2 \subseteq M_1$ and $i_2 < \ell$ such that:

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For pairs with oscillation $2, 3$ **and** 4**:** We use moreover the infinite Ramsey theorem and diagonalization processes.

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ω^n and triplets

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Theorem

Given $n, \ell > 1$ *and* $\omega^n < \alpha < \omega_1$. *If* $m = \left[\sum_{i=1}^{n} \binom{2i+1}{i+1}\right] - n$ *then*

$$\alpha \to (\text{top } \omega^n + 1)^2_{\ell,m}.$$

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Theorem

Given $\omega^2 < \alpha < \omega_1$ *and* $\ell > 1$ *we have*

$$\alpha \to (\text{top } \omega^2 + 1)^3_{\ell,71}.$$

ω^n and triplets

Theorem

Given $n, \ell > 1$ and $\omega^n < \alpha < \omega_1$. If $m = \left[\sum_{i=1}^{n} \binom{2i+1}{i+1}\right] - n$ then

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- Optimal values for m satisfy $2n \leq m \leq \left[\sum_{i=1}^{n} \binom{2i+1}{i+1}\right] - n$,
- In fact, $m = \left[\sum_{i=1}^{n} \binom{2i+1}{i+1}\right] - n$ is optimal for $\omega^n < \alpha < \omega^\omega$.

Theorem

Given $\omega^2 < \alpha < \omega_1$ and $\ell > 1$ we have

$$\alpha \to (\text{top } \omega^2 + 1)^3_{\ell,71}.$$

Moreover, 71 is optimal for every $\omega^2 < \alpha < \omega^\omega$.

Thank you!

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