Triangular Bases for Strata of Algebraic Groups

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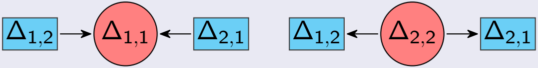
Outline

- Work with $\Bbbk = \mathbb{C}$ (or $\mathbb{C}[q^{\pm\frac{1}{2}}]$). Toy model:

- $G = SL_2 := \{g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} | \Delta_{12,12}(g) = 1\}$
- $\mathbb{C}[G] = \mathbb{C}[\Delta_{1,1}, \Delta_{1,2}, \Delta_{2,1}, \Delta_{2,2}]/(\Delta_{1,1}\Delta_{2,2} = 1 + \Delta_{1,2}\Delta_{2,1})$
  - Basis:$\{$mono. in $\Delta_{1,1}, \Delta_{1,2}, \Delta_{2,1}\} \cup \{$mono. in $\Delta_{2,2}, \Delta_{1,2}, \Delta_{2,1}\}$ (cluster monomials)
- $G^{w_0,w_0} := \{g \in SL_2 | \Delta_{1,2}(g) \neq 0, \Delta_{2,1}(g) \neq 0\}$

- (Localized) upper cluster algebra $\boldsymbol{U} = \mathbb{C}[\Delta_{1,1}^{\pm}, \Delta_{1,2}^{\pm}, \Delta_{2,1}^{\pm}] \cap \mathbb{C}[\Delta_{2,2}^{\pm}, \Delta_{1,2}^{\pm}, \Delta_{2,1}^{\pm}] = \mathbb{C}[G^{w_0,w_0}]$
  - Seeds (local charts):



- (Partially compactified) upper cluster algebra $\overline{\boldsymbol{U}} = \mathbb{C}[\Delta_{1,1}^{\pm}, \Delta_{1,2}, \Delta_{2,1}] \cap \mathbb{C}[\Delta_{2,2}^{\pm}, \Delta_{1,2}, \Delta_{2,1}] = \mathbb{C}[G]$

[FZ02] Fomin-Zelevinsky invented cluser algebras to study

- total positivity [Lus94]
- dual canonical bases $\mathbf{B}^*$ of quantum groups [Lus90, Lus91][Kas91]

Expect:

- for many varieties $\mathcal{A}$ from Lie theory, $\Bbbk[\mathcal{A}] = \overline{\boldsymbol{U}}$ (or $\boldsymbol{U}$)
- $\Bbbk[\mathcal{A}]$ has a basis: analog of $\mathbf{B}^*$, contains all cluster monomials.

FZ-Conj. [FZ02] (Kac-Moody N^w [Kim12][GLS13][GY16])

$\forall$ quantum coord. ring $\Bbbk[N]$, quantum cluster monomials $\subset \mathbf{B}^*$.

Proof: [Qin20b] $\mathbf{B}^*$ is the *common triangular basis* $\Longrightarrow$ FZ-Conj.

- Symmetric [Qin17], [KKKO18] (symmetric Kac-Moody);
- All cases [Qin20b]. (p-canonical bases [McN21])

- G: connected, simply connected, linear algebraic group
- $G^{u,v} = B_+ u B_+ \cap B_- v B_-$ double Bruhat cell
- (Quantized) coordinate ring $\Bbbk[G^{u,v}] = \boldsymbol{U}$ [BFZ05][GY20]
- $\mathbb{C}$[double Bott-Salmeson cell] $= \boldsymbol{U}$ [SW21]
- $\mathbb{C}[SL_n] = \overline{\boldsymbol{U}}$ [FWZ20]

Result 1 [Qin22]

$\Bbbk[G^{u,v}]$ and $\mathbb{C}$[double Bott-Salmeson cell] possess the common triangular bases. The bases are positive when the Cartan datum is symmetric.

Result 2 [Qin22]

$\mathbb{C}[G] = \overline{\boldsymbol{U}}$, and the statements as in Result 1 are still true.

Conjecture

- Quantized $\Bbbk[G] = \overline{\boldsymbol{U}}$.
- Its common triangular basis is the global crytal basis [Kas93].

