Thermodynamic Formalism on Generalized Markov Shifts

Research Project jointly with Eric O. Endo (NYU-Shanghai)
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Partially supported by CNPq, CAPES and FAPESP

Symbolic Dynamical Systems - Oaxaca - 2019

Outline

1. Main references
2. Generalized shift spaces
   - Countable Markov Shifts and Exel-Laca Algebras
   - X_A - the candidate to replace Σ_A
   - Renewal Shift
3. Thermodynamic Formalism
4. Conformal measures on X_A in (and out) of Σ_A

Main references

- Video of Ruy Exel's talk at the youtube channel of ICM 2018.

For those who want to see more algebraic aspects of the results: groupoids, equivalence relations, $\mathbb{C}^*$-algebras...

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- **Conformal Measures on Generalized Renault-Deaconu Groupoids**. [RB, R. Exel, T. Raszeja, R. Frausino]

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- **Infinite DLR Measures and Volume-Type Phase Transitions on Countable Markov Shifts**. [E. R. Beltrán, RB, E.O. Endo]

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Generalized $=$ Locally compact version of Σ_A, denoted by X_A:

- X_A locally compact polish space. (in many cases compact)
- Σ_A is dense in X_A.
- $Y_A = X_A \backslash \Sigma_A$ is a set of finite words of the shift, it is also dense in X_A. (empty words are possible)
- When Σ_A is locally compact, then $\Sigma_A = X_A$.

Renewal shift

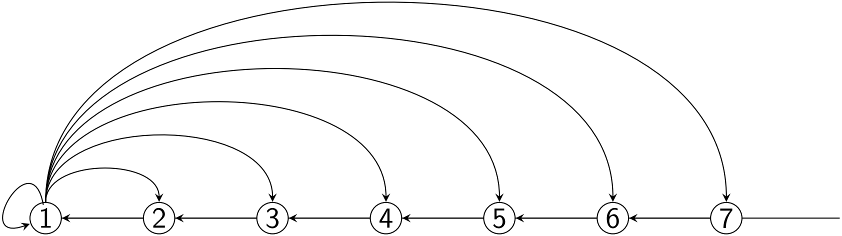

Figure: The Renewal shift Σ_A

The generalized renewal shift X_A

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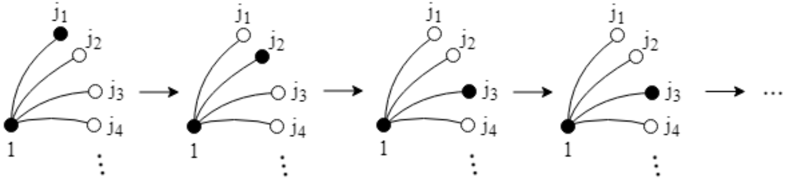
$Y_A = \{\textit{finite words ending in } 1\} \cup \{\xi_0\}$, where ξ_0 is the empty word.



Main reference:

- R. Exel and M. Laca, *Cuntz-krieger algebras for infinite matrices*. J. Reine Angew. Math., **512**, 119-172, (1999).

Partial isometries satisfiyng:

(EL1) $S_i^*S_i$ and $S_j^*S_j$ commute for every $i, j \in \mathbb{N}$;

(EL2) $S_i^*S_j = 0$ whenever $i \neq j$;

(EL3) $(S_i^*S_i)S_j = A(i,j)S_j$ for all $i, j \in \mathbb{N}$;

(EL4) for every pair X, Y of finite subsets of $\mathbb{N}$ such that the quantity

$$A(X,Y,j) := \prod_{x\in X} A(x,j) \prod_{y \in Y} (1 - A(y,j)), j \in \mathbb{N}$$

is non-zero only for a finite number of j's, we have

$$\left(\prod_{x\in X} S_x^* S_x\right)\left(\prod_{y\in Y}(1 - S_y^*S_y)\right) = \sum_{j\in\mathbb{N}} A(X,Y,j)S_jS_j^*.$$

Exel-Laca algebras

For each $s \in \mathbb{N}$, consider the following operators on $\mathfrak{B}(\ell^2(\Sigma_A))$,

$$T_s(\delta_x) = \begin{cases} \delta_{sx} \text{ if } A(s, x_0) = 1, \\ 0 \text{ otherwise;} \end{cases} \quad \text{with} \quad T_s^*(\delta_x) = \begin{cases} \delta_{\sigma(x)} \text{ if } x \in [s], \\ 0 \text{ otherwise,} \end{cases}$$

where $\{\delta_x\}_{x \in \Sigma_A}$ is the canonical basis.

