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BIRS

Contents

Five-day Workshop Reports

Chapter 1

Dynamics, Probability, and Conformal Invariance (05w5009)

March 12, 2005 – March 17, 2005

Organizer(s): Ilia Binder (University of Toronto), Peter W. Jones (Yale University), Steffen Rohde (University of Washington), Michael Yampolsky (University of Toronto)

The study of dynamics in the plane has recently seen a surge in interest due to three recent breakthroughs: the Sullivan-McMullen-Lyubich proof of the Feigenbaum Universality, the introduction by O. Schramm of SLE processes, and the work of S. Smirnov on percolation. The fields of Holomorphic Dynamics, SLE, and Conformal Field Theory (CFT) are now seen to be closely linked, the glue being provided by renormalization arguments, conformal mappings, Brownian Motion, and other methods related to Conformal Invariance. Indeed, there is an emerging field where these different dynamical processes, as well as more classical areas in conformal mappings, are unified into a more general theory. Though it is still early in the game, much progress has been made. The workshop has brought together leading experts from the areas of SLE, Holomorphic Dynamics, Probability Theory, and Conformal Mappings to present the latest developments in these areas and search for further unification of the fields.

Holomorphic Dynamics of Rational Maps and Kleinian Groups.

Hyperbolic geometry in 3 dimensions has experienced some very exciting progress recently. The main recent achievement is the completion of the program of Minsky of proving Thurston's Ending Lamination Conjecture (ELC) by J. Brock, D. Canary, and Y. Minsky [15, 3] (both in the incompressible-boundary case). The Conjecture had the same place in the field as the MLC (Mandelbrot set is Locally Connected) Conjecture occupies in Holomorphic Dynamics. It is a rigidity statement which postulates that combinatorial invariants (ending laminations) uniquely describe the geometry of the 3 manifolds. Another exciting recent progress is the proof of the Tameness Conjecture by Agol [1] and Calegari & Gabai [4]. It implies, in particular, the Ahlfors' Conjecture: If the limit set of a finitely generated Kleinian group has no interior, then its area is zero.

These achievements are of particular interest to holomorphic dynamicists as both have analogues in the dynamics of rational maps (see below) which remain open. This, of course, is largely due to the fact that the geometric objects (hyperbolic 3-manifolds) provide an additional set of tools to the study of the dynamics of Kleinian groups, which are at best still being developed in the dynamics of rational maps. However, the intuition coming from Kleinian groups has historically played a very important role in Holomorphic Dynamics. **Yair Minsky's** mini-course was a major event of the workshop. In his lectures Minsky has outlined the proof of ELC, and tried to present the material in the form understandable to complex analysts and dynamicists.

In dynamics of rational maps the counterpart of the Ahlfors' Conjecture would state that the Julia set of a rational map is either equal to the sphere (that is has non-empty interior), or has area zero. Given parallels between the two fields, it is both exciting and unexpected that the feeling in rational dynamics is now that this statement may be false.

A decade ago A. Douady has initiated a program to the end of constructing a quadratic polynomial whose Julia set has positive measure. **A. Chéritat** [32] has recently been able to push through a large part of this program, and gave a lecture on his results. The Douady's program consists in approximating the candidate quadratic polynomial by a sequence of carefully chosen quadratics with parabolic periodic orbits.

Each step of approximation is done through two stages. If we write the rotation number of a parabolic point using the digits of its continued fraction expansion as $[a_0, a_1, \ldots, a_n, \infty]$, the first stage consists of perturbing the parabolic point to a nearby Siegel disk with rotation number

$$[a_0, a_1, \ldots, a_n, \text{ very large } N, 1, 1, 1, 1, \ldots];$$

and the second stage with going back to a parabolic

$$[a_0, a_1, \ldots, a_n, \text{ very large } N, 1, 1, 1, 1, \ldots, 1, \infty].$$

Geometrically, each successive approximation should correspond to removing some thin cusps from the filled Julia set – the hope is to bound the area of what is left from below. The limit filled Julia set would then have a Cremer point. Such a filled Julia set would coincide with its Julia set and have a positive measure.

Chéritat has shown that the second stage of an approximation step may be carried out with an arbitrarily small loss of measure. Moreover, due to the work of X. Buff and A. Chéritat, making the loss arbitrarily small at the first stage boils down to several conjectures about *cylinder renormalization*. The latter was introduced by M. Yampolsky for proving the hyperbolicity of renormalization of critical circle maps. Geometrically, this renormalization boils down to successive blow-ups of the golden-mean Siegel Julia set. A convergence result for this procedure has been established earlier by McMullen; what is required now is a proof of the hyperbolic properties of the limiting fixed point.

The appearance of renormalization-type arguments is common for this class of problems: for example, Shishikura [19] used a parabolic renormalization procedure to demonstrate the existence of quadratic Julia sets of Hausdorff Dimension 2 used in his proof of $\text{HDim}(\partial M) = 2$. The one-dimensional renormalization theory (see e.g. [13]) has seen a spectacular progress since the works of Douady, Hubbard, and Sullivan which related it to Holomorphic Dynamics, culminating in a proof of the Feigenbaum Universality by Sullivan, McMullen, and Lyubich [11, 14, 21]. Many important problems of scaling invariance and universality still remain open, however, even in the setting of One-Dimensional Dynamics.

In particular, a renormalization hyperbolicity result for Siegel disks which would imply positive measure is still missing. However, a lot of numerical evidence exists in its favour, and moreover, the recent unpublished work of Shishikura opens an approach for settling this conjecture. In the light of the recent proof of Ahlfors' Conjecture, the existence of positive measure Julia sets would truly be surprising.

Of course, renormalization has been the main tool for the attack at the MLC Conjecture. This counterpart of ELC in the dynamics of quadratic polynomials, and its higher degree generalization, the Fatou Conjecture are arguably the main open problems in Holomorphic Dynamics.

In the early 1990's Yoccoz proved that MLC holds at all parameter values c in the boundary of the Mandelbrot set which are at most finitely many times renormalizable. His proof also showed that the corresponding Julia sets are also locally connected, provided all periodic orbits are repelling. Several partial results exist for infinitely renormalizable values of c, however, MLC is still not known in full generality. A particular example of an infinitely-renormalizable quadratic for which MLC is still open is the celebrated Feigenbaum quadratic polynomial. This particular map is infinitely renormalizable with the same combinatorial type (a particular case of *sattelite type*.

M. Lyubich has spoken on a new progress on this front in his recent joint works with J. Kahn [7, 8]. In these works the authors have introduced a new analytic tool for study of Julia sets, which they call a *Quasi-Additivity Law*. This law is a statement about extremal lenghts of families of curves, in the vein of the classical Grötsch Inequality, which is strongly motivated with the analogy with Kleinian groups. They use this law to prove that the Julia set $J(f)$ of at most finitely renormalizable unicritical polynomial $f : z \mapsto z^d + c$ with all periodic points repelling is locally connected, thus providing a higher-degree analogue of the results of Yoccoz. The theorem of Yoccoz was a major step towards MLC, and with the new tool, further progress can be expected. In particular, it is likely that the Quasi-Addittivity Law will lead to MLC for certain infinitely renormalizable values of c of satellite type, thereby bringing the conjecture closer to completion.

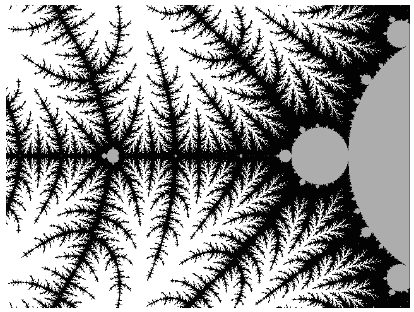

Figure 1.1: The Mandelbrot set near the Feigenbaum parameter: “hairy”, but locally connected?

As we have seen above, Cremer Julia sets are conjecturally capable of having extreme measure-theoretical properties. Shishikura [19] has established earlier that these Julia sets can have Hdim $= 2$; and it is well-known that they are bad topologically, in particular, never locally connected. All the more surprising that the recent work of I. Binder, M. Braverman, and M. Yampolsky shows that these sets are *algorithmically computable*. That is, an algorithm may be produced to, given the value of the parameter c, draw such Julia sets on the computer screen with an arbitrary magnification. This nothwithstanding the fact that informative pictures of Cremer Julia sets have never been produced. **M. Braverman** reported on this work, as well as on his earlier result with M. Yampolsky demonstrating the existence of non-computable Julia sets in the quadratic family. These results led to a lively discussion, as a number of natural questions follow. Computability results for limit sets of Kleinian groups are not yet known. And in the quadratic case, the size of the set of values c for which the Julia set is uncomputable is interesting – and in particular, whether some such values are actually computable reals themselves.

Random shapes and conformal invariance

SLE

One of the central topics of the workshop was Stochastic (or Schramm) Loewner evolution (SLE) (see [18, 17]). It is a process defined by using one-dimensional Brownian motion as the driving parameter in Loewner's differential equation. There is one free parameter in SLE, which is the speed of the Brownian driving process. Thus the whole family of conformally invariant processes, SLE_κ, is defined. Introduction to the properties of SLE and the general overview of the subject was given by **Oded Schramm**, the inventor of SLE in the first talk of his three-lecture mini-course.

