Limit theorems for quadratic functionals of heavy-tailed long-memory processes

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Self-Similarity, Long-Range Dependence and Extremes

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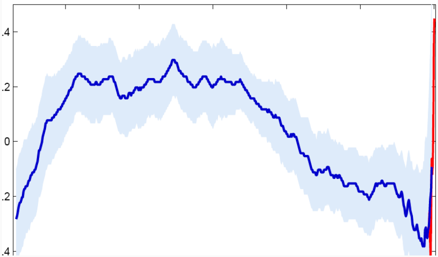


Joint work with
R. Lachièze-Rey (University Paris Decartes), M. Podolskij (Aarhus University),
T. Grønbæk (Aarhus University).

Estimation of variance: short-range dependence

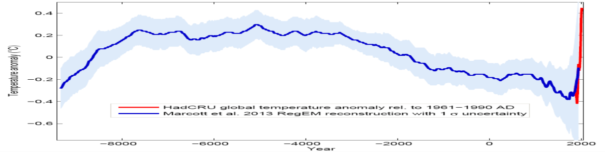
Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. with $\mathbb{E}[X_1] = 0$ and $\sigma^2 := \mathbb{E}[X_1^2] < \infty$, and set

$$QV_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

1. QV_n is the standard estimate of σ^2.
2. QV_n is consistent for σ^2 by the LLN,
   i.e. $QV_n \xrightarrow{\mathbb{P}} \sigma^2$.
3. QV_n is asymptotic normal if $\mathbb{E}[X_1^4] < \infty$ by the CLT,
   i.e. $\sqrt{n}(QV_n - \sigma^2) \xrightarrow{w} N(0, \rho^2)$.

The above properties also holds for many other short-range dependence models.

Long-range dependence data



The temperature on earth the last 10,000 years.

Theorem (The Birkhoff–Khinchin theorem)

If $(X_n)_{n\in\mathbb{N}}$ is a stationary sequence such that $\mathbb{E}[|X_1|] < \infty$. Then,

1. $$\frac{1}{n}\sum_{j=1}^{n} X_j \xrightarrow{a.s.} r.v.$$
2. *If $(X_n)_{n\in\mathbb{N}}$ is ergodic then*

$$\frac{1}{n}\sum_{j=1}^{n} X_j \xrightarrow{a.s.} \mathbb{E}[X_1].$$

Gaussian sequences with long-range dependence

If $(X_i)_{i\in\mathbb{N}}$ is a stationary Gaussian sequence with

$$\gamma(n) := \mathrm{Cov}(X_0, X_n) \sim cn^{2(H-1)} \qquad n \to \infty, \qquad H \in (0,1).$$

$H < 1/2 \qquad \longrightarrow \qquad$ short-range dependence,

$H > 1/2 \qquad \longrightarrow \qquad$ long-range dependence.

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$H < 1/2 \qquad \longrightarrow \qquad$ short-range dependence,

$H > 1/2 \qquad \longrightarrow \qquad$ long-range dependence.

$QV_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ is a consistent estimator for σ^2 by the Birkhoff–Khinchin ergodic theorem.

However, QV_n is not always asymptotic normal:

Theorem (Rosenblatt, Breuer and Major, Taqqu)

1. $H < 3/4$: $\quad \sqrt{n}(QV_n - \sigma^2) \xrightarrow{w} N(0, \rho^2)$.
2. $H > 3/4$: $\quad n^{2(1-H)}(QV_n - \sigma^2) \xrightarrow{w}$ *Rosenblatt r.v.*

For $H > 3/4$, we obtain a slower convergence rate and a non-Gaussian limit.

Quadratic variation of Gaussian processes with long-range dependence ✓

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Main question of the talk:

What is the behaviour of the quadratic variation of non-Gaussian (stable) processes with long-range dependence?

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What is the behaviour of the quadratic variation of non-Gaussian (stable) processes with long-range dependence?

First we will review some structural results of stationary processes.

