Topological dimension and Baire category

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Topological dimension of $\mathbb{R}^n$

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A topological space X has dimension $\leq n$ if it admits a basis of sets whose boundaries have dimension $\leq n-1$, and $\dim X = -1 :\Leftrightarrow X = \emptyset$.

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Can "injective" be replaced with "injective on a large set"?

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Proof.

- Measure μ lives on a countable disjoint union of Cantor sets and they embed everywhere.
- Finite union of Cantor sets is closed, so these embeddings continuously extend (Tietze) to the whole X. □

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What about $n \geq 2$? I don't know, do you?

For the rest of the talk we will discuss two proofs of the $n = 1$ case and possible approaches/counter-examples for $n \geq 2$.

Reduction to compact domain and injectivity points

The conjecture is equivalent to its restriction to a compact domain:

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For $f : X \to Y$, call $x \in X$ an injectivity point of f if $f^{-1}(f(x)) = \{x\}$. Call f generically absolutely injective if it has comeager many injectivity points.

Splitting the question into two

We split our question into two subquestions.

Question 1

Is every generically injective continuous $f : \mathbb{I}^{n+1} \to \mathbb{R}^n$ generically absolutely injective?

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Question 1 is tightly connected to the Kuratowski–Ulam property.

The forward Kuratowski–Ulam property

Recall the classical Kuratowski–Ulam theorem:

Theorem (Kuratowski–Ulam)

For second-countable spaces Y, Z and any Baire measurable $A \subseteq Y \times Z$,

$$A \text{ is comeager in } Y \times Z \iff (\forall^* y \in Y)\ A_y \text{ is comeager in } Z.$$

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Rewrite in terms of the projection function $\pi : Y \times Z \to Y$:

$$A \text{ is comeager in } Y \times Z \iff (\forall^* y \in Y)\ A \cap \pi^{-1}(y) \text{ is comeager in } \pi^{-1}(y).$$

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Definition

For topological spaces X, Y, say that $f : X \to Y$ has the KU property if for every Baire measurable $A \subseteq X$,

$$A \text{ is comeager in } X \iff (\forall^* y \in Y)\ A \cap f^{-1}(y) \text{ is comeager in } f^{-1}(y).$$

Generic injectivity vs. generic true injectivity

Proposition

Let X, Y be Polish and $f : X \to Y$ continuous with the KU *property.*

$$f \text{ is generically injective} \iff f \text{ is generically absolutely injective.}$$

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When does a function have the KU property?

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Corollary

Any nonsingular (doesn't map nonempty open to a point) continuous map $f : \mathbb{I}^n \to \mathbb{R}$ has the KU *property.*

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Any nonsingular (doesn't map nonempty open to a point) continuous map $f : \mathbb{I}^n \to \mathbb{R}$ has the KU property.

Proof. f maps connected to connected, so open balls to nontrivial intervals, hence nonempty open to somewhere dense. □

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- In particular, there is $x \in \mathbb{I}^{n+1}$ such that $f(x) \in \mathrm{Int}(I)$.
- $I \setminus f(x)$ is disconnected, but its f-preimage is just $\mathbb{I}^{n+1} \setminus \{x\}$, which is still connected! □

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- Therefore, $f(C \cap \mathbb{I}^n_p)$ is nonmeager in $\mathbb{R}$, for comeager-many $p \in \mathbb{I}$.
- But the sets $f(C \cap \mathbb{I}^n_p)$ are disjoint for different $p \in \mathbb{I}$.
- We have obtained a disjoint family of continuum-many nonmeager sets, a contradiction. □

Generalize to $n \geq 2$?

To generalize the last proof, we need to:

1. Prove the KU property (only forward direction would be enough) for generically injective maps $f : \mathbb{I}^{n+1} \twoheadrightarrow f(\mathbb{I}^{n+1}) \subseteq \mathbb{R}^n$.

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Granted these, the same argument as in the last proof would imply

$$\nexists \text{ generically injective continuous } f : \mathbb{I}^{n+1} \to \mathbb{R}^n.$$

Generalize to $n \geq 2$?

