On the tail behavior of a class of multivariate conditionally heteroskedastic processes

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Motivations

Multivariate GARCH models

- GARCH models are very popular in econometrics to deal with extremes.
- The univariate GARCH models are well understood thanks to Kesten's theory, see Buraczewski et al. (2016).
- What about the most multivariate extensions?

Few literature

CCC-GARCH are treated in Starica (1999), factor-GARCH in Basrak and Segers (2009). For all these models the tails of the margins are asymptotically equivalent and regularly varying (multivariate regularly varying).

What about the popular BEKK-GARCH models?

Existence of a stationary solution

The BEKK-ARCH Process

Definition (BEKK-ARCH (or BEKK$(1, 0, n)$), Engle and Kroner (1995))

Let $X_t \in \mathbb{R}^d$ satisfying

$$\begin{aligned} X_t &= H_t^{1/2} Z_t, \\ H_t &= C + \sum_{i=1}^{n} A_i X_{t-1} X_{t-1}^\intercal A_i^\intercal, \end{aligned}$$

with $Z_t \sim i.i.d.N(0, I_d)$, C a $d \times d$ positive definite matrix, $A_1, \ldots, A_n \in \mathbb{R}^{d \times d}$.

- Includes Scalar BEKK: $n = 1$ and $A_1 = aI_d$ with $a \in \mathbb{R}$.
- Includes Diagonal BEKK: $n = 1$ and A_1 diagonal.
- Contains the case of stacked independent univariate ARCH(1) processes: Let $n = d$, C diagonal, and $A_i = a_i e_i e_i^\intercal$ with $a_i \in \mathbb{R}$, $i = 1, \ldots, d$.

The SRE Representation

Recall that

$$
\begin{aligned}
X_t &= H_t^{1/2} Z_t, \quad Z_t \sim i.i.d. N(0, I_d), \\
H_t &= C + \sum_{i=1}^{n} A_i X_{t-1} X_{t-1}^{\intercal} A_i^{\intercal}.
\end{aligned}
$$

Remark (Caporin & Mc Aleer, 2008)

Exploiting that Z_t is Gaussian, we obtain the stochastic recurrence equation (SRE) representation for X_t:

$$X_t = M_t X_{t-1} + Q_t,$$

with

$$M_t = \sum_{i=1}^{n} m_{it} A_i$$

and $(m_{it} : t \in \mathbb{Z})$ is an *i.i.d.* process mutually independent of $(m_{jt} : t \in \mathbb{Z})$ for $i \neq j$, with $m_{it} \sim N(0, 1)$. Moreover $(Q_t : t \in \mathbb{Z})$ is an *i.i.d.* process with $Q_t \sim N(0, C)$ mutually independent of $(m_{it} : t \in \mathbb{Z})$ for all $i = 1, ..., n$.

Geometric Ergodicity

Recall that

$$X_t = M_t X_{t-1} + Q_t, \quad Q_t \sim i.i.d.N(0, C),$$
$$M_t = \sum_{i=1}^{n} m_{it} A_i, \quad m_{it} \sim i.i.d.N(0, 1).$$

Exploiting the SRE representation we obtain the following result:

Theorem

Let $(X_t : t = 0, 1, ...)$ be a BEKK-ARCH process. Suppose that

$$\inf_{k \in \mathbb{N}} \left\{ \frac{1}{k} E\left[\log \left(\left\| \prod_{t=1}^{k} M_t \right\| \right) \right] \right\} < 0.$$

Then $(X_t : t = 0, 1, ...)$ is geometrically ergodic, and for the associated stationary solution, $E[\|X_t\|^s] < \infty$ for some $s > 0$.

Note the following special case:

When $n = 1$, the condition corresponds to

$$\rho(A_1)^2 < 3.56...,$$

which is similar to Nelson's stationarity condition for univariate ARCH(1).

Tail properties

Tail properties

Recall that

$$X_t = M_t X_{t-1} + Q_t, \quad Q_t \sim i.i.d.N(0, C),$$
$$M_t = \sum_{i=1}^{n} m_{it} A_i, \quad m_{it} \sim i.i.d.N(0, 1).$$

In order to determine the tail behavior of X_t, we exploit the SRE representation and apply existing results for $\mathbb{R}^d$-valued SREs:

- Kesten's theory for SRE's satisfying certain irreducibility and density conditions (**ID BEKK**). (Alsmeyer and Mentemeier, 2012):
  Essentially, M_t should have a suitable Lebesgue density. which is strictly positive in a neighborhood around I_d.
- Results of Buraczewski et al. (2009), where M_t is a similarity (**Similarity BEKK**). This includes scalar BEKK.

