L-packets via types and covers

joint work with Guy Henniart and Shaun Stevens

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Introduction

- smooth complex representations of reductive p-adic groups over F local non archimedean field.
- cuspidal and supercuspidal coincide, we say cuspidal for irreducible cuspidal.
- case of symplectic groups with p odd.
- Langlands functoriality predicts that representations of $\mathrm{Sp}(2n, F)$ have a Langlands parameter : a morphism $\varphi : W_F \times \mathrm{SL}(2, F) \to \mathrm{SO}(2n+1, \mathbb{C})$. Representations having the same Langlands parameter form an L-paquet.
- Aim : describe the Langlands parameter of a cuspidal representation of $\mathrm{Sp}(2n, F)$.

C. Blondel, G. Henniart and S. Stevens, *Jordan blocks of cuspidal representations of symplectic groups,* Algebra Number Theory 12 (2018), no. 10, 2327–2386.

Outline

L-packets via Mœglin's work

Reductibility of parabolic induction

$\updownarrow$

Jordan blocks

$\updownarrow$

L-packets

Colette Mœglin, *Paquets stables des séries discrètes accessibles par endoscopie tordue ; leur paramètre de Langlands,*
in Automorphic forms and related geometry : assessing the legacy of I. I. Piatetski-Shapiro (New Haven, CT, 2012), edited by J. W. Cogdell et al., Contemp. Math. 614, Amer. Math. Soc., Providence, RI, 2014.

Parabolic induction considered

F nonarchimedean local field,
$\mathfrak{o}$, $\mathfrak{p}$, k, ϖ, $q = |k|$, p **odd.**

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$M \simeq \mathrm{GL}(N_\rho, F) \times G_0$ maximal Levi subgroup of G

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Note : $\rho|\det|^s$ self-dual $\iff$ $s = 0$ or $s = \frac{1}{2}\frac{e_\rho}{N_\rho}\frac{2i\pi}{\log q}$

where for χ unramified character of $F^\times$:

$$\rho \simeq \rho \otimes \chi \circ \det \iff \chi^{\frac{N_\rho}{e_\rho}} = 1 \quad (\tfrac{N_\rho}{e_\rho} \text{ torsion number}).$$

Reducibility points of $\mathrm{Ind}_P^G\, \rho|\det|^s \otimes \pi$

$$\{s \in \mathbb{R} / \ I(\pi, \rho, s) \text{ is reducible}\} = \{\pm s_0\},\ s_0 \in \mathbb{R}. \quad \text{(Silberger)}$$

if π is generic : $s_0 \in \{0, \pm\frac{1}{2}, \pm 1\}$;

otherwise : $s_0 \in \frac{1}{2}\mathbb{Z}$. (Mœglin-Tadić)

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Red(π) set of pairs (ρ, s) such that

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$\mathrm{Jord}(\pi)$ set of pairs (ρ, a) with $a \geq 1$ integer, such that

$$\exists\, s \,/\, (\rho, s) \in \mathrm{Red}(\pi) \text{ and } a_\rho - a \in 2\mathbb{Z}_{\geq 0}.$$

Jordan blocks and Langlands parameters

Langlands parameter for $\pi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}(2k+1, \mathbb{C})$

$$\bigoplus_{(\rho,a)\in \mathrm{Jord}(\pi)} L(\rho) \otimes \mathrm{St}_a$$

–> Langlands parameter for π determined by Red(π).

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- find the finite number of self-dual cuspidal representations ρ of $\mathrm{GL}(N_\rho, F)$ such that :

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Approach : use types and covers

$\mathrm{Ind}_P^G\ \rho|\det|^s \otimes \pi$ via types and covers

types (J_π, λ_π) in G_0 for π and $(\tilde{J}_\rho, \tilde{\lambda}_\rho)$ in L_ρ for ρ

$$\updownarrow$$

G-cover (J, λ) of $(\tilde{J}_\rho \times J_\pi, \tilde{\lambda}_\rho \otimes \lambda_\pi)$

$$\updownarrow$$

reducibility of Hecke algebras modules

$$\mathrm{Hom}_{\mathcal{H}(M, \tilde{\lambda}_\rho \otimes \lambda_\pi)}(\mathcal{H}(G, \lambda), X)$$

Colin J. Bushnell and Philip C. Kutzko. *Smooth representations of reductive p-adic groups : structure theory via types.*
Proc. London Math. Soc. (3), 77(3) :582–634, 1998.