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Definition (Exel-Laca algebra)

The Exel-Laca algebra $\mathcal{O}_A$ is the subalgebra of $\widetilde{\mathcal{O}}_A$ which is the unital C^*-algebra generated by the partial isometries T_s, $s \in \mathbb{N}$.

There exists a collection of projections indexed by the free group generated by $\mathbb{N}$: $e_g := T_g T_g^*$, $\quad g \in \mathbb{F}_{\mathbb{N}}$ reduced word.

These elements commute each other.

The set X_A

Consider $\mathcal{D}_A := C^*(\{e_g : g \in \mathbb{F}_\mathbb{N}\})$ the commutative unital C^*-subalgebra of $\mathcal{O}_A$ generated by the projections.

Definition

Given an irreducible transition matrix A on the alphabet $\mathbb{N}$, define the sets

$$X_A := \operatorname{spec} \mathcal{D}_A \quad \text{and} \quad \widetilde{X}_A := \operatorname{spec} \widetilde{\mathcal{D}}_A$$

where the second one is only considered in the case that $\mathcal{O}_A$ is non-unital.

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On the weak* topology it is well known that X_A is at least locally compact and that $\widetilde{X}_A$ is always compact.

Gelfand's Theorem $\Rightarrow \mathcal{D}_A := C_0(X_A)$ and $\widetilde{\mathcal{D}}_A := C(\widetilde{X}_A)$

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When we have finite number of symbols we have $\mathcal{D}_A := C(\Sigma_A)$.

X_A is a locally compact version of Σ_A.

Actually, when Σ_A is locally compact we have $X_A = \Sigma_A$.

The set X_A

$X_A \subset \{0,1\}^{\mathbb{F}} = \{0,1\}^{\mathbb{F}_{\mathbb{N}}}$

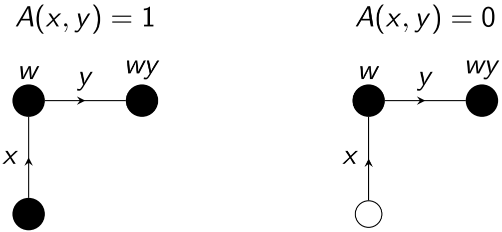
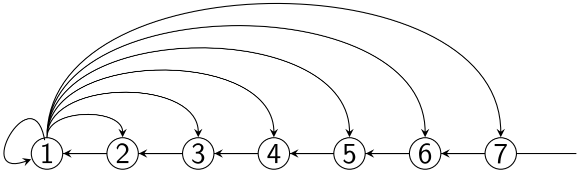


Figure: The black dots represents that the configuration ξ is filled.

$$\left\{ \begin{array}{l} \xi \in \{0,1\}^{\mathbb{F}} : \xi_e = 1,\ \xi \text{ connected}, \\ \text{if } \xi_\omega = 1, \text{ then there exists at most one } y \in \mathbb{N} \text{ s.t. } \xi_{\omega y} = 1, \\ \text{if } \xi_\omega = \xi_{\omega y} = 1,\ y \in \mathbb{N}, \text{ then for all } x \in \mathbb{N}\ (\xi_{\omega x^{-1}} = 1 \iff A(x,y) = 1) \end{array} \right.$$