Convention

- A seed $\mathbf{t} = (B, (x_i)_{i \in I})$:
  - $I = I_{\text{uf}} \sqcup I_{\text{f}}$ (unfrozen, frozen)
  - $B = (b_{ij})$: $I \times I$ skew-symmetrizable integer matrix
    - Skew-symmetric $B \iff$ quiver Q s.t. $b_{ij} = |i \to j| - |j \to i|$
  - $x_i =$ cluster variables, $i \in I$. $x^{\underline{m}} = \prod x_i^{m_i}$
  - $y_k = \prod x_i^{b_{ik}}$, $k \in I_{\text{uf}}$. $y^{\underline{n}} = \prod y_k^{n_k}$
- For any $k \in I_{\text{uf}}$, mutation μ_k generates a new seed $\mu_k(\mathbf{t})$
- Iterate mutations $\Longrightarrow$ more seeds, cluster variables $\Longrightarrow$ $\boldsymbol{U}$, $\overline{\boldsymbol{U}}$
- We assume $\mathbf{t}$ can be quantized as in [BZ05]
  - $\iff B_{I_{\text{uf}}, I}$ is of full-rank [GSV03, GSV05]
  - q-twisted product $*$

$\underline{g}$-pointed Functions: Replace Module Characters

- Choose a seed $\mathbf{t}$

Comparison

- Cluster monomials take the form [CC06][FZ07][DWZ10]

$$x^{\underline{g}} \cdot \sum_{\underline{n} \geq 0} c_{\underline{n}} y^{\underline{n}}, \ c_0 = 1$$

  - called $\underline{g}$-**pointed** [Qin17]
- Highest weight modules of $U_q(\widehat{\mathfrak{g}})$ have characters $\chi(S(\underline{w})) = Y^{\underline{w}} \cdot \sum_{\underline{v} \geq 0} c_{\underline{v}} A^{-\underline{v}}, \ c_0 = 1, \ c_{\underline{v}} = \dim(\text{eigen space})$

[Qin17] introduced
Dominance order $\prec_{\mathbf{t}}$: $\underline{g}$ is the highest deg of $\underline{g}$-pointed func.

- on quiver varieties: partial order of strata [Nak11]
- in monoidal category $\mathscr{M}$: interpreted via degrees of R-matrices [KK19]

Cluster algebras do not have standard bases (or PBW bases)

- For $i \in I$, $\deg(x_i) = f_i$ (i-th unit vector)
  - $x_i = CC(T_i)$, rigid T_i in a cluster category
  - $x_i = [S_i]$, simple S_i in a monoidal category
- For $k \in I_{uf}$, define pointed func $\mathbb{I}_k$ s.t $\deg \mathbb{I}_k = -f_k \mod \mathbb{Z}^{I_f}$
  - $\mathbb{I}_k :=$ q. cluster variable for (almost) all well-known cluster alg
  - $\mathbb{I}_k :=$ quantum theta function [GHKK18][DM21]
  - $\mathbb{I}_k = CC_q(T_k[1])$. [1]: shift functor (use Calabi-Yau reduction)
  - $\mathbb{I}_k = [\mathscr{D}(S_k)]$ (right) dual of S_k

Distinguished Functions (Standard Monomials)

$\mathbf{I}_{\underline{m},\underline{m}'}(\mathbf{t})$ are $\underline{g}$-pointed functions

$$q^\alpha \prod_{j\in I_f} x_j^{m_j} * \prod_{k\in I_{uf}} x_k^{m_k} * \prod_{k\in I_{uf}} \mathbb{I}_k^{m'_k}$$

where $\alpha \in \frac{\mathbb{Z}}{2}$, $m_j \in \mathbb{Z}$, $m_k, m'_k \geq 0$.

- **Reduced** if $m_k m'_k = 0\ \forall k \in I_{uf}$: denoted as $\mathbf{I}_{\underline{g}}(\mathbf{t})$.

