Hybrid High-Order methods: Overview and recent advances

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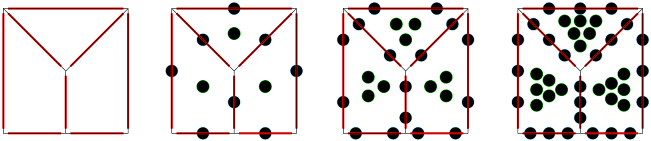
Outline

1. **HHO in a nutshell**
2. **Main ideas: elliptic PDEs**
   - [Di Pietro, AE, Lemaire, CMAM, 14] for diffusion
   - [Di Pietro, AE, CMAME, 15] for linear elasticity (genuinely symmetric, locking-free, no post-processing)
3. **Building bridges**
   - [Cockburn, Di Pietro, AE, M2AN, 16]
   - HHO methods are HDG methods with another viewpoint and some advantages
4. **Unfitted meshes: cutHHO**
   - [Burman, AE, SINUM, 18]

HHO in a nutshell

- HHO methods attach discrete unknowns to **mesh faces**
  - one polynomial of order $k \geq 0$ on each mesh face
- HHO methods also use **cell unknowns**
  - elimination by static condensation (local Schur complement)



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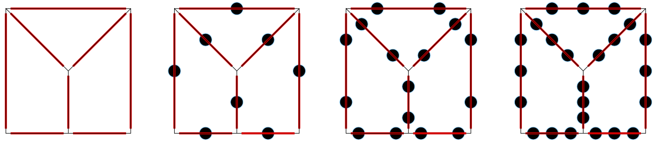
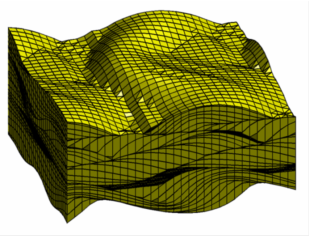
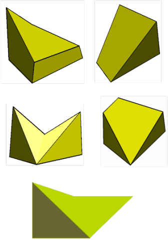


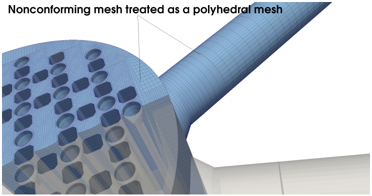
Main assets

- **General meshes** are supported
  - polygonal/polyhedral cells, hanging nodes
- **Physical fidelity**
  - local conservation
  - robustness (dominant advection, quasi-incompressible elasticity...)
- **Attractive computational costs**
  - energy-error decays as $O(h^{k+1})$ using face polynomials of order $k \geq 0$
  - more compact stencil than for vertex-based methods (esp. in 3D)
  - global system size $k^2\#(\text{faces})$ vs. $k^3\#(\text{cells})$ for dG
- **Genericity**
  - construction independent of space dimension
  - open-source HHO library on Github [Cicuttin, Di Pietro, AE 18]
- Industrial collaborations: EDF, CEA, BRGM

Motivations for polyhedral methods (Courtesy IFPEN, EDF R&D)







Related low-order methods

- **Mimetic Finite Differences (MFD)**
  - [Brezzi, Lipnikov, Shashkov 05]
- **Hybrid Finite Volumes**
  - [Droniou, Eymard, Gallouet, Herbin 06-10]
- **Non-conforming FEM**
  - [Crouzeix, Raviart 73]
- **Unified settings**
  - Gradient Schemes [Droniou, Eymard, Gallouet, Herbin 10, 13]
  - Compatible Discrete Operator (CDO) schemes [Bonelle, AE 14]

Related high-order methods

- **Hybridizable DG (HDG)**
  - [Cockburn, Gopalakrishnan, Lazarov 09]
  - Weak Galerkin [Wang & Ye 13], equivalent to HDG [Cockburn 16]
- **Non-conforming Virtual Elements** (ncVEM)
  - [Lipnikov, Manzini 14; Ayuso, Lipnikov, Manzini 16]
- HDG, HHO and ncVEM are very closely related
  - [Cockburn, Di Pietro, AE, 16]

Recent developments in HHO methods

- **Transport and flows**
  - Péclet-robust advection-diffusion [Di Pietro, Droniou, AE 15]
  - Stokes [Di Pietro, AE, Linke, Schieweck 16], NS [Di Pietro, Krell 18]
  - viscoplastic fluids [Cascavita, Bleyer, Chateau, AE 18]
  - fractured porous media [Chave, Di Pietro, Formaggia 18]
- **Nonlinear mechanics**
  - small defs [Botti, Di Pietro, Sochala 17]
  - hyperelasticity [Abbas, AE, Pignet 18]
  - elastoplasticity [Abbas, AE, Pignet 18]
- **Obstacle problems** [Cicuttin, AE, Gudi 18; Cascavita, Chouly, AE 18]
- **Spectral approximation** [Calo, Cicuttin, Deng, AE, 18]

Main ideas

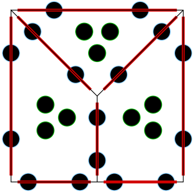
- **Poisson model problem**
- Let $f \in L^2(D)$ and let $D \subset \mathbb{R}^d$ be a Lipschitz polyhedron
- Find $u \in V := H_0^1(D)$ s.t.

$$(\nabla u, \nabla w)_{\boldsymbol{L}^2(D)} = (f, w)_{L^2(D)} \quad \forall w \in V$$

- Other BC's can be considered as well

Devising HHO methods

- Devising from **primal formulation** using two ideas
- Local reconstruction operator to build a higher-order field in each cell from cell and face unknowns
- Local stabilization operator to connect cell and face unknowns

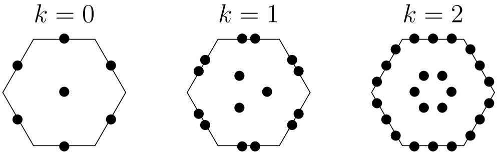


Local viewpoint

- Consider a **mesh** $\mathcal{T} = \{T\}$ of D and a **polynomial degree** $k \geq 0$
  - broken polynomial space $\mathbb{P}^k(\mathcal{F}_{\partial T})$ (one poly. on each face of T)
- For all $T \in \mathcal{T}$, the discrete unknowns are

$$(v_T, v_{\partial T}) \in \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T})$$

- Examples in hexagonal cell



Reconstruction operator

$$\mathrm{R}_T^{k+1} : \underbrace{\mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T})}_{\text{cell and face unknowns}} \longrightarrow \underbrace{\mathbb{P}^{k+1}(T)}_{\text{higher-order polynomial}}$$