The set Y_A: finite words on the Renewal shift



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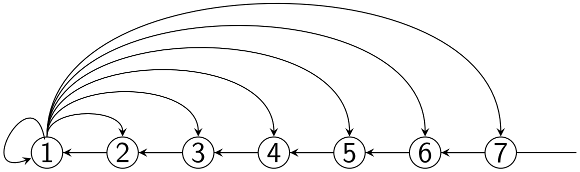
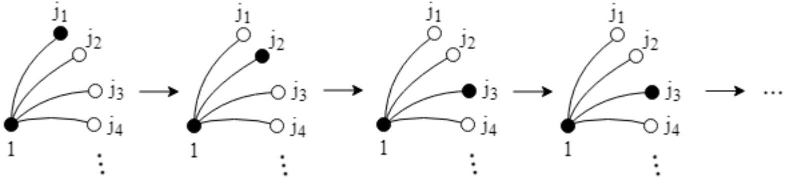
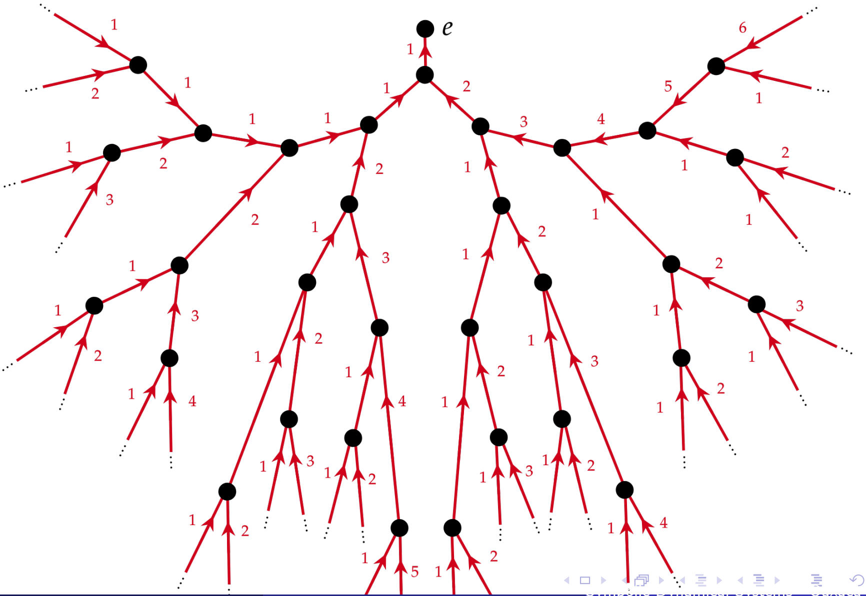




Finite words ending with the symbol 1 are the elements of Y_A for the renewal shift.

The empty word on the Renewal shift



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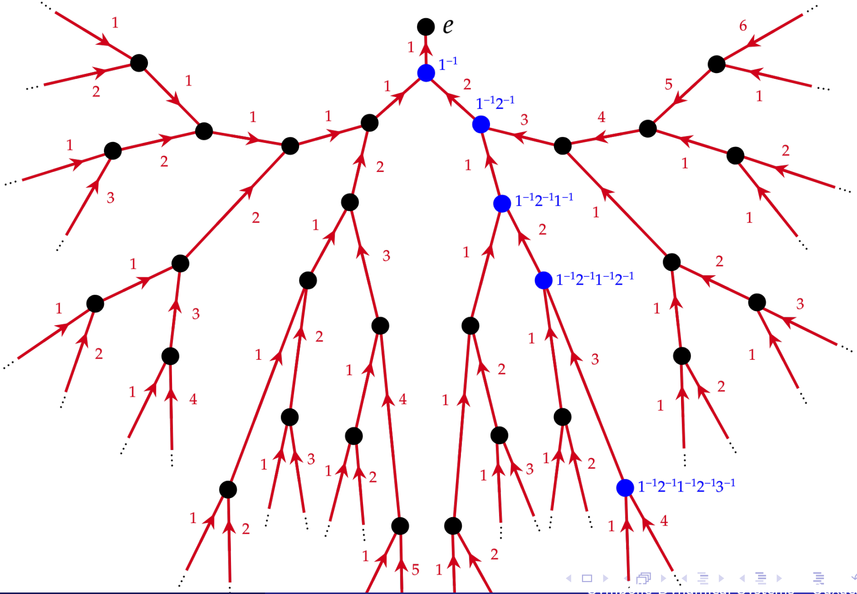

Thermodynamic Formalism

- The shift map σ is partially defined on X_A, we can not apply the shift on empty words.

