LSD: a robust and efficient finite element method for solving elliptic PDEs

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Numerical Analysis of Coupled and Multi-Physics Problems with Dynamic Interfaces
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Continuous problem with heterogeneous coefficients

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \mathcal{A} \nabla u \cdot \nabla v \, d\boldsymbol{x} = \int_\Omega g v \, d\boldsymbol{x} \quad \text{for all } v \in H_0^1(\Omega)$$

where

- $\Omega \subset \mathbb{R}^d$ polygonal
- $g \in L^2(\Omega)$ piece-wise constant (for simplicity)
- $\mathcal{A} \in [L^\infty(\Omega)]_{\text{sym}}^{d \times d}$ uniformly coercive

Hurdles: low regularity, high-contrast and multiscale

Let

$$a(u,v) \stackrel{\text{def}}{=} \int_\Omega \mathcal{A} \nabla u \cdot \nabla v \, d\boldsymbol{x}, \qquad (g,v) \stackrel{\text{def}}{=} \int_\Omega g v \, d\boldsymbol{x}$$

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Messages

Two main points of the talk

1. "smart" space decomposition
   - allow local or "quasi-local" static condensation
2. spectral decomposition of spaces
   - based on local eigenvalue problems
   - make the scheme robust w.r.t. high contrast

Finite Elements

- $\mathcal{T}_H$ nice partition of Ω
- $V_H = \{v \in H_0^1(\Omega) : v$ piece-wise linear$\}$
- $u_H \in V_H$ such that

$$a(u_H, v_H) = (g, v_H) \quad \text{for all } v_H \in V_H$$

- error analysis:

$$\|u - u_H\|_{H^1(\Omega)} \leq CH|u|_{H^2(\Omega)}$$

- $|u|_{H^2(\Omega)}$ and C not good for multiscale problems

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High Contrast

Consider

$$u^\epsilon = \underset{v \in H_0^1(0,1)}{\arg\min} \int_0^1 \frac{1}{2} |a(x/\epsilon) v'(x)|^2 - v(x)\, dx.$$

where

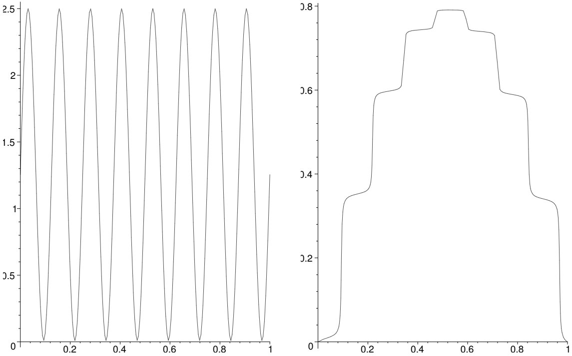

Figure: $a(\cdot/\epsilon)$ and exact solution for $\epsilon = 1/8$ (contrast=250)

High Contrast

Homogenization and finite elements fail:

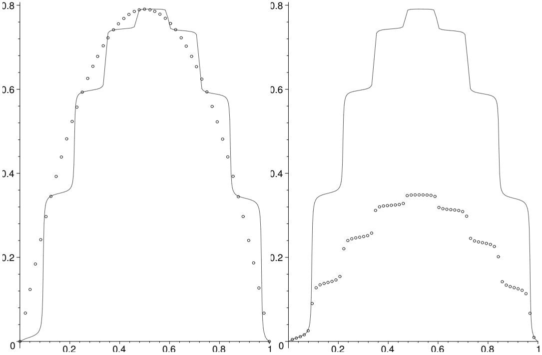

Figure: Comparison between the exact, homogenized solutions and finite element approximation for $h = 1/64$, for $\epsilon = 1/8$.

High Contrast

Even for homogeneous media, finite elements fail under high contrast:

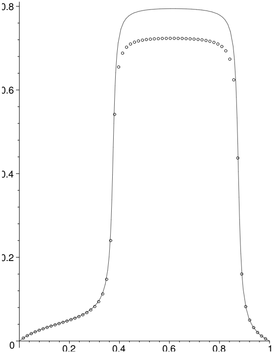

Figure: Exact and finite element solutions for $\epsilon = 1/2$ and $h = 1/64$

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Primal hybrid formulation

Spaces

- $H^1(\mathcal{T}_H) = \{v \in L^2(\Omega) : v|_\tau \in H^1(\tau),\ \tau \in \mathcal{T}_H\}$
- $\Lambda(\mathcal{T}_H) = \{\prod_{\tau\in\mathcal{T}_H} \boldsymbol{\sigma}\cdot\boldsymbol{n}^\tau|_{\partial\tau} : \boldsymbol{\sigma} \in H(\mathrm{div};\Omega)\}$