Two subclasses of stationary stable processes

1. (X_n) is called a **moving average** if it is on the form
$$X_n = \int_{\mathbb{R}} \phi(n-s)\, dL_s$$
where (L_s) is a stable Lévy process, and $\phi : \mathbb{R} \to \mathbb{R}$ a deterministic function.
2. (X_n) is called a **harmonizable process** if it is on the form
$$X_n = \int_{\mathbb{R}} e^{ins}\, \Lambda(ds),$$
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Stationary Gaussian processes:

1. Any stationary process is harmonizable.
2. A stationary process is a moving average if and only if its spectral measure is absolutely continuous.

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Stationary non-Gaussian stable processes:

1. The class of moving averages and harmonizable processes are disjoint.
2. Moving averages are ergodic, harmonizable processes are not.

Self-similar processes

A process (Y_t) is **self-similar** if

"scaling of time equals scaling space in distribution", i.e. for some H

$$(Y_{at}) \stackrel{\mathcal{D}}{=} (a^H Y_t) \qquad \text{for all } a > 0.$$

The parameter H is called index of self-similarity or Hurst index.

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Gaussian processes: The only self-similar Gaussian processes with stationary increments are the fractional Brownian motions B^H, $H \in (0,1)$,

$$\begin{aligned} B_t^H &\stackrel{\mathcal{D}}{=} \int_{\mathbb{R}} \{(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\}\, dB_s && (\text{"moving average rep."}) \\ &\stackrel{\mathcal{D}}{=} \int_{\mathbb{R}} \frac{e^{its}-1}{is} |s|^{1/2-H}\, d\tilde{B}_s && (\text{"harmonizable rep."}) \end{aligned}$$

where (B_s) is a Brownian motion, and $(\tilde{B}_s)$ is a "$\mathbb{C}$-valued Brownian motion".

Non-Gaussian stable processes:

The class of self-similar stable processes with stationary increments is huge[1].

Key examples includes:

1. **linear fractional stable motion**

$$Y_t = \int_{\mathbb{R}} \{(t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}\}\, dL_s$$

where (L_s) is an α-stable Lévy process.

[1] V. Pipiras and M. Taqqu (2002). The structure of self-similar stable mixed moving averages. *Ann. Probab.* **30**.

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2. **harmonizable fractional stable motion**

$$Y_t = \int_{\mathbb{R}} \frac{e^{its}-1}{is} |s|^{1-H-1/\alpha}\, dL_s,$$

   where (L_s) is a $\mathbb{C}$-valued α-stable rotational invariante Lévy process.

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3. **Mittag–Leftler fractional stable motion**

$$Y_t = \int_{\mathbb{R}\times\Omega'} L_t^x(\omega')\, \Lambda(dx, d\omega'),$$

   where $(L_t^x)_{t\in\mathbb{R}_+, x\in\mathbb{R}}$ is the local time for a symmetric stable Lévy process defined on a *new* probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, and Λ is a symmetric α-stable random measure on $S = \mathbb{R}\times\Omega'$.

[1] V. Pipiras and M. Taqqu (2002). The structure of self-similar stable mixed moving averages. *Ann. Probab.* **30**.

A stationary sequence (X_n) is called a **fractional noise** if it is on the form

$$X_n = Y_n - Y_{n-1}$$

where (Y_t) is a self-similar process with stationary increments.

Key examples fractional noises:

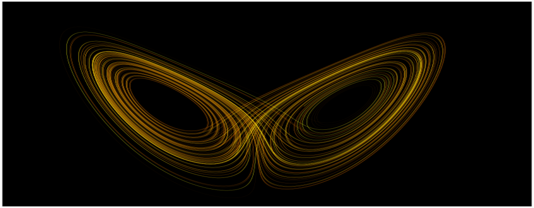
1. The linear fractional stable noise
2. The harmonizable fractional stable noise
3. The Mittag–Leftler fractional stable noise

The general structure of stationary stable processes

Rosiński (1995), *Ann. Probab.*:

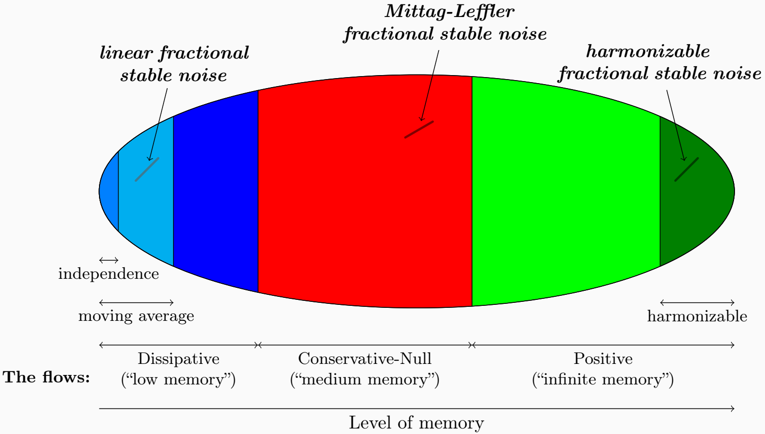
$(X_n)_{n\in\mathbb{N}}$ stationary stable $\longleftrightarrow$ Dynamic system (+ co-cycle)

Non-singular flow on a measure space



Dynamic system: Flow of the Lorenz ODE

Structure of stationary stable processes.



The flows for the two extremes

Moving average

The flows are translations on $\mathbb{R}$;

$\phi_n \colon x \mapsto x + n$.

$\lambda =$ Lebesgue measure.

Harmonizable processes

Identify flow on $\mathbb{R}$; $\phi_n \colon x \mapsto x$.

All dynamic properties are determined by the co-cycle.

$QV_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2$

Theorem (Gnedenko and Kolmogorov)

Let (X_j) be i.i.d. α-stable r.v. Then as $n \to \infty$,

$$n^{1-2/\alpha} QV_n \xrightarrow{w} Z_0,$$

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Theorem (B., Lachièze-Rey and Podolskij '17*)
Let (X_j) be the linear fractional α-stable noise. Then as $n \to \infty$,

$$n^{1-2/\alpha} QV_n \xrightarrow{w} Z_0,$$

where Z_0 is a totally right-skewed $\alpha/2$-stable r.v.

Remark:

1. The Birkhoff–Khinchin theorem do not apply due to $\mathbb{E}[X_j^2] = \infty$.
2. The proof is very different than the Gnedenko and Kolmogorov result, due the dependence.

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* *Ann. Probab.* 2017. Vol. 45

Important ingredients in the proof

1. The result is proved in the high-frequency setting (by self-similarity)

$$QV_n \stackrel{\mathcal{D}}{=} n^{1-2H} \sum_{j=1}^{n} (X_{i/n} - X_{(i-1)/n})^2$$

   where one can use a more "pathwise approach".

2. Rounding result of Tukey '38:

   Let

   2.1 Z be an absolutely continuous r.v.
   2.2 $\{x\} := x - \lfloor x \rfloor \in [0, 1)$ denote the fractional part of $x \in \mathbb{R}$.

   Then,

$$\{nZ\} \stackrel{w}{\to} U \sim \mathcal{U}([0, 1]).$$

1. Assume that L has only one jump occurring at a random time T, which has a density on the interval $(0,1)$.
2. Let j_n be the random index satisfying $T \in [(j_n-1)/n, j_n/n)$.
3. Observe that

$$X_{l/n} - X_{(l-1)/n} = \begin{cases} 0, & l < j_n \\ \Delta L_T \left(\left(\frac{j_n+l}{n} - T \right)_+^{H-1/\alpha} - \left(\frac{j_n+l-1}{n} - T \right)_+^{H-1/\alpha} \right), & l \geq j_n. \end{cases}$$

4. By Tukey '38 we obtain

$$n^{1/2-H}(X_{(j_n+l)/n} - X_{(j_n+l-1)/n}) \xrightarrow{w} \Delta L_T \left((l+U)_+^{\alpha} - (l-1+U)_+^{\alpha} \right), \qquad l \geq 0.$$

Thus,

$$n^{1-2H} \sum_{j=1}^{n} (X_{j/n} - X_{(j-1)/n})^2 \xrightarrow{w} |\Delta L_T|^p \sum_{l=0}^{\infty} |(l+U)_+^{\alpha} - (l-1+U)_+^{\alpha}|^p .$$

What happens when (L_t) has more than one jump?

