Interval Arithmetic: Fundamentals, History, and Semantics

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BIRS Casa Oaxaca Seminar, November 13, 2016

Outline

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- The logical definition leads to operational definitions:

$$\begin{aligned}
\boldsymbol{x} + \boldsymbol{y} &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}], \\
\boldsymbol{x} - \boldsymbol{y} &= [\underline{x} - \overline{y}, \overline{x} - \underline{y}], \\
\boldsymbol{x} \times \boldsymbol{y} &= [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}] \\
\frac{1}{\boldsymbol{x}} &= [\frac{1}{\overline{x}}, \frac{1}{\underline{x}}] \qquad \text{if } \underline{x} > 0 \text{ or } \overline{x} < 0 \\
\boldsymbol{x} \div \boldsymbol{y} &= \boldsymbol{x} \times \frac{1}{\boldsymbol{y}}
\end{aligned}$$

(There are alternatives for $\times$ and $\div$ more efficient for certain architectures.)

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Example (interval dependence)

If $f(x) = (x+1)(x-1)$, then

$$\boldsymbol{f}([-2,2]) = \big([-2,2]+1\big)\big([-2,2]-1\big) = [-1,3][-3,1] = [-9,3],$$

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- The interval $[-9,3]$ represents the exact range of $\tilde{f}(x,y) = (x+1)(y-1)$ over the rectangle $x \in [-2,2]$, $y \in [-2,2]$ (when x and y vary independently).

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- If we use downward rounding to compute the lower end point and upward rounding to compute the upper end point, the result of each elementary operation contains the exact range of that operation.
- Hence, an interval evaluation of an expression on a machine mathematically rigorously contains the range of the expression.

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  - *Note: The converse is not true.*

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Kaucher arithmetic, modal arithmetic etc.: Algebraically completes interval arithmetic with additive inverses. It has uses, but interpretation of the results is more complicated, sometimes depending on monotonicity properties.

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- Different definitions for the operation's result and different interpretations are appropriate in different contexts. (More to be said later.)

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- This has been carefully considered and defined in an exception-tracking framework in the IEEE 1788-2015 standard for interval arithmetic.

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- See https://arxiv.org/abs/1501.02155v1 and https://en.wikipedia.org/wiki/Kepler_conjecture.

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2001 Warwick Tucker (in dissertation work) used normal form theory and interval arithmetic to solve Stephen Smale's 14-th problem, namely, that the Lorenz equations have a strange attractor that persists under perturbations of the coefficients in the differential equations.

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Additional work

The R. E. Moore Prize for application of interval arithmetic has been awarded to various researchers for proving certain mathematical conjectures. See `http://www.cs.utep.edu/interval-comp/honors.html`. Among these are:

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2016 Banhelyi, Csendes, Krisztin, and Neumaier for Global attractivity of the zero solution for Wright's equation (a model in population genetics)

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3. Incorporation of range bounds to rigorously enclose solution sets to differential equations in sophisticated mathematically rigorous ODE integrators.

2. Stadtherr et al Correction of major errors in widely used tables of vapor-liquid equilibria.
3. Berz et al Proof of stability of the beam, given assumed tolerances on the geometry and magnets, of the once-proposed superconducting supercollider (and the software continues to be used for other cyclotrons).

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  In early work (1988) yours truly used Fortran-77-based software to show the set of published solutions to a manipulator problem posed by Alexander Morgan at General Motors was incorrect. This led to discovery of an incorrectly-given coefficient in the paper and to improvement in the software in use at General Motors.

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: prior to digital computers) “The Algebra of Many-Valued Quantities.” The focus is on an arithmetic on limits, where $\liminf_{x\to x_0} f(x)$ and $\limsup_{x\to x_0} f(x)$ are distinct (such as in in generalized gradients of nonsmooth functions). Ranges and roundoff error do not seem to have been the primary motivation.

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Mieczyslaw Warmus (*Calculus of Approximations*, 1956) The motivation is apparently to provide a sound theoretical backing to numerical computation.

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- 1887, 1879, and 1854 French work where explicit formulas for the elementary operations and rigorous error bounds were given;

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- an 1896 article "On computing with inexact numbers" (in German) in the *Journal for Junior Highschool Studies*, giving the impression interval computations were standard fare in middle schools;
- 1887, 1879, and 1854 French work where explicit formulas for the elementary operations and rigorous error bounds were given;
- An 1809 work by Gauß in Latin where explicit computation of error bounds, including rounding errors, appears.

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- It is made clear that interval computations promise rigorous bounds on the exact result, even when finite (rounded) computer arithmetic is used.

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- Founded the *Institute for Reliable Computing* at Hamburg, educating students and developing software.

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- A many other salient Russian IA researchers and teachers are Boris Dobronets, Sergey Shary, Irina Dugarova, Nikolaj Glazunov, Grigory Menshikov, ...

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Constraint propagation: Interpretation in equality constraints

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  - Note that $\pm\sqrt{[-3, 0]}$ represents the set of all x_2 with $x_1 \in [1, 2]$ satisfying the constraint; no problem here.

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  - $[1, \sqrt{2}]$ must be replaced by $[1, \infty)$; this depends on $\ge$ and monotonicity of $\sqrt{\cdot}$.

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Constraint propagation: Interpretation in inequality constraints

Which bounds to use and the sense can be confusing.