Hybrid formulation: $u \in H^1(\mathcal{T}_H)$, $\lambda \in \Lambda(\mathcal{T}_H)$ solve

$$\sum_{K\in\mathcal{T}_h}\int_K \mathcal{A}\,\boldsymbol{\nabla}\,u\cdot\boldsymbol{\nabla}\,v\,d\boldsymbol{x} + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\lambda v\,d\boldsymbol{x} = \int_\Omega gv\,d\boldsymbol{x} \quad \forall v \in H^1(\mathcal{T}_H)$$

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}\mu u\,d\boldsymbol{x} = 0 \quad \forall \mu \in \Lambda(\mathcal{T}_H)$$

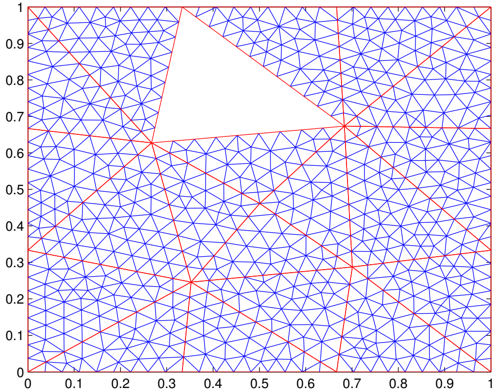
Comments

- first eqtn: equilibrium and $\lambda = -(\mathcal{A}\,\boldsymbol{\nabla}\,u)\boldsymbol{n}^K$ on ∂K
- second eqtn: continuity and Dirichlet boundary condition
- Pian & Tong, 69'-71'; Raviart & Thomas 77'

It's a large problem…

practical implementation:

- introduce sub-mesh $\mathcal{T}_h$
- $\mathcal{F}_h$ be a partition of the faces of elements in $\mathcal{T}_H$
- $\Lambda_h = \{\mu_h \in \Lambda(\mathcal{T}_H) : \mu_h|_{F_h}$ is const on faces $F_h \in \mathcal{F}_h\}$



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Assume that

$$|u - u_h|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} = |\lambda - \lambda_h|_{\Lambda(\mathcal{T}_H)} \leq \mathscr{H},$$

for some desired precision $\mathscr{H}$

"Classical" way out: hybridization based on fluxes

- FETI, MHM, HDG, M^{3}FEM, MRCM…

Hybridization example

Decompose (FETI & MHM): $H^1(\mathcal{T}_H) = \mathbb{P}^0(\mathcal{T}_H) \oplus \widetilde{H}^1(\mathcal{T}_H)$

- $\mathbb{P}^0(\mathcal{T}_H)$ - constant by parts (dim $=$ # coarse elements)
- Λ_h - flux space (dim $\sim H^{-d}h^{-d+1}$)

Then $u_h = u_h^0 + T\lambda_h$ where

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h}\int_{\partial K}\lambda_h T\mu_h\,d\boldsymbol{x} + \sum_{K\in\mathcal{T}_h}\int_{\partial K}u_h^0\mu_h\,d\boldsymbol{x} &= 0 && \forall \mu_h \in \Lambda_h \\
\sum_{K\in\mathcal{T}_h}\int_{\partial K}\lambda_h v_h^0\,d\boldsymbol{x} &= \int_\Omega g v_h^0\,d\boldsymbol{x} && \forall v \in \mathbb{P}^0(\mathcal{T}_H)
\end{aligned}$$

where T is local $\mathcal{A}$-harmonic extension (Neumann-to-Dirichlet)

Not good enough yet:

- system size $\sim$ flux space. Still large...
- goal: final system depending *only* on H

Decompose the fluxes!