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- We will assume the potential $F : U \subseteq\to \mathbb{R}$ at least continuous.

Generalizations

Definition (Ruelle transformation)

For a given continuous potential $F : U \to \mathbb{R}$ and inverse of the temperature $\beta > 0$, the Ruelle transformation $L_{-\beta F}$ is given by

$$L_{-\beta F} : C_c(U) \to C_c(X),$$
$$f \mapsto L_{-\beta F}(f)(x) := \sum_{x=\sigma(y)} e^{-\beta F(y)} f(y).$$

Generalizations

Definition (Eigenmeasure associated to the Ruelle Transformation)

Given the Borel σ-algebra $\mathbb{B}$ on X, $\sigma : U \to X$ the shift map, $F : U \to \mathbb{R}$ a continuous potential and $\beta > 0$. A measure μ on $\mathbb{B}$ is said to be a eigenmeasure associated with the Ruelle transformation $L_{-\beta F}$ when

$$\int_X L_{-\beta F}(f)(x)d\mu(x) = \int_U f(x)d\mu(x), \quad (1)$$

for all $f \in C_c(U)$.

Definition (Conformal measure - Denker-Urbański)

Let $(X, \mathcal{F})$ be a measurable space, $\sigma : U \subseteq X \to X$ a measurable endomorphism and $D : U \to [0, \infty)$ also measurable. A set $A \subseteq U$ is called special if $A \in \mathcal{F}$ and $\sigma_A := \sigma \restriction_A : A \to \sigma(A)$ is injective. A measure μ in X is said to be D-conformal in the sense of Denker-Urbański if

$$\mu(\sigma(A)) = \int_A D d\mu, \tag{2}$$

for all special sets A.

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As an example, in the Markov shifts, a Borel set contained in a cylinder set is special.

Definition

Let X be a locally compact Hausdorff and second countable topological space. Let $\sigma : U \subseteq X \to X$ a local homeomorphism. Given a borel measure μ on X we define the measure $\mu \odot \sigma$ on U by

$$\mu \odot \sigma(E) := \sum_{i \in \mathbb{N}} \mu(\sigma(E_i)).$$

For all measurable $E \subseteq U$, where the E_i are pairwise disjoint measurable sets such that $\sigma \restriction E_i$ is injective, for each i, and $E = \sqcup_i E_i$.

Conformal measures in the sense of Sarig

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The measure $\mu \odot \sigma$ is well defined since there always exists at least one family of E_i's. Moreover, the definition independs of the choice of the E_i's.

Conformal measures in the sense of Sarig

Definition (Conformal measure - Sarig)

A measure μ in X is called $(\beta F, \lambda)$-conformal in the sense of Sarig if there exists $\lambda > 0$ such that

$$\frac{d\mu \odot \sigma}{d\mu}(x) = \lambda e^{-\beta F(x)} \quad x \in U.$$

When we are in the standard thermodynamic formalism $X_A = \Sigma_A$ and the potential is regular enough, $\lambda = e^{P_G(\beta F)}$ where $P_G(\varphi)$ is the Gurevich's pressure of the potential φ.

