SO(3) monopoles and relations between Donaldson and Seiberg-Witten invariants

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Outline

Introduction and main results

Introduction I

In his article [52], Witten (1994)

- Gave a formula expressing the Donaldson series in terms of Seiberg-Witten invariants for *standard* four-manifolds,
- Outlined an argument based on supersymmetric quantum field theory, his previous work [51] on topological quantum field theories (TQFT), and his work with Seiberg [44, 45] explaining how to derive this formula.

(We call a four-manifold standard if it is closed, connected, oriented, and smooth with odd $b^+ \geq 3$ and $b_1 = 0$.)

In a later article [35], Moore and Witten

- extended the scope of Witten's previous formula by allowing four-dimensional manifolds with $b_1 > 0$ and $b^+ = 1$, and

Introduction II

- provided more details underlying the derivation of these formulae using supersymmetric quantum field theory.

Using similar supersymmetric quantum field theoretic ideas methods, Marinõ, Moore, and Peradze (1999) also showed that a certain low-degree polynomial part of the Donaldson series always vanishes [31, 32], a consequence of their notion of superconformal simple type.

Marinõ, Moore, and Peradze noted that this vanishing would confirm a conjecture (attributed to Fintushel and Stern) for a lower bound on the number of (Seiberg-Witten) basic classes of a four-dimensional manifold.

Introduction III

Soon after the Seiberg-Witten invariants were discovered, Pidstrigatch and Tyurin (1994) proposed a method [42] to prove Witten's formula using a classical field theory paradigm via the space of SO(3) monopoles which simultaneously extend the

- Anti-self-dual SO(3) connections, defining Donaldson invariants, and
- U(1) monopoles, defining Seiberg-Witten invariants.

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The Pidstrigatch-Tyurin SO(3)-monopole paradigm is intuitively appealing, but there are also significant technical difficulties in such an approach.

In this lecture, we summarize some ideas in our proofs that, for all standard four-manifolds, the SO(3)-monopole paradigm and

$$\begin{aligned} \textit{Seiberg-Witten simple type} &\implies \textit{Superconformal simple type}, \\ \textit{Superconformal simple type} &\implies \textit{Witten's Conjecture}. \end{aligned}$$

Taken together, these implications prove

- Marinõ, Moore, and Peradze's Conjecture on superconformal simple type and Fintushel and Stern's Conjecture on the lower bound on the number of basic classes, and

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- Witten's Conjecture on the relation between Donaldson and Seiberg-Witten invariants.

It is unknown whether all four-manifolds have Seiberg-Witten simple type.

More details can be found in

1. P. M. N. Feehan and T. G. Leness, *Gluing maps for* $\mathrm{SO}(3)$ *monopoles and invariants of smooth four-manifolds*, in preparation; based on recent work and arXiv:math/9812060 and arXiv:math/9907107.

and our two articles

1. P. M. N. Feehan and T. G. Leness, *The* $\mathrm{SO}(3)$ *monopole cobordism and superconformal simple type*, arXiv:1408.5307.

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2. P. M. N. Feehan and T. G. Leness, *Superconformal simple type and Witten's conjecture*, arXiv:1408.5085.

These are in turn based on methods described earlier in our

1. P. M. N. Feehan and T. G. Leness, *A general* $\mathrm{SO}(3)$*-monopole cobordism formula relating Donaldson and Seiberg-Witten invariants*, Memoirs of the American Mathematical Society, in press, arXiv:math/0203047.
2. P. M. N. Feehan and T. G. Leness, *Witten's conjecture for many four-manifolds of simple type*, Journal of the European Mathematical Society **17** (2015), 899–923.

Additional useful references include

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1. P. M. N. Feehan and T. G. Leness, PU(2) *monopoles. I: Regularity, compactness and transversality*, Journal of Differential Geometry 49 (1998), 265–410.
2. P. M. N. Feehan, *Generic metrics, irreducible rank-one* PU(2) *monopoles, and transversality*, Communications in Analysis & Geometry 8 (2000), 905–967.
3. P. M. N. Feehan and T. G. Leness, PU(2) *monopoles and links of top-level Seiberg-Witten moduli spaces*, Journal fr die Reine und Angewändte Mathematik 538 (2001), 57–133.
4. P. M. N. Feehan and T. G. Leness, PU(2) *monopoles. II: Top-level Seiberg-Witten moduli spaces and Witten's conjecture in low degrees*, Journal für die Reine und Angewändte Mathematik 538 (2001), 135–212.

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5. P. M. N. Feehan and T. G. Leness, SO(3) *monopoles, level-one Seiberg-Witten moduli spaces, and Witten's conjecture in low degrees*, Topology and its Applications 124 (2002), 221–326.

Statements of main results I

A closed, oriented four-manifold X has an *intersection form*,

$$Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}.$$

One lets $b^{\pm}(X)$ denote the dimensions of the maximal positive or negative subspaces of the form Q_X on $H_2(X;\mathbb{Z})$ and

$$e(X) := \sum_{i=0}^{4}(-1)^i b_i(X) \quad \text{and} \quad \sigma(X) := b^+(X) - b^-(X)$$

denote the *Euler characteristic* and *signature* of X, respectively.

Statements of main results II

We define the characteristic numbers,

$$(1)\qquad \begin{aligned} c_1^2(X) &:= 2e(X) + 3\sigma(X), \\ \chi_h(X) &:= (e(X) + \sigma(X))/4, \\ c(X) &:= \chi_h(X) - c_1^2(X). \end{aligned}$$

Recall that a four-manifold X is standard if it is closed, connected, oriented, and smooth with odd $b^+(X) \geq 3$ and $b_1(X) = 0$.

(The methods we will describe allow $b^+(X) = 1$ and $b_1(X) > 0$.)

For a standard four-manifold, the Seiberg-Witten invariants comprise a function,

$$SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z},$$

Statements of main results III

on the set of spin^c structures on X.

The set of Seiberg-Witten basic classes, $B(X)$, is the image under $c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z})$ of the support of SW_X, that is

$$B(X) := \{K \in H^2(X;\mathbb{Z}) : K = c_1(\mathfrak{s}) \text{ with } SW_X(\mathfrak{s}) \neq 0\},$$

and is finite.

X has Seiberg-Witten simple type if $K^2 = c_1^2(X)$, $\forall\, K \in B(X)$.

(Here, $c_1(\mathfrak{s})^2 = c_1^2(X) \iff$ the moduli space, $M_{\mathfrak{s}}$, of Seiberg-Witten monopoles has dimension zero.)

Statements of main results IV

According to Kronheimer and Mrowka [26, Theorem 1.7 (a)], the Donaldson series of a standard four-manifold of simple type (in their sense), for any $w \in H^2(X;\mathbb{Z})$, is given by

$$(2) \qquad \mathbf{D}_X^w(h) = e^{Q_X(h)/2} \sum_{K \in H^2(X;\mathbb{Z})} (-1)^{(w^2+K\cdot w)/2} \beta_X(K) e^{\langle K,h\rangle},$$

as an equality of analytic functions of $h \in H_2(X;\mathbb{R})$, where

$$(3) \qquad \beta_X : H^2(X;\mathbb{Z}) \to \mathbb{Q},$$

is a function such that $\beta_X(K) \neq 0$ for at most finitely many classes, K, which are integral lifts of $w_2(X) \in H^2(X;\mathbb{Z}/2\mathbb{Z})$ (the Kronheimer-Mrowka basic classes).

Statements of main results V

Conjecture 1.1 (Witten's Conjecture)

Let X be a standard four-manifold. If X has Seiberg-Witten simple type, then X has Kronheimer-Mrowka simple type, the Seiberg-Witten and Kronheimer-Mrowka basic classes coincide, and for any $w \in H^2(X;\mathbb{Z})$,

$$(4) \quad \mathbf{D}_X^w(h) = 2^{2-(\chi_h - c_1^2)} e^{Q_X(h)/2} \times \sum_{\mathfrak{s} \in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}), h\rangle}, \quad \forall\, h \in H_2(X;\mathbb{R}).$$

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As defined by Mariño, Moore, and Peradze, [32, 31], a manifold X has superconformal simple type if $c(X) \leq 3$ or $c(X) \geq 4$ and for $w \in H^2(X; \mathbb{Z})$ characteristic,

$$\boxed{SW_X^{w,i}(h) = 0 \quad \text{for } i \leq c(X) - 4} \tag{5}$$

and all $h \in H_2(X; \mathbb{R})$, where

$$\boxed{SW_X^{w,i}(h) := \sum_{\mathfrak{s} \in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s}) \cdot w)} SW_X(\mathfrak{s}) \langle c_1(\mathfrak{s}), h \rangle^i}$$

From [8], we have the

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Theorem 1.2 (All standard four-manifolds with Seiberg-Witten simple type have superconformal simple type)

(See F and Leness [8, Theorem 1.1].) If X is a standard four-manifold of Seiberg-Witten simple type, then X has superconformal simple type.

Marinõ, Moore, and Peradze had previously shown [32, Theorem 8.1.1] that if the set of Seiberg-Witten basic classes, $B(X)$, is non-empty and X has superconformal simple type, then

$$\boxed{|B(X)/\{\pm 1\}| \geq [c(X)/2].} \tag{6}$$

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For example, suppose X is the K3 surface.

Because $c(X) \leq 3$, the K3 surface is superconformal simple type by our definition.

It is known that $B(X) = \{0\}$, so $|B(X)/\{\pm 1\}| = 1$, while

$$b_1(X) = 0, \quad b^+(X) = 3, \quad b^-(X) = 19,$$

which gives $e(X) = 24$, $\sigma(X) = -16$, $c_1^2(X) = 0$, $\chi_h(X) = 2$, and $c(X) = 2$, so $[c(X)/2] = 1$ and equality holds in (6).