Triangular Bases: Kazhdan-Lusztig Type Bases

- The **triangular functions** $\mathbf{L}_{\underline{g}}(\mathbf{t}) :=$ unique **Laurent series**
  1. $\overline{\mathbf{L}_{\underline{g}}} = \mathbf{L}_{\underline{g}}$ under the involution $\overline{q} = q^{-1}$
  2. $\mathbf{L}_{\underline{g}} \in \mathbf{I}_{\underline{g}} + \sum_{\underline{g}' \prec_{\mathbf{t}} \underline{g}} q^{-\frac{1}{2}}\mathbb{Z}[q^{-\frac{1}{2}}]\mathbf{I}_{\underline{g}'}$ (infinite sum)

  Computed by infinite steps of Kazhdan-Lusztig algorithm

- If $\boldsymbol{U}$ has a basis $\mathbf{L}$: $\forall \mathbf{t}$, $\mathbf{L} = \{\mathbf{L}_{\underline{g}}(\mathbf{t}), \forall \underline{g}\}$ and satisfies

$$\mathbf{I}_{\underline{m},\underline{m}'}(\mathbf{t}) \in \mathbf{L}_{\underline{g}} + \sum_{\underline{g}' \prec_{\mathbf{t}} \underline{g}} q^{-\frac{1}{2}}\mathbb{Z}[q^{-\frac{1}{2}}]\mathbf{L}_{\underline{g}'}$$

it is called the **(common) triangular basis**.

- For $A \subset \boldsymbol{U}$, if $\mathbf{L} \cap A$ is its basis, it is still called the **triangular basis**.

[BZ14]: $\mathbf{L}^{BZ}_{\underline{g}}(\mathbf{t})$ for acyclic $\mathbf{t}$. [Qin16, Qin20a] $\mathbf{L}^{BZ}_{\underline{g}}(\mathbf{t}) = \mathbf{L}_{\underline{g}}(\mathbf{t})$.

Triangular Bases: Properties and Observations

- **L** contains all cluster monomials
- **L** generalizes $\mathbf{B}^*$ (a motivation of cluster theory)
- **L** is naturally parameterized by the tropical points of the (Langlands dual) cluster variety (FG-conjecture [FG06])

- [HL10] proposed monoidal categorification of cluster algebras
  - $\forall$ known monoidal categorification [Qin17][KKKO18][KKOP21][CW19], $\{\text{simples}\} = \mathbf{L}$
- **L** is related to categorification and/or geometric representation theory (like previous Kazhdan-Lusztig type bases)
- $\forall$ known cases, **L** has positive structure constants when B is skew-symmetric.

Triangular Bases: Crystal-Like Structure?

- The triangularity of $\mathbf{L}$ can be characterized as:

$$\forall \mathbf{t},\ x_i * \mathbf{L}_{\underline{g}} \in q^{\alpha}\mathbf{L}_{\underline{g}+f_i} + \sum_{\underline{g}' \prec_{\mathbf{t}} \underline{g}+f_i} q^{\alpha} \cdot q^{-\frac{1}{2}}\mathbb{Z}[q^{-\frac{1}{2}}]\mathbf{L}_{\underline{g}'}$$

  - [Qin20a] Similar statement holds if we work with the $\prec_{\mathbf{t}}$-lowest Laurent degree (codegree)
- This is an analog of Leclerc's conjecture for $\mathbf{B}^*$ [Lec03] (proved by [KKKO18])
  - $x_i * (\ )$ acts like a crystal operator.

Freezing Operators

- Choose any seed $\mathbf{t}$. Take any Laurent series of the form

$$z = x^{\underline{m}} \cdot \sum_{\underline{n} \in \mathbb{N}^{I_{\mathrm{uf}}}} c_{\underline{n}} y^{\underline{n}}$$

- Given $F \subset I_{\mathrm{uf}}$, freeze F in $\mathbf{t}$ $\dashrightarrow$ seed $\mathbf{t}'$

The freezing operator sends $y_k \mapsto 0$ in z for $k \in F$:

$$\mathfrak{f}_{\underline{m}}(z) := x^{\underline{m}} \cdot \sum_{n_k = 0 \forall k \in F} c_{\underline{n}} y^{\underline{n}}$$

- If z has the leading degree $\deg z = m$, we abbreviate $\mathfrak{f}(z) = \mathfrak{f}_{\underline{m}}(z)$