Covers

$$\rho = \text{c-Ind}_{N(\tilde{J}_\rho)}^{L_\rho} \overline{\tilde{\lambda}_\rho} \text{ (Bushnell-Kutzko) ; } \pi = \text{c-Ind}_{J_\pi}^{G_0} \lambda_\pi \text{ (Stevens).}$$

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(J, λ) is a G-cover of $(\tilde{J}_\rho \times J_\pi, \tilde{\lambda}_\rho \otimes \lambda_\pi)$ if

- for $P = MU$ parabolic subgroup of G with Levi M

$$J = (J \cap U^-)\,(J \cap M)\,(J \cap U) \text{ with } J \cap M = \tilde{J}_\rho \times J_\pi$$

$$\lambda \text{ trivial on } J \cap U^- \text{ and } J \cap U \text{ and } \lambda_{|J \cap M} \simeq \tilde{\lambda}_\rho \otimes \lambda_\pi$$

$$\begin{pmatrix} \mathrm{I}_{N_\rho} & 0 & 0 \\ - & \mathrm{I}_{2k} & 0 \\ - & - & \mathrm{I}_{N_\rho} \end{pmatrix} \begin{pmatrix} g & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & \bar{g}^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{I}_{N_\rho} & * & * \\ 0 & \mathrm{I}_{2k} & * \\ 0 & 0 & \mathrm{I}_{N_\rho} \end{pmatrix} \longmapsto \tilde{\lambda}_\rho(g) \otimes \lambda_\pi(z)$$

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- a strong technical condition, that ensures :

(J, λ) is a type for $[M, \rho \otimes \pi]_G$.

Types and Hecke algebras

$$\mathcal{H}(G,\lambda) = \{f : G \to \mathrm{End}(V_\lambda) \mid f \text{ compactly supported and } \forall g \in G,\ \forall j,k \in J,\ f(jgk) = \lambda(j)f(g)\lambda(k)\}$$

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This provides an equivalence of categories

$$M_\lambda : \mathrm{Rep}^{[M,\rho\otimes\pi]}(G) \longrightarrow \mathrm{Mod} - \mathcal{H}(G,\lambda)$$
$$\sigma \longmapsto \sigma_\lambda = \mathrm{Hom}_J(\lambda,\sigma)$$

Right action of $f \in \mathcal{H}(G,\lambda)$ on $\phi \in \sigma_\lambda$:

$$\phi.f(v) = \int_G \sigma(g^{-1})\, \phi(f(g)v)dg \quad (v \in V_\lambda).$$

Fundamental commutative diagram

(J, λ) G-cover of $(\tilde{J}_\rho \times J_\pi, \tilde{\lambda}_\rho \otimes \lambda_\pi)$ provides

$$t : \mathcal{H}(M, \tilde{\lambda}_\rho \otimes \lambda_\pi) \hookrightarrow \mathcal{H}(G, \lambda)$$

on modules : $\quad t_*(X) = \mathrm{Hom}_{\mathcal{H}(M, \tilde{\lambda}_\rho \otimes \lambda_\pi)}(\mathcal{H}(G, \lambda), X)$.

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The following diagram commutes :

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–> Study reducibility of $M_\lambda(\mathrm{I}nd_P^G \rho|\det|^s \otimes \pi)$:

$$\mathrm{Hom}_{\mathcal{H}(M,\tilde{\lambda}_\rho\otimes\lambda_\pi)}(\mathcal{H}(G,\lambda), (\rho|\det|^s \otimes \pi)_{\tilde{\lambda}_\rho\otimes\lambda_\pi}).$$

The $\mathcal{H}(M, \tilde{\lambda}_\rho \otimes \lambda_\pi)$-module $(\rho|\det|^s \otimes \pi)_{\tilde{\lambda}_\rho \otimes \lambda_\pi}$

$\rho = \text{c-Ind}_{N(\tilde{J}_\rho)}^{L_\rho} \overline{\tilde{\lambda}_\rho}$, $\overline{\tilde{\lambda}_\rho}$ extension of $\tilde{\lambda}_\rho$ to $N(\tilde{J}_\rho) = \varpi_E^{\mathbb{Z}} \tilde{J}_\rho$, with E extension of F attached to $\tilde{\lambda}_\rho$, $e(E/F) = e_\rho$.

