p-adic L-functions obtained by Eisenstein measure for unitary groups II: Katz's p-adic L-function

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$$k + d_j \ge 1 \ge 0 \ge -d_j \qquad \text{or} \qquad -d_j \ge 1 \ge 0 \ge k + d_j$$

$$\left(\gamma_\infty \left(s, \chi, \psi_{\mathbb{R}} \circ \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\right) \sim \prod_{j=1}^{r} \frac{\Gamma\left(1 - s - \min\{k + d_j, -d_j\}\right)}{\Gamma\left(s + \max\{k + d_j, -d_j\}\right)}\right).$$

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The cases addressed in the main theorem in Katz's paper are

$$k\ge 1 \text{ and } d_j\ge 0$$

(half of the critical points satisfying $k+d_j\ge 1\ge 0\ge -d_j$, or more precisely those to the right of the center).

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Assume that all places of F above p split in K.

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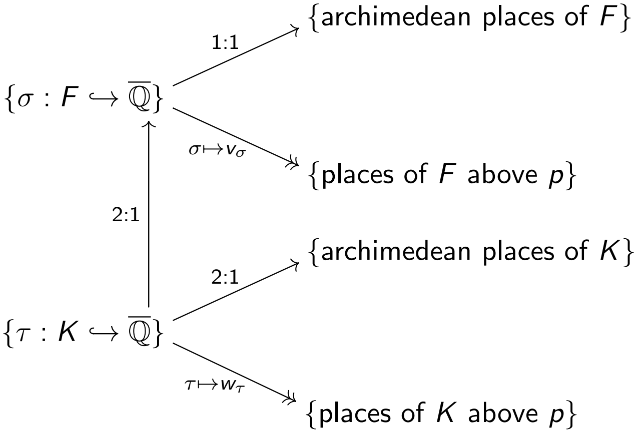

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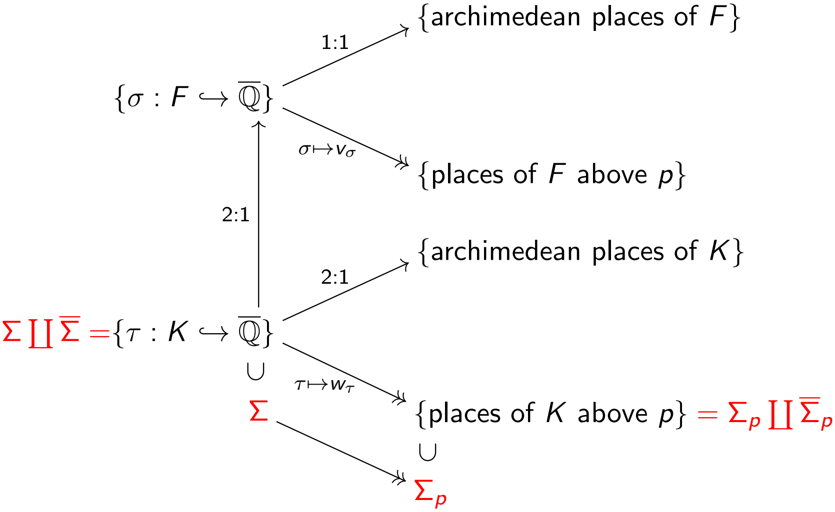

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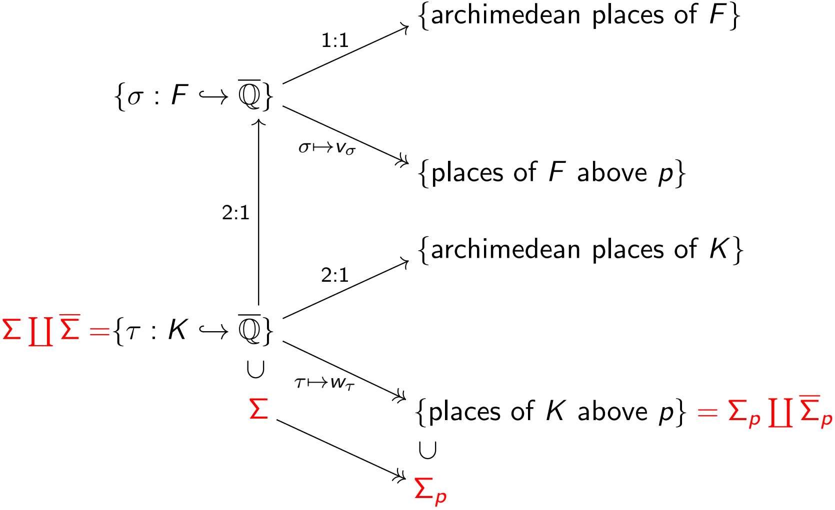

Note: Σ = half of embeddings, $1:1$ to archimedean places
Σ_p = set of places, half of places above p in K

p-adic avatar of Hecke characters

Given $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$, algebraic Hecke character of ∞-type

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define its p-adic avatar as

$$\chi_{p\text{-adic}} : K^\times \backslash \mathbb{A}_{K,f}^\times \longrightarrow \overline{\mathbb{Q}}_p^\times$$
$$x \longmapsto \chi_f(x) \cdot \prod_{w \in \Sigma_p} \prod_{\substack{\tau \in \Sigma \\ w_\tau = w}} \tau(x_w)^{k+d_\tau} \tau(x_{\overline{w}})^{-d_\tau}.$$

Some auxiliary data

Fix additive character $\psi : \mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}} \to \mathbb{C}^{\times}$ to be the one (1) fixed by $\widehat{\mathbb{Z}}$, (2) sending $x_\infty \in \mathbb{R}$ to $e^{2\pi i \cdot x_\infty}$.

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Fix an ideal $\mathfrak{c} = \prod_{w \in S^{p\infty}} \mathcal{P}_w^{c_w} \subset \mathcal{O}_K$, $c_w \in \mathbb{Z}_{\geq 1}$.

Let

$$U_{\mathfrak{c}}^p = \prod_{w \nmid S} \mathcal{O}_{K_w}^\times \times \prod_{w \in S^{p\infty}} 1 + \mathcal{P}^{c_w} \mathcal{O}_{K_w}.$$

Goal

Define a p-adic measure $\mathfrak{L}_{\mathfrak{c}}$ on $K^\times \backslash \mathbb{A}_{K,f}^\times / U_{\mathfrak{c}}^p$ ($\mathbb{Z}_p$-rank equal to $[F:\mathbb{Q}] + 1 +$ Leopoldt defect) with interpolation property:

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If $\chi : K^\times \backslash \mathbb{A}_K^\times / U_{\mathfrak{c}}^p \to \mathbb{C}^\times$ is algbraic with ∞-type $(k + d_\tau, -d_\tau)_{\tau \in \Sigma}$ and $k \geq 1$, $d_\tau \geq 0$, then

$$\mathfrak{L}_{\mathfrak{c}}(\chi_{p\text{-adic}}) = \text{period} \cdot E_p(0, \chi) \cdot L^S(0, \chi),$$

with

$$E_p(s, \chi) = \prod_{w \in \Sigma_p} \gamma_w \left(0, \chi, \psi_p \circ \mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1}$$

as conjectured by Coates–Perrin-Riou.

Doubling method formula

The doubling method (Garrett, Piatetski-Shapiro–Rallis, Shimura ...) specialized to the case

$$\mathrm{U}(1) \times \mathrm{U}(1) \hookrightarrow \mathrm{U}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

provides a nice integral representation to study the critical L-values we are interested in.

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For each $g_f \in G(\mathbb{A}_f)$, we can also define a map

$$\iota_{G,g_f} : \mathrm{U}(1, \mathbb{A}_F) \times \mathrm{U}(1, \mathbb{A}_F) \hookrightarrow G(\mathbb{A}_F) \xrightarrow{\text{right mult by } g_f} G(\mathbb{A}_F).$$

$$\left(\iota_{G,g_f}((a, b)(a', b')) = \iota_G(a, b)\iota_{G,g_f}(a', b').\right)$$

Later $g_f = \bigotimes_{v|p} \begin{pmatrix} \frac{1}{2} & \sqrt{-D} \\ -\frac{\sqrt{-D}^{-1}}{2} & 1 \end{pmatrix}_v$.

Doubling method formula

Let $\chi_0 = \chi|\cdot|_{\mathbb{A}_K}^{-\frac{k}{2}}$ (unitary character), inducing a character on the Siegel parabolic subgroup of G

$$Q = \left\{ \begin{pmatrix} a & \\ & \bar{a}^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : a \in K^\times, x \in F \right\} \subset G.$$

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$$f(s, \chi_0) = \otimes f_v(s, \chi_0) \in \mathrm{Ind}_{Q(\mathbb{A}_F)}^{G(\mathbb{A}_F)} \chi_0 |\cdot|^s \text{ (normalized induction)},$$

define the Siegel Eisenstein series as

$$E(g, f(s, \chi_0)) = \sum_{\gamma \in Q(F) \backslash G(F)} f(s, \chi_0)(\gamma g), \qquad g \in G(\mathbb{A}_F).$$

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This is a special case of doubling method formula for trivial representation on $\mathrm{U}(1)$.

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&= \int_{\mathrm{U}(1,\mathbb{A}_F)} f(s,\chi_0)\left(\iota_{G,g_f}(a,1)\right) da \\
&= \prod_v Z_v\left(f_v(s,\chi_0), \mathrm{triv}\right)
\end{aligned}$$

with
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In general $Q(F)\backslash G(F)/(\mathrm{Im}\,\iota_G)(F)$ has more than one orbit, some analysis is needed to throw away orbits other than 1.

Doubling method formula

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$(*)$ is a purely local problem and is via either directly computing the locla zeta integral $Z_v\left(f_v(s,\chi_0), \mathrm{triv}\right)$ (as we will see later), or citing results from general theory of doubling method for place v where everything is unramified.

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$\Rightarrow$ One can p-adically interpolate L-values by p-adically interpolating Eisenstein series.

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(2) Compute the β-th Fourier coefficients ($\beta \in F$) of

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(5) Compute local zeta integrals to get exact evaluation formulas.

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(1) For each χ of our interest, pick $f_v(s,\chi_0) \in \mathrm{Ind}_{Q(\mathbb{A}_F)}^{G(\mathbb{A}_F)} \chi_0 |\cdot|^s$ (only dependent on $\chi_0 = \chi|\cdot|_{\mathbb{A}_K}^{-\frac{k}{2}}$).

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(3)(4) mostly about geometry

Choice of $f_v(s,\chi_0)$: unramified & archimedean

Set $m_{D,v} = \left[\frac{\mathrm{val}_v(D)}{2}\right]$.

	$f_v\left(s,\chi_0^{-1}\right)$
$v \notin S^F$, split or ramified or inert & $v \nmid 2$	$f_v(s,\chi_0)(g) = f_v^{\mathrm{ur}}(s,\chi_0)\left(\begin{pmatrix}\varpi_v & \\ & \varpi_v^{-1}\end{pmatrix}^{-m_{D,v}} g \begin{pmatrix}\varpi_v & \\ & \varpi_v^{-1}\end{pmatrix}^{m_{D,v}}\right)$
$v \mid 2$ inert	$f_v(s,\chi_0)(g) = f_v^{\mathrm{ur}}(s,\chi_0)\left(\begin{pmatrix}1 & \frac{1}{2} \\ 0 & \frac{1}{2}\end{pmatrix}^{-1}\begin{pmatrix}\varpi_v & \\ & \varpi_v^{-1}\end{pmatrix}^{-m_{D,v}} g \begin{pmatrix}\varpi_v & \\ & \varpi_v^{-1}\end{pmatrix}^{m_{D,v}}\begin{pmatrix}1 & \frac{1}{2} \\ 0 & \frac{1}{2}\end{pmatrix}\right)$
$v=\sigma=\tau\|_F$ archimedean, $\tau \in \Sigma$	$f_v(s,\chi_0)\left(g = \left(\begin{smallmatrix}a & b \\ c & d\end{smallmatrix}\right)\right) = (\det g)^{k+2d_\tau}\left(c\tau(\sqrt{-D})+d\right)^{-k-2d_\tau}\left\|c\tau(\sqrt{-D})+d\right\|_{\mathbb{C}}^{-s-\frac{1}{2}+\frac{k}{2}+d_\tau}$

Unramified sections

Recall the embedding

$$\mathrm{U}(1)\times\mathrm{U}(1) \hookrightarrow \underbrace{\mathrm{U}\begin{pmatrix}1 & \\ & -1\end{pmatrix}}_{\substack{\text{local zeta integrals computed in references of}\\ \text{representation theory}}} \xrightarrow[\mathcal{R}]{\begin{pmatrix}1 & \sqrt{-D}\\ 1 & -\sqrt{-D}\end{pmatrix}^{-1} * \begin{pmatrix}1 & \sqrt{-D}\\ 1 & -\sqrt{-D}\end{pmatrix}} \underbrace{G = \mathrm{U}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}}_{\text{Siegel Eisenstein series usually studied}}.$$

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The conjugation by $\begin{pmatrix} \varpi_v & \\ & \varpi_v^{-1} \end{pmatrix}^{m_{D,v}}$ is to balance the fact that D might not be an optimal choice. The conjugation by $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$ at $v \mid 2$ inert is to deal with $\det \begin{pmatrix} 1 & \sqrt{-D} \\ 1 & -\sqrt{-D} \end{pmatrix} = -2\sqrt{-D}$.

