Concentration for Coulomb gases and Coulomb transport inequalities

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Outline of the talk

- Coulomb gases : definition and known results
- Concentration inequalities
- Outline of the proof and Coulomb transport inequalities

Coulomb gases ($d \geq 2$)

We consider the Poisson equation

$$\Delta g = -c_d \delta_0.$$

The fundamental solution is given by

$$g(x) := \begin{cases} -\log|x| & \text{for } d = 2, \\ \frac{1}{|x|^{d-2}} & \text{for } d \geq 3. \end{cases}$$

A gas of N particles interacting according to the Coulomb law would have an energy given by

$$H_N(x_1, \dots, x_N) := \sum_{i \neq j} g(x_i - x_j) + N \sum_{i=1}^{N} V(x_i).$$

We denote by $\mathbb{P}^N_{V,\beta}$ the Gibbs measure on $(\mathbb{R}^d)^N$ associated to this energy :

$$\mathrm{d}\mathbb{P}^N_{V,\beta}(x_1,\ldots,x_N) = \frac{1}{Z^N_{V,\beta}} \mathrm{e}^{-\frac{\beta}{2} H_N(x_1,\ldots,x_N)} \mathrm{d}x_1,\ldots,\mathrm{d}x_N$$

Example (Ginibre) : let M_N be an N by N matrix with iid entries with law $\mathcal{N}_{\mathbb{C}}(0, \frac{1}{N})$, then the eigenvalues have joint law $\mathbb{P}^N_{|x|^2,2}$ with

$$\mathrm{d}\mathbb{P}^N_{|x|^2,2}(x_1,\ldots,x_N) \sim \prod_{i<j} |x_i - x_j|^2 \mathrm{e}^{-N\sum_{i=1}^N |x_i|^2}$$

Global asymptotics of the empirical measure

Our main subject of study is the empirical measure

$$\hat{\mu}_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}.$$

One can rewrite

$$\begin{aligned} H_N(x_1, \ldots, x_N) &= N^2 \mathcal{E}_V^{\neq}(\hat{\mu}_N) \\ &:= N^2 \left(\iint_{x \neq y} g(x-y) \hat{\mu}_N(\mathrm{d}x) \hat{\mu}_N(\mathrm{d}y) + \int V(x) \hat{\mu}_N(\mathrm{d}x) \right). \end{aligned}$$

More generally, one can define, for any $\mu \in \mathcal{P}(\mathbb{R}^d)$,

$$\mathcal{E}_V(\mu) := \iint \left(g(x-y) + \frac{1}{2} V(x) + \frac{1}{2} V(y) \right) \mu(\mathrm{d}x) \mu(\mathrm{d}y).$$

If V is admissible, there exists a unique minimizer μ_V of the functional $\mathcal{E}_V$ and it is compactly suppported.

If V is continuous, one can check that almost surely $\hat{\mu}_N$ converges weakly to μ_V.

A large deviation principle, due to Chafaï, Gozlan and Zitt is also available : for d a distance that metrizes the weak topology (for example Fortet-Mourier) one has in particular

$$\frac{1}{N^2}\log \mathbb{P}^N_{V,\beta}(\mathrm{d}(\hat{\mu}_N,\mu_V)\geq r) \xrightarrow[N\to\infty]{} -\frac{\beta}{2}\inf_{\mathrm{d}(\mu,\mu_V)\geq r}(\mathcal{E}_V(\mu)-\mathcal{E}_V(\mu_V)).$$

What about concentration ?

Local behavior extensively studied by Sandier, Serfaty, Rougerie, Petrache, Leblé, Bauerschmidt, Bourgade, Nikula, Yau etc. using several variations of the concept of renormalized energy.

Concentration estimates

We will consider both the bounded Lipschitz distance d_{BL} and the Wassertein W_1 distance, where we recall that

$$\mathrm{d}_{BL}(\mu,\nu) = \sup_{\substack{\|f\|_\infty \le 1 \\ \|f\|_{Lip} \le 1}} \int f \mathrm{d}(\mu-\nu); \, W_1(\mu,\nu) = \sup_{\|f\|_{Lip}\le 1} \int f \mathrm{d}(\mu-\nu)$$

Theorem

If V is $\mathcal{C}^2$ and V and ΔV satisfy some growth conditions,then there exist $a>0$, $b\in\mathbb{R}$, $c(\beta)$ such that for all $N\ge 2$ and for all $r>0$,

$$\mathbb{P}^N_{V,\beta}(d(\hat{\mu}_N,\mu_V)\ge r) \le e^{-a\beta N^2 r^2 + \mathbf{1}_{d=2}\frac{\beta}{4} N \log N + b\beta N^{2-\frac{2}{d}} + c(\beta) N}$$

A few remarks :

