Constructions of p-adic L-functions and admissible measures for Hermitian modular forms

Alexei PANTCHICHKINE
Institut Fourier, University Grenoble-Alpes

Special Values of Automorphic L-functions and Associated p-adic L-Functions (18w5053)

October 1-5, 2018
Oaxaca, Mexico

Abstract

For a prime p and a positive integer n, the standard zeta function $L_F(s)$ is considered, attached to an Hermitian modular form $F = \sum_H A(H)q^H$ on the Hermitian upper half plane $\mathcal{H}_n$ of degree n, where H runs through semi-integral positive definite Hermitian matrices of degree n, i.e. $H \in \Lambda_n(\mathcal{O})$ over the integers $\mathcal{O}$ of an imaginary quadratic field K, where $q^H = \exp(2\pi i \mathrm{Tr}(HZ))$. Analytic p-adic continuation of their zeta functions constructed by A.Bouganis in the ordinary case (in [Bou16] is presently extended to the admissible case via growing p-adic measures. Previously this problem was solved for the Siegel modular forms, [CourPa], [BS00]. Present main result is stated in terms of the Hodge polygon $P_H(t) : [0, d] \to \mathbb{R}$ and the Newton polygon $P_N(t) = P_{N,p}(t) : [0, d] \to \mathbb{R}$ of the zeta function $L_F(s)$ of degree $d = 4n$. Main theorem gives a p-adic analytic interpolation of the L values in the form of certain integrals with respect to Mazur-type measures.

p-adic zeta functions of modular forms

Since the p-adic zeta function of Kubota-Leopoldt was constructed by p-adic interpolation of zeta-values $\zeta(1-k) = -B_k/k (k \geq 1)$ [KuLe64], also p-adic zeta functions of various modular forms were constructed, such as p-adic interpolation of the special values

$$L_\Delta(s,\chi) = \sum_{n=1}^{\infty} \chi(n)\tau(n)n^{-s}, \ (s = 1, 2, \cdots, 11), \quad \Delta = \sum_{n=1}^{\infty} \tau(n)q^n,$$

for the Ramanujan function $\tau(n)$ twisted by Dirichlet characters $\chi : (\mathbb{Z}/p^r\mathbb{Z})^* \to \mathbb{C}^*$. Interpolation done in the elliptic and Hilbert modular cases by Yu.I.Manin and B.Mazur, via modular symbols and p-adic integration, see [Ma73], [Ma76]).

In the Siegel modular case $\mathrm{Sp}(2n, \mathbb{Z})$ the p-adic standard zeta functions were constructed in [Pa88], [Pa91] via Rankin-Selberg Andrianov's identity (n even), and [BS00] via doubling method.

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Hermitian modular group $\Gamma_{n,K}$ and the standard zeta function $\mathcal{Z}(s,\mathbf{f})$ (definitions)

Let $\theta = \theta_K$ be the quadratic character attached to K, $n' = [\frac{n}{2}]$.

$$\Gamma_{n,K} = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(\mathcal{O}_K) | M\eta_n M^* = \eta_n \right\}, \ \eta_n = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix},$$

$$\mathcal{Z}(s,\mathbf{f}) = \left(\prod_{i=1}^{2n} L(2s-i+1, \theta^{i-1}) \right) \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

(defined via Hecke's eigenvalues: $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}, \mathfrak{a} \subset \mathcal{O}_K$)

$= \prod_{\mathfrak{q}} \mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}$(an Euler product over primes $\mathfrak{q} \subset \mathcal{O}_K$,

with $\deg \mathcal{Z}_{\mathfrak{q}}(X) = 2n$, the Satake parameters $t_{i,\mathfrak{q}}, i = 1, \cdots, n$),

$$\boxed{\mathcal{D}(s,\mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})}$$ (Motivically normalized standard zeta function

with a functional equation $s \mapsto \ell - s$; $\mathrm{rk} = 4n$, and motivic weight $\ell - 1$).

Main result: p-adic interpolation of all critical values $\mathcal{D}(s,\mathbf{f},\chi)$ normalized by $\times\Gamma_{\mathcal{D}}(s)/\Omega_{\mathbf{f}}$, in the critical strip $n \le s \le \ell - n$ for all $\chi \bmod p^r$ in both bounded or unbounded case, i.e. when the product $\alpha_{\mathbf{f}} = \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i}\right) p^{-n(n+1)}$ is not a p-adic unit.

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The idea of motivic normalization: Ikeda's lifting [Ike08]

The standard Gamma factor of Ikeda's lifting, denoted by $\mathbf{f}$, of an elliptic modular form f extends to a general (not necessarily lifted) Hermitian modular form $\mathbf{f}$ of weight ℓ, used as a pattern, namely

$S_{2k+1}(\Gamma_0(D),\theta) \ni f \rightsquigarrow \mathbf{f} = Lift(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n})$, if $n = 2n'$ is even (E)

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the standard $L-$ function of $\mathbf{f} = Lift^{(n)}(f)$ is $\mathcal{Z}(s,\mathbf{f}) =$

$$\prod_{i=1}^{n} L(s+k+n'-i+(1/2), f)L(s+k+n'-i+(1/2), f, \theta) \quad \text{[Ike08]}$$

$$= \prod_{i=0}^{n-1} L(s+\ell/2-i-(1/2), f)L(s+\ell/2-i-(1/2), f, \theta).$$

because in the lifted case $k+n' = \ell/2$, and the Gamma factor of the standard zeta function with the symmetry $s \mapsto 1-s$ becomes (see p.58) $\Gamma_{\mathcal{Z}}(s) = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s+\ell/2-i-(1/2))^2$. This Gamma factor suggests the following motivic normalization $\mathcal{D}(s) = \mathcal{Z}(s-(\ell/2)+(1/2))$ with the Gamma factor

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and the L-function $\mathcal{D}(s)$ satisfies the symmetry $s \mapsto \ell - s$ of motivic weight $\ell - 1$ with the slopes $2\cdot 0, 2\cdot 1, \dots 2\cdot(n-1)$, $2\cdot(\ell-n), \cdots, 2\cdot(\ell-1)$, so that Deligne's critical values are at $s = n, \dots, s = \ell - n$.

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General zeta functions: critical values and coefficients

More general zeta functions are Euler products of degree d

$$\mathcal{D}(s,\chi)=\sum_{n=1}^{\infty}\chi(n)a_n n^{-s}=\prod_p \frac{1}{\mathcal{D}_p(\chi(p)p^{-s})},\ \Lambda_{\mathcal{D}}(s,\chi)=\Gamma_{\mathcal{D}}(s)\mathcal{D}(s,\chi),$$

where $\deg \mathcal{D}_p(X)=d$ for all but finitely many p, and $\mathcal{D}_p(0)=1$.

