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An SDDP-like algorithm for infinite horizon multistage stochastic programmes

Regan Baucke

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We are familiar with the deterministic control problem

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\min_{u} \quad & \sum_{t=0}^{\infty} \gamma^t C\,(x_t, u_t), \quad \gamma \in [0,1), \\
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1. The naïve approach
2. Lipschitz considerations
3. Our algorithm
4. To the stochastic case

We can approximate our infinite horizon problem with:

$$\begin{aligned} \min_{u} \quad & \sum_{t=0}^{T} \gamma^t C(x_t, u_t), \quad \gamma \in [0,1), \\ \text{s.t.} \quad & u_t \in \mathcal{U}(x_t),\ \forall t, \\ & x_t \in \mathcal{X},\ \forall t, \\ & x_{t+1} = f(x_t, u_t),\ \forall t. \end{aligned}$$

We define the following Bellman operator

$$\mathbb{B}_t[V](x) := \min_{\substack{u \in \mathcal{U}(x), \\ f(x,u) \in \mathcal{X}}} \gamma^t C(x,u) + V(f(x,u)),$$

and "future-state" operator

$$\mathbb{F}_t[V](x) := f(x, u^{\#}(x)).$$

We can now write out our optimisation problem in terms of these Bellman operators.

$$\begin{cases} V_T = 0, \\ V_t = \mathbb{B}_t[V_{t+1}], \ \forall t < T. \end{cases}$$

We wish to compute V_0 for an initial state x_0.

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- Can be difficult to control the truncation error
- Failure to exploit self-similarity of the formulation

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- Direct compatibility with existing SDDP algorithms

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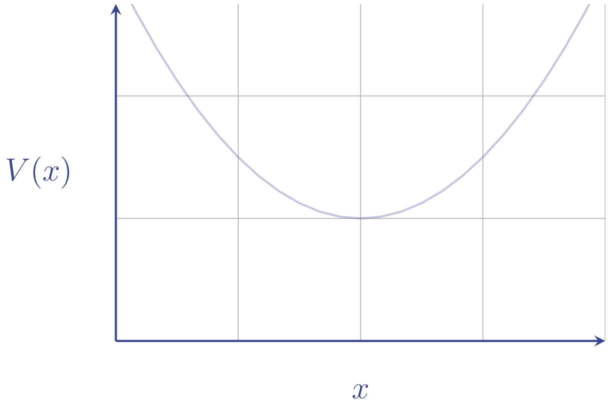
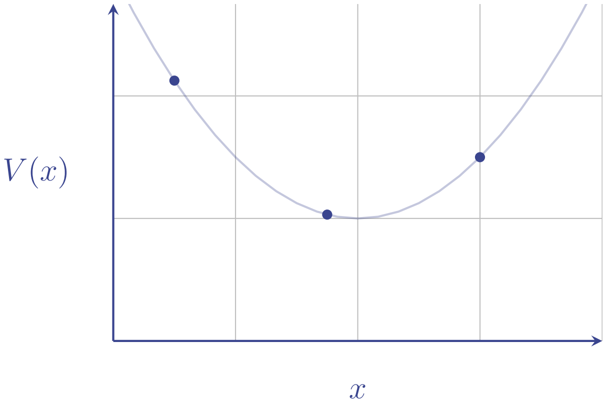


$V(x)$
x



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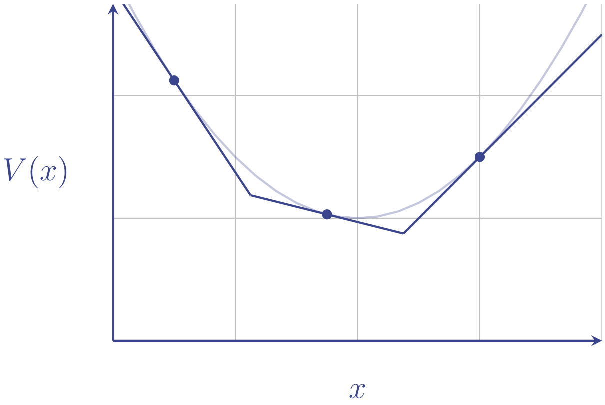

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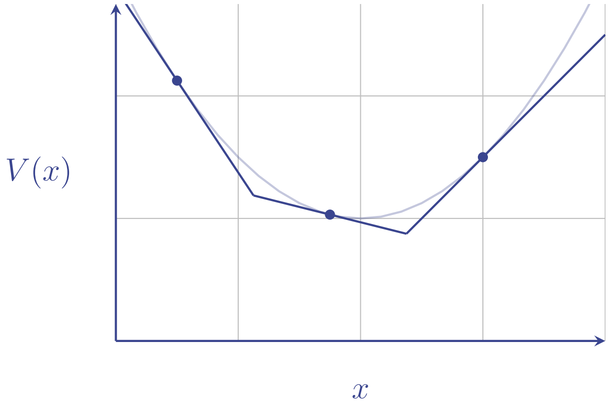
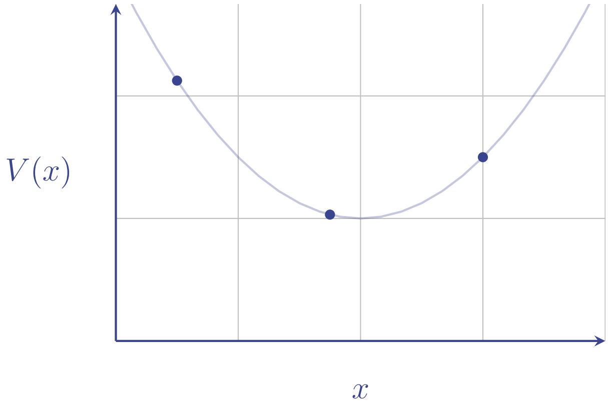