Archimedean sections

For an archimedean place $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$ and $\tau \in \Sigma$, $\tau|_F = \sigma$,

$$\chi_\sigma(x) = x^{k+d_\tau}\overline{x}^{-d_\tau}.$$

We want a section in $\mathrm{Ind}_{Q(\mathbb{R})}^{G(\mathbb{R})}\chi_\sigma$ fixed by the right translation of $\iota_G(\mathrm{U}(1,\mathbb{R}),1)$.

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	action by $\iota_G\,(\mathrm{U}(1) \times \mathrm{U}(1))$	weight (in usual sense) for Siegel Eisenstein series
$(\det g)^{k+d_\tau} \left(c\tau(\sqrt{-D}) + d\right)^{-k}$	$(d_\tau,\ k + d_\tau)$	k
$f_\sigma(s,\chi_0)\|_{s=\frac{k}{2}-\frac{1}{2}}(g) =$ $(\det g)^{k+2d_\tau} \left(c\tau(\sqrt{-D}) + d\right)^{-k-2d_\tau} \|c(\tau(\sqrt{-D}) + d\|_{\mathbb{C}}^{d_\tau}$	$(0,\ k + 2d_\tau)$	$k + 2d_\tau$

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Sections at places in S_f, "big cell" sections

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Later we will see that the local Fourier coefficient indexed by $\beta \in F$ for the "big cell" section is simply $\widehat{\alpha}_v(\beta)$.

Choice of $f_v(s,\chi_0)$: $v \in S_f$

Set $\chi^F = \chi|_{\mathbb{A}_F}$.

		$\alpha_{\chi_0,v}$
$v \mid p$, $v = w\overline{w}$, $w \in \Sigma_p$		$\left(\chi_{0,\overline{w}}(2\sqrt{-D})\mathbb{1}_{\mathcal{O}_{F_v}^\times} \cdot \chi_{0,w}\right)^\wedge$
$v \in S_f^F$, $v \nmid p\infty$, $v = w\overline{w}$, split	$w \in S$, $\overline{w} \notin S$	$\chi_{0,v}^F \left(\varpi_v^{-m_{D,v}}\sqrt{-D}\right)^{-1} \chi_{0,\overline{w}}(\varpi_v)^{-c_w} \lvert\varpi_v\rvert_v^{-c_w(s+\frac{1}{2})}$ $\times \mathbb{1}_{\varpi_v^{-m_{D,v}}\sqrt{-D}(1+\varpi_v^{c_w}\mathcal{O}_{F_v})}$
	$w, \overline{w} \in S$	$\chi_{0,v}^F(2)\lvert\varpi_v\rvert_v^{-2s\max\{c_w,c_{\overline{w}}\}} \cdot \mathbb{1}_{\varpi^{-\max\{c_w,c_{\overline{w}}\}}\mathcal{O}_{F_v}^\times} \cdot \chi_{0,v}^{F,-1}$
$v \in S_f^F$ inert		$\chi_{0,v}^F(2)\lvert\varpi_v\rvert_v^{-2s\,c_w}\dfrac{\sharp\,\mathrm{U}(1,\kappa_v)}{\sharp\kappa_v^\times} \cdot \mathbb{1}_{\varpi^{-c_w}\mathcal{O}_{F_v}^\times} \cdot \chi_{0,v}^{F,-1}$

(p-adic) Fourier coefficients of Siegel Eisenstein series

Given $\beta \in F$, let

$$E_\beta(g, f(s, \chi_0)) = \int_{F \backslash \mathbb{A}_F} E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, f(s, \chi_0)\right) \psi\left(-\mathrm{Tr}_{\mathbb{A}_F/\mathbb{A}_\mathbb{Q}} \beta x\right) dx.$$

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For p-adic interpolation, it suffices to look at

$$E_\beta\left(g, f(s,\chi_0)\right)\Big|_{s=\frac{k}{2}-\frac{1}{2}},$$

for $g = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in G(\mathbb{A}_F)$ with $a_v = 1$ for all $v \mid p$.

(p-adic) Fourier coefficients of Siegel Eisenstein series

Our choice of sections at places above $p \Rightarrow$

- If $\beta = 0$, then $E_\beta(g, f(s, \chi_0)) = 0$ if g_v is upper triangular for some $v | p$.
- If $\beta \neq 0$, then

$$E_\beta(g, f(s, \chi_0)) = \prod_v W_{\beta,v}(g_v, f_v(s, \chi_0))$$

$$W_{\beta,v}(g_v, f_v(s, \chi_0)) = \int_{F_v} f_v(s, \chi_0)\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & x \end{smallmatrix}\right) g\right) \psi_{q_v}\left(-\mathrm{Tr}_{F_v/\mathbb{Q}_{q_v}} \beta x\right) dx.$$

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Our choice of archimedean sections sections $\Rightarrow$

- $W_{\beta,\sigma}(g_\sigma, f_\sigma(s,\chi_0))|_{s=\frac{k}{2}-\frac{1}{2}} = 0$ unless $\sigma(\beta) > 0$.

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In the following, we assume

$$a_v = 1 \text{ for all } v \mid p, \qquad \sigma(\beta) > 0 \text{ for all } \sigma : F \hookrightarrow \mathbb{R}.$$

Define

$$d_v(s, \chi_0) := L_v(2s + 1, \chi_0^F).$$

Fourier coefficient: unramified places

$\mathfrak{d}$ = differential ideal of F.

	$W_{\beta,v}\left(\begin{pmatrix} a_v & \\ & d_v \end{pmatrix}, f_v(s,\chi_0)\right)$
$v \notin S^F$ split, $v = w\overline{w}$	$d_v(s,\chi_0)^{-1} \cdot \chi_{0,w}(d_v)\|d_v\|_v^{s-\frac{1}{2}} \cdot \chi_{0,\overline{w}}(a_v)^{-1}\|a_v\|_v^{-s+\frac{1}{2}}$ $\times\, \mathbb{1}_{\mathfrak{d}_v^{-1}}(\varpi_v^{2m_{D,v}} d_v^{-1}\beta a_v) \sum_{j=0}^{\mathrm{val}_v(d_v^{-1}\beta a_v \cdot \mathfrak{d}_v)} \left(\chi_{0,v}^F(\varpi_v)\|\varpi_v\|_v^{2s}\right)^{j-2m_{D,v}}$
$v \notin S^F$ inert & $v \nmid 2$ or ramified, $w \mid v$	$d_v(s,\chi_0)^{-1} \cdot \chi_{0,w}(\overline{a}_v)^{-1}\|a_v\|_v^{-s+\frac{1}{2}}$ $\times\, \mathbb{1}_{\mathfrak{d}_v^{-1}}(\varpi_v^{2m_{D,v}} \overline{a}_v\beta a_v) \sum_{j=0}^{\mathrm{val}_v(\overline{a}_v\beta a_v \cdot \mathfrak{d}_v)} \left(\chi_{0,v}^F(\varpi_v)\|\varpi_v\|_v^{2s}\right)^{j-2m_{D,v}}$
$v \mid 2$ inert	$d_v(s,\chi_0)^{-1} \cdot \chi_{0,w}(2\overline{a}_v)^{-1}\|2a_v\|_v^{-s+\frac{1}{2}} \cdot \psi_2(-\mathrm{Tr}_{F_v/\mathbb{Q}_2}\beta/2)$ $\times\, \mathbb{1}_{\mathfrak{d}_v^{-1}}(2\varpi_v^{2m_{D,v}} \overline{a}_v\beta a_v) \sum_{j=0}^{\mathrm{val}_v(2\overline{a}_v\beta a_v \cdot \mathfrak{d}_v)} \left(\chi_{0,v}^F(\varpi_v)\|\varpi_v\|_v^{2s}\right)^{j-2m_{D,v}}$

Fourier coefficients: archimedean places

$\sigma = \tau|_F$, $\tau \in \Sigma$. Denote

$$f_\sigma^{t_1,t_2,t_3}\left(g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = (\det g)^{t_1} \left(c\tau(\sqrt{-D}) + d\right)^{t_2} |c\tau(\sqrt{-D}) + d|_{\mathbb{C}}^{t_3}$$

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Then

$$f_\sigma(s, \chi_0)|_{s=\frac{k}{2}-\frac{1}{2}} = f_\sigma^{k+2d_\tau,\, -k-2d_\tau,\, d_\tau}$$

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Well known: $\sigma(\beta) > 0$ for all σ,

$$W_{\beta,\sigma}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), f_\sigma^{t_1,t_2,0}\right) = \frac{(2\pi i)^{-t_2}}{\Gamma(-t_2)}\sigma(\beta)^{-t_2-1}\cdot(\det g)^{t_1}\left(c\tau(\sqrt{-D}) + d\right)^{t_2}$$
$$\times\ e^{2\pi i\cdot\sigma(\beta)\left(a\tau(\sqrt{-D})+b\right)\left(c\tau(\sqrt{-D})+d\right)^{-1}}.$$

Fourier coefficients: archimedean places

$\sigma = \tau|_F$, $\tau \in \Sigma$. Denote

$$f_\sigma^{t_1,t_2,t_3}\left(g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = (\det g)^{t_1}\left(c\tau(\sqrt{-D}) + d\right)^{t_2} |c\tau(\sqrt{-D}) + d|_{\mathbb{C}}^{t_3}$$

Then

$$f_\sigma(s,\chi_0)|_{s=\frac{k}{2}-\frac{1}{2}} = f_\sigma^{k+2d_\tau,\,-k-2d_\tau,\,d_\tau}$$

Well known: $\sigma(\beta) > 0$ for all σ,

$$W_{\beta,\sigma}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), f_\sigma^{t_1,t_2,0}\right) = \frac{(2\pi i)^{-t_2}}{\Gamma(-t_2)}\sigma(\beta)^{-t_2-1} \cdot (\det g)^{t_1}\left(c\tau(\sqrt{-D}) + d\right)^{t_2}$$
$$\times\, e^{2\pi i \cdot \sigma(\beta)\left(a\tau(\sqrt{-D})+b\right)\left(c\tau(\sqrt{-D})+d\right)^{-1}}.$$

How to obtain $W_{\beta,\sigma}$ for $t_3 \neq 0$ from results on $t_3 = 0$?

