A parameterized Douglas-Rachford algorithm

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Outline

1. Setup
2. Properties of α-Douglas-Rachford algorithm.
3. A numerical experiment of solving $0 \in Ax + Bx$.
4. Solving a primal-dual problem with mixtures composite and parallel-sum type monotone operators.
5. A numerical experiment of solving primal-dual problem.

Setup

The Euclidean space $\mathbb{R}^m$ has an inner product $\langle \cdot, \cdot \rangle$, and norm $\| \cdot \|$.

Assume that

$$A, B \text{ are maximally monotone operator on } \mathbb{R}^m$$

and

$$f, g : \mathbb{R}^m \to (-\infty, +\infty] \text{ are proper, lower semicontinuous and convex.}$$

Goal: Find $x \in \operatorname{zer}(A+B)$, i.e.,

$$0 \in Ax + Bx.$$

[1] A is monotone if $\langle x - y, u - v \rangle \geq 0$ for all $(x, u), (y, v) \in \operatorname{gra} A$. A is maximally monotone if there is no monotone operator that properly contains it.

[2] The set of zeros of M is: $\operatorname{zer} M := \{x \in \mathbb{R}^m : 0 \in Mx\}$.

The connection to the optimization problem

If we assume $\operatorname{dom} f \cap \operatorname{intdom} g \neq \emptyset$, and $A = \partial f, B = \partial g$.

Solving the problem: Find $x \in \mathbb{R}^m$ such that

$$x \in \operatorname{zer}(A + B), \tag{1}$$

means solving the optimization problem: Find $x \in \mathbb{R}^m$ such that

$$x \in \operatorname{Argmin}\{f + g\}. \tag{2}$$

[1] $\partial f(x) := \{v \in \mathbb{R}^m : f(y) \geq f(x) + \langle v, y - x \rangle \text{ for all } y \in \mathbb{R}^m\}$.

The Douglas-Rachford splitting operator

The Douglas-Rachford splitting operator, introduced by Lions and Mercier, associated with the maximally monotone operators A, B is

$$D_{A,B} = \frac{\operatorname{Id} - R_B + 2J_A R_B}{2} = \frac{1}{2}\operatorname{Id} + \frac{1}{2} R_A R_B,$$

where J_A and R_A denote the resolvent and the reflected resolvent of A, defined by

$$J_A := (\operatorname{Id} + A)^{-1}, \quad R_A := 2J_A - \operatorname{Id},$$

respectively. We recall that J_A is firmly nonexpansive and R_A is nonexpansive.

[1] An operator T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$.

[2] T is firmly nonexpansive if $\|Tx - Ty\|^2 + \|(\operatorname{Id} - T)x - (\operatorname{Id} - T)y\|^2 \leq \|x - y\|^2$.

The Douglas-Rachford algorithm

Fact 1 (Lions-Mercier, 1979)

Suppose $\operatorname{zer}(A+B) \neq \emptyset$*. Let* $x_0 \in \mathbb{R}^m$ *be the starting point. Set*

$$(\forall n \in \mathbb{N}) \quad \begin{cases} y_n = J_B x_n \\ z_n = J_A(2y_n - x_n) \\ x_{n+1} = x_n + (z_n - y_n). \end{cases} \tag{DR}$$

Then there exists $x \in \operatorname{Fix} R_A R_B$ *such that the following hold:*

(i) $J_B x \in \operatorname{zer}(A+B)$.

(ii) $(y_n - z_n)_{n=1}^{+\infty}$ *converges to 0.*

(iii) $(x_n)_{n=1}^{+\infty}$ *converges to* x*.*

(iv) $(y_n)_{n=1}^{+\infty}$ *converges to* $J_B x$*.*

(v) $(z_n)_{n=1}^{+\infty}$ *converges to* $J_B x$*.*

[1] The fixed points set is $\operatorname{Fix} T = \{x \in \mathbb{R}^m : Tx = x\}$.

Question: What happens if we change the parameter 2 into α, where $\alpha \in [1, 2)$?

Outline

Theorem 2

Let

$$R_A^\alpha = \alpha J_A - \mathrm{Id}, \qquad R_B^\alpha = \alpha J_B - \mathrm{Id}\,.$$

Then R_A^α and R_B^α are nonexpansive if $\alpha \in [1,2)$.

Theorem 3

If $0 \in \mathrm{int}(\mathrm{dom}\, A - \mathrm{dom}\, B)$, then $\mathrm{zer}(A + B + \gamma\, \mathrm{Id}) \neq \emptyset$ when $\gamma \in \mathbb{R}_{++}$.