$$\underline{V}(x) = \min_{\mu \in \mathbb{R}} \quad \mu$$
$$\text{s.t.} \quad \mu \geq V(\hat{x}) + \langle d_{\hat{x}}, x - \hat{x} \rangle, \ \forall \hat{x},$$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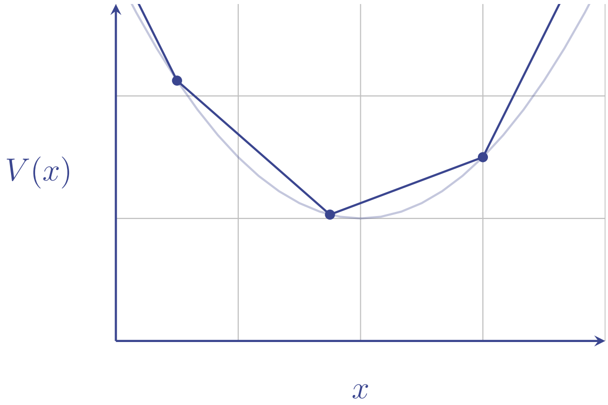
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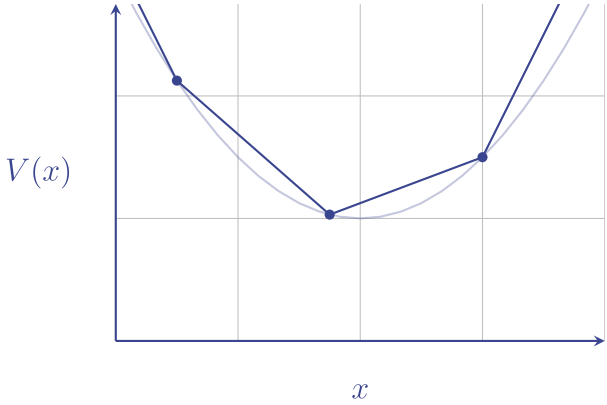

$$\bar{V}(x) = \min_{\mu \in \mathbb{R}, \lambda \in \mathbb{R}^x} \quad \mu + \langle \lambda, x \rangle$$
$$\text{s.t.} \quad \mu + \langle \lambda, \hat{x} \rangle \geq V(\hat{x}), \ \forall \hat{x}.$$
$$||\lambda||_* \leq \alpha.$$

Outline of DDP, which refines bounding functions $\underline{V}_t^k \leq V_t \leq \bar{V}_t^k$

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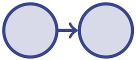

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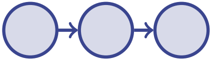

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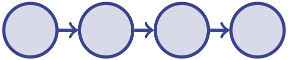

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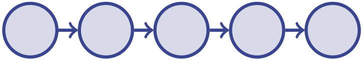

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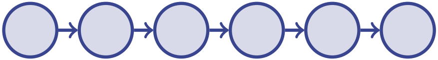

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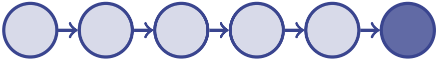

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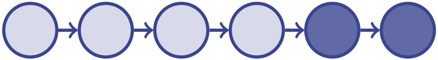

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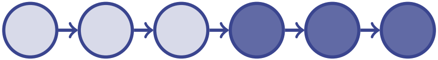

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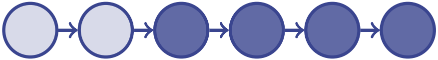

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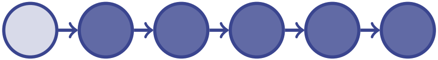

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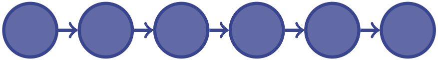

Theorem

$$\lim_{k\to\infty} \bar{V}_t^k(x_t^k) - \underline{V}_t^k(x_t^k) = 0,\ \forall t \leq T$$

Proof.

1. Value functions are Lipschitz-continuous (relies on a finiteness of T)
2. Bounding functions have nice properties

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We define the following Bellman operator

$$\mathbb{B}[V](x) := \min_{\substack{u \in \mathcal{U}(x), \\ f(x,u) \in \mathcal{X}}} C(x,u) + \gamma \times V(f(x,u)),$$

and "future-state" operator

$$\mathbb{F}[V](x) := f(x, u^{\#}(x)).$$

We can now write out our optimisation problem in terms of these Bellman operators.

$$V = \mathbb{B}[V].$$

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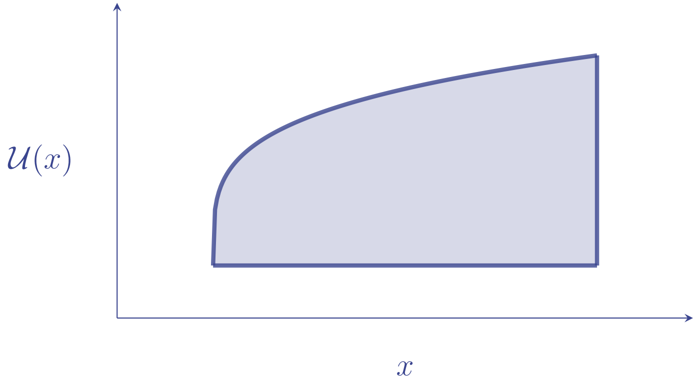