Differential operator/Lie algebra action

Recall $\mathcal{R} : \mathrm{U}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \to \mathrm{U}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. Let

$$L_+ = \mathcal{R}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{-D} \\ 1 & -\sqrt{-D} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{-D} \\ 1 & -\sqrt{-D} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{-D}}{2} \\ -\frac{\sqrt{-D}^{-1}}{2} & -\frac{1}{2} \end{pmatrix}.$$

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The action of $L_+ \in \mathrm{Lie}\, G$ by right translation is $\left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$

$$L_+ = \frac{1}{2}(a\sqrt{-D} - b)(\sqrt{-D}^{-1}\frac{\partial}{\partial a} + \frac{\partial}{\partial b}) + \frac{1}{2}(c\sqrt{-D} - d)(\sqrt{-D}^{-1}\frac{\partial}{\partial c} + \frac{\partial}{\partial d}).$$

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In particular,

$$\tau(L_+)^{d} \cdot f_\sigma^{t_1,t_2,t_3} = \frac{\Gamma(-t_2 - t_3 + d)}{\Gamma(-t_2 - t_3)} \cdot f_\sigma^{t_1+d,\, t_2-2d,\, t_3+d}$$

Differential operator/Lie algebra action

	action by $\iota_G(\mathrm{U}(1)\times\mathrm{U}(1))$	weight (in usual sense) for Siegel Eisenstein series
$(\det g)^{k+d_\tau}\left(c\tau(\sqrt{-D})+d\right)^{-k}$	$(d_\tau,\ k+d_\tau)$	k
$(\det g)^{k+2d_\tau}\left(c\tau(\sqrt{-D})+d\right)^{-k-2d_\tau}\lvert c(\tau(\sqrt{-D})+d\rvert_{\mathbb{C}}^{d_\tau}$	$(0,\ k+2d_\tau)$	$k+2d_\tau$

$\tau(L_+)^{d_\tau}$

$$f_\sigma(s,\chi_0)|_{s=\frac{k}{2}-\frac{1}{2}}(g) = (\det g)^{k+2d_\tau}\left(c\tau(\sqrt{-D})+d\right)^{-k-2d_\tau}\lvert c(\tau(\sqrt{-D})+d\rvert_{\mathbb{C}}^{d_\tau}$$

Differential operator/Lie algebra action

	action by $\iota_G\left(\mathrm{U}(1)\times\mathrm{U}(1)\right)$	weight (in usual sense) for Siegel Eisenstein series
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$\tau(L_+)^{d_\tau}$ (arrow from the first row to the second row)

$$f_\sigma(s,\chi_0)|_{s=\frac{k}{2}-\frac{1}{2}}(g) = (\det g)^{k+2d_\tau} \left(c\tau(\sqrt{-D})+d\right)^{-k-2d_\tau} \lvert c(\tau(\sqrt{-D})+d\rvert_{\mathbb{C}}^{d_\tau}$$

$\Rightarrow$

$$f_\sigma(s,\chi_0)|_{s=\frac{k}{2}-\frac{1}{2}} = f_\sigma^{k+2d_\tau,\,-k-2d_\tau,\,d_\tau} = \frac{\Gamma(k)}{\Gamma(k+d_\tau)}\cdot\tau(L_+)^{d_\tau}\cdot f_\sigma^{k+d_\tau,\,-k,\,0}$$

$$W_{\beta,\sigma}\left(g, f_\sigma(s,\chi_0)\right)|_{s=\frac{k}{2}-\frac{1}{2}} = \frac{\Gamma(k)}{\Gamma(k+d_\tau)}\cdot\tau(L_+)^{d_\tau}\cdot W_{\beta,\sigma}\left(g, f_\sigma^{k+d_\tau,\,-k,\,0}\right)$$

Fourier coefficients: archimedean places

$$W_{\beta,\sigma}\left(g, f_\sigma(s,\chi_0)\right)\big|_{s=\frac{k}{2}-\frac{1}{2}} = \frac{\Gamma(k)}{\Gamma(k+d_\tau)} \cdot \tau(L_+)^{d_\tau} \cdot W_{\beta,\sigma}\left(g, f_\tau^{k+d_\tau,\,-k,\,0}\right)$$

$$W_{\beta,\sigma}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), f_\tau^{k+d_\tau,\,-k,\,0}\right) = \frac{(2\pi i)^k}{\Gamma(k)}\sigma(\beta)^{k-1} \cdot (\det g)^{k+d_\tau}\left(c\tau(\sqrt{-D})+d\right)^{-k}$$

$$\times\, e^{2\pi i\cdot\sigma(\beta)\left(a\tau(\sqrt{-D})+b\right)\left(c\tau(\sqrt{-D})+d\right)^{-1}}$$

Fourier coefficients: archimedean places

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$$W_{\beta,\sigma}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), f_\tau^{k+d_\tau,\,-k,\,0}\right) = \frac{(2\pi i)^k}{\Gamma(k)} \sigma(\beta)^{k-1} \cdot (\det g)^{k+d_\tau} \left(c\tau(\sqrt{-D}) + d\right)^{-k}$$
$$\times\ e^{2\pi i\cdot\sigma(\beta)\left(a\tau(\sqrt{-D})+b\right)\left(c\tau(\sqrt{-D})+d\right)^{-1}}$$

$$L_+\left(\overline{c\sqrt{-D}+d}\right) = 0, \qquad L_+(c\sqrt{-D}+d) = (-\det g)\left(\overline{c\sqrt{-D}+d)}\right),$$
$$L_+\left(e^{2\pi i\cdot\beta\,\frac{a\sqrt{-D}+b}{c\sqrt{-D}+d}}\right) = 2\pi i\cdot\beta\cdot 2\sqrt{-D}\det g\,(c\sqrt{-D}+d)^{-2}\, e^{2\pi i\cdot\beta\,\frac{a\sqrt{-D}+b}{c\sqrt{-D}+d}}$$

Fourier coefficients: archimedean places

$$W_{\beta,\sigma}\left(g, f_\sigma(s,\chi_0)\right)\big|_{s=\frac{k}{2}-\frac{1}{2}} = \frac{\Gamma(k)}{\Gamma(k+d_\tau)} \cdot \tau(L_+)^{d_\tau} \cdot W_{\beta,\sigma}\left(g, f_\tau^{k+d_\tau,\,-k,\,0}\right)$$

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$$\Longrightarrow W_{\beta,\sigma}\left(g, f_\sigma(s,\chi_0)\right)\big|_{s=\frac{k}{2}-\frac{1}{2}}$$
$$= \frac{(2\pi i)^{k+d_\tau}}{\Gamma(k+2d_\tau)} \cdot \left(2\tau(\sqrt{-D})\right)^{d_\tau} \sigma(\beta)^{k+d_\tau-1} \cdot (\det g)^{k+2d_\tau} \left(c\tau(\sqrt{-D})+d\right)^{-k-2d_\tau}$$
$$\times \left(1 + \lambda_1 \frac{|c\tau(\sqrt{-D})+d|_{\mathbb{C}}}{4\pi\sigma(\sqrt{D})} + \cdots + \lambda_{d_\tau}\left(\frac{|c\tau(\sqrt{-D})+d|_{\mathbb{C}}}{4\pi\sigma(\sqrt{D})}\right)^{d_\tau}\right)$$
$$\times\, e^{2\pi i\cdot\sigma(\beta)\left(a\tau(\sqrt{-D})+b\right)\left(c\tau(\sqrt{-D})+d\right)^{-1}}$$

Fourier coefficients: archimedean places

$$W_{\beta,\sigma}\left(g, f_\tau(s,\chi_0)\right)\Big|_{s=\frac{k}{2}-\frac{1}{2}}$$

$$= \frac{(2\pi i)^{k+d_\tau}}{\Gamma(k+2d_\tau)} \left(2\tau(\sqrt{-D})\right)^{d_\tau} \sigma(\beta)^{k+d_\tau-1} (\det g)^{k+2d_\tau} \left(c\tau(\sqrt{-D})+d\right)^{-k-2d_\tau}$$

$$\times \left(1+\lambda_1 \frac{|c\tau(\sqrt{-D})+d|_{\mathbb{C}}}{4\pi\sigma(\sqrt{D})} + \cdots + \lambda_{d_\tau}\left(\frac{|c\tau(\sqrt{-D})+d|_{\mathbb{C}}}{4\pi\sigma(\sqrt{D})}\right)^{d_\tau}\right)$$

$$\times\ e^{2\pi i\cdot\sigma(\beta)\left(a\tau(\sqrt{-D})+b\right)\left(c\tau(\sqrt{-D})+d\right)^{-1}}$$

Fourier coefficients: archimedean places

$$W_{\beta,\sigma}\left(g, f_\tau(s,\chi_0)\right)\big|_{s=\frac{k}{2}-\frac{1}{2}}$$

$$= \underbrace{\frac{(2\pi i)^{k+d_\tau}}{\Gamma(k+2d_\tau)}}_{\substack{\text{divide to normalize}\\ \text{Eis series}\\ \text{appearing in period}}} \cdot \left(2\tau(\sqrt{-D})\right)^{d_\tau} \sigma(\beta)^{k+d_\tau-1}(\det g)^{k+2d_\tau}\left(c\tau(\sqrt{-D})+d\right)^{-k-2d_\tau}$$

$$\times\left(1+\lambda_1\frac{|c\tau(\sqrt{-D})+d|_{\mathbb{C}}}{4\pi\sigma(\sqrt{D})}+\cdots+\lambda_{d_\tau}\left(\frac{|c\tau(\sqrt{-D})+d|_{\mathbb{C}}}{4\pi\sigma(\sqrt{D})}\right)^{d_\tau}\right)$$

$$\times\, e^{2\pi i\cdot\sigma(\beta)\left(a\tau(\sqrt{-D})+b\right)\left(c\tau(\sqrt{-D})+d\right)^{-1}}$$

Fourier coefficients: archimedean places

$$W_{\beta,\sigma}\left(g, f_\tau(s,\chi_0)\right)\big|_{s=\frac{k}{2}-\frac{1}{2}}$$

$$= \underbrace{\frac{(2\pi i)^{k+d_\tau}}{\Gamma(k+2d_\tau)}}_{\substack{\text{divide to normalize}\\ \text{Eis series}\\ \text{appearing in period}}} \cdot \underbrace{\left(2\tau(\sqrt{-D})\right)^{d_\tau} \sigma(\beta)^{k+d_\tau-1}}_{\substack{\text{appearing in } p\text{-adic}\\ q\text{-expansion}}} (\det g)^{k+2d_\tau} \left(c\tau(\sqrt{-D})+d\right)^{-k-2d_\tau}$$

$$\times \left(1 + \lambda_1 \frac{|c\tau(\sqrt{-D})+d|_{\mathbb{C}}}{4\pi\sigma(\sqrt{D})} + \cdots + \lambda_{d_\tau}\left(\frac{|c\tau(\sqrt{-D})+d|_{\mathbb{C}}}{4\pi\sigma(\sqrt{D})}\right)^{d_\tau}\right.$$

$$\times\, e^{2\pi i\cdot\sigma(\beta)\left(a\tau(\sqrt{-D})+b\right)\left(c\tau(\sqrt{-D})+d\right)^{-1}}$$

Fourier coefficients: archimedean places

$$W_{\beta,\sigma}\left(g, f_\tau(s,\chi_0)\right)\big|_{s=\frac{k}{2}-\frac{1}{2}}$$

$$= \underbrace{\frac{(2\pi i)^{k+d_\tau}}{\Gamma(k+2d_\tau)}}_{\substack{\text{divide to normalize} \\ \text{Eis series} \\ \text{appearing in period}}} \cdot \underbrace{\left(2\tau(\sqrt{-D})\right)^{d_\tau} \sigma(\beta)^{k+d_\tau-1}}_{\substack{\text{appearing in } p\text{-adic} \\ q\text{-expansion}}} (\det g)^{k+2d_\tau} \left(c\tau(\sqrt{-D})+d\right)^{-k-2d_\tau}$$

$$\times \left(1 + \underbrace{\lambda_1 \frac{|c\tau(\sqrt{-D})+d|_{\mathbb{C}}}{4\pi\sigma(\sqrt{D})} + \cdots + \lambda_{d_\tau}\left(\frac{|c\tau(\sqrt{-D})+d|_{\mathbb{C}}}{4\pi\sigma(\sqrt{D})}\right)^{d_\tau}}_{\text{irrelevant/disappearing when viewed as } p\text{-adic form}}\right)$$

$$\times\, e^{2\pi i\cdot\sigma(\beta)\left(a\tau(\sqrt{-D})+b\right)\left(c\tau(\sqrt{-D})+d\right)^{-1}}$$

Fourier coefficients: $v \in S_f^F$:

Easy computation shows

$$W_{\beta,v}\left(\begin{pmatrix} a_v & \\ & d_v \end{pmatrix}, f^{\alpha_v}(s,\chi_0)\right)$$

$$= \widehat{\alpha}_v(d_v^{-1}\beta a_v) \cdot \begin{cases} \chi_{0,w}(d_v)|d_v|_v^{s-\frac{1}{2}}\, \chi_{0,\overline{w}}(a_v)^{-1}|a_v|_v^{-s+\frac{1}{2}}, & v \text{ split} \\ \chi_{0,w}(\overline{a}_v)^{-1}|a_v|_w^{-s+\frac{1}{2}}, & v \text{ inert} \end{cases}$$

where

$$\widehat{\alpha}_v(x) = \int_{F_v} \alpha_v(y)\, \psi_{q_v}\left(-\mathrm{Tr}_{F_v/\mathbb{Q}_{q_v}} xy\right)\, dy.$$

β-th coefficient in p-adic q-expansion

Combining above results at each place, we get formula for the β-th Fourier coefficient in the p-adic q-expansion of

$$\boxed{\frac{\prod \Gamma(k+2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}} \cdot d^{S^F}(s,\chi_0)\, E\left(\cdot, f(s,\chi_0)\right)\big|_{s=\frac{k}{2}-\frac{1}{2}}}$$

(nearly holomorphic form viewed as p-adic form)

at the (p-adic) cusp corresponding to $\left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right)_f$.