The SLE paths are conjectured to be the scaling limits of various natural random processes in the plane, such as the interface of critical percolation, the Ising model or the self avoiding walk. Some such statements have been recently proved by several authors: Smirnov [20] for Critical Percolation on the triangular lattice ($\kappa = 6$); Lawler, Schramm, and Werner [10] for the Uniform Spanning Tree ($\kappa = 8$), and Loop Erased Random Walk ($\kappa = 2$). Quite a few other statements of this sort remain unproven. The direction of the research is currently extremely active. Two of the talks of the mini-course series by Oded Schramm were devoted to the problem. First, he discussed the so-called harmonic explorer process, which, as proven by the speaker and Scott Sheffield, converges to SLE_4. Using the result, they establish that the level lines of the discrete Gaussian Free Field also converge to SLE_4. It was also explained how one can find SLE_κ for $\kappa \neq 4$ in the Gaussian Free Field.

Although the conjectures about the value of κ for scaling limits of different lattice models are widely believed, it is not clear in a few cases which particular form of the SLE is obtained in the limit – there can be different parameterizations, boundary conditions, etc. To understand this situation in a few specific cases, Monte Carlo simulations of the two dimensional self-avoiding walk (SAW) were discussed at the talk by **T. Kennedy** entitled *Monte Carlo comparisons of the self-avoiding walk and SLE*. This simulations have given support to the conjecture that the scaling limit of the SAW is SLE with parameter 8/3. These past simulations treated the SAW and SLE as subsets of the plane, i.e., the parameterization of the curves was ignored. In this talk the speaker considered the SAW and SLE as parameterized curves and compared things that depend on the parameterization.

Another very active area of research is understanding of the fine geometric properties of the SLE. It is known that for $\kappa \leq 4$ SLE is almost surely a simple path (Rohde and Schramm [17]), for $\kappa > 4$ SLE is not a simple path almost surely, but is still generated by a curve called *trace*(Rohde and Schramm for $\kappa \neq 8$, Lawler, Schramm, and Werner for $\kappa = 8$). The estimate for the upper bound, $1 + \kappa/8$ on the Hausdorff dimension of SLE trace was established by S. Rohde and O. Schramm. It was shown by V. Beffara that the Hausdorff dimension of the SLE_κ-trace is actually equal to $1 + \kappa/8$. On the other hand, the conjecture that the dimension of the boundary of the hull when $\kappa > 4$ is equal to $1 + 2/\kappa$ still remains open.

Normalized Schwarzian derivatives of SLE maps and other geometric properties of the SLE boundary were discussed by **Nam-Gyu Kang** in his talk *Boundary Behavior of SLE*. He showed that the normalized (pre-)Schwarzian derivatives of SLE maps with higher order terms are continuous square integrable martingales with second moment obeying the Duplantier duality. Also he showed that they have correlations that decay exponentially in the hyperbolic distance. The BMO space, or the space of functions of bounded mean oscillation, is the appropriate substitute for L^∞ in many results concerning singular integrals. This notion can be modified in the setting of continuous martingales. The normalized (pre-)Schwarzian derivatives of SLE maps with negligible terms are BMO martingales. As a corollary, they satisfy the John-Nirenberg inequality. This result may lead to an estimate on the lower bound for the Hausdorff dimension of the boundary of SLE hull. The results obtained by Kang allows to make a formal argument for the lower bound.

Stas Smirnov in his talk entitled *Conformally invariant fractals* discussed some recent progress and techniques in the study of the fine geometric properties of the SLE. In particular, he explained the multifractal analysis of harmonic measure on SLE. In a joint work with D. Beliaev the speaker derived a formula for one of the multifractal spectra, the so-called integral means spectrum, of the SLE. The spectrum reflects the behavior of the Riemann map for SLE near the boundary. Using these calculations one can see that the fine behavior of harmonic measure of the boundary, predicted by B. Duplantier, is very plausible.

Many geometric properties of the SLE were predicted by theoretical physicists (for example Cardy, Duplantier).

Overview of the physics point of view on SLE (see [6]) was given in the talk by **Bertrand Duplantier** entitled *Conformal fractal geometry and Quantum Gravity*. More specifically, he discussed the fractal geometry of conformally-invariant (CI) scaling curves. He focused on deriving critical exponents associated with interacting random paths, by exploiting an underlying quantum gravity (QG) structure, which uses KPZ maps relating exponents in the plane to those on a random lattice, i.e., in a fluctuating metric. This was accomplished within the framework of conformal field theory (CFT), with applications to well-recognized critical models, like $O(N)$ and Potts models, and to the Stochastic Löwner Evolution (SLE). Two fundamental ingredients of the QG construction are relating bulk and Dirichlet boundary exponents, and establishing additivity rules for QG boundary conformal dimensions associated with mutually-avoiding random sets. These rules are established from the general structure of correlation functions of arbitrary interacting random sets on a random lattice, as derived from random matrix theory. The physics derivation of the multifractal spectra was also discussed.

An essential role in the derivation is played by the Quantum Gravity, i.e. the theory of random two-dimensional Riemann surfaces, and especially by Knizhnik-Polyakov-Zamolodchikov (KPZ) equation [9]. It would be extremely important both for SLE theory and for String Theory to obtain the rigorous mathematical justification of the Quantum Gravity and KPZ.

The talk of **Angel** was devoted to the construction of the rigorous theory of discrete random Riemann surfaces.

One can also consider the talk of **J. Dubedat** entitled *Commutation of SLEs* related to this program. In the talk he discusses questions pertaining to the definition of several SLEs in a domain (i.e. several random curves). In particular, the speaker derived infinitesimal commutation conditions, discussed some solutions, and show how to lift these infinitesimal relations to global relations in simple cases. All these relations agree to what is predicted by the means of Quantum Gravity, and they give some insights on how Quantum Gravity can be defined using SLE.

The workshop finished with an informal talk by **Peter Jones**. In the talk he presented his new result related to the welding problem for the SLE. Peter Jones proposed a family of (random) homeomorphisms of the circle which are conjectured to be the welding homeomorphisms of SLE. The family of the homeomorphisms is related to the Gaussian Free Field on the unit circle. He also discussed the connection of this new family with some previous conjectures.

Other random shapes

Other random shapes were discussed during the workshop.

One of the most important questions in the Geometric function theory is understanding of the extremal behavior of the multifractal spectra. The answer to the question would incorporate the Makarov's and Jones-Wolff's dimension theorem, affirm the famous Brennan's conjecture, and answer many classical questions related to the coefficient growth problem for the univalent functions. It is known that the extremal behavior of the spectra is the same for general and for the simply-connected domains, and that this behavior "almost" occur on Julia sets. Nice upper estimates on the spectra were obtained by H. Hedenmalm and S. Shimorin using the technique of Bergman spaces.

D. Beliaev in the joint work with S. Smirnov proposed a new class of random fractals, so-called *Random Snowflakes*. It is proven that the almost extremal behavior of the integral means spectrum also occur for the class of objects. Because of the stochastic nature of the random snowflakes, the explicit calculations of the multifractal spectra for them are much easier to control. Using the random snowflakes new rigorous lower estimates on multifractal spectra are obtained. The estimates are now extremely close to the conjectured values.

A generalization of the simple random walk and SLE to two- and higher-dimensional processes is another active area of research. One of such analogies was given by **Rick Kenyon** in the talk entitled *Simple random surfaces*. The talk was devoted to the speaker's joint work in progress with David Brydges and Jessica Young. They consider a natural model of random immersed surfaces in a (finite or infinite) 2-complex. This is in many ways a natural generalization of the simple random walk. Although little is known about this model, certain expectations can be computed using the Green's function on 1-forms.

While SLE provide at least conjectural limit for various two dimensional lattice model, nothing like this exists in higher dimensions. **G. Slade** in the talk entitled *Scaling limits and super-Brownian motion* explained how critical percolation and related models can be described by super-Brownian motion, in high spatial dimensions. The talk provided a survey of several results and gave all the necessary background on super-Brownian motion.

Complex Analysis

Holomorphic Dynamics, the analytic theory of Kleinian groups, and SLE have their roots in classical complex analysis and geometric function theory. In this section we report on some developments and talks that are dealing with fundamental questions from complex analysis. They are not necessarily directly related to the topics described above, but in most cases the relevance to the central theme of the workshop is very obvious.

Joan Lind discussed how properties of the driving term in the Loewner equation affect the geometry of solutions to the Loewner equation. Since the Schramm-Loewner evolution is the Loewner equation driven by one-dimensional Brownian motion, this can be viewed as the ”deterministic” counterpart to the path properties of SLE. A natural space of driving terms is the space of Hoelder continuous functions with exponent 1/2, with the Hoelder norm c replacing the speed κ in SLE. It was shown that for $c < 4$ the Loewner equation always generates simple curves whereas for $c \geq 4$ selfintersections and even topologically wild compacts can occur. This phase transition at $c = 4$ is the deterministic counterpart to the phase transition in SLE at $\kappa = 4$ from simple to non-simple curves. In her talk she also illustrated by means of examples that there is no simple other phase transition that would correspond to the transition at $\kappa = 8$ from ”swallowing curves” to ”space filling” curves.