In many cases algebraicity of the zeta values was proven as

$$\frac{\mathcal{D}^*(s_0,\chi)}{\Omega_{\mathcal{D}}^{\pm}}\in\mathbb{Q}(\{\chi(n),a_n\}_n),\ \text{where } \mathcal{D}^*(s,\chi) \text{ is normalized by } \Gamma_{\mathcal{D}},$$

at critical points $s_0\in\mathbb{Z}_{crit}$ as linear combinations of coefficients a_n dividing out periods $\Omega_{\mathcal{D}}^{\pm}$, where $\mathcal{D}^*(s_0,\chi)=\Lambda_{\mathcal{D}}(s_0,\chi)$ if $h^{\ell,\ell}=0$.

In p-adic analysis, the Tate field is used $\mathbb{C}_p=\hat{\bar{\mathbb{Q}}}_p$, the completion of an algebraic closure $\bar{\mathbb{Q}}_p$, in place of $\mathbb{C}$. Let us fix embeddings

$$\begin{cases} i_p:\bar{\mathbb{Q}}\hookrightarrow\mathbb{C}_p \\ i_\infty:\bar{\mathbb{Q}}\hookrightarrow\mathbb{C}, \end{cases}$$

and try to continue analytically these zeta values to $s\in\mathbb{Z}_p$, χ mod p^r.

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Main result stated with Hodge/Newton polygons of $\mathcal{D}(s)$

The Hodge polygon $P_H(t) : [0, d] \to \mathbb{R}$ of the function $\mathcal{D}(s)$ and the Newton polygon $P_{N,p}(t) : [0, d] \to \mathbb{R}$ at p are piecewise linear:

The Hodge polygon of pure weight w has the slopes j of $length_j = h^{j,w-j}$ given by Serre's Gamma factors of the functional equation of the form $s \mapsto w + 1 - s$, relating $\Lambda_{\mathcal{D}}(s, \chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s, \chi)$ and $\Lambda_{\mathcal{D}^\rho}(w + 1 - s, \bar{\chi})$, where ρ is the complex conjugation of a_n, and $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathcal{D}^\rho}(s)$ equals to the product $\Gamma_{\mathcal{D}}(s) = \prod_{j \le \frac{w}{2}} \Gamma_{j,w-j}(s)$, where

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The Newton polygon at p is the convex hull of points $(i, \mathrm{ord}_p(a_i))$ $(i = 0, \ldots, d)$; its slopes λ are the p-adic valuations $\mathrm{ord}_p(\alpha_i)$ of the inverse roots α_i of $\mathcal{D}_p(X) \in \bar{\mathbb{Q}}[X] \subset \mathbb{C}_p[X]$: $length_\lambda = \sharp\{i \mid \mathrm{ord}_p(\alpha_i) = \lambda\}$.

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The Hodge polygon of pure weight w has the slopes j of $length_j = h^{j,w-j}$ given by Serre's Gamma factors of the functional equation of the form $s \mapsto w + 1 - s$, relating $\Lambda_{\mathcal{D}}(s, \chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s, \chi)$ and $\Lambda_{\mathcal{D}^\rho}(w + 1 - s, \bar{\chi})$, where ρ is the complex conjugation of a_n, and $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathcal{D}^\rho}(s)$ equals to the product $\Gamma_{\mathcal{D}}(s) = \prod_{j \leq \frac{w}{2}} \Gamma_{j,w-j}(s)$, where

$$\Gamma_{j,w-j}(s) = \begin{cases} \Gamma_{\mathbb{C}}(s-j)^{h^{j,w-j}}, & \text{if } j < w, \\ \Gamma_{\mathbb{R}}(s-j)^{h_+^{j,j}} \Gamma_{\mathbb{R}}(s-j+1)^{h_-^{j,j}}, & \text{if } 2j = w, \end{cases} \text{ where}$$

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), \Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s),$$
$$h^{j,j} = h_+^{j,j} + h_-^{j,j}, \sum_j h^{j,w-j} = d.$$

The Newton polygon at p is the convex hull of points $(i, \mathrm{ord}_p(a_i))$ $(i = 0, \dots, d)$; its slopes λ are the p-adic valuations $\mathrm{ord}_p(\alpha_i)$ of the inverse roots α_i of $\mathcal{D}_p(X) \in \bar{\mathbb{Q}}[X] \subset \mathbb{C}_p[X]$: $length_\lambda = \sharp\{i \mid \mathrm{ord}_p(\alpha_i) = \lambda\}$.

p-adic analytic interpolation of $\mathcal{D}(s, \mathbf{f}, \chi)$

The result expresses the zeta values as integrals with respect to p-adic Mazur-type measures. These measures are constructed from the Fourier coefficients of Hermitian modular forms, and from eigenvalues of Hecke operators on the unitary group.

Pre-ordinary case: $P_H(t) = P_{N,p}(t)$ **at** $t = \frac{d}{2}$ The integrality of measures is proven by T.Bouganis [Bou16], representing $\mathcal{D}^*(s,\chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s,\chi)$ as a Rankin-Selberg type integral at critical points $s = m$. Coefficients of modular forms in this integral satisfy Kummer-type congruences and produce certain bounded measures $\mu_{\mathcal{D}}$ from integral representations and Petersson product, [CourPa]. For the case of p inert in K, see [Bou16].

Admissible case: $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2}) > 0$ The zeta distributions are unbounded, but their sequence produce h-admissible (growing) measures of Amice-Vélu-type, allowing to integrate any continuous characters $y \in \mathrm{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*) = \mathcal{Y}_p$. A general result is used on the existence of h-admissible (growing) measures from binomial congruences for the coefficients of Hermitian modular forms. Their p-adic Mellin transforms $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y(x) d\mu_{\mathcal{D}}(x)$, $\mathcal{L}_{\mathcal{D}} : \mathcal{Y}_p \to \mathbb{C}_p$ give p-adic analytic interpolation of growth $\log_p^h(\cdot)$ of the L-values: the values $\mathcal{L}_{\mathcal{D}}(\chi x_p^m)$ are integrals given by $i_p\left(\frac{\mathcal{D}^*(m, \mathbf{f}, \chi)}{\Omega_{\mathbf{f}}}\right) \in \mathbb{C}_p$.

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A Hermitian modular form of weight ℓ with character σ

is a holomorphic function $\mathbf{f}$ on $\mathcal{H}_n$ $(n \geq 2)$ such that $\mathbf{f}(g\langle Z\rangle) = \sigma(g)\mathbf{f}(Z)j(g,Z)^\ell$ for any $g \in \Gamma_{n,K}$. Here σ be a character of $\Gamma_K^{(n)}$, trivial on $\left\{\begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix}\right\}$, and for $Z \in \mathcal{H}_n$, put $g\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$, $j(g,Z) = \det(CZ+D)$.

Fourier expansions: a semi-integral Hermitian matrix is a Hermitian matrix $H \in (\sqrt{-D_K})^{-1}M_n(\mathcal{O})$ whose diagonal entries are integral. Denote the set of semi-integral Hermitian matrices by $\Lambda_n(\mathcal{O})$, the subset of its positive definite elements is $\Lambda_n(\mathcal{O})^+$, with $\mathcal{O} = \mathcal{O}_K$.