- thanks to the large deviation results of CGZ, we know that we are in the right scale
- new even for Ginibre (can we use the Gaussian nature of the entries ?)
- Possible rewriting : there exist $u, v > 0$, such that for all $N \geq 2$, if
$$r \geq \begin{cases} v\sqrt{\frac{\log N}{N}} & \text{if } d = 2 \\ v\,N^{-1/d} & \text{if } d = 3, \end{cases}$$
$$\mathbb{P}^N_{V,\beta}(\mathrm{d}(\hat{\mu}_N, \mu_V) \geq r) \leq \mathrm{e}^{-uN^2r^2}.$$
- non optimal local laws can be deduced

Outline of the proof

Special case when $V = \delta_K$, for K a compact set of $\mathbb{R}^d$.
First ingredient : lower bound on the partition function. There exists C such that

$$Z_{V,\beta}^N \geq \mathrm{e}^{-\frac{\beta}{2} N^2 \mathcal{E}_V(\mu_V) - NC}.$$

For $A \subset (\mathbb{R}^d)^N$,

$$\begin{aligned}
\mathbb{P}_{V,\beta}^N(A) &= \frac{1}{Z_{V,\beta}^N} \int_A \mathrm{e}^{-\frac{\beta}{2} H_N(x_1,\ldots,x_N)} \mathrm{d}x_1 \ldots \mathrm{d}x_N \\
&\leq \mathrm{e}^{NC} \int_A \mathrm{e}^{-\frac{\beta}{2} N^2 (\mathcal{E}_V^{\neq}(\hat{\mu}_N) - \mathcal{E}_V(\mu_V))} \mathrm{d}x_1 \ldots \mathrm{d}x_N \\
&\leq \mathrm{e}^{NC} \mathrm{e}^{-\frac{\beta}{2} N^2 \inf_A (\mathcal{E}_V^{\neq}(\hat{\mu}_N) - \mathcal{E}_V(\mu_V))} (\mathrm{vol} K)^N
\end{aligned}$$

We want to take $A := \{\mathrm{d}(\hat{\mu}_N, \mu_V) \geq r\}$.

Coulomb transport inequalities

We aim at an inequality of the type : for any $\mu \in \mathcal{P}(\mathbb{R}^d)$,

$$\mathrm{d}(\mu, \mu_V)^2 \leq C_V(\mathcal{E}_V(\mu) - \mathcal{E}_V(\mu_V)).$$

This inequality is the Coulomb counterpart of Talagrand $\mathbf{T}_1$ inequality : ν satisfies $\mathbf{T}_1$ iif there exists $C > 0$ such that for any $\mu \in \mathcal{P}(\mathbb{R}^d)$,

$$W_1(\mu, \nu)^2 \leq CH(\mu|\nu).$$

To point out what is specific to the Coulombian nature of the interaction, we will show the following local version of our inequality :

Proposition *For any compact set D of $\mathbb{R}^d$, there exists C_D such that for any $\mu, \nu \in \mathcal{P}(D)$ such that $\mathcal{E}(\mu) < \infty$ and $\mathcal{E}(\nu) < \infty$,*

$$W_1(\mu, \nu)^2 \leq C_D \mathcal{E}(\mu - \nu).$$

Proof of the Proposition

If μ and ν have their support in D, there exists D_+ such that

$$W_1(\mu, \nu) = \sup_{\substack{\|f\|_{Lip} \leq 1 \\ f \in \mathcal{C}(D_+)}} \int f \mathrm{d}(\mu - \nu)$$

By a density argument, one can asumme that $\eta := \mu - \nu$ has a smooth density h, let $U^\eta := g * h$. From the Poisson equation, we know that for any smooth function φ,

$$\int \Delta\varphi(y) g(y) \mathrm{d}y = -c_d \varphi(0).$$

Choosing $\varphi(y) = h(x - y)$, we get that

$$\int \Delta h(x - y) g(y) \mathrm{d}y = -c_d h(x)$$

But we also have

$$\int \Delta h(x - y) g(y) \mathrm{d}y = \int \Delta g(x - y) h(y) \mathrm{d}y = \Delta U^\eta(x).$$

Therefore, for any Lipschitz function with support in D_+

$$\int f \mathrm{d}\eta = -\frac{1}{c_d} \int f(x) \Delta U^\eta(x) \mathrm{d}x = -\frac{1}{c_d} \int \nabla f(x) \cdot \nabla U^\eta(x) \mathrm{d}x$$

We can now conclude as

$$\begin{aligned} \left| \int \nabla f(x) \cdot \nabla U^\eta(x) \mathrm{d}x \right| &\leq \int_{D_+} |\nabla f| \cdot |\nabla U^\eta| \leq \int_{D_+} |\nabla U^\eta| \\ &\leq \left(\mathrm{vol}(D_+) \int |\nabla U^\eta|^2 \right)^{1/2}. \end{aligned}$$

But

$$\int |\nabla U^\eta|^2 = c_d \mathcal{E}(\eta).$$

Thank you for your attention !