Proof theory of CAT(κ)-spaces

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A CAT(κ)-space is a geodesic space whose triangles $\Delta(x_1, x_2, x_3)$ are thinner than their comparison triangles in the space M_κ^2 which is the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ equipped with

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$$\mathbf{d_{M^2_\kappa}(x, y) := \frac{1}{\sqrt{\kappa}} \arccos(\langle x, y \rangle),}$$

i.e

$$\mathbf{\forall t \in [0, 1] \ \left(d(x_1, (1-t)x_2 + tx_3) \le d_{M^2_\kappa}(\overline{x_1}, (1-t)\overline{x_2} + t\overline{x_3}) \right),}$$

whenever $x_1, x_2, x_3 \in X$ and $\overline{x_1}, \overline{x_2}, \overline{x_3} \in \mathbb{S}^2$ with $d(x_1, x_2) + d(x_2, x_3) + d(x_1, x_3) < 2D_\kappa$, where $D_\kappa := \pi/\sqrt{\kappa}$, and

$$\mathbf{d(x_i, x_j) = d_{M^2_\kappa}(\overline{x_i}, \overline{x_j}) \ \ (i, j \in \{1, 2, 3\}.}$$

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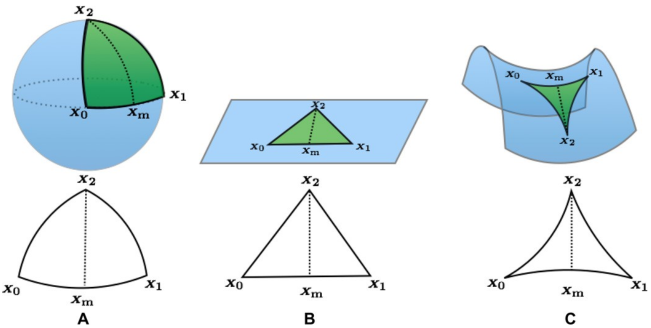
Example: $(\mathbb{S}^2, d_{M_1^2})$ is a CAT(1)-space.

Fig. 2 An intuitive understanding of curvature.



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A nonlinear ergodic theorem

Consider first a **Hilbert space** X and a closed convex subset C. For a sequence (λ_n) in $[0,1]$ define the Halpern iteration of a nonexpansive mapping $T : C \to C$ starting from x_0 by

$$\mathbf{x_{n+1} = \lambda_{n+1} x_0 + (1 - \lambda_{n+1}) T x_n.}$$

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Under suitable conditions on (λ_n) that allow for the choice $\lambda_n := 1/(n+1)$ Wittmann proved in 1992:

Theorem

If T has a fixed point then (x_n) is strongly convergent to the fixed point closest to x_0.

- If T is linear and $\lambda_n := 1/(n+1)$, then (x_n) coincides with the Cesàro means and so gives the Mean Ergodic Theorem.

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- Safarik JMAA 2012 gave a **full quantitative analysis** of Wittmann's result.

Back to the Halpern's iteration

- In K. Adv.Math.2011, a **quadratic rate of asymptotic regularity** for $\|x_n - Tx_n\| \to 0$ and a **primitive recursive rate of metastability** for (x_n) were extracted in **Hilbert space**.

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- In K./Leuștean Adv.Math.2012, similar results were extracted from a proof due to Saejung 2010 who had generalized Wittmann's result to **CAT(0)-spaces**.

Proof mining ergodic theorems in CAT(κ)-space

In Leuştean/Nicolae ETDS 2016, rates of asymptotic regularity and metastability on (x_n) were obtained for **CAT(κ)-spaces** $(\kappa > 0, diam(X) < \pi/2\sqrt{\kappa})$ by generalizing the approach for the CAT(0)-case thereby also re-proving the generalization of Saejung's result itself to CAT(κ)-spaces due to Piątek 2011.

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While the rate of metastability extracted in Leuștean/Nicolae ETDS 2016 is very complicated, the rate of asymptotic regularity (for $\lambda_n = 1/(n+1)$) is

$$\mathbf{exp}\left(\lceil\frac{1}{\cos(M\sqrt{\kappa})}\rceil\lceil\frac{8M}{\varepsilon}+2\rceil\ln 4\right),$$

where $diam(X) \leq M < \pi/2\sqrt{\kappa}$.