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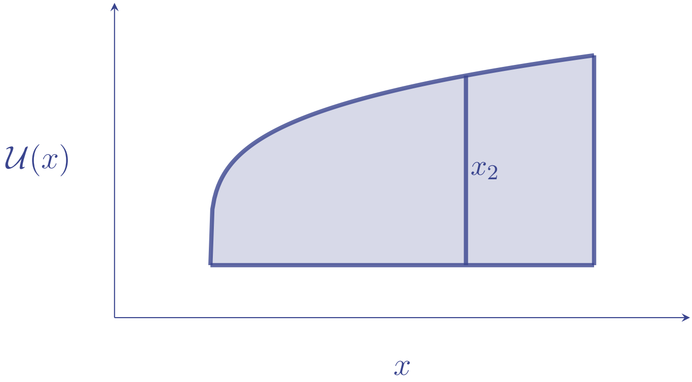

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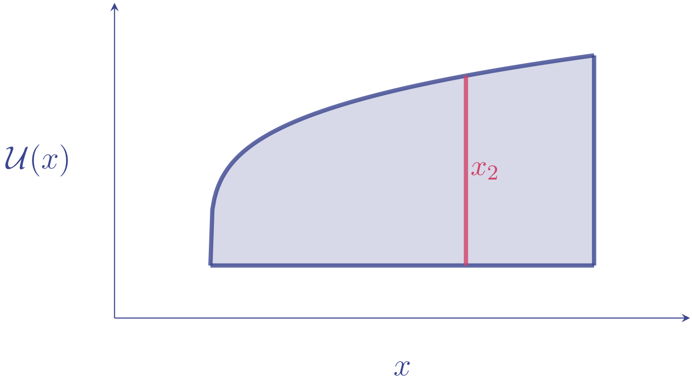

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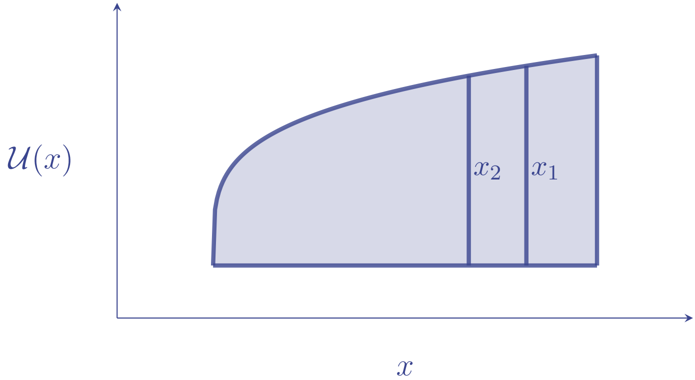

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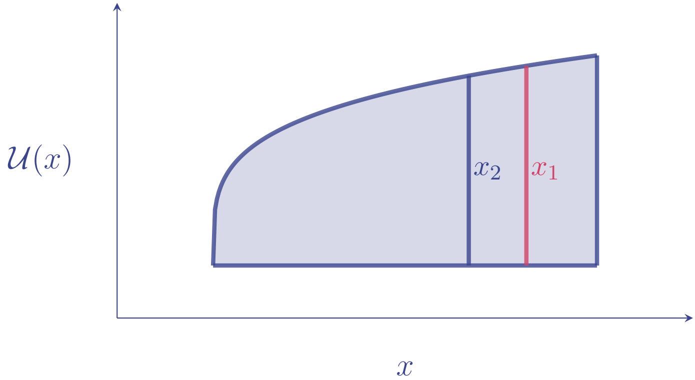

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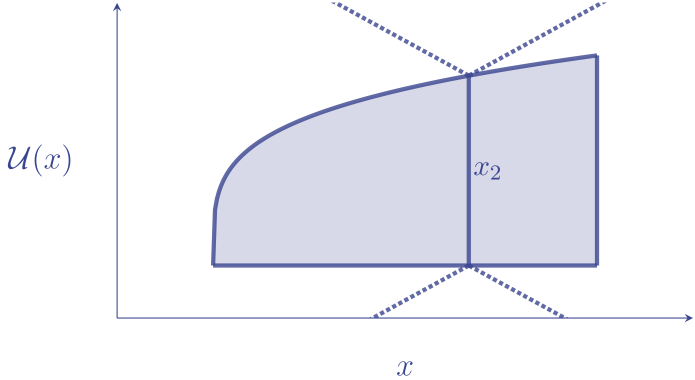

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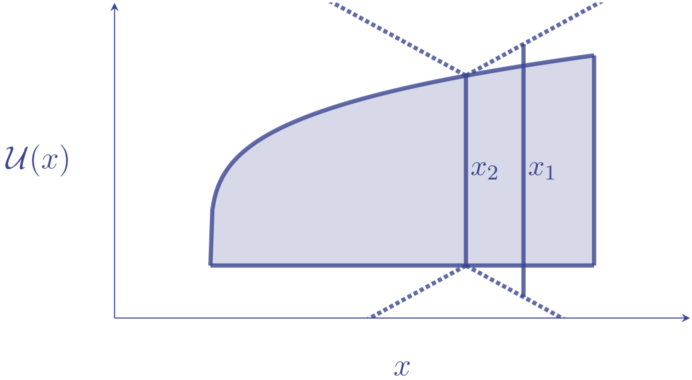