Another topic very closely related to the Loewner equation was discussed by **Don Marshall**. He (and independently Rainer Kühnau) discovered in the early 1980’s an elementary algorithm for computing conformal maps (see [12]). The algorithm is fast and accurate, but convergence was not known. Given points $z_0, , z_n$ in the plane, the algorithm computes an explicit conformal map of the unit disk onto a region bounded by a smooth curve γ with $z_0, , z_n \in \gamma$. Marshall reported on joint work with S. Rohde, proving convergence for Jordan regions in the sense of uniformly close boundaries, and gave corresponding uniform estimates on the closed disc for the mapping functions. Improved estimates are obtained if the data points lie on a smooth or a K-quasicircle. The algorithm was discovered as an approximate method for conformal welding, however it can also be viewed as a discretization of the Loewner differential equation.

A central topic of complex analysis is quasiconformal mappings. Quasiconformal mappings appear naturally in the deformation theory of Riemann surfaces and are an indispensible tool in Kleinian groups. Since their introduction to complex dynamics in the proof of Sullivan’s no wandering domain theorem, they have become one of the most powerful tools in dynamics. They are the main tool in the work of Marshall-Rohde and of Lind, as well as a cornerstone of the work of Peter Jones described above. **Kari Astala** and **Daniel Meyer** both talked about exciting developments related to the theory of quasiconformal maps. Astala described his deep joint work with Päivärinta [2], solving the Calderon’s inverse conductivity problem: In tomography, or inverse problems in general, one aims to determine the structure of an object from indirect observations. Such methods have a variety of immediate applications, ranging e.g. from medical imaging to different industrial processes. A typical example is to determine the (conductivity) structure of a body from (electrical) measurements on the boundary. From the mathematical point of view this question has a clear and precise formulation, asking if the Dirichlet-to-Neumann boundary data determines the coefficients of a differential operator in the interior of a domain. In his talk, Astala discussed recent joint work with L. Päivärinta, solving the problem in two dimensions. Complex analysis, quasiconformal methods and, in particular, the function theoretic view to elliptic PDE’s developed by Bers, are unavoidable for the solution in its full generality.

Self-similar sets in two dimensions often can be quasisymmetrically (quasiconformally) mapped to standard sets: For instance, limit sets of quasifuchsian groups and Julia sets of hyperbolic rational maps are quasiconformal circles (if they are topological circles). The powerful tools to prove such statements, an explicit geometric characterization of quasicircles (the Ahlfors three-point condition) and the $\lambda-$ Lemma about holomorphic motions, are not available in dimensions higher than two. Already in three dimensions, there are self-similar surfaces (such as the product of the van koch snowflake with the real line, known as ”Rickman’s rug”) that cannot be quasisymmetrically parametrized by the plane. Daniel Meyer discussed quasisymmetric parametrizations of fractal surfaces in three dimensions. A Quasisphere is the image of the sphere under a quasiconformal map (of $\mathbf{R^3}$). The largest known class of quasispheres are called snowballs, they are topologically 2-dimensional analogues of the snowflake curve. For those surfaces the qc-embedding can be constructed explicitly. Many questions about the mapping behavior can be answered, at least numerically. For instance, Meyer showed that the ”harmonic measure on a snowball”, i.e. the image of Lebesgue measure of the sphere under the quasisymmetric parametrization, has dimension strictly smaller than the dimension of the snowball. This can be viewed as an analog of the celebrated Makarov theorem concerning harmonic measure of simply connected planar domains, and its higher dimensional generalization by Bourgain. The question if the dimension is greater than two (reminiscent of Wolff’s example) was raised, at this point a positive answer is suggested by numerical results.

Nick Makarov reported on joint work with H. Hedenmalm on the quantum Hele-Shaw flow. The Hele-Shaw flow is closely related to the Loewner differential equation and describes the geometry of a growing ”cell”. It is used to model the interface between two fluids of different viscosity (such as water and oil). It appears as the formal limit of diffusion limited aggregation DLA.

Michel Zinsmeister talked about joint work with S.Rohde. The physicists Hastings and Levitov proposed a stochastic model for Laplacian growth, based on compositions of ”random” conformal maps, depending on a parameter $0 \leq \alpha \leq 2$. For $\alpha = 2$, the process is a version of DLA. SLE can also be viewed as (the scaling limit of) random compositions of conformal maps, but the difference is that in SLE the growth is restricted to a specified boundary point, whereas in the Hastings-Levitov model HL(α) the growth is uniformly distributed with respect to harmonic measure. Incidentally, the special case HL(0) was considered in the late 1980’s by

Richard Rochberg and his son and called ”stochastic Loewner evolution”. As no result was published, the name did not stick. Indeed, the celebrated work of Hastings and Levitov appeared about ten years after Rochberg’s unpublished work. Zinsmeister proved some rigorous results about HL(α). In particular, he proved that the scaling limit for $\alpha = 0$ exists, he described this limit in terms of the Loewner equation, and proved that the Hausdorff dimension of the random set is 1 almost surely. For α near two, he explained how the Carleson-Makarov formalism can be adopted to the current setting to obtain nontrivial lower bounds for the dimension of the cluster. He also discussed the formal limit of the model and its relation to the Hele-Shaw equation.

A. Poltoratski described joint work with N. Makarov. He generalized the definition of Toeplitz operators to larger spaces of analytic functions. After that he studied the problem of injectivity of Toeplitz operators in these spaces. It turns out that many problems of classical analysis, such as distributions of zeros of entire functions (Levinson), completeness of bases of reproducing kernels (Beurling-Malliavin), spectral problems for the Schroedinger and string operators (Krein, Marchenko, ...), naturally become a part of the picture. One can use the Toeplitz approach together with some of the recent advances in complex and harmonic analysis to give shorter proves and further generalizations to the classical results.

List of Participants

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Chapter 2

Computational Fuel Cell Dynamics-III (05w5073)

March 19–24, 2005

Organizer(s): Keith Promislow (Michigan State University), Jean St-Pierre (Ballard Power Systems), Brian Wetton (University of British Columbia)

Report Prepared By: Paul Chang[1]

Introduction

Replacing today's fossil fuel economy with a hydrogen one would alleviate much of today's environmental and political problems. The transport and consumption of fossil fuels has contributed to oil spills, fossil fuel scarcity issues, political instability in the middle east, etc. Hydrogen consumption, on the other hand, would not since hydrogen can be produced by electrolyzing water and the latter is abundant and ubiquitous. Moreover, Proton Exchange Membrane (PEM) fuel cells (a key component of the hydrogen economy) produce only water as its byproduct, and therefore greenhouse gases and other air pollutants would cease to be produced.

Key challenges remain, however, in the transition to a hydrogen economy. Infrastructure for producing and distributing hydrogen needs to be established. An economical means for storing hydrogen needs to be developed. And, if PEM fuel cells are to supplant the internal combustion engine, PEM fuel cells need to be as (if not more) durable, efficient, economical, and powerful as the latter. Our community aims to meet this last challenge.

Cost-effective and rapid improvement of current fuel cell designs requires computationally fast and accurate fuel cell models; a pure trial-and-error approach is clearly expensive and slow. The development of fuel cell models requires the talents of a diverse group of scientists and engineers: chemists and physicists are needed to understand the fundamental chemical and mechanical processes and their interactions, mathematicians are needed to develop fast and stable numerical algorithms to solve the governing model equations, engineers are needed to implement these models to optimize fuel cell design which in turn directs future model development, and finally experimentalists are required to validate these models. Many members of our community are able to play one or more of these roles, but since few are experts in all roles, it is clear that a high degree of collaboration is needed.

The CFCD workshops hosted by Ballard Power Systems and PIMS at Simon Fraser University in June 2001, and at BIRS in April 2003, gave focus to these activities. These meetings brought together a diverse mix of scientists and engineers to exchange expertise and to find common ground, and provided future research directions. The CFCD III workshop is a continuation of these efforts, providing a forum where the

[1] University of British Columbia

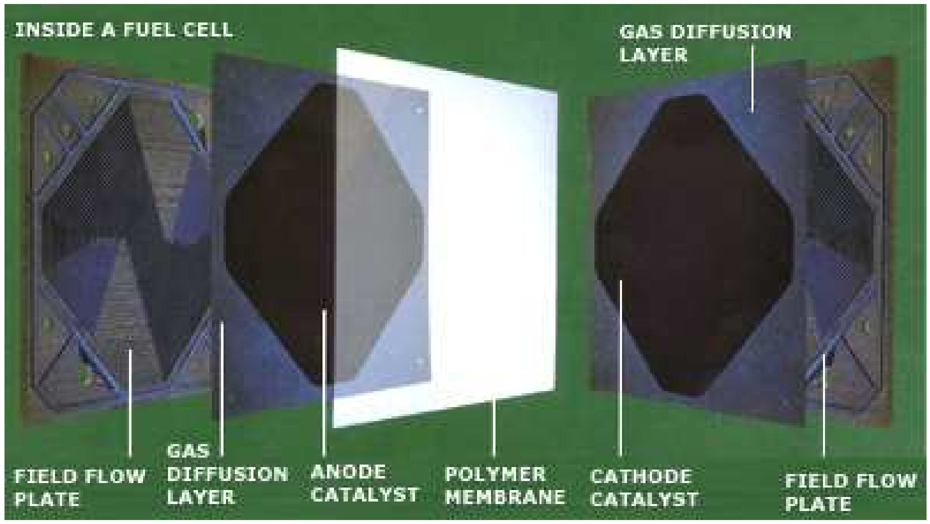

Figure 2.1: The layers comprising a fuel cell. [5]

latest fuel cell knowledge and technologies can be shared, and focusing the multi-disciplinary efforts of its participants. This important work will certainly lead to the development of a new generation of analytical and computational tools for PEM fuel cell design, and ultimately the realization of the hydrogen economy.