β-th coefficient in p-adic q-expansion

β-th coefficient of $\frac{\prod \Gamma(k+2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}} d^{S^F}(s,\chi_0)\, E(\cdot, f(s,\chi_0))|_{s=\frac{k}{2}-\frac{1}{2}}$ at $\begin{pmatrix} a & \\ & d \end{pmatrix}_f (\, a_v = 1,\ v|p)$

$$\prod_{v|2 \text{ inert}} \chi^F_{p\text{-adic},v}(2)^{-1}|2|_v\, \psi_v(-\mathrm{Tr}_{F_v/\mathbb{Q}_2}\beta/2) \cdot \chi_{p\text{-adic}}(d_f^p)|d_f^p|^{-1}_{\mathbb{A}_F^p}$$

$$\times \prod_{v\nmid S^F} \mathbb{1}_{\mathcal{O}_{F_v}}(\varpi^{l_v+2m_{D,v}}\beta) \sum_{j=1}^{l_v} \left(\chi^F_{p\text{-adic},v}(\varpi_v)|\varpi_v|_v^{-1}\right)^{j-2m_{D,v}} \quad \left(l_v = \begin{cases} \mathrm{val}_v\left(2d_v^{-1}\beta a_v\, \mathfrak{d}_v\right), & v|2 \text{ inert} \\ \mathrm{val}_v\left(d_v^{-1}\beta a_v\, \mathfrak{d}_v\right), & \text{otherwise} \end{cases} \right.$$

$$\times \prod_{\substack{v|p \\ w|v,\, w\in\Sigma_p}} \chi_w(d_v)\, \widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta\right) \prod_{\substack{\sigma=\tau|_F : F\hookrightarrow\overline{\mathbb{Q}} \\ v_\sigma = v,\ \tau\in\Sigma}} \left(2\tau(\sqrt{-D})\right)^{d_\tau} \sigma(\beta)^{k+d_\sigma-1}$$

$$\times \prod_{v\in S^F,\ v\nmid p\infty} \widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta a_{\chi_0,v}\right)$$

β-th coefficient in p-adic q-expansion

$$\beta\text{-th coefficient of } \frac{\prod \Gamma(k+2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}} d^{S^F}(s,\chi_0)\, E(\cdot, f(s,\chi_0))\big|_{s=\frac{k}{2}-\frac{1}{2}} \text{ at } \begin{pmatrix} a & \\ & d \end{pmatrix}_f \left(a_v = 1,\ v|p \right)$$

$$\prod_{v|2 \text{ inert}} \chi^F_{p\text{-adic},v}(2)^{-1}|2|_v\, \psi_v(-\mathrm{Tr}_{F_v/\mathbb{Q}_2}\beta/2) \cdot \chi_{p\text{-adic}}(d_f^p)|d_f^p|^{-1}_{\mathbb{A}_F^p}$$

$$\times \prod_{v\nmid S^F} \mathbb{1}_{\mathcal{O}_{F_v}}(\varpi^{l_v+2m_{D,v}}\beta) \sum_{j=1}^{l_v} \left(\chi^F_{p\text{-adic},v}(\varpi_v)|\varpi_v|_v^{-1}\right)^{j-2m_{D,v}} \quad \left(l_v = \begin{cases} \mathrm{val}_v\left(2d_v^{-1}\beta a_v\, \mathfrak{d}_v\right), & v|2 \text{ inert} \\ \mathrm{val}_v\left(d_v^{-1}\beta a_v\, \mathfrak{d}_v\right), & \text{otherwise} \end{cases} \right.$$

$$\times \prod_{\substack{v|p \\ w|v,\, w\in\Sigma_p}} \underbrace{\chi_w(d_v)\, \widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta\right)}_{\text{factor from Fourier coefficient at } v|p} \prod_{\substack{\sigma=\tau|_F: F\hookrightarrow\overline{\mathbb{Q}} \\ v_\sigma = v,\ \tau\in\Sigma}} \underbrace{\left(2\tau(\sqrt{-D})\right)^{d_\tau} \sigma(\beta)^{k+d_\sigma-1}}_{\text{factor from Fourier coefficient at archimedean place } \sigma}$$

$$\times \prod_{v\in S^F,\ v\nmid p\infty} \underbrace{\widehat{\alpha}_v\left(d_v^{-1}\beta a_v\right)}_{\text{factors from Fourier coefficient at } v\in S^F,\ v\nmid p\infty}$$

Choice of $\widehat{\alpha}_{\chi_0,v}$, $v \mid p$

The criterion for choosing $\widehat{\alpha}_{\chi_0,v}$, $v \mid p$, is to make

$$\chi_{p\text{-adic}} \mapsto \chi_w(d_v)\ \widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta\right) \prod_{\substack{\sigma=\tau|_F : F \hookrightarrow \overline{\mathbb{Q}} \\ v_\sigma = v,\ \tau\in\Sigma}} \left(2\tau(\sqrt{-D})\right)^{d_\tau} \sigma(\beta)^{k+d_\sigma-1}$$

interpolated by a p-adic measure on $K^\times \backslash \mathbb{A}_{K,f}^\times / U_{\mathfrak{c}}^p$.

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interpolated by a p-adic measure on $K^\times \backslash \mathbb{A}_{K,f}^\times / U_{\mathfrak{c}}^p$.

Rewrite the above expression as

$$\chi_w(d_v^{-1}\beta)^{-1}\ \widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta\right) \chi_w(\beta) \prod_{\substack{\sigma=\tau|_F : F\hookrightarrow\overline{\mathbb{Q}} \\ v_\sigma = v,\ \tau\in\Sigma}} \left(2\tau(\sqrt{-D})\right)^{d_\tau} \sigma(\beta)^{k+d_\sigma-1}$$

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interpolated by a p-adic measure on $K^\times \backslash \mathbb{A}_{K,f}^\times / U_{\mathfrak{c}}^p$.

Rewrite the above expression as

$$\chi_w(d_v^{-1}\beta)^{-1}\ \widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta\right) \chi_w(\beta) \prod_{\substack{\sigma=\tau|_F : F\hookrightarrow\overline{\mathbb{Q}} \\ v_\sigma=v,\ \tau\in\Sigma}} \left(2\tau(\sqrt{-D})\right)^{d_\tau} \sigma(\beta)^{k+d_\sigma-1}$$

Consider the δ-measure at $\beta \in K_w$ and $2\sqrt{-D} \in K_{\overline{w}}$,

$$\delta_{\beta,w}\left(\chi_{p\text{-adic}}\right) = \chi_{p\text{-adic},w}(\beta) = \chi_w\left(\beta\right) \prod_{\substack{\sigma : F\hookrightarrow\overline{\mathbb{Q}} \\ v_\sigma=v}} \sigma(\beta)^{k+d_\sigma-1}$$

$$\delta_{2\sqrt{-D},\overline{w}}\left(\chi_{p\text{-adic}}\right) = \chi_{p\text{-adic},\overline{w}}(2\sqrt{-D}) = \chi_{\overline{w}}(2\sqrt{-D}) \prod_{\substack{\tau : K\hookrightarrow\overline{\mathbb{Q}} \\ w_\tau=w,\ \tau\in\Sigma}} \left(2\tau(\sqrt{-D})\right)^{d_\tau}.$$

Choice of $\widehat{\alpha}_{\chi_0,v}$, $v \mid p$

The criterion for choosing $\widehat{\alpha}_{\chi_0,v}$, $v \mid p$, is to make

$$\chi_{p\text{-adic}} \mapsto \chi_w(d_v)\ \widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta\right) \prod_{\substack{\sigma=\tau|_F : F\hookrightarrow\overline{\mathbb{Q}} \\ v_\sigma = v,\ \tau\in\Sigma}} \left(2\tau(\sqrt{-D})\right)^{d_\tau} \sigma(\beta)^{k+d_\sigma-1}$$

interpolated by a p-adic measure on $K^\times\backslash\mathbb{A}_{K,f}^\times/U_{\mathfrak{c}}^p$.
Rewrite the above expression as

$$\chi_w(d_v^{-1}\beta)^{-1}\ \widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta\right) \chi_w(\beta) \prod_{\substack{\sigma=\tau|_F : F\hookrightarrow\overline{\mathbb{Q}} \\ v_\sigma = v,\ \tau\in\Sigma}} \left(2\tau(\sqrt{-D})\right)^{d_\tau} \sigma(\beta)^{k+d_\sigma-1}$$

Consider the δ-measure at $\beta \in K_w$ and $2\sqrt{-D} \in K_{\overline{w}}$,

$$\delta_{\beta,w}\left(\chi_{p\text{-adic}}\right) = \chi_{p\text{-adic},w}(\beta) = \chi_w(\beta) \prod_{\substack{\sigma : F\hookrightarrow\overline{\mathbb{Q}} \\ v_\sigma = v}} \sigma(\beta)^{k+d_\sigma-1}$$

$$\delta_{2\sqrt{-D},\overline{w}}\left(\chi_{p\text{-adic}}\right) = \chi_{p\text{-adic},\overline{w}}(2\sqrt{-D}) = \chi_{\overline{w}}(2\sqrt{-D}) \prod_{\substack{\tau : K\hookrightarrow\overline{\mathbb{Q}} \\ w_\tau = w,\ \tau\in\Sigma}} \left(2\tau(\sqrt{-D})\right)^{d_\tau}.$$

$\Rightarrow$ want: $\chi_w(d_v^{-1}\beta)^{-1}\ \widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta\right) = \chi_{\overline{w}}(2\sqrt{-D})$

$\Rightarrow$ a natural choice: $\boxed{\widehat{\alpha}_{\chi_0,v} = \chi_{\overline{w}}(2\sqrt{-D}) \cdot \mathbb{1}_{\mathcal{O}_{F_v}^\times} \cdot \chi_w}$

Choice of $\widehat{\alpha}_{\chi_0,v}$, $v \mid p$

want: $\chi_w(d_v^{-1}\beta)^{-1}\ \widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta\right) = \chi_{\overline{w}}(2\sqrt{-D})$

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Essentially we want $\widehat{\alpha}_{\chi_0,v} \sim \chi_w$. However, In order to make χ_w a Schwartz function on F_v, one must cut off in the direction towards both 0 and infinity. One natural choice for this cut-off is by $\mathbb{1}_{\mathcal{O}_{F_v}^\times}$.

β-th coefficient in p-adic q-expansion

$$\prod_{v|2 \text{ inert}} \chi^F_{p\text{-adic},v}(2)^{-1}|2|_v\, \psi_v(-\mathrm{Tr}F_v/\mathbb{Q}_2\beta/2) \cdot \chi_{p\text{-adic}}(d^p_f)|d^p_f|^{-1}_{\mathbb{A}^p_F}$$

$$\times \prod_{v \nmid S^F} \mathbb{1}_{\mathcal{O}_{F_v}}\left(\varpi^{l_v+2m_{D,v}} \cdot \beta\right) \sum_{j=1}^{l_v} \left(\chi^F_{p\text{-adic},v}(\varpi_v)|\varpi_v|^{-1}_v\right)^{j-2m_{D,v}}$$

$$\times \prod_{\substack{v|p \\ w|v, w\in\Sigma_p}} \chi_{p\text{-adic},\overline{w}}(2\sqrt{-D}) \cdot \mathbb{1}_{\mathcal{O}^\times_{F_v}}\left(d_v^{-1}\beta\right) \cdot \chi_{p\text{-adic},w}(\beta)$$

$$\times \prod_{v\in S^{F,p\infty}} \widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta a_v\right)$$

β-th coefficient in p-adic q-expansion

$$\prod_{v|2 \text{ inert}} \chi^F_{p\text{-adic},v}(2)^{-1}|2|_v\, \psi_v(-\mathrm{Tr} F_v/\mathbb{Q}_2\beta/2)\cdot \chi_{p\text{-adic}}(d_f^p)|d_f^p|^{-1}_{\mathbb{A}_F^p}$$
$$\times \prod_{v\nmid S^F} \mathbb{1}_{\mathcal{O}_{F_v}}\left(\varpi^{l_v+2m_{D,v}}\cdot\beta\right)\sum_{j=1}^{l_v}\left(\chi^F_{p\text{-adic},v}(\varpi_v)|\varpi_v|_v^{-1}\right)^{j-2m_{D,v}}$$
$$\times \prod_{\substack{v|p\\ w|v,w\in\Sigma_p}} \chi_{p\text{-adic},\overline{w}}(2\sqrt{-D})\cdot\mathbb{1}_{\mathcal{O}^\times_{F_v}}\left(d_v^{-1}\beta\right)\cdot\chi_{p\text{-adic},w}(\beta)$$
$$\times \prod_{v\in S^{F,p\infty}} \widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta a_v\right)$$