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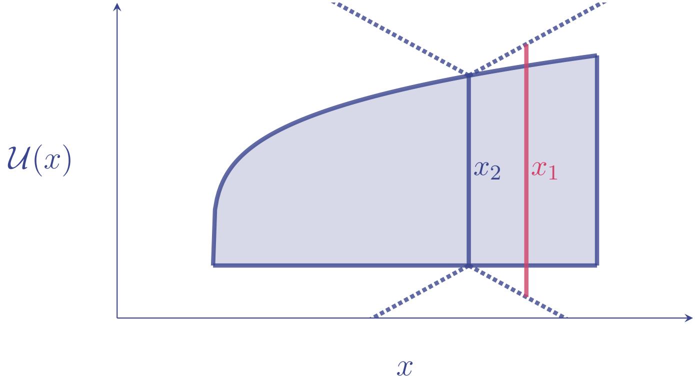

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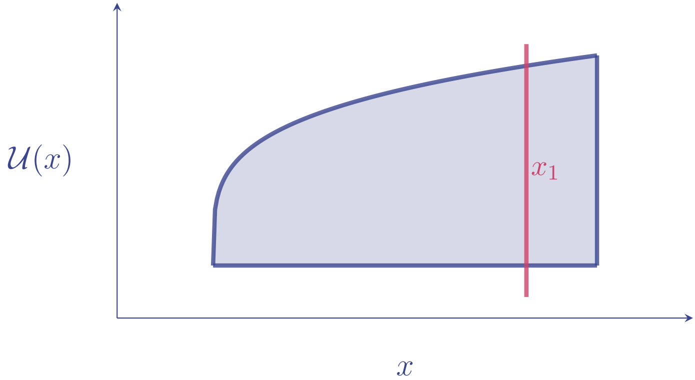

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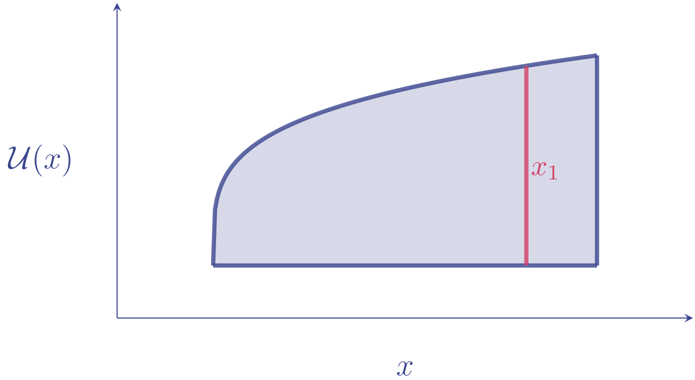

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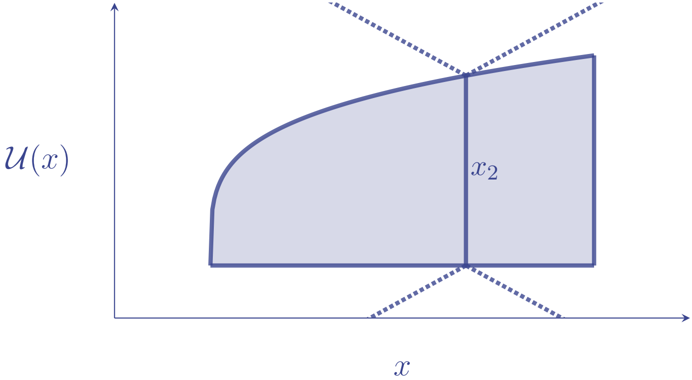

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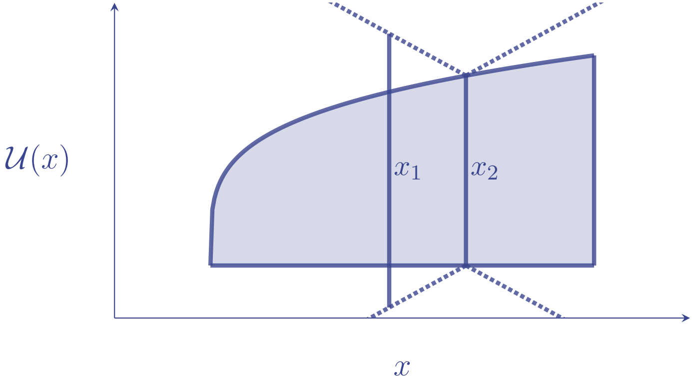

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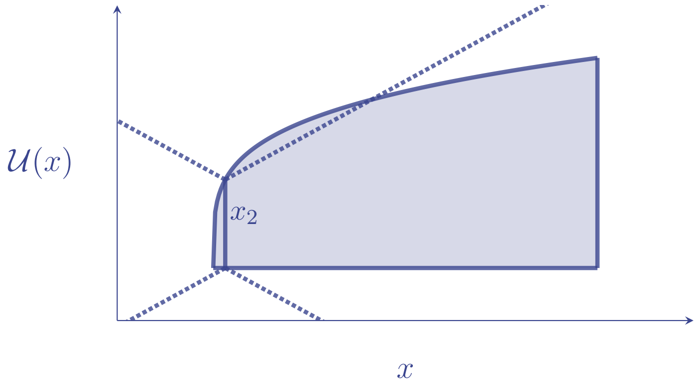