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Theorem 1.2 and [32, Theorem 8.1.1] therefore yield a proof of the following result, first conjectured by Fintushel and Stern [15].

Corollary 1.3 (Lower bound for the number of basic classes)

(See F and Leness [8, Corollary 1.2]) Let X be a standard four-manifold of Seiberg-Witten simple type. If $B(X)$ is non-empty and $c(X) \geq 3$, then the number of basic classes obeys the lower bound (6).

From [10], we have the

Statements of main results X

Theorem 1.4 (Superconformal simple type $\implies$ Witten's Conjecture holds for all standard four-manifolds)

(See F and Leness [10, Theorem 1.2].) If a standard four-manifold with has superconformal simple type, then it satisfies Witten's Conjecture 1.1.

Combining Theorems 1.2 and 1.4 thus yields the following

Corollary 1.5 (Witten's Conjecture holds for all standard four-manifolds)

(See F and Leness [10, Corollary 1.3] or [8, Corollary 1.4].) If X is a standard four-manifold of Seiberg-Witten simple type then X satisfies Witten's Conjecture 1.1.

Further results and future directions I

Kronheimer and Mrowka applied our SO(3)-monopole cobordism formula (Theorem 3.3) to give their first of two proofs of Property P for knots in [27].

Property P asserts that $+1$ surgery on a non-trivial knot K in S^3 yields a three-manifold which is not a homotopy sphere.

In parallel to their role in confirming the relationship between the Donaldson and Seiberg-Witten invariants of four-manifolds, one could use SO(3) monopoles to explore the relationship between the instanton (Yang-Mills) and monopole (Seiberg-Witten) Floer homologies of three-manifolds.

Review of Donaldson and Seiberg-Witten invariants

Seiberg-Witten invariants

Seiberg-Witten invariants I

Detailed expositions of the theory of Seiberg-Witten invariants, introduced by Witten in [52], are provided in [28, 36, 39].

These invariants define an integer-valued map with finite support,

$$SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z},$$

on the set of spin^c structures on X.

Aside: A spin^c structure, $\mathfrak{s} = (W^\pm, \rho_W)$ on X, consists of a pair of complex rank-two bundles $W^\pm \to X$ and a Clifford multiplication map $\rho = \rho_W : T^*X \to \mathrm{Hom}_{\mathbb{C}}(W^\pm, W^\mp)$ such that [25, 29, 43]

$$(7) \qquad \rho(\alpha)^* = -\rho(\alpha) \quad \text{and} \quad \rho(\alpha)^*\rho(\alpha) = g(\alpha, \alpha)\,\mathrm{id}_W,$$

Seiberg-Witten invariants II

for all $\alpha \in C^\infty(T^*X)$, where $W = W^+ \oplus W^-$ and g denotes the Riemannian metric on T^*X.

The Clifford multiplication ρ induces canonical isomorphisms $\Lambda^\pm \cong \mathfrak{su}(W^\pm)$, where $\Lambda^\pm = \Lambda^\pm(T^*X)$ are the bundles of self-dual and anti-self-dual two-forms, with respect to the Riemannian metric g on T^*X.

Any two spin connections on W differ by an element of $\Omega^1(X; i\mathbb{R})$, since the induced connection on $\mathfrak{su}(W) \cong \Lambda^2$ is determined by the Levi-Civita connection for the metric g on T^*X.

Consider a spin connection, B, on W and section $\Psi \in C^\infty(W^+)$.

Seiberg-Witten invariants III

We call a pair (B, Ψ) a Seiberg-Witten monopole if

$$
\begin{aligned}
\operatorname{Tr}(F_B^+) - \tau\rho^{-1}(\Psi \otimes \Psi^*)_0 - \eta &= 0, \\
D_B\Psi + \rho(\vartheta)\Psi &= 0,
\end{aligned}
\tag{8}
$$

where, writing $\mathfrak{u}(W^+) = i\underline{\mathbb{R}} \oplus \mathfrak{su}(W^+)$,

- $F_B^+ \in C^\infty(\Lambda^+ \otimes \mathfrak{u}(W^+))$ is the self-dual component of the curvature F_B of B, and
- $\operatorname{Tr}(F_B^+) \in C^\infty(\Lambda^+ \otimes i\underline{\mathbb{R}})$ is the trace part of F_B^+,
- $D_B = \rho \circ \nabla_B : C^\infty(W^+) \to C^\infty(W^-)$ is the Dirac operator defined by the spin connection B,
- The perturbation terms τ and ϑ are as in our version of the forthcoming SO(3)-monopole equations (17),

Seiberg-Witten invariants IV

- $\eta \in C^\infty(i\Lambda^+)$ is an additional perturbation term,
- The quadratic term $\Psi \otimes \Psi^*$ lies in $C^\infty(i\mathfrak{u}(W^+))$ and $(\Psi \otimes \Psi^*)_0$ denotes the traceless component lying in $C^\infty(i\mathfrak{su}(W^+))$, so $\rho^{-1}(\Psi \otimes \Psi^*)_0 \in C^\infty(i\Lambda^+)$.

In the usual presentation of the Seiberg-Witten equations, one takes $\tau = \mathrm{id}_{\Lambda^+}$ and $\vartheta = 0$, while η is a generic perturbation.

However, in order to identify solutions to the Seiberg-Witten equations (8) with reducible solutions to the forthcoming SO(3)-monopole equations (17), one needs to employ the perturbations given in equation (8) and choose

$$\eta = F^+_{A_\Lambda}, \tag{9}$$

Seiberg-Witten invariants V

where A_Λ is the fixed unitary connection on the line bundle $\det^{\frac{1}{2}}(V^+)$ with Chern class denoted by $c_1(\mathfrak{t}) = \Lambda \in H^2(X;\mathbb{Z})$ and represented by the real two-form $(1/2\pi i)F_{A_\Lambda}$, where $V = W \otimes E$ and $V^\pm = W^\pm \otimes E$.

Here, E is a rank-two, Hermitian bundle over X arising in definitions of anti-self-dual SO(3) connections and SO(3) monopoles.

Given a spin^c structure, $\mathfrak{s}$, one may construct a moduli space, $M_\mathfrak{s}$, of solutions to the Seiberg-Witten monopole equations, modulo gauge equivalence.

Seiberg-Witten invariants VI

The space, $M_{\mathfrak{s}}$, is a compact, finite-dimensional, oriented, smooth manifold (for generic perturbations of the Seiberg-Witten monopole equations) of dimension

$$\dim M_{\mathfrak{s}} = \frac{1}{4}\left(c_1(\mathfrak{s})^2 - 2\chi - 3\sigma\right), \tag{10}$$

and contains no zero-section points $[B, 0]$.

When $M_{\mathfrak{s}}$ has odd dimension, the Seiberg-Witten invariant, $SW_X(\mathfrak{s})$, is defined to be zero.

When $M_{\mathfrak{s}}$ has dimension zero, then $SW_X(\mathfrak{s})$, is defined by counting the number of points in $M_{\mathfrak{s}}$.

Seiberg-Witten invariants VII

When $M_{\mathfrak{s}}$ has even positive dimension d_s, one defines

$$SW_X(\mathfrak{s}) := \langle \mu_{\mathfrak{s}}^{d_s/2}, [M_{\mathfrak{s}}] \rangle,$$

where $\mu_{\mathfrak{s}} = c_1(\mathbb{L}_{\mathfrak{s}})$ is the first Chern class of the universal complex line bundle over the configuration space of pairs.

If $\mathfrak{s} \in \mathrm{Spin}^c(X)$, then $c_1(\mathfrak{s}) := c_1(W^+) \in H^2(X;\mathbb{Z})$ and $c_1(\mathfrak{s}) \equiv w_2(X) \pmod 2 \in H^2(X;\mathbb{Z}/2\mathbb{Z})$, where $w_2(X)$ is second Stiefel-Whitney class of X.

One calls $c_1(\mathfrak{s})$ a Seiberg-Witten basic class if $SW_X(\mathfrak{s}) \neq 0$. Define

$$B(X) = \{c_1(\mathfrak{s}) : SW_X(\mathfrak{s}) \neq 0\}. \tag{11}$$

Seiberg-Witten invariants VIII

If $H^2(X;\mathbb{Z})$ has 2-torsion, then $c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z})$ is not injective and as we will work with functions involving real homology and cohomology, we define

$$(12) \qquad SW_X' : H^2(X;\mathbb{Z}) \ni K \mapsto \sum_{\mathfrak{s}\in c_1^{-1}(K)} SW_X(\mathfrak{s}) \in \mathbb{Z}.$$

With the preceding definition, Witten's Formula (4) is equivalent to

$$(13) \quad \mathbf{D}_X^w(h) = 2^{2-(\chi_h - c_1^2)} e^{Q_X(h)/2} \times \sum_{K\in B(X)} (-1)^{\frac{1}{2}(w^2+K\cdot w)} SW_X'(K) e^{\langle K,h\rangle}.$$

A four-manifold, X, has Seiberg-Witten simple type if $SW_X(\mathfrak{s}) \neq 0$ implies that $c_1^2(\mathfrak{s}) = c_1^2(X)$ (or, in other words, $\dim M_{\mathfrak{s}} = 0$).

Donaldson invariants

Donaldson invariants I

For $w \in H^2(X;\mathbb{Z})$, the Donaldson invariant is a linear function,

$$D_X^w : \mathbb{A}(X) \to \mathbb{R},$$

where $\mathbb{A}(X) = \mathrm{Sym}(H_{\mathrm{even}}(X;\mathbb{R}))$, the symmetric algebra.