Proton Exchange Membrane Fuel Cells and Modelling Activities

PEM fuel cells generate power by consuming hydrogen and oxygen. As earlier mentioned, hydrogen can be produced by electrolyzing water, but in cases where pure hydrogen fuel is unavailable, it can be obtained by processing available fuels including natural gas, propane, diesel, methanol, etc. Oxygen is drawn directly from air. A PEM unit cell consists of a polymer membrane sandwiched between a pair of gas diffusion layers sandwiched between a pair of bipolar plates (See Figure 2.1). The polymer membrane is usually made of Nafion and the gas diffusion layers are often teflonated carbon fibre paper. The bipolar plates are usually made of graphite. At the interface between the gas diffusion layer and membrane lies a catalyst layer which facilitates the power-generating electrochemical reactions. The catalyst is usually Platinum, but because Platinum is such an expensive component of the fuel cell, other catalyst materials are being developed as possible replacements.

Channels are carved in the bipolar plates which deliver hydrogen (on the anode side) and oxygen (on the cathode side) to the reaction sites. The channel configuration can be straight, serpentine, or cross-flow. The hydrogen diffuses through the gas diffusion layer to the anode catalyst sites where it disassociates into two protons and two electrons. The electrolyte membrane, being a good protonic and poor electronic conductor, allows the protons to diffuse to the cathode side while the electrons are conducted through the bipolar plates through an external circuit where useful work can be performed. The protons and electrons then meet with the oxygen, which has diffused through the cathode diffusion layer, at the cathode catalyst sites where water and heat is produced. The net electrochemical reaction is simply

$$2H_2 + O_2 \rightarrow 2H_2O. \tag{2.1}$$

A key advantage of PEM fuel cells is its operation at low temperatures (around $70°C$). However, this necessitates a catalyst layer as the activation potential for the electrochemical reactions is much too high at these temperatures. Reducing Platinum loadings at the electrodes, or the complete replacement thereof, is a priority for the fuel cell community due to high cost of Platinum; but to do so without degrading the power output capabilities of the fuel cell requires an understanding of the fundamental processes which drive the catalyzed reactions. Existing approaches to modelling the catalyst layer include interface models in which the

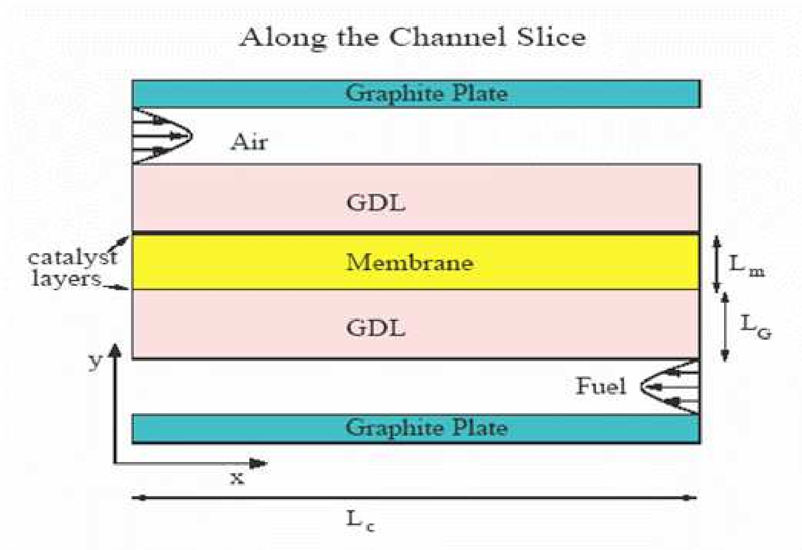

Figure 2.2: A 2D unit cell slice.

layer is infinitesimally thin, microscopic or single-pore models in which the layer consists of pores filled with gas and pores filled with electrolyte and catalyst, and agglomerate models in which the layer is composed of spherical agglomerates of carbon grains supporting Platinum. Research in this area is active and ongoing.

The electrolyte membrane is a complex polymer comprised of Teflon spines from which typically hydrophilic SO_3 groups extend. These are arranged in a nanoscale configuration which facilitates the selective diffusivity of the membrane, enabling the fuel cell to perform close to the thermodynamic limit for efficiency. While the membrane must be well hydrated to allow the protons to cross over, the overproduction of liquid water may saturate the surrounding porous electrodes, flood the gas channels, and lead to a pronounced drop in local power density. The control of the motion and distribution of liquid water in both the nano-structure of the membrane and the surrounding fibrous electrodes is referred to as water management, and is critical to effective cell operation. The understanding of water management is also key to optimizing fuel cell design.

Many efforts have been undertaken to develop fuel cell models which incorporate these effects. These models can be roughly classified as either fully three dimensional, or reduced dimensional where quantities are averaged in one or more directions. There are also models which look at specific aspects of the fuel cell.

In recent years, several large computational fluid dynamics (CFD) code vendors have become interested in developing comprehensive three dimensional fuel cell computational models. Some examples are the modules developed by CFX [2], StarCD [4], and the more academic FEMLAB [3]. These CFD codes provide convenient 3D meshing and visualization tools and robust solvers for the traditional fluid dynamics elements of fuel cell models. These codes also provide a platform for validated models of elements unique to fuel cells to be integrated into the “big picture”. However, preliminary models suggest that the delicate balance of temperature, condensation and liquid water transport in the gas diffusion layers will be difficult to capture accurately in these general packages. It is apparent that larger scale problems such as electrical coupling of unit cells in the stack and long time transients will have to be handled by specialized codes.

Reduced dimensional models exploit the high aspect ratio of the unit cell, roughly 1000 to 1 down channel versus thru membrane, and solve for quantities averaged in a particular direction. For instance, unit cell models which assume a straight channel design and average in the cross channel direction are comprised of two one-dimensional models for transport along the channel and thru the membrane electrode assembly, coupled through their boundary conditions. With such simplified geometries, these models are computationally speedier than their CFD counterparts, yet certain higher dimensional effects may not be captured with these models. A 2D slice of such a unit cell model is shown in Figure 2.2. It should be mentioned that this type of fuel cell, which uses pure hydrogen, is one of many design approaches.

Currently, there are condensation and two phase flow models for gas diffusion layers. These are based on hydrophobicity and capillary forces combined with porosity and permeability factors associated with the gas diffusion layer. This coupling of forces leads to difficulties in predicting water formation within the various regions of the gas diffusion layer and catalyst areas. These parameters are extremely difficult to measure and to correlate to model results.

Studies of water mobility and proton motion through the Nafion membrane and similar PEM products have also been conducted. Some of the questions of interest here have been considered by researchers of biological membranes. Various effects can be considered, ranging from molecular level models, hydraulic pumping, nano-technology and capillary forces.

Presentation Highlights

The following is a summary of the presentations presented at our workshop.

- *Bernhard Andreaus (Simon Fraser University):* Performs kinetic Monte Carlo simulations of CO oxidation on supported catalyst particles in the nanometer range. The goal is to improve our understanding of the catalyst structure and the prevailing kinetic mechanisms, which can help us improve catalyst utilization and optimize rates of current generation.

- *Daniel Baker (General Motors):* AC impedance tools have the potential of isolating the various contributions to the fuel cell polarization curve. Daniel Baker presented some findings in the low frequency range (much less than 1 KHz), and showed that the impedance spectra in this frequency range offers a very sensitive tool for measuring gas-phase transport resistance. Of particular interest is a low-frequency inductive effect that becomes observable at frequencies less than around 1 Hz. On another note, General Motors will build an environmental chamber for car testing at the University of Ontario Institute of Technology. It will include temperature and relative humidity control. Quoted as the best facility in North America when completed.

- *Jay Benziger (Princeton University):* Recent studies at Princeton have discovered that multiple steady states and autonomous oscillations occur in PEM fuel cells due to a positive feedback between the resistance of the polymer membrane and the water production in the fuel cell. It was also discovered that additional steady state multiplicity arises from the coupling of the mechanical properties of the polymer electrolytes and their electrical and chemical properties. Control of the construction of PEM fuel cells is key: if the sealing pressure is too low the membrane-electrode contact is poor, whereas if the sealing pressure is too high water is squeezed out of the membrane thus increasing membrane resistance. A series of experiments that show the effects of water inventory on the dynamics of fuel cell performance was presented, as well as a lumped parameter model of a differential PEM fuel cell. A model explaining these experimental results was also developed by Keith Promislow.