The equivalences of conformality notions

Theorem (R. B., R. Exel, R. Frausino, T. Raszeja - 2019+)

Let X be locally compact, Hausdorff and second countable space, $U \subseteq X$ open and $\sigma : U \to X$ a local homeomorphism. Let μ be a finite measure on the Borel sets of X. For a given continuous potential $F : U \to \mathbb{R}$, the following are equivalent:

(i) *μ is $e^{\beta F}$-conformal measure in the sense of Denker-Urbański;*

(ii) *μ is an eigenmeasure (fix point) associated to the Ruelle transformation $L_{-\beta F}$;*

(iii) *μ is $(-\beta F, 1)$-conformal in the sense of Sarig.*

Some results for the classical case and the standard symbolic space:

Theorem (Sarig - CMP - 2001)

Let Σ_A be the renewal shift and let $F : \Sigma_A \to \mathbb{R}$ be a weakly Hölder continuous function such that $\sup F < \infty$.
Then there exists $0 < \beta_c \leq \infty$ such that:

(i) *For $0 < \beta < \beta_c$, there exists a $(-\beta F, e^{P(\beta F)})$ conformal measure in the sense of Sarig.*

(ii) *For $\beta_c < \beta$, there is no $(-\beta F, e^{P(\beta F)})$ conformal measures in the sense of Sarig.*

Phase Transitions

Volume-Type phase transitions:

Theorem (RB, E.R. Beltrán, E.O. Endo, 2019+)

Let Σ_A be the renewal shift and let $F : \Sigma_A \to \mathbb{R}$ be a weakly Hölder continuous function such that $\sup F < \infty$. For $\beta > 0$, consider ν_β be the eigenmeasure associated to the potential βF. Let $\beta_c \in (0, +\infty]$ from the previous theorem. Then, there exists $\tilde{\beta}_c \in (0, \beta_c]$ such that:

(i) *For $0 < \beta < \tilde{\beta}_c$, ν_β is finite.*

(ii) *For $\tilde{\beta}_c < \beta < \beta_c$, ν_β is infinite.*

$$\tilde{\beta}_c = \sup\left\{\beta \in (0, \beta_c] : \limsup_{n\to\infty} \frac{1}{n}\sum_{j=2}^{n} \phi\left(\gamma_j\right) < \frac{P_G(\beta\phi)}{\beta}\right\}$$

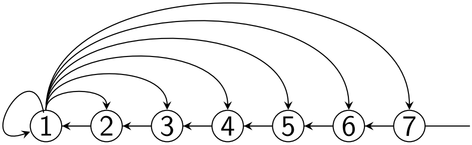
where $\gamma_j = \overline{(j, j-1, j-2, ..., 1)}$.

There are examples where $\tilde{\beta}_c < \beta_c$ and $\tilde{\beta}_c = \beta_c$.
Consider the constant potential $\phi \equiv c$ with $c \in \mathbb{R}$. It is easy to see $\tilde{\beta}_c = \beta_c = +\infty$. When $\phi(x) = x_0 - x_1$, we have $\tilde{\beta}_c = \log 2$ and $\beta_c = +\infty$.

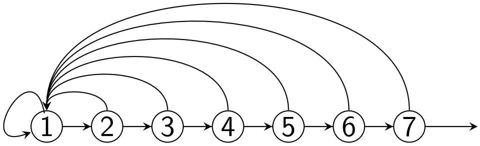
Corolary

Let Σ_A be the renewal shift and let $F : \Sigma_A \to \mathbb{R}$ be a weakly Hölder continuous function such that $\sup F < \infty$. For $0 < \beta$ small enough the eigenmeasure is finite.

Warning! Differences between renewal shift and reversal renewal shift:



$X_A \neq \Sigma_A$, we have finite words on X_A and the eigenmeasure (in the standard formalism) is finite $\beta > 0$ small enough.



$X_A = \Sigma_A$ and the eigenmeasure can be infinite for every $\beta > 0$.

Phase Transitions on Generalized Shift Spaces

Theorem (RB, R. Exel, R. Frausino, T. Raszeja - 2019+)

Consider the space X_A *associate with the renewal shift and potential* $F : X_A \setminus \{\xi^0\} \to \mathbb{R}$ *in the form*

$$F(\omega) = \beta f(\omega_0),$$

where $\beta > 0$ *is the inverse of the temperature and* $f : X_A \setminus \{\xi^0\} \to \mathbb{R}$ *depends on the first coordinate. Suppose that* f *is bounded and a non-negative function on* $X_A \setminus \{\xi^0\}$*. We let* $M > 0$ *be a lower bound. We have the results:*