From our choice of $\alpha_{\chi_0,v}$, calculation of Fourier transform gives

$$\widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta a_v\right)$$
$$=\begin{cases}\chi_v^F(\varpi_v^{-m_{D,v}}\sqrt{-D})\,\chi_{\overline{w}}(\varpi_v)^{-c_w}|\varpi_v|_v^{c_w}\cdot\psi_{q_v}\left(-\mathrm{Tr}_{F_v/\mathbb{Q}_{q_v}}\beta\right)\mathbb{1}_{\varpi_v^{-c_w}\mathfrak{d}_v^{-1}}(\beta), & w\in S,\ \overline{w}\notin S\\ \chi_v^F(2)\cdot\epsilon\left(0,\chi_v^F,\overline{\psi}_{q_v}\circ\mathrm{Tr}_{F_v/\mathbb{Q}_{q_v}}\right)\cdot\mathbb{1}_{\varpi^{\max\{c_w,c_{\overline{w}}\}-c_{\chi,v}}\mathfrak{d}_v^{-1}\mathcal{O}^\times_{F_v}}(\beta)\cdot\chi_v^F(\mathfrak{d}_v\beta), & w,\overline{w}\in S,\ w\neq\overline{w}\\ \chi_v^F(2)\,\frac{\sharp\,\mathrm{U}(1,\kappa_v)}{\sharp\kappa_v^\times}\cdot\epsilon\left(0,\chi_v^F,\overline{\psi}_{q_v}\circ\mathrm{Tr}_{F_v/\mathbb{Q}_{q_v}}\right)\cdot\mathbb{1}_{\varpi^{c_w-c_{\chi,w}}\mathfrak{d}_v^{-1}\mathcal{O}^\times_{F_v}}(\mathfrak{d}_v\beta), & w\in S,\ w=\overline{w}\end{cases}$$

β-th coefficient in p-adic q-expansion

$$\prod_{v|2\text{ inert}} \chi^F_{p\text{-adic},v}(2)^{-1}|2|_v\,\psi_v(-\mathrm{Tr}F_v/\mathbb{Q}_2\beta/2)\cdot\chi_{p\text{-adic}}(d_f^p)|d_f^p|^{-1}_{\mathbb{A}_F^p}$$

$$\times\prod_{v\nmid S^F}\mathbb{1}_{\mathcal{O}_{F_v}}\left(\varpi^{l_v+2m_{D,v}}\cdot\beta\right)\sum_{j=1}^{l_v}\left(\chi^F_{p\text{-adic},v}(\varpi_v)|\varpi_v|_v^{-1}\right)^{j-2m_{D,v}}$$

$$\times\prod_{\substack{v|p\\ w|v,w\in\Sigma_p}}\chi_{p\text{-adic},\overline{w}}(2\sqrt{-D})\cdot\mathbb{1}_{\mathcal{O}_{F_v}^\times}\left(d_v^{-1}\beta\right)\cdot\chi_{p\text{-adic},w}(\beta)$$

$$\times\prod_{v\in S^{F,p\infty}}\widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta a_v\right)$$

From our choice of $\alpha_{\chi_0,v}$, calculation of Fourier transform gives

$$\widehat{\alpha}_{\chi_0,v}\left(d_v^{-1}\beta a_v\right)$$
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Every term can be realized as δ-measure at β, powers of ϖ_v ...
Their products can be achieved by convolution of p-adic measures.

Choice of $\alpha_{\chi_0,v}$:

- At $v \mid p$, we choose $\widehat{\alpha}_{\chi_0,v}$ such that it aligns with factor from archimedean Fourier coefficient to constitute a p-adic measure on $K^\times \backslash \mathbb{A}_{K,f}^\times / U_{\mathfrak{c}}^p$.
- At $v \in S^{F,p\infty}$, we choose $\alpha_{\chi_0,v}$ so that the local zeta integrals give the desired factors. (In general case, this has not yet been achieved. A coarse "volume section" is used.)

Eisenstein measure and p-adic L-function

By our choice of $f(s,\chi_0) \in \mathrm{Ind}_{Q(\mathbb{A})}^{G(\mathbb{Q})} \chi_0|\cdot|_{\mathbb{A}_K}^{s}$, the Eisenstein series on $G = \mathrm{U}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$$E\left(\cdot, f(s,\chi_0)\right)\big|_{s=\frac{k}{2}-\frac{1}{2}}$$

is nearly holomorphic (of weight $(k+2d_\tau)_{\tau\in\Sigma}$).

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$$\frac{\prod \Gamma(k+2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}} d^{S^F}(s,\chi_0)\, E(\cdot, f(s,\chi_0))|_{s=\frac{k}{2}-\frac{1}{2}} \tag{1}$$

as p-adic form.

The above discussion shows that when χ varies among all algebraic characters on $K^\times\backslash\mathbb{A}_K^\times/U_{\mathfrak{c}}^p$ with $\chi_\infty \sim (k+d_\tau, -d_\tau)_{\tau\in\Sigma}$, $k\geq 1, d_\tau \geq 0$,

Eisenstein measure and p-adic L-function

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as p-adic form.

The above discussion shows that when χ varies among all algebraic characters on $K^\times\backslash\mathbb{A}_K^\times/U_{\mathfrak{c}}^p$ with $\chi_\infty \sim (k+d_\tau, -d_\tau)_{\tau\in\Sigma}$, $k\geq 1, d_\tau \geq 0$, the β-th Fourier coefficient in the p-adic q-expansion of (1) at $\begin{pmatrix} a & \\ & d \end{pmatrix}_f$ ($\beta \in F$, $a_v = 1$ for all $v \mid p$) is interpolated by a p-adic measure on $K^\times\backslash\mathbb{A}_{K,f}^\times/U_{\mathfrak{c}}^p$.

Eisenstein measure and p-adic L-function

By our analysis of the Fourier coefficients and the q-expansion principle for p-adic forms on $\mathrm{U}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$,

$$\left\{ \begin{array}{l} \frac{\prod \Gamma(k+2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}} d^{S^F}(s,\chi_0) \\ \times \; E\left(\cdot, f(s,\chi_0)\right)\big|_{s=\frac{k}{2}-\frac{1}{2}} \end{array} \;\middle|\; \begin{array}{l} \chi : K^\times\backslash\mathbb{A}_K^\times/U_{\mathfrak{c}}^p \to \mathbb{C}^\times \\ \text{algebraic with infinite type} \\ (k+d_\tau, -d_\tau)_{\tau\in\Sigma},\; k\geq 1, d_\tau \geq 0 \end{array} \right\}$$

$\rightsquigarrow$ interpolating q-expansions

$$\mu_{\mathcal{E},\mathfrak{c}} \in \mathcal{M}eas\left(K^\times\backslash\mathbb{A}_{K,f}^\times/U_{\mathfrak{c}}^p,\; V_{G,\Gamma_1(\mathfrak{c}),p\text{-adic}}\right)$$

p-adic measures valued in $V_{G,\Gamma_1(\mathfrak{c}),p\text{-adic}}$, the space of p-adic forms on $G = U\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ of tame level $\Gamma_1(\mathfrak{c})$

Eisenstein measure and p-adic L-function

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$$\left\{ \begin{array}{l|l} \frac{\prod \Gamma(k+2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}} d^{S^F}(s,\chi_0) & \chi : K^\times \backslash \mathbb{A}_K^\times / U_\mathfrak{c}^p \to \mathbb{C}^\times \\ \times\ E\left(\cdot, f(s,\chi_0)\right)\big|_{s=\frac{k}{2}-\frac{1}{2}} & \text{algebraic with infinite type} \\ & (k+d_\tau, -d_\tau)_{\tau\in\Sigma},\ k\geq 1, d_\tau \geq 0 \end{array} \right\}$$

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p-adic measures valued in $V_{G,\Gamma_1(\mathfrak{c}),p\text{-adic}}$, the space of p-adic forms on $G = U\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ of tame level $\Gamma_1(\mathfrak{c})$

$\rightsquigarrow$ via our picked $\widetilde{\iota}_G : [U(1)] \hookrightarrow [G]_{\mathrm{ord}}$

$$\mathrm{Res}\left(\mu_{\mathcal{E},\mathfrak{c}}\right) \in \mathcal{M}eas\left(K^\times \backslash \mathbb{A}_{K,f}^\times / U_\mathfrak{c}^p,\ C_c^\infty\left(\mathrm{U}(1,F)\backslash U(1,\mathbb{A}_F)/U_\mathfrak{c}\right)\right)$$

full level at p, finit set

Eisenstein measure and p-adic L-function

By our analysis of the Fourier coefficients and the q-expansion principle for p-adic forms on $\mathrm{U}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$,

$$\left\{ \begin{array}{l|l} \frac{\prod \Gamma(k+2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}} d^{S^F}(s,\chi_0) & \chi : K^\times\backslash\mathbb{A}_K^\times/U_\mathfrak{c}^p \to \mathbb{C}^\times \\ & \text{algebraic with infinite type} \\ \times\ E\left(\cdot, f(s,\chi_0)\right)\big|_{s=\frac{k}{2}-\frac{1}{2}} & (k+d_\tau, -d_\tau)_{\tau\in\Sigma},\ k\geq 1, d_\tau\geq 0 \end{array} \right\}$$

$\rightsquigarrow$ interpolating q-expansions

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p-adic measures valued in $V_{G,\Gamma_1(\mathfrak{c}),p\text{-adic}}$, the space of p-adic forms on $G = U\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ of tame level $\Gamma_1(\mathfrak{c})$

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$$\mathrm{Res}\left(\mu_{\mathcal{E},\mathfrak{c}}\right) \in \mathcal{M}eas\left(K^\times\backslash\mathbb{A}_{K,f}^\times/U_\mathfrak{c}^p,\ C_c^\infty\left(\mathrm{U}(1,F)\backslash U(1,\mathbb{A}_F)/U_\mathfrak{c}\right)\right)$$

full level at p, finit set

$\rightsquigarrow$ sum over $\mathrm{U}(1,F)\backslash U(1,\mathbb{A}_F)/U_\mathfrak{c}$

$$\mathfrak{L}_\mathfrak{c} \in \mathcal{M}eas\left(K^\times\backslash\mathbb{A}_{K,f}^\times/U_\mathfrak{c}^p,\ \overline{\mathbb{Q}}_p\right)$$

Evaluations of $\mathfrak{L}_{\mathfrak{c}}$

We pick $\quad \widetilde{\iota}_G = \iota_{G,g_f}, \quad g_f = \bigotimes_{v|p} \begin{pmatrix} \frac{1}{2} & \sqrt{-D} \\ -\frac{\sqrt{-D}^{-1}}{2} & 1 \end{pmatrix}_v.$

Evaluations of $\mathfrak{L}_{\mathfrak{c}}$

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Assume that the Hecke character $\chi : K^\times \backslash \mathbb{A}_K^\times / U_{\mathfrak{c}}^p \to \mathbb{C}^\times$ is algebraic with ∞-type $(k + d_\tau, -d_\tau)_{\tau \in \Sigma}$, $k \geq 1, d_\tau \geq 0$, and p-adic avatar $\chi_{p\text{-adic}} : K^\times \backslash \mathbb{A}_{K,f}^\times / U_{\mathfrak{c}}^p \to \overline{\mathbb{Q}}_p^\times$.