A Hermitian modular form $\mathbf{f}$ is called a cusp form if it has a Fourier expansion of the form $\mathbf{f}(Z) = \sum_{H \in \Lambda_n(\mathcal{O})^+} A(H)q^H$. Denote the space of cusp forms of weight ℓ with character σ by $\mathcal{S}_\ell(\Gamma_{n,K},\sigma)$.

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The standard zeta function of a Hermitian modular form

For all integral ideals $\mathfrak{a} \subset \mathcal{O}$ let $T(\mathfrak{a})$ denotes the Hecke operator associated to it as in [Shi00], page 162, using the action of double cosets $\Gamma\xi\Gamma$ with $\xi = \mathrm{diag}(\hat{D}, D)$, $(\det(D)) = (\alpha)$, $\hat{D} = (D^*)^{-1}$, $\alpha \in \mathfrak{a}$.

Consider a non-zero Hermitian modular form $\mathbf{f} \in \mathcal{M}_\ell(\Gamma)$, for a (congruence) subgroup $\Gamma \subset \Gamma_{n,K}$, and assume $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ with $\lambda(\mathfrak{a}) \in \mathbb{C}$ for all integral ideals $\mathfrak{a} \subset \mathcal{O}$. Then

$$\mathcal{Z}(s, \mathbf{f}) = \left(\prod_{i=1}^{2n} L(2s - i + 1, \theta^{i-1})\right) \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

the sum is over all integral ideals of $\mathcal{O}_K$.

This series has an Euler product representation $\mathcal{Z}(s, \mathbf{f}) = \prod_{\mathfrak{q}} (\mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s}))^{-1}$, where the product is over all prime ideals of $\mathcal{O}_K$, $\mathcal{Z}_{\mathfrak{q}}(X)$ is the numerator of the series $\sum_{r \geq 0} \lambda(\mathfrak{q}^r) X^r \in \mathbb{C}(X)$, computed by Shimura as follows.

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Euler factors of the standard zeta function, [Shi00], p. 171

The Euler factors $\mathcal{Z}_{\mathfrak{q}}(X)$ in the Hermitian modular case at the prime ideal $\mathfrak{q}$ of $\mathcal{O}_K$ are

(i) $$\mathcal{Z}_{\mathfrak{q}}(X) = \prod_{i=1}^{n} \Big((1 - N(\mathfrak{q})^{n-1} t_{\mathfrak{q},i} X)(1 - N(\mathfrak{q})^{n} t_{\mathfrak{q},i}^{-1} X) \Big)^{-1},$$

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The standard motivic-normalized zeta $\mathcal{D}(s, \mathbf{f}, \chi)$

The standard zeta function of $\mathbf{f}$ is defined by means of the p-parameters as the following Euler product:

$$\mathcal{D}(s, \mathbf{f}, \chi) = \prod_p \prod_{i=1}^{2n} \left\{ \left(1 - \frac{\chi(p)\alpha_i(p)}{p^s}\right) \left(1 - \frac{\chi(p)\alpha_{4n-i}(p)}{p^s}\right) \right\}^{-1},$$

where χ is an arbitrary Dirichlet character. The p–parameters $\alpha_1(p), \ldots, \alpha_{4n}(p)$ of $\mathcal{D}(s, \mathbf{f}, \chi)$ for p not dividing the level C of the form $\mathbf{f}$ are related to the the $4n$ characteristic numbers

$$\alpha_1(p), \; \cdots \;, \; \alpha_{2n}(p), \; \alpha_{2n+1}(p), \; \cdots \;, \; \alpha_{4n}(p)$$

of the product of all $\mathfrak{q}$-factors $\mathcal{Z}_{\mathfrak{q}}(N\mathfrak{q}^{(\ell-1)/2)}X)^{-1}$ for all $\mathfrak{q}|p$, which is a polynomial of degree $4n$ of the variable $X = p^{-s}$ (for almost all p) with coefficients in a number field $T = T(\mathbf{f})$.

There is a relation between the two normalizations $\mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f}) = \mathcal{D}(s, \mathbf{f})$ explained in [Ha97] for general zeta functions.

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Description of the Main theorem

Let $\Omega_{\mathbf{f}}$ be a period attached to an Hermitian cusp eigenform $\mathbf{f}$, $\mathcal{D}(s,\mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})$ the standard zeta function, and

$$\alpha_{\mathbf{f}} = \alpha_{\mathbf{f},p} = \left(\prod_{\mathfrak{q}|p}\prod_{i=1}^{n} t_{\mathfrak{q},i}\right) p^{-n(n+1)}, \quad h = \mathrm{ord}_p(\alpha_{\mathbf{f},p}),$$

The number $\alpha_{\mathbf{f}}$ turns out to be an eigenvalue of Atkin's type operator $U_p : \sum_H A_H q^H \mapsto \sum_H A_{pH} q^H$ on some $\mathbf{f_0}$, and $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2})$.

Definition. Let M be a $\mathcal{O}$-module of finite rank where $\mathcal{O} \subset \mathbb{C}_p$. For $h \geq 1$, consider the following $\mathbb{C}_p$-vector spaces of functions on $\mathbb{Z}_p^*$: $\mathcal{C}^h \subset \mathcal{C}^{loc-an} \subset \mathcal{C}$. Then

- a continuous homomorphism $\mu : \mathcal{C} \to M$ is called a (bounded) measure M-valued measure on $\mathbb{Z}_p^*$.
- $\mu : \mathcal{C}^h \to M$ is called an h admissible measure M-valued measure on $\mathbb{Z}_p^*$ measure if the following growth condition is satisfied

$$\left|\int_{a+(p^v)} (x-a)^j d\mu\right|_p \leq p^{-v(h-j)}$$

for $j = 0, 1, ..., h-1$, and et $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ be the space of definition of p-adic Mellin transform

Theorem ([Am-V], [MTT]) For an h-admissible measure μ, the Mellin transform $\mathcal{L}_\mu : \mathcal{Y}_p \to \mathbb{C}_p$ exists and has growth $o(\log^h)$ (with infinitely many zeros).

Description of the Main theorem

Let $\Omega_{\mathbf{f}}$ be a period attached to an Hermitian cusp eigenform $\mathbf{f}$, $\mathcal{D}(s, \mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})$ the standard zeta function, and

$$\alpha_{\mathbf{f}} = \alpha_{\mathbf{f},p} = \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i} \right) p^{-n(n+1)}, \quad h = \mathrm{ord}_p(\alpha_{\mathbf{f},p}),$$

The number $\alpha_{\mathbf{f}}$ turns out to be an eigenvalue of Atkin's type operator $U_p : \sum_H A_H q^H \mapsto \sum_H A_{pH} q^H$ on some $\mathbf{f_0}$, and $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2})$.