Decompose: $\Lambda_h = \Lambda^{\text{const}} \oplus \widetilde{\Lambda}_h^f$

- Λ^{const} - constant on edges (dim $= \#$ edges)
- $\widetilde{\Lambda}_h^f$ - zero average on edges (dim $\sim h^{-1}\#$ edges)
- $\lambda_h = \lambda_h^{\text{const}} + \tilde{\lambda}_h^f$

Then $u_h = u_h^0 + T\lambda_h$ and $\lambda_h = \lambda_h^{\text{const}} + \tilde{\lambda}_h^f$

$$
\begin{aligned}
(\lambda_h^{\text{const}}, v^0)_{\partial\mathcal{T}_H} &= -(g, v^0)_{\mathcal{T}_H} \\
(\tilde{\mu}_h^f, T\lambda_h^{\text{const}} + T\tilde{\lambda}_h^f)_{\partial\mathcal{T}_H} &= 0 \\
(\mu^{\text{const}}, T\lambda_h^{\text{const}} + T\tilde{\lambda}_h^f)_{\partial\mathcal{T}_H} + (\mu^{\text{const}}, u_h^0)_{\partial\mathcal{T}_H} &= 0
\end{aligned}
$$

One large system: solve $\tilde{\lambda}_h^f$ w.r.t. λ_h^{const}

Static Condensation

Let $PT : \Lambda_h \to \tilde{\Lambda}_h^f$ such that, given $\lambda \in \Lambda_h$,

$$(\tilde{\mu}_h^f, TPT\lambda)_{\partial\mathcal{T}_H} = (\tilde{\mu}_h^f, T\lambda)_{\partial\mathcal{T}_H} \quad \text{for all } \tilde{\mu}_h^f \in \tilde{\Lambda}_h^f$$

Then $\tilde{\lambda}_h^f = -PT\lambda_h^{\text{const}}$, and it's enough to solve:

$$\begin{aligned} \left((I - PT)\mu_h^{\text{const}}, T(I - PT)\lambda_h^{\text{const}}\right)_{\partial\mathcal{T}_H} + (\mu_h^{\text{const}}, u_h^0)_{\partial\mathcal{T}_H} &= 0 \\ (\lambda_h^{\text{const}}, v^0)_{\partial\mathcal{T}_H} &= -(g, v^0)_{\mathcal{T}_H} \end{aligned}$$

Good:

- problem size $\sim \#$ coarse space
- "elliptic system"

Not good:

- P is not local
- solving P is as hard as the whole problem

Localize

Let $w \in H^1(\mathcal{T}_H)$ with local support

- then $PT\lambda$ decays exponentially!!!
- solve $PT\lambda$ in a patch of j elements around support of λ (we call it $P^jT\lambda$)
- replace P by P^j in the final system

$$
\begin{aligned}
\big((I - P^jT)\mu_h^{\text{const}}, T(I - P^jT)\lambda_h^{\text{const}}\big)_{\partial\mathcal{T}_H} + (\mu_h^{\text{const}}, u_h^0)_{\partial\mathcal{T}_H} &= 0 \\
(\lambda_h^{\text{const}}, v^0)_{\partial\mathcal{T}_H} &= -(g, v
\end{aligned}
$$

Only good news

- problems size $\sim \#$ coarse space
- elliptic system

Energy decay

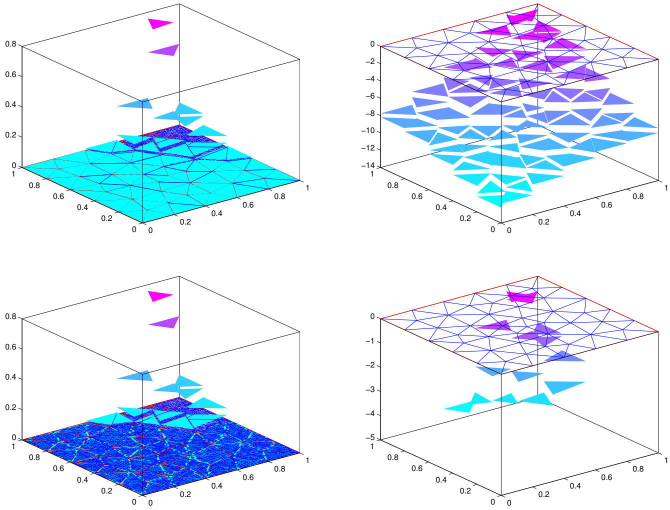

Figure: Energy decay of $PT\lambda$ (top) and $P^jT\lambda$ (bottom); real energy (left) and log of energy (right)