For $h \in H_2(X;\mathbb{R})$ and a generator $x \in H_0(X;\mathbb{Z})$, one defines $D_X^w(h^{\delta-2m}x^m) = 0$ unless

$$\delta \equiv -w^2 - 3\chi_h(X) \pmod 4. \tag{14}$$

Given (14), then $D_X^w(h^{\delta-2m}x^m)$ is (heuristically) defined by pairing

1. A cohomology class $\mu(z)$ of dimension 2δ on the configuration space of SO(3) connections on $\mathfrak{su}(E)$, corresponding to the degree-δ element $z = h^{\delta-2m}x^m \in \mathbb{A}(X)$, and

Donaldson invariants II

2. A fundamental class $[\bar{M}_\kappa^w(X)]$ defined by the Uhlenbeck compactification of a moduli space $M_\kappa^w(X)$ of anti-self-dual SO(3) connections on $\mathfrak{su}(E)$, where $\kappa = -\frac{1}{4}p_1(\mathfrak{su}(E))$ and E is a rank-two Hermitian bundle with $w = c_1(E)$.

See Donaldson [2], Donaldson and Kronheimer [3], Friedman and Morgan [16], Kronheimer and Mrowka [26], and Morgan and Mrowka [37] for precise definitions of $D_X^w(h^{\delta-2m}x^m)$.

Suppose A is a unitary connection on a Hermitian vector bundle E over X and $\hat{A}$ is the induced SO(3) connection on $\mathfrak{su}(E)$.

One calls $\hat{A}$ *anti-self-dual* (with respect to the metric, g, on X) if

$$F_{\hat{A}}^+ = 0,$$

Donaldson invariants III

where $F_{\hat{A}}$ is the curvature of $\hat{A}$ and $F_{\hat{A}}^{+}$ is its self-dual component with respect to the splitting, $\Lambda^2(T^*X) = \Lambda^+ \oplus \Lambda^-$.

We denote $\kappa = -\frac{1}{4}p_1(\mathfrak{su}(E))$ and $w = c_1(E)$ and write $M_\kappa^w(X)$ for the moduli space of gauge-equivalence classes of anti-self-dual SO(3) connections on $\mathfrak{su}(E)$.

A four-manifold has Kronheimer-Mrowka simple type if for all $w \in H^2(X;\mathbb{Z})$ and all $z \in \mathbb{A}(X)$ one has

$$D_X^w(x^2 z) = 4D_X^w(z). \tag{15}$$

Donaldson invariants IV

This equality implies that the Donaldson invariants are determined by the Donaldson series (see Kronheimer and Mrowka [26, Section 2]), the formal power series

$$\mathbf{D}_X^w(h) := D_X^w((1+\tfrac{1}{2}x)e^h), \quad \forall\, h \in H_2(X;\mathbb{R}), \tag{16}$$

and computed using their formula (2).

More generally (see Kronheimer and Mrowka [24]), a four-manifold X has *finite type* or *type* τ if

$$D_X^w((x^2-4)^\tau z) = 0,$$

for some $\tau \in \mathbb{N}$ and all $z \in \mathbb{A}(X)$.

Donaldson invariants V

Kronheimer and Mrowka conjectured [24] that all standard four-manifolds X have finite type and state an analogous formula for the series $\mathbf{D}_X^w(h)$.

Proofs of different parts of their conjecture have been given by Frøyshov [17, Corollary 1], Muñoz [40, Corollary 7.2 & Proposition 7.6], and Wieczorek [50, Theorem 1.3].

SO(3)-monopole cobordism

SO(3)-monopole equations I

The SO(3)-monopole equations take the form,

$$\begin{aligned} \mathrm{ad}^{-1}(F_{\hat{A}}^+) - \tau\rho^{-1}(\Phi \otimes \Phi^*)_{00} &= 0, \\ D_A\Phi + \rho(\vartheta)\Phi &= 0. \end{aligned} \tag{17}$$

where

- A is a spin connection on $V = W \otimes E$ and E is a Hermitian, rank-two bundle,
- $\Phi \in C^\infty(W^+ \otimes E)$,
- $F_{\hat{A}}^+ \in C^\infty(\Lambda^+ \otimes \mathfrak{so}(\mathfrak{su}(E)))$ is the self-dual component of the curvature $F_{\hat{A}}$ of the induced SO(3) connection, $\hat{A}$, on $\mathfrak{su}(E)$,
- $\mathrm{ad}^{-1}(F_{\hat{A}}^+) \in C^\infty(\Lambda^+ \otimes \mathfrak{su}(E))$,

SO(3)-monopole equations II

- $D_A = \rho \circ \nabla_A : C^\infty(V^+) \to C^\infty(V^-)$ is the Dirac operator,
- $\tau \in C^\infty(\mathrm{GL}(\Lambda^+))$ and $\vartheta \in C^\infty(\Lambda^1 \otimes \mathbb{C})$ are perturbation parameters.

Aside: For $\Phi \in C^\infty(V^+)$, we let Φ^* denote its pointwise Hermitian dual and let $(\Phi \otimes \Phi^*)_{00}$ be the component of $\Phi \otimes \Phi^* \in C^\infty(i\mathfrak{u}(V^+))$ which lies in the factor $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$ of the decomposition,

$$i\mathfrak{u}(V^+) \cong \underline{\mathbb{R}} \oplus i\mathfrak{su}(V^+).$$

The Clifford multiplication ρ defines an isomorphism $\rho : \Lambda^+ \to \mathfrak{su}(W^+)$ and thus an isomorphism

$$\rho = \rho \otimes \mathrm{id}_{\mathfrak{su}(E)} : \Lambda^+ \otimes \mathfrak{su}(E) \cong \mathfrak{su}(W^+) \otimes \mathfrak{su}(E).$$

SO(3)-monopole equations III

Note also that

$$\mathrm{ad} : \mathfrak{su}(E) \to \mathfrak{so}(\mathfrak{su}(E))$$

is an isomorphism of real vector bundles.

We fix, once and for all, a smooth, unitary connection A_Λ on the square-root determinant line bundle, $\det^{\frac{1}{2}}(V^+)$, and require that our unitary connections A on $V = V^+ \oplus V^-$ induce the resulting unitary connection on $\det(V^+)$,

$$A^{\det} = 2A_\Lambda \text{ on } \det(V^+), \tag{18}$$

where $A^{\det}$ is the connection on $\det(V^+)$ induced by $A|_{V^+}$.

SO(3)-monopole equations IV

If a unitary connection A on V induces a connection $A^{\det} = 2A_\Lambda$ on $\det(V^+)$, then it induces the connection A_Λ on $\det^{\frac{1}{2}}(V^+)$.

We let $\mathscr{M}_\mathfrak{t}$ denote the moduli space of solutions to the SO(3)-monopole equations (17) moduli gauge-equivalence, where $\mathfrak{t} = (\rho, W^\pm, E)$.

The moduli space, $\mathscr{M}_\mathfrak{t}$, of SO(3) monopoles contains the

- Moduli subspace of anti-self-dual SO(3) connections, M_κ^w, identified with the subset of equivalence classes of SO(3) monopoles, $[A, 0]$, with $\Phi \equiv 0$, and

SO(3)-monopole equations V

- Moduli subspaces, $M_{\mathfrak{s}}$, of Seiberg-Witten monopoles, identified with subsets of equivalence classes of SO(3) monopoles, $[A_1 \oplus A_2, \Phi_1 \oplus 0]$, where the connections, A on E, become reducible with respect to different splittings, $E = L_1 \oplus L_2$, and A_i is a $U(1)$ connection on L_i, and $\mathfrak{s} = (\rho, W^{\pm} \oplus L_1)$.

We let $\mathscr{M}_{\mathfrak{t}}^{*,0}$ denote the complement in $\mathscr{M}_{\mathfrak{t}}$ of these subspaces of zero-section and reducible SO(3) monopoles.

For generic perturbations, $\mathscr{M}_{\mathfrak{t}}^{*,0}$ is a (finite-dimensional) smooth, orientable manifold (see F [4], F and Leness [11], or Teleman [49]).

SO(3)-monopole equations VI

For $\mathfrak{t} = (W^{\pm} \otimes E, \rho)$ and $\ell \geq 0$, define $\mathfrak{t}(\ell) := (W^{\pm} \otimes E_\ell, \rho)$, where

$$c_1(E_\ell) = c_1(E), \quad c_2(E_\ell) = c_2(E) - \ell.$$

We define the space of ideal SO(3) monopoles by

$$I^N \mathscr{M}_{\mathfrak{t}} := \bigsqcup_{\ell=0}^{N} \left(\mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^\ell(X) \right),$$

where $\mathrm{Sym}^\ell(X)$ is the symmetric product of X and N is a large integer (depending on perturbations and geometry of X and E).

The Uhlenbeck compactification, $\bar{\mathscr{M}}_{\mathfrak{t}}$, is the closure of $\mathscr{M}_{\mathfrak{t}}$ in $I^N \mathscr{M}_{\mathfrak{t}}$ (see F and Leness [11]).