Definition. Let M be a $\mathcal{O}$-module of finite rank where $\mathcal{O} \subset \mathbb{C}_p$. For $h \geq 1$, consider the following $\mathbb{C}_p$-vector spaces of functions on $\mathbb{Z}_p^*$: $\mathcal{C}^h \subset \mathcal{C}^{loc-an} \subset \mathcal{C}$. Then

- a continuous homomorphism $\mu : \mathcal{C} \to M$ is called a (bounded) measure M-valued measure on $\mathbb{Z}_p^*$.
- $\mu : \mathcal{C}^h \to M$ is called an *h* admissible measure M-valued measure on $\mathbb{Z}_p^*$ measure if the following growth condition is satisfied

$$\left| \int_{a+(p^v)} (x-a)^j d\mu \right|_p \leq p^{-v(h-j)}$$

for $j = 0, 1, ..., h-1$, and et $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ be the space of definition of p-adic Mellin transform

Theorem ([Am-V], [MTT]) For an h-admissible measure μ, the Mellin transform $\mathcal{L}_\mu : \mathcal{Y}_p \to \mathbb{C}_p$ exists and has growth $o(\log^h)$ (with infinitely many zeros).

Main Theorem.

Let f be a Hermitian cusp eigenform of degree $n \geq 2$ and of weight $\ell > 4n+2$. There exist distributions $\mu_{\mathcal{D},s}$ for $s = n, \cdots, \ell - n$ with the properties:

i) for all pairs (s, χ) such that $s \in \mathbb{Z}$ with $n \leq s \leq \ell - n$,

$$\int_{\mathbb{Z}_p^*} \chi d\mu_{\mathcal{D},s} = A_p(s,\chi) \frac{\mathcal{D}^*(s, \mathbf{f}, \overline{\chi})}{\Omega_{\mathbf{f}}}$$

(under the inclusion i_p), with elementary factors $A_p(s,\chi) = \prod_{\mathfrak{q}|p} A_{\mathfrak{q}}(s,\chi)$ including a finite Euler product, Satake parameters $t_{\mathfrak{q},i}$, gaussian sums, the conductor of χ; the integral is a finite sum.

(ii) if $\mathrm{ord}_p \left((\prod_{\mathfrak{q}|p} \prod_{i=1}^n t_{\mathfrak{q},i}) p^{-n(n+1)} \right) = 0$ then the above distributions $\mu_{\mathcal{D},s}$ are bounded measures, we set $\mu_{\mathcal{D}} = \mu_{\mathcal{D},s^*}$ and the integral is defined for all continuous characters $y \in \mathrm{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*) =: \mathcal{Y}_p$.

Their Mellin transforms $\mathcal{L}_{\mu_{\mathcal{D}}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, $\mathcal{L}_{\mu_{\mathcal{D}}} : \mathcal{Y}_p \to \mathbb{C}_p$, give bounded p-adic analytic interpolation of the above L-values to on the $\mathbb{C}_p$-analytic group $\mathcal{Y}_p$; and these distributions are related by:

$$\int_X \chi d\mu_{\mathcal{D},s} = \int_X \chi x^{s^*-s} d\mu_{\mathcal{D}}^*, \; X = \mathbb{Z}_p^*, \text{ where } s^* = \ell - n, \; s_* = n.$$

Main theorem (continued)

(iii) in the admissible case assume that

$0 < h \le \dfrac{s^* - s_* + 1}{2} = \dfrac{\ell + 1 - 2n}{2}$, where

$h = \mathrm{ord}_p \left((\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i}) p^{-n(n+1)} \right) > 0$, Then there exist

h–admissible measures $\mu_{\mathcal{D}}$ whose integrals $\int_{\mathbb{Z}_p^*} \chi x_p^{s^*-s} d\mu_{\mathcal{D}}$ are given

by $i_p \left(A_p(s,\chi) \dfrac{\mathcal{D}^*(s,\mathbf{f},\overline{\chi})}{\Omega_{\mathbf{f}}} \right) \in \mathbb{C}_p$ with $A_p(s,\chi)$ as in (i); their

Mellin transforms $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, belong to the type $o(\log x_p^h)$.

(iv) the functions $\mathcal{L}_{\mathcal{D}}$ are determined by (i)-(iii).

Remarks.

(a) Interpretation of s^*: the smallest of the "big slopes" of P_H

(b) Interpretation of $s_* - 1$: the biggest of the "small slopes" of P_H.

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Eisenstein series and congruences (KEY POINT!)

The (Siegel-Hermite) Eisenstein series $E_{2\ell,n,K}(Z)$ of weight 2ℓ, character $\det^{-\ell}$, is defined in [Ike08] by

$$E_{2\ell,n,K}(Z) = \sum_{g \in \Gamma_{n,K,\infty} \backslash \Gamma_{n,K}} (\det g)^{\ell} j(g,Z)^{-2\ell} \text{ (converges for } \ell > n).$$

The normalized Eisenstein series is given by
$\mathcal{E}_{2\ell,n,K}(Z) = 2^{-n} \prod_{i=1}^{n} L(i-2\ell, \theta^{i-1}) \cdot E_{2\ell,n,K}(Z)$.

If $H \in \Lambda_n(\mathcal{O})^+$, then the H-th Fourier coefficient of $\mathcal{E}_{2\ell}^{(n)}(Z)$ is polynomial over $\mathbb{Z}$ in variables $\{p^{\ell-(n/2)}\}_p$, and equals

$$|\gamma(H)|^{\ell-(n/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, p^{-\ell+(n/2)}), \gamma(H) = (-D_K)^{[n/2]} \det H.$$

Here, $\tilde{F}_p(H,X)$ is a certain Laurent polynomial in the variables $\{X_p = p^{-s}, X_p^{-1}\}_p$ over $\mathbb{Z}$. This polynomial is a key point in proving congruences for the modular forms in a Rankin-Selberg integral. Also, for a certain congruence subgroup $C = \Gamma_{\mathfrak{c}}$, $s \in \mathbb{C}$ and a Hecke ideal character ψ mod $\mathfrak{c}$, the series is defined

$$E(Z,s,\ell,\psi) = \sum_{g \in C_\infty \backslash C} \psi(g)(\det g)^{\ell} j(g,Z)^{-2\ell} |(\det g) j(g,Z)|^{-s}.$$

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An integral representation of Rankin-Selberg type

The integral representation of Rankin-Selberg type in the Hermitian modular case: is stated for the level $\mathfrak{c}$ moodular forms:

Theorem 4.1 (Shimura, Klosin), see [Bou16], p.13.