- Let $(v_T, v_{\partial T}) \in \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T})$
- Then $\mathrm{R}_T^{k+1}(v_T, v_{\partial T}) \in \mathbb{P}^{k+1}(T)$ solves, $\forall w \in \mathbb{P}^{k+1}(T)$,

$$(\nabla \mathrm{R}_T^{k+1}(v_T, v_{\partial T}), \nabla w)_{\boldsymbol{L}^2(T)} = -(v_T, \Delta w)_{L^2(T)} + (v_{\partial T}, \boldsymbol{n}_T \cdot \nabla w)_{L^2(\partial T)}$$

  - well-posed local Neumann pb. (with $(\mathrm{R}_T^{k+1}(v_T, v_{\partial T}) - v_T, 1)_{L^2(T)} = 0$)
  - local stiffness matrix in $\mathbb{P}^{k+1}(T)$
  - **fully parallelizable**
- Note that $\mathrm{R}_T^{k+1}(v_T, v_{T|\partial T}) = v_T$
  - no order pickup if trace of cell values coincides with face values

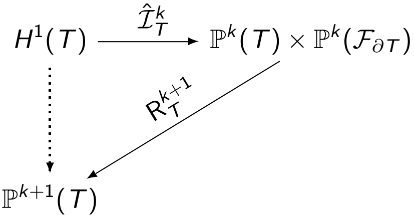
Reduction and approximation operators

- Reconstruction operator $\mathrm{R}_T^{k+1} : \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) \to \mathbb{P}^{k+1}(T)$
- Reduction operator $\hat{\mathcal{I}}_T^k : H^1(T) \to \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T})$ s.t.

$$\hat{\mathcal{I}}_T^k(v) := (\Pi_T^k(v), \Pi_{\partial T}^k(v))$$

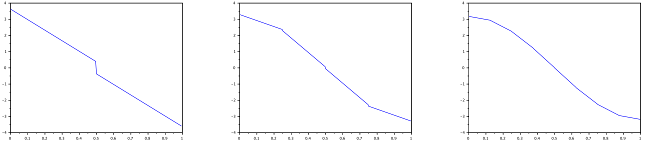
with L^2-orthogonal projectors onto $\mathbb{P}^k(T)$ and $\mathbb{P}^k(\mathcal{F}_{\partial T})$ resp.

- $\mathrm{R}_T^{k+1} \circ \hat{\mathcal{I}}_T^k : H^1(T) \to \mathbb{P}^{k+1}(T)$ acts as an approximation operator

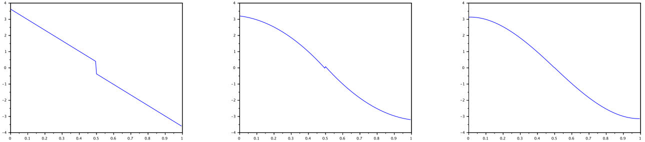


Numerical illustration

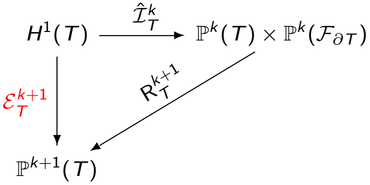
- *h*-approximation of $\cos(\pi x)$, $N = 2, 4, 8$, $k = 0$



- *p*-approximation of $\cos(\pi x)$, $N = 2$, $k = 0, 1, 2$



Elliptic projector



- We have $\mathrm{R}_T^{k+1} \circ \hat{\mathcal{I}}_T^k = \mathcal{E}_T^{k+1}$, where $\mathcal{E}_T^{k+1} : H^1(T) \to \mathbb{P}^{k+1}(T)$ is the elliptic projector
  - $(\nabla(\mathcal{E}_T^{k+1}(v) - v), \nabla w)_{\boldsymbol{L}^2(T)} = 0$, $\forall w \in \mathbb{P}^{k+1}(T)$
  - $(\mathcal{E}_T^{k+1}(v) - v, 1)_{L^2(T)} = 0$

A short proof

- Let $v \in H^1(T)$
- For all $w \in \mathbb{P}^{k+1}(T)$, we have

$$
\begin{aligned}
&(\nabla \mathrm{R}_T^{k+1}(\hat{\mathcal{I}}_T^k(v)), \nabla w)_{\boldsymbol{L}^2(T)} \\
&\qquad = -(\Pi_T^k(v), \Delta w)_{L^2(T)} + (\Pi_{\partial T}^k(v), \boldsymbol{n}_T \cdot \nabla w)_{L^2(\partial T)} \\
&\qquad = -(v, \Delta w)_{L^2(T)} + (v, \boldsymbol{n}_T \cdot \nabla w)_{L^2(\partial T)} \\
&\qquad = (\nabla v, \nabla w)_{\boldsymbol{L}^2(T)}
\end{aligned}
$$

$$\Longrightarrow \mathrm{R}_T^{k+1}(\hat{\mathcal{I}}_T^k(v)) = \mathcal{E}_T^{k+1}(v)$$

Stabilization

- $\{\nabla \mathrm{R}_T^{k+1}(v_T, v_{\partial T}) = \mathbf{0}\} \not\Rightarrow \{v_T = v_{\partial T} = \mathrm{cst}\}$
- Connect cell and face unknowns by LS penalty on $v_{\partial T} - v_{T|\partial T}$

$$\mathrm{S}_{\partial T}^k(v_T, v_{\partial T}) := \tilde{\mathrm{S}}_{\partial T}^k(v_{\partial T} - v_{T|\partial T})$$
$$:= \Pi_{\partial T}^k\Big(\underbrace{(v_{\partial T} - v_{T|\partial T})}_{\text{HDG-like term}} - \underbrace{(I - \Pi_T^k)\mathrm{R}_T^{k+1}(0, v_{\partial T} - v_{T|\partial T})_{|\partial T}}_{\textbf{HHO high-order correction}}\Big)$$

- Note that $S_{\partial T}^k(v_T, v_{T|\partial T}) = 0$
  - stabilization vanishes if trace of cell values coincides with face values
- Local mass matrices in $\mathbb{P}^k(T)$ and $\mathbb{P}^k(\mathcal{F}_{\partial T})$, **fully parallelizable**

$\mathbb{P}^{k+1}$-polynomial consistency

- Recall elliptic projector $\mathcal{E}_T^{k+1} : H^1(T) \to \mathbb{P}^{k+1}(T)$
- Recall reduction operator s.t. $\hat{\mathcal{I}}_T^k(v) = (\Pi_T^k(v), \Pi_{\partial T}^k(v))$
- For all $v \in H^1(T)$, we have

$$\mathsf{S}_{\partial T}^k(\hat{\mathcal{I}}_T^k(v)) = (\Pi_T^k - \Pi_{\partial T}^k)(v - \mathcal{E}_T^{k+1}(v))_{|\partial T}$$