Theorem 4

Let $\alpha \in [1,2)$, and $0 \in \mathrm{int}(\mathrm{dom}\, A - \mathrm{dom}\, B)$. Let $T = R_A^\alpha R_B^\alpha$. Then

1. *T is nonexpansive.*
2. $J_B(\mathrm{Fix}\, T) = \mathrm{zer}(A + B + (2-\alpha)\,\mathrm{Id})$.
3. *Consequently,* $\mathrm{Fix}\, T \neq \emptyset$.

[1] $0 \in \mathrm{int}(\mathrm{dom}\, A - \mathrm{dom}\, B)$ implies $A + B$ is maximally monotone.

The α-Douglas-Rachford splitting operator

Changing the parameter 2 of the algorithm (DR) into α, where $\alpha \in [1, 2)$, we propose the α-DR algorithm

$$\begin{cases} y_n = J_B x_n \\ z_n = J_A(\alpha y_n - x_n) \\ x_{n+1} = x_n + (z_n - y_n). \end{cases} \tag{α-DR}$$

We call it α-Douglas-Rachford splitting operator:

$$D^{\alpha}_{A,B} = (1 - \frac{1}{\alpha}) \operatorname{Id} + \frac{1}{\alpha} R^{\alpha}_{A} R^{\alpha}_{B}.$$

$D^{\alpha}_{A,B}$ is an averaged operator.

Remark

Let $D \subseteq \mathbb{R}^m$, $T : D \to \mathbb{R}^m$, and $\gamma \in [0, 1]$. T is called $\gamma-$ *averaged*, if there exists a nonexpansive operator $N : D \to \mathbb{R}^m$ such that $T = (1 - \gamma) \operatorname{Id} + \gamma N$.

α-Douglas-Rachford algorithm

Theorem 5

Let $\alpha \in (1, 2)$ *and* $0 \in \operatorname{int}(\operatorname{dom} A - \operatorname{dom} B)$. *Let* $x_0 \in \mathbb{R}^m$ *be the starting point. Set*

$$\begin{cases} y_n = J_B x_n \\ z_n = J_A(\alpha y_n - x_n) \\ x_{n+1} = x_n + (z_n - y_n). \end{cases} \tag{α-DR}$$

Then there exists $x \in \operatorname{Fix} R_A^\alpha R_B^\alpha$ *such that the following hold:*

- (i) $J_B x = \operatorname{zer}(A + B + (2 - \alpha) \operatorname{Id})$.
- (ii) $(y_n - z_n)_{n=1}^{+\infty}$ *converges to 0.*
- (iii) $(x_n)_{n=1}^{+\infty}$ *converges to* x.
- (iv) $(y_n)_{n=1}^{+\infty}$ *converges to* $J_B x$.
- (v) $(z_n)_{n=1}^{+\infty}$ *converges to* $J_B x$.

The Krasnosel'skiĭ–Mann algorithm plays an important role.

Fact 6

Let D be a nonempty closed convex subset of $\mathbb{R}^m$, let $T : D \to D$ be a nonexpansive operator such that $\operatorname{Fix} T \neq \emptyset$, where the fixed points set

$$\operatorname{Fix} T = \{x \in \mathbb{R}^m : Tx = x\}.$$

Let $(\lambda_n)_{n=1}^{+\infty}$ be a sequence in $[0, 1]$ such that $\sum_{n=1}^{+\infty} \lambda_n(1 - \lambda_n) = +\infty$, and let $x_0 \in D$. Set

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = x_n + \lambda_n(Tx_n - x_n).$$

Then the following hold:

1. $(Tx_n - x_n)_{n=1}^{+\infty}$ *converges to 0.*
2. $(x_n)_{n=1}^{+\infty}$ *converges to a point in* $\operatorname{Fix} T$.

Proof of Theorem 4

(i) Let $T = R_A^\alpha R_B^\alpha$, we proved that $\operatorname{Fix} T \neq \emptyset$ and $J_B(\operatorname{Fix} T) = \operatorname{zer}(A + B + (2-\alpha)\operatorname{Id})$. Therefore, there exists $x = R_A^\alpha R_B^\alpha x$ such that

$$J_B x = \operatorname{zer}(A + B + (2-\alpha)\operatorname{Id}).$$

(ii) From

$$\begin{cases} y_n = J_B x_n \\ z_n = J_A(\alpha y_n - x_n) \\ x_{n+1} = x_n + (z_n - y_n), \end{cases}$$

it follows that

$$z_n - y_n = \frac{1}{\alpha}(Tx_n - x_n).$$

Therefore, $z_n - y_n \to 0$.