- *Peter Berg (University of Ontario IT) and Arian Novruzi (University of Ottawa):* Presented a dry, non-isothermal, macroscopic model for the catalyst layer. The model couples variables for these three phases: 1) electric potential for the Carbon/Platinum, 2) oxygen and water vapor concentrations and pressure in the pores, and 3) proton concentration, water content, electric potential in the membrane.

- *Uwe Beuscher (W. L. Gore and Associates, Inc.):* A detailed model is under development for studying the material and structural properties of the membrane, catalyst layer, and gas diffusion layer. The Gore Electrode Model (GEM) is a one-dimensional description of all essential processes in the PEM fuel cell. Transport processes that are considered include proton transport in the catalyst layers and membrane, electron and gas transport in the catalyst layers and gas diffusion layers, and water transport in all these domains. Need for degradation modeling mentioned.

- *Viola Birss (University of Calgary):* Developing non-noble metal ORR catalysts using sol-gel synthesis, a simple and low cost approach known to yield nanoparticulate composite materials. These new catalysts have demonstrated very good ORR activity in acidic solutions after adsorption on carbon and subsequent heat treatment, with a maximum in performance and minimum in H_2O_2 generation after prepartion at $700°C$.

- *Radu Bradean (Ballard Power Systems):* Presented a model for controlling the MEA water content. Such a model is used to provide input into the design of operating strategies of automotive fuel cell stacks. The measurements of MEA water content during fuel cell operation, stack purging after shutdown, and natural cooling after shutdown is reasonably predicted by the model.

- *Felix Buechi (Paul Scherrer Institute):* Presented a fast 1+1D model used for parameter space analysis of along-the-channel current and species distribution. The model accounts for heat transport in the MEA and along the channel, and has been validated against experimental data in a wide parameter space. Measured electrical interactions with a two cell stack and a non straight channel design.

- *Brian Carnes (University of Victoria):* Presented a general model, named BFM2, for the transport of water and protons within PEMs. It rigorously accounts for multicomponent transport using the Binary Friction Model for transport in a porous medium. The model was shown to provide an excellent fit to experimental conductivity data. Mentioned the need for ionomer (different properties than manufactured membranes) and membrane property measurements including the direct relationship between conductivity and water content at different operating conditions.

- *Paul Chang (University of British Columbia):* Presented a stack model which accounts for electrical and thermal coupling effects between unit cells. This model is comprised of a four parameter 1+1D unit cell model which was validated using a significantly large and varied data set. A two dimensional end plate model is also included. Runs with simulated anomalies were presented, where a unit cell received substantially less coolant flux and oxidant flux than its neighbours. Experimental validation presented by Gwang-Soo Kim.

- *Juergen Fuhrmann (Weierstrass Institute Berlin):* Presented a model for Direct Methanol Fuel Cells using the control volume method. The model includes fully resolved catalytic reaction chains, evaporation/condensation/dissolution reactions, two-phase flow of water and a gas mixture in a hydrophilic-hydrophobic porous medium, and Stefan Maxwell diffusion.

- *Herwig Haas (Ballard Power Systems):* PEM fuel cell models often lack validation in respect to predicted MEA water distributions. Two experimental methods have been developed at Ballard which can serve to validate these models. These methods were presented and discussed.

- *Erin Kimball (Princeton University):* Presented a simplified lumped parameter Stirred Tank Reactor model for the kinetics and mass transport in a differential PEM fuel cell; this model captures the dynamic water balance in response to changes in load, feed, and temperature. Highlighted how the model matches dynamic results from a differential PEM fuel cell, and what it predicts for more complex flow patterns.

- *Hyunchul Ju (Pennsylvania State University):* Presented a model for two-phase flow (of water) which accounts for catalyst active area reduction due to liquid water coverage, liquid water transport through hydrophobic porous media, and liquid water droplets emerging at the gas diffusion layer/channel interface. Emphasis on understanding water transport and effects on flooding.

- *John Kenna (Ballard Power Systems):* Gave an overview of Ballard Power System fuel cell products and simulation models, and how stack requirements are managed with the use of bounded design space analysis tools. The bounded design space methodology allows the interaction of multiple variables as well as the effect of advancing technology to be clearly visualized. Introduction of design space tools and using DOE hydrogen energy roadmap has helped focus Ballard's simulation and modeling efforts towards meeting their targets.

- *Gwang-Soo Kim (Ballard Power Systems):* Presented experimental results which elucidated the electrical and thermal cell interactions which occur in a stack. Specific anomalies were introduced for this purpose. For electrical interactions, different bus plate materials and a partially inactive cell was introduced. For thermal interactions, the geometry of the coolant flow field channel in a bipolar plate was modified. Results were compared with model predictions.

- *Andrei Kulikovsky (Research Center Jlich):* Presented a 1+1D model of PEM and direct methanol fuel cells. The direct methanol fuel cell model reveals a new effect where, for infinitely small total current, a “bridge” of finite local current density forms near the inlet of the oxygen gas channel. This bridge forms only in the presence of methanol crossover, and short-circuits the electrodes. This phenomenon explains a well known effect of mixed potential in direct methanol fuel cells.

- *Xianguo Li (University of Waterloo):* Presented a fuel cell stack model which takes into account a variety of factors. A new flow field design was also proposed. Need for significant data was mentioned including cell voltage and pressure drop measurements under significant mass transfer control (low stoichiometries, wide range of temperatures and relative humidities including over-saturation).

- *Chun Liu (Pennsylvania State University):* Introduced a general energetic variational procedure for modeling the free interfacial motions in complex fluids. The method employs a phase field approach to capture the moving free interfaces, and gives a natural coupling between the flow field and the different interfacial properties.

- *Simon Liu (National Research Council Canada):* Presented an overview of PEM fuel cell modelling activities at NRC. The capabilities of commercial modeling software are illustrated by means of several engineering case studies the authors have conducted in the past four years, involving computational fluid dynamics, computational solid mechanics, computation electrochemical engineering, and computational materials. Need for an increased level of activity in two phase flow modelling was mentioned.

- *Graeme Milton (University of Utah):* Outlined the basic theory of linear composite materials and their effective properties. Discussed approximation schemes such as average field approximations, effective medium schemes, differential schemes, and asymptotic methods. A brief overview of the subject of bounds on the effective properties of composites, and the optimal microstructures which achieve them. Authored a book on the subject [1].

- *John Pharoah (Queens University):* Presented a gas diffusion layer model and investigated the effects of several properties of the gas diffusion layer on fuel cell performance, including thermal conductivity, mass diffusivity, and relative permeability. A new method for the determination of anistropic transport coefficients was outlined, and the results were compared to currently used values.

- *Keith Promislow (Michigan State University):* Presented a model of ignition dynamics and bistable operation of a Stirred Tank Reactor PEM fuel cell. In dry inlet gas operation, the positive feedback between current, water production, and membrane resistance leads to two stable “ignited” states, which correspond to a uniform current distribution or a partially extinguished cell with localized current production. Comparison with experimental data gathered by Jay Benziger.

- *Isaac Rubinstein (Ben Gurion University):* Over-limiting conductance is a phenomenon where steady state current higher than the limiting one is readily passed through a cation exchange membrane. Electro-convection driven by nonequilibrium electroosmotic slip at the solution/membrane interface was suggested as a mechanism drawing together the overlimiting phenomena at cation exchange membranes. Numerical calculations and experimental results were shown which support this case.

- *Tobias Schaeffer (City University of New York):* Based on the work of Grimshaw et al., Tobias Schaeffer presented a 1D transient model for membrane swelling and contraction, and the effects these changes have on membrane hydration. Results were compared with a simple ex situ type test with a membrane immersed in a solution.

- *Juergen Schumacher (Fraunhofer Institute for Solar Energy Systems):* Overview of different modeling approaches at the Fraunhofer Institute for Solar Energy Systems at the unit cell, stack and system scales. These models include a two dimensional non-isothermal model for planar self-breathing fuel cells (validated with experimental results), a dynamic two-phase flow model for unit cells, and a simplified dynamic stack model with energy, mass, and charge transfer phenomena. Fuel cell system modeling using the Colsim package of Fraunhofer ISE was also presented, which includes a fuel cell stack model, models for reformers, power inverters, heat storage units, pumps, compressors, valves, and controllers.

- *Sirivatch Shimpalee (University of South Carolina):* Presented a model which relates the electrical conductivity of the gas diffusion layer to fuel cell performance. Relative in-plane to thru-plane electrical conductivity including contact resistance are experimentally measured, and the interaction of flow-field geometry with the gas diffusion layer is also studied.

- *Jean St.-Pierre (Ballard Power Systems):* Presented a simplified 1D unit cell model for low cell voltages which elucidates our understanding of unit cell behaviour in the mass transfer limited regime. This model was validated and can be used to extract mass transfer coefficients from full size unit cells. Criteria were also defined to ensure model applicability.

- *John Stockie (Simon Fraser University):* Previous work has shown that mass transport limitations in the catalyst layer, rather than the gas diffusion layer, is responsible for limiting current density behaviour. A catalyst layer model which captures this effect is presented, and results are compared to existing results from both experiments and simulations in the literature.