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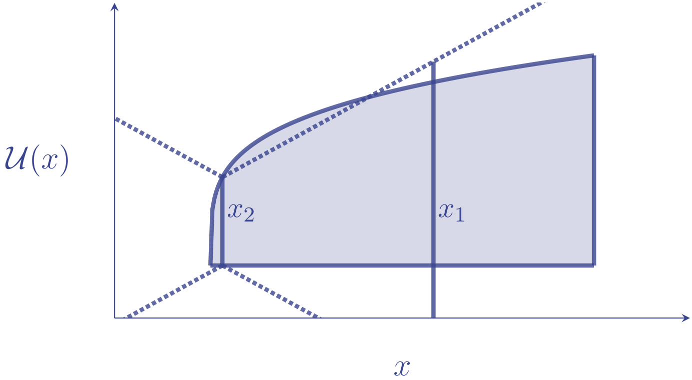

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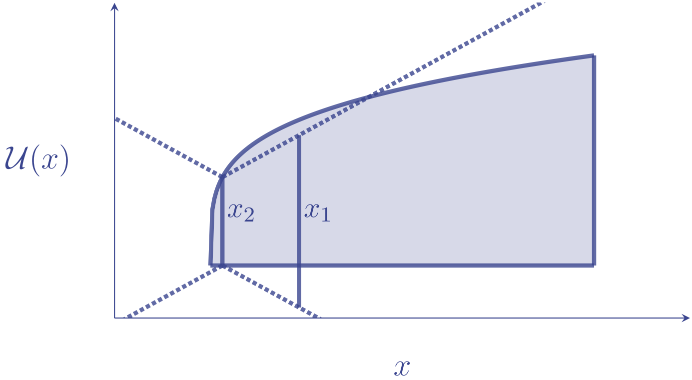

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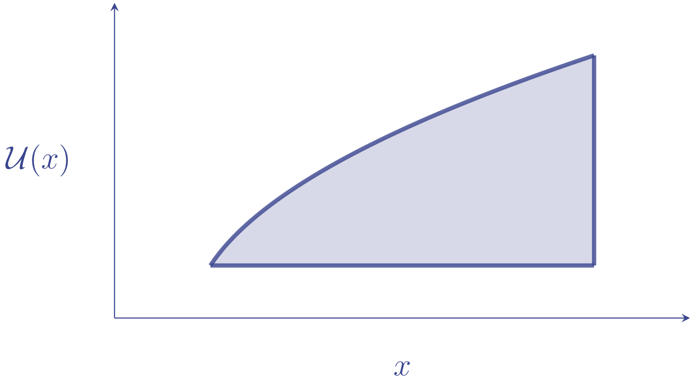

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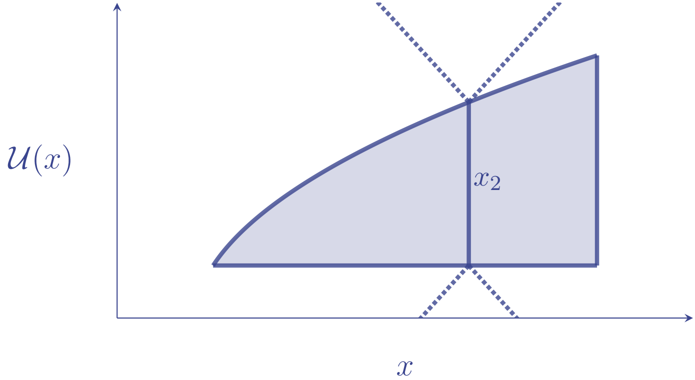

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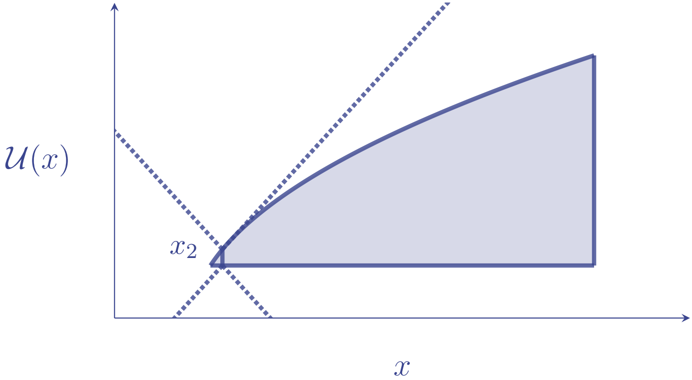

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Lemma (K. Hinderer 2005)

Suppose $\mathcal{U}, C$ *and* f *have Lipschitz constants* $L^{\mathcal{U}}, L^{C}, L^{f}$*, respectively. Then*

$$\textit{Lip}\big(\mathbb{B}(H)\big) \leq L^{C}(1 + L^{\mathcal{U}}) + \gamma L^{f}(1 + L^{\mathcal{U}})\textit{Lip}(H)$$

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Corollary

Suppose $\mathcal{U}, C$ *and* f *have Lipschitz constants* $L^{\mathcal{U}}, L^{C}, L^{f}$*, respectively. If* $\gamma L^{f}(1 + L^{\mathcal{U}}) < 1$ *then*

$$\lim_{n \to \infty} \mathit{Lip}\big(\mathbb{B}^n(0)\big) \leq \frac{L^{C}(1 + L^{\mathcal{U}})}{1 - \gamma L^{f}(1 + L^{\mathcal{U}})}$$

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$$L^V \leq \frac{L^C(1 + L^{\mathcal{U}})}{1 - \gamma L^f(1 + L^{\mathcal{U}})}$$

Similar to DDP, we are going to generate a sequence of functions $\underline{V}^k \leq V \leq \bar{V}^k$.