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Note that the rate is **exponential** in ε while it was **quadratic in the CAT(0)-case**.

Convex feasibility problems in CAT(κ) spaces

Let $\mathbf{X}$ be a CAT(κ)-space ($\kappa > 0$) with $\mathbf{diam(X) \leq M < \pi/(2\sqrt{\kappa})}$ and $\mathbf{C_1, \ldots, C_k \subseteq X}$ be closed convex subsets with $\bigcap_{i=1}^{k} \mathbf{C_i} \neq \emptyset$,
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Then $\mathbf{F = Fix(T) = \bigcap_{i=1}^{k} C_i}$ (Ariza-Ruiz,López-Acedo,Nicolae JOTA 2015).

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Ariza-Ruiz/López-Acedo/Nicolae:

1) $\mathbf{(x_n)}$ and $\mathbf{T}$ are **asymptotically regular**: $\mathbf{d(x_n, Tx_n)} \to \mathbf{0}$.

2) if X is **compact**, then $\mathbf{(x_n)}$ **converges** to a point in $\mathbf{F}$.

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Metric projections in CAT(κ)-spaces X with $diam(X) < \pi/2\sqrt{\kappa}$ are examples for this: hence **explicit rate $\Phi(\varepsilon)$** s.t.

$$\forall \mathbf{x} \in \mathbf{X}\, \forall \mathbf{g} \in \mathbb{N}^{\mathbb{N}}\, \forall \varepsilon > \mathbf{0}\, \exists \mathbf{n} \leq \boldsymbol{\Phi}(\varepsilon)\, \forall \mathbf{j} \in [\mathbf{n}, \mathbf{n} + \mathbf{g}(\mathbf{n})]\ \Big(\bigwedge_{\mathbf{i}=1}^{\mathbf{k}} \mathbf{x_j} \in \mathbf{C}_{\mathbf{i},\varepsilon}\Big),$$

where $\mathbf{C}_{\mathbf{i},\varepsilon} := \{\mathbf{y} \in \mathbf{X} : \exists \mathbf{z} \in \mathbf{C_i}\, (\mathbf{d}(\mathbf{y}, \mathbf{z}) < \varepsilon)\}$.

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Also: **rate of metastability** Ψ for (x_n) if C_k is a **totally bounded**.

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- Even uniformly strongly quasi-nonexpansive (Bruck) with modulus (Kohlenbach)

$$\omega(\varepsilon) := \frac{\varepsilon^2 \cdot \beta}{2d} \text{ with } \beta := \frac{1}{2}(\pi - 2\sqrt{\kappa}\delta\tan(\sqrt{\kappa}\delta),$$

where $0 < \delta < D_\kappa - diam(X)$ and $d \geq D_\kappa$, i.e.

$$\forall \varepsilon > 0\, \forall x \in X\, \forall p \in C\ (d(x,p) - d(P_C(x), p) < \omega(\varepsilon) \rightarrow d(x, P_C(x)) < \varepsilon).$$

Formal systems for analysis with abstract spaces X

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

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$\mathcal{A}^{\omega}[X, d, \ldots]$ results by adding constants $d_X, \ldots$ with axioms expressing that (X, d) is a nonempty metric space.

A warning concerning equality

Extensionality rule (**only!**):

$$\frac{s =_\rho t}{r(s) =_\tau r(t)},$$

where only $x =_{\mathbb{N}} y$ primitive equality predicate but for $\rho \to \tau$

$$x^X =_X y^X :\equiv d_X(x, y) =_{\mathbb{R}} 0_{\mathbb{R}},$$
$$x =_{\rho\to\tau} y :\equiv \forall v^\rho (s(v) =_\tau t(v)).$$

A novel form of majorization

y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$ and a^X of type X:

$$\mathbf{x}^{\mathbb{N}} \gtrsim^{\mathbf{a}}_{\mathbb{N}} \mathbf{y}^{\mathbb{N}} :\equiv \mathbf{x} \geq \mathbf{y}$$
$$\mathbf{x}^{\mathbb{N}} \gtrsim^{\mathbf{a}}_{\mathbf{X}} \mathbf{y}^{\mathbf{X}} :\equiv \mathbf{x} \geq \mathbf{d}(\mathbf{y}, \mathbf{a}).$$