(i) *If* $\beta > \frac{\log 2}{M}$*, there exists a unique* $e^{\beta f}$*-conformal measure* μ_β *that vanishes in* Σ_A*.*

(ii) *If* $\beta < \frac{\log 2}{\|f\|_\infty}$ *there are no* $e^{\beta f}$*-conformal measures that vanish in* Σ_A*.*

Phase Transitions

Corolary (RB, R. Exel, R. Frausino, T. Raszeja - 2019+)

Let $f \equiv 1$. *Then, for the constant* $\beta_c = \log 2$, *the result follows:*

(1) *For* $\beta > \beta_c$ *we have a unique* e^{β}*-conformal probability measure that vanishes on* Σ_A.

(2) *For* $\beta \leq \beta_c$ *there is no* e^{β}*- conformal probability measure that vanishes on* Σ_A.

(3) $\lim_{\beta \to \beta_c} \mu_{\beta} = \mu_{\beta_c}$ *(weak convergence) where* μ_{β_c} *lives on* Σ_A *and it is a conformal measure in the classical framework.*

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$$\mu_{\beta_c}[\alpha] = 2^{-|\alpha|}, \text{ where } \alpha \text{ is a word ending in } 1.$$

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M. Denker and M. Yuri (2015) + M. Denker and M. Urbański (1991)
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Pressure of F **at a point** $x \in X_A$ (can be a finite word)

$$P(F,x) := \limsup_{n\to\infty} \frac{1}{n} \log(Z_n(F,x)),$$

Pressure(s)

Theorem (M. Denker and M. Yuri)

Let X_A be compact, $F : U \to \mathbb{R}$ a continuous potential such that the Ruelle operator L_F is well defined. Suppose there exists a $x \in X_A$ such that $P(F, x) < \infty$ then there exists an eigenmeasure (probability) m for L_F with eigenvalue $e^{P(F,x)}$.

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Example: $F : U \to \mathbb{R}$

$F(x) = \log(x_0) - \log(x_0 + 1)$, well defined on all $x \in U$.

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Example: $F : U \to \mathbb{R}$
$F(x) = \log(x_0) - \log(x_0 + 1)$, well defined on all $x \in U$.
Take $x = 1\xi_0$, then $P(\beta F, x) = P_G(\beta F) < \infty$ for every $\beta > 0$ and:

- There exists a probability m_β such that $L^*_{\beta F} m_\beta = e^{P_G(\beta F)} m_\beta \ \forall \beta > 0$.
- By O. Sarig there exists $\beta_c \in (1, 2)$ such that for $\beta > \beta_c$ there is no eigenmeasure which comes from the standard RPF theorem.

Technical slides

In addition, define projection operator $Q_s := T_s^* T_s$, given by

$$Q_s(\delta_\omega) = \begin{cases} \delta_\omega \text{ if } \omega \in \sigma([s]); \\ 0 \text{ otherwise.} \end{cases}$$

Consider the free group $\mathbb{F} = \mathbb{F}_{\mathbb{N}}$ and the map

$$\begin{aligned} T : \mathbb{F}_{\mathbb{N}} &\to \widetilde{\mathcal{O}}_A, \\ s &\mapsto T_s, \\ s^{-1} &\mapsto T_{s^{-1}} := T_s^*. \end{aligned}$$

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Also, for any word g in $\mathbb{F}$, take its reduced form $g = x_1 ... x_n$ and define that T realizes the mapping

$$g \mapsto T_g := T_{x_1} \cdots T_{x_n},$$

and that $T_e = 1$.

Technical slides

Consider the projections

$$e_g := T_g T_g^*, \quad g \in \mathbb{F} \text{ reduced word.}$$

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Such elements commute each other. Consider the following commutative unital C^*-subalgebra of $\mathcal{O}_A$:

$$\mathcal{D}_A := C^*(\{e_g : g \in \mathbb{F}\})$$

J. Renault, *Cuntz-like algebras*, Operator theoretical methods. (Timioara, 1998), Theta Found (2000) 371-386.