Proof continued

(iii) Since $1 < \alpha < 2$, $(x_n)_{n=1}^{+\infty}$ converges to x.

(iiii) In $\mathbb{R}^m$, by using that J_B is Lipschitz continuous, we get

$$\lim_{n\to+\infty} y_n = \lim_{n\to+\infty} J_B(x_n) = J_B(\lim_{n\to+\infty} x_n) = J_B x.$$

(iiiii) Combining result (ii) and result (iv), we have
$z_n = (z_n - y_n) + y_n \to 0 + J_B x$, i.e., $z_n \to J_B x$.

The α-Douglas-Rachford algorithm with $\alpha \to 2$

Theorem 7

Let $0 \in \operatorname{int}(\operatorname{dom} A - \operatorname{dom} B)$ *and* $\operatorname{zer}(A+B) \neq \emptyset$. *Let* $(\alpha_k)_{k=1}^{+\infty}$ *be an increasing sequence in* $[1,2)$ *such that* $\lim_{k\to+\infty} \alpha_k = 2$. *Set*

$$\begin{cases} y_n = J_B x_n \\ z_n = J_A(\alpha_k y_n - x_n) \\ x_{n+1} = x_n + (z_n - y_n). \end{cases} \tag{α-DR}$$

Then for any fixed α_k, *there exists a corresponding* $x_k^* \in \operatorname{Fix} R_A^{\alpha_k} R_B^{\alpha_k}$ *such that* $J_B x_k^* = \operatorname{zer}(A + B + (2-\alpha_k)\operatorname{Id})$, *and the following hold:*

- (a) $\lim_{\alpha_k \to 2} J_B x_k^* = \mathrm{P}_{\operatorname{zer}(A+B)}(0)$.
- (b) *For any fixed* α_k, $(x_n)_{n=1}^{+\infty}$ *converges to its corresponding* x_k^*.
- (c) *Suppose* $(x_k^*)_{k=1}^{+\infty}$ *is a convergent sequence with limit* x^*. *Then* $J_B x^* \in \operatorname{zer}(A+B)$, *and* $\|J_B x^*\| \le \|y\|$ *for any* $y \in \operatorname{zer}(A+B)$.

Proof

(a) $J_B x_k^* = \operatorname{zer}(A + B + (2-\alpha_k)\operatorname{Id})$ implies

$$0 \in (A+B)J_B x_k^* + (2-\alpha_k)(J_B x_k^* - 0).$$

Because A, B are maximally monotone and $0 \in \operatorname{int}(\operatorname{dom} A - \operatorname{dom} B)$, $A + B$ is maximally monotone. As $\operatorname{zer}(A+B) \neq \emptyset$, we have

$$J_B x_k^* \to \mathrm{P}_{\operatorname{zer}(A+B)}(0) \text{ as } (2-\alpha_k) \downarrow 0.$$

That is,

$$\lim_{\alpha_k \to 2} J_B x_k^* = \mathrm{P}_{\operatorname{zer}(A+B)}(0).$$

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[1] Fact Let $x \in \mathbb{R}^m$. Then the inclusions $(\forall \gamma \in (0,1)) \quad 0 \in Ax_\gamma + \gamma(x_\gamma - x)$ define a unique curve $(x_\gamma)_{\gamma \in (0,1)}$. Moreover, exactly one of the following holds:

1. $\operatorname{zer} A \neq \emptyset$ and $x_\gamma \to P_{\operatorname{zer} A} x$ as $\gamma \downarrow 0$.
2. $\operatorname{zer} A = \emptyset$ and $\|x_\gamma\| \to +\infty$ as $\gamma \downarrow 0$.

Proof continued

(b) Once α_k is fixed, we have $(x_n)_{n=1}^{+\infty}$ converges to x_k^* by Theorem 4(iii).

(c) In $\mathbb{R}^m$, by using that J_B is Lipschitz continuous, we get

$$\lim_{k\to+\infty} J_B(x_k^*) = J_B(\lim_{k\to+\infty} x_k^*) = J_B(x^*).$$

As we already proved $\lim_{k\to+\infty} J_B(x_k^*) = \mathrm{P}_{\mathrm{zer}(A+B)}(0)$, we have

$$J_B(x^*) = \mathrm{P}_{\mathrm{zer}(A+B)}(0).$$

Therefore, $J_B x^* \in \mathrm{zer}(A+B)$, and $\|J_B x^*\| \leq \|y\|$ for any $y \in \mathrm{zer}(A+B)$.

