Scale invariant set functions arising from general iterative schemes

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Iterated random functions

- Let f_n, $n \geq 1$, be i.i.d. random Lipschitz functions on a Polish space.
- Let L denote the (random) Lipschitz constant of a generic function f.
- Forward iterations

$$\xi_n = f_n(f_{n-1}(\cdots f_1(z_0)\cdots)), \quad n \geq 1,$$

  build a Markov chain.
- Backward iterations

$$\xi_n = f_1(f_2(\cdots f_n(z_0)\cdots)), \quad n \geq 1,$$

  converge almost surely if $\mathbf{E}L < \infty$, $\mathbf{E}\log L < 0$, and

$$\mathbf{E}\rho(f(z_0), z_0) < \infty$$

  for some z_0, see Diaconis & Freedman (1999).

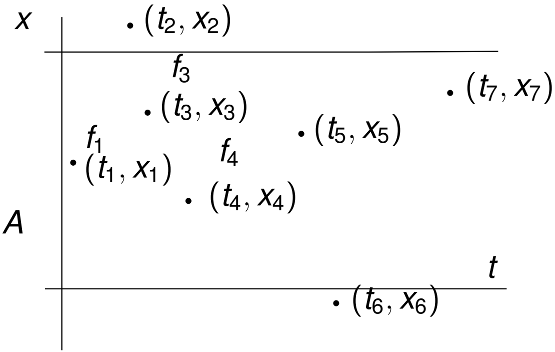
Sieving the functions

- Leave some functions out.
- Let $\{(t_i, x_i, f_i), i \geq 1\}$ be a Poisson process on $\mathbb{R}_+ \times \mathbb{R}_+$ of intensity $dt \otimes \mu$, independently marked by i.i.d. random Lipschitz functions.
- Consider $\{(t_{i_k}, x_{i_k}, f_{i_k}) : \ k \geq 1, x_{i_k} \in A\}$ and fix z_0.
- The backward iterations restricted to $x_i \in A$

  $$\zeta_n(A) = f_{i_1}(\cdots f_{i_n}(z_0)\cdots) \to \zeta(A)$$

  a.s. as $n \to \infty$.
- $\zeta(A)$ is a random set-function, often $A = [0, x]$.

Leaving some functions out



$$f_1(f_3(f_4(f_5(f_7(\cdots\cdots)))))$$

Properties of the limit

- The distribution of $\zeta(A)$ does not depend on A if $\mu(A) \in (0, \infty)$.
- The values of ζ on disjoint sets are independent.
- The process $\zeta_x = \zeta([0, x])$, $x > 0$, is scale invariant; the same holds for $\zeta(A)$ as function of A, that is,

$$(\zeta(A_1), \ldots, \zeta(A_m)) \overset{d}{\sim} (\zeta(cA_1), \ldots, \zeta(cA_m))$$

for all $c > 0$.

Continuity properties

Theorem

- *If $A_n \uparrow A$, then $\zeta(A_n) \to \zeta(A)$ a.s.*
- *If $A_n \downarrow A$, then $\zeta(A_n) \to \zeta(A)$ a.s.*

Proof.

Let (t_{i_k}, x_{i_k}) are such that $x_{i_k} \in A$, and let

$$N_n = \inf\{k : \; x_{i_k} \notin A_n\}.$$

Then

$$\zeta(A_n) = f_{i_1} \circ \cdots \circ f_{i_{N_n - 1}}(z_n)$$
$$\zeta(A) = f_{i_1} \circ \cdots \circ f_{i_{N_n - 1}}(z_n').$$

Note that $N_n \uparrow \infty$. □

Scale-invariant process

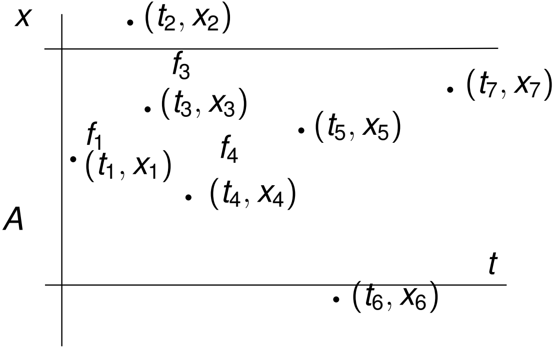
Theorem

The process $\zeta_x = \zeta([0, x])$ is continuous at any fixed x, is càdlàg and not pathwise continuous.

Proof.