For $\alpha \in (0,1)$ we can do rough estimates to allow (L_t) to jump more.

The $\alpha \in (1,2)$ case is more complicated:

1. We need precise conditions for when a stochastic $(X_t)_{t\in T}$ has finite supremum: $\sup_{t\in T} |X_t| < \infty$ a.s.
2. A Gaussian process $(X_t)_{t\in T}$ has finite supremum if and only if there exists a majorizing measure for the metric space (T, d) with $d(s,t) = \|X_s - X_t\|_{L^2}$, i.e. for each probability measure μ we have
$$\mathbb{E}\Big[\sup_{t\in T} |X_t|\Big] \leq K \sup_{t\in T} \int_0^D \sqrt{\log \frac{1}{\mu(B(t,\epsilon))}}\, d\epsilon.$$
3. For our proof we use the majorizing measure techniques of Marcus and Rosiński* for infinitely divisible processes to show boundedness of a family of random variables $(R_{i,n})_{i,n\in\mathbb{N}}$.

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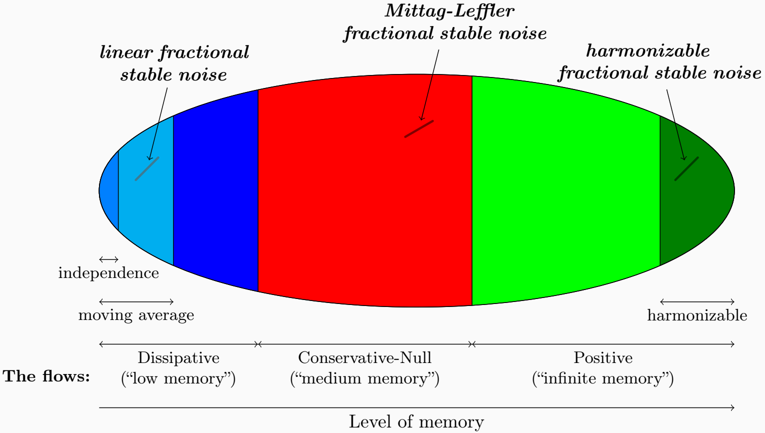
*M. Marcus and J. Rosiński (2005). *Journal of Theoretical Probability* **18**.

Structure of stationary stable processes.



What is the quadratic variation of the other extreme?

Infinitely divisible harmonizable processes

Let (X_j) be a Lévy driven harmonizable process of the form

$$X_j = \int_{\mathbb{R}} e^{ijs} g(s)\, dL_s, \qquad QV_n = \frac{1}{n} \sum_{j=1}^{n} \|X_j\|^2,$$

where $(L_t)_{t\in\mathbb{R}}$ is a rotational invariant Lévy process indexed by $\mathbb{R}$.

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Theorem (B., Podolskij and Grønbæk)

As $n \to \infty$ we have

$$QV_n \xrightarrow{\mathbb{P}} U_0$$

where U_0 is an infinitely divisible r.v. of the form

$$U_0 = \int_{\mathbb{R}} |g(s)|^2 d([L^1]_s + [L^2]_s).$$

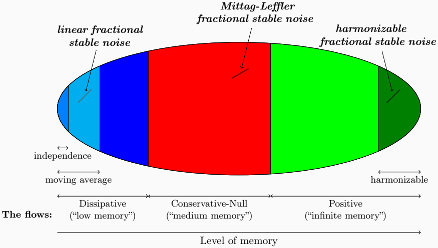
Corollary (B., Podolskij and Grønbæk)

Let (X_j) be the harmonizable fractional α-stable noise. Then as $n \to \infty$,

$$QV_n \xrightarrow{\mathbb{P}} U_0$$

where U_0 is a totally right-skewed $\alpha/2$-stable r.v.