Question: Where should we compute cuts?

The 'Sticky' algorithm, refining bounding functions $\underline{V}^k \leq V \leq \bar{V}^k$

For a given iteration k:

1. generate new point using $x^k \in \mathbb{F}[\underline{V}^{k-1}](x^{k-1})$
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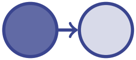

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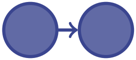

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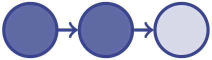

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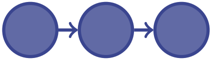

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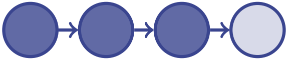

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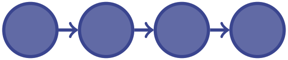

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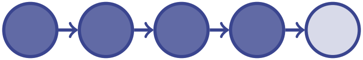

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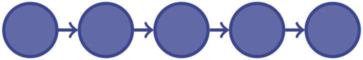

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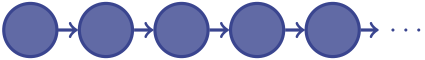

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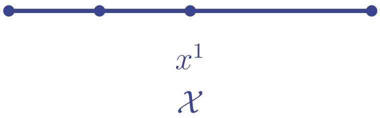

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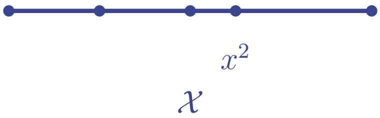

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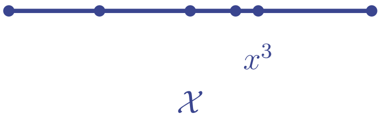

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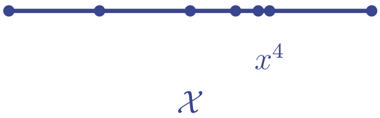

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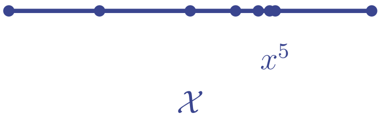

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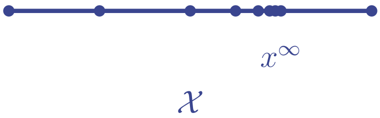

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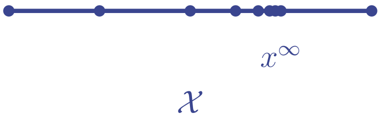

This begs the question, what type of convergence are we looking for?

It would be nice to have convergence on points generated by the policy:

$$\lim_{k\to\infty} \bar{V}^k(x_t^k) - \underline{V}^k(x_t^k) = 0, \forall t \in \mathbb{N}.$$

The key to this type of convergence is to start from x_0 at each iteration and generate longer and longer state trajectories as $k \to \infty$. Let T_k be how 'far' we look out at iteration k.

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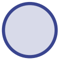

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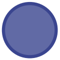

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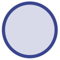

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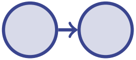

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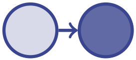

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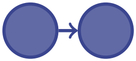

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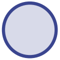

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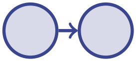

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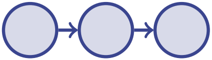

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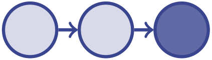

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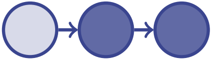

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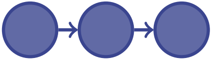

The 'Yo-yo' algorithm, refining bounding functions $\underline{V}^k \leq V \leq \bar{V}^k$

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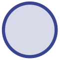

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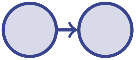

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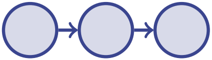

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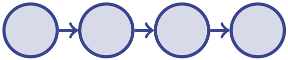

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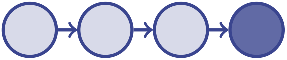

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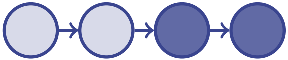

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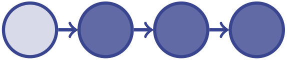

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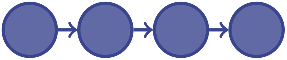

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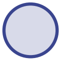

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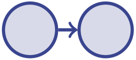

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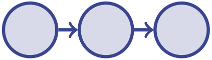

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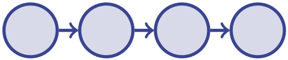

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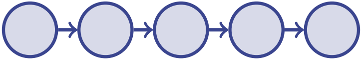

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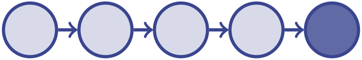

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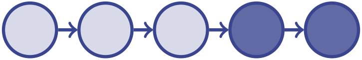

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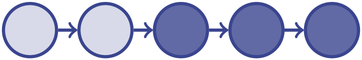

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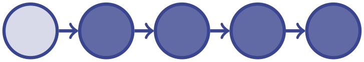

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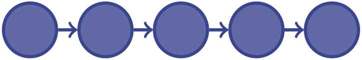

Theorem

$$\limsup_{k\to\infty} T_k = \infty \implies \lim_{k\to\infty} \bar{V}^k(x_t^k) - \underline{V}^k(x_t^k),\ \forall t \in \mathbb{N}.$$

Proof.