If $x_n \downarrow x$ and $x'_n \uparrow x$, then $\zeta([0, x_n]) \downarrow \zeta([0, x])$ and $\zeta([0, x'_n]) \uparrow \zeta([0, x)) = \zeta([0, x])$ a.s.

Discontinuous at the point x_i with the smallest t_i. □



x
(t_2, x_2)
f_3
(t_7, x_7)
(t_3, x_3)
(t_5, x_5)
f_1
f_4
(t_1, x_1)
(t_4, x_4)
A
t
(t_6, x_6)

Decomposition by the first entry point

- Consider two sets A_1 and A_2.
- Let (t_*, x_*, f_*) be such that t_* is the smallest for all $x_i \in (A_1 \cup A_2)$. Then

$$(\zeta(A_1), \zeta(A_2)) \overset{d}{\sim} (f_*(\zeta(A_1)), f_*(\zeta(A_2)))\mathbf{1}_{\{x_* \in A_1 \cap A_2\}} + (\zeta(A_1), f_*(\zeta(A_2)))\mathbf{1}_{\{x_* \in A_2 \setminus A_1\}} + (f_*(\zeta(A_1)), \zeta(A_2))\mathbf{1}_{\{x_* \in A_1 \setminus A_2\}}.$$

- If $A_1 = [0, x]$ and $A_2 = [0, y]$ with $y \geq x$, then

$$y\mathbf{E}(\zeta_x \zeta_y) = x\mathbf{E}(f(\zeta_x)f(\zeta_y)) + (y - x)\mathbf{E}(\zeta_x f(\zeta_y)).$$

  where $\zeta_x = \zeta([0, x])$ and $\zeta_y = \zeta([0, y])$.

Finite interval

- Assume $x \in [0, 1]$ and consider

$$f_1(f_2(f_3(\cdots)))$$

- Let $\{U_n, n \geq\}$ be i.i.d. uniform. For each $x \in (0, 1]$, leave in the iteration the functions with $U_i \leq x$.
- The process ζ_x, $x \in (0, 1]$, satisfies

$$\zeta_x \overset{d}{\sim} (f(\zeta_x)\mathbf{1}_{\{U \leq x\}} + \zeta_x \mathbf{1}_{\{U > x\}}), \quad x \in (0, 1].$$

- Equivalently, possible to modify the iteration as

$$\zeta_x \overset{d}{\sim} \begin{cases} f(\zeta_x) & \text{if } \ x \leq U \\ \zeta_x & \text{oherwise} \end{cases}$$

being an iteration that acts on functions.

Example: perpetuities

- Let $f(z) = Mz + Q$.
- Converges if $\mathbf{E}\log|M| \in (-\infty, 0)$ and $\mathbf{E}\log^{+}|Q| < \infty$, see Goldie & Maller (2000).
- Assume $\mathbf{E}|M| < 1$. Then $\zeta_x = \zeta([0, x])$ satisfies

$$\mathbf{E}\zeta_x = \frac{\mathbf{E}Q}{1 - \mathbf{E}M}, \quad \mathbf{E}(\zeta_x\zeta_y) = \frac{x\mathbf{E}Q^2}{(1 - \mathbf{E}M)y + (\mathbf{E}M - \mathbf{E}M^2)x}$$

- Thus, $\tilde{\zeta}_s = \zeta([0, e^s])$, $s \in \mathbb{R}$, is a stationary process with the covariance

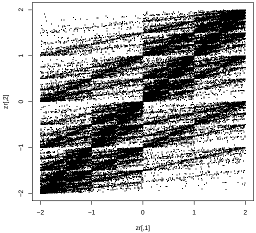
$$\mathbf{E}(\tilde{\zeta}_s\tilde{\zeta}_0) = \frac{a}{ce^{|s|} + 1}$$

Bernoulli convolutions

- Consider $f(z) = \frac{1}{2}z + Q$, where Q equally likely takes values ± 1.
- Then ζ_x is uniformly distributed on $[-2, 2]$ for all x, and

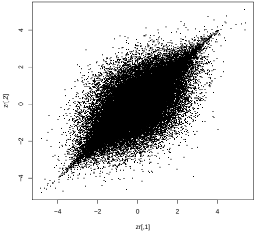
$$\mathbf{E}(\zeta_x \zeta_y) = \frac{4x}{2y + x}, \quad \mathbf{E}(\tilde{\zeta}_s \tilde{\zeta}_0) = \frac{4}{2e^{|s|} + 1}.$$