$\pi = \text{c-Ind}_{J_\pi}^{G_0} \lambda_\pi$.

Intertwining of $\tilde{\lambda}_\rho \otimes \lambda_\pi$ supported on $\Pi_E^{\mathbb{Z}}(\tilde{J}_\rho \times J_\pi)$ with

$$\Pi_E = \begin{pmatrix} \varpi_E^{-1} & 0 & 0 \\ 0 & \mathrm{I}_{2k} & 0 \\ 0 & 0 & \bar{\varpi}_E \end{pmatrix}$$

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$\mathcal{H}(M, \tilde\lambda_\rho \otimes \lambda_\pi) \simeq \mathbb{C}[Z, Z^{-1}]$ with Supp $Z = \Pi_E(\tilde J_\rho \times J_\pi)$

commutative algebra : $\dim_{\mathbb{C}} \left((\rho|\det|^s \otimes \pi)_{\tilde\lambda_\rho\otimes\lambda_\pi} \right) = 1$

Character given by $\xi_s(Z) = |\det \varpi_E|^s \, \xi_0(Z)$.

$\mathrm{Hom}_{\mathcal{H}(M,\tilde\lambda_\rho\otimes\lambda_\pi)}(\mathcal{H}(G,\lambda),(\rho|\det|^s\otimes\pi)_{\tilde\lambda_\rho\otimes\lambda_\pi})$.

$\mathcal{H}(G,\lambda)$ is a free $\mathcal{H}(M,\tilde\lambda_\rho\otimes\lambda_\pi)$-module of rank the cardinality of

$$N_G([M,\rho\otimes\pi])/M=\{g\in N_G(M)/[\rho\otimes\pi]^g\simeq[\rho\otimes\pi]\}/M.$$

Rank 1 if $\rho|\det|^s$ never selfdual : induced repr. always irreducible.

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Here ρ self-dual hence

$\mathcal{H}(G,\lambda)$ is a free $\mathcal{H}(M,\tilde{\lambda}_\rho\otimes\lambda_\pi)$-module of rank 2.

Actually for suitable $K\supset J$ maximal compact subgroup :

$$\mathcal{H}(G,\lambda)\overset{\text{v.s.}}{\simeq}\mathcal{H}(K,\lambda)\otimes t\ (\ \mathcal{H}(M,\tilde{\lambda}_\rho\otimes\lambda_\pi)\ )$$

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Reducibility <–> one-dim ^{al} repr. of $\mathcal{H}(G,\lambda)$ extending ξ_s.

Reducibility points (real parts)

$\mathcal{H}(G,\lambda)$ has two generators with quadratic relations, say T_0 and T_1 with $T_0 T_1$ a scalar multiple of $t(Z)$.

Need to find them,

- either exactly, by hand calculation ;
- or through Hecke algebras of finite reductive groups

Example : hand calculation in $\mathrm{Sp}(4, F)$

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Laure Blasco et Corinne Blondel. *Algèbres de Hecke et séries principales généralisées de* $\mathrm{Sp}_4(F)$, Proc. London Math. Soc. (3) 85 (2002).

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Two generators T_w and T_s supported respectively on (the J-double cosets of)

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{I}_2 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ and } s = \begin{pmatrix} 0 & 0 & \varpi \\ 0 & \mathrm{I}_2 & 0 \\ -\varpi^{-1} & 0 & 0 \end{pmatrix}, \quad ws = -\Pi$$

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Compute $T_w^2 = b_w T_w + c_w$, $T_s^2 = b_s T_s + c_s$.

(Calculation involving systems of representatives of $J/J \cap wJw^{-1}$, an explicit realisation of the type for π, Gauss sums...)