Least norm solution of convex feasibility

Theorem 8

Let $C_1, C_2 \subseteq \mathbb{R}^m$ be two closed convex suets such that $C_1 \cap \operatorname{ri} C_2 \neq \emptyset$ or $\operatorname{ri} C_1 \cap C_2 \neq \emptyset$. Then for every $1 < \alpha_k < 2$, the α_k-DR algorithm

$$\begin{cases} y_n = \mathrm{P}_{C_2}(x_n) \\ z_n = \mathrm{P}_{C_1}(\alpha_k y_n - x_n) \\ x_{n+1} = x_n + (z_n - y_n). \end{cases} \tag{3}$$

generates a sequence $(x_n)_{n=1}^{+\infty}$ such that:

1. $x_n \to x^*$.
2. $P_{C_2} x^*$ *is the least norm point of* $C_1 \cap C_2$.

Remark 2.1

The scheme is different from Dykstra's alternating projection algorithm.

Outline

Example 1

Let $f = I_{C_1}, g = I_{C_2}$, where C_1 is a circle centred at $(5, 0)$ with radius 2, and C_2 is a box centred at $(3, 1.5)$ with radius 1. Let $A = \partial f, B = \partial g$, the problem we want to solve is:

$$0 \in N_{C_1}(x) + N_{C_2}(x). \tag{4}$$

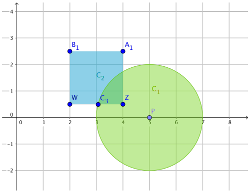

Figure: The plot of Example 1

[1] Let C be a set in $\mathbb{R}^m$. The indicator function is

$$I_C : \mathbb{R}^m \to [-\infty, +\infty] : x \mapsto \begin{cases} 0, & \text{if } x \in C; \\ +\infty & \text{otherwise.} \end{cases}$$

[2] Let C be a nonempty convex set in $\mathbb{R}^m$ and $x \in \mathbb{R}^m$. Then

$$N_C(x) = \begin{cases} \{u \in \mathbb{R}^m | \sup\langle C - x, u\rangle \le 0\}, & \text{if } x \in C; \\ \emptyset & \text{otherwise.} \end{cases}$$

Theoretical results

Let α_k be a increasing convergent sequence in $[1, 2)$ such that $\lim_{k \to +\infty} \alpha_k = 2$. Then the following holds:

1. The inclusion problem: For any fixed α_k, find $x \in \mathbb{R}^2$ such that

$$0 \in N_{C_1}(x) + N_{C_2}(x) + (2 - \alpha_k)(x) \tag{5}$$

   is reduced to (4) as $\alpha_k \to 2$.

2. The problem (5) can be solved by the α-Douglas-Rachford algorithm.

Numerical result

With $x_0 = (5, 1)$ and the stopping criteria being $\|x_{n+1} - x_n\| < \epsilon = 10^{-5}$, we obtain:

Table: α_k-DR: optimization point y^*, $\|y^*\|$.

α_k	y^*	$\|y^*\|$
1	(3.0635,0.5)	3.104
$2 - \frac{1}{10}$	(3.0635,0.5)	3.104
$2 - \frac{1}{50}$	(3.0635,0.5)	3.104
$2 - \frac{1}{100}$	(3.0635,0.5)	3.104
$2 - \frac{1}{1000}$	(3.0635,0.5)	3.104
$2 - \frac{1}{10000}$	(3.0635,0.5)	3.104

Numerical result

However, when we use the classic Douglas-Rachford algorithm to solve (4), the answer changes if we choose different starting point.

Table: DR: starting point x_0, optimization point y^*, $\|y^*\|$.

x_0	y^*	$\|y^*\|$
(5,1)	(4,0.8944)	4.0988
(-3,1)	(3.0785,0.5548)	3.1281
(-4,-6)	(4,0.5)	4.0311
(10,-20)	(4,0.5)	4.0311

1. As $\alpha_k \to 2$, the optimization result which is gotten by the α-Douglas-Rachford algorithm converges to the smallest norm solution of (4).
2. When using Douglas-Rachford algorithm to solve (4), the answer changes if we choose different starting point. However, the selection of starting points has no influence on the result when we use the α-Douglas-Rachford algorithm.