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By our construction,

$$\mathfrak{L}_{\mathfrak{c}}\left(\chi_{p\text{-adic}}\right) = \left(2\pi i \frac{\Omega_p}{\Omega_\infty}\right)^{k[F:\mathbb{Q}]+2\sum d_\tau} \frac{\prod \Gamma(k + 2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}} \cdot d^{S^F}(s, \chi_0) \times \int_{\mathrm{U}(1,F)\backslash \mathrm{U}(1,\mathbb{A}_F)} E\left(\iota_G(a,1), f(s,\chi_0)\right) \, da \Bigg|_{s=\frac{k}{2}-\frac{1}{2}} .$$

Evaluations of $\mathfrak{L}_{\mathfrak{c}}$

We pick $\quad \widetilde{\iota}_G = \iota_{G,g_f}, \quad g_f = \bigotimes_{v|p} \begin{pmatrix} \frac{1}{2} & \sqrt{-D} \\ -\frac{\sqrt{-D}^{-1}}{2} & 1 \end{pmatrix}_v .$

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$$\mathfrak{L}_{\mathfrak{c}}(\chi_{p\text{-adic}}) = \left(2\pi i \frac{\Omega_p}{\Omega_\infty}\right)^{k[F:\mathbb{Q}]+2\sum d_\tau} \frac{\prod \Gamma(k+2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}} \cdot d^{S^F}(s, \chi_0) \times \int_{\mathrm{U}(1,F)\backslash \mathrm{U}(1,\mathbb{A}_F)} E\left(\iota_G(a,1), f(s,\chi_0)\right) da \Big|_{s=\frac{k}{2}-\frac{1}{2}} .$$

In fact

$$\mathfrak{L}_{\mathfrak{c}}(\chi_{p\text{-adic}}) \cdot e_{p\text{-adic-HT}}^{k[F:\mathbb{Q}]+2\sum d_\tau} = \text{red part} \cdot e_{\mathrm{Tate}}^{k[F:\mathbb{Q}]+2\sum d_\tau},$$

Evaluations of $\mathfrak{L}_{\mathfrak{c}}$

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Assume that the Hecke character $\chi : K^\times \backslash \mathbb{A}_K^\times / U_{\mathfrak{c}}^p \to \mathbb{C}^\times$ is algebraic with ∞-type $(k + d_\tau, -d_\tau)_{\tau \in \Sigma}$, $k \geq 1, d_\tau \geq 0$, and p-adic avatar $\chi_{p\text{-adic}} : K^\times \backslash \mathbb{A}_{K,f}^\times / U_{\mathfrak{c}}^p \to \overline{\mathbb{Q}}_p^\times$.

By our construction,

$$\mathfrak{L}_{\mathfrak{c}}\left(\chi_{p\text{-adic}}\right) = \left(2\pi i \frac{\Omega_p}{\Omega_\infty}\right)^{k[F:\mathbb{Q}]+2\sum d_\tau} \frac{\prod \Gamma(k + 2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}} \cdot d^{S^F}(s, \chi_0) \times \int_{\mathrm{U}(1,F)\backslash \mathrm{U}(1,\mathbb{A}_F)} E\left(\iota_G(a,1), f(s,\chi_0)\right) da \Big|_{s=\frac{k}{2}-\frac{1}{2}} .$$

In fact

$$\mathfrak{L}_{\mathfrak{c}}(\chi_{p\text{-adic}}) \cdot e_{p\text{-adic-HT}}^{k[F:\mathbb{Q}]+2\sum d_\tau} = \text{red part} \cdot e_{\mathrm{Tate}}^{k[F:\mathbb{Q}]+2\sum d_\tau},$$

$$e_{\mathrm{Tate}} = 2\pi i \cdot e_{\text{upper-half-plane}},$$

$$e_{\mathrm{CM}} = \Omega_\infty \cdot e_{\text{upper-half-plane}} = \Omega_p \cdot e_{p\text{-adic-HT}}$$

Evaluations of $\mathfrak{L}_{\mathfrak{c}}$

$$\mathfrak{L}_{\mathfrak{c}}\left(\chi_{p\text{-adic}}\right) = \left(2\pi i \frac{\Omega_p}{\Omega_\infty}\right)^{k[F:\mathbb{Q}]+2\sum d_\tau} \frac{\prod \Gamma(k+2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}} \cdot d^{S^F}(s,\chi_0)$$

$$\times \int_{\mathrm{U}(1,F)\backslash \mathrm{U}(1,\mathbb{A}_F)} E\left(\iota_G(a,1), f(s,\chi_0)\right)\, da \Bigg|_{s=\frac{k}{2}-\frac{1}{2}} .$$

Evaluations of $\mathfrak{L}_{\mathfrak{c}}$

$$\mathfrak{L}_{\mathfrak{c}}\left(\chi_{p\text{-adic}}\right) = \left(2\pi i \frac{\Omega_p}{\Omega_\infty}\right)^{k[F:\mathbb{Q}]+2\sum d_\tau} \frac{\prod \Gamma(k+2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}} \cdot d^{S^F}(s,\chi_0)$$
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Recall **Doubling method formula**

$$\int_{\mathrm{U}(1,F)\backslash \mathrm{U}(1,\mathbb{A}_F)} E\left(\iota_G(a,1), f(s,\chi_0)\right)\, da = \prod_v Z_v\left(f_v(s,\chi_0), \mathrm{triv}\right),$$
$$Z_v\left(f_v(s,\chi_0), \mathrm{triv}\right) = \int_{\mathrm{U}(1,F_v^\times)} f_v(s,\chi_0)\left(\iota_D(a,1)\right)\, da.$$

Evaluations of $\mathfrak{L}_{\mathfrak{c}}$

$$\mathfrak{L}_{\mathfrak{c}}\left(\chi_{p\text{-adic}}\right) = \left(2\pi i \frac{\Omega_p}{\Omega_\infty}\right)^{k[F:\mathbb{Q}]+2\sum d_\tau} \frac{\prod \Gamma(k+2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}} \cdot d^{S^F}(s,\chi_0)$$

$$\times \int_{\mathrm{U}(1,F)\backslash \mathrm{U}(1,\mathbb{A}_F)} E\left(\iota_G(a,1), f(s,\chi_0)\right)\, da \Big|_{s=\frac{k}{2}-\frac{1}{2}}.$$

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$$Z_v\left(f_v(s,\chi_0), \mathrm{triv}\right) = \int_{\mathrm{U}(1,F_v^\times)} f_v(s,\chi_0)\left(\iota_D(a,1)\right)\, da.$$

$\Rightarrow$ Formulas for interpolation property of $\mathfrak{L}_{\mathfrak{c}}$ reduce to computing local zeta integrals.

Local zeta integrals

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right)=\int_{\mathrm{U}(1,F_v^\times)} f_v(s,\chi_0)\left(\iota_D(a,1)\right)\,da$$

Local zeta integrals

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \int_{\mathrm{U}(1,F_v^\times)} f_v(s,\chi_0)\left(\iota_D(a,1)\right)\, da$$

- $v \notin S^F$, $v = w\overline{w}$ split
- $v \notin S^F$ inert, $w \mid v$
  - $v \nmid 2$
  - $v \mid 2$ (extra conjugation by $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$ in $f_v(s,\chi_0)$)
- $v \notin S^F$, $v = w^2$ ramified

Local zeta integrals

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right)=\int_{\mathrm{U}(1,F_v^\times)} f_v(s,\chi_0)\left(\iota_D(a,1)\right)\,da$$

- $v \notin S^F$, $v = w\overline{w}$ split
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- $v \notin S^F$, $v = w^2$ ramified
- v archimedean

Local zeta integrals

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \int_{\mathrm{U}(1,F_v^\times)} f_v(s,\chi_0)\left(\iota_D(a,1)\right)\,da$$

- $v \notin S^F$, $v = w\overline{w}$ split
- $v \notin S^F$ inert, $w \mid v$
  - $v \nmid 2$
  - $v \mid 2$ (extra conjugation by $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$ in $f_v(s,\chi_0)$)
- $v \notin S^F$, $v = w^2$ ramified
- v archimedean
- $v \mid p$

Local zeta integrals

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- $v \in S^{F,p\infty}$, $v = w\overline{w}$ split

Local zeta integrals

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- $v \in S^{F,p\infty}$, $v = w\overline{w}$ split
  - $w \in S$, $\overline{w} \notin S$
  - $w, \overline{w} \in S$

Local zeta integrals

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  - $w, \overline{w} \in S$
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Local zeta integrals

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- $v\notin S^F$, $v=w\overline{w}$ split

$$\boxed{\frac{1-\chi_{0,w}(\varpi_v)|\varpi_v|_v^{s+\frac{1}{2}}}{(1-\chi_{0,w}(\varpi_v)|\varpi_v|_v^{s+\frac{1}{2}})(1-\chi_{0,\overline{w}}(\varpi_v)|\varpi_v|_v^{s+\frac{1}{2}})}=\frac{L_w(s+\frac{1}{2},\chi_0)L_{\overline{w}}(s+\frac{1}{2},\chi_0)}{d_v(s,\chi_0)}}$$

- $v\notin S^F$ inert, $w\mid v$
  - $v\nmid 2$
  - $v\mid 2$ (extra conjugation by $\begin{pmatrix}1&\frac{1}{2}\\0&\frac{1}{2}\end{pmatrix}$ in $f_v(s,\chi_0)$)
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Local zeta integrals

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$$\boxed{\frac{1-\chi_{0,w}(\varpi_v)|\varpi_v|_v^{s+\frac{1}{2}}}{(1-\chi_{0,w}(\varpi_v)|\varpi_v|_v^{s+\frac{1}{2}})(1-\chi_{0,\overline{w}}(\varpi_v)|\varpi_v|_v^{s+\frac{1}{2}})} = \frac{L_w(s+\frac{1}{2},\chi_0)L_{\overline{w}}(s+\frac{1}{2},\chi_0)}{d_v(s,\chi_0)}}$$

- $v \notin S^F$ inert, $w \mid v$ $\boxed{1 = \frac{L_w(s+\frac{1}{2},\chi_0)}{d_v(s,\chi_0)}}$
  - $v \nmid 2$
  - $v \mid 2$ (extra conjugation by $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$ in $f_v(s,\chi_0)$)
- $v \notin S^F$, $v = w^2$ ramified $\boxed{\frac{1}{2} = \frac{1}{2}\cdot\frac{L_w(s+\frac{1}{2},\chi_0)}{d_v(s,\chi_0)}}$
- v archimedean

Local zeta integrals

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right)=\int_{\mathrm{U}(1,F_v^\times)} f_v(s,\chi_0)\left(\iota_D(a,1)\right)\,da$$

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- $v \notin S^F$ inert, $w \mid v$ $\boxed{1=\frac{L_w(s+\frac{1}{2},\chi_0)}{d_v(s,\chi_0)}}$
  - $v \nmid 2$
  - $v \mid 2$ (extra conjugation by $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$ in $f_v(s,\chi_0)$)
- $v \notin S^F$, $v = w^2$ ramified $\boxed{\frac{1}{2}=\frac{1}{2}\cdot\frac{L_w(s+\frac{1}{2},\chi_0)}{d_v(s,\chi_0)}}$

  The conjugation by $\begin{pmatrix} \varpi_v & \\ & \varpi_v^{-1} \end{pmatrix}^{m_{D,v}}$ and $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$ (for $v \mid 2$ inert) in $f_v(s,\chi_0)$ simplifies computation and helps produce desired factors
- v archimedean

Local zeta integrals

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right)=\int_{\mathrm{U}(1,F_v^\times)} f_v(s,\chi_0)\left(\iota_D(a,1)\right)\,da$$

- $v\notin S^F$, $v=w\overline{w}$ split

$$\boxed{\frac{1-\chi_{0,w}(\varpi_v)|\varpi_v|_v^{s+\frac12}}{(1-\chi_{0,w}(\varpi_v)|\varpi_v|_v^{s+\frac12})(1-\chi_{0,\overline{w}}(\varpi_v)|\varpi_v|_v^{s+\frac12})}=\frac{L_w(s+\frac12,\chi_0)L_{\overline{w}}(s+\frac12,\chi_0)}{d_v(s,\chi_0)}}$$

- $v\notin S^F$ inert, $w\mid v$ $\boxed{1=\dfrac{L_w(s+\frac12,\chi_0)}{d_v(s,\chi_0)}}$
  - $v\nmid 2$
  - $v\mid 2$ (extra conjugation by $\begin{pmatrix}1&\frac12\\0&\frac12\end{pmatrix}$ in $f_v(s,\chi_0)$)
- $v\notin S^F$, $v=w^2$ ramified $\boxed{\dfrac12=\dfrac12\cdot\dfrac{L_w(s+\frac12,\chi_0)}{d_v(s,\chi_0)}}$
- v archimedean $\boxed{1}$

Local zeta integrals

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right)=\int_{\mathrm{U}(1,F_v^\times)} f_v(s,\chi_0)\left(\iota_D(a,1)\right)\,da$$

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$$\boxed{\frac{1-\chi_{0,w}(\varpi_v)|\varpi_v|_v^{s+\frac{1}{2}}}{(1-\chi_{0,w}(\varpi_v)|\varpi_v|_v^{s+\frac{1}{2}})(1-\chi_{0,\overline{w}}(\varpi_v)|\varpi_v|_v^{s+\frac{1}{2}})}=\frac{L_w(s+\frac{1}{2},\chi_0)L_{\overline{w}}(s+\frac{1}{2},\chi_0)}{d_v(s,\chi_0)}}$$