Let $0 \neq \mathbf{f} \in \mathcal{M}_\ell(\Gamma_\mathfrak{c}, \psi))$ of scalar weight ℓ, ψ mod $\mathfrak{c}$, such that $\forall \mathfrak{a}, \mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$, and assume that $2\ell \geq n$, then there exists $\mathcal{T} \in S_+ \cap \mathrm{GL}_n(K)$ and $\mathcal{R} \in \mathrm{GL}_n(K)$ such that

$$\Gamma((s))\psi(\det(\mathcal{T}))\mathcal{Z}(s + 3n/2, \mathbf{f}, \chi) =$$
$$\Lambda_\mathfrak{c}(s + 3n/2, \theta\psi\chi) \cdot C_0\langle \mathbf{f}, \theta_\mathcal{T}(\chi)\mathcal{E}(\bar{s} + n, \ell - \ell_\theta, \chi^\rho\psi)\rangle_{C''},$$

where $\mathcal{E}(Z, s, \ell - \ell_\theta, \psi)_{C''}$ is a normalized group theoretic (or adelic) Eisenstein series with components as above of level $\mathfrak{c}''$ divisible by $\mathfrak{c}$, and weight $\ell - \ell_\theta$. Here $\langle\cdot,\cdot\rangle_{C''}$ is the normalized Petersson inner product associated to the congruence subgroup C'' of level $\mathfrak{c}''$.

$$\Gamma((s)) = (4\pi)^{-n(s+h)}\Gamma_n^\iota(s + h), \Gamma_n^\iota(s) = \pi^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \Gamma(s - j),$$

where $h = 0$ or 1, C_0 the index of a subgroup.

Proof of the Main Theorem (ii): Kummer congruences

Let us se the notation $\mathcal{D}_p^{alg}(m, \mathbf{f}, \chi) = A_p(s, \chi)\frac{\mathcal{D}^*(m, \mathbf{f}, \chi)}{\Omega_\mathbf{f}}$

The integrality of measures is proven representing $\mathcal{D}_p^{alg}(m, \chi)$ as Rankin-Selberg type integral at critical points $s = m$. Coefficients of modular forms in this integral satisfy Kummer-type congruences and produce bounded measures $\mu_\mathcal{D}$ whose construction reduces to congruences of Kummer type between the Fourier coefficients of modular forms, see also [Bou16]. Suppose that we are given infinitely many "critical pairs" (s_j, χ_j) at which one has an integral representation $\mathcal{D}_p^{alg}(s_j, \mathbf{f}, \chi_j) = A_p(s, \chi)\frac{\langle \mathbf{f}, h_j \rangle}{\Omega_\mathbf{f}}$ with all $h_j = \sum_\mathcal{T} b_{j,\mathcal{T}} q^\mathcal{T} \in \mathcal{M}$ in a certain finite-dimensional space $\mathcal{M}$ containing $\mathbf{f}$ and defined over $\bar{\mathbb{Q}}$. We prove the following Kummer-type congruences:

$$\forall x \in \mathbb{Z}_p^*, \sum_j \beta_j \chi_j x^{k_j} \equiv 0 \mod p^N \Longrightarrow \sum_j \beta_j \mathcal{D}_p^{alg}(s_j, \mathbf{f}, \chi) \equiv 0 \mod p^N$$

$$\beta_j \in \bar{\mathbb{Q}}, k_j = s^* - s_j, \text{ where } s^* = \ell - n \text{ in our case.}$$

Computing the Petersson products of a given modular form $\mathbf{f}(Z) = \sum_H a_H q^H \in \mathcal{M}_*(\bar{\mathbb{Q}})$ by another modular form $h(Z) = \sum_H b_H q^H \in \mathcal{M}_*(\bar{\mathbb{Q}})$ uses a linear form $\ell_\mathbf{f} : h \mapsto \frac{\langle \mathbf{f}, h \rangle}{\langle \mathbf{f}, \mathbf{f} \rangle}$ defined over a subfield $k \subset \bar{\mathbb{Q}}$.

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Admissible Hermitian case

Let $\mathbf{f} \in \mathcal{S}_\ell(C, \psi)$ be a Hecke eigenform for the congruence subgroup $C = \Gamma_c$ of level $\mathfrak{c}$. Let $\mathfrak{q}$ be a prime of K over p, which is inert over $\mathbb{Q}$. Then we say that $\mathbf{f}$ is pre-ordinary at $\mathfrak{q}$ if there exists an eigenform $0 \neq \mathbf{f}_0 \in \mathcal{M}_{\{p\}} \subset \mathcal{S}_\ell(Cp, \psi)$ with Satake parameters $t_{\mathfrak{q},i}$ such that

$$\left\| \left(\prod_{i=1}^{n} t_{\mathfrak{q},i} \right) N(\mathfrak{q})^{-\frac{n(n+1)}{2}} \right\|_p = 1,$$

where $\|\|_p$ the normalized absolute value at p.

The admissible case corresponds to

$$\left\| \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i} \right) p^{-n(n+1)} \right\|_p = p^{-h} \text{ for a positive } h > 0.$$

An interpretation of h as the difference $h = P_{N,p}(d/2) - P_H(d/2)$ comes from the above explicit relations.

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Existence of h-admissible measures

of Amice-Vï¿½lu-type gives an unbounded p-adic analytic interpolation of the L-values of growth $\log_p^h(\cdot)$, using the Mellin transform of the constructed measures. This condition says that the product $\prod_{i=1}^n t_{\mathfrak{p},i}$ is nonzero and divisible by a certain power of p in $\mathcal{O}$:

$$\mathrm{ord}_p\left(\prod_{\mathfrak{q}|p}\left(\prod_{i=1}^{n} t_{\mathfrak{q},i}\right)p^{-n(n+1)}\right) = h.$$

We use an easy condition of admissibility of a sequence of modular distributions Φ_j on $X = \mathbb{Z}_p^*$ with values in the semigroup algebra $\mathcal{O}[[q]] = \mathcal{O}[[q^H]]_{H\in\Lambda(\mathcal{O})^+}$ as in Theorem 4.8 of [CourPa]. It suffices to check congruences of the type (with $\varkappa = 4$)

$$U^{\varkappa v}\Big(\sum_{j'=0}^{j}\binom{j}{j'}(-a_p^0)^{j-j'}\Phi_{j'}(a+(p^v)\Big) \in Cp^{vj}\mathcal{O}[[q]]$$

for all $j = 0, 1, \ldots, \varkappa h - 1$. Here $s = s^* - j'$, $\Phi_{j'}(a+(p^v))$ a certain convolution of two Hermitian modular forms, i.e.

$$\Phi_{j'}(\chi) = \theta(\chi)\cdot\mathcal{E}(s,\chi)$$

of a Hermitian theta series $\theta(\chi)$ and an Eisenstein series $\mathcal{E}(s,\chi)$ with any Dirichlet character χ mod p^r. We use a general sufficient condition of admissibility of a sequence of modular distributions Φ_j on $X = \mathbb{Z}_p^*$ with values in $\mathcal{O}[[q]]$ as in Theorem 4.8 of [CourPa].

