Hypersurfaces and Isoperimetric Inequalities in Cartan-Hadamard Manifolds

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This is usually expressed as an inequality between the area A and boundary length L of a plane domain:

Theorem (The Isoperimetric Inequality)

Let Ω be a bounded domain in $\mathbb{R}^2$, with area A and boundary length L.

Then $L^2 \geq 4\pi A$, with equality precisely if Ω is a disk.

In higher dimensional Euclidean spaces, the ball is again the unique domain which maximizes volume for perimeter:

Theorem (The Isoperimetric Inequality)

Let Ω be a bounded domain in $\mathbb{R}^n$, of volume $|\Omega|$ and perimeter $|\partial\Omega|$.

Then $|\partial\Omega|^n \geq \frac{\Sigma_{n-1}^n}{B_n^{n-1}} |\Omega|^{n-1}$, with equality precisely if Ω is a ball.

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Σ_{n-1} denotes the area of the unit $(n-1)$-sphere and B_n the volume of the unit n-ball.

More generally, the same statement is true in complete, simply connected spaces of constant curvature – spheres $\mathbb{S}_\kappa^n$ and hyperbolic spaces $\mathbb{H}_\kappa^n$ as well as Euclidean space:

The unique domain with the largest volume for a fixed perimeter is a geodesic ball.

Generalizations of the Isoperimetric Inequality

A generalization of the isoperimetric inequality allows any closed hypersurface (or curve in the plane), non necessarily embedded, in place of the boundary $\partial\Omega$ of a domain Ω.

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Let M be a closed curve in $\mathbb{R}^2$, not necessarily simple, of length L.

Let $\Omega_1, \ldots, \Omega_k$ be the bounded components of the complement of M and let w_i be the winding number of M about Ω_i.

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Theorem (Hurwitz [Hu1901], see also Osserman [Os78])

$$L^2 \geq 4\pi \left(\sum_{i=1}^{k} w_i |\Omega_i| \right). \tag{1}$$

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Let M be a closed curve in $\mathbb{R}^2$, not necessarily simple, of length L.

Let $\Omega_1, \ldots, \Omega_k$ be the bounded components of the complement of M and let the winding number of M about Ω_i be w_i.

Theorem (Banchoff-Pohl [BP71], see also [Pohl68])

$$L^2 \geq 4\pi \left(\sum_{i=1}^{k} w_i^2 |\Omega_i| \right). \tag{3}$$

Equality holds if and only if M is a circle, or several coincident circles, each traversed in the same direction some number of times.

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This can also be expressed as an integral in terms of the winding number of M:

Let $w(M, p)$ be the winding number of M about a point p in $\mathbb{R}^2$:

$$L^2 \geq 4\pi \int_{\mathbb{R}^2} w^2(M, p) dp, \tag{4}$$

with equality if and only if M is a circle, possibly with multiplicity as above.

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Let $f : M^m \to \mathbb{R}^N$ be an immersion of a closed, oriented manifold M in Euclidean space. For $x, y \in M$, let $r(x,y) = |f(x) - f(y)|$.

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$$\iint_{M\times M} \frac{1}{r^{m-1}} dVol_{M\times M} \geq K_{n,m} \int_{H_{n-m-1}} \lambda^2(M,E) dE. \tag{5}$$

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Note that for a hypersurface $f : M^{n-1} \to \mathbb{R}^n$, this says:

$$\iint_{M\times M} \frac{1}{r^{n-2}} dVol_{M\times M} \geq K_n^\star \int_{\mathbb{R}^n} w^2(M,p)dp, \tag{6}$$

where $w(M,p)$ is the winding number of M about a point p.

The conjecture that the isoperimetric inequality holds in complete, simply connected Riemannian manifolds with non-positive sectional curvature has appeared in the work of Aubin [Aub76], Burago-Zalgaller [BZ13] and Gromov [Gr07].

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let Ω be a bounded domain in a complete, simply connected Riemannian n-manifold $\mathscr{H}^n$ with non-positive sectional curvature.

Then $|\partial\Omega|^n \geq \frac{\Sigma_{n-1}^n}{B_n^{n-1}}|\Omega|^{n-1}$, with equality precisely if $(\Omega, \partial\Omega)$ is isometric to a ball (B^n, S^{n-1}) in $\mathbb{R}^n$.