- $v\notin S^F$ inert, $w\mid v$ $\boxed{1=\frac{L_w(s+\frac{1}{2},\chi_0)}{d_v(s,\chi_0)}}$
  - $v\nmid 2$
  - $v\mid 2$ (extra conjugation by $\begin{pmatrix}1 & \frac{1}{2}\\ 0 & \frac{1}{2}\end{pmatrix}$ in $f_v(s,\chi_0)$)
- $v\notin S^F$, $v=w^2$ ramified $\boxed{\frac{1}{2}=\frac{1}{2}\cdot\frac{L_w(s+\frac{1}{2},\chi_0)}{d_v(s,\chi_0)}}$
- v archimedean $\boxed{1}$ (The archimedean local zeta integral is particularly simple here because $U(1,\mathbb{R})$ is compact. However, it becomes difficult if the unitary group is non-compact at infinity)

Local zeta integrals at $v \mid p$

Local zeta integrals at $v \mid p$

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v = w\overline{w},\ w\in\Sigma_p$$

Local zeta integrals at $v \mid p$

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\ \chi_{0,w},\ \psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v = w\overline{w},\ w\in\Sigma_p$$

- $$f_v(s,\chi_0)\left(g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = f^{\alpha_{\chi_0,v}}(s,\chi_0)(g)$$
  $$= \chi_{0,\overline{w}}^{-1}(c^{-1})\,\chi_{0,w}(c^{-1}\det g)\,|c^{-2}\det g|_v^{s+\frac{1}{2}}\cdot\alpha_{\chi_0,v}(c^{-1}d)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$$

Local zeta integrals at $v \mid p$

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\ \chi_{0,w},\ \psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v = w\overline{w},\ w\in\Sigma_p$$

- $$f_v(s,\chi_0)\left(g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right) = f^{\alpha_{\chi_0,v}}(s,\chi_0)(g)$$
  $$= \chi_{0,\overline{w}}^{-1}(c^{-1})\,\chi_{0,w}(c^{-1}\det g)\,|c^{-2}\det g|_v^{s+\frac{1}{2}}\cdot\alpha_{\chi_0,v}(c^{-1}d)\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$$
- $$\widehat{\alpha}_{\chi_0,v} = \chi_{\overline{w}}(2\sqrt{-D})\cdot\mathbb{1}_{\mathcal{O}_{F_v}^\times}\cdot\chi_w,$$
  $$\alpha_{\chi_0,v}(x) = \chi_{\overline{w}}(2\sqrt{-D})\cdot\int_{F_v}\mathbb{1}_{\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}xy\right)\,dy$$
  $$= \chi_{\overline{w}}(2\sqrt{-D})\cdot\chi_{0,w}^{-1}(x)\,|x|_v^{-1}\int_{F_v}\mathbb{1}_{x\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$

Local zeta integrals at $v \mid p$

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v = w\overline{w},\ w\in\Sigma_p$$

- $$f_v(s,\chi_0)\left(g=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = f^{\alpha_{\chi_0,v}}(s,\chi_0)(g)$$
  $$= \chi_{0,\overline{w}}^{-1}(c^{-1})\,\chi_{0,w}(c^{-1}\det g)\,|c^{-2}\det g|_v^{s+\frac{1}{2}}\cdot\alpha_{\chi_0,v}(c^{-1}d)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$$
- $$\widehat{\alpha}_{\chi_0,v} = \chi_{\overline{w}}(2\sqrt{-D})\cdot\mathbb{1}_{\mathcal{O}_{F_v}^\times}\cdot\chi_w,$$
  $$\alpha_{\chi_0,v}(x) = \chi_{\overline{w}}(2\sqrt{-D})\cdot\int_{F_v}\mathbb{1}_{\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}xy\right)\,dy$$
  $$= \chi_{\overline{w}}(2\sqrt{-D})\cdot\chi_{0,w}^{-1}(x)\,|x|_v^{-1}\int_{F_v}\mathbb{1}_{x\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$
- $$\widetilde{\iota}_{G,v}(a,1) = \begin{pmatrix} 1 & \sqrt{-D} \\ 1 & -\sqrt{-D}\end{pmatrix}^{-1}\begin{pmatrix} a & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & \sqrt{-D} \\ 1 & -\sqrt{-D}\end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{2} & \sqrt{-D} \\ -\frac{\sqrt{-D}^{-1}}{2} & 1\end{pmatrix}$$
  $$= \begin{pmatrix} \frac{1}{2} & a\sqrt{-D} \\ -\frac{\sqrt{-D}^{-1}}{2} & a\end{pmatrix}$$

Local zeta integrals at $v \mid p$:

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v=w\overline{w},\ w\in\Sigma_p$$

- $$f_v(s,\chi_0)\,(g) = \chi_{0,\overline{w}}^{-1}(c^{-1})\,\chi_{0,w}(c^{-1}\det g)\,|c^{-2}\det g|_v^{s+\frac{1}{2}}\alpha_{\chi_0,v}(c^{-1}d)\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$$
- $$\alpha_{\chi_0,v}(x) = \chi_{\overline{w}}(2\sqrt{-D})\cdot\chi_{0,w}^{-1}(x)\,|x|_v^{-1}\int_{F_v}\mathbb{1}_{x\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$
- $$\widetilde{\iota}_{G,v}(a,1) = \begin{pmatrix} \frac{1}{2} & a\sqrt{-D} \\ -\frac{\sqrt{-D}^{-1}}{2} & a \end{pmatrix}$$

Local zeta integrals at $v \mid p$:

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v = w\overline{w},\ w\in\Sigma_p$$

- $$f_v(s,\chi_0)\,(g) = \chi_{0,\overline{w}}^{-1}(c^{-1})\,\chi_{0,w}(c^{-1}\det g)\,|c^{-2}\det g|_v^{s+\frac{1}{2}}\alpha_{\chi_0,v}(c^{-1}d)\,\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$$

- $$\alpha_{\chi_0,v}(x) = \chi_{\overline{w}}(2\sqrt{-D})\cdot\chi_{0,w}^{-1}(x)\,|x|_v^{-1}\int_{F_v}\mathbb{1}_{x\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$

- $$\widetilde{\iota}_{G,v}(a,1) = \begin{pmatrix} \frac{1}{2} & a\sqrt{-D} \\ -\frac{\sqrt{-D}^{-1}}{2} & a\end{pmatrix}$$

$\Rightarrow$

$$f_v(s,\chi_0)\,(\widetilde{\iota}_{G,v}(a,1)) = \chi_{0,\overline{w}}(-2\sqrt{-D})\,\chi_{0,w}(-2a\sqrt{-D})\,|a|^{s+\frac{1}{2}}\cdot\alpha_{\chi_0,v}(-2a\sqrt{-D})$$

Local zeta integrals at $v \mid p$:

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right)=\chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v=w\overline{w},\ w\in\Sigma_p$$

- $$f_v(s,\chi_0)\,(g)=\chi_{0,\overline{w}}^{-1}(c^{-1})\,\chi_{0,w}(c^{-1}\det g)\,|c^{-2}\det g|_v^{s+\frac{1}{2}}\alpha_{\chi_0,v}(c^{-1}d)\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$$
- $$\alpha_{\chi_0,v}(x)=\ \chi_{\overline{w}}(2\sqrt{-D})\cdot\chi_{0,w}^{-1}(x)\,|x|_v^{-1}\int_{F_v}\mathbb{1}_{x\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$
- $$\widetilde{\iota}_{G,v}(a,1)=\begin{pmatrix}\frac{1}{2} & a\sqrt{-D}\\ -\frac{\sqrt{-D}-1}{2} & a\end{pmatrix}$$

$\Rightarrow$

$$f_v(s,\chi_0)\,(\widetilde{\iota}_{G,v}(a,1))=\chi_{0,\overline{w}}(-2\sqrt{-D})\,\chi_{0,w}(-2a\sqrt{-D})\,|a|^{s+\frac{1}{2}}\cdot\alpha_{\chi_0,v}(-2a\sqrt{-D})$$

$$=\chi_{0,\overline{w}}(-2\sqrt{-D})\,\chi_{0,w}(-2a\sqrt{-D})\,|a|^{s+\frac{1}{2}}\cdot\chi_{\overline{w}}(2\sqrt{-D})\cdot\chi_{0,w}^{-1}(-2a\sqrt{-D})\,|a|_v^{-1}$$

$$\times\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$

Local zeta integrals at $v \mid p$:

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v=w\overline{w},\ w\in\Sigma_p$$

- $$f_v(s,\chi_0)\,(g) = \chi_{0,\overline{w}}^{-1}(c^{-1})\,\chi_{0,w}(c^{-1}\det g)\,|c^{-2}\det g|_v^{s+\frac{1}{2}}\alpha_{\chi_0,v}(c^{-1}d)\,\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$$
- $$\alpha_{\chi_0,v}(x) = \chi_{\overline{w}}(2\sqrt{-D})\cdot\chi_{0,w}^{-1}(x)\,|x|_v^{-1}\int_{F_v}\mathbb{1}_{x\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$
- $$\widetilde{\iota}_{G,v}(a,1) = \begin{pmatrix} \frac{1}{2} & a\sqrt{-D} \\ -\frac{\sqrt{-D}^{-1}}{2} & a\end{pmatrix}$$

$\Rightarrow$

$$f_v(s,\chi_0)\,(\widetilde{\iota}_{G,v}(a,1)) = \chi_{0,\overline{w}}(-2\sqrt{-D})\,\chi_{0,w}(-2a\sqrt{-D})\,|a|^{s+\frac{1}{2}}\cdot\alpha_{\chi_0,v}(-2a\sqrt{-D})$$

$$= \chi_{0,\overline{w}}(-2\sqrt{-D})\,\chi_{0,w}(-2a\sqrt{-D})\,|a|^{s+\frac{1}{2}}\cdot\chi_{\overline{w}}(2\sqrt{-D})\cdot\chi_{0,w}^{-1}(-2a\sqrt{-D})\,|a|_v^{-1}$$

$$\times\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$

$$= \chi_{0,\overline{w}}(-1)\,|a|^{s-\frac{1}{2}}\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$

Local zeta integrals at $v \mid p$:

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v = w\overline{w},\ w\in\Sigma_p$$

$$f_v(s,\chi_0)\left(\widetilde{\iota}_{G,v}(a,1)\right) = \chi_{0,\overline{w}}(-1)\,|a|^{s-\frac{1}{2}}\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$

.

Local zeta integrals at $v \mid p$:

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ \mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v=w\overline{w},\ w\in\Sigma_p$$

$$f_v(s,\chi_0)\left(\widetilde{\iota}_{G,v}(a,1)\right) = \chi_{0,\overline{w}}(-1)\,|a|^{s-\frac{1}{2}}\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^{\times}}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$

Therefore,

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \int_{F_v^{\times}} f_v(s,\chi_0)\left(\widetilde{\iota}_{G,v}(a,1)\right)da$$

Local zeta integrals at $v \mid p$:

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v=w\overline{w},\ w\in\Sigma_p$$

$$f_v(s,\chi_0)\left(\widetilde{\iota}_{G,v}(a,1)\right) = \chi_{0,\overline{w}}(-1)\,|a|^{s-\frac{1}{2}}\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$

Therefore,

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \int_{F_v^\times} f_v(s,\chi_0)\left(\widetilde{\iota}_{G,v}(a,1)\right)da$$

$$=\chi_{0,\overline{w}}(-1)\int_{F_v^\times}|a|^{s-\frac{1}{2}}\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy\,da$$

Local zeta integrals at $v \mid p$:

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\quad v = w\overline{w},\ w\in\Sigma_p$$

$$f_v(s,\chi_0)\left(\widetilde{\iota}_{G,v}(a,1)\right) = \chi_{0,\overline{w}}(-1)\,|a|^{s-\frac{1}{2}}\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$

Therefore,

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \int_{F_v^\times} f_v(s,\chi_0)\left(\widetilde{\iota}_{G,v}(a,1)\right)da$$

$$= \chi_{0,\overline{w}}(-1)\int_{F_v^\times}|a|^{s-\frac{1}{2}}\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy\,da$$

$$= \chi_{0,\overline{w}}(-1)\int_{F_v^\times}\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\cdot|y|^{s-\frac{1}{2}}\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy\,da$$

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Therefore,

$$\begin{aligned}
Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) &= \int_{F_v^\times} f_v(s,\chi_0)\left(\widetilde{\iota}_{G,v}(a,1)\right)da \\
&= \chi_{0,\overline{w}}(-1)\int_{F_v^\times}|a|^{s-\frac{1}{2}}\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy\,da \\
&= \chi_{0,\overline{w}}(-1)\int_{F_v^\times}\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\cdot|y|^{s-\frac{1}{2}}\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy\,da \\
&= \chi_{0,\overline{w}}(-1)\int_{F_v}|y|^{s-\frac{1}{2}}\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy\int_{F_v^\times}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\,da
\end{aligned}$$