Proof of the Main Theorem (iii): (admissible case)

Using a Rankin-Selberg integral representation for $\mathcal{D}^{alg}(s, \mathbf{f}, \chi)$ and an eigenfunction $\mathbf{f}_0$ of Atkin's operator $U(p)$ of eigenvalue $\alpha_\mathbf{f}$ on $\mathbf{f}_0$ the Rankin-Selberg integral of $\mathcal{F}_{s,\chi} := \theta(\chi) \cdot \mathcal{E}(s, \chi)$ gives

$$\mathcal{D}^{alg}(s, \mathbf{f}, \chi) = \frac{\langle \mathbf{f}_0, \theta(\chi) \cdot \mathcal{E}(s, \chi) \rangle}{\langle \mathbf{f}, \mathbf{f} \rangle} \text{ (the Petersson product on } G = GU(\eta_n))$$

$$= \alpha_\mathbf{f}^{-v} \frac{\langle \mathbf{f}_0, U(p^v)(\theta(\chi) \cdot \mathcal{E}(s, \chi)) \rangle}{\langle \mathbf{f}, \mathbf{f} \rangle} = \alpha_\mathbf{f}^{-v} \frac{\langle f_0, U(p^v)(\mathcal{F}_{s,\chi}) \rangle}{\langle \mathbf{f}, \mathbf{f} \rangle}.$$

Modification in the admissible case: instead of Kummer congruences, to estimate p-adically the integrals of test functions: $M = p^v$:

$$\int_{a+(M)} (x-a)^j d\mathcal{D}^{alg} := \sum_{j'=0}^{j} \binom{j}{j'} (-a)^{j-j'} \int_{a+(M)} x^{j'} d\mathcal{D}^{alg}, \text{ using}$$

the orthogonality of characters and the sequence of zeta distributions

$$\int_{a+(M)} x^j d\mathcal{D}^{alg} = \frac{1}{\sharp(\mathcal{O}/M\mathcal{O})^\times} \sum_{\chi \bmod M} \chi^{-1}(a) \int_X \chi(x) x^j d\mathcal{D}^{alg},$$

$\int_X \chi d\mathcal{D}^{alg}_{s^*-j} = \mathcal{D}^{alg}(s^*-j, f, \chi) =: \int_X \chi(x) x^j d\mathcal{D}^{alg}$.

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Congruences between the coefficients of the Hermitian modular forms

In order to integrate any locally-analytic function on X, it suffices to check the following binomial congruences for the coefficients of the Hermitian modular form $\mathcal{F}_{s^*-j,\chi} = \sum_\xi v(\xi, s^* - j, \chi)q^\xi$: for $v \gg 0$, and a constant C

$$\frac{1}{\sharp(\mathcal{O}/M\mathcal{O})^\times} \sum_{j'=0}^{j} \binom{j}{j'} (-a)^{j-j'} \sum_{\chi \bmod M} \chi^{-1}(a) v(p^v \xi, s^* - j', \chi) q^\xi$$

$$\in Cp^{vj}\mathcal{O}[[q]] \quad \text{(This is a quasimodular form if } j' \neq s^*\text{)}$$

The resulting measure $\mu_{\mathcal{D}}$ allows to integrate all continuous characters in $\mathcal{Y}_p = \mathrm{Hom}_{cont}(X, \mathbb{C}_p^*)$, including Hecke characters, as they are always locally analytic.

Its p-adic Mellin transform $\mathcal{L}_{\mu_{\mathcal{D}}}$ is an analytic function on $\mathcal{Y}_p$ of the logarithmic growth $\mathcal{O}(\log^h)$, $h = \mathrm{ord}_p(\alpha)$.

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Proof of the main congruences

Thus the Petersson product in $\ell_{\mathbf{f}}$ can be expressed through the Fourier coeffcients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coeffcients:
$\ell_{\mathcal{T}_i} : h \mapsto b_{\mathcal{T}_i} (i = 1, \ldots, n)$. It follows that $\ell_{\mathbf{f}}(h) = \sum_i \gamma_i b_{\mathcal{T}_i}$, where $\gamma_i \in k$.
Using the expression for $\ell_f(h_j) = \sum_i \gamma_{i,j} b_{j,\mathcal{T}_i}$, the above congruences reduce to

$$\sum_{i,j} \gamma_{i,j} \beta_j b_{j,\mathcal{T}_i} \equiv 0 \mod p^N.$$

The last congruence is done by an elementary check on the Fourier coefficients $b_{j,\mathcal{T}_i}$.
The abstract Kummer congruences are checked for a family of test elements.
In the admissible case it suffices to check binomial congruences for the Fourier coefficients as above in place of Kummer congruences.

Appendix A. Rewriting the local factor at p with character θ

Notice that if θ is the quadratic character attached to $K/\mathbb{Q}$ then

$$(1-\alpha_p X)(1-\alpha_p\theta(p)X) = \begin{cases} (1-\alpha_p X)^2 & \text{if } \theta(p)=1, p\mathfrak{r}=\mathfrak{q}_1\mathfrak{q}_2, N(\mathfrak{q}_i)=p,\\ (1-\alpha_p^2X^2), & \text{if } \theta(p)=-1, p\mathfrak{r}=\mathfrak{q}, N(\mathfrak{q})=p^2,\\ (1-\alpha_p X) & \text{if } \theta(p)=0, p\mathfrak{r}=\mathfrak{q}^2, N(\mathfrak{q})=p. \end{cases}$$

Thus, if $X=p^{-s}$, $X^2=p^{-2s}$, $N(\mathfrak{q})=p$, $\mathcal{Z}_{\mathfrak{q}}(X)^{-1}$

$$= \begin{cases} \prod_{i=1}^{2n}(1-N(\mathfrak{q}_1)^{2n}t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1}X)(1-N(\mathfrak{q}_2)^{-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}X), & \text{if } \theta(p)=1,\\ \prod_{i=1}^{n}(1-N(\mathfrak{q})^{n-1}t_{\mathfrak{q},i}X^2)(1-N(\mathfrak{q})^{n}t_{\mathfrak{q},i}^{-1}X^2), & \text{if } \theta(p)=-1,\\ \prod_{i=1}^{n}(1-N(\mathfrak{q})^{n-1}t_{\mathfrak{q},i}X)(1-N(\mathfrak{q})^{n}t_{\mathfrak{q},i}^{-1}X), & \text{if } \theta(p)=0. \end{cases}$$

$$= \begin{cases} \prod_{i=1}^{n}(1-\gamma_{p,i}X)^2\prod_{i=1}^{n}(1-\delta_{p,i}X)^2 & \text{if } \theta(p)=1,\ \text{i.e. } p\mathfrak{r}=\mathfrak{q}_1\mathfrak{q}_2,\\ \prod_{i=1}^{n}(1-\alpha_{p,i}^2X^2)\prod_{i=1}^{n}(1-\beta_{p,i}^2X^2), & \text{if } \theta(p)=-1,\ \text{i.e. } p\mathfrak{r}=\mathfrak{q},\\ \prod_{i=1}^{n}(1-\alpha'_{p,i}X)\prod_{i=1}^{n}(1-\beta'_{p,i}X) & \text{if } \theta(p)=0,\ \text{i.e. } p\mathfrak{r}=\mathfrak{q}^2, \end{cases}$$

where $\alpha'_{p,i}=p^{n-1}t_{\mathfrak{q},i}$, $\beta'_{p,i}p^nt_{\mathfrak{q},i}^{-1}$, $\gamma_{p,i}=p^{2n}t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1}$, $p^{-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}$. It follows that $\prod_{\mathfrak{q}|p}\mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-n-(1/2)}X)=X^{4n}+\cdots$