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The conjecture is known to be true in dimensions 2, 3 and 4, by work of Weil [We26] and Beckenbach-Radó [BR33], Kleiner [Kl91] and Croke [Cr84] respectively, and is open in dimensions ≥ 5.

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Then:

$$\iint_{M\times M} \frac{1}{r^{n-2}} dVol_{M\times M} \geq K_n^{\star} \int_{\mathscr{H}} w^2(M,p)dp, \tag{7}$$

with equality if and only if M is the boundary, possibly with multiplicity, of a domain Ω isometric to a ball in $\mathbb{R}^n$.

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with equality if and only if M is the boundary, possibly with multiplicity, of a domain Ω isometric to a ball in $\mathbb{R}^n$.

In particular, if M is a closed curve of length L in a Cartan-Hadamard surface $\mathscr{H}^2$, $\Omega_1, \ldots, \Omega_k$ are the bounded components of the complement of M and w_i is the winding number of M about Ω_i,

$$L^2 \geq 4\pi \left(\sum_{i=1}^{k} w_i^2 |\Omega_i| \right), \tag{8}$$

with equality only for boundaries of flat disks. This 2-dimensional case has also been proven by Howard [How98].

Proof outline:

1.) The space of oriented geodesics in a Cartan-Hadamard manifold $\mathscr{H}^n$ is canonically a symplectic manifold, diffeomorphic to the tangent bundle of the $(n-1)$-sphere.

This space can be defined as the quotient of the unit tangent bundle $U(\mathscr{H})$ by the geodesic flow.

We will denote this space $\mathscr{G}$ and its symplectic form dI.

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We will denote this space $\mathscr{G}$ and its symplectic form dI.

2.) Given an immersion $f : M^m \to \mathscr{H}^n$ of a closed, oriented manifold M in $\mathscr{H}$, we have an a.e.-defined secant mapping $\mathscr{S} : M \times M \to \mathscr{G}$.

This mapping sends $(x, y) \in M \times M$ with $f(x) \neq f(y)$ to the oriented geodesic from $f(x)$ to $f(y)$.

3.) Let $f : M^{n-1} \to \mathscr{H}^n$ be an immersion of a closed, oriented hypersurface in a Cartan-Hadamard manifold, with secant mapping $\mathscr{S} : M \times M \to \mathscr{G}$.

One can show, via the double fibration of the unit tangent bundle $U(\mathscr{H})$ over $\mathscr{H}$ and $\mathscr{G}$, that:

$$\int_{\mathscr{H}} w^2(M,p)dp = C_n \int_{M\times M} r\, \mathscr{S}^*(dl)^{n-1}. \tag{9}$$

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This can be used to show:

$$\iint_{M\times M} \frac{1}{r^{n-2}} dVol_{M\times M} \geq C_n \int_{M\times M} r\,\mathscr{S}^*(dl)^{n-1}, \tag{10}$$

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with equality only if M is the boundary of a domain isometric to a flat ball.

The inequality in (10) can be adapted for submanifolds M of $\mathscr{H}$ of arbitrary codimension:

Theorem (H., [Hois21])

Let $f : M^m \to \mathscr{H}^n$ be an immersion of a closed, oriented m-manifold in a Cartan-Hadamard space $\mathscr{H}^n$. Let $r(x, y)$ be the chordal distance function on $M \times M$ as above.

Let $\mathscr{S} : M \times M \to \mathscr{G}$ be the secant mapping to the space of geodesics in $\mathscr{H}$ and dI the canonical symplectic form on $\mathscr{G}$.

Then:

$$\iint_{M \times M} \frac{1}{r^{m-1}} dVol_{M \times M} \geq \mathscr{C}_m \int_{M \times M} r \, \mathscr{S}^*(dI)^m. \tag{11}$$

Equality holds if and only if M is the boundary, possibly with multiplicity, of an embedded, totally geodesic submanifold $\mathscr{D}^{m+1}$ isometric to a disk in $\mathbb{R}^{m+1}$.