SO(3)-monopole equations VII

Because $\dim \mathscr{M}_{\mathfrak{t}(\ell)} = \dim \mathscr{M}_{\mathfrak{t}} - 6\ell$, one has

$$\dim \mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^{\ell}(X) = \dim \mathscr{M}_{\mathfrak{t}} - 2\ell.$$

For each level, ℓ, in the range $0 \leq \ell \leq N$, the top ($\ell = 0$) and lower level ($\ell \geq 1$) subspaces of $\bar{\mathscr{M}}_{\mathfrak{t}}$,

$$\bar{\mathscr{M}}_{\mathfrak{t}} \cap \left(\mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^{\ell}(X)\right), \quad 0 \leq \ell \leq N,$$

may contain (ideal) Seiberg-Witten moduli subspaces,

$$\mathscr{M}_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X) \subset \mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^{\ell}(X).$$

SO(3)-monopole cobordism formula for link pairings I

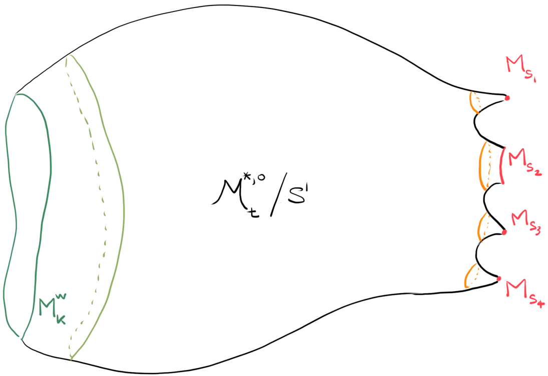
Our forthcoming SO(3)-monopole cobordism formula (20) is proved by evaluating the pairings of cup products of suitable *cohomology classes* on $\bar{\mathscr{M}}_{\mathfrak{t}}$ with (or intersecting *geometric representatives* of those classes with) the

1. Link $\mathbf{L}_{\mathfrak{t}}^{\mathrm{asd}}$ of the moduli subspace of anti-self-dual SO(3) connections, $\bar{M}_{\kappa}^{w}$, potentially giving multiples of the Donaldson invariant,
2. Links $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of the Seiberg-Witten moduli subspaces, $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$, giving sums of multiples of the Seiberg-Witten invariants.

SO(3)-monopole cobordism formula for link pairings II

The following figure illustrates the SO(3)-monopole cobordism between codimension-one links in $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ of $\bar{M}_\kappa^w$ and $M_{\mathfrak{s}_i} \times \mathrm{Sym}^\ell(X)$.

SO(3)-monopole cobordism formula for link pairings III



SO(3)-monopole cobordism formula for link pairings IV

We shall use a definition of a link of a stratum in smoothly stratified space, following Mather [33] and Goresky-MacPherson [19].

We need only consider the relatively simple case of a stratified space with two strata since the lower strata in

$$\mathscr{M}_{\mathfrak{t}} \cong \mathscr{M}_{\mathfrak{t}}^{*,0} \sqcup M_{\kappa}^{w} \cup \bigsqcup_{\mathfrak{s}} M_{\mathfrak{s}} \tag{19}$$

do not intersect when $\mathscr{M}_{\mathfrak{t}}$ *contains no reducible, zero-section solutions*.

SO(3)-monopole cobordism formula for link pairings V

The finite union in (19) over $\mathfrak{s}$ is over the subset of all spin^c structures for which $M_{\mathfrak{s}}$ is non-empty and for which there is a splitting $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s}'$.

Aside: The space, Z, in the forthcoming Definition 3.1 is a *smoothly stratified space* (with two strata) in the sense of Morgan, Mrowka, and Ruberman [38, Chapter 11].)

SO(3)-monopole cobordism formula for link pairings VI

Definition 3.1 (Link of a stratum in a smoothly stratified space)

Let Z be a closed subset of a smooth, Riemannian manifold M, and suppose that $Z = Z_0 \cup Z_1$, where Z_0 and Z_1 are locally closed, smooth submanifolds of M and $Z_1 \subset \bar{Z}_0$.

Let N_{Z_1} be the normal bundle of $Z_1 \subset M$ and let $\mathscr{O}' \subset N_{Z_1}$ be an open neighborhood of the zero section $Z_1 \subset N_{Z_1}$ such that there is a diffeomorphism γ, commuting with the zero section of N_{Z_1} (so $\gamma|_{Z_1} = \mathrm{id}_{Z_1}$), from $\mathscr{O}'$ onto an open neighborhood $\gamma(\mathscr{O}')$ of $Z_1 \subset M$.

Let $\mathscr{O} \Subset \mathscr{O}'$ be an open neighborhood of the zero section $Z_1 \subset N_{Z_1}$, where $\bar{\mathscr{O}} = \mathscr{O} \cup \partial\mathscr{O} \subset \mathscr{O}'$ is a smooth manifold-with-boundary.

Then $L_{Z_1} := Z_0 \cap \gamma(\partial\mathscr{O})$ is a *link of Z_1 in Z_0*.

SO(3)-monopole cobordism formula for link pairings VII

The compactification $\bar{\mathscr{M}}_{\mathfrak{t}}^{*,0}/S^1$ defines a compact cobordism, stratified by smooth oriented manifolds, between

$$\mathbf{L}_{\mathfrak{t}}^{\mathrm{asd}} \quad \text{and} \quad \bigsqcup_{\mathfrak{s}\in\overline{\mathrm{Red}}(\mathfrak{t})} \mathbf{L}_{\mathfrak{t},\mathfrak{s}}.$$

For $\delta + \eta_c = \frac{1}{2}\dim \mathbf{L}_{\mathfrak{t}}^{\mathrm{asd}}$, this cobordism gives the following equality (see F and Leness [6, Equation (1.6.1)]),

$$\boxed{\begin{aligned}&\#\left(\bar{\mathscr{V}}(h^{\delta-2m}x^m)\cap\bar{\mathscr{W}}^{\eta_c}\cap\mathbf{L}_{\mathfrak{t}}^{\mathrm{asd}}\right)\\ &\quad = -\sum_{\mathfrak{s}\in\overline{\mathrm{Red}}(\mathfrak{t})}(-1)^{\frac{1}{2}(w^2-\sigma)+\frac{1}{2}(w^2+(w-c_1(\mathfrak{t}))\cdot c_1(\mathfrak{s}))}\\ &\qquad\qquad \times\#\left(\bar{\mathscr{V}}(h^{\delta-2m}x^m)\cap\bar{\mathscr{W}}^{\eta_c}\cap\mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right).\end{aligned}} \tag{20}$$

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SO(3)-monopole cobordism formula for link pairings VIII

The left-hand side of (20) is obtained by computing the intersection number for geometric representatives on $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ with the link of the moduli subspace $\bar{M}_\kappa^w$ of anti-self-dual SO(3) connections.

The right-hand side of (20) is obtained by computing the intersection numbers for geometric representatives on $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ with the links of the moduli subspaces $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ of ideal Seiberg-Witten monopoles appearing in $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$.

SO(3)-monopole cobordism and Donaldson invariants I

It will be more convenient to have Witten's Formula (4) expressed at the level of the Donaldson polynomial invariants rather than the Donaldson power series which they form.

Let $B'(X)$ be a fundamental domain for the action of $\{\pm 1\}$ on $B(X)$.

SO(3)-monopole cobordism and Donaldson invariants II

Lemma 3.2 (Donaldson invariants implied by Witten's formula)

(See F and Leness [12, Lemma 4.2].) Let X be a standard four-manifold. Then X satisfies equation (4) and has Kronheimer-Mrowka simple type if and only if the Donaldson invariants of X satisfy $D_X^w(h^{\delta-2m}x^m) = 0$ for $\delta \not\equiv -w^2 - 3\chi_h \pmod 4$ and for $\delta \equiv -w^2 - 3\chi_h \pmod 4$ satisfy

$$(21) \quad D_X^w(h^{\delta-2m}x^m) = \sum_{\substack{i+2k \\ =\delta-2m}} \sum_{K\in B'(X)} (-1)^{\varepsilon(w,K)}\nu(K) \times \frac{SW_X'(K)(\delta-2m)!}{2^{k+c(X)-3-m}k!i!}\langle K,h\rangle^i Q_X(h)^k,$$

SO(3)-monopole cobordism and Donaldson invariants III

Lemma 3.2 (Donaldson invariants implied by Witten's formula)

where

(22)
$$\varepsilon(w, K) := \frac{1}{2}(w^2 + w \cdot K),$$

and

(23)
$$\nu(K) = \begin{cases} \frac{1}{2} & \text{if } K = 0, \\ 1 & \text{if } K \neq 0. \end{cases}$$

The SO(3)-monopole cobordism formula given below provides an expression for the Donaldson invariant in terms of the Seiberg-Witten invariants.

SO(3)-monopole cobordism and Donaldson invariants IV

Theorem 3.3 (Cobordism formula for Donaldson invariants)

(See F and Leness [6, Main Theorem].) Let X be a standard four-manifold of Seiberg-Witten simple type. Assume further that $w, \Lambda \in H^2(X;\mathbb{Z})$ and $\delta, m \in \mathbb{N}$ satisfy

$$w - \Lambda \equiv w_2(X) \pmod{2}, \tag{24a}$$

$$I(\Lambda) = \Lambda^2 + c(X) + 4\chi_h(X) > \delta, \tag{24b}$$

$$\delta \equiv -w^2 - 3\chi_h(X) \pmod{4}, \tag{24c}$$

$$\delta - 2m \geq 0. \tag{24d}$$

Then, for any $h \in H_2(X;\mathbb{R})$ and positive generator $x \in H_0(X;\mathbb{Z})$,

SO(3)-monopole cobordism and Donaldson invariants V

Theorem 3.3 (Cobordism formula for Donaldson invariants)

$$D_X^w(h^{\delta-2m}x^m) = \sum_{K\in B(X)} (-1)^{\frac{1}{2}(w^2-\sigma)+\frac{1}{2}(w^2+(w-\Lambda)\cdot K)} SW_X'(K) \times f_{\delta,m}(\chi_h(X), c_1^2(X), K, \Lambda)(h), \tag{25}$$

where the map,

$$f_{\delta,m}(h) : \mathbb{Z}\times\mathbb{Z}\times H^2(X;\mathbb{Z})\times H^2(X;\mathbb{Z}) \to \mathbb{R}[h],$$

takes values in the ring of polynomials in the variable h with

SO(3)-monopole cobordism and Donaldson invariants VI

Theorem 3.3 (Cobordism formula for Donaldson invariants)

real coefficients, is universal (independent of X) and is given by

$$(26)\quad \begin{aligned} &f_{\delta,m}(\chi_h(X), c_1^2(X), K, \Lambda)(h) \\ &:= \sum_{\substack{i+j+2k \\ =\delta-2m}} a_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m)\,\langle K, h\rangle^i \langle \Lambda, h\rangle^j Q_X(h)^k. \end{aligned}$$

For each triple, $i, j, k \in \mathbb{N}$, the coefficients,

$$a_{i,j,k} : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{N} \to \mathbb{R},$$

are universal (independent of X) real analytic functions of the variables $\chi_h(X)$, $c_1^2(X)$, $c_1(\mathfrak{s})\cdot\Lambda$, Λ^2, and m.