1. We are always looking further and further ahead and $\gamma \in [0, 1)$.
2. Value functions are Lipschitz-continuous
3. Bounding functions have nice properties

□

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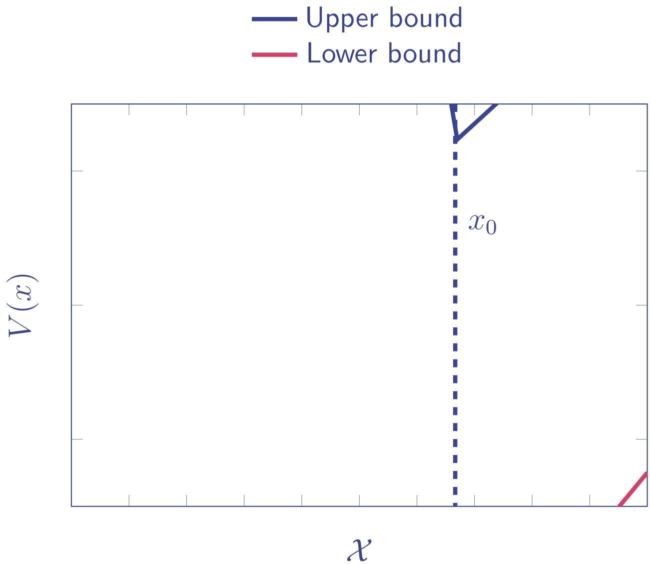
Upper bound
Lower bound
x_0
$V(x)$
$\mathcal{X}$

Now we have a discrete, independent noise in the dynamics, $\omega \in \Omega$
We define a new Bellman operator

$$\mathbb{B}[V](x) := \min_{\substack{u \in \mathcal{U}(x), \\ f_\omega(x,u) \in \mathcal{X}}} C_n(x,u) + \gamma \sum_{\omega \in \Omega} P(\omega) V(f_\omega(x,u)),$$

With a new 'future-state' operator

$$\mathbb{F}[V](x,\omega) := f_\omega(x, u^\#(x))$$

We seek a function

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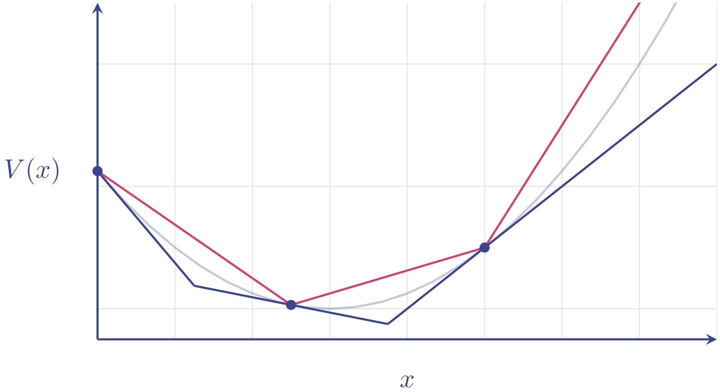
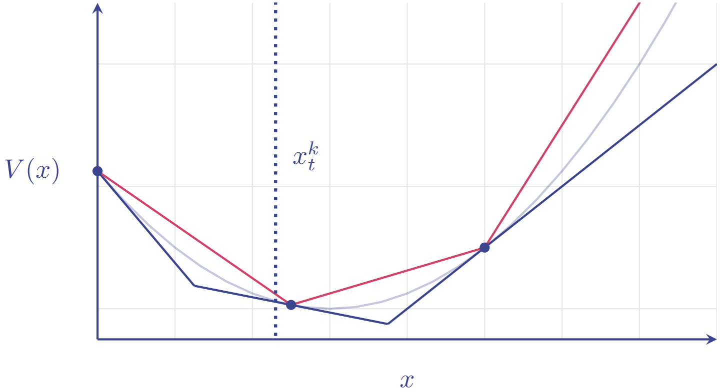
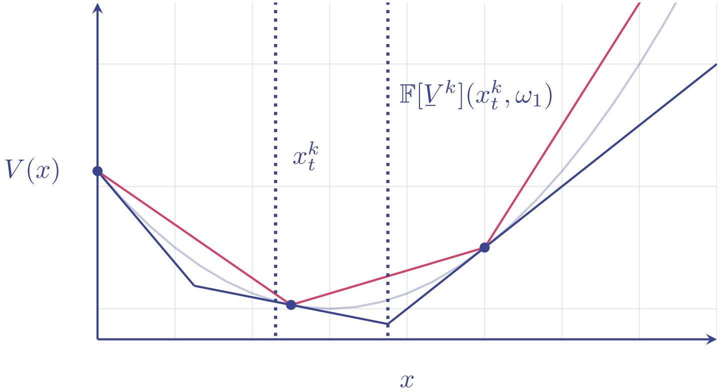
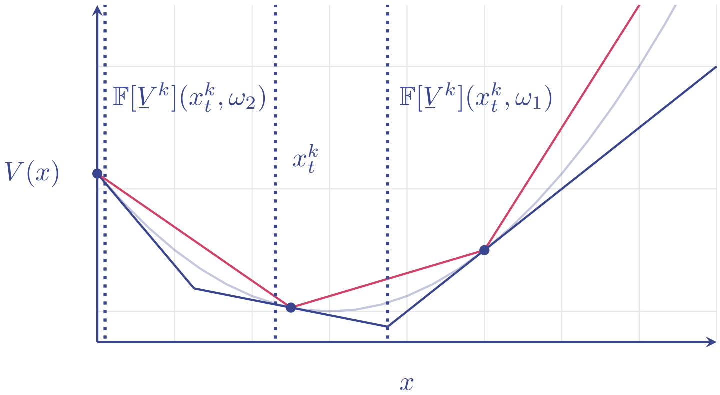