Quadratic variation of α-stable processes: $QV_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2$



i.i.d.-case and

the linear fractional stable noise

Normalization factor: $n^{1-2/\alpha}$

Convergence form: in law

Limiting distribution: $\alpha/2$-stable

harmonizable fractional stable noise

Normalization factor: non

Convergence form: in probability

Limiting distribution: $\alpha/2$-stable

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1. The heavy dependence structure of harmonizable processes has great impact, even on the first order asymptotic theory.
2. "The harmonizable stable noise behaves as if it was integrable."

Key ideas of the proof:

Let $X_n = \int_{\mathbb{R}} e^{ins} g(s)\, dL_s$ be a harmonizable process.

Key decomposition:

$$\|X_n\|^2 = U_0 + V_n,$$

where

1. U_0 is a positive infinitely divisible r.v. **not depending on** n
2. V_n is a second-order multiple integral of the form

$$V_n = 2\Re\Big(\int_{\mathbb{R}} \int_{-\infty}^{s-} e^{in(s-u)} g(s)\overline{g(u)}\, d\overline{L_u}\, dL_s \Big).$$

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Hence,

$$QV_n = U_0 + \frac{1}{n}\sum_{j=1}^{n} V_j.$$

We show that $\frac{1}{n}\sum_{j=1}^{n} V_j \xrightarrow{\mathbb{P}} 0$, which finish the proof.

The proof uses the Kallenberg and Szulga (1989)* theory for multiple infinitely divisible integrals. (Remark that $V_n \not\to 0$ in probability.)

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* Probab. Theory Related Fields 83(1-2)

Convergence rate for Q_n: The $H > 3/4$ case

Theorem (B., Podolskij and Grønbæk)

Let (X_j) be the harmonizable fractional α-stable noise.

For $H > 3/4$, we have as $n \to \infty$,

$$n^{2(1-H)}\Big(QV_n - U_0\Big) \xrightarrow{w} R_\alpha,$$

where

$$R_\alpha = 2\Re\Big(\int_{\mathbb{R}^2} \frac{e^{i(s-u)} - 1}{i(s-u)} |su|^\gamma \, d\overline{L}_u \, dL_s\Big).$$

Remark:

1. We call R_α a stable Rosenblatt r.v., due to its similarities with the spectral representation of the standard Rosenblatt r.v.
2. The convergence rate $n^{2(1-H)}$ is the same as in the Gaussian case.

Proof

By the "key decomposition" we have

$$QV_n - U_0 = 2\Re\Big(\int_{\mathbb{R}}\int_{-\infty}^{s-}\Big(\frac{1}{n}\sum_{j=1}^{n} e^{in(s-u)}\Big)|su|^{\gamma}\, d\overline{L_u}\, dL_s\Big)$$
$$= 2\Re\Big(\int_{\mathbb{R}}\int_{-\infty}^{s-}\Big(\frac{1-e^{in(s-u)}}{n(1-e^{i(s-u)})}\Big)|su|^{\gamma}\, d\overline{L_u}\, dL_s\Big),$$

which is used to show

$$n^{2(1-H)}\Big(QV_n - U_0\Big) \xrightarrow{w} R_\alpha.$$

Back to the linear fractional stable noise: a second order limit theory?

Theorem (B., Lachièze-Rey and Podolskij '17)
Let (X_j) be the linear fractional α-stable noise. Then as $n \to \infty$,

$$n^{1-2/\alpha} QV_n \xrightarrow{w} Z_0, \tag{1}$$

where Z_0 is a totally right-skewed $\alpha/2$-stable r.v.

1. Since convergence in probability do not hold for (1) we can not obtain a second theory, contrarily to harmonizable processes.
2. How can we avoid this situation?

Power variation

1. For $p > 0$ consider the power variation

$$V(p)_n = \frac{1}{n} \sum_{j=1}^{n} |X_j|^p,$$

   and note $QV_n = V(2)_n$.

