Convex Feasibility via Monotropic Programming

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Dedicated to Jonathan M. Borwein

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Joint work with

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Outline

1. Problem Formulation
2. Monotropic Programming
3. Preliminaries
4. Facts
5. Analysis of Consistency

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The problem formulation

Let H be a Hilbert space and let $C_n, n = 1, ..., m$ be convex closed subsets of H. The convex feasibility problem is to find some point

$$x \in \bigcap_{n=1}^{m} C_n \qquad (CFP)$$

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The *CFP* has wide ranging applications:

- medical imaging, computerised tomography, signal processing.
- Partial differential equations (Dirichlet problem), complex analysis (Bergman kernels, conformal mappings);
- Subgradient algorithms with application in solution of convex inequalities, minimization of convex nonsmooth functions.

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- CFP equivalent to problem involving only two convex and closed sets in $H^m = H \times \ldots \times H$ consisting of m copies of H,with the additional advantage that one of these sets is a linear subspace
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Monotropic Model (Minty, 1960) (Rockafellar, 1970, 1981, 1998)

$$\min \sum_{i=i}^{m} f_i(x_i) \qquad (P)$$

subject to $(x_1, \ldots, x_m) \in S$,

- $f_i : H_i \to \mathbb{R} \cup \{+\infty\}$ proper, convex,
- $S \subseteq \prod_{i=1}^{m} H_i$ is a closed linear subspace

(P) will be our primal model.
(P) has a very symmetric dual problem:

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$$f^*(v) := \sup_{x \in H}\{\langle v, x\rangle - f(x)\}$$

- The *subdifferential* of f at x is defined by

$$\partial f(x) := \{v \in H \mid \langle v, y - x\rangle \leq f(y) - f(x), \text{ for all } y \in H\},$$

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Basic Ingredients (II):

- For $C \subset H$, the *indicator function* of C is $\iota_C(x) := 0$ if $x \in C$ and $\iota_C(x) := +\infty$ otherwise.

- The the *support function* of C is

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for $v \in H$

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For $\psi_1, \psi_2 : H \to \mathbb{R} \cup \{+\infty\}$, their *infimal convolution* is defined by

$$(\psi_1 \Box \psi_2)(z) := \inf_{z_1+z_2=z} \{\psi_1(z_1) + \psi_2(z_2)\}.$$

For $f : H \to \mathbb{R} \cup \{+\infty\}$ recall that the *epigraph* is the set

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Fact (B.-Jeyakumar, 2005):

$C, D \subset H$ closed convex:

$$C \cap D \neq \emptyset \iff (0, -1) \notin \mathrm{cl}\,(\mathrm{epi}\,\sigma_C + \mathrm{epi}\,\sigma_D)$$

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Primal for CFP:

Our problem is (recall we reduced the problem to 2 sets):

$$\text{find } (x, y) \in C_1 \times C_2 \subset H \times H, \text{ such that } x = y$$

which can be formulated as

$$\min_{(x,y)\in S} d_{C_1}(x) + d_{C_2}(y) \qquad (P)$$

where $S = \{(x, y) \in H^2 \ : \ x = y\}$.

Dual for CFP:

Using monotropic formulation we obtain its dual:

$$\sup_{(v,w)\in S^{\perp}} \; -d^*_{C_1}(v) - d^*_{C_2}(w) \qquad (D)$$

where $S^{\perp} = \{(u,v) \in H^2 \;:\; u + v = 0\}$.

What do we know about this primal-dual pair?

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Duality facts:

Pro 15.22 and Theo 19.1 from Bauschke-Combettes book yield:

$v(P) = v(D)$ and (D) always has a solution

In this situation, (x, y) solves (P) and (u, v) solves (D).

$$\Updownarrow$$

$$\begin{array}{ll} (x, y) \in S, & (u, v) \in S^{\perp} \\ u \in \partial d_{C_1}(x) & v \in \partial d_{C_2}(y) \end{array}$$

Proof not very direct!

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$d_C^*(v) = \sigma_C(v) + \imath_B(v)$ yields:

$$\sup_{v \in H} - d_{C_1}^*(v) - d_{C_2}^*(-v) =$$

$$= -\min_{t \in [0,1]} t \underbrace{\left(\inf_{\|v\| \le 1} \sigma_{C_1}(v) + \sigma_{C_2}(-v) \right)}_{\Phi(1)},$$

which gives an equivalent reformulation of the dual in terms of $\Phi(1)$. Always $\Phi(1) \le 0$. Value $\Phi(1)$ gives important information:

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Consistency results for CFP:

1. $\Phi(1) < 0 \iff 0 \notin \mathrm{cl}\,(C_2 - C_1)$. So $C_1 \cap C_2 = \emptyset$.

2. $\Phi(1) = 0 \iff 0 \in \mathrm{cl}\,(C_2 - C_1)$. This leads to two cases:

   2.1 If $(\sigma_{C_1} \square \sigma_{C_2})$ is lsc at 0, then $C_1 \cap C_2 \neq \emptyset$.

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   2.2 If $\boxed{(\sigma_{C_1} \Box \sigma_{C_2}) \text{ is not lsc at } 0}$ then $C_1 \cap C_2 = \emptyset$, $\exists$ (possibly improper) closed separating hyperplane.
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Characterization of Consistency:

Assume that $(\sigma_{C_1} \Box \sigma_{C_2})(0) > -\infty$. Then $(\sigma_{C_1} \Box \sigma_{C_2})$ is proper, and TFSAE:

(i) $C_1 \cap C_2 \neq \emptyset$,

(ii) $(\sigma_{C_1} \Box \sigma_{C_2})$ is lsc at 0,

(iii) $\{0\} \times \mathbb{R} \cap \operatorname{epi}(\sigma_{C_1} \Box \sigma_{C_2}) = \{0\} \times \mathbb{R}_+$

Consequently, if $\operatorname{epi} \sigma_{C_1} + \operatorname{epi} \sigma_{C_2}$ is closed, then $C_1 \cap C_2 \neq \emptyset$.

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Consistency for CFP in the critical case $v(D) = 0$:

Recall that (D) always has solutions. Assume $v(D) = 0$. Then:

(a) If $v = 0$ is unique solution of $(D) \iff C_1 \cap C_2 \neq \emptyset$.

(b) (D) has multiple solutions if and only if $C_1 \cap C_2 = \emptyset$. In this situation, every nonzero dual solution induces a possibly improper separation of the sets.

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Inconsistency for CFP in critical case $d(C_1, C_2) = 0$. TFSAE:

(i) (P) has no solution.

(ii) $0 \in \mathrm{cl}(C_1 - C_2) \setminus (C_1 - C_2)$.

(iii) $\sigma_{C_1} \Box \sigma_{C_2}$ is not lsc at 0.

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