SO(3)-monopole cobordism and Donaldson invariants VII

The left-hand side of the SO(3)-monopole cobordism formula (25) is obtained by computing the intersection number for geometric representatives on $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ with the link of the moduli subspace $\bar{M}_\kappa^w$ of anti-self-dual SO(3) connections.

One uses the fiber-bundle structure of the link over $\bar{M}_\kappa^w$ to compute the intersection number and show that this is equal to a multiple of the Donaldson invariant, $D_X^w(h^{\delta-2m}x^m)$.

The right-hand side of the SO(3)-monopole cobordism formula (25) is obtained by computing the intersection numbers for geometric representatives on $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ with the links of the moduli subspaces $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ of ideal Seiberg-Witten monopoles appearing in $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$.

SO(3)-monopole cobordism and Donaldson invariants VIII

One uses the fiber-bundle structure of the link over each Seiberg-Witten moduli space, $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$, to compute the intersection number and show that this is equal to a multiple of a Seiberg-Witten invariant, $SW'_X(K)$, for each $K \in H^2(X;\mathbb{Z})$ with $c_1(\mathfrak{s}) = K$.

SO(3)-monopole cobordism and algebraic geometry I

Aside: When X is a complex projective surface, T. Mochizuki [34] proved a formula (see Göttsche, Nakajima, and Yoshioka [21, Theorem 4.1]) expressing the Donaldson invariants in a form similar to our SO(3)-monopole cobordism formula (Theorem 3.3).

The coefficients in Mochizuki's formula are given as the residues of a generating function for integrals of $\mathbb{C}^*$-equivariant cohomology classes over the product of Hilbert schemes of points on X.

In [21, p. 309], Göttsche, Nakajima, and Yoshioka conjecture that the coefficients in Mochizuki's formula (which are meaningful for any standard four-manifold) and in our SO(3)-monopole cobordism formula are the same.

SO(3)-monopole cobordism and algebraic geometry II

Göttsche, Nakajima, and Yoshioka prove an explicit formula for complex projective surfaces relating Donaldson invariants and Seiberg-Witten invariants of standard four-manifolds of Seiberg-Witten simple type using an instanton-counting formula due to Nekrasov and verify Witten's Conjecture for complex projective surfaces.

In [21, p. 323], Göttsche, Nakajima, and Yoshioka discuss the relationship between their approach, Mochizuki's formula, and our SO(3)-monopole cobordism formula.

See also [20, pp. 344–347] for a related discussion concerning their wall-crossing formula for the Donaldson invariants of a four-manifold with $b^+ = 1$.

SO(3)-monopole cobordism and SW invariant relations I

In addition to its application to deriving a formula for Donaldson invariants, the SO(3)-monopole cobordism may also be used to derive relations among Seiberg-Witten invariants, as in the forthcoming Theorem 3.4.

SO(3)-monopole cobordism and SW invariant relations II

Theorem 3.4 (SO(3)-monopole cobordism formula vanishing)

(See F and Leness [8, Theorem 3.3].) Let X be a standard four-manifold of Seiberg-Witten simple type. Assume that $m, n \in \mathbb{N}$ satisfy

$$n \leq 2\chi_h(X), \tag{27a}$$

$$1 < n, \tag{27b}$$

$$0 \leq c(X) - n - 2m - 1. \tag{27c}$$

We abbreviate the coefficients in equation (25) in Theorem 3.3 by

$$a_{i,0,k} := a_{i,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, 2\chi_h(X) - n). \tag{28}$$

SO(3)-monopole cobordism and SW invariant relations III

Theorem 3.4 (SO(3)-monopole cobordism formula vanishing)

Then, for $A = c(X) - n - 2m - 1$ and $w \in H^2(X;\mathbb{Z})$ characteristic,

$$0 = \sum_{k=0}^{2\chi_h(X)-n} a_{A+2k,0,2\chi_h(X)-n-k} SW_X^{w,A+2k}(h) Q_X(h)^{2\chi_h(X)-n+k}. \tag{29}$$

To show that equation (29) is non-trivial, we now demonstrate, in a computation similar to that due to Kotschick and Morgan [23, Theorem 6.1.1], that the coefficient of the term in (29) including the highest power of Q_X is non-zero.

SO(3)-monopole cobordism and SW invariant relations IV

Proposition 3.5 (Leading-order term in the SO(3)-monopole cobordism formula (29) for link pairings)

(See F and Leness [8, Proposition 4.1].) Continue the notation and assumptions of Theorem 3.4. In addition, assume that there is $K \in B(X)$ with $K \neq 0$. Let m and n be non-negative integers satisfying the conditions (27). Define $A := c(X) - n - 2m - 1$, and $\delta := c(X) + 4\chi_h(X) - 3n - 1$, and $\ell = 2\chi_h(X) - n$. Then

$$a_{A,0,\ell}(\chi_h(X), c_1^2(X), 0, 0, m, \ell) = (-1)^{m+\ell} 2^{\ell-\delta} \frac{(\delta - 2m)!}{\ell! A!}. \tag{30}$$

SO(3)-monopole cobordism and SW invariant relations V

The proof of Proposition 3.5 requires knowledge of the topology of an open neighborhood of

$$M_{\mathfrak{s}} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_{\mathfrak{t}}/S^1,$$

where $\mathbf{x} \in \mathrm{Sym}^\ell(X)$.

While we generally need to consider arbitrary *lower levels*, $\ell \geq 0$, in the Uhlenbeck compactification,

$$\bar{\mathscr{M}}_{\mathfrak{t}} \subset \bigsqcup_{\ell=0}^{N} \mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^\ell(X),$$

SO(3)-monopole cobordism and SW invariant relations VI

the proof of Proposition 3.5 only requires us to consider points,

$$\mathbf{x} \in \mathrm{Sym}^{\ell}(X),$$

in the *top stratum* of $\mathrm{Sym}^{\ell}(X)$ (all points distinct).

We can also show that

SO(3)-monopole cobordism and SW invariant relations VII

Proposition 3.6 (Vanishing coefficients in SO(3)-monopole cobordism formula for link pairings)

(See F and Leness [8, Proposition 5.1].) Continue the hypothesis and notation of Theorem 3.4. In addition, assume $c(X) \geq 3$ *and*

$$n \equiv 1 \pmod 2. \tag{31}$$

Then for $p \geq c(X) - 3$ *and* $k \geq 0$ *an integer such that* $p + 2k = c(X) + 4\chi_h(X) - 3n - 1 - 2m$,

$$a_{p,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, 2\chi_h - n) = 0.$$

SO(3)-monopole cobordism and SW invariant relations VIII

We prove Theorem 1.2 (simple type $\implies$ superconformal simple type) by applying the computations of the coefficients in

- Proposition 3.5 (formula for $a_{A,0,\ell}(\chi_h(X), c_1^2(X), 0, 0, m, \ell)$),
- Proposition 3.6 ($a_{p,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, 2\chi_h - n) = 0$),

to the vanishing sum formula (29), namely

$$\boxed{0 = \sum_{k=0}^{2\chi_h(X)-n} a_{A+2k,0,2\chi_h(X)-n-k} SW_X^{w,A+2k}(h) Q_X(h)^{2\chi_h(X)-n+k}.}$$

Superconformal simple type and Witten's formula

Blow-up formulae I

We recall versions of the blow-up formulae for Donaldson and Seiberg-Witten invariants.

These formulae are used to

- Verify invariance of Witten's Formula (4) under blow-ups,
- Verify invariance of the Superconformal Simple Type property under blow-ups,
- Eliminate the need to consider certain difficult cases,
- Enrich useful families of example manifolds for which Donaldson and Seiberg-Witten invariants are known.

Blow-up formulae II

Let $\widetilde{X} \to X$ be the blow-up of X at one point, let $e \in H_2(\widetilde{X};\mathbb{Z})$ be the fundamental class of the exceptional curve, and let $e^* \in H^2(\widetilde{X};\mathbb{Z})$ be the Poincaré dual of e.

Using the direct sum decomposition of the homology and cohomology of $\widetilde{X} = X\#\overline{\mathbb{CP}}^2$, we can consider both the homology and cohomology of X as subspaces of those of $\widetilde{X}$,

$$H_\bullet(X) \subset H_\bullet(\tilde{X}) \quad \text{and} \quad H^\bullet(X) \subset H^\bullet(\tilde{X}).$$

Denote $\tilde{w} := w + e^*$. The simplest blow-up formula for Donaldson invariants (see Kotschick [22] or Leness [30] for SO(3) invariants and Ozsváth [41] for SU(2) invariants) gives

$$\boxed{D_X^w(h^{\delta-2m}x^m) = D_{\tilde{X}}^{\tilde{w}}(h^{\delta-2m}ex^m).} \tag{32}$$

Blow-up formulae III

Versions of the blow-up formula for Seiberg-Witten invariants have been established by Fintushel and Stern [14], Nicolaescu [39, Theorem 4.6.7], and Frøyshov [18, Theorem 14.1.1] (in increasing generality).