Outline

Combettes', Bot-Hendrich's primal-dual framework

Assume that

$$L : \mathbb{R}^m \to \mathbb{R}^m \text{ is a nonzero bounded linear invertible operator,}$$

and

$$r \in \mathbb{R}^m.$$

The primal problem: find a point $\bar{x} \in \mathbb{R}^m$ such that

$$0 \in A\bar{x} + L^\star(B \Box D)(L\bar{x} - r) \tag{P}$$

One can solve the primal-dual problem instead: find a point $(x, v) \in \mathbb{R}^m \times \mathbb{R}^m$ such that

$$\begin{cases} -L^\star v \in Ax \\ v \in (B \Box D)(Lx - r). \end{cases} \tag{PD}$$

[1] The parallel sum of B, D is defined as $B \Box D : \mathbb{R}^m \to 2^{\mathbb{R}^m}$, and

$$B \Box D = (B^{-1} + D^{-1})^{-1}.$$

Fact 9 (Bot and Hendrich' 2013, Combettes' 2013)

Define three set-valued operators M, Q and S as follows:

$$M : \mathcal{K} \to 2^{\mathcal{K}} : (x, v) \mapsto (Ax, r + B^{-1}v);$$
$$Q : \mathcal{K} \to 2^{\mathcal{K}} : (x, v) \mapsto (0, D^{-1}v);$$
$$S : \mathcal{K} \to \mathcal{K} : (x, v) \mapsto (L^\star v, -Lx).$$

Moreover, define an bounded linear operator

$$V : \mathcal{K} \to \mathcal{K} : (x, v) \mapsto (\frac{x}{\tau} - \frac{1}{2}L^\star v, \frac{v}{\sigma} - \frac{1}{2}Lx),$$

where $\tau, \sigma \in \mathbb{R}_{++}$, *and* $\tau\sigma\|L\|^2 < 4$.

Fact continued

Finally, define two operators on $\mathcal{K}V$:

$$\boldsymbol{A} := V^{-1}(\frac{1}{2}S + Q),$$
$$\boldsymbol{B} := V^{-1}(\frac{1}{2}S + M).$$

Here, the space $\mathcal{K}V$ is an inner product space with $\langle x, y\rangle_{\mathcal{K}V} = \langle x, Vy\rangle_{\mathcal{K}}$. Then any

$$(\bar{x}, \bar{v}) \in \operatorname{zer}(\boldsymbol{A} + \boldsymbol{B})$$

is a pair of primal-dual solution to problem(PD) and vice versa.

[1] Bot and Hendrich also showed:

- V^{-1} exists.
- $\boldsymbol{A}$ and $\boldsymbol{B}$ are maximally monotone on $\mathcal{K}V$, and $\operatorname{zer}(\boldsymbol{A} + \boldsymbol{B}) = \operatorname{zer}(M + S + Q)$.

When $\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B}) \neq \emptyset$, they used the Douglas-Rachford algorithm to get the solution of the problem with primal inclusion (P) together with dual inclusion (PD) :
Let $x_0 \in \mathbb{R}^m$ be the starting point. Set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} y_n = J_{\boldsymbol{B}} x_n \\ z_n = J_{\boldsymbol{A}}(2y_n - x_n) \\ x_{n+1} = x_n + (z_n - y_n). \end{cases}$$

Then there exists $x \in \operatorname{Fix} R_{\boldsymbol{A}} R_{\boldsymbol{B}}$ such that $J_{\boldsymbol{B}} x \in \operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$, and $(x_n)_{n=1}^{+\infty}$ converges to x.

The α-version primal-dual problem

Recall the construction of M, Q, S, V, $\boldsymbol{A}$ and $\boldsymbol{B}$. Let $\alpha \in [1,2)$, and for any $\beta \in \mathbb{R}$, define $B \overset{\beta}{\square} D = (B^{-1} + D^{-1} + \beta \operatorname{Id})^{-1}$. Then the following two inclusion problems are equivalent:

1. Find $(x, v) \in \mathbb{R}^m \times \mathbb{R}^m$ such that $(x, v) \in \operatorname{zer}(\boldsymbol{A} + \boldsymbol{B} + (2-\alpha)\operatorname{Id})$.
2. Solve the problem with primal inclusion: find $x \in \mathbb{R}^m$ such that

$$0 \in Ax + \frac{2-\alpha}{\tau}x + \frac{\alpha}{4-\alpha}\boldsymbol{L}^\star \circ (B \overset{\frac{2-\alpha}{\sigma}}{\square} D) \circ (\boldsymbol{L}x - r) \qquad (\alpha \text{ P})$$

   where $\boldsymbol{L} = \frac{4-\alpha}{2}L$, $\tau \in \mathbb{R}_{++}$ and $\sigma \in \mathbb{R}_{++}$, together with the dual inclusion: find (x, v) such that

$$\begin{cases} -\frac{\alpha}{4-\alpha}\boldsymbol{L}^\star v \in Ax + \frac{(2-\alpha)}{\tau}x \\ v \in (B \overset{\frac{2-\alpha}{\sigma}}{\square} D)(\boldsymbol{L}x - r). \end{cases} \qquad (\alpha \text{ PD})$$

When $0 \in \operatorname{int}(\operatorname{dom} \boldsymbol{A} - \operatorname{dom} \boldsymbol{B})$,

$$\operatorname{zer}(\boldsymbol{A} + \boldsymbol{B} + (2-\alpha)\operatorname{Id})$$

can be solved by using the α-Douglas-Rachford algorithm:

Let $x_0 \in \mathbb{R}^m \times \mathbb{R}^m$ be the starting point. Set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} y_n = J_{\boldsymbol{B}} x_n \\ z_n = J_{\boldsymbol{A}}(\alpha y_n - x_n) \\ x_{n+1} = x_n + (z_n - y_n). \end{cases}$$

Then there exists $x \in \operatorname{Fix} R_{\boldsymbol{A}}^{\alpha} R_{\boldsymbol{B}}^{\alpha}$ such that $J_{\boldsymbol{B}} x \in \operatorname{zer}(\boldsymbol{A} + \boldsymbol{B} + (2-\alpha)\operatorname{Id})$, and $(x_n)_{n=1}^{+\infty}$ converges to x.

The α-Douglas-Rachford algorithm can be used to solve the α-primal-dual problem with primal inclusion: find $x \in \mathbb{R}^m$ such that

$$0 \in Ax + \frac{2-\alpha}{\tau} x + \frac{\alpha}{4-\alpha} \boldsymbol{L}^\star \circ (B \overset{\frac{2-\alpha}{\sigma}}{\square} D) \circ (\boldsymbol{L}x - r) \qquad (\alpha \text{ P})$$

where $\boldsymbol{L} = \frac{4-\alpha}{2} L$, $\tau \in \mathbb{R}_{++}$ and $\sigma \in \mathbb{R}_{++}$, together with the primal-dual inclusion: find (x, v) such that

$$\begin{cases} -\frac{\alpha}{4-\alpha} \boldsymbol{L}^\star v \in Ax + \frac{(2-\alpha)}{\tau} x \\ v \in (B \overset{\frac{2-\alpha}{\sigma}}{\square} D) \circ (\boldsymbol{L}x - r). \end{cases} \qquad (\alpha \text{ D})$$

Theorem 10

Recall that

$$M : \mathcal{K} \to 2^{\mathcal{K}} : (x, v) \mapsto (Ax, r + B^{-1}v);$$
$$Q : \mathcal{K} \to 2^{\mathcal{K}} : (x, v) \mapsto (0, D^{-1}v);$$
$$S : \mathcal{K} \to \mathcal{K} : (x, v) \mapsto (L^\star v, -Lx);$$
$$V : \mathcal{K} \to \mathcal{K} : (x, v) \mapsto (\frac{x}{\tau} - \frac{1}{2}L^\star v, \frac{v}{\sigma} - \frac{1}{2}Lx),$$

where $\tau, \sigma \in \mathbb{R}_{++}$, *and* $\tau\sigma\|L\|^2 < 4$. *And*

$$\boldsymbol{A} := V^{-1}(\frac{1}{2}S + Q).$$
$$\boldsymbol{B} := V^{-1}(\frac{1}{2}S + M).$$

Then $\operatorname{dom} D^{-1} = \mathbb{R}^m$ *implies*

$$0 \in \operatorname{int}(\operatorname{dom} \boldsymbol{A} - \operatorname{dom} \boldsymbol{B}).$$

In particular, $\operatorname{dom} D^{-1} = \mathbb{R}^m$ *if* $D = N_{\{0\}}$, *or* $D = \operatorname{Id}$.

The least norm primal-dual solution

We can use α-Douglas-Rachford algorithm

$$\begin{cases} y_n = J_{\boldsymbol{B}} x_n \\ z_n = J_{\boldsymbol{A}}(\alpha_k y_n - x_n) \\ x_{n+1} = x_n + (z_n - y_n). \end{cases} \tag{6}$$

to find the solution of $\operatorname{zer}(\boldsymbol{A} + \boldsymbol{B} + (2 - \alpha_k)\operatorname{Id})$.