Consequently, $\mathsf{S}_{\partial T}^k(\hat{\mathcal{I}}_T^k(p)) = 0,\ \forall p \in \mathbb{P}^{k+1}(T)$

$$\begin{aligned} \mathsf{S}_{\partial T}^k(\hat{\mathcal{I}}_T^k(v)) &= \Pi_{\partial T}^k(\Pi_T^k(v) - \Pi_{\partial T}^k(v) + (I - \Pi_T^k)(\mathcal{E}_T^{k+1}(v))) \\ &= \Pi_{\partial T}^k(\Pi_T^k(v - \mathcal{E}_T^{k+1}(v)) - (\Pi_{\partial T}^k(v) - \mathcal{E}_T^{k+1}(v))) \\ &= \Pi_T^k(v - \mathcal{E}_T^{k+1}(v)) - \Pi_{\partial T}^k(v - \mathcal{E}_T^{k+1}(v)) \end{aligned}$$

since $\Pi_{\partial T}^k \Pi_T^k = \Pi_T^k$ and $\Pi_{\partial T}^k \Pi_{\partial T}^k = \Pi_{\partial T}^k$

- Without the higher-order term, $\mathsf{S}_{\partial T}^k(\hat{\mathcal{I}}_T^k(p)) = 0$ only for $p \in \mathbb{P}^k(T)$

Local stability and boundedness

- Local bilinear form (with $\tau_{\partial T|F} \sim h_F^{-1}$ for all $F \in \mathcal{F}_{\partial T}$)

$$\hat{a}_T((v_T, v_{\partial T}), (w_T, w_{\partial T})) := \underbrace{(\nabla \mathrm{R}_T^{k+1}(v_T, v_{\partial T}), \nabla \mathrm{R}_T^{k+1}(w_T, w_{\partial T}))_{\boldsymbol{L}^2(T)}}_{\text{Galerkin/reconstruction}}$$
$$+ \underbrace{(\tau_{\partial T} \mathrm{S}_{\partial T}^k(v_T, v_{\partial T}), \mathrm{S}_{\partial T}^k(w_T, w_{\partial T}))_{L^2(\partial T)}}_{\text{stabilization}}$$

- **Local stability and boundedness**:

$$\hat{a}_T((v_T, v_{\partial T}), (v_T, v_{\partial T})) \sim |(v_T, v_{\partial T})|^2_{\mathcal{H}^1(T)}$$

with the local H^1-like seminorm

$$|(v_T, v_{\partial T})|^2_{\mathcal{H}^1(T)} = \|\nabla v_T\|^2_{\boldsymbol{L}^2(T)} + \|\tau_{\partial T}^{\frac{1}{2}}(v_T - v_{\partial T})\|^2_{L^2(\partial T)}$$

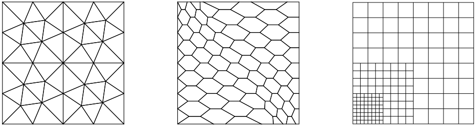
Note that $|(v_T, v_{\partial T})|_{\mathcal{H}^1(T)} = 0$ implies $v_T = v_{\partial T} = \text{cst}$

Variations on the cell unknowns

- Let $k \geq 0$ be the degree of the face unknowns
- Let $l \geq 0$ be the degree of the cell unknowns
- The equal-order case is $l = k$
- One can choose $l = k - 1$ $(k \geq 1)$ and achieve the **same stability and approximation properties**
- One can also choose $l = k + 1$
  - no further gain in stability/approximation
  - **simplified stabilization** $\tilde{S}^k_{\partial T}(v_{\partial T} - v_{T|\partial T}) = \Pi^k_{\partial T}(v_{\partial T} - v_{T|\partial T})$, but more cell unknowns to eliminate
  - cf. [Lehrenfeld, Schöberl 10] for HDG

Assembling the discrete problem (1)

- Mesh $\mathcal{M} = \{\mathcal{T}, \mathcal{F}\}$, cells collected in $\mathcal{T} = \{T\}$, faces in $\mathcal{F} = \{F\}$



- The (global) discrete unknowns are in

$$(v_{\mathcal{T}}, v_{\mathcal{F}}) \in \hat{\mathcal{V}}^k_{\mathcal{M}} := \mathcal{V}^k_{\mathcal{T}} \times \mathcal{V}^k_{\mathcal{F}} := \mathbb{P}^k(\mathcal{T}) \times \mathbb{P}^k(\mathcal{F})$$

  - one polynomial of order k per cell (or $l \in \{k-1, k, k+1\}$)
  - one polynomial of order k per face

- Let $(v_{\mathcal{T}}, v_{\mathcal{F}}) \in \hat{\mathcal{V}}^k_{\mathcal{M}}$; the discrete unknowns attached to a cell $T \in \mathcal{T}$ and its faces $F \in \mathcal{F}_{\partial T}$ are denoted $(v_T, v_{\partial T})$

Assembling the discrete problem (2)

- To enforce homogeneous Dirichlet BCs, we restrict to $\hat{\mathcal{V}}^k_{\mathcal{M},0}$
  - global unknowns **attached to boundary faces are set to zero**
- The discrete problem is: Find $(u_\mathcal{T}, u_\mathcal{F}) \in \hat{\mathcal{V}}^k_{\mathcal{M},0}$ s.t.

$$\hat{a}_\mathcal{M}((u_\mathcal{T}, u_\mathcal{F}), (w_\mathcal{T}, w_\mathcal{F})) := \sum_{T\in\mathcal{T}} \hat{a}_T((u_T, u_{\partial T}), (w_T, w_{\partial T})) = \sum_{T\in\mathcal{T}} (f, w_T)_{L^2(T)}$$

for all $(w_\mathcal{T}, w_\mathcal{F}) \in \hat{\mathcal{V}}^k_{\mathcal{M},0}$

- Simple assembly by summing **local contributions**

Local conservation

- For all $T \in \mathcal{T}$, we define the **numerical flux trace**

$$\phi_{\partial T} := -\nabla \mathrm{R}_T^{k+1}(u_T, u_{\partial T})\cdot \boldsymbol{n}_T + \alpha_{\partial T}^{\mathrm{HHO}}(u_{T|\partial T} - u_{\partial T}) \in \mathbb{P}^k(\mathcal{F}_{\partial T})$$

with $\alpha_{\partial T}^{\mathrm{HHO}} := \tilde{\mathrm{S}}_{\partial T}^{k*}(\tau_{\partial T}\tilde{\mathrm{S}}_{\partial T}^{k})$ (self-adjoint non-negative boundary operator)

- We have the **local cell balance**

$$(\nabla \mathrm{R}_T^{k+1}(u_T, u_{\partial T}), \nabla p)_{\boldsymbol{L}^2(T)} + (\phi_{\partial T}, p)_{L^2(\partial T)} = (f, p)_{L^2(T)}$$