- *Henning Struchtrup (University of Victoria):* Presented a simplified conductivity model, named BFCM, for perfluorosulfonic acid membranes to investigate the unknown parameters in the general transport model BFM2 (See Brian Carnes). This model was shown to provide a more consistent fit to 1100 EW Nafion than other established models, and was able to predict the conductivity of a Dow and Membrane C membrane.

- *John Van Zee (University of South Carolina):* Presented experimental data relating PEM fuel cell performance to rapid changes in the voltage. This dynamic behaviour depends on the type of flow-field and the voltage range of the voltage change. Overshoot and undershoot of the steady state current density profile were observed for fixed flowrates when the fuel stoichiometry varied between 1.2 and 1.1. The dimensionless peak current and percentage of overshoot current is shown to depend on starting cell voltage and the range of voltage change. These peaks are limited primarily by oxygen, even though operating conditions are close to fuel starved conditions.

- *Adam Weber (Lawrence Berkeley National Laboratory):* Presented a model for transport in PEMs. It is based on a physical model that is semi-phenomenological and takes into account Schroeder's paradox. The model addresses two different transport mechanisms, vapor- and liquid-equilibrated, as well as the simultaneous occurrence of both modes. The model thus bridges the gap between one- and two-phase macroscopic models currently used in the literature.

- *Brian Wetton (University of British Columbia):* An overview of PEM fuel cell operation is given, with emphasis on stack design. Some of the fundamental scientific questions related to device performance are outlined, and a summary of modelling approaches and the use of modelling in the application is given.

- *Ziheng Zheng (University of New Brunswick):* A new Magnetic Resonance Imaging (MRI) methodology was presented to measure membrane gas phase diffusion coefficients. The MRI challenges of low spin density and short gas phase relaxation times, especially for hydrogen gas, have been successfully overcome with a modified one-dimensional, Single-Point Ramped Imaging with T1 Enhancement (SPRITE) measurement. The diffusion coefficients of both hydrogen gas and sulfur hexafluoride were measured in a model polymeric membrane, which is of potential interest as a gas separator in metal hydride batteries.

- *Christoph Ziegler (Fraunhofer Institute for Solar Energy Systems):* Presented a dynamic, two-phase flow model which accounts for Schroeder's paradox. Cyclo-voltammograms are simulated and measured, and a hysteresis effect is found in the measured IV-curves. This is likely due to the accumulation of liquid water at the cathode side of the cell.

Personal Remarks from the Organizers

There are other notable fuel cell meetings: the Gordon Conference on Fuel Cells, the American Society of Mechanical Engineering meetings on Fuel Cell Science, and sessions at the larger Electrochemical Society meetings. There are also several possibilities for general meetings on industrial mathematics: the SIAM annual meetings and the larger ICIAM meetings every four years. However, at both of these kinds of meeting, the mathematical researcher with a focus on the fuel cell application is an outsider. The CFCD series of meetings at BIRS is a chance for this activity to be at the centre, with participation of experts in mathematical areas that will be used in the next generation of models, and application experts to identify where modelling activity should be focused. BIRS provides a really wonderful opportunity for these communities to meet.

We would like to thank the staff and directorship of BIRS for their enthusiastic support of our workshop. Banff was the perfect setting to hold this workshop: the majestic scenery, the recreational facilities, the food attracted many top-notch participants who otherwise might not have come. Given the opportunity, we would welcome the chance to hold our next meeting at BIRS again.

List of Participants

Andreaus, Bernhard (Simon Fraser University)
Baker, Daniel (General Motors, Fuel Cell Applications)
Benziger, Jay (Princeton University)
Berg, Peter (University of Ontario)
Beuscher, Uwe (W.L. Gore & Associates, Inc.)
Birss, Viola (University of Calgary)
Bradean, Radu (Ballard Power Systems)
Buchi, Felix (Paul Scherrer Institute)
Carnes, Brian (University of Victoria)
Chang, Paul (University of British Columbia)
Fuhrmann, Juergen (Weierstrass Institute Berlin)
Haas, Herwig (Ballard Power Systems)
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Novruzi, Arian (University of Ottawa)
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Promislow, Keith (Michigan State University)
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Xue, Guangri (Pennsylvania State University)
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Ziegler, Christoph (Fraunhofer Institute for Solar Energy Systems)

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[5] www.fuelcelltoday.com.

Chapter 3

Representations of Kac-Moody Algebras and Combinatorics (05w5064)

March 26–31, 2005

Organizer(s): Vyjayanthi Chari (University of California, Riverside), Gerald Cliff (University of Alberta), Peter Littelmann (University of Wuppertal), Nicolai Reshetikhin (University of California, Berkeley)

The particular focus of this workshop was on the combinatorial aspects of representation theory. It brought together senior mathematicians working in the representation theory of Kac-Moody algebras with students and postdoctoral fellows who are in the initial stages of their career in this field. The participants represented the field quite well, in subjects ranging from the algebraic aspects of the representation theory of infinite–dimensional algebras, the combinatorial aspects of the crystal base theory and the path model, the geometric aspects of quiver varieties and the mathematical physics aspects of the Bethe Ansatz. Towards the end of the conference a good picture emerged of the development and the interplay between the different aspects of the subject.

We outline the main developments which were presented and discussed in the workshop.

Algebraic Aspects

The study of Kac–Moody Lie algebras began in the 1970's and were a natural generalization of the theory of semisimple Lie algebras. A Kac–Moody Lie algebra, [K] of rank n is defined by an $n \times n$ integer valued matrix $A = (a_{ij})$ (called the generalized Cartan matrix) satisfying the conditions: $a_{ii} = 2$ and for $i \neq j$ $a_{ij} = 0 \iff a_{ji} = 0$. Such matrices were classified by Vinberg and were shown to be satisfy one of the following mutually exclusive conditions: (a) the matrix is positive definite, (b) the matrix is positive semidefinite and and every proper principal minor is positive definite, (c) there exists a vector v of positive integers such that Av is a negative vector. In case (a) the associated Lie algebra is a semisimple finite dimensional Lie algebra while in the other cases the Lie algebra is infinite dimensional. If the matrix A satisfies condition (b) or (c) it is said to be of affine or indefinite type respectively.

The affine Lie algebras and the representation theory of these algebras is widely studied and is motivated by important applications in physics. One such application comes from the underlying symmetry of two dimensional conformal field theories which also led to the study of vertex algebras. Another application is that of the quantized universal enveloping algebras in the theory of quantum integrable systems. Both these applications in turn, brought important ideas to the study of the representation theory of these algebras. The talks in the conference that focussed on the algebraic side were given by Bakalov, Brundan, Greenstein, Hernandez, Loktev and de Moura. The talks dealt with representations of affine Lie algebras and its applications to physics. The representation theory or indeed the structure of an arbitrary Kac–Moody algebra is in general poorly understood. However in the last two years some progress has been made in understanding the representation theory of certain infinite families of Kac–Moody Lie algebras and S. Viswanath reported on this new development.

All the talks discussed so far focussed on the integrable representations of the Kac–Moody algebra which are the analogs of the finite–dimensional representations of a semisimple Lie algebra. A closely related category is the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. C. Stroppel presented some new results for the category $\mathcal{O}$ associated to a Kac-Moody algebra and indicated a very intriguing new connection with knot invariants.

Vertex operator algebras

These were introduced by R. Borcherds as an algebraic tool to study the underlying the operator product expansion operation in conformal field theory. They were instrumental in the proof of monster moonshine conjecture. Chiral algebras which are a generalization of vertex algebras were introduced later and play an important role in the geometric Langlands program. An algebraic structure that emerged from the study of chiral algebras in conformal field theory are the Lie conformal algebras and its higher dimensional analogs the Lie Pseudoalgebras, [3]. B. Bakalov discussed the classification of Lie Pseudoalgebras and the relation to solutions of the classical Yang–Baxter equations.

Representations of indefinite type Kac–Moody algebras

For many years there was limited progress in the representation theory of Kac-Moody algebras associated to a generalized Cartan matrix of indefinite type. The best studied amongst these were the algebras of hyperbolic type and even there, results are hard to come by. More recently, M. Kleber and S. Viswanath identified infinite families of algebras of indefinite type whose representation theory parallels and in fact generalizes that of sl_n. Roughly speaking, the algebras they consider are obtained by "extending" the Dynkin diagram of a Kac–Moody algebra by a tail which is the Dynkin diagram of sl_n. Clearly the finite–dimensional algebras of classical type belong to this picture. But now, one can also allow the infinite series coming from the simple laced exceptional algebras, this includes the hyperbolic Lie algebra E_{10} which has been studied by mathematicians and physicists. In his lecture, S. Viswanath discussed the representation ring of these algebras and showed that the tensor product of the integrable representations decomposed in a stable way: namely as the length of the tail went to infinity, the multiplicity of the isotypical components remained the same. Using this, he explained how to define a stable product on a suitable vector space, analogous to the ring of symmetric functions coming from the representation theory of sl_n.

Ben Webster, a graduate student at Berkeley and one of the participants of the conference noticed that this stabilization feature can be explained clearly using quiver realizations of representations of Kac–Moody algebras. His preprint is now available on the archive, [29].