The smallest norm solution of $\operatorname{zer}(\boldsymbol{A} + \boldsymbol{B})$ gives the smallest norm primal-dual solution:

$$\begin{cases} -L^\star v \in Ax \\ v \in (B \square D)(Lx - r). \end{cases} \tag{PD}$$

The algorithm

The algorithm (6) can be rewritten as

$$\begin{cases} y_{1n} = J_{\tau A}(x_{1n} - \frac{\tau}{2} L^\star x_{2n}) \\ y_{2n} = J_{\sigma B^{-1}}(x_{2n} - \frac{\sigma}{2} L x_{1n} + \sigma L y_{1n}) \\ w_{1n} = \alpha_k y_{1n} - x_{1n} \\ w_{2n} = \alpha_k y_{2n} - x_{2n} \\ z_{1n} = w_{1n} - \frac{\tau}{2} L^\star w_{2n} \\ z_{2n} = J_{\sigma D^{-1}}(w_{2n} - \frac{\sigma}{2} L w_{1n} + \sigma L z_{1n}) \\ x_{1n+1} = x_{1n} + (z_{1n} - y_{1n}) \\ x_{2n+1} = x_{2n} + (z_{2n} - y_{2n}), \end{cases} \tag{7}$$

where $x_n = (x_{1n}, x_{2n}), y_n = (y_{1n}, y_{2n})$.

Outline

Example 2

Let $f = \iota_{C_1}, g = \iota_{C_2}$, where C_1 is a circle centred at $(5, 0)$ with radius 2, and C_2 is a box centred at $(3, 1.5)$ with radius 1. Let $A = \partial f, B = \partial g$, We aim to find the least norm primal-dual solution:

$$\begin{cases} -v \in N_{C_1}(x) \\ v \in N_{C_2}(x), \end{cases} \tag{8}$$

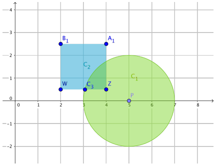

Figure: The plot of Example 2

We can solve (8) by the α-Douglas-Rachford method.

[1] $0 \in N_{C_1}(x) + (N_{C_2} \Box N_{\{0\}})(x)$ is equivalent to $0 \in N_{C_1}(x) + N_{C_2}(x)$.

Theoretical results

Let α_k be a increasing convergent sequence in $[1,2)$ such that $\lim_{k\to+\infty} \alpha_k = 2$. For each α_k, let $\boldsymbol{L} = \frac{4-\alpha_k}{2}\,\mathrm{Id}$. Then the following holds:

1. The problem with primal inclusion: find $x \in \mathbb{R}^n$ such that

$$0 \in N_{C_1}(x) + \frac{2-\alpha_k}{\tau}x + \frac{\alpha_k}{4-\alpha_k}\boldsymbol{L}^\star(N_{C_2}\Box\frac{\sigma}{2-\alpha_k}\,\mathrm{Id})(\boldsymbol{L}x), \qquad (9)$$

   where $\tau \in \mathbb{R}_{++}$, $\sigma \in \mathbb{R}_{++}$, and $\tau\sigma < 4$, together with the primal-dual inclusion: find (x, v) such that

$$\begin{cases} -\frac{\alpha_k}{4-\alpha_k}\boldsymbol{L}^\star v \in N_{C_1}(x) + \frac{2-\alpha_k}{\tau}x \\ v \in (N_{C_2}\Box\frac{\sigma}{2-\alpha_k}\,\mathrm{Id})(\boldsymbol{L}x) \end{cases} \qquad (10)$$

   reduces to (8) as $\alpha_k \to 2$.

2. The problem with primal-dual inclusion (10) can be solved by the α-Douglas-Rachford algorithm.

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[1] $(N_{C_2} \overset{\frac{2-\alpha_k}{\sigma}}{\Box} N_{\{0\}})$ is equivalent to $(N_{C_2}\Box\frac{\sigma}{2-\alpha_k}\,\mathrm{Id})$

Numerical result

Numerical result of (10) by using α-Douglas-Rachford algorithm with $\sigma = 2$, $\tau = 3/2$, and starting point $x_0 = (5, 1)$, $v_0 = (0, 0)$.

Table: Six fixed $\alpha_k = 2 - 1/k$, optimal point y_1^* and y_2^*, and the case $\alpha = 2$.