$V(x)$
x



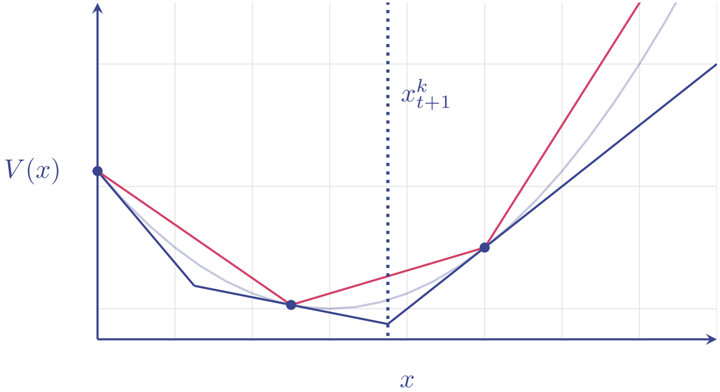
$V(x)$
x_t^k
x



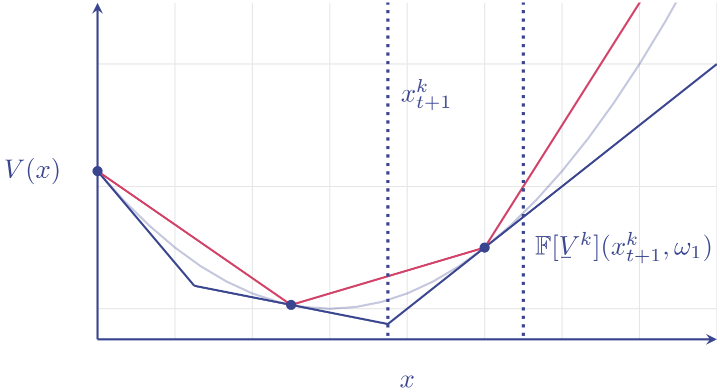
$\mathbb{F}[\underline{V}^k](x_t^k, \omega_1)$
x_t^k
$V(x)$
x

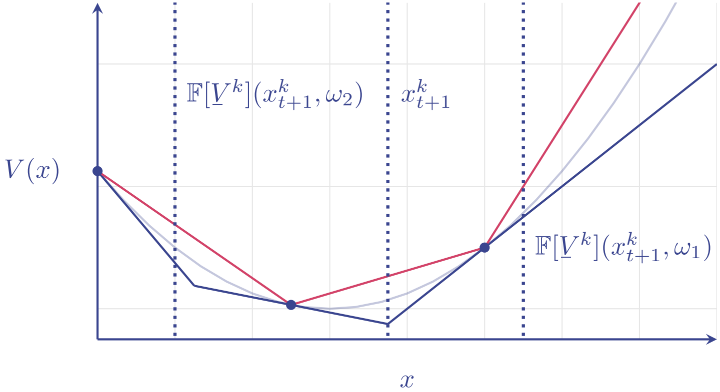


$\mathbb{F}[\underline{V}^k](x_t^k, \omega_2)$
$\mathbb{F}[\underline{V}^k](x_t^k, \omega_1)$
x_t^k
$V(x)$
x



x_{t+1}^k
$V(x)$
x


x_{t+1}^k
$V(x)$
$\mathbb{F}[\underline{V}^k](x_{t+1}^k, \omega_1)$
x



$\mathbb{F}[\underline{V}^k](x^k_{t+1}, \omega_2)$
x^k_{t+1}
$V(x)$
$\mathbb{F}[\underline{V}^k](x^k_{t+1}, \omega_1)$
x

The 'Yo-yo' algorithm, refining bounding functions $\underline{V}^k \leq V \leq \bar{V}^k$

For a given iteration k:

1. generate state trajectory from $t = 0$ to $t = T_k$ using $x_{t+1}^k \in \mathbb{F}[\underline{V}^{k-1}](x_t^k, \omega^*)$
2. where $\omega^* = \arg\max_{\omega} \bar{V}^k(\mathbb{F}[\underline{V}^{k-1}](x_t^k, \omega)) - \underline{V}^k(\mathbb{F}[\underline{V}^{k-1}](x_t^k, \omega))$
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Theorem

$$\limsup_{k\to\infty} T_k = \infty \implies$$
$$\lim_{k\to\infty} \bar{V}^k(x_t^k(\omega)) - \underline{V}^k(x_t^k(\omega)), \ \forall t \in \mathbb{N}, \forall \omega \in \{\Omega\}^t.$$

Proof.

1. Previous theorem
2. Always focusing on the 'scenario' with the largest gap.

□

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Conclusions

- Verify the conditions of Lipschitz continuity of V
- Adapting SDDP to the infinite horizon case
- Proof of convergence

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