  - test discrete pb. with $((p\delta_{T,T'})_{T'\in\mathcal{T}}, (0)_{F'\in\mathcal{F}})$, $\forall p \in \mathbb{P}^k(T)$

- We have the **flux equilibration condition**

$$\phi_{\partial T_1|F} + \phi_{\partial T_2|F} = 0, \quad F = \partial T_1 \cap \partial T_2$$

  - test discrete pb. with $((0)_{T'\in\mathcal{T}}, (q\delta_{F,F'})_{F'\in\mathcal{F}})$, $\forall q \in \mathbb{P}^k(F)$

Algebraic realization

- Ordering cell unknowns first and then face unknowns, we obtain the linear system

$$\begin{bmatrix} \mathbf{A}_{\mathcal{T}\mathcal{T}} & \mathbf{A}_{\mathcal{T}\mathcal{F}} \\ \mathbf{A}_{\mathcal{F}\mathcal{T}} & \mathbf{A}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\mathcal{T}} \\ \mathbf{U}_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\mathcal{T}} \\ \mathbf{0} \end{bmatrix}$$

- The system matrix is **SPD**
- Local elimination of cell unknowns
  - $\mathbf{A}_{\mathcal{T}\mathcal{T}}$ is block-diagonal $\rightarrow$ one can solve the Schur complement system in terms of face unknowns
  - size $\sim k^2\#(\text{faces})$
  - compact stencil (two faces interact only if they belong to same cell)
  - can be interpreted as a **global transmission problem** [Cockburn 16]

Error analysis

- Stability and $\mathbb{P}^{k+1}$-consistency give $O(h^{k+1})$ energy-error estimate

$$\Big(\sum_{T\in\mathcal{T}} \|\nabla(u - \mathrm{R}_T^{k+1}(u_T, u_{\partial T}))\|^2_{L^2(T)}\Big)^{\frac12} \le c\Big(\sum_{T\in\mathcal{T}} h_T^{2(k+1)}|u|^2_{H^{k+2}(T)}\Big)^{\frac12}$$

- Under (full) elliptic regularity, $O(h^{k+2})$ L^2-error estimate

$$\Big(\sum_{T\in\mathcal{T}} \|\Pi_T^k(u) - u_T\|^2_{L^2(T)}\Big)^{\frac12} \le c\,h\Big(\sum_{T\in\mathcal{T}} h_T^{2(k+1)}|u|^2_{H^{k+2}(T)}\Big)^{\frac12}$$

Polyhedral mesh regularity

- (Usual) assumption that each mesh cell is an agglomeration of **finitely many, shape-regular simplices**; we assume planar faces
- Polynomial approximation in polyhedral cells in Sobolev norms
  - Poincaré–Steklov inequality:

$$\|v - \Pi_T^0(v)\|_{L^2(T)} \le C_{\mathrm{PS}} h_T \|\nabla v\|_{\boldsymbol{L}^2(T)}, \quad \forall v \in H^1(T)$$

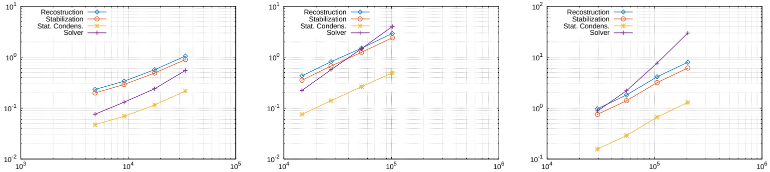
    - $C_{\mathrm{PS}} = \frac{1}{\pi}$ for convex T [Poincaré 1894; Steklov 1897; Bebendorf 03]
    - on polyhedral cells, combine PS on simplices with multiplicative trace inequality [Veeser, Verfürth 12; AE, Guermond 16]

$$\|v\|_{L^2(\partial T)} \le C_{\mathrm{MT}} \left(h_T^{-\frac{1}{2}} \|v\|_{L^2(T)} + \|v\|_{L^2(T)}^{\frac{1}{2}} \|\nabla v\|_{\boldsymbol{L}^2(T)}^{\frac{1}{2}} \right), \quad \forall v \in H^1(T)$$

  - higher-order polynomial approximation using Morrey's polynomial
  - this argument avoids a star-shapedness assumption on cells
  - both PS and MT inequalities allow for some **face degeneration** (see also [Cangiani, Georgoulis, Houston 14; Dong, PhD Thesis 2016])

Implementation

- Disk++ library, **open-source** Github distribution under MPL license
  - library description in [Cicuttin, Di Pietro, AE 18]
- Generic programming: "write once, run on any kind of mesh and in any space dimension"
  - other examples: `deal.II` [Bangerth et al.], `DUNE` [Bastian et al.], `FreeFEM++` [Hecht], `Feel++` [Prud'homme et al.]
- Profiling example on tet meshes ($k \in \{0, 1, 2\}$)



Building bridges

- We can bridge the viewpoints of **HHO, HDG & ncVEM**
  - see [Cockburn, Di Pietro, AE 16]
- **Usual presentation of HDG**
  - approximate the triple $(\boldsymbol{\sigma}, u, \lambda)$, with $\boldsymbol{\sigma} = -\nabla u$, $\lambda = u_{|\mathcal{F}}$
  - $(\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}, \lambda_{\mathcal{F}}) \in \boldsymbol{S}_{\mathcal{T}} \times V_{\mathcal{T}} \times V_{\mathcal{F}}$ with local spaces $\boldsymbol{S}_T$, V_T, V_F
  - discrete HDG problem: $\forall(\boldsymbol{\tau}_T, w_T, \mu_F) \in \boldsymbol{S}_T \times V_T \times V_F$,

$$(\boldsymbol{\sigma}_T, \boldsymbol{\tau}_T)_{\boldsymbol{L}^2(T)} - (u_T, \nabla\cdot\boldsymbol{\tau}_T)_{L^2(T)} + (\lambda_{\partial T}, \boldsymbol{\tau}_T\cdot\boldsymbol{n}_T)_{L^2(\partial T)} = 0$$

$$-(\boldsymbol{\sigma}_T, \nabla w_T)_{\boldsymbol{L}^2(T)} + (\phi_{\partial T}, w_T)_{L^2(\partial T)} = (f, w_T)_{L^2(T)}$$

$$(\phi_{\partial T_1} + \phi_{\partial T_2}, \mu_F)_{L^2(F)} = 0, \quad F = \partial T_1 \cap \partial T_2$$

with the numerical flux trace

$$\phi_{\partial T} = \boldsymbol{\sigma}_T\cdot\boldsymbol{n}_T + \alpha_{\partial T}^{\mathrm{HDG}}(u_{T|\partial T} - \lambda_{\partial T})$$