Appendix A (continued).Relations between $\alpha_i(p)$ and $t_{i,\mathfrak{q}}$

were studied and explained by M.Harris [Ha97] for general Hermitian zeta functions $\mathcal{Z}(s, \mathbf{f})$ of type introduced in [Shi00], using reprsentation theory of unitary groups and Deligne's approach to L-functions, see [De79], in terms of a n-dimensional Galois representations $\rho_\lambda : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}(M_{\mathbf{f},\lambda}) \cong \mathrm{GL}_n(E_\lambda)$ over a completion E_λ of a number field E containing K and the Hecke eigenvalues of a vector-valued Hermitian modular form $\mathbf{f}$:

$$\mathcal{Z}(s - n' - \frac{1}{2}, \mathbf{f}) = \mathcal{D}(s, \mathbf{f}) = L(s, M_{\mathbf{f},\lambda} \boxtimes M(\psi))$$

for an algebraic Hecke ideal character ψ as above of the infinity type m_ψ, see [GH16], p.20. Here the symbol $L(s, M_{\mathbf{f},\lambda} \boxtimes M(\psi))$ denotes the Rankin-Selberg type convolution (it corresponds to tensor product of Galois representations). Notice that $L(s, M_{\mathbf{f},\lambda})$ is of degree $2n$, and $L(s, M_{\mathbf{f},\lambda} \boxtimes M(\psi))$ is of degree $4n$ because $L(s, \psi) = L(s, R(\psi))$ is of degree 2.
Moreover, M.Harris suggested a general description of $\mathcal{D}(s)$ with given Gamma factors and analytic properties as some $\mathcal{D}(s, \mathbf{f})$ some under natural conditions on Gamma factors, giving higher versions of Shimura-Taniyama-Weil conjecture (i.e. higher Wiles' modularity theorem). This can be stated also over a totally real field F (instead of $\mathbb{Q}$), and its quadratic totally imaginary extension K, see [GH16], [Pa94].

Appendix B. Shimura's Theorem: algebraicity of critical values in Cases Sp and UT, p.234 of [Shi00]

Let $\mathbf{f} \in \mathcal{V}(\bar{\mathbb{Q}})$ be a non zero arithmetical automorphic form of type Sp or UT. Let χ be a Hecke character of K such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{\ell}|x_{\mathbf{a}}|^{-\ell}$ with $\ell \in \mathbb{Z}^{\mathbf{a}}$, and let $\sigma_0 \in 2^{-1}\mathbb{Z}$. Assume, in the notations of Chapter 7 of [Shi00] on the weights k_v, μ_v, ℓ_v, that

$$\begin{array}{ll} \text{Case Sp} & 2n+1-k_v+\mu_v \leq 2\sigma_0 \leq k_v-\mu_v, \\ & \text{ where } \mu_v = 0 \text{ if } [k_v]-l_v \in 2\mathbb{Z} \\ & \text{ and } \mu_v = 1 \text{ if } [k_v]-l_v \notin 2\mathbb{Z};\ \sigma_0-k_v+\mu_v \\ & \text{ for every } v \in \mathbf{a} \text{ if } \sigma_0 > n \text{ and} \\ & \sigma_0-1-k_v+\mu_v \in 2\mathbb{Z} \text{ for every } v \in \mathbf{a} \text{ if } \sigma_0 \leq n. \\ \text{Case UT} & 4n-(2k_{v\rho}+\ell_v) \leq 2\sigma_0 \leq m_v - |k_v-k_{v\rho}-\ell_v| \\ & \text{ and } 2\sigma_0-\ell_v \in 2\mathbb{Z} \text{ for every } v \in \mathbf{a}. \end{array}$$

Appendix B. Shimura's Theorem (continued)

Further exclude the following cases

(A) Case Sp $\sigma_0 = n+1, F = \mathbb{Q}$ and $\chi^2 = 1$;

(B) Case Sp $\sigma_0 = n + (3/2), F = \mathbb{Q}; \chi^2 = 1$ and $[k] - \ell \in 2\mathbb{Z}$

(C) Case Sp $\sigma_0 = 0, \mathfrak{c} = \mathfrak{g}$ and $\chi = 1$;

(D) Case Sp $0 < \sigma_0 \le n, \mathfrak{c} = \mathfrak{g}, \chi^2 = 1$ and the conductor of χ is $\mathfrak{g}$;

(E) Case UT $2\sigma_0 = 2n+1, F = \mathbb{Q}, \chi_1 = \theta,$ and $k_v - k_{v\rho} = \ell_v$;

(F) Case UT $0 \le 2\sigma_0 < 2n, \mathfrak{c} = \mathfrak{g}, \chi_1 = \theta^{2\sigma_0}$ and the conductor of χ is $\mathfrak{r}$

Then

$$\mathcal{Z}(\sigma_0, \mathbf{f}, \chi)/\langle \mathbf{f}, \mathbf{f}\rangle \in \pi^{n|m|+d\varepsilon}\bar{\mathbb{Q}},$$

where $d = [F : \mathbb{Q}]$, $|m| = \sum_{v \in \mathbf{a}} m_v$, and

$$\varepsilon = \begin{cases} (n+1)\sigma_0 - n^2 - n, & \text{Case Sp}, k \in \mathbb{Z}^{\mathbf{a}}, \text{ and } \sigma_0 > n_0), \\ n\sigma_0 - n^2, & \text{Case Sp}, k \notin \mathbb{Z}^{\mathbf{a}}, \text{or} \sigma_0 \le n_0), \\ 2n\sigma_0 - 2n^2 + n & \text{Case UT} \end{cases}$$

Notice that $\pi^{n|m|+d\varepsilon} \in \mathbb{Z}$ in all cases; if $k \notin \mathbb{Z}^{\mathbf{a}}$, the above parity condition on σ_0 shows that $\sigma_0 + k_v \in \mathbb{Z}$, so that $n|m| + d\varepsilon \in \mathbb{Z}$.

Appendix C. Examples of Hermitian cusp forms

The Hermitian Ikeda lift, [Ike08]. Assume $n = 2n'$ even.

Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in \mathcal{S}_{2k+1}(\Gamma_0(D_K), \chi)$ be a primitive form, whose L-function is given by

$$L(f,s) = \prod_{p \nmid D_K} (1 - a(p)p^{-s} + \theta(p)p^{2k-2s})^{-1} \prod_{p | D_K} (1 - a(p)p^{-s})^{-1}.$$

For each prime $p \nmid D_K$, define the Satake parameter $\{\alpha_p, \beta_p\} = \{\alpha_p, \theta(p)\alpha_p^{-1}\}$ by

$$(1 - a(p)X + \theta(p)p^{2k}X^2) = (1 - p^k\alpha_p X)(1 - p^k\beta_p X)$$

For $p | D_K$, we put $\alpha_p = p^{-k}a(p)$. Put

$$A(H) = |\gamma(H)|^k \prod_{p|\gamma(H)} \tilde{F}_p(H; \alpha_p), H \in \Lambda_n(\mathcal{O})^+$$

$$\mathbf{f}(Z) = \sum_{H \in \Lambda_n(O)^+} A(H)q^H, Z \in \mathcal{H}_{2n}.$$

Appendix C (continued).The first theorem (even case)

Theorem 5.1 (Case E) of [Ike08] Assume that $n = 2n'$ is even. Let $f(\tau)$, $A(H)$ and $\mathbf{f}(Z)$ be as above. Then we have $\mathbf{f} \in \mathcal{S}_{2k+2n'}(\Gamma_K^{(n)}, \det^{-k-n'})$.

In the case when n is odd, consider a similar lifting for a normalized Hecke eigenform $n = 2n' + 1$ is odd. Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N$ $\in \mathcal{S}_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a primitive form, whose L-function is given by

$$L(f, s) = \prod_p (1 - a(p)p^{-s} + p^{2k-1-2s})^{-1}.$$

For each prime p, define the Satake parameter $\{\alpha_p, \alpha_p^{-1}\}$ by

$$(1 - a(p)X + p^{2k-1}X^2) = (1 - p^{k-(1/2)}\alpha_p X)(1 - p^{k-(1/2)}\alpha^{-1}X).$$

Put

$$A(H) = |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H; \alpha_p), H \in \Lambda_n(\mathcal{O})^+$$

$$\mathbf{f}(Z) = \sum_{H \in \Lambda_n(O)^+} A(H)q^H, Z \in \mathcal{H}_n.$$

Appendix C (continued). The second theorem (odd case)

Theorem 5.2 (Case O) of [Ike08]. Assume that $n = 2n' + 1$ is odd. Let $f(\tau)$, $A(H)$ and $\mathbf{f}(Z)$ be as above. Then we have $\mathbf{f} \in \mathcal{S}_{2k+2n'}(\Gamma_K^{(n)}, \det^{-k-n'})$.

The lift $Lift^{(n)}(f)$ of f is a common Hecke eigenform of all Hecke operators of the unitary group, if it is not identically zero (Theorem 13.6).

Theorem 18.1 of [Ike08]. Let n, n', and f be as in Theorem 5.1 or as in Theorem 5.2. Assume that $Lift^{(n)}(f) \neq 0$. Let $L(s, Lift^{(n)}(f), st)$ be the L-function of $Lift^{(n)}(\mathbf{f})$ associated to $st : {}^L\mathcal{G} \to \mathrm{GL}_{4n}(\mathbb{C})$. Then up to bad Euler factors, $L(s, Lift^{(n)}(f), st)$ is equal to

$$\prod_{i=1}^{n} L(s + k + n' - i + \frac{1}{2}, f) L(s + k + n' - i + \frac{1}{2}, f, \theta).$$

Moreover, the $4n$ charcteristic roots of $L(s, Lift^{(n)}(f), st)$ given as follows: for $i = 1, \cdots, n$

$$\alpha_p p^{-k-n'+i-\frac{1}{2}}, \alpha_p^{-1} p^{-k-n'+i-\frac{1}{2}}, \theta(p)\alpha_p p^{-k-n'+i-\frac{1}{2}}, \theta(p)\alpha_p^{-1} p^{-k-n'+i-\frac{1}{2}}$$

Functional equation of the lift (thanks to Sho Takemori!)

There are two cases [Ike08]: the even case (E) and the odd case

$$(O): \begin{cases} f \in S_{2k+1}(\Gamma_0(D),\theta), \mathbf{f} = Lift^{(n)}(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}) & (E) \\ (\text{of even degree } n = 2n' \text{ and of weight } 2k+2n') & \\ f \in S_{2k}(\mathrm{SL}(\mathbb{Z})), \mathbf{f} = Lift^{(n)}(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}) & (O) \\ (\text{of odd degree } n = 2n'+1 \text{ and of weight } 2k+2n'). & \end{cases}$$

Then, up to bad Euler factors, the standard L-function of $\mathbf{f} = Lift^{(n)}(f)$ is given by $\mathcal{Z}(s,\mathbf{f})=$
$\prod_{i=1}^{n} L(s+k+n'-i+\frac{1}{2}, f)L(s+k+n'-i+\frac{1}{2}, f, \theta)$
Let us denote $t(s,i) = s+k+n'-i+\frac{1}{2}$ then

$$= \begin{cases} \prod_{i=1}^{2n'} L(s+k+n'-i+\frac{1}{2}, f)L(s+k+n'-i+\frac{1}{2}, f,\theta) & (E) \\ \prod_{i=1}^{n'} L(t(s,i), f)L(t(s,n+1-i), f) & \\ L(t(s,i), f,\theta)L(t(s,n+1-i), f,\theta) & \\ \prod_{i=1}^{2n'+1} L(s+k+n'-i+\frac{1}{2}, f) & \\ \times L(s+k+n'-i+\frac{1}{2}, f,\theta) & (O) \\ = L(s+k-\frac{1}{2}, f)L(s+k-\frac{1}{2}, f,\theta) & \\ \prod_{i=1}^{n'} L(t(s,i), f)L(t(s,n+1-i), f) & \\ L(t(s,i), f,\theta)L(t(s,n+1-i), f,\theta). & \end{cases}$$

The Gamma factor $\Gamma_{\mathcal{Z}}(s)$ of Ikeda's lift

In the even case $t(1-s, n+1-i) = t(1-s, 2n'+1-i)$ $=(2k+1) - t(s,i)$. The Hecke functional equation $s \mapsto 2k+1-s$ in all symmetric terms of the product, gives the functional equation of the standard L-function of the form $s \mapsto 1-s$, and the gamma factor is then

$$\prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s+k+n'-i+1/2)^2 = \Gamma_{\mathcal{D}}(s+n'+\frac{1}{2}).$$

In the odd case $n = 2n'+1$ when $f \in S_{2k}(SL_2(\mathbb{Z}))$, the $Lift(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n})$. By $2k - t(s,i) = t(1-s, n+1-i)$, the standard L functions has functional equation of the form $s \mapsto 1-s$ and the gamma factor is the same.

Hence the Gamma factor of Ikeda's lifting, denoted by $\mathbf{f}$, of an elliptic modular form f and used as a pattern, extends to a general (not necessarily lifted) Hermitian modular form $\mathbf{f}$ of even weight ℓ, which equals in the lifted case to $\ell = 2k+2n'$, where $k = (\ell - 2n')/2 = \ell/2 - n' = \ell/2 - n'$, when the Gamma factor of the standard zeta function with the symmetry $s \mapsto 1-s$ becomes (see p.58) $\prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s+\ell/2-n'+n'-i+(1/2))^2 =$ $\prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s+\ell/2-i+(1/2))^2 = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s+\ell/2-i-(1/2))^2$.

Thanks for your attention!

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