Freezing Operators: Properties

- $\forall$ pointed Laurent series z_1, z_2, we have
$$\mathfrak{f}(z_1 * z_2) = \mathfrak{f}(z_1) * \mathfrak{f}(z_2);$$
$$\mathfrak{f}_{\deg z_1}(z_1 + z_2) = \mathfrak{f}_{\deg z_1}(z_1) + \mathfrak{f}_{\deg z_1}(z_2) \text{ if } \deg z_2 \preceq_{\mathbf{t}} \deg z_1.$$
- $\mathfrak{f}$ sends localized cluster monomials of $\boldsymbol{U}(t)$ to localized cluster monomials of $\boldsymbol{U}(t')$
- $\mathfrak{f}$ sends theta func. to theta func.

Theorem [Qin22]

Assume that $\boldsymbol{U}(\mathbf{t})$ possesses the (common) triangular basis $\mathbf{L}$, then $\mathfrak{f}(\mathbf{L})$ is the (common) triangular basis for $\boldsymbol{U}(\mathbf{t}')$.

Coefficient Change & Similarity

- Allow relabeling vertices and $q \mapsto q^{\alpha}$ in the following.

Definition ([Qin14, Qin17])

- Two seeds $\mathbf{t}, \mathbf{t}'$ are similar if they share the same unfrozen submatrix: $B_{I_{\text{uf}}, I_{\text{uf}}} = B'_{I_{\text{uf}}, I_{\text{uf}}}$. Denote $\mathbf{t} \sim \mathbf{t}'$.
    - We can also define similarity between quantum seeds
- Take $\underline{m}$-pointed Laurent series $z = x^{\underline{m}} \cdot F_z$ for $\mathbf{t}$ and $\underline{m}'$-pointed Laurent series $z' = x^{\underline{m}'} \cdot F_{z'}$ for $\mathbf{t}'$. They are similar if $\text{pr}_{I_{\text{uf}}} \underline{m} = \text{pr}_{I_{\text{uf}}} \underline{m}'$ and $F_z = F_{z'}$.

- $\mathbf{t} \sim \mathbf{t}' \implies \boldsymbol{U}(\mathbf{t})$ and $\boldsymbol{U}(\mathbf{t}')$ share similar structures
    - Localized cluster monomials are similar.
    - If S is a well-behaved basis for $\boldsymbol{U}(t)$, then the similar elements form a basis for $\boldsymbol{U}(t')$.
- If $\mathbf{t} \sim \mathbf{t}'$, $\mu_k \mathbf{t} \sim \mu_k \mathbf{t}'$.

Coefficient Change: Cartesian Product

- Coefficient ring $R(t) = \Bbbk[x_j^{\pm}]_{j \in I_{\mathrm{f}}(t)} \xrightarrow{\pi^*} \boldsymbol{U}(t)$
- $\mathbf{t}^{\mathrm{prin}}$: the seed of principal coefficients associated to $\mathbf{t}$
  - $\forall k \in I_{\mathrm{uf}}$, a framing (frozen) vertex $k' \to k$
  - $B(\mathbf{t}^{\mathrm{prin}}) = \begin{pmatrix} B_{I_{\mathrm{uf}}, I_{\mathrm{uf}}} & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$
- Assume $\boldsymbol{U}(t^{\mathrm{prin}})$ has an $R(t^{\mathrm{prin}})$-basis

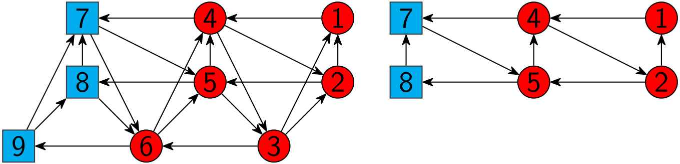
$$S = \{s_{\underline{g}} = x(t^{\mathrm{prin}})^{\underline{g}} F_{s_{\underline{g}}}(y_k(t^{\mathrm{prin}})) | \underline{g} \in \mathbb{Z}^{I_{\mathrm{uf}}}\}.$$