HHO meets HDG

- HDG method specified through $\boldsymbol{S}_T$, V_T, V_F and $\alpha_{\partial T}^{\mathrm{HDG}}$
  - $\boldsymbol{S}_T = \mathbb{P}^k(T;\mathbb{R}^d)$, $V_T = \mathbb{P}^k(T)$, $V_F = \mathbb{P}^k(F)$, $\alpha_{\partial T}^{\mathrm{HDG}}$ acts pointwise
- HHO as an HDG method
  - $\boldsymbol{S}_T = \nabla\mathbb{P}^{k+1}(T)$, V_T, V_F as above, $\alpha_{\partial T}^{\mathrm{HHO}} = \tilde{\mathrm{S}}_{\partial T}^{k*}(\tau_{\partial T}\tilde{\mathrm{S}}_{\partial T}^{k})$
  - 1st HDG eq: $\boldsymbol{\sigma}_T = -\nabla \mathrm{R}_T^{k+1}(u_T, \lambda_{\partial T})$
  - 2nd HDG eq: HHO tested with $(w_T, 0)$
  - 3rd HDG eq: HHO tested with $(0, \mu_F)$
- Comments
  - HHO uses **smaller flux space** (avoids curl-free functions)
  - HHO uses **nonlocal stabilization** for polyhedral high-order CV
  - alternative route for HDG: space triplets using ***M*-decompositions** [Cockburn, Fu, Sayas 16]

HHO meets ncVEM

- For (conforming) VEM, see [Beirão da Veiga, Brezzi, Marini, Russo, 13]
- Consider the (finite-dimensional) virtual space

$$V^{k+1}(T) = \{v \in H^1(T) \mid \Delta v \in \mathbb{P}^k(T),\ \boldsymbol{n}_T \cdot \nabla v \in \mathbb{P}^k(\mathcal{F}_{\partial T})\}$$

  - $\mathbb{P}^{k+1}(T) \subsetneq V^{k+1}(T)$; other functions are not explicitly known
  - recall reduction operator $\hat{\mathcal{I}}_T^k(v) = (\Pi_T^k(v), \Pi_{\partial T}^k(v))$; then

$$\hat{\mathcal{I}}_T^k : V^{k+1}(T) \longleftrightarrow \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) \text{ is an isomorphism}$$

- Let $\varphi \in V^{k+1}(T)$
  - $\mathcal{E}_T^{k+1}(\varphi) = \mathrm{R}_T^{k+1}(\hat{\mathcal{I}}_T^k(\varphi))$ is computable from the dof's $\hat{\mathcal{I}}_T^k(\varphi)$ of φ
  - same for $\check{S}_{\partial T}^k(\varphi) = S_{\partial T}^k(\hat{\mathcal{I}}_T^k(\varphi))$
- We have $\check{a}_T(\varphi, \psi) = \hat{a}_T(\hat{\mathcal{I}}_T^k(\varphi), \hat{\mathcal{I}}_T^k(\psi))$ with

$$\check{a}_T(\varphi, \psi) = (\nabla \mathcal{E}_T^{k+1}(\varphi), \nabla \mathcal{E}_T^{k+1}(\psi))_{\boldsymbol{L}^2(T)} + (\tau_{\partial T} \check{S}_{\partial T}^k(\varphi), \check{S}_{\partial T}^k(\psi))_{L^2(\partial T)}$$

Elliptic interface problem

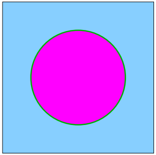
- Let Ω be a Lipschitz polyhedron in $\mathbb{R}^d$ s.t.

$$\overline{\Omega} = \overline{\Omega^1} \cup \overline{\Omega^2}, \qquad \Gamma = \partial\Omega^1 \cap \partial\Omega^2$$

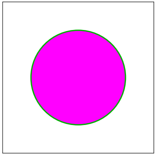
- We consider the following elliptic interface problem:

$$\begin{aligned} -\nabla\cdot(\kappa\nabla u) &= f &\quad \text{on } \Omega^1 \cup \Omega^2 \\ [\![u]\!]_\Gamma &= g_\mathrm{D} &\quad \text{on } \Gamma \\ [\![\kappa\nabla u]\!]_\Gamma\cdot\boldsymbol{n}_\Gamma &= g_\mathrm{N} &\quad \text{on } \Gamma \end{aligned}$$

  - $f \in L^2(\Omega)$, $g_\mathrm{D} \in H^{\frac{1}{2}}(\Gamma)$, $g_\mathrm{N} \in L^2(\Gamma)$, Dirichlet BCs on $\partial\Omega$
  - each subdomain has a specific diffusivity $\kappa^i = \kappa_{|\Omega^i}$, $i \in \{1, 2\}$



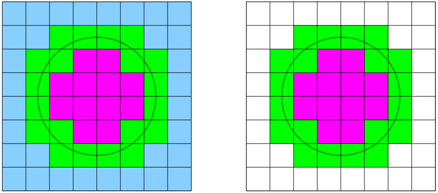
interface pb.



fictitious domain pb.

Unfitted meshes

- The domain Ω is meshed without fitting the interface Γ
  - uncut cells in Ω^1
  - uncut cells in Ω^2
  - cut cells overlapping Ω^1 and Ω^2



- For cut cells, we set $T^i = T \cap \Omega^i$, $i \in \{1,2\}$
  - a degenerate cut arises when $\min(|T^1|,|T^2|) \ll \max(|T^1|,|T^2|)$
- The highly-contrasted case arises when $\min(\kappa^1,\kappa^2) \ll \max(\kappa^1,\kappa^2)$

Robust unfitted methods

- Cut-robust unfitted conforming FEM [Hansbo, Hansbo 02]
  - use function pairs to approximate solution in cut cells
  - consistent Nitsche's penalty method [Nitsche 71]
  - cut-dependent averaging for consistency terms
  - **cut-robustness, but not κ-robustness ...**
  - DG version with *hp*-analysis [Massjung 12]

- Ghost penalty [Burman 10]
  - diffusion-dependent averaging [Dryja 03; Burman, Zunino 06; Ern et al. 09] for **κ-robustness**
  - **cut-robustness** achieved by additional patch-based stabilization

- Alternative route to **cut-robustness** by cell-agglomeration
  - eliminate degenerate cells by local agglomeration $\rightarrow$ polyhedral cells
  - well suited to DG setting [Johansson, Larson 13]; for conforming FEM on quadrilaterals with hanging nodes, see [Huang, Wu, Xuo 17]