Representations of affine Kac–Moody algebras

Affine Kac-Moody algebras are one of the most important and well studied class of Kac-Moody algebras. The main reason for this is that they can be realized as the universal (one–dimensional) central extension of the Lie algebra of Laurent maps from to a semisimple Lie algebra. The representation theory of the affine algebras are "controlled" by the center, and there is striking difference between the representations where the center acts by a positive integer (positive level representations) and those where the center acts trivially (level zero representations). One outcome of the workshop was a very good understanding and formalizing of the connection between these two families of representations.

The irreducible finite–dimensional representations of quantized affine algebras play a key role in the theory of quantum integrable systems. The structure of these representations is quite complicated and there are a number of approaches to studying them, [1], [4], [5], [7], [9], [10], [27]. Two of these approaches have had significant success recently and were discussed in the conference. One is the approach of q–characters, an idea that was introduced by Frenkel and Reshetikhin and further studied by Frenkel and Mukhin. D. Hernandez discussed his work on q–characters and showed how his methods could be used to solve a conjecture on the structure of a particular family of modules, the so-called Kirillov–Reshetikhin modules. This also allowed him to establish that these characters solved a system of equations called the Q–system arising in the study of integrable systems. A. deMoura (joint work with V. Chari) presented an alternate approach to defining the q–characters which leads to a parametrization of the blocks (generalizing results of [8]) in the category of finite–dimensional representations of quantized affine algebras.

A second approach is to consider the $q = 1$ limit of representations of quantum affine algebras which led to the idea of Weyl modules in this context. These modules were defined and initially studied by Chari and Pressley who also had a conjecture on the structure of these modules. About the same time, B. Feigin and S. Loktev defined the notion of a fusion product of finite–dimensional representations of a simple Lie algebra and showed that these could be regarded as modules for the polynomial valued subalgebra of the affine algebra. S. Loktev discussed (joint work with various coauthors) in his lecture the relationship between the Weyl modules and fusion products and also generalizations of Weyl modules to other algebras. J. Greenstein (joint work with Chari) discussed extensions in the category of finite dimensional representations of affine Lie algebras and a new realization of current algebras.

A new connection between the theory of Yangians [D] and W-algebras was also presented during the conference. W-algebras are endomorphism algebras of certain induced modules for a finite dimensional reductive complex Lie algebra. There is a natural way to associate to nilpotent element such an algebra (Slodowy slice). In fact, the coordinate ring of the Slodowy slice is isomorphic to the an appropriate associated graded version of the W-algebra. In special cases it has been observed before that in this way one gets the Yangian of level ℓ. J. Brundan reported on his joint work with A. Kleschev, where they consider arbitrary nilpotent matrices. They describe a presentation of these algebras which leads to a generalization of the of Yangians, the so-called shifted Yangians. Because of the Schur-Weyl duality or rather it's quantized version, they obtain also a close connection with the degenerate cyclotomic Hecke algebra and representations of the Lie algebra gl_∞

Projective functors in the Bernstein–Bernstein–Gelfand category $\mathcal{O}$ [2] are the functors obtained as direct summands of the functors given by tensoring with finite dimensional representations. Such functors have been classified by Gelfand and Bernstein.

A different approach to this problem was presented during the meeting by C. Stroppel. The advantage of the approach is that it not only recovers the known results but also can be easily generalized to the Kac-Moody case. Further, the approach by deformation theory also opens a new and very interesting connection to knot and tangle invariants

Combinatorial Representation theory

Macdonald polynomials play an important role in representation theory and govern in many cases the combinatorial aspects of a theory. In his talk M. Haiman explained the latest developments in this field. In particular, he explained a new combinatorial formula for Macdonald polynomials. The advantage of this formula is that fact that it gives deep insights into the structure of these polynomials and provides a new approach to understanding the charge formula of Lascoux and Schützenberger. M. Shimozono gave a talk on finding a Schubert calculus on affine Grassmanians and explained the importance of this in enumerative algebraic geometry. Roughly speaking the idea is to find a pairing between the Schubert bases of the cohomology and homology of the affine Grassmanian associated to sl_n. Using this he and his collaborators hope to find the structure constants of the homology. This should give the decomposition of the fusion product of positive level representations generalizing the the Littlewood Richardson rule for the tensor product of finite dimensional representations of sl_n.

The theory of crystal bases developed independently by Kashiwara [2], [16] and Lusztig [23] has become a very important tool in many aspects of representation theory. The associated graph reflects in many ways important properties of the representation. Different aspects of the theory of crystals were discussed at the meeting: in the case of finite dimensional representations of affine Kac-Moody algebras for example, it is only conjectured [17] that crystal bases exist in the general case. Proofs for the existence in special cases need case by case considerations. The other important point is that of constructing combinatorial models of these graphs, [20], [21], [26]. Of course, different models of the same graph may be particularly adapted to different properties, so a third important point is to understand the relationship between different existing models

K. Misra gave a report on further development of the second problem mentioned above. He presented results on a joint work with Kashiwara, Okado and Yamada. They construct perfect crystals for the integrable highest weight $D_4^{(3)}$-modules of level $k > 0$. These perfect crystals are finite graphs, but the graphs for the infinite dimensional integrable highest weight modules can then be constructed as semi-infinite tensor products of these graphs.

The crystal graph can be also very helpful in constructing bases of the representation spaces. One case was reported on the conference by A. Premat. The aim was to construct a monomial basis for Demazure modules. Of course, there is the global / canonical basis by Kashiwara and Lusztig, but which is in general not always easy to compute in an explicit way. Using a combinatorial model for the crystal basis by Young diagrams, she reported that the transition matrix between the monomial basis constructed by her and global bases are upper triangular with ones in the diagonal.

An important step in developing a crystal graph theory for finite dimensional representations of untwisted affine Kac-Moody algebras was presented by D. Sagaki and S. Naito. The set of Lakshmibai-Seshadri paths makes sense for affine Kac-Moody algebras even in the case where λ is not a weight in the Tits cone. Suppose λ is of level zero and dominant integral for the underlying finite dimensional Lie algebra. They show that after the projection on the space modulo the imaginary root one can endow this set with the structure of a crystal graph. In fact, this set has a tensor product decomposition, it is the product of the corresponding sets for the fundamental weights. Since these are combinatorial models for the crystal graph of quantum Weyl modules, it follows that in this way they provide a uniform way to get a combinatorial way for the quantum Weyl modules of all untwisted affine Kac-Moody algebras.

Another successful tool to obtain combinatorial models for crystal bases / crystal graphs for irreducible highest weight crystals of quantum (affine) algebras.

The Young walls consist of colored blocks with various shapes that are built on a given ground-state wall and can be viewed as generalizations of Young diagrams. The rules for building Young walls and the action of Kashiwara operators are given explicitly in terms of combinatorics of Young walls. The crystal graph of a basic representation is characterized as the set of all reduced proper Young walls. The character of a basic representation can be computed easily by counting the number of colored blocks that have been added to the ground-state wall.

This theory has been developed by Seok-Jin Kang, J. H. Kwon, J.-A. Kim, H. Lee, D.-U. Shin and others. A report on the present state of the theory was given and a possible connection between modular representation theory and crystal bases.

A first step in the understanding of the connection between the Kyoto path model for representations of affine quantum algebras and the path model by Littelmann was presented by P. Magyar. In the case of the basic level-one representation, he derives a direct connection between the two path models by generalizing the path model to a class of semi-infinite concatenations of paths, called skeins.

Let $\mathfrak{g}$ be a simple complex Lie algebra and denote by $\widehat{\mathfrak{g}}$ the affine Kac–Moody algebra associated to the extended Dynkin diagram. It is a natural approach to understand the infinite dimensional highest weight representations of $\widehat{\mathfrak{g}}$ by first studying them as $\mathfrak{g}$–modules. To do so, one needs restriction formulas. A natural filtration by finite dimensional subspaces of such a representation is given by its $\mathfrak{g}$–stable Demazure modules.

In the case where $V = V(\ell\Lambda_0)$ corresponds to a multiple of the highest weight of the vacuum representation (and some more general cases), G. Fourier (joint work with P. Littelmann) presented a very effective approach. In this case the Demazure modules are indexed by dominant coweights, and it was explained that the Demazure module decomposes as $\mathfrak{g}$-module into a tensor product of Demzure modules corresponding to fundamental coweights. This decomposition can be viewed as the natural generalization and uniform formulation of many partial results know before.

For these "smallest modules" an explicit decomposition is given in the classical case (and in many non-classical cases). In fact, it turns out that as $\mathfrak{g}$-module they are isomorphic to some Kirillov–Reshetikhin-modules.

As a consequence one can give a description of the $\mathfrak{g}$-module structure of $V(\Lambda)$ for an arbitrary dominant weight as a semi-infinite tensor product of finite dimensional $\mathfrak{g}$–modules.

The Bethe Ansatz

The Bethe Ansatz is a method to obtain eigenvectors for a certain set of operators. The corresponding Bethe vectors correspond then in the general case to certain parameters satisfying the Bethe equations. There are two methods to obtain these vectors, one coming from the crystal base theory and another method to obtain to obtain eigenvectors comes from representation theory of affine Lie algebras.