Then we have the following Cartesian products:

$$\begin{array}{ccccc} \operatorname{Spec} \boldsymbol{U}(t) & \xrightarrow{f} & \operatorname{Spec} \boldsymbol{U}(t^{\mathrm{prin}}) & \leftarrow & \operatorname{Spec} \boldsymbol{U}(t') \\ \downarrow \pi & & \pi \downarrow & & \downarrow \\ \operatorname{Spec} R(t) & \xrightarrow{f} & \operatorname{Spec} R(t^{\mathrm{prin}}) & \leftarrow & \operatorname{Spec} R(t') \end{array}$$

- $\forall k \in I_{\mathrm{uf}}$, $f^*(x_k) := x_k$, $f^*(x_{k'}) := \prod_{j \in I_{\mathrm{f}}(t)} x_j^{b_{jk}}$.
- $f^*(S)$ is an $R(t)$-basis of $\boldsymbol{U}(t)$, $f^*(s_{\underline{g}}) = x(t)^{\underline{g}} F_{s_{\underline{g}}}(y_k(t))$.

Example



t for $\Bbbk[N^{c^3}]$, $c = s_1 s_2 s_3$ [GLS11]

t′ for $\mathscr{C}_2$ [HL10] (subcategory of $U_q(\widehat{\mathfrak{sl}_3})$ mod)

- $z = x_4^{-1} x_7 (1 + y_4 + y_2 y_4 + y_3 y_4 + 2 y_2 y_3 y_4 + y_2^2 y_3 y_4 + 2 y_1 y_2 y_3 y_4 + 2 y_1 y_2^2 y_3 y_4 + y_1^2 y_2^2 y_3 y_4)$
- Freeze 3, 6 in **t**: $\mathfrak{f}(z) = x_4^{-1} x_7 (1 + y_4 + y_2 y_4)$
- Similar element in **t**′: $z' = x_4^{-1} x_7 (1 + y_4 + y_2 y_4)$

$\Bbbk[N^{c^N}]$ has **L** [Qin17][KKKO18]. Freezing and coefficient change: $\Longrightarrow$ q-deformed $K_0(\mathscr{C}_{N-1})$ has **L** [Qin17].

Double Bott-Salmeson Cells

- $A = (a_{i,j})_{i,j\in[1,r]}$ symmetrizable generalized Cartan matrix
- Generalized braid group $\mathrm{Br} = \langle s_i \rangle_{i\in[1,r]}$:
  - $s_i s_j = s_j s_i$, if $a_{ij}a_{ji} = 0$
  - $s_i s_j s_i = s_j s_i s_j$, if $a_{ij}a_{ji} = 1$
  - $(s_i s_j)^m = (s_j s_i)^m$, if $m = a_{ij}a_{ji} = 2, 3$
- For $\underline{j} = (j_1, \ldots, j_r)$, $s_{\underline{j}} := s_{j_1} \ldots s_{j_r}$

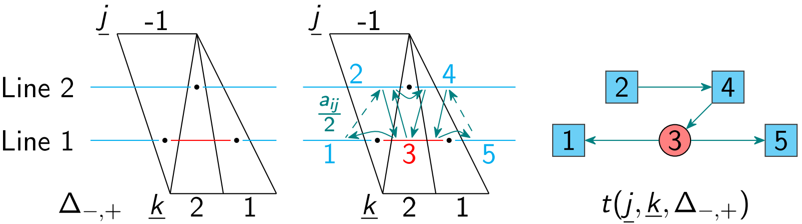
[SW21] For any double Bott-Samelson cell, we have

$$\mathbb{C}[\mathrm{Conf}_{s_{\underline{k}}}^{s_{\underline{j}}}(\mathcal{A}_{\mathrm{sc}})] = \boldsymbol{U}(\mathbf{t}(\underline{j}, \underline{k}, \Delta))$$

- If $(s_{\underline{j}'}, s_{\underline{k}'}) = (s_{\underline{j}}, s_{\underline{k}})$, $\mathbf{t}(\underline{j}', \underline{k}', \Delta')$ can be obtained from $\mathbf{t}(\underline{j}, \underline{k}, \Delta)$ by mutations [SW21]