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Results ($\chi = \tilde{\lambda}_\rho = \rho_{|\mathfrak{o}^\times}$, W space of λ_π)

(2.17) TABLEAU. *Cas non ramifié, $n = 1$ et $\dim W = \frac{1}{2}(q-1)$:*

$$\chi = 1 : \begin{cases} T_w^2 = -q(q-1)T_w + q^3, \\ T_s^2 = (q-1)T_s + q; \end{cases} \qquad \chi \neq 1 : \begin{cases} T_w^2 = \pm(q^2-1)q^{1/2}T_w + q^3, \\ T_s^2 = q. \end{cases}$$

Cas non ramifié, n impair et $\dim W \neq \frac{1}{2}(q-1)$:

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Cas ramifié :

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$\mathcal{H}(G, \lambda)$ by generators and relations

Theorem (Miyauchi-Stevens 2014)

$\mathcal{H}(G, \lambda)$ is generated by T_0 and T_1, supported respectively on double cosets Jw_0J and Jw_1J, with relations

$$(T_0 - q^{r_0})(T_0 + 1) = 0, \quad (T_1 - q^{r_1})(T_1 + 1) = 0, \quad r_0, r_1 \in \mathbb{Z}.$$

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- 4 repr. of dimension 1 (with multiplicities).

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Proposition (Blondel 2012)

The real parts of the reducibility points are

$$\frac{\pm 1}{N_\rho / e_\rho} \frac{r_0 + r_1}{2}, \quad \frac{\pm 1}{N_\rho / e_\rho} \frac{r_0 - r_1}{2}$$

Strategy for $\mathrm{Red}(\pi) = \{(\rho, s)/I(\pi, \rho, s) \text{ reducible}, s \geq 1\}$

Recall : types (J_π, λ_π) in G_0 for π and $(\tilde{J}_\rho, \tilde{\lambda}_\rho)$ in L_ρ for ρ.

Don't want / don't need to compute relations for all $(\tilde{J}_\rho, \tilde{\lambda}_\rho)$.

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Don't want / don't need to compute relations for all $(\tilde{J}_\rho, \tilde{\lambda}_\rho)$.

- Guess a reasonable set of good candidates.
- Compute relations and real parts of reducibility points for those.
- If we find "enough" real parts ≥ 1 we are done.

  "enough" : $\sum_{(\rho,a)\in \text{Jord}(\pi)} a N_\rho = 2k + 1$

Inertial Jordan set

Say $\{s_1, s_2\}$ non-negative real parts of reducibility points.
Does reducibility actually occur :

- for $\rho|\det|^{s_1}$ or for $\left(\rho|\det|^{\frac{1}{2}\frac{e_\rho}{N_\rho}\frac{2i\pi}{\log q}}\right)|\det|^{s_1}$?
- for $\rho|\det|^{s_2}$ or for $\left(\rho|\det|^{\frac{1}{2}\frac{e_\rho}{N_\rho}\frac{2i\pi}{\log q}}\right)|\det|^{s_2}$?

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We determine the *Inertial Jordan set* *IJord*(π) :
multiset of pairs $([\rho], a)$ with $(\rho, a) \in$ Jord(π).

Determination of the inertial Jordan set

- structure of λ_π –> semi-simple objects,
- related simple objects and ""simple"" cuspidal representations π_i,
- relate IJord(π) to the IJord(π_i)
- compute IJord(π_i) using Lusztig's results.

Skew semi-simple data for $\pi = \text{c-Ind}_{J_\pi}^{G_0} \lambda_\pi$

1. a skew semi-simple stratum $(\Lambda, -, -, \beta)$ in $G_0 = \mathrm{Sp}_F(V)$:

$$V = V_1 \perp \cdots \perp V_r, \ \dim V = 2k, \ \dim V_i = 2k_i,$$

on each V_i a skew simple stratum $(\Lambda_i, -, -, \beta_i)$.

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2. a skew semi-simple character θ_π on H_π^1 :

$$\begin{array}{ccccc} J_\pi & \supset & J_\pi^1 & \supset & H_\pi^1 \\ \downarrow \kappa_\pi \ \substack{\text{p-primary} \\ \text{beta-extension}} & & \downarrow \eta_\pi \ \text{unique} & & \downarrow \theta_\pi \\ \mathrm{GL}_v(\mathbb{C}) & \supset & \mathrm{GL}_v(\mathbb{C}) & \supset & \mathrm{GL}_1(\mathbb{C}) \end{array}$$

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On $H_\pi^1 \cap \mathrm{Sp}_F(V_i)$ we get a skew simple character

$$\theta_{\pi,i} : H_{\pi,i}^1 \longrightarrow \mathrm{GL}_1(\mathbb{C})$$

Level zero data for $\pi = \text{c-Ind}_{J_\pi}^{G_0} \lambda_\pi$

Let $E = F[\beta]$ and $E_i = F[\beta_i]$ (a field), so $E = E_1 \oplus \cdots E_r$.