The following is a special case of their results.

Blow-up formulae IV

Theorem 4.1 (Blow-up formula for Seiberg-Witten invariants)

Let X be a standard four-manifold and let $\widetilde{X} = X\#\overline{\mathbb{CP}}^2$ be its blow-up. Then $\widetilde{X}$ has Seiberg-Witten simple type if and only if that is true for X. If X has Seiberg-Witten simple type, then

$$B(\widetilde{X}) = \{K \pm e^* : K \in B(X)\}, \tag{33}$$

where $e^* \in H^2(\widetilde{X};\mathbb{Z})$ is the Poincaré dual of the exceptional curve, and if $K \in B(X)$, then

$$SW'_{\widetilde{X}}(K \pm e^*) = SW'_X(K).$$

The significance of Theorem 4.1 lies in its universality; more general versions, with more complicated statements, hold without the assumption of simple type.

Refinements of SO(3)-monopole cobordism formula I

We shall rewrite the SO(3)-monopole cobordism formula (25) for $D_X^w(h^{\delta-2m}x^m))$ as a sum over $B'(X) \subset B(X)$, a fundamental domain for the action of $\{\pm 1\}$.

To this end, we define (compare [12, Equation (4.4)])

$$
\begin{aligned}
b_{i,j,k}&(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m) \\
&:= (-1)^{c(X)+i} a_{i,j,k}(\chi_h(X), c_1^2(X), -K\cdot\Lambda, \Lambda^2, m) \\
&\qquad + a_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m),
\end{aligned}
$$

where $a_{i,j,k}$ are the coefficients appearing in the expression (26) for

$$f_{\delta,m}(\chi_h(X), c_1^2(X), K, \Lambda)(h)$$

in the SO(3)-monopole cobordism formula (25) for $D_X^w(h^{\delta-2m}x^m)$.

Refinements of SO(3)-monopole cobordism formula II

To simplify the orientation factor in (25), we define

$$(34)\quad \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m) := (-1)^{\frac{1}{2}(\Lambda^2+\Lambda\cdot K)} b_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m).$$

Observe that

$$(35)\quad \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X), -K\cdot\Lambda, \Lambda^2, m) = (-1)^{c(X)+i+\Lambda\cdot K} \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m).$$

We now rewrite the SO(3)-monopole cobordism formula (25) as a sum over $B'(X)$.

Refinements of SO(3)-monopole cobordism formula III

Lemma 4.2 (SO(3)-monopole cobordism formula on fundamental domain)

(See F and Leness [10, Lemma 3.4].) Assume the hypotheses of Theorem 3.3 (the SO(3)*-monopole cobordism formula). Denote the coefficients in* (35) *more concisely by*

$$\tilde{b}_{i,j,k}(K\cdot\Lambda) := \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m).$$

Then, for $\varepsilon(w,K) = \frac{1}{2}(w^2 + w\cdot K)$ *as in* (22) *and* $\nu(K)$ *as in* (23),

$$\begin{aligned} D_X^w(h^{\delta-2m}x^m) = \sum_{K\in B'(X)} \sum_{\substack{i+j+2k\\=\delta-2m}} \nu(K)(-1)^{\varepsilon(w,K)} SW_X'(K) \\ \times \tilde{b}_{i,j,k}(K\cdot\Lambda)\langle K,h\rangle^i\langle\Lambda,h\rangle^j Q_X(h)^k. \end{aligned} \tag{36}$$

Fintushel-Park-Stern family of example manifolds I

To determine the coefficients $\tilde{b}_{i,j,k}$ appearing in the SO(3)-monopole cobordism formula (36) for $D_X^w(h^{\delta-2m}x^m)$, we compare

- Witten's formula (21) for $D_X^w(h^{\delta-2m}x^m)$, and
- SO(3)-monopole cobordism formula (36) for $D_X^w(h^{\delta-2m}x^m)$,

on manifolds where Witten's Conjecture 1.1 is known to hold.

In [13], Fintushel, Park and Stern constructed families of symplectic manifolds with one basic class (as rational blow-downs of elliptic surfaces).

We use the Fintushel-Park-Stern manifolds to create a family of standard four-manifolds, X_q, for $q = 2, 3, \ldots$, obeying the

Fintushel-Park-Stern family of example manifolds II

following conditions (see F and Leness [12, Section 4.2] and [10, Section 4.3]):

1. X_q satisfies Witten's Conjecture 1.1;
2. For $q = 2, 3, \ldots,$ one has $\chi_h(X_q) = q$ and $c(X_q) = 3$;
3. $B'(X_q) = \{K\}$ with $K \neq 0$;
4. For each q, there are classes $f_1, f_2 \in H^2(X_q; \mathbb{Z})$ satisfying

(37a) $f_1 \cdot f_2 = 1, \quad f_i^2 = 0,$ and $f_i \cdot K = 0$ for $i = 1, 2$,

(37b) $\{f_1, f_2, K\}$ linearly independent subset of $H^2(X_q; \mathbb{R})$,

(37c) Restriction of Q_{X_q} to $\operatorname{Ker} f_1 \cap \operatorname{Ker} f_2 \cap \operatorname{Ker} K$ is non-zero.

Fintushel-Park-Stern family of example manifolds III

Let $X_q(n)$ be the blow-up of X_q at n points,

$$X_q(n) := X_q \# \underbrace{\overline{\mathbb{CP}}^2 \# \cdots \# \overline{\mathbb{CP}}^2}_{n \text{ copies}}. \tag{38}$$

Then $X_q(n)$ is a standard four-manifold of Seiberg-Witten simple type and satisfies Witten's Conjecture 1.1 (see F and Leness [12, Theorem 2.7] or [10, Theorem 2.3]), with

$$\chi_h(X_q(n)) = q, \quad c_1^2(X_q(n)) = q - n - 3, \quad \text{and} \quad c(X_q(n)) = n + 3. \tag{39}$$

We will consider both the homology and cohomology of X_q as subspaces of those of $X_q(n)$.

Fintushel-Park-Stern family of example manifolds IV

Let $e_u^* \in H^2(X_q(n); \mathbb{Z})$ be the Poincaré dual of the u-th exceptional class, $1 \leq u \leq n$.

Let $\pi_u : (\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{Z}/2\mathbb{Z}$ be projection onto the u-th factor.

For $\varphi \in (\mathbb{Z}/2\mathbb{Z})^n$, we define

$$(40) \qquad K_\varphi := K + \sum_{u=1}^{n} (-1)^{\pi_u(\varphi)} e_u^* \quad \text{and} \quad K_0 := K + \sum_{u=1}^{n} e_u^*.$$

By the blow-up formula for Seiberg-Witten invariants (see Frøyshov [18, Theorem 14.1.1])

$$(41) \qquad B'(X_q(n)) = \{K_\varphi : \varphi \in (\mathbb{Z}/2\mathbb{Z})^n\},$$

Fintushel-Park-Stern family of example manifolds V

and, for all $\varphi \in (\mathbb{Z}/2\mathbb{Z})^n$,

$$SW'_{X_q(n)}(K_\varphi) = SW'_{X_q}(K). \tag{42}$$

Because $X_q(n)$ has Seiberg-Witten simple type, we have

$$K_\varphi^2 = c_1^2(X_q(n)) \quad \text{for all } \varphi \in (\mathbb{Z}/2\mathbb{Z})^n. \tag{43}$$

In addition, because $K \neq 0$, we see that

$$0 \notin B'(X_q(n)). \tag{44}$$

For $n \geq 2$, the set $B'(X_q(n))$ is not a linearly independent subset of $H^2(X_q(n);\mathbb{R})$ but can be replaced by linearly independent subset, $\{K \pm e_1^*, e_2^*, \ldots, e_n^*\}$ to give the

Fintushel-Park-Stern family of example manifolds VI

Lemma 4.3 (Donaldson invariants of $X_q(n)$ via SO(3)-monopole cobordism)

(See F and Leness [10, Lemma 4.6].) For $n, q \in \mathbb{Z}$ with $n \geq 1$ and $q \geq 2$, let $X_q(n)$ be the manifold defined in (38). *For $\Lambda, w \in H^2(X_q; \mathbb{Z})$ and $\delta, m \in \mathbb{N}$ satisfying $\Lambda - w \equiv w_2(X_q) \pmod 2$ and $\delta - 2m \geq 0$, define $\tilde{w}, \tilde{\Lambda} \in H^2(X_q(n); \mathbb{Z})$ by*

$$\tilde{w} := w + \sum_{u=1}^{n} w_u e_u^* \quad \text{and} \quad \tilde{\Lambda} := \Lambda + \sum_{u=1}^{n} \lambda_u e_u^*, \tag{45}$$

where $w_u, \lambda_u \in \mathbb{Z}$ and $w_u + \lambda_u \equiv 1 \pmod 2$ for $u = 1, \ldots, n$. We assume that

$$\Lambda^2 > \delta - (n+3) - 4q + \sum_{u=1}^{n} \lambda_u^2, \tag{46a}$$

$$\delta \equiv -w^2 + \sum_{u=1}^{n} w_u^2 - 3q \pmod 4. \tag{46b}$$

Fintushel-Park-Stern family of example manifolds VII

Lemma 4.3 (Donaldson invariants of $X_q(n)$ via SO(3)-monopole cobordism)