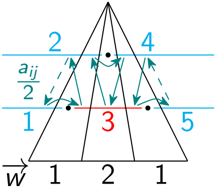
Example: Seeds for Double Bott-Salmeson cells

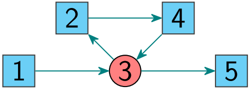
- $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $(\underline{j}, \underline{k}) = ((1),(2,1))$.
  - Choose Δ for a trapezoid (letters of $\underline{j}$ viewed as negative)



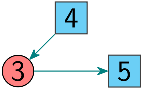
Unipotent Cells

- Weyl group $W := \mathrm{Br}/(s_i^2 = e, \forall i)$
- $\overrightarrow{w}$ reduced word of $w \in W$
- $\Bbbk[N^w] = \boldsymbol{U}(\mathbf{t}^+(\overrightarrow{w}))$, $N^w = N \cap B_- w B_-$
  - $\mathbf{t}^+(\overrightarrow{w})$: obtained from $t(\emptyset, \overrightarrow{w}, \Delta)$ by removing the left open intervals





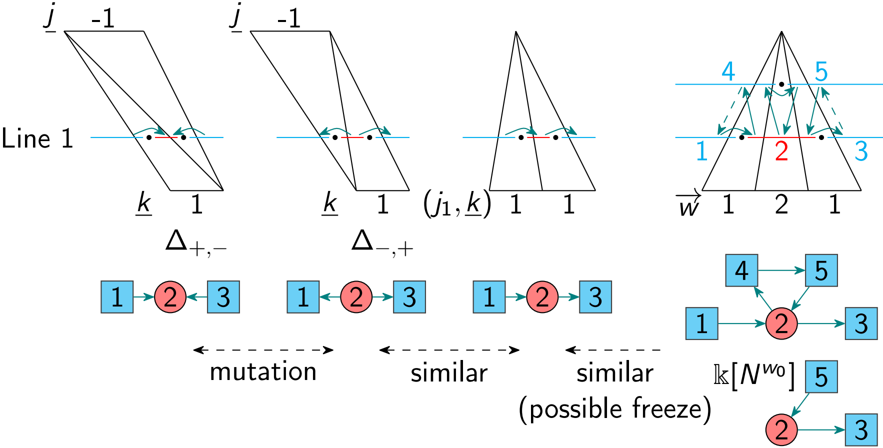
$t(\emptyset, \overrightarrow{w}, \Delta)$



$t^+(\overrightarrow{w})$

Oversimplified Example: $\Bbbk[G^{u,v}]$ from $\Bbbk[N^w]$

- If $\underline{j}, \underline{k}$ are reduced word $[s_{\underline{j}}], [s_{\underline{k}}] \in W$
  - $\Bbbk[G^{[s_{\underline{j}}],[s_{\underline{k}}]}] = \boldsymbol{U}(t(\underline{j}, \underline{k}, \Delta_{+,-}))$ [BFZ05][GY20]



Double Bott-Salmeson cells from Unipotent Cells

- Given $(\underline{j}, \underline{k})$
  - Extend size r matrix A to size $r+1$ matrix $\widetilde{A}$
  - Insert letters $r+1$ to $(\underline{j}^{op}, \underline{k}) \Longrightarrow$ reduced word $\overrightarrow{w}$

$t(\underline{j}, \underline{k}, \Delta_{-,+})$ can be obtained from $t^{+}(\overrightarrow{w})$ by mutations, freezing and coefficient change

1. Reflection (coefficient change) $t(\underline{j}, \underline{k}, \Delta_{-,+}) \sim t((j_2, \ldots), (j_1, \underline{k}), \Delta')$
2. Mutations $t((j_2, \ldots), (j_1, \underline{k}), \Delta') \dashrightarrow t((j_2, \ldots), (j_1, \underline{k}), \Delta_{-,+})$
3. Repeat, until obtain $t(\emptyset, (\underline{j}^{op}, \underline{k}), \Delta)$
4. $t(\emptyset, (\underline{j}^{op}, \underline{k}), \Delta)$ is obtained from $t(\emptyset, \overrightarrow{w}, \widetilde{\Delta})$ by freezing and then deleting the vertices on Line $r+1$.