Energy decay

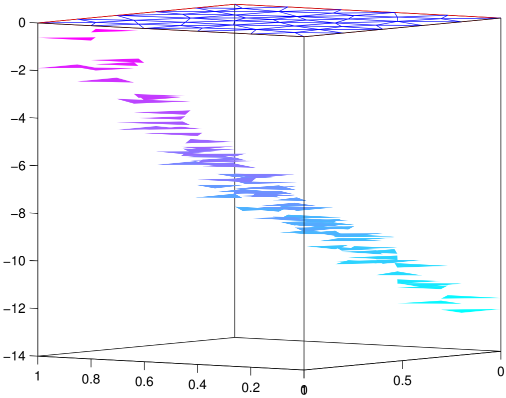

Figure: Log plot of energy of $PT\lambda$. Log of decay should be linear

Error estimate

In energy norm:

$$|u - u_h^j|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \leq \mathscr{H} + cj^d e^{-\frac{j-2}{c\beta_{H/h}}} \beta_{H/h} \|g\|_{L^2(\Omega)}$$

where $\beta_{H/h} = 1 + \log(H/h)$

Comments

- two parts: $\mathscr{H}$ (accuracy) and exponential term
- if $h \to 0$ then $\beta_{H/h}$ grows
- if patch size j grows, error $\to \mathscr{H}$
- needs minimum regularity, i.e., $u \in H^1(\Omega)$
- c depends on the contrast