Local zeta integrals at $v \mid p$:

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\quad v=w\overline{w},\ w\in\Sigma_p$$

$$f_v(s,\chi_0)\left(\widetilde{\iota}_{G,v}(a,1)\right) = \chi_{0,\overline{w}}(-1)\,|a|^{s-\frac{1}{2}}\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy$$

Therefore,

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Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) &= \int_{F_v^\times} f_v(s,\chi_0)\left(\widetilde{\iota}_{G,v}(a,1)\right)da\\
&=\chi_{0,\overline{w}}(-1)\int_{F_v^\times}|a|^{s-\frac{1}{2}}\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy\,da\\
&=\chi_{0,\overline{w}}(-1)\int_{F_v^\times}\int_{F_v}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\cdot|y|^{s-\frac{1}{2}}\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy\,da\\
&=\chi_{0,\overline{w}}(-1)\int_{F_v}|y|^{s-\frac{1}{2}}\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy\int_{F_v^\times}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(y)\,da\\
&=\chi_{0,\overline{w}}(-1)\,\gamma_w\left(-s+\frac{1}{2},\,\chi_{0,w}^{-1},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)
\end{aligned}$$

Local zeta integrals at $v \mid p$:

$$\boxed{Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v = w\overline{w},\ w\in\Sigma_p}$$

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_{0,\overline{w}}(-1)\,\gamma_w\left(-s+\frac{1}{2},\,\chi_{0,w}^{-1},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)$$

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$$\begin{aligned} Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) &= \chi_{0,\overline{w}}(-1)\,\gamma_w\left(-s+\frac{1}{2},\,\chi_{0,w}^{-1},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right) \\ &= \chi_{0,\overline{w}}(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\overline{\psi}_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1} \end{aligned}$$

Local zeta integrals at $v \mid p$:

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right)=\chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v=w\overline{w},\ w\in\Sigma_p$$

$$\begin{aligned}
Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) &= \chi_{0,\overline{w}}(-1)\,\gamma_w\left(-s+\frac{1}{2},\,\chi_{0,w}^{-1},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)\\
&= \chi_{0,\overline{w}}(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\overline{\psi}_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1}\\
&= \chi_{0,\overline{w}}(-1)\,\chi_{0,w}(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1}
\end{aligned}$$

Local zeta integrals at $v \mid p$:

$$Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right)=\chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1},\ v=w\overline{w},\ w\in\Sigma_p$$

$$\begin{aligned}Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right)&=\chi_{0,\overline{w}}(-1)\,\gamma_w\left(-s+\frac{1}{2},\,\chi_{0,w}^{-1},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)\\&=\chi_{0,\overline{w}}(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\overline{\psi}_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1}\\&=\chi_{0,\overline{w}}(-1)\,\chi_{0,w}(-1)\,\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1}\end{aligned}$$

$$\gamma_w\left(s+\frac{1}{2},\,\chi_{0,w},\,\psi_p\circ\mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1}=\left(\int_{F_v}|y|^{s-\frac{1}{2}}\,\chi_{0,w}(y)\cdot\psi_p\left(\mathrm{Tr}_{F_v/\mathbb{Q}_p}y\right)dy\right)^{-1}$$

$$=\begin{cases}\frac{L_w(s+\frac{1}{2},\chi_{0,w})}{L_w(-s+\frac{1}{2},\chi_{0,w}^{-1})}, & \chi_{0,w}\ \text{unram}\\ \left(\chi_{0,w}(\varpi_v)|\varpi_v|_v^{s+\frac{1}{2}}\right)^{-c_{\chi_{0,w}}-\mathrm{val}_v(\mathfrak{d}_v)}\cdot\ \text{Gauss sum of}\ \chi_{0,w}|_{\mathcal{O}_{F_v}^\times}, & \chi_{w,0}\ \text{ram}\end{cases}$$

Local zeta integrals at $v \in S^F$, $v \nmid p\infty$

Local zeta integrals at $v \in S^F$, $v \nmid p\infty$

- $v = w\overline{w}$ split, $w \in S$, $\overline{w} \notin S$:

$$\alpha_{\chi,v} = \chi_{0,v}^F\left(\varpi_v^{-m_{D,v}}\sqrt{-D}\right)^{-1}\chi_{0,\overline{w}}(\varpi_v)^{-c_w}\,|\varpi_v|_v^{-c_w(s+\frac{1}{2})}\cdot\mathbb{1}_{\varpi_v^{-m_{D,v}}\sqrt{-D}\,(1+\varpi_v^{c_w}\mathcal{O}_{F_v})}$$

$$\boxed{Z_v\left(f^{\alpha_{\chi,v}}(s,\chi_0),\mathrm{triv}\right) = L_{\overline{w}}\left(s+\frac{1}{2},\chi_0\right)}$$

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$$\boxed{Z_v\left(f^{\alpha_{\chi,v}}(s,\chi_0), \mathrm{triv}\right) = L_{\overline{w}}\left(s + \frac{1}{2}, \chi_0\right)}$$

- $v = w\overline{w}$ split, $w, \overline{w} \in S$:

$$\alpha_{\chi,v} = \chi_{0,v}^F(2)\,|\varpi_v|_v^{\max\{c_w, c_{\overline{w}}\}} \cdot \mathbb{1}_{\varpi^{-\max\{c_w,c_{\overline{w}}\}} \mathcal{O}_{F_v}^\times} \cdot \chi_v^{F,-1}$$

$$\boxed{Z_v\left(f^{\alpha_{\chi,v}}(s,\chi_0), \mathrm{triv}\right) = 1}$$

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$$\boxed{Z_v\left(f^{\alpha_{\chi,v}}(s,\chi_0),\mathrm{triv}\right) = 1}$$

- v inert:

$$\alpha_{\chi,v} = \chi_{0,v}^F(2)\, |\varpi_v|_v^{c_w} \frac{\sharp\,\mathrm{U}(1,\kappa_v)}{\sharp\kappa_v^\times} \cdot \mathbb{1}_{\varpi^{-2c_w}\mathcal{O}_{F_v}^\times} \cdot \chi_v^{F,-1}$$

$$\boxed{Z_v\left(f^{\alpha_{\chi,v}}(s,\chi_0),\mathrm{triv}\right) = 1}$$

Local zeta integrals: summary

- $v \notin S^F$, $\eta_v = \begin{cases} 1, & v \text{ split or inert} \\ \frac{1}{2}, & v \text{ ramified} \end{cases}$:

$$\boxed{Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \eta_v \cdot d_v(s,\chi_0)^{-1}\, L_w\left(s+\frac{1}{2},\chi_0\right) L_{\overline{w}}\left(s+\frac{1}{2},\chi_0\right)}$$

- v archimedean: $\boxed{Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = 1}$

- $v \mid p$, $v = w\overline{w}$, $w \in \Sigma_p$:

$$\boxed{Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = \chi_v^F(-1)\,\gamma_w\left(s+\frac{1}{2},\ \chi_{0,w},\ \psi_p \circ \mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1}}$$

- $v \in S^F$, $v = w\overline{w}$ split, $w \in S$, $\overline{w} \notin S$:

$$\boxed{Z_v\left(f^{\alpha_{\chi,v}}(s,\chi_0),\mathrm{triv}\right) = L_{\overline{w}}\left(s+\frac{1}{2},\chi_0\right)}$$

- $v \in S^{F,\,p\infty}$, $v = w\overline{w}$ split, $w,\overline{w} \in S$, or inert:

$$\boxed{Z_v\left(f_v(s,\chi_0),\mathrm{triv}\right) = 1}$$

Evaluations of $\mathfrak{L}_{\mathfrak{c}}$

Doubling method formula + results on local zeta integrals $\Rightarrow$

$$\int_{\mathrm{U}(1,F)\backslash \mathrm{U}(1,\mathbb{A}_F)} E\left(\iota_G(a,1), f(s,\chi_0)\right) da = \prod_v Z_v\left(f_v(s,\chi_0), \mathrm{triv}\right)$$

$$= \frac{2^{-\sharp\, \mathrm{ram}_{K/F}}}{d^{S_F}(s,\chi_0)^{-1}} \prod_{v|p} \chi_v^F(-1) \prod_{w\in\Sigma_p} \gamma_w\left(s+\frac{1}{2},\, \chi_0,\, \psi_p\circ \mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1} L^S\left(s+\frac{1}{2},\chi_0\right).$$

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Recall that by construction,

$$\mathfrak{L}_{\mathfrak{c}}\left(\chi_{p\text{-adic}}\right) = \left(2\pi i \frac{\Omega_p}{\Omega_\infty}\right)^{k[F:\mathbb{Q}]+2\sum d_\tau} \frac{\prod \Gamma(k+2d_\tau)}{(2\pi i)^{k[F:\mathbb{Q}]+\sum d_\tau}}\, d^{S^F}(s,\chi_0)$$

$$\times \int_{\mathrm{U}(1,F)\backslash \mathrm{U}(1,\mathbb{A}_F)} E\left(\iota_G(a,1), f(s,\chi_0)\right)\, da \Bigg|_{s=\frac{k}{2}-\frac{1}{2}},$$

if $\chi : K^\times\backslash\mathbb{A}_K^\times/U_{\mathfrak{c}}^p \to \mathbb{C}^\times$ with $\chi_\infty \sim (k+d_\tau, -d_\tau)_{\tau\in\Sigma}$, $k\geq 1, d_\tau \geq 0$.

Evaluations of $\mathfrak{L}_\mathfrak{c}$

For $\chi : K^\times \backslash \mathbb{A}_K^\times / U_\mathfrak{c}^p \to \mathbb{C}^\times$ algebraic with $\chi_\infty \sim (k + d_\tau, -d_\tau)_{\tau \in \Sigma}$, $k \geq 1, d_\tau \geq 0$,

$$\begin{aligned}
&\mathfrak{L}_\mathfrak{c}\left(\chi_{p\text{-adic}}\right) \\
&= \left(\frac{\Omega_p}{\Omega_\infty}\right)^{k[F:\mathbb{Q}]+2\sum d_\tau} (2\pi i)^{\sum d_\tau} \prod_{\tau \in \Sigma} \Gamma(k + 2d_\tau) \cdot 2^{-\sharp \mathrm{ram}_{K/F}} \prod_{v|p} \chi_v^F(-1) \\
&\quad \times \prod_{w \in \Sigma_p} \gamma_w \left(\frac{k}{2}, \chi_0, \psi_p \circ \mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1} L^S\left(\frac{k}{2}, \chi_0\right)
\end{aligned}$$

Evaluations of $\mathfrak{L}_{\mathfrak{c}}$

For $\chi : K^\times \backslash \mathbb{A}_K^\times / U_{\mathfrak{c}}^p \to \mathbb{C}^\times$ algebraic with $\chi_\infty \sim (k + d_\tau, -d_\tau)_{\tau \in \Sigma}$, $k \geq 1, d_\tau \geq 0$,

$$\begin{aligned}
&\mathfrak{L}_{\mathfrak{c}}\left(\chi_{p\text{-adic}}\right) \\
&= \left(\frac{\Omega_p}{\Omega_\infty}\right)^{k[F:\mathbb{Q}]+2\sum d_\tau} (2\pi i)^{\sum d_\tau} \prod_{\tau\in\Sigma} \Gamma(k+2d_\tau) \cdot 2^{-\sharp\,\mathrm{ram}_{K/F}} \prod_{v|p} \chi_v^F(-1) \\
&\quad \times \prod_{w\in\Sigma_p} \gamma_w\left(\frac{k}{2},\, \chi_0,\, \psi_p \circ \mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1} L^S\left(\frac{k}{2}, \chi_0\right) \\
&= \left(\frac{\Omega_p}{\Omega_\infty}\right)^{k[F:\mathbb{Q}]+2\sum d_\tau} (2\pi i)^{\sum d_\tau} \prod_{\tau\in\Sigma} \Gamma(k+2d_\tau) \cdot 2^{-\sharp\,\mathrm{ram}_{K/F}} \prod_{v|p} \chi_v^F(-1) \\
&\quad \times \prod_{w\in\Sigma_p} \gamma_w\left(0,\, \chi,\, \psi_p \circ \mathrm{Tr}_{F_v/\mathbb{Q}_p}\right)^{-1} L^S\left(0, \chi\right)
\end{aligned}$$