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Example:

$$\mathbf{f}^* \gtrsim^{\mathbf{a}}_{\mathbf{X}\to\mathbf{X}} \mathbf{f} \equiv \forall \mathbf{n} \in \mathbb{N}, \mathbf{x} \in \mathbf{X}[\mathbf{n} \geq \mathbf{d}(\mathbf{a}, \mathbf{x}) \to \mathbf{f}^*(\mathbf{n}) \geq \mathbf{d}(\mathbf{a}, \mathbf{f}(\mathbf{x}))].$$

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Example:

$$\mathbf{f^* \gtrsim^a_{X \to X} f \equiv \forall n \in \mathbb{N}, x \in X[n \geq d(a, x) \to f^*(n) \geq d(a, f(x))].}$$

$f : X \to X$ is **nonexpansive (n.e.)** if $\mathbf{d(f(x), f(y)) \leq d(x, y)}$.

Then $\lambda n.n + b \gtrsim^a_{X \to X} f$, if $d(a, f(a)) \leq b$.

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Normed linear case: $a := 0_X$.

The formal system $\mathcal{A}^\omega[X, d, W, CAT(\kappa)]$

We extend $\mathcal{A}^\omega[X, d]$ by a constant $W_X^{X \to X \to \mathbb{N}^\mathbb{N} \to X}$ satisfying the axioms

$\forall x^X, y^X, z^X$

(W1) $\forall \lambda^{\mathbb{N}^\mathbb{N}} \big(d_X(z, W_X(x, y, \lambda)) \leq_\mathbb{R} (1 -_\mathbb{R} \tilde{\lambda}) \cdot_\mathbb{R} d_X(z, x) +_\mathbb{R} \tilde{\lambda} \cdot_\mathbb{R} d_X(z, y)\big)$,

(W2) $\forall \lambda_1^{\mathbb{N}^\mathbb{N}}, \lambda_2^{\mathbb{N}^\mathbb{N}} \big(d_X(W_X(x, y, \lambda_1), W_X(x, y, \lambda_2)) =_\mathbb{R} |\tilde{\lambda}_1 -_\mathbb{R} \tilde{\lambda}_2|_\mathbb{R} \cdot_\mathbb{R} d_X(x, y)\big)$,

(W3) $\forall \lambda^{\mathbb{N}^\mathbb{N}} \big(W_X(x, y, \lambda) =_X W_X(y, x, 1_\mathbb{R} -_\mathbb{R} \lambda)\big)$,

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i.e. (X, d, W) is a space of hyperbolic type (see Goebel/Kirk, K.) and - instead of (W4) used in K.2005 to define the class of (W)-hyperbolic spaces - we now have the axiom

The formal system $\mathcal{A}^\omega[X, d, W, CAT(\kappa)]$

We extend $\mathcal{A}^\omega[X, d]$ by a constant $W_X^{X \to X \to \mathbb{N}^\mathbb{N} \to X}$ satisfying the axioms

$\forall x^X, y^X, z^X$
(W1) $\forall \lambda^{\mathbb{N}^\mathbb{N}} \big(d_X(z, W_X(x, y, \lambda)) \le_\mathbb{R} (1 -_\mathbb{R} \tilde{\lambda}) \cdot_\mathbb{R} d_X(z, x) +_\mathbb{R} \tilde{\lambda} \cdot_\mathbb{R} d_X(z, y)\big)$,
(W2) $\forall \lambda_1^{\mathbb{N}^\mathbb{N}}, \lambda_2^{\mathbb{N}^\mathbb{N}} \big(d_X(W_X(x, y, \lambda_1), W_X(x, y, \lambda_2)) =_\mathbb{R} |\tilde{\lambda}_1 -_\mathbb{R} \tilde{\lambda}_2|_\mathbb{R} \cdot_\mathbb{R} d_X(x, y)\big)$,
(W3) $\forall \lambda^{\mathbb{N}^\mathbb{N}} \big(W_X(x, y, \lambda) =_X W_X(y, x, 1_\mathbb{R} -_\mathbb{R} \lambda)\big)$,