Examples

Consider $\Omega = [0,1]^2$, $g = 1$, and the mesh

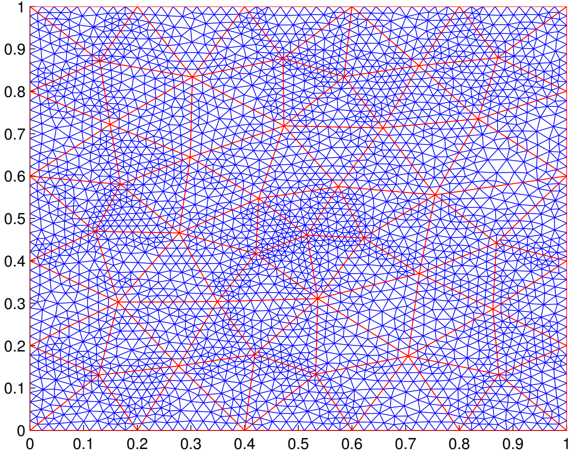

Figure: Mesh with $H = 0.2$ and $h \sim H/8$

High-contrast case

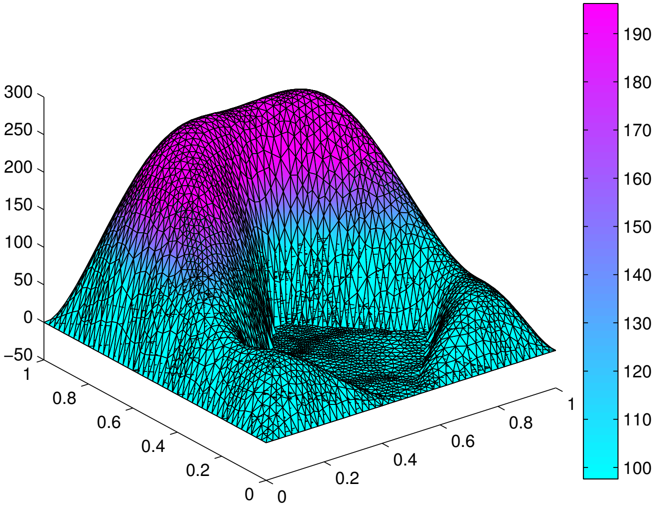

Figure: “Exact” solution u_h

High-contrast case

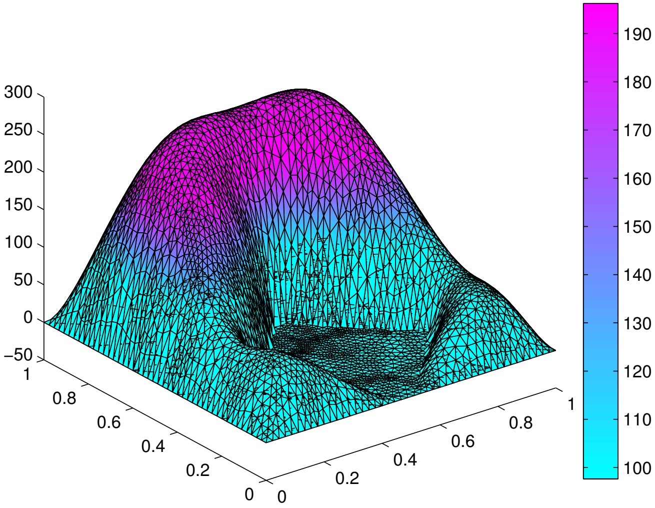

Figure: Approximate solution with patch of four elements

High-contrast case

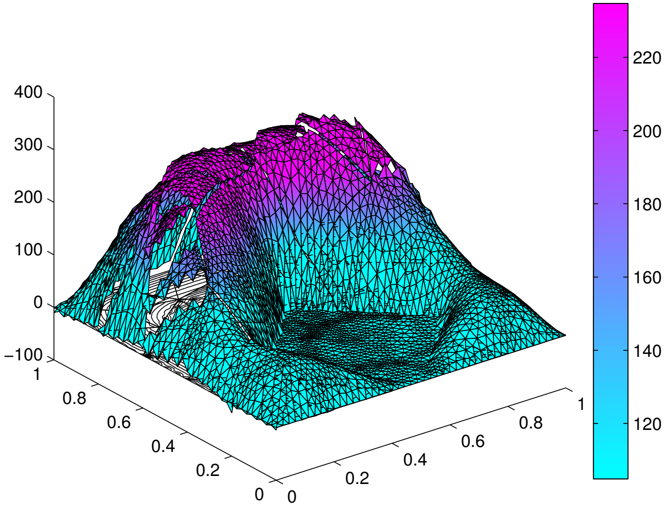

Figure: Approximate solution with patch of three elements

Contents

Spectral Decomposition

- Fix a face $F \in \partial\mathcal{T}_H$, shared by elements τ and τ'
- For the element τ, let $F_\tau^c = \partial\tau \backslash F$.
- Let $T : \Lambda_h \to \Lambda_h'$ be Neumann-to-Dirichlet operator on $\partial\tau$:

$$T\lambda_h^\tau = \begin{pmatrix} T_{FF}^\tau & T_{F^cF}^\tau \\ T_{FF^c}^\tau & T_{F^cF^c}^\tau \end{pmatrix} \begin{pmatrix} \lambda_F^\tau \\ \lambda_{F^c}^\tau \end{pmatrix}$$

- Find eigenpairs $(\alpha_i^F, \tilde{\mu}_{h,i}^F) \in (\mathbb{R}, \tilde{\Lambda}_h^F)$ such that

$$\left(\tilde{\nu}_h^F, (T_{FF}^\tau + T_{FF}^{\tau'})\tilde{\mu}_{h,i}^F\right)_F = \alpha_i^F \left(\tilde{\nu}_h^F, (\hat{T}_{FF}^\tau + \hat{T}_{FF}^{\tau'})\tilde{\mu}_{h,i}^F\right)_F$$

  for all $\tilde{\nu}_h^F \in \tilde{\Lambda}_h^F$ ($\hat{T}_{FF}^\tau$ is the Schur complement)
- $1 \leq \alpha_1^F \leq \alpha_2^F \leq \alpha_3^F \leq \dots$

Spectral Decomposition

- Choose $\alpha_{\text{stab}} \geq 1$
- Decompose $\tilde{\Lambda}_h^F := \tilde{\Lambda}_h^{F,\triangle} \oplus \tilde{\Lambda}_h^{F,\Pi}$ such that

$$\tilde{\Lambda}_h^{F,\triangle} := \text{span}\{\tilde{\mu}_{h,i}^F : \alpha_i^F < \alpha_{\text{stab}}\},$$
$$\tilde{\Lambda}_h^{F,\Pi} := \text{span}\{\tilde{\mu}_{h,i}^F : \alpha_i^F \geq \alpha_{\text{stab}}\}.$$

- Define

$$\tilde{\Lambda}_h^{\Pi} = \{\tilde{\mu}_h \in \widetilde{\Lambda}_h^f : \tilde{\mu}_h|_F \in \tilde{\Lambda}_h^{F,\Pi} \text{ for all } F \in \partial\mathcal{T}_H\},$$
$$\tilde{\Lambda}_h^{\triangle} = \{\tilde{\mu}_h \in \widetilde{\Lambda}_h^f : \tilde{\mu}_h|_F \in \tilde{\Lambda}_h^{F,\triangle} \text{ for all } F \in \partial\mathcal{T}_H\}.$$