Denote $x := \tilde{K}_\varphi \cdot \tilde{\Lambda}$ and, for $i, j, k \in \mathbb{N}$ satisfying $i + j + 2k + 2m = \delta$, write

$$\tilde{b}_{i,j,k}(x) = \tilde{b}_{i,j,k}(\chi_h(X_q(n)), c_1^2(X_q(n)), x, \tilde{\Lambda}^2, m).$$

Then, for $x_0 = K_0 \cdot \tilde{\Lambda}$ where K_0 is defined in (40),

Fintushel-Park-Stern family of example manifolds VIII

Lemma 4.3 (Donaldson invariants of $X_q(n)$ via SO(3)-monopole cobordism)

(47)
$$
\begin{aligned}
&\sum_{\substack{i_1+\cdots+i_n+2k\\=\delta-2m}} \frac{(\delta-2m)!}{2^{k+n-m}k!i_1!\cdots i_n!} p^{\tilde{w}}(i_2,\dots,i_n)\left(\prod_{u=2}^n \langle e_u^*, h\rangle^{i_u}\right) Q_{X_q(n)}(h)^k\\
&\quad\times\left(\langle K+e_1^*,h\rangle^{i_1}+(-1)^{w_1}\langle K-e_1^*,h\rangle^{i_1}\right)\\
&=\sum_{\substack{i_1+\cdots+i_n+j+2k\\=\delta-2m}} \binom{i_1+\cdots+i_n}{i_1,\dots,i_n}\langle\tilde{\Lambda},h\rangle^j\left(\prod_{u=2}^n \langle e_u^*, h\rangle^{i_u}\right) Q_{X_q(n)}(h)^k\\
&\quad\times\left(\nabla_{2\lambda_2}^{i_2+w_2}\cdots\nabla_{2\lambda_n}^{i_n+w_n}\tilde{b}_{i,j,k}(x_0)\langle K+e_1^*,h\rangle^{i_1}\right.\\
&\qquad\left.+(-1)^{w_1}\nabla_{2\lambda_2}^{i_2+w_2}\cdots\nabla_{2\lambda_n}^{i_n+w_n}\tilde{b}_{i,j,k}(x_0+2\lambda_1)\langle K-e_1^*,h\rangle^{i_1}\right),
\end{aligned}
$$

Fintushel-Park-Stern family of example manifolds IX

Lemma 4.3 (Donaldson invariants of $X_q(n)$ via SO(3)-monopole cobordism)

are both equal to the following multiple of the Donaldson invariant,

$$\frac{(-1)^{\varepsilon(\tilde{w},\varphi_0)}}{SW'_{X_q}(K)} D^{\tilde{w}}_{X_q(n)}(h^{\delta-2m}x^m),$$

where $\tilde{\Lambda}$ is as defined in (45) and

$$p^{\tilde{w}}(i_2,\dots,i_n) = \begin{cases} 0 & \text{if } \exists u \text{ with } 2 \le u \le n \text{ and } w_u + i_u \equiv 1 \pmod 2, \\ 2^{n-1} & \text{if } w_u + i_u \equiv 0 \pmod 2 \ \forall\, u \text{ with } 2 \le u \le n. \end{cases} \tag{48}$$

Fintushel-Park-Stern family of example manifolds X

For a function $f : \mathbb{Z} \to \mathbb{R}$ and $p, q \in \mathbb{Z}$, we define

$$(\nabla_p^q f)(x) := f(x) + (-1)^q f(x+p), \quad \forall\, x \in \mathbb{Z}.$$

The forthcoming Proposition 4.4 determines the coefficients $\tilde{b}_{i,j,k}$ with $i \geq c(X) - 3$.

However, the forthcoming condition (49a) in Proposition 4.4 prevents an immediate determination of the coefficients $\tilde{b}_{i,j,k}$ with $i < c(X) - 3$.

Fintushel-Park-Stern family of example manifolds XI

Proposition 4.4 (Computation of coefficients $\tilde{b}_{i,j,k}$ for $i \geq c(X) - 3$)

(See F and Leness [12, Proposition 4.8] or [10, Proposition 4.7].) Let $n > 0$ and $q \geq 2$ be integers. If x, y are integers and i, j, k, m are non-negative integers satisfying, for $A := i + j + 2k + 2m$,

$$i \geq n, \tag{49a}$$

$$y > A - 4q - 3 - n, \tag{49b}$$

$$A \geq 2m, \tag{49c}$$

$$x \equiv y \equiv 0 \pmod 2, \tag{49d}$$

then the coefficients $\tilde{b}_{i,j,k}(\chi_h, c_1^2, \Lambda \cdot K, \Lambda^2, m)$ defined in (34) are given by

$$\tilde{b}_{i,j,k}(q, q-3-n, x, y, m) = \begin{cases} \dfrac{(A-2m)!}{k!i!} 2^{m-k-n} & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases}$$

Blow-up trick I

The following lemma allows us to ignore the coefficients $\tilde{b}_{0,j,k}$ in the formula (36) for $D_X^w(h^{\delta-2m}x^m)$ for the purpose of proving Theorem 1.4 and Corollary 1.5.

Blow-up trick II

Lemma 4.5 (Eliminating the coefficients $\tilde{b}_{0,j,k}$ in the formula (36) for $D_X^w(h^{\delta-2m}x^m)$)

(See F and Leneess [10, Lemma 3.5].) Continue the notation and hypotheses of Lemma 4.2. Then,

$$D_X^w(h^{\delta-2m}x^m) = \sum_{K\in B'(X)} \sum_{\substack{i+j+2k\\=\delta-2m}} (-1)^{\varepsilon(w,K)} SW_X'(K) \frac{2(i+1)}{(\delta-2m+1)} \times \tilde{b}_{i+1,j,k}(\chi_h(X), c_1^2(X)-1, K\cdot\Lambda, \Lambda^2, m) \times \langle K, h\rangle^i \langle \Lambda, h\rangle^j Q_X(h)^k. \tag{50}$$

Difference equations and superconformal simple type I

Proposition 4.4 yields the coefficients $\tilde{b}_{i,j,k}$ in the SO(3)-monopole cobordism formula (36) for $D_X^w(h^{\delta-2m}x^m)$ with

$$i \geq c(X) - 3 > 0$$

but not those coefficients $\tilde{b}_{i,j,k}$ with

$$0 \leq i < c(X) - 3,$$

when $c(X) - 3 > 0$.

Lemma 4.5 allows us to *ignore* the coefficients $\tilde{b}_{i,j,k}$ with $i = 0$, that is, $\tilde{b}_{0,j,k}$ when proving Theorem 1.4 and Corollary 1.5.

Difference equations and superconformal simple type II

However, again using the fact that the Fintushel-Park-Stern four-manifolds, $X_q(n)$, satisfy Witten's Conjecture 1.1, then

- Witten's formula (21) for $D_X^w(h^{\delta-2m}x^m)$, and
- SO(3)-monopole cobordism formula (36) for $D_X^w(h^{\delta-2m}x^m)$,

when applied to the manifolds $X_q(n)$, will *also* show that, when $1 \leq i < c(X) - 3$, the coefficients $\tilde{b}_{i,j,k}$ not determined by Proposition 4.4 satisfy a homogeneous difference equation in the parameter $K \cdot \Lambda$ and thus can be written as *polynomials* in $K \cdot \Lambda$.

Difference equations and superconformal simple type III

Proposition 4.6 (Difference equation for $\tilde{b}_{i,j,k}$ with $1 \leq i < c(X) - 3$)

(See F and Leness [10, Proposition 4.9].) Let $n > 1$ and $q \geq 2$ be integers. If x, y are integers and p, j, k, m are non-negative integers satisfying, for $A := p + j + 2k + 2m$,

$$1 \leq p \leq n - 1, \tag{51a}$$

$$y > A - 4q - n - 3, \tag{51b}$$

$$y \equiv A - (n+3) \pmod 4, \tag{51c}$$

$$x - y \equiv 0 \pmod 2, \tag{51d}$$

and we abbreviate $\tilde{b}_{p,j,k}(x) = \tilde{b}_{p,j,k}(q, q - n - 3, x, y, m)$, then

$$\left(\nabla^{1}_{4}\right)^{n-p} \tilde{b}_{p,j,k}(x) = 0. \tag{52}$$

Difference equations and superconformal simple type IV

We next record an algebraic consequence of superconformal simple type which will allow us to show that Witten's Formula (4) holds even without determining the coefficients $\tilde{b}_{i,j,k}$ with $i < c(X) - 3$ in the SO(3)-monopole cobordism formula (36) for $D_X^w(h^{\delta-2m}x^m)$.

Difference equations and superconformal simple type V

Lemma 4.7 (An algebraic consequence of superconformal simple type)

(See F and Leness [10, Lemma 5.1].) Let X be a standard four-manifold of superconformal simple type. Assume $0 \notin B(X)$. If $w \in H^2(X,\mathbb{Z})$ is characteristic and $j, u \in \mathbb{N}$ satisfy $j + u < c(X) - 3$ and $j + u \equiv c(X) \pmod 2$, then

$$(53) \qquad \sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} SW'_X(K) \langle K, h_1 \rangle^j \langle K, h_2 \rangle^u = 0,$$

for any $h_1, h_2 \in H_2(X;\mathbb{R})$.

Proposition 4.6 and the difference equation for the coefficients $\tilde{b}_{i,j,k}$ allow us to write the $\tilde{b}_{i,j,k}$ as polynomials in $\Lambda \cdot K$.