α_k	y_1^*	y_2^*	$\sqrt{\|y_1\|^2 + \|y_2\|^2}$
1	(3.0053,0.1460)	(1.0160,-0.5621)	3.2251
$2 - \frac{1}{10}$	(3.0565,0.4721)	(0,-0.0852)	3.0939
$2 - \frac{1}{50}$	(3.0622,0.4949)	(0,-0.0172)	3.1020
$2 - \frac{1}{100}$	(3.0629,0.4975)	(0,-0.0086)	3.1030
$2 - \frac{1}{1000}$	(3.0634,0.4997)	1.0e-03 *(0,-0.8606)	3.1039
$2 - \frac{1}{10000}$	(3.0635,0.5000)	1.0e-04 *(0,-0.8607)	3.1040
$\alpha = 2$	(3.6259,0.6339)	(0,0)	3.6809

Numerical result

Numerical result of (10) by using α-Douglas-Rachford algorithm with $\sigma = 1$, $\tau = 1$, and the same starting point $x_0 = (5, 1)$, $v_0 = (0, 0)$.

Table: Six fixed $\alpha_k = 2 - 1/k$, optimal point y_1^* and y_2^*, and the case $\alpha = 2$.

α_k	y_1^*	y_2^*	$\sqrt{\|y_1\|^2 + \|y_2\|^2}$
1	(3.0014,0.0740)	(0.5021,-0.3890)	3.0687
$2 - \frac{1}{10}$	(3.0546,0.4642)	(0,-0.1256)	3.0922
$2 - \frac{1}{50}$	(3.0621,0.4945)	(0,-0.0258)	3.1019
$2 - \frac{1}{100}$	(3.0628,0.4974)	(0,-0.0129)	3.1030
$2 - \frac{1}{1000}$	(3.0634,0.4997)	(0,-0.0013)	3.1039
$2 - \frac{1}{10000}$	(3.0635,0.5000)	1.0e-03 *(0,-0.1291)	3.1040
$\alpha = 2$	(3.7500,0.7500)	(0,0)	3.8243

Numerical result

Numerical result of (10) by using α-Douglas-Rachford algorithm with $\sigma = 1$, $\tau = 1$, and with another starting point $x_0 = (-4, -6)$, $v_0 = (0, 0)$.

Table: Six fixed $\alpha_k = 2 - 1/k$, optimal point y_1^* and y_2^*, and the case $\alpha = 2$.

α_k	y_1^*	y_2^*	$\sqrt{\|y_1\|^2 + \|y_2\|^2}$
1	(3.0014,0.0740)	(0.5021,-0.3890)	3.0687
$2 - \frac{1}{10}$	(3.0546,0.4642)	(0,-0.1256)	3.0922
$2 - \frac{1}{50}$	(3.0621,0.4945)	(0,-0.0258)	3.1019
$2 - \frac{1}{100}$	(3.0628,0.4974)	(0,-0.0129)	3.1030
$2 - \frac{1}{1000}$	(3.0634,0.4997)	(0,-0.0013)	3.1039
$2 - \frac{1}{10000}$	(3.0635,0.5000)	1.0e-03 *(0,-0.1291)	3.1040
$\alpha = 2$	(3.3945,0.6448)	(0,0)	3.4552

1. If we let $y^* = (3.0635, 0.5000)$ and $w^* = (0, 0)$, tables 3, 4, and 5 all shows that when $\alpha_k \to 2$, we have the smallest norm primal-dual solution (y^*, w^*), where y^* solves the primal and w^* solves the dual.

2. Once we fix the value of k with fixed τ and σ, the result we get by using α-Douglas-Rachford algorithm does not change if we change its starting point.

3. In three tables 3, 4, and 5, $\alpha = 2$ gives different y_1^* is because

$$\begin{cases} -v \in N_{C_1}(x) \\ v \in N_{C_2}(x), \end{cases} \tag{11}$$

has multiple solutions.

Possible future works

1. If we change the space from $\mathbb{R}^n$ to a more general space, like $\mathcal{H}$, a general Hilbert space, does the α-Douglas-Rachford algorithm have the same results and properties?

2. More numerical experiments on the α-Douglas-Rachford algorithm for higher dimensions and practical applications are required.

3. Consider $T_{\alpha,\beta,\gamma} = (1-\gamma)\,\mathrm{Id} + \gamma R_A^{\beta} R_B^{\alpha}$?

4. A comparison to Aragón Artacho's recent work?

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Thank you!