To generalize the last proof, we need to:

1. Prove the KU property (only forward direction would be enough) for generically injective maps $f : \mathbb{I}^{n+1} \twoheadrightarrow f(\mathbb{I}^{n+1}) \subseteq \mathbb{R}^n$.
   - This will turn generically injective into generically absolutely injective.
2. Prove that for any generically absolutely injective continuous $g : \mathbb{I}^n \to \mathbb{R}^n$, the image $f(\mathbb{I}^n)$ has nonempty interior.
   - This is equivalent to $\dim(f(\mathbb{I}^n)) = n$ and this is where dimension theory should enter the picture.

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However, I have some discouraging examples regarding both parts...

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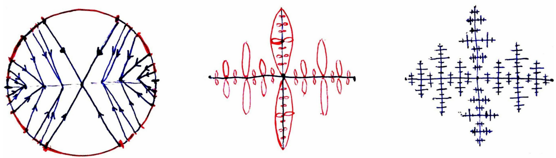

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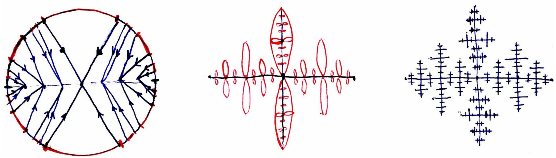
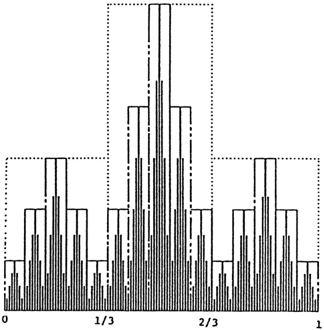


- I think this example can be used to build a counter-example to the conjecture for $n = 2$.

W/o connectivity KU epically fails: the ultimate mohawk

There is a compact space $H \subseteq 2^{\mathbb{N}} \times [0,1] \subseteq \mathbb{I}^2$, the hairy arc, such that



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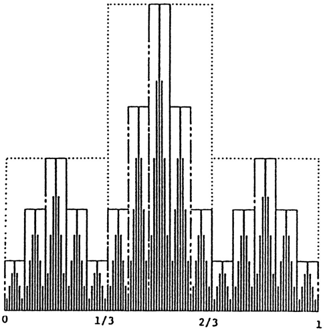

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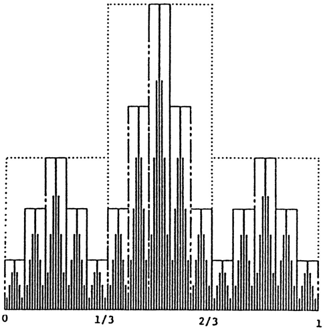

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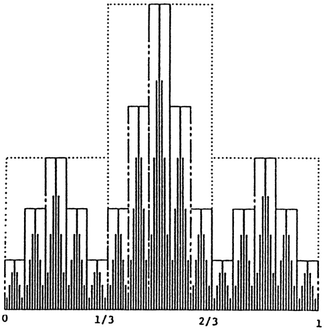

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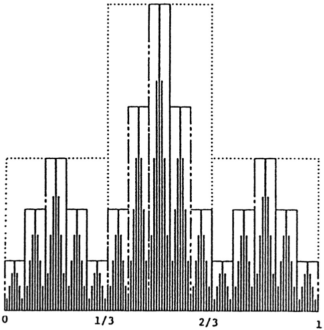

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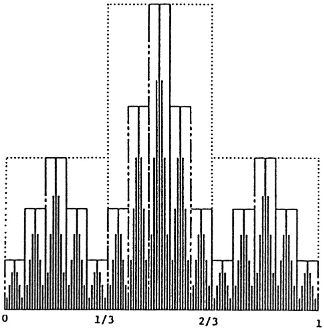

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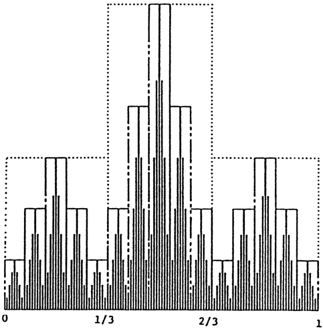

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Suggestion. ask your hair stylist to give you a haircut such that the tips of your hair amount to 99% of its total volume.

THANK YOU