Decompositions

Decompose: $\Lambda_h = \Lambda_h^{\text{const}} \oplus \widetilde{\Lambda}_h^{0,\Pi} \oplus \tilde{\Lambda}_h^{\triangle}$, where

- Λ_h^{const} - edge-wise constant fluxes (small dim.)
- $\widetilde{\Lambda}_h^{\Pi}$ - "high energy" fluxes (slow decay)
- $\widetilde{\Lambda}_h^{\triangle}$ - "low energy" fluxes (fast decay)

Define orthogonal projection: $P^{\triangle}T : \Lambda_h \to \widetilde{\Lambda}_h^{\triangle}$

- If λ_h has local support then $P^{\triangle}T\lambda_h$ decays exponentially
- define $P^{\triangle,j}T$ on patches
- $\tilde{\lambda}_h^{\triangle,j} = -P^{\triangle,j}(T\lambda^{\text{const}} + T\tilde{\lambda}^{0,\Pi,j})$
- get a method with size $\sim$ #edges
- contrast free

Error

In energy norm:

$$|u - u_h^{\mathsf{LSD},j}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \leq \mathscr{H} + cj^{2d}d^4\alpha^2_{\text{stab}}e^{-\frac{[(j-3)/2]}{1+d^2\alpha_{\text{stab}}}}(\mathscr{H}^2 + C^2_{P,G} + c_p^2 H^2)$$

where c_p, $C_{P,G}$ are the local and global Poincaré constants

Comments

- two components: $\mathscr{H}$ (target) and exponential part
- if $h \to 0$ then $\mathscr{H}$ decreases but costs increase
- if patch size j grows, error$\to \mathscr{H}$
- faster decay with smaller α_{stab}
- needs minimum regularity

High-contrast case

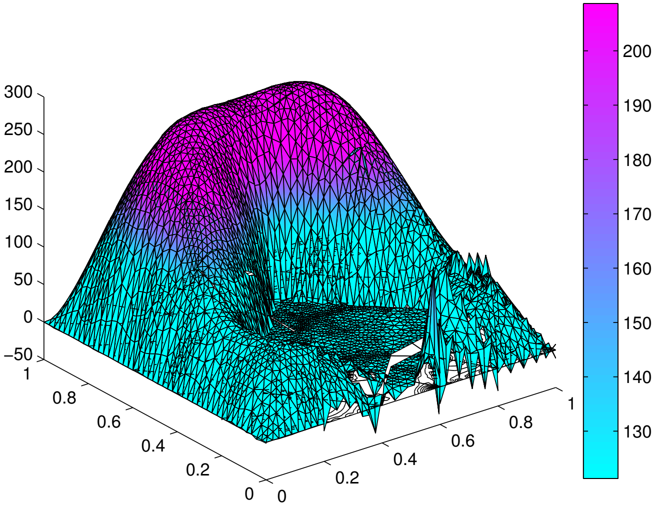

Figure: LSD solution with $\alpha_{\text{stab}} = 2$ and one patch

High-contrast case

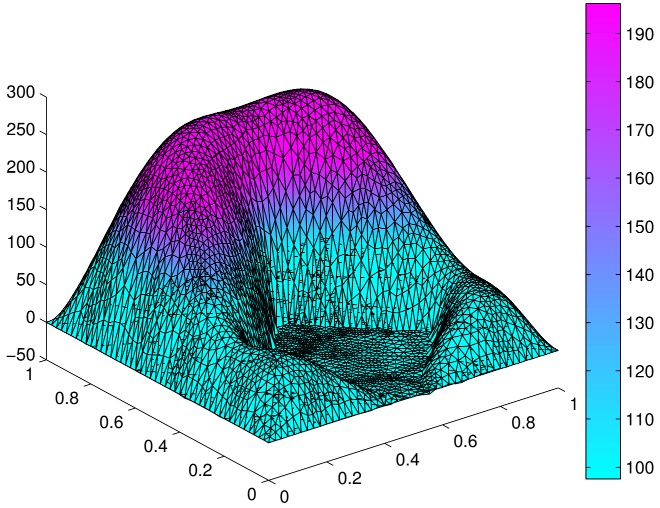

Figure: LSD solution with $\alpha_{\text{stab}} = 1.2$ and one patch

Contents

Finally

Final remarks

- method based on primal hybrid formulation
- needs minimum regularity
- discrete flux decomposition
- exponential decay allows localization of global problems
- the choice of spaces depend on an appropriate spectral decomposition

References

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Gracias!!

Thank you