- Consider an inequality constraint $x_1^2 - x_2^2 \le 1$ within the box $([-3,3],[-0.1,1])$.
  - If we solve for x_1, we obtain

$$x_1 \le [1, \sqrt{2}] \quad \textit{or} \quad x_1 \le [-\sqrt{2}, -1].$$

  - Here, our conclusion is that $x_1 \in [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$, and the computation and logic are straightforward.
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  - $[1, \sqrt{2}]$ must be replaced by $[1, \infty)$; this depends on $\ge$ and monotonicity of $\sqrt{\cdot}$.
  - The interpretation of the interval arithmetic result is different for $\ge$ than for $\le$.

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A contrasting context with inequalities:

Vertex and half-plane representation of a simplex

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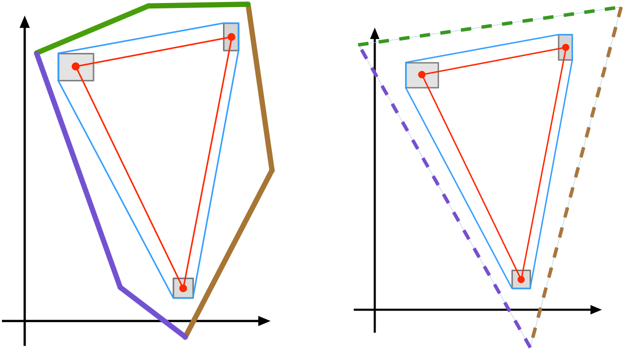
- the feasible set of $Ax \geq b$ encloses $\mathcal{S}$ for ***any*** $A \in \boldsymbol{A}$.

Simplex Representations

Illustration

(box sizes were exaggerated for clarity)



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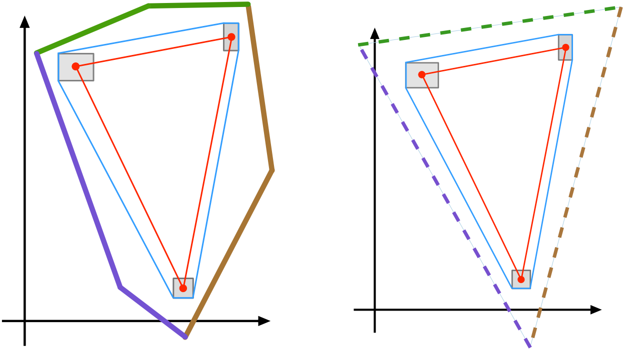

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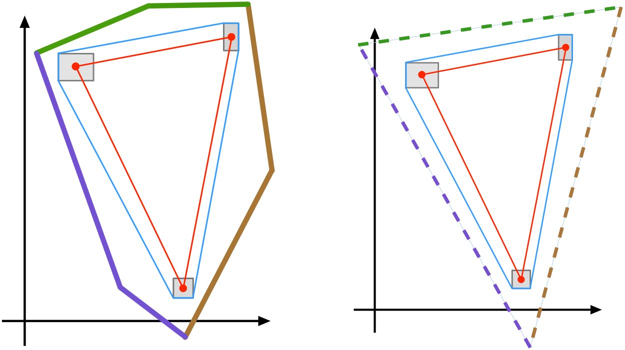

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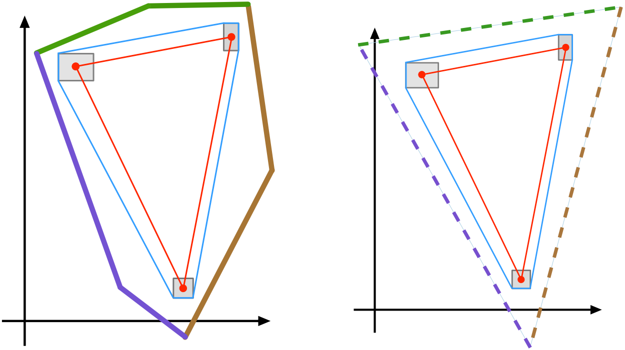

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- (Thank you, Sam Karhbet.)

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If g is a continuous mapping from a compact convex set $\boldsymbol{x}$ into itself, there is a fixed-point $x \in \boldsymbol{x}$ of g, i.e. $g(x) = x$.

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Example (thank you, John Pryce)

Consider $g(x) = \sqrt{x-1} + 0.9$, with a fixed point at $x \approx 1.0127$ and $x \approx 1.7873$.

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- if $\boldsymbol{x} = [0, 1]$, $\sqrt{x-1} = \sqrt{[-1, 0]}$ evaluates to $[0, 0]$, so $\boldsymbol{g}(\boldsymbol{x}) = [0.9, 0.9] \subset \boldsymbol{x}$, for an incorrect conclusion.

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- There are situations where a condition must hold for every element of a computed interval, and other situations where a any element of a computed interval (or interval vector) may be chosen.
- Simple partial evaluation ignores continuity conditions that are necessary for rigorous existence / uniqueness proofs.

IEEE 1788-2015

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- Thank you, John Pryce, IEEE 1788 technical editor and a leader in development of the decoration system.

Conforming

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Conformance in Progress

`ValidatedNumerics.jl` (Julia) by David P. Sanders and Luis Benet (UNAM)
See `https://github.com/dpsanders/ValidatedNumerics.jl`