CutHHO

- Our **objectives** are
  - extend the cell-agglomeration idea of [Johansson, Larson 13] using general meshes (even unstructured triangulations)
  - achieve both cut- and κ-robustness
- We neglect quadrature errors due to geometry approximation
  - [Burman, Hansbo, Larson 17]: Taylor expansions
  - [Lehrenfeld, Reusken 17]: isoparametric level-set function

Unfitted HHO

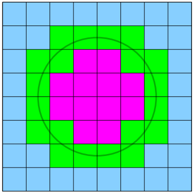
- We consider a mesh $\mathcal{T}$ of Ω that does not fit the interface Γ

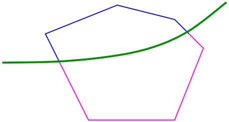
$$\begin{array}{ll} \text{cut cells} & \mathcal{T}^{\Gamma} = \{T \in \mathcal{T} \mid \mathrm{mes}_{d-1}(T \cap \Gamma) > 0\} \\ \text{uncut cells} & \mathcal{T}^{\setminus\Gamma} = \mathcal{T}^{1} \cup \mathcal{T}^{2} \end{array}$$

  with $\mathcal{T}^i = \{T \in \mathcal{T} \mid T \subset \Omega^i\}$, $i \in \{1,2\}$

- For a cut cell $T \in \mathcal{T}^{\Gamma}$, we define

$$T^i = T \cap \Omega^i, \qquad T^{\Gamma} = T \cap \Gamma, \qquad \partial T^i = (\partial T)^i \cup T^{\Gamma}$$



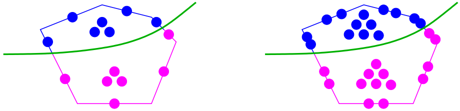


Discrete HHO unknowns

- For unfitted FEM, the solution is approximated in a cut cell by a **pair of polynomials**, one attached to each Ω^i [Hansbo, Hansbo 02]
- For the unfitted HHO method, the solution is approximated in a cut cell by a **pair of HHO unknowns**, one attached to each Ω^i

$$\hat{V}_T = (V_T, V_{\partial T}) = ((v_{T^1}, v_{T^2}), (v_{(\partial T)^1}, v_{(\partial T)^2})) \in \hat{\mathcal{X}}_T$$

with $\hat{\mathcal{X}}_T = \Big(\mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2)\Big) \times \Big(\mathbb{P}^{k}(\mathcal{F}_{(\partial T)^1}) \times \mathbb{P}^{k}(\mathcal{F}_{(\partial T)^2})\Big)$



- We do not introduce HHO unknowns on T^Γ

Nitsche's mortaring

- To fix the ideas, let us assume that $\kappa^1 < \kappa^2$
- The Nitsche mortaring bilinear form is defined s.t.

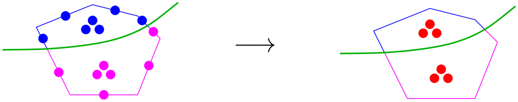
$$n_T(V,W) = \sum_{i\in\{1,2\}} \int_{T^i} \kappa^i \nabla v^i \cdot \nabla w^i + \int_{T^\Gamma} \eta \frac{\kappa^1}{h_T} [\![V]\!]_\Gamma [\![W]\!]_\Gamma - \int_{T^\Gamma} (\kappa \nabla v)^1 \cdot \boldsymbol{n}_\Gamma [\![W]\!]_\Gamma + (\kappa \nabla w)^1 \cdot \boldsymbol{n}_\Gamma [\![V]\!]_\Gamma$$

for all $V = (v^1, v^2), W = (w^1, w^2)$ in $H^1(T^1) \times H^1(T^2)$

- The penalty parameter η is to be taken **large enough**
  - cut-robust minimum value
  - depends on discrete trace inequality

Reconstruction and stabilization

R_T^{k+1} :



- Let $\hat{V}_T = (V_T, V_{\partial T}) \in \hat{\mathcal{X}}_T$, then $\mathrm{R}_T^{k+1}(\hat{V}_T) \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2)$
- We solve for all $Z \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2)$,

$$n_T(\mathrm{R}_T^{k+1}(\hat{V}_T), Z) = n_T(V_T, Z) - \sum_{i\in\{1,2\}} \int_{(\partial T)^i} (v_{T^i} - v_{(\partial T)^i}) \boldsymbol{n}_T \cdot \kappa^i \nabla z^i$$

  - well-posed local Neumann pb. owing to coercivity of n_T
  - local Nitsche's mortaring matrix, fully parallelizable
  - the reconstructions in T^1 and T^2 are **built simultaneously**

- The stabilization bilinear form is here of Lehrenfeld–Schöberl type

$$s_T(\hat{V}_T, \hat{W}_T) = \sum_{i\in\{1,2\}} \kappa^i h_T^{-1} \int_{(\partial T)^i} \Pi^k_{(\partial T)^i}(v_{T^i} - v_{(\partial T)^i})(w_{T^i} - w_{(\partial T)^i})$$

Assembling the discrete problem (1)

- On all cut cells $T \in \mathcal{T}^{\Gamma}$, we consider $\hat{V}_T = (V_T, V_{\partial T})$ with $V_T = (v_{T^1}, v_{T^2})$, $V_{\partial T} = (v_{(\partial T)^1}, v_{(\partial T)^2})$, and we set

$$\hat{a}_T^{\Gamma}(\hat{V}_T, \hat{W}_T) = n_T(\mathrm{R}_T^{k+1}(\hat{V}_T), \mathrm{R}_T^{k+1}(\hat{W}_T)) + s_T(\hat{V}_T, \hat{W}_T)$$

$$\hat{\ell}_T^{\Gamma}(\hat{W}_T) = \sum_{i \in \{1,2\}} \int_{T^i} f w_{T^i} + \int_{T^{\Gamma}} (g_{\mathrm{N}} w_{T^2} + g_{\mathrm{D}} \phi_T(W_T))$$

with $\phi_T(W_T) = -\kappa^1 \nabla w_{T^1} \cdot \boldsymbol{n}_{\Gamma} + \eta \kappa^1 h_T^{-1} [\![W_T]\!]_{\Gamma}$ for consistency reasons

- On all the uncut cells $T \in \mathcal{T}^{\setminus\Gamma}$, we consider $\hat{v}_T = (v_T, v_{\partial T})$ (as before), and we set

$$\hat{a}_T^{\setminus\Gamma}(\hat{v}_T, \hat{w}_T) = a_T(\mathrm{R}_T^{k+1}(\hat{v}_T), \mathrm{R}_T^{k+1}(\hat{w}_T)) + s_T(\hat{v}_T, \hat{w}_T)$$

$$\hat{\ell}_T^{\setminus\Gamma}(\hat{w}_T) = \int_T f w_T$$

Assembling the discrete problem (2)