G_{E_i} : centralizer of β_i in $\mathrm{Sp}_F(V_i)$, $G_E = G_{E_1} \times \cdots \times G_{E_r}$.

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$$J_{\pi,i} = P_{E_i}(\Lambda_i)\, J^1_{\pi,i} \text{ with } P_{E_i}(\Lambda_i) = P(\Lambda_i) \cap G_{E_i}$$

$$J_{\pi,i}/J^1_{\pi,i} \simeq P_{E_i}(\Lambda_i)/P^1_{E_i}(\Lambda_i) = \mathcal{G}_i \text{ finite reductive group.}$$

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$$J_\pi = \left(P_{E_1}(\Lambda_1) \times \cdots \times P_{E_r}(\Lambda_r)\right)\, J^1_\pi$$
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Finally $\lambda_\pi = \kappa_\pi \otimes \tau_\pi$ with τ_π cuspidal representation of $\mathcal{G}_\pi$

$\tau_\pi = \tau_1 \otimes \cdots \otimes \tau_r$ $\quad$ τ_i cuspidal rep. of $\mathcal{G}_i$

Hecke algebras of finite reductive groups

Recall the cover (J, λ) :

$$\begin{pmatrix} \mathrm{I}_{N_\rho} & 0 & 0 \\ - & \mathrm{I}_{2k} & 0 \\ - & - & \mathrm{I}_{N_\rho} \end{pmatrix} \begin{pmatrix} g & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & \bar{g}^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{I}_{N_\rho} & * & * \\ 0 & \mathrm{I}_{2k} & * \\ 0 & 0 & \mathrm{I}_{N_\rho} \end{pmatrix} \longmapsto \tilde{\lambda}_\rho(g) \otimes \lambda_\pi(z)$$

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For (J,λ) : related semi-simple stratum $(\Sigma, -, -, \alpha)$ with
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Theorem (Stevens 2008)

There are skew semi-simple strata $(\mathfrak{M}_t, -, -, \alpha)$, $t = 0, 1$,

$$
\begin{array}{ccc}
P_{F[\alpha]}(\Sigma)/P^1_{F[\alpha]}(\Sigma) & \simeq \mathcal{G}_\rho \times \mathcal{G}_\pi & \tau = \tilde{\tau}_\rho \otimes \tau_\pi \\
\cap \ \text{Levi} & & \text{inflation} \\
P_{F[\alpha]}(\Sigma)/P^1_{F[\alpha]}(\mathfrak{M}_t) & & \tilde{\tau}_\rho \otimes \tau_\pi \\
\cap \ \text{parabolic} & & \\
P_{F[\alpha]}(\mathfrak{M}_t)/P^1_{F[\alpha]}(\mathfrak{M}_t) & = \mathcal{G}_{\mathfrak{M}_t} & \text{Ind } \tilde{\tau}_\rho \otimes \tau_\pi
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and signature characters $\epsilon_{\mathfrak{M}_t}$ *of* $\mathcal{G}_\rho \times \mathcal{G}_\pi$ *such that*

$$\mathcal{H}(\mathcal{G}_{\mathfrak{M}_t}, \epsilon_{\mathfrak{M}_t}(\tilde{\tau}_\rho \otimes \tau_\pi)) \overset{\text{alg}}{\hookrightarrow} \mathcal{H}(G, \lambda) \quad \text{(preserving support)}.$$

Reduction to the simple case

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Assume : $(\tilde{\Gamma}, -, -, \gamma) \oplus (\Lambda_1, -, -, \beta_1)$ simple stratum.
Then $\mathcal{G}_\rho \times \mathcal{G}_\pi = \mathcal{G}_\rho \times \mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_r$
Levi subgroup of $\mathcal{G}_{\mathfrak{M}_t} = \mathcal{G}_{\mathfrak{M}_t^1} \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_r$

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–> Two quadratic relations $(T_t - q^{r_t})(T_t + 1) = 0, t = 0, 1$;

–> the integers r_0 $(\epsilon_{\mathfrak{M}_0}(\tilde{\tau}_\rho \otimes \tau_1))$ and r_1 $(\epsilon_{\mathfrak{M}_1}(\tilde{\tau}_\rho \otimes \tau_1))$

–> real parts of reducibility points

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Compare with Red(π_1), $\pi_1 = \text{c-Ind}_{J_1}^{\mathrm{Sp}_F(V_1)} \lambda_1$, $\lambda_1 = \kappa_1 \otimes \tau_1$.