i.e. (X, d, W) is a space of hyperbolic type (see Goebel/Kirk, K.) and - instead of (W4) used in K.2005 to define the class of (W)-hyperbolic spaces - we now have the axiom

$$(W5): \ \forall x^X, y^X, z^X \forall \lambda^{\mathbb{N}^\mathbb{N}} \ (d_X(W_X(x, z, \lambda), W_X(y, z, \lambda)) \le d_X(x, y)),$$

which expresses that $d(W(x, z, \lambda), W(y, z, \lambda)) \le d(x, y)$ for all $\lambda \in [0, 1]$ and $x, y, z \in X$.

Next we add new constants c_κ of type $\mathbb{N} \to \mathbb{N}$ (written as κ) and $\overline{N}_\kappa$ of type $\mathbb{N}$ together with the following axioms

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$$(\kappa 3) \begin{cases} \forall a^X, b^X, p^X, q^X\ \forall n^{\mathbb{N}} \Big(d_X(a,p), d_X(b,q) >_{\mathbb{R}} \frac{1}{n+1} \to \\ \frac{\cos(\sqrt{\kappa} d_X(p,q)) + \cos(\sqrt{\kappa} d_X(a,p)) \cos(\sqrt{\kappa} d_X(b,q))}{\sin(\sqrt{\kappa} d_X(a,p)) \sin(\sqrt{\kappa} d_X(b,q))} \\ - \frac{\left(\cos(\sqrt{\kappa} d_X(a,p)) + \cos(\sqrt{\kappa} d_X(b,p))\right)\left(\cos(\sqrt{\kappa} d_X(b,q)) + \cos(\sqrt{\kappa} d_X(a,q))\right)}{\left(1+\cos(\sqrt{\kappa} d_X(a,b)\right) \sin(\sqrt{\kappa} d_X(a,p)) \sin(\sqrt{\kappa} d_X(b,q))} \leq_{\mathbb{R}} 1 \Big), \end{cases}$$

expressing that X satisfies the 'upper four point κ-quadrilateral cos-condition cosq_κ condition' (I.D. Berg, I.G. Nikolaev 2015).

Proposition (K./Nicolae, to appear in Studia Logica)

Let (X,d) be a metric space, $W : X \times X \times [0,1] \to X$ be a mapping, $\kappa \in (0,\infty)$ and $N_\kappa \in \mathbb{N}$.

The full set-theoretic type structure $\mathcal{S}^{\omega,X}$ is a model of $\mathcal{A}^\omega[X,d,W,\mathrm{CAT}(\kappa)]$ (in the sense of K.2008) iff (X,d) is a CAT(κ)-space with $\kappa \geq 1/(N_\kappa + 1)$ and $diam(X) \leq \pi/(2\sqrt{\kappa})$ and W is defined via the unique geodesic joining x, y.

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We next show that the proof-theoretic bound extraction theorems due to K. TAMS 2005 for CAT(0)-spaces (among others) can be adapted to the CAT(κ)-case.

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We next show that the proof-theoretic bound extraction theorems due to K. TAMS 2005 for CAT(0)-spaces (among others) can be adapted to the CAT(κ)-case.

The extraction is based on a monotone version (K.1996) of (an extension of) Gödel's functional ('Dialectica') interpretation.

A proof-theoretic bound extraction theorem

Theorem (K./Nicolae, to appear in Studia Logica)

Let σ, ρ be types of degree $\mathbb{N} \to \mathbb{N}$ and τ be a type of degree $(1, X)$ (e.g. $\tau = X, \mathbb{N} \to \mathbb{N}, X \to X$).

Let $s^{\sigma \to \rho}$ be a closed term of $\mathcal{A}^\omega[X, d, W, CAT(\kappa)]$ and $A_\exists(x^\sigma, y^\rho, z^\tau, v^\mathbb{N})$ be an $\exists$-formula containing only x, y, z, u.