- The global discrete unknowns in Ω^i are in

$$\hat{\mathcal{X}}_h^i = \mathbb{P}^{k+1}(\mathcal{T}^i) \times \mathbb{P}^k(\mathcal{F}^i), \quad i \in \{1,2\}$$

- The global discrete unknowns in Ω are in $\hat{\mathcal{X}}_h = \mathcal{X}_h^1 \times \mathcal{X}_h^2$
  - subspace $\hat{\mathcal{X}}_{h0}$ with Dirichlet BCs enforced on face unknowns in $\partial\Omega$
- Find $\hat{U}_h \in \hat{\mathcal{X}}_{h0}$ s.t. $\hat{a}_h(\hat{U}_h, \hat{W}_h) = \hat{\ell}_h(\hat{W}_h)$, $\forall \hat{W}_h \in \hat{\mathcal{X}}_{h0}$, with

$$\hat{a}_h(\hat{V}_h, \hat{W}_h) = \sum_{T\in\mathcal{T}^{\setminus\Gamma}} \hat{a}_T^{\setminus\Gamma}(\hat{v}_T, \hat{w}_T) + \sum_{T\in\mathcal{T}^\Gamma} \hat{a}_T^\Gamma(\hat{V}_T, \hat{W}_T)$$

$$\hat{\ell}_h(\hat{W}_h) = \sum_{T\in\mathcal{T}^{\setminus\Gamma}} \hat{\ell}_T^{\setminus\Gamma}(w_T) + \sum_{T\in\mathcal{T}^\Gamma} \hat{\ell}_T^\Gamma(\hat{W}_T)$$

  - global SPD system matrix
  - solve **Schur complement for face unknowns**

Main results

- **Stability** and **error estimates** for cutHHO
- Our estimates depend on three (main) parameters
  - $\rho \in (0, 1)$ quantifying polyhedral mesh regularity
  - $\gamma \in (0, 1)$ quantifying how well the mesh resolves the interface
  - $\delta \in (0, 1)$ quantifying how well the interface cuts the mesh cells
  - depend on polynomial degree $k \geq 0$ and space dimension $d \geq 2$
- γ can be bounded away from zero by mesh refinement
  - we assume that the interface Γ is of class C^2
- δ can be bounded away from zero by local cell agglomeration
- Precise geometric bounds in [Burman, AE 18]

Discrete trace inequality

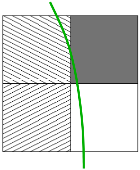
- Let $\delta \in (0,1)$; the cell $T \in \mathcal{T}^\Gamma$ is δ-regular if

$$\forall i \in \{1,2\}, \quad \exists \boldsymbol{x}_{T^i} \in T^i = T \cap \Omega^i, \quad B(\boldsymbol{x}_{T^i}, \delta h_T) \subset T^i \qquad (1)$$

- Let $\ell \in \mathbb{N}$; there is $c_{\mathrm{dtr}} = c_{\mathrm{dtr}}(\rho, \delta, \ell)$ s.t. for all δ-regular cut cell $T \in \mathcal{T}^\Gamma$, all $v \in \mathbb{P}_d^\ell(T^i)$ and $i \in \{1,2\}$,

$$\|v\|_{L^2(\partial T^i)} \leq c_{\mathrm{dtr}}\, h_T^{-\frac{1}{2}} \|v\|_{L^2(T^i)}$$

- δ-regularity achieved by **local cell agglomeration**
  - if (1) fails for T^1, we look for a neighbor T' of T, $T \cap T' \neq \emptyset$ s.t. (1) holds for $(T \cup T')^1$
  - agglomerated cell is of size $O(h_T)$ and is not necessarily connected!



Stability

- Assume that the Nitsche stability parameter is s.t. $\eta \geq 4c_{\mathrm{dtr}}^2$
- Stability of Nitsche's mortaring: $\forall V_T \in \mathbb{P}_d^{k+1}(T^1) \times \mathbb{P}_d^{k+1}(T^2)$,

$$n_T(V_T, V_T) \geq \frac{1}{2}|V_T|_{n_T}^2, \qquad |V_T|_{n_T}^2 = \sum_{i\in\{1,2\}} \kappa^i \|\nabla v_{T^i}\|_{\boldsymbol{L}^2(T^i)}^2 + \eta \frac{\kappa^1}{h_T} \|[\![V_T]\!]_\Gamma\|_{L^2(T^\Gamma)}^2$$

- HHO stability on cut cells: $\forall \hat{V}_T \in \hat{\mathcal{X}}_T$,

$$\hat{a}_T^\Gamma(\hat{V}_T, \hat{V}_T) \gtrsim |\hat{V}_T|_{\hat{a}_T}^2, \qquad |\hat{V}_T|_{\hat{a}_T}^2 = |V_T|_{n_T}^2 + s_T(\hat{V}_T, \hat{V}_T)$$

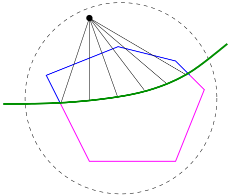
- HHO stability on uncut cells (as before): $\forall v_T \in \hat{\mathcal{X}}_T$,

$$\hat{a}_T^{\backslash\Gamma}(\hat{v}_T, \hat{v}_T) \gtrsim |\hat{v}_T|_{\hat{a}_T}^2, \qquad |\hat{v}_T|_{\hat{a}_T}^2 = \kappa_T \|\nabla v_T\|_{\boldsymbol{L}^2(T)}^2 + s_T(\hat{v}_T, \hat{v}_T)$$

- Global stability norm: $|\hat{V}_h|_{\hat{a}_h}^2 := \sum_{T\in\mathcal{T}^{\backslash\Gamma}} |\hat{v}_T|_{\hat{a}_T}^2 + \sum_{T\in\mathcal{T}^\Gamma} |\hat{V}_T|_{\hat{a}_T}^2$

Multiplicative trace inequality

- There is $\gamma \in (0,1)$ s.t., for all $T \in \mathcal{T}^\Gamma$,
  - there is a ball $T^\dagger$ s.t. $T \subset T^\dagger$ and $\gamma h_{T^\dagger} \leq h_T$
  - there is $\boldsymbol{x} \in T^\dagger$ s.t. the fan $\{t\boldsymbol{x} + (1-t)\boldsymbol{y},\ t \in [0,1],\ \boldsymbol{y} \in T^\Gamma\} \subset T^\dagger$ and for each $\boldsymbol{y} \in T^\Gamma$, its segment cuts T^Γ only once



- There is $c_{\mathrm{mtr}} = c_{\mathrm{mtr}}(\rho, \gamma)$ s.t. for all $T \in \mathcal{T}^\Gamma$ and $v \in H^1(\Omega)$,

$$\max_{i \in \{1,2\}} \|v\|_{L^2(\partial T^i)} \leq c_{\mathrm{mtr}} \left(h_T^{-\frac{1}{2}} \|v\|_{L^2(T^\dagger)} + \|v\|_{L^2(T^\dagger)}^{\frac{1}{2}} \|\nabla v\|_{L^2(T^\dagger)}^{\frac{1}{2}} \right)$$