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Nice try !

The reduction step : good candidates

Assumptions on $(\tilde{\Gamma}, -, -, \gamma)$, $(\tilde{J}_\rho, \tilde{\lambda}_\rho)$ in $L_\rho = \mathrm{GL}_F(W)$

- $\beta_1 \mapsto \gamma$ defines an isomorphism $F[\beta_1] \to F[\gamma]$;
- $\tilde{\theta}_{\pi,1}$ the self-dual simple char. of $\tilde{H}^1_{\pi,1}$ extending $\theta_{\pi,1}$, $\tilde{\theta}_\gamma$ the transfer to $\tilde{H}^1_\rho$ of $\tilde{\theta}_{\pi,1}$, we ask : $\tilde{\theta}_\rho = (\tilde{\theta}_\gamma)^2$.

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There is a character χ_1 of $k^\times_{E_1}$, $\chi_1^2 = 1$, such that for any ρ with $\tilde{\theta}_\rho \rightleftharpoons (\tilde{\theta}_{\pi,1})^2$: $\epsilon_{\mathfrak{M}_0}\epsilon_{\mathfrak{M}_0^1} = \epsilon_{\mathfrak{M}_1}\epsilon_{\mathfrak{M}_1^1} = \chi_1 \circ \det{}_{k_{E_1}}$.

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- $r_0 = r_0\left(\epsilon_{\mathfrak{M}_0^1}(\chi_1.\tilde{\tau}_\rho \otimes \tau_1)\right)$, $r_1 = r_1\left(\epsilon_{\mathfrak{M}_1^1}(\chi_1.\tilde{\tau}_\rho \otimes \tau_1)\right)$

Main theorems

Theorem (B-H-S 2019)

$$IJord(\pi, \mathbf{\Theta}^2_{\pi,1}) = \; IJord(\pi_1, \mathbf{\Theta}^2_{\pi,1})_{\chi_1}$$

with $\mathbf{\Theta}_{\pi,1}$ *endoclass of* $\tilde{\theta}_{\pi,1}$, $\pi_1 = c\text{-}Ind_{J_{\pi,1}}^{Sp(V_1)} \kappa_{\pi,1} \otimes \tau_1$.

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Note : $\beta_1 = 0 \iff \pi_1$ of depth zero, case done in Jaime Lust and Shaun Stevens, *On depth zero L-packets for classical groups.* Proc. Lond. Math. Soc. (3) 121 (2020).

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Note : $\beta_1 = 0 \iff \pi_1$ of depth zero, case done in Jaime Lust and Shaun Stevens, *On depth zero L-packets for classical groups.* Proc. Lond. Math. Soc. (3) 121 (2020).

Convention : one of the β_i's is 0, possibly with $V_i = \{0\}$.
Recall : $\sum_{(\rho,a)\in \mathrm{Jord}(\pi)} a N_\rho = 2k + 1$

Main theorems

Theorem (B-H-S 2019)

$$IJord(\pi, \mathbf{\Theta}^2_{\pi,1}) = IJord(\pi_1, \mathbf{\Theta}^2_{\pi,1})_{\chi_1}$$

with $\mathbf{\Theta}_{\pi,1}$ *endoclass of* $\tilde{\theta}_{\pi,1}$, $\pi_1 = c\text{-}Ind_{J_{\pi,1}}^{Sp(V_1)} \kappa_{\pi,1} \otimes \tau_1$.