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When the ambient space $\mathscr{H}^n$ is Euclidean space $\mathbb{R}^n$, one can show:

$$\mathscr{C}_m \int_{M\times M} r\, \mathscr{S}^*(dI)^m = K_{n,m} \int_{H_{n-m-1}} \lambda^2(M,E) dE \tag{12}$$

The result above therefore gives a generalization of the Banchoff-Pohl inequality to Cartan-Hadamard manifolds.

Question

Let $f : M^{n-2} \to \mathscr{H}^n$ be an immersion of a closed, oriented manifold of codimension 2 in a Cartan-Hadamard manifold.

For an oriented geodesic γ in $\mathscr{H}$, let $\lambda(M, \gamma)$ be the linking number of M about γ.

Let $dVol_{\mathscr{G}}$ be the canonical measure on the space of geodesics $\mathscr{G}$, given by the top power dl^{n-1} of the canonical symplectic form.

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Is it true that:

$$\iint_{M\times M} \frac{1}{r^{n-3}} \geq \mathscr{K}_n \int_{\mathscr{G}} \lambda^2(M,\gamma) dVol_{\mathscr{G}}, \tag{13}$$

with equality if and only if M is the boundary of a flat, totally geodesic disk of codimension 1?

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Teufel has proven this is the case for $n = 3$ [Te93].

The Generalized Cartan-Hadamard Conjecture

A stronger version of the Cartan-Hadamard conjecture states that if a complete, simply connected Riemannian manifold $\mathscr{H}^n$ has sectional curvature bounded above by a non-positive κ, domains in $\mathscr{H}$ satisfy the isoperimetric inequality of the space of constant curvature κ.

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The results above extend to the case where the sectional curvature of a Cartan-Hadamard manifold is bounded above by a negative κ. These results then show that the equivalent of the generalized Cartan-Hadamard conjecture holds for the Banchoff-Pohl inequality.

The Generalized Cartan-Hadamard Conjecture

Yau has proven an isoperimetric inequality for Cartan-Hadamard manifolds with a negative upper curvature bound:

Theorem (Yau [Yau75])

Let Ω be a domain in a complete, simply connected Riemannian manifold $\mathscr{H}^n$ with the sectional curvature of $\mathscr{H}^n$ bounded above by $\kappa < 0$. Then:

$$|\partial\Omega| > (n-1)\sqrt{|\kappa|}|\Omega|. \tag{14}$$

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We also prove the following sharp, quantitative version of Yau's isoperimetric inequality:

Theorem (H. [Hois21])

Let $f : M^{n-1} \to \mathscr{H}^n$ be an immersion of a closed, oriented hypersurface in a Cartan-Hadamard manifold $\mathscr{H}$, with the sectional curvature of $\mathscr{H}$ bounded above by $\kappa < 0$. Let $w(M,p)$ be the winding number of M about a point $p \in \mathscr{H}$. Then:

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$$|M|^2 - \left((n-1)\sqrt{|\kappa|}\int_{\mathscr{H}} |w(M,p)|dp\right)^2 \geq \iint_{M\times M} \Psi^n_\kappa(r,|\nabla r|)dVol_{M\times M} > 0. \tag{16}$$

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Equality holds if and only if M is the boundary, possibly with multiplicity, of a domain isometric to a ball in a hyperbolic space of sectional curvature κ.

The function $\Psi^n_\kappa(r,|\nabla r|)$ is an explicit, non-negative, analytic function of the chordal distance r, and the norm of the gradient of r as a function on $M \times M$. $\Psi^n_\kappa(r,|\nabla r|)$ is related to the volume of a geodesic ball in hyperbolic n-space.

For closed curves in Cartan-Hadamard surfaces with curvature bounded above by $\kappa < 0$, this can be combined with the earlier results to give:

Theorem (H. [Hois21], see also Howard [How98])

Let M be a closed curve in a complete, simply connected surface $\mathscr{H}^2$ with curvature bounded above by $\kappa < 0$. Let L be the length of M and $w(M,p)$ the winding number of M about $p \in \mathscr{H}$. Then:

$$L^2 \geq 4\pi \int_{\mathscr{H}} w^2(M,p)dp + |\kappa| \left(\int_{\mathscr{H}} |w(M,p)|dp \right)^2. \tag{17}$$

Equality holds if and only if M is the boundary, possibly with multiplicity, of a domain isometric to a disk in the hyperbolic plane of curvature κ.

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