Under the stationarity condition, X_t is multivariate regularly varying with index $\alpha > 0$.

Tail properties

Note that:

- The regular variation is in the Kesten sense, each component of X_t has the same tail index, $\alpha > 0$:
  For $i = 1, ..., d$, $P(X_{t,i} > x) \sim c_i x^{-\alpha}$ as $x \to \infty$ for some constant $c_i > 0$.
- The ID and Similarity BEKK processes are not that interesting from an empirical point of view.

What can be said about the Diagonal BEKK processes?

Tail properties: Diagonal BEKK

Consider the Diagonal BEKK process:

$$\begin{aligned} X_t &= M_t X_{t-1} + Q_t, \quad Q_t \sim i.i.d.N(0, C), \\ M_t &= m_t A, \quad m_t \sim i.i.d.N(0,1), \end{aligned}$$

where A is diagonal with non-zero diagonal elements, $A_{11}, ..., A_{dd} > 0$.

Theorem (Goldie (1991))

Then each marginal of X_t satisfies the SRE

$$X_{t,i} = m_t A_{ii} X_{t-1,i} + Q_{t,i}, \quad i = 1, ..., d.$$

It holds that

$$P(X_{t,i} > x) \sim c_i x^{-\alpha_i}, \quad x \to \infty,$$

with $c_i > 0$ and $\alpha_i > 0$ depending on A_{ii}.

Hence, in general the tail indices differ along the components of X_t!

Asymptotic independence tails

Consider diagonal terms A_{ii} that are different so that

$$X_{t,i} = m_t A_{ii} X_{t-1,i} + Q_{t,i}, \quad i = 1, ..., d.$$

satisfies

$$P(X_{t,i} > x) \sim c_i x^{-\alpha_i}, \quad x \to \infty$$

for different α_i.

Theorem (Mentemeier & W.)

For any $i \neq j$ we have

$$\lim_{x \to \infty} u \, \mathbb{P}\big(X_{t,i} > x^{1/\alpha_1}, \; X_{t,j} > x^{1/\alpha_2}\big) \; = \; 0.$$

Thus X_t is non-standard regularly varying with spectral measure degenerate on the axis $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$.

Tail properties: ID BEKK

Suppose that $d = 2$ and

$$X_t = M_t X_{t-1} + Q_t, \quad Q_t \sim i.i.d. N(0, C),$$
$$M_t = \sum_{i=1}^{4} A_i m_{it}, \quad m_{it} \sim i.i.d. N(0, 1),$$

with

$$A_1 = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 \\ a_2 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & a_3 \\ 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 \\ 0 & a_4 \end{pmatrix},$$

and

$$a_1, a_2, a_3, a_4 \neq 0.$$

Under the stationarity condition X_t is multivariate regularly varying with $\alpha > 0$.

Tail properties: Similarity BEKK

Suppose that

$$X_t = M_t X_{t-1} + Q_t, \quad Q_t \sim i.i.d.N(0, C),$$
$$M_t = m_{1t} A_1, \quad m_{1t} \sim i.i.d.N(0,1),$$

where $A_1 = aO$, with $a > 0$ and O an orthogonal matrix.

Then M_t is a similarity with probability one.

Under the stationarity condition, X_t is multivariate regularly varying with $\alpha > 0$.

Tail properties: Diagonal BEKK

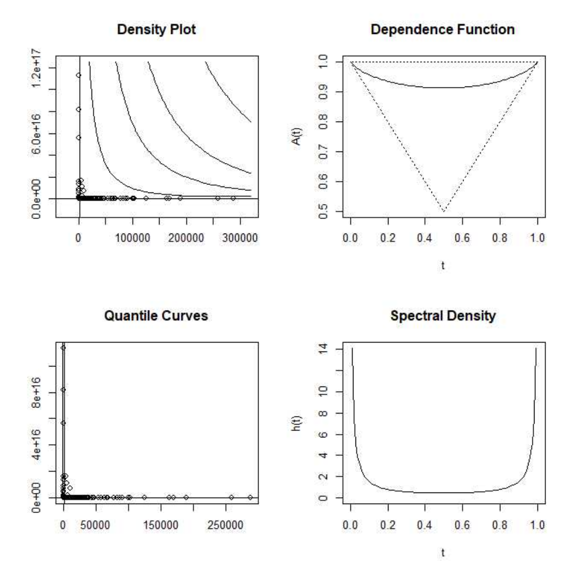
Suppose that

$$X_t = M_t X_{t-1} + Q_t, \quad Q_t \sim i.i.d. N(0, C),$$
$$M_t = m_{1t} A_1, \quad m_{1t} \sim i.i.d. N(0, 1),$$

where A_1 is a diagonal matrix with distinct coeffcients on the diagonal.