If

$$\mathcal{A}^\omega[X, d, W, CAT(\kappa)] \vdash \forall x^\sigma \, \forall y \leq_\rho s(x) \, \forall z^\tau \, \exists v^\mathbb{N} A_\exists(x, y, z, v),$$

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If

$$\mathcal{A}^\omega[\mathbf{X}, \mathbf{d}, \mathbf{W}, \mathbf{CAT}(\kappa)] \vdash \forall \mathbf{x}^\sigma \, \forall \mathbf{y} \leq_\rho \mathbf{s}(\mathbf{x}) \, \forall \mathbf{z}^\tau \, \exists \mathbf{v}^{\mathbb{N}} \mathbf{A}_\exists(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}),$$

then one can extract a (subrecursively) computable functional $\Phi : \mathcal{S}_\sigma \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $x \in \mathcal{S}_\sigma$ and all $b, N \in \mathbb{N}$

$$\forall \mathbf{y} \leq_\rho \mathbf{s}(\mathbf{x}) \forall \mathbf{z}^\tau \, \exists \mathbf{v} \leq \mathbf{\Phi}(\mathbf{x}, \mathbf{b}, \mathbf{N}) \, \mathbf{A}_\exists(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v})$$

holds in any (non-empty) CAT(κ)-space (X, d) with $1/(N+1) \leq \kappa \leq b$ and $diam(X) \leq \pi/(2\sqrt{\kappa})$.

The characterization of CAT(κ)-spaces via the quadrilateral cos-condition due to Berg and Nikolaev is given by a purely universal axiom which is trivially admissable in the proof-theoretic bound extraction metatheorems (since it is its own monotone functional interpretation).

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The monotone functional interpretation of that characterization (which has the form $\forall(\forall \rightarrow \forall)$), however, asks for a uniform quantitative version.

We next show that such a uniform quantitative version already follows from the seemingly weaker qualitative one.

Definition

Let (X, d) be a CAT(κ)-space with $\kappa > 0$ and $\text{diam}(X) \leq \pi/(2\sqrt{\kappa})$. Take $x_1, x_2, x_3 \in X$. Having $\delta > 0$, a δ-comparison triangle for $\Delta(x_1, x_2, x_3)$ is a triangle $\Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in M_κ^2 such that

$$\left| d(x_i, x_j) - d_{M_\kappa^2}(\overline{x_i}, \overline{x_j}) \right| \leq \frac{\delta}{\sqrt{\kappa}} \quad \text{for } i, j \in \{1, 2, 3\}.$$

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Proposition (K./Nicolae, to appear in Studia Logica)

In the setting above, for every $\varepsilon \in (0,1)$ there exists $\delta := \frac{\varepsilon^2}{108} \sin \frac{\varepsilon^2}{36}$ such that for every δ-comparison triangle $\Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ we have that

$$\forall t \in [0,1] \left(d(x_1, (1-t)x_2 + tx_3) \leq d_{M^2_\kappa}(\overline{x_1}, (1-t)\overline{x_2} + t\overline{x_3}) + \frac{\varepsilon}{\sqrt{\kappa}} \right).$$

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This version can be stated as a universal axiom with $(1-t)x_2 + tx_3$ to be understood as $W_X(x_2, x_3, t)$. One easily shows that the comparison inequality stated just for the geodesic selected by W implies the uniqueness of the geodesic.

Since comparison triangles are (up to isometry) unique one could also state the characterization of $\mathrm{CAT}(\kappa)$-spaces in the form of a so-called axiom Δ which can be freely added to the formal system:

$$(*) \begin{cases} \forall x_1, x_2, x_3 \in X \, \exists \overline{x_1}, \overline{x_2}, \overline{x_3} \in B_1(0) \, \forall t \in [0,1] \\ \quad \Big(\bigwedge_{i,j \in \{1,2,3\}} (\|\overline{x_i}\|_E = 1 \wedge d(x_i, x_j) = d_{M^2_\kappa}(\overline{x_i}, \overline{x_j})) \wedge \\ \quad\quad d(x_1, (1-t)x_2 + tx_3) \le d_{M^2_\kappa}(\overline{x_1}, (1-t)\overline{x_2} + t\overline{x_3}) \Big). \end{cases}$$

The quantitative formulation can be viewed as mining the uniqueness proof by which $(*)$ implies the official characterization.