$\Bbbk[N^w] = \boldsymbol{U}(t^{+}(\overrightarrow{w}))$ has $\mathbf{L} \Longrightarrow$ So does $\boldsymbol{U}(t(\underline{j}, \underline{k}, \Delta_{-,+}))$.

Algebraic Groups

- $\mathbb{C}[G^{w_0,w_0}] = \boldsymbol{U}$. $G = \overline{G^{w_0,w_0}}$.

$\mathbb{C}[G] \subset \overline{\boldsymbol{U}}$.
Proof: $f \in \mathbb{C}[G^{w_0,w_0}]$ is contained in $\mathbb{C}[G] \Longrightarrow$ regular on $\{x_j = 0\} \subset G$.

$\mathbb{C}[G] = \overline{\boldsymbol{U}}$.
Proof: a comparison up to codim 2 in G.

- $\forall$ j frozen, $\exists$ double Bruhat cell V_j open dense in $\{x_j = 0\}$
  - $\mathbb{C}[V_j]$= localization of $\mathbb{C}[G]/(x_j)$
  - Already know that $\mathbb{C}[V_j] = \boldsymbol{U}'$.
- Show $\boldsymbol{U}' =$ localization of $\overline{\boldsymbol{U}}/(x_j)$

Take the triangular basis $\mathbf{L} \subset \boldsymbol{U}$, then $\mathbf{L} \cap \mathbb{C}[G]$ spans $\mathbb{C}[G]$.
Proof: $\forall j$ frozen, $\exists$ an optimized seed t_j ($b_{jk} \geq 0$ $\forall k \in I_{\mathrm{uf}}$)

HAPPY BIRTHDAY, BERNARD!

References I

[BFZ05] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), no. 1, 1–52.

[BZ05] Arkady Berenstein and Andrei Zelevinsky, *Quantum cluster algebras*, Adv. Math. **195** (2005), no. 2, 405–455, `arXiv:math/0404446v2`.

[BZ14] ______, *Triangular bases in quantum cluster algebras*, International Mathematics Research Notices **2014** (2014), no. 6, 1651–1688, `arXiv:1206.3586`.

[CC06] Philippe Caldero and Frédéric Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv. **81** (2006), no. 3, 595–616, `arXiv:math.RT/0410187`.

[CW19] Sabin Cautis and Harold Williams, *Cluster theory of the coherent Satake category*, Journal of the American Mathematical Society **32** (2019), no. 3, 709–778, `arXiv:1801.08111`.

References II

[DM21] Ben Davison and Travis Mandel, *Strong positivity for quantum theta bases of quantum cluster algebras*, Inventiones mathematicae (2021), 1–119, arXiv:1910.12915.

[DWZ10] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky, *Quivers with potentials and their representations II: Applications to cluster algebras*, J. Amer. Math. Soc. **23** (2010), no. 3, 749–790, arXiv:0904.0676.

[FG06] Vladimir Fock and Alexander Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. (2006), no. 103, 1–211, arXiv:math/0311149.

[FWZ20] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky, *Introduction to Cluster Algebras. Chapter 6*, arXiv:2008.09189.

[FZ02] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I: Foundations*, Journal of the American Mathematical Society **15** (2002), no. 2, 497–529, arXiv:math/0104151.

References III

[FZ07] ———, *Cluster algebras IV: Coefficients*, Compositio Mathematica **143** (2007), 112–164, arXiv:math/0602259v3.

[GHKK18] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich, *Canonical bases for cluster algebras*, Journal of the American Mathematical Society **31** (2018), no. 2, 497–608, arXiv:1411.1394.

[GLS11] Christof Geiß, Bernard Leclerc, and Jan Schröer, *Kac-Moody groups and cluster algebras*, Advances in Mathematics **228** (2011), no. 1, 329–433, arXiv:1001.3545.

[GLS13] ———, *Cluster structures on quantum coordinate rings*, Selecta Mathematica **19** (2013), no. 2, 337–397, arXiv:1104.0531.

[GSV03] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein, *Cluster algebras and Poisson geometry*, Mosc. Math. J. **3** (2003), no. 3, 899–934, 1199, {Dedicated to Vladimir Igorevich Arnold on the occasion of his 65th birthday}.