Theorem (B-H-S 2019)

$$\sum_{([\rho],m) \in IJord(\pi_1, \mathbf{\Theta}^2_{\pi,1})} m \deg(\rho) = \begin{cases} 2k_1 + 1 & \text{if } \beta_1 = 0, \\ 2k_1 & \text{otherwise.} \end{cases}$$

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Corollary (B-H-S 2019)

$$IJord(\pi) = \bigcup_{i=1}^{r} IJord(\pi_i, \mathbf{\Theta}^2_{\pi,i})_{\chi_i}$$

The ramification theorem (char $F = 0$)

Theorem (B-H-S 2019)

The self-dual endo-parameter of θ depends only on π :

$$e_G(\pi) = \sum_{i=1}^{r} \frac{dim_F(V_i)}{[F[\beta_i] : F]} \, \boldsymbol{\Theta}_{\pi,i}$$

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The diagram

$$\begin{array}{ccc} \mathit{Cusp}\,(G_0) & \overset{transfer}{\longrightarrow} & \mathit{Irr}\ \mathrm{GL}_{2k+1}(F) \\ \mathrm{e}_G \downarrow & & \downarrow \mathrm{e}_{2k+1} \\ \mathcal{E}\mathcal{E}^{\mathrm{sd}}_{2k}(F) & \overset{\delta_{2k}}{\longrightarrow} & \mathcal{E}\mathcal{E}_{2N+1}(F) \end{array}$$

commutes, where e_{2k+1} maps an irreducible rep. to the sum of the endoparameters of its cuspidal support and

$$\delta_{2k}\left(\sum_{j=1}^{s} a_j \mathbf{\Theta}_j\right) = \sum_{j=1}^{s} a_j \mathbf{\Theta}_j^2 + \mathbf{\Theta}_0$$

Unitary groups, strongly ramified case, setup

$\tilde{G} = \mathrm{GL}_F(V)$, F/F_0 quadratic ramified extension,
G unitary group of a non degenerate hermitian form on V.

–> Same framework applies.

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Cuspidal rep. $\pi = c\text{-}\mathrm{Ind}_J^G \lambda$, $\lambda = \kappa(\theta) \otimes \tau$, such that :

- underlying stratum $(\Lambda, -, -, \beta)$ simple, $[F[\beta] : F] = \dim V$,
- $F[\beta]/F[\beta]_0$ (quadratic) ramified, $F[\beta]/F$ tamely ramified.

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Look for base change of π as $\tilde{\pi} = c\text{-}\mathrm{Ind}_{\tilde{\boldsymbol{J}}}^{\tilde{G}} \tilde{\kappa} \otimes \tilde{\tau}$ with :

- $\widetilde{\boldsymbol{J}} = F[\beta]^\times \widetilde{J} = F[\beta]^\times \widetilde{J}^1$,
- unique self-dual simple character $\tilde{\theta}$ extending θ^2 to $\tilde{H}^1$,
- $\tilde{\kappa}_{|\tilde{J}}$ p-primary beta-extension of $\tilde{\theta}$ and $\tilde{\kappa}(\varpi_{F[\beta]}) = 1$,
- $\tilde{\tau}$ tamely ramified self-dual character of $F[\beta]^\times$.

Unitary groups, strongly ramified case, result

We compute $T_w^2 = b_w T_w + c_w, T_s^2 = b_s T_s + c_s$.

C. Blondel and Geo Kam-Fai Tam, *Base change for ramified unitary groups : the strongly ramified case*. J. Reine Angew. Math. 774 (2021).

Unitary groups, strongly ramified case, result

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Assume $\tilde{\tau}|_{k_E^\times} = \chi^{f(E/F)-1}$ with $\begin{cases} E = F[\beta], \\ \chi \text{ the quadratic character of } k_E^\times. \end{cases}$

The computation of b_s involves a Gauss sum of sign ϵ_s :

$$\sum_{X \in \mathfrak{W}_z} \prod_{j=0}^{d} \psi \circ \mathrm{tr}_{A/F}(\varpi_E (c_j X_{j+1} - X_{j+1} c_j)^\alpha X_{j+1}),$$

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Theorem (Blondel-Tam 2021)

Assume $\tilde{\tau}|_{k_E^\times} = \chi^{f(E/F)-1}$ and $\tilde{\tau}(\varpi_E) = \tau(-1)\epsilon_s$. The points of reducibility of $\mathrm{Ind}\ \tilde{\pi}|\det|^s \otimes \pi$ are

$$\pm 1 \quad \textit{and} \quad \frac{\pi i}{\log q_E}$$

hence $\tilde{\pi}$ is the base change of π.

C. Blondel and Geo Kam-Fai Tam, *Base change for ramified unitary groups : the strongly ramified case*. J. Reine Angew. Math. 774 (2021).

Thank you !