- The joint distributions are of the fractal type, e.g. $(\zeta_{0.7}, \zeta_1)$

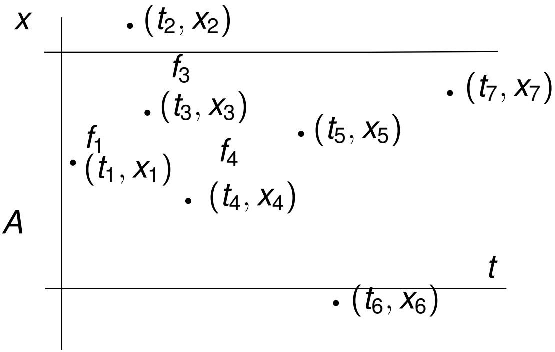


Gaussian additive term

- Consider $f(z) = \lambda z + Q$ with deterministic λ and Gaussian Q.
- Then ζ_x is Gaussian for all x.
- The covariance are similar to the case of Bernoulli convolutions.
- The joint distributions are not Gaussian, e.g. $(\zeta_{0.7}, \zeta_1)$, $\lambda = 1/2$:



Leaving some functions out



$$f_1(f_3(f_4(f_5(f_7(\cdots\cdots)))))$$

Interpretation in terms of empirical cdf

- Consider $f(z) = \lambda z + Q$.
- Then

$$\begin{aligned}\zeta_x &= \sum_{n=1}^{\infty} \lambda^{\mathbf{1}_{\{U_1 \le x\}} + \cdots + \mathbf{1}_{\{U_{n-1} \le x\}}} Q_n \mathbf{1}_{\{U_n \le x\}} \\ &= \sum_{n=1}^{\infty} \lambda^{(n-1)\widehat{F}_{n-1}(x)} Q_n \mathbf{1}_{\{U_n \le x\}},\end{aligned}$$

Self-decomposability

- If $\{\Gamma_i, i \geq 1\}$ is Poisson process on $(0, \infty)$, and $\{\varepsilon_i\}$ are i.i.d., then
$$\zeta = \sum_i e^{-\Gamma_i} \varepsilon_i$$
is self-decomposable.
- It appears from iterating $f(z) = Mz + \varepsilon$ for the uniformly distributed M, so that $M = e^{-\xi}$.
- After sieving, on $[0, 1]$,
$$\zeta_x = \sum_i e^{-\xi_1 \mathbf{1}_{U_1 \leq x} - \cdots - \xi_i \mathbf{1}_{U_i \leq x}} \varepsilon_{i+1} \mathbf{1}_{U_{i+1} \leq x} = \int_0^\infty e^{-t} dL_x(t),$$
where, for every fixed $x \in (0, 1]$, $L_x : \mathbb{R}_+ \to \mathbb{R}$ is a Lévy process.

Construction of processes by iterations

The sieving machinery can be applied to any iterative scheme that almost surely converges, so for backwards iterations.

$$f_1(f_2(f_3(\cdots(z_0)\cdots)))$$

Example: numerical continued fractions

- Let $f(z) = 1/(z + \xi)$, $z > 0$, where ξ is Gamma distributed.
- We obtain a continued fraction.
- Then ζ_x has the inverse Gaussian distribution for each $x > 0$, see Letac & Seshadri (1983).
- Generalisation: products of random matrices.

Continued fractions with the Gamma process

- Let ξ_t, $t \geq 0$, be the Gamma process, and let $\xi_t^{(i)}$ be its independent copies.
- Construct continued fraction

$$\zeta_t = \frac{1}{\xi_t^{(1)} + \frac{1}{\xi_t^{(2)} + \frac{1}{\xi_t^{(3)} + \cdots}}}$$

  Then ζ_t has inverse Gaussian distribution for each $t > 0$ (with parameter depending on t), but no independent increments.
- Joint distributions

$$\begin{cases} \frac{1}{\zeta_t} \overset{d}{\sim} \zeta_t + \xi_t \\ \frac{1}{\zeta_{t+s}} \overset{d}{\sim} \zeta_{t+s} + \xi_t + \gamma \end{cases}$$

  where $\gamma = \xi_{t+s} - \xi_t$.

Iterating Poisson processes

- Let N_t, $t \geq 0$, be the Poisson process.
- Then

$$\zeta_t = \cfrac{1}{N_t^{(1)} + \cfrac{1}{N_t^{(2)} + \cfrac{1}{N_t^{(3)} + \cdots}}}$$

  is a Markov process.
- Reason: from the value of ζ_t it is possible to recover all $N_t^{(i)}$.