Approximation in cut cells

- Let $E^i : H^1(\Omega^i) \to H^1(\Omega)$, $i \in \{1,2\}$, be stable extension operators
- We approximate the exact pair $U^{\mathrm{ex}} = (u^1, u^2)$ in $T \in \mathcal{T}^\Gamma$ by

$$J_T^{k+1}(U^{\mathrm{ex}}) := (\Pi_{T^\dagger}^{k+1}(E^1(u^1))_{|T^1}, \Pi_{T^\dagger}^{k+1}(E^2(u^2))_{|T^2}) \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2)$$

- **Local approximation.** Assume $U^{\mathrm{ex}} \in H^{k+2}(\Omega^1) \times H^{k+2}(\Omega^2)$; then

$$\|J_T^{k+1}(U^{\mathrm{ex}}) - U^{\mathrm{ex}}\|_{*T} \lesssim \sum_{i\in\{1,2\}} (\kappa^i)^{\frac{1}{2}} h_T^{k+1} |E^i(u^i)|_{H^{k+2}(T^\dagger)}$$

  where

$$\|V\|_{*T}^2 = \sum_{i\in\{1,2\}} \kappa^i \Big(\|\nabla v^i\|_{T^i}^2 + h_T \|\nabla v^i\|_{(\partial T)^i}^2 + h_T^{-1} \|v^i\|_{(\partial T)^i}^2 \Big)$$
$$+ \kappa^1 \Big(h_T \|\nabla v^1\|_{T^\Gamma}^2 + h_T^{-1} \|[\![V]\!]_\Gamma\|_{T^\Gamma}^2 \Big) + \kappa^2 h_T \|\nabla v^2\|_{T^\Gamma}^2$$

- use multiplicative trace inequality and standard approximation properties for L^2-projectors in $T^\dagger$

Consistency/boundedness

- Define on cut cells $T \in \mathcal{T}^{\Gamma}$,

$$\hat{J}_T^{k+1}(U^{\mathrm{ex}}) = (J_T^{k+1}(U^{\mathrm{ex}}), (\Pi^k_{(\partial T)^1}(u^1), \Pi^k_{(\partial T)^2}(u^2))) \in \hat{\mathcal{X}}_T$$

- Recall on uncut cells $T \in \mathcal{T}^{\backslash\Gamma}$, with $u^{\mathrm{ex}} = u^i$, $T \in \Omega^i$, the local approximation by $j_T^{k+1}(u^{\mathrm{ex}}) = \Pi_T^{k+1}(u^{\mathrm{ex}})$ and define

$$\hat{\jmath}_T^{k+1}(u^{\mathrm{ex}}) = (j_T^{k+1}(u^{\mathrm{ex}}), \Pi^k_{\partial T}(u^{\mathrm{ex}})) \in \hat{\mathcal{X}}_T$$

- Define the **consistency error** s.t., $\forall \hat{W}_h \in \hat{\mathcal{X}}_{h0}$,

$$\mathcal{F}(\hat{W}_h) = \sum_{T \in \mathcal{T}^{\backslash\Gamma}} \hat{a}_T^{\backslash\Gamma}(\hat{\jmath}_T^{k+1}(u^{\mathrm{ex}}) - \hat{u}_T, \hat{w}_T) + \sum_{T \in \mathcal{T}^{\Gamma}} \hat{a}_T^{\Gamma}(\hat{J}_T^{k+1}(U^{\mathrm{ex}}) - \hat{U}_T, \hat{W}_T)$$

- Assume $U^{\mathrm{ex}} \in H^s(\Omega^1) \times H^s(\Omega^2)$, $s > \frac{3}{2}$. Then,

$$\frac{|\mathcal{F}(\hat{W}_h)|}{|\hat{W}_h|_{\hat{a}_h}} \lesssim \left(\sum_{T \in \mathcal{T}^{\backslash\Gamma}} \|j_T^{k+1}(u^{\mathrm{ex}}) - u^{\mathrm{ex}}\|^2_{*T} + \sum_{T \in \mathcal{T}^{\Gamma}} \|J_T^{k+1}(U^{\mathrm{ex}}) - U^{\mathrm{ex}}\|^2_{*T} \right)^{\frac{1}{2}}$$

Error estimate

- Assume $U^{\rm ex} = (u^1, u^2) \in H^s(\Omega^1) \times H^s(\Omega^2)$, $s > \frac{3}{2}$. Then,

$$\mathcal{E} := \sum_{T \in \mathcal{T}\setminus\Gamma} \kappa_T \|\nabla(u^{\rm ex} - u_T)\|_T^2 + \sum_{T \in \mathcal{T}^\Gamma} \sum_{i \in \{1,2\}} \kappa^i \|\nabla(U^{\rm ex} - U_T)^i\|_{T^i}^2$$

$$+ \sum_{T \in \mathcal{T}^\Gamma} \frac{\kappa^1}{h_T} \|g_{\rm D} - [\![U^{\rm ex}]\!]_\Gamma\|_{T^\Gamma}^2 + \frac{h_T}{\kappa^2} \|g_{\rm N} - [\![\kappa \nabla U^{\rm ex}]\!]_\Gamma \cdot \boldsymbol{n}_\Gamma\|_{T^\Gamma}^2$$

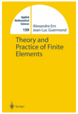
$$\lesssim \sum_{T \in \mathcal{T}\setminus\Gamma} \|j_T^{k+1}(u_T^{\rm ex}) - u_T^{\rm ex}\|_{*T}^2 + \sum_{T \in \mathcal{T}^\Gamma} \|J_T^{k+1}(U^{\rm ex}) - U_T^{\rm ex}\|_{*T}^2$$

- Moreover, if $U^{\rm ex} \in H^{k+2}(\Omega^1) \times H^{k+2}(\Omega^2)$, then

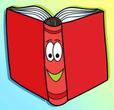
$$\mathcal{E} \lesssim \sum_{i \in \{1,2\}} \kappa^i h^{2(k+1)} |u^i|_{H^{k+2}(\Omega^i)}^2$$

Conclusions

- HHO methods offer **physical fidelity**, **robustness** and **competitive costs** for a wide range of problems
- `Disk++` **library**, Github open-source distribution (MPL license)
- New Finite Element book(s) with J.-L. Guermond (Fall 2018)
  - 10 chapters of 50 pages $\rightarrow$ 65 chapters of 14 pages with exercices



$\Longrightarrow$



Thank you for your attention