References IV

[GSV05] ______, *Cluster algebras and Weil-Petersson forms*, Duke Math. J. **127** (2005), no. 2, 291–311.

[GY16] K.R. Goodearl and M.T. Yakimov, *Quantum cluster algebra structures on quantum nilpotent algebras*, Memoirs of the American Mathematical Society **247** (2016), no. 1169, `arXiv:1309.7869`.

[GY20] ______, *The Berenstein-Zelevinsky quantum cluster algebra conjecture*, Journal of the European Mathematical Society **22** (2020), no. 8, 2453–2509, `arXiv:1602.00498`.

[HL10] David Hernandez and Bernard Leclerc, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), no. 2, 265–341, `arXiv:0903.1452`.

[Kas91] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), no. 2, 465–516.

[Kas93] ______, *Global crystal bases of quantum groups*, Duke Math. J. **69** (1993), no. 2, 455–485, Available from: `http://dx.doi.org/10.1215/S0012-7094-93-06920-7`, `doi:10.1215/S0012-7094-93-06920-7`. MR 1203234 (94b:17024)

[Kim12] Yoshiyuki Kimura, *Quantum unipotent subgroup and dual canonical basis*, Kyoto J. Math. **52** (2012), no. 2, 277–331, `arXiv:1010.4242`, `doi:10.1215/21562261-1550976`.

[KK19] Masaki Kashiwara and Myungho Kim, *Laurent phenomenon and simple modules of quiver Hecke algebras*, Compositio Mathematica **155** (2019), no. 12, 2263–2295, `arXiv:1811.02237`.

[KKKO18] S.-J. Kang, M. Kashiwara, M. Kim, and S.-j. Oh, *Monoidal categorification of cluster algebras*, J. Amer. Math. Soc. **31** (2018), 349–426, `arXiv:1412.8106`, `doi:https://doi.org/10.1090/jams/895`.

References VI

[KKOP21] Masaki Kashiwara, Myungho Kim, Se-jin Oh, and Euiyong Park, *Monoidal categorification and quantum affine algebras II*, arXiv:2103.10067.

[Lec03] B. Leclerc, *Imaginary vectors in the dual canonical basis of $U_q(\mathfrak{n})$*, Transform. Groups **8** (2003), no. 1, 95–104.

[Lus90] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498.

[Lus91] ______, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), no. 2, 365–421.

[Lus94] ______, *Total positivity in reductive groups*, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 531–568.

[McN21] Peter J McNamara, *Cluster monomials are dual canonical*, arXiv:2112.04109.

[Nak11] Hiraku Nakajima, *Quiver varieties and cluster algebras*, Kyoto J. Math. **51** (2011), no. 1, 71–126, arXiv:0905.0002v5.

References VII

[Qin14] Fan Qin, *t-analog of q-characters, bases of quantum cluster algebras, and a correction technique*, International Mathematics Research Notices **2014** (2014), no. 22, 6175–6232, `arXiv:1207.6604`, `doi:10.1093/imrn/rnt115`.

[Qin16] ______, *Compare triangular bases of acyclic quantum cluster algebras*, `arXiv:1606.05604`.

[Qin17] ______, *Triangular bases in quantum cluster algebras and monoidal categorification conjectures*, Duke Mathematical Journal **166** (2017), no. 12, 2337–2442, `arXiv:1501.04085`.

[Qin20a] ______, *An analog of Leclerc's conjecture for bases of quantum cluster algebras*, SIGMA. Symmetry, Integrability and Geometry: Methods and Applications **16** (2020), 122, `arXiv:2004.12466`.

[Qin20b] ______, *Dual canonical bases and quantum cluster algebras*, `arXiv:2003.13674`.

[Qin22] ______, *Bases for strata of algebraic groups via operations in cluster theory*, in preparation (2022).

References VIII

[SW21] Linhui Shen and Daping Weng, *Cluster structures on double Bott–Samelson cells*, Forum of Mathematics, Sigma, vol. 9, Cambridge University Press, 2021, arXiv:1904.07992.