Under the stationarity condition, X_t is non-standard regularly varying with different tails indices α_i.

$$A_{11} = 1, \quad A_{22} = 2; \quad \Longrightarrow \quad \alpha_1 = 1, \quad \alpha_2 \approx 0.3102022477.$$



VSRV and tail chain

Vector scaling regular variation (VSRV)

We introduce the notion of vector scaling regular variation (VSRV):

Definition: VSRV

Let $X_t \in \mathbb{R}^d$. Suppose that

- for some $\alpha_i > 0$, $c_i > 0$, $P(|X_{t,i}| > x) \sim c_i x^{-\alpha_i}, \quad x \to \infty, \quad$ for $i = 1, .., d$,
- X_t is non-standard regularly varying in the sense of Resnick (2007).

Then the distribution of X_t is said to be VSRV.

Remarks:

- The "vector scaling" is due to the non-standard regular variation:
  There exists $(x(s) : s \geqslant 0)$ with $x(s) := (x_1(s), \dots, x_d(s))^\intercal \in \mathbb{R}^d$ and a Radon measure μ with non-null marginals, such that

$$sP(x(s)^{-1} \odot X_t \in \cdot) \to \mu(\cdot) \quad \text{vaguely}, \qquad s \to \infty.$$

- We show that the VSRV X_t has a spectral decomposition $Y\Theta_0 \sim \mu(\cdot; |x| > 1)$, $P(Y > y) = y^{-1}$, $y > 1$, Y independent of $\Theta_0 \in \mathbb{S}^{d-1}$.

Tail chain

Adapted from Perfekt (1997). Assume $X_t \in \mathbb{R}^d$ VSRV so that $P(|X_{t,i}| > x) \sim c_i x^{-\alpha_i}$, $x \to \infty$ and define

$$\|x\|_\alpha = \left|(c_i^{-1}|x_i|^{\alpha_i})_{1 \leqslant i \leqslant d}\right|.$$

Theorem (Pedersen & W.)

Let $X_t \in \mathbb{R}^d$ constitue a stationary VSRV SRE. The tail chain (Θ_t) satisfying $\Theta_t = M_t \Theta_{t-1}$, $t \geqslant 1$ is such that

$$P(\|X_0\|_\alpha^{-1}(X_0, \dots, X_t) \in \cdot \mid \|X_0\|_\alpha > x) \to P((\Theta_0, \dots, \Theta_t) \in \cdot), \qquad x \to \infty.$$

Similar tail process in the multivariate and non-standard regularly varying cases.

Let (X_t) be a stationary BEKK-ARCH process and VSRV.

Define the sample covariance matrix.

$$\Sigma_n := \frac{1}{n}\sum_{t=1}^{n} X_t X_t^{\intercal}.$$

Stable limit theory

With $\alpha_{i,j} = \frac{\alpha_i \alpha_j}{\alpha_i + \alpha_j}$ and assume there exists $p > 0$ such that

$$\lim_{n\to\infty} \mathbb{E}[\|M_1 \cdots M_n\|^p]^{1/n} < 1,$$

then we have

$$\Big(\min(\sqrt{n}, n^{1-1/\alpha_{i,j}}) \times (\Sigma_n - E[\Sigma_n]\mathbf{1}_{\alpha_{i,j}>1})_{i,j}\Big)_{1\leqslant j\leqslant i\leqslant d} \xrightarrow{d} S, \qquad n\to\infty,$$

where $S_{i,j}$ is a $\min(\alpha_{i,j}, 2)$-stable random variable for $1 \leqslant i \leqslant j \leqslant d$.

Conclusion and work in progress

Conclusion:

- Exploit a SRE representation of BEKK-ARCH.
- Mild conditions for geometric ergodicity.
- Tail properties. Vector scaling regular variation.
- Stable limit theory

Ongoing research:

- Tail behavior of more general processes.
- Hidden regular variation for Diagonal BEKK.
- QML estimation.

Thanks for your attention!