2. Since any moving average is ergodic the Birkhoff–Khinchin theorem implies:

3. *Let (X_j) be the linear fractional stable noise and $p < \alpha$. Then,*

$$V(p)_n \xrightarrow{a.s.} \mathbb{E}[|X_1|^p].$$

4. What is the convergence rate for $V(p)_n$?

"Classical" results of the form

$$a_n \sum_{j=1}^{n} Y_j \xrightarrow{\mathcal{D}} U \qquad n \to \infty,$$

where $(Y_i)_{i\geq 1}$ is a stationary sequence which satisfies one of the following

1. i.i.d.
2. martingale difference
3. Markov chain
4. strongly mixing

are *never* applicable.

Theorem (Breuer–Major [1], **Taqqu** [2]**)**

Suppose that X is the fractional Gaussian noise Hurst index $H \in (0,1)$.

(i) *For $H \in (0, 3/4)$,*

$$\sqrt{n}\Big(V(p)_n - \mathbb{E}[|X_1|^p]\Big) \xrightarrow{w} \mathcal{N}(0, v_p).$$

(ii) *When $H \in (3/4, 1)$ it holds that*

$$n^{2(1-H)}\Big(V(p)_n - \mathbb{E}[|X_1|^p]\Big) \xrightarrow{w} \textit{Rosenblatt r.v.}$$

Remark: The asymptotics for $V(p)_n$ is analogue to that of QV_n.

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[1] Breuer and Major (1983). *Journal of Multivariate Analysis* 13.

[2] Taqqu (1979). *Z. Wahrsch. Verw. Gebiete* 50.

The non-Gaussian case

Theorem (B., Lachièze-Rey and Podolskij)

Suppose that (X_j) is the k-order linear fractional stable noise with Hurst index $H \in (1/\alpha, k)$. Let $p < \alpha/2$.

(a): For $H < k - 1/\alpha$, we obtain

$$\sqrt{n}\Big(V(p)_n - \mathbb{E}[|X_1|^p]\Big) \xrightarrow{w} \mathcal{N}(0, v^2).$$

(b): For $H > k - 1/\alpha$, it holds that

$$n^{\frac{(k-H)\alpha}{(k-H)\alpha+1}}\Big(V(p)_n - \mathbb{E}[|X_1|^p]\Big) \xrightarrow{w} S_{(k-H)\alpha+1}$$

where $S_{(k-H)\alpha+1}$ is a totally right skewed $((k-H)\alpha+1)$-stable random variable with mean zero.

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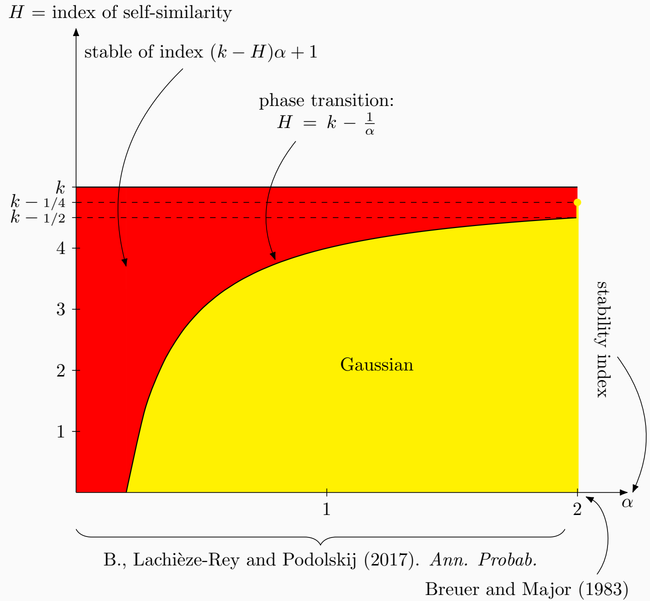
where $S_{(k-H)\alpha+1}$ is a totally right skewed $((k-H)\alpha+1)$-stable random variable with mean zero.

Remark: For $\alpha > 1$, case (a), also follows by

V. Pipiras and M. Taqqu, and P. Abry (2007). *Bernoulli* **13**.

Stochastic fluctiation of the power variation of the k-order linear fractional stable noise



Thank you for your attention!