Difference equations and superconformal simple type VI

Corollary 4.8 (Coefficients $\tilde{b}_{i,j,k}$ as polynomials in $\Lambda \cdot K$)

(See F and Leness [10, Corollary 4.11].) Continue the assumptions of Proposition 4.6. In addition, assume

1. *There is a class $K_0 \in B(X)$ such that $\Lambda \cdot K_0 = 0$;*
2. *For all $K \in B(X)$, we have $\Lambda \cdot K \equiv 0 \pmod 4$.*

Then for $1 \leq i \leq n-1$, the function $\tilde{b}_{i,j,k}$ is a polynomial of degree $n-1-i$ in $\Lambda \cdot K$ and thus

$$\tilde{b}_{i,j,k}(q, q-n-3, K \cdot \Lambda, \Lambda^2, m) = \sum_{u=0}^{n-1-i} \tilde{b}_{u,i,j,k}(q, q-n-3, \Lambda^2, m) \langle K, h_\Lambda \rangle^u, \tag{54}$$

where $h_\Lambda = \mathrm{PD}[\Lambda]$ is the Poincaré dual of Λ and if $u \equiv n+i \pmod 2$, then

$$\tilde{b}_{u,i,j,k}(q, q-n-3, \Lambda^2, m) = 0. \tag{55}$$

Difference equations and superconformal simple type VII

We combine Corollary 4.8 with Lemma 4.7 to show that, for manifolds of superconformal simple type, the coefficients $\tilde{b}_{i,j,k}$ with $i \leq c(X) - 4$ do not contribute to the SO(3)-monopole cobordism expression (36) for the Donaldson invariant $D_X^w(h^{\delta-2m}x^m)$.

This proves Theorem 1.4 (that Superconformal Simple Type $\Longrightarrow$ Witten's Formula).

Construction of local and global gluing maps and obstruction sections for SO(3) monopoles

Local gluing maps for SO(3) monopoles I

When $\ell \geq 1$, the construction of links of Seiberg-Witten moduli subspaces,

$$M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X) \subset \mathscr{M}_{\mathfrak{t}},$$

and the computation of intersection numbers for intersections of geometric representatives of cohomology classes on $\mathscr{M}_{\mathfrak{t}}^{*,0}$ with those links requires the construction of a (global) SO(3)-monopole gluing map (and obstruction section of an obstruction bundle, since gluing is always obstructed in the case of SO(3) monopoles).

We summarize the steps in the construction of the local SO(3)-monopole gluing map and obstruction section and proofs of their properties and hence completing the verification of

Local gluing maps for SO(3) monopoles II

Theorem 5.1 (Properties of local SO(3)-monopole gluing maps)

The local gluing map, constructed in [9], gives a continuous parametrization of a neighborhood of $M_{\mathfrak{s}} \times \Sigma$ in $\bar{\mathscr{M}}_{\mathfrak{t}}$ for each smooth stratum $\Sigma \subset \mathrm{Sym}^{\ell}(X)$.

These local gluing maps are the analogues for SO(3) monopoles of the local gluing maps for anti-self-dual SO(3) connections constructed by Taubes in [46, 47, 48], Donaldson [1], and Donaldson and Kronheimer in [3].

Local gluing maps for SO(3) monopoles III

Local splicing (or pregluing) map

This map is a smooth embedding from the local gluing data parameter space — a finite-dimensional, open, Riemannian manifold — into the configuration space of gauge-equivalence classes of SO(3) pairs.

The image of the map is given by gauge-equivalence classes of approximate SO(3) monopoles, $[A, \Phi]$, defined by a "cut-and-paste" construction.

We splice anti-self-dual SU(2) connections from S^4 onto background SO(3) monopoles on X (elements of $\mathscr{M}_{\mathfrak{t}(\ell)}$) at points in the support of

$$\mathbf{x} \in \Sigma \subset \mathrm{Sym}^{\ell}(X)$$

Local gluing maps for SO(3) monopoles IV

to form gauge-equivalence classes of SO(3) pairs, $[A, \Phi]$, which are close to the stratum

$$\mathscr{M}_{\mathfrak{t}(\ell)} \times \Sigma \subset \bar{\mathscr{M}}_{\mathfrak{t}}.$$

See F and Leness [5, 6, 9].

Local gluing maps for SO(3) monopoles V

Local gluing map

This is a smooth map from the gluing data parameter space defined by a single stratum,

$$\Sigma \subset \mathrm{Sym}^{\ell}(X),$$

into the configuration space of SO(3) pairs.

The image of the map is given by gauge-equivalence classes of *extended* SO(3)*-monopoles*, $[A + a, \Phi + \phi]$, obtained by solving the *extended* SO(3)*-monopole equations* for the perturbations, (a, ϕ),

$$\Pi^{\perp}_{A,\phi,\mu}\mathfrak{S}(A + a, \Phi + \phi) = 0,$$

Local gluing maps for SO(3) monopoles VI

rather than the SO(3)-monopole equations directly,

$$\mathfrak{S}(A + a, \Phi + \phi) = 0,$$

since Coker $D\mathfrak{S}(A, \Phi) = \operatorname{Ran}\Pi_{A,\Phi,\mu}$ is non-zero, where $\mu > 0$ is a "small-eigenvalue" cut-off parameter.

With respect to local coordinates and bundle trivializations, these equations comprise an elliptic, quasi-linear, partial integro-differential system.

The gauge-equivalence classes of true SO(3) monopoles are given by the zero-locus of a local, smooth section of a finite-rank *local Kuranishi obstruction bundle* over the gluing data parameter space.

Local gluing maps for SO(3) monopoles VII

defined by L^2-orthogonal projection onto finite-dimensional, "small-eigenvalue" vector spaces (see [5, 9]).

Smooth embedding property of the local gluing map

One must compute the differential of the gluing map and prove that the differential is injective.

Surjectivity of the local gluing map

Every extended SO(3) monopole close enough to the Uhlenbeck boundary of $\mathscr{M}_\mathfrak{t}$ must lie in the image of the local gluing map.

Local gluing maps for SO(3) monopoles VIII

Continuity of the local gluing map and obstruction section

The gluing map and obstruction section must extend continuously to the compactification of the local gluing data space, which includes the Uhlenbeck compactification of moduli spaces of anti-self-dual connections on S^4.

Global gluing maps for SO(3) monopoles I

Building a global gluing map and obstruction section from the local gluing maps and obstruction sections

Theorem 5.1 describes a neighborhood of $M_{\mathfrak{s}} \times \Sigma$ in $\bar{\mathscr{M}}_{\mathfrak{t}}$ for $\Sigma \subset \mathrm{Sym}^{\ell}(X)$ a smooth stratum while the proof of Theorem 3.3 (general SO(3)-monopole cobordism formula) requires a description of a neighborhood of the union of these strata, $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$.

In [6], we proved how the local gluing data parameter spaces, splicing maps, obstruction bundles, and obstruction sections given by Theorem 5.1 for different $\Sigma \subset \mathrm{Sym}^{\ell}(X)$ fit together and extend over the Uhlenbeck compactification, $\bar{\mathscr{M}}_{\mathfrak{t}}$.

Global gluing maps for SO(3) monopoles II

The splicing maps are deformed so that they obey a type of "cocycle condition" — to give *global* splicing maps and obstruction sections, solving the overlap problem identified by Kotschick and Morgan for gluing SO(3) anti-self-connections in [23].

Using this construction, we computed the expressions for the intersection number yielding the SO(3)-monopole cobordism formula (25) and completing the proof of Theorem 3.3.

Analytical difficulties in gluing SO(3) monopoles I

Finally, we indicate some of the analytical difficulties in the construction of the gluing maps and obstruction sections that are particular to SO(3)-monopoles:

1. *Small-eigenvalue obstructions to gluing.*

   The Laplacian, $d^1_{A,\Phi} d^{1,*}_{A,\Phi}$, constructed from the differential, $d^1_{A,\Phi} = D\mathfrak{S}(A,\Phi)$, of the SO(3)-monopole map $\mathfrak{S}$, at an almost SO(3) monopole, (A,Φ), has small eigenvalues tending to zero when A bubbles and $\mathfrak{S}(A,\Phi)$ tends to zero.

   This phenomenon occurs for SO(3) monopoles because the

   - Dirac operators, when coupled with an anti-self-dual connections over S^4, always have non-trivial cokernels and

Analytical difficulties in gluing SO(3) monopoles II

- Seiberg-Witten monopoles need not be smooth points of their ambient moduli space of background SO(3) monopoles.

2. *Bubbling curvature component in Bochner-Weitzenböck formulae.*

   A key ingredient employed by Taubes in his solution to the anti-self-dual equation in [46, 47] is his use of the Bochner-Weitzenböck formula for the Laplacian, $d_A^+ d_A^{+,*}$, constructed from the differential, d_A^+, of the map, $A \mapsto F_A^+$, at an approximate anti-self-dual connection.

   While Taubes' Bochner-Weitzenböck formula only involves the *small*, self-dual curvature component, F_A^+, our

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Analytical difficulties in gluing SO(3) monopoles III

Bochner-Weitzenböck formula for $d^1_{A,\Phi} d^{1,*}_{A,\Phi}$ also involves the *large* anti-self-dual curvature component, F^-_A.

3. *Seiberg-Witten moduli spaces of positive dimension and spectral flow.*

   When $\dim M_{\mathfrak{s}} > 0$, one cannot fix a single, uniform positive upper bound for the small eigenvalues of $d^1_{A,\Phi} d^{1,*}_{A,\Phi}$, due to spectral flow as the point $[A, \Phi]$ varies in an open neighborhood of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in the local gluing data parameter space.

These are addressed in our preprints [9, 5, 7].

Thank you for your attention!

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