Min-max

- Let $f(z) = \min(z, \xi)$ or $f(z) = \max(z, \xi)$ with some probabilities and a random variable ξ.
- Was discussed in 2012 with Bernardo D'Auria and Sid Resnick.
- However f has the Lipschitz constant $L = 1$; this does not suffice for the a.s. convergence of the backward iterations
- One has the convergence in distribution for forward iterations.

Sieving forward iterations

- The same sieving idea can be applied to forward iterations.
- Recursion: if $A_1 \subset A_2$, then

$$(\zeta(A_1), \zeta(A_2)) \overset{d}{\sim} \begin{cases} (f(\zeta(A_1)), f(\zeta(A_2))), & x \in A_1, \\ (\zeta(A_1), f(\zeta(A_2))), & x \in A_2 \setminus A_1. \end{cases}$$

Distributions of random sets

- There is a shortage of distributions of random sets.
- One can try to obtain new distributions by applying iterative schemes.
- There are natural scale-invariant random closed sets, e.g. $\{t : \; w_t = 0\}$ – zero set of the Wiener process.

Random fractals

- Iterated function system: $S_1, \ldots, S_k$, and

$$K = \bigcup_{i=1}^{k} S_i(K).$$

- For example, the Cantor set appears if $S_1(z) = z/3$ and $S_2(z) = (z+2)/3$.
- Let $f(\cdot)$ be the Lipschitz map on the space of compact sets, such that $f(K)$ is $S_1(K)$, $S_2(K)$, or $S_1(K) \cup S_2(K)$ with equal probabilities.
- The limit is a random fractal set, where at each step, one deletes the mid third, the first two-third or the last two-third with equal probabilities.
- Sieving produces a set-valued process of this type.

Set-valued perpetuities

- Let
$$f(Z) = MZ + Q,$$
where $M > 0$, and Z, Q are convex bodies.
- The limit provides a set-valued process with self-decomposable (for Minkowski sums) univariate distributions.
- Set-valued autoregression:
$$X_n = e^{-\beta_n} X_{n-1} + Q_n, \quad n \geq 1.$$

Set-valued continued fractions

- Let X_0 be any random (or deterministic) convex body.
- Let Y_n, $n \geq 1$, be a sequence of i.i.d. random convex bodies distributed as Y.
- Define
$$X_{n+1} = (X_n + Y_{n+1})^o,$$
where
$$K^o = \{u : \ h_K(u) \leq 1\}$$
is the polar body to K.
- If $Y_n = [0, \xi_n]$ in $\mathbb{R}$, one obtains conventional continued fractions.
$$\frac{1}{Y_3 + \frac{1}{Y_2 + \frac{1}{Y_1}}} \quad \text{cf} \quad \frac{1}{Y_1 + \frac{1}{Y_2 + \frac{1}{Y_3}}}$$
- Deterministic set-valued continued fractions: IM (2016).

Almost sure convergence: backward iterations

Theorem (IM 2016)

$\rho_H(K^o, L^o) \leq \max(\|K^o\|, \|L^o\|)^2 \rho_H(K, L)$.

Theorem

Assume that $Y \supset B_\zeta$ *with* $\zeta > 0$ *a.s. and*

$$\mathbf{E}\zeta^{-2} < \infty, \quad \mathbf{E}\log\zeta > 0.$$

Then the backwards iterations converge almost surely.

Convergence in distribution: forward iterations

- The Markov chain X_n, $n \geq 0$, is obtained by iteration of monotone transformations.

Theorem

Assume that X_n is a.s. compact, contains a neighbourhood of the origin for all sufficiently large n, and

$$\delta_1 = \mathbf{P}\{X_{2k-1} \subset rB\} > 0$$

for some $r < 1$ and $k \geq 1$ and that

$$\delta_2 = \mathbf{P}\{Y_1 \subset (r^{-1} - r)B\} > 0 \,.$$

Then X_n converges in distribution to a random convex body X which a.s. contains a neighbourhood of the origin and satisfies $X^o \overset{d}{\sim} X + Y$.

Example

- Y_1 a.s. contains a neighbourhood of the origin, and

$$\mathbf{P}\{Y_1 \supset rB\} > 0,$$
$$\mathbf{P}\{Y_1 \subset (r - r^{-1})B\} > 0$$

  for some $r > 1$.
- $Y_1, Y_2, \ldots$ are i.i.d. segments in the plane such that $Y_1 + Y_2$ a.s. contains a neighbourhood of the origin.