E. Mukhin showed that the Bethe equation for the nonhomogenous Gaudin model could be solved by certain orthogonal polynomials. He also addressed a similar problem for other models, namely the trignometric model and the XXZ model. All these involve looking at suitable finite dimensional representations of affine Lie algebras.

Another problem on this subject was addressed by Anne Schilling. In the case of a given spin model, the Bethe vectors are indexed by certain rigged configurations, whereas the solutions obtained by representation theory are indexed by elements of a crystal graph. So it is natural to ask for the relationship between these to methods.

A. Kirillov and N. Reshetikhin provided a combinatorial bijection between certain restricted rigged configurations and highest weights in crystal. This bijection was generalized later by A. Kirillov, A. Schilling and M. Shimozono. A. Schilling gave a report on this subject and presented the latest development: An extension of the bijection above to a bijection between the rigged configurations parameterizing the Bethe vectors and the crystals parameterizing the eigenvectors obtained by representation theoretic methods. This result was obtained by defining a crystal graph structure on the set of rigged configurations.

The tensor product of evaluation representations of affine Kac-Moody algebras lifts to the fusion product of integrable modules. The fusion tensor product induces the grading of the multiplicity spaces for the decomposition of tensor product of irreducible modules over the underlying simple Lie algebra. Poincaré polynomials for graded multiplicity spaces can be regarded as generalizations Kostka-Foulkes polynomials.

The structure of these polynomials is closely related to the structure of irreducible characters of corresponding affine Kac-Moody algebras. The latest progress in this direction was reported by R. Kedem.

Talks

Speaker: Bojko Bakalov

Title: Lie Pseudoalgebras

Abstract: One of the algebraic structures that has emerged recently in the study of the operator product expansions of chiral fields in conformal field theory is that of a Lie conformal algebra. A Lie pseudoalgebra is a "higher-dimensional" generalization of the notion of a Lie conformal algebra. On the other hand, Lie pseudoalgebras can be viewed as Lie algebras in certain pseudo-tensor categories.

I will review the classification of finite simple Lie pseudoalgebras, and I will discuss their relationship to solutions of the classical Yang-Baxter equation and to linear Poisson brackets. I will also describe the irreducible representations of the Lie pseudoalgebra $W(\mathfrak{d})$, which is closely related to the Lie-Cartan algebra W_N of vector fields, where $N = \dim \mathfrak{d}$. (Based on a joint work with A. D'Andrea and V. G. Kac.)

Speaker: Jon Brundan

Yangians, Whittaker modules and cyclotomic Hecke algebras.

There has recently been some progress in understanding some algebras introduced originally by Kostant in 1978. These algebras can be viewed as quantizations of the Slodowy slice associated to a nilpotent orbit in a semisimple Lie algebra. In type A, it turns out that these quantizations of the Slodowy slice are closely related to the Yangian of the Lie algebra gl_n. Actually, they are generalizations of the Yangians which we call shifted Yangians.

In recent work with A. Kleshchev, we have worked out the combinatorics of the finite dimensional representations of shifted Yangians. The approach uses in an essential way a theorem of Skryabin relating representations of these algebras to certain categories of generalized Whittaker modules. In particular, we are able to reprove and generalize the known results about representations of Yangians, all as a direct application of the Kazhdan-Lusztig conjecture.

There is also a close connection between shifted Yangians and the degenerate cyclotomic Hecke algebras, thanks to a Schur-Weyl duality which interpolates between the classical Schur-Weyl duality and Drinfeld's affine analogue of it. This leads to a natural representation theoretic construction of some higher level Fock spaces for the Lie algebra gl_∞, complete with their dual canonical bases.

Speaker: Jacob Greenstein

An application of free Lie algebras to current algebras

We realize the current algebra of a Kac-Moody algebra as a quotient of a semi-direct product of the Kac-Moody Lie algebra and the free Lie algebra of the Kac-Moody algebra. We use this realization to study the representations of the current algebra. In particular we see that every ad-invariant ideal in the symmetric algebra of the Kac-Moody algebra gives rise in a canonical way to a representation of the current algebra. These representations include certain well-known families of representations of the current algebra of a simple Lie algebra. Another family of examples, which are the classical limits of the Kirillov-Reshetikhin modules, are also obtained explicitly by using a construction of Kostant. Finally we study extensions in the category of finite dimensional modules of the current algebra of a simple Lie algebra.

Speaker: Mark Haiman

Title: A combinatorial formula for Macdonald polynomials

Abstract: I'll explain recent joint work with Jim Haglund and Nick Loehr, in which we prove a combinatorial formula for the Macdonald polynomial $\tilde{H}_\mu(x; q, t)$ which had been conjectured by Haglund. Such a combinatorial formula had been sought ever since Macdonald introduced his polynomials in 1988.

The new formula has various pleasant consequences, including the expansion of Macdonald polynomials in terms of LLT polynomials, a new proof of the charge formula of Lascoux and Schutzenberger for Hall-Littlewood polynomials, and a new proof (and more general version) of Knop and Sahi's combinatorial formula for Jack polynomials.

In general, our formula doesn't yet give a new proof of the positivity theorem for Macdonald polynomials, because it expresses them in terms of monomials, rather than Schur functions. However, it does yield a new combinatorial expression for the Schur function expansion when the partition μ has parts ≤ 2, and there is hope to extend this result.

Speaker: David Hernandez

Title:The Kirillov-Reshetikhin conjecture and solutions of T-systems.

In this talk we present a proof of the Kirillov-Reshetikhin conjecture for all untwisted quantum affine algebras : we prove that the characters of Kirillov-Reshetikhin modules solve the Q-system, and so we get explicit formulas for the characters of their tensor products. Moreover we establish exact sequences involving tensor products of Kirillov-Reshetikhin modules and prove that their q-characters solve the T-system. For simply-laced cases these results were first obtained by Nakajima with geometric arguments which are not available in general. The proof we present is different and purely algebraic, and so can be extended uniformly to non simply-laced cases.

Speaker: Seok-Jin Kang

Title: Combinatorics of Young walls and crystal bases.

We will discuss the construction of irreducible highest weight crystals using Young walls. We will also discuss the possible connection between modular representation theory and crystal bases.

Speaker: Rinat Kedem

Title: Constructions of affine Lie algebra modules via graded tensor products via generalized Kostka polynomials.

The graded tensor product is a tensor product of finite-dimensional g-modules, endowed with a g-equivariant grading. This grading is related to the action of the loop algebra on the "fusion product" of representations of conformal field theory, and was originally defined by Feigin and Loktev. A conjecture, which has been proven in some special cases is that the graded multiplicity of an irreducible g-module in the graded tensor product is related to the Kostka polynomial or one of its generalized or level-restricted versions.

I will discuss how this graded tensor product allows us to construct integrable modules in two very different ways. One is in terms of the inductive limit of the graded tensor product of an infinite number of g-modules. The other is a generalization of the semi-infinite construction of Feigin and Stoyanovksy, which allows us to compute the characters of arbitrary highest weight integrable modules. This last requires use of the inverse of the matrix of generalized Kostka polynomials, and hence gives an interesting alternating sum expression for characters corresponding to non-rectangular highest weights in terms of rectangular ones.

Speaker: Sergei Loktev

Title: Weyl modules over sl_r-valued currents

Abstract: We discuss Weyl modules over sl_r-valued currents in one and two variable.

For one–dimensional currents a construction of basis, proposed by V.Chari and the speaker, will be described. If there will be enough time, the relation to Demazure modules and fusion modules will be discussed.

For two–dimensional currents relation to the space of diagonal coinvariants and parking functions, observed by B.Feigin and the speaker, will be explained.

Speaker: Kailash Misra

Title: Perfect crystal for $D_4^{(3)}$

Abstract: The crystal base theory developed by Kashiwara and independently by Lusztig provides an important combinatorial tool to study the representations of symmetrizable Kac-Moody algebras. It is known that the crystal base for affine Kac-Moody Lie algebras can be concretely realized as a subset of the semi-infinite tensor products of perfect crystals. In this talk we will present a perfect crystal for the integrable highest weight $D_4^{(3)}$-module of level $k > 0$. This is a joint work with Kashiwara, Okado and Yamada.

Speaker: Adriano A Moura

Title: Blocks of Finite Dimensional Representations of Classical and Quantum Affine Algebras.

Abstract: It is well known that the category of finite dimensional representations of classical or quantum affine algebras is not semisimple. To understand its block decomposition in the quantum case, P. Etingof and the speaker introduced the notion of Elliptic Characters. However, the original definition using analytic properties of the R-matrix imposed some un-natural restrictions to the problem (—q— should be different from 1). In particular, it was unclear how to compute the classical limit of the block decomposition. In this talk based on joint work with V. Chari we present a definition of Elliptic Characters from the point of view of the Braid Group action and the theory of q-Characters. This allow us to obtain the block decomposition for generic q as well as for q=1.

SPEAKER: Evgeny Mukhin

TITLE: Multiple orthogonal polynomials in Bethe Ansatz.

ABSTRACT: We show that the Bethe Ansatz equation for the non-homogeneous sl_n Gaudin model and two finite dimensional representations one of which is a symmetric power of vector representation, is solved in term of zeroes of multiple orthogonal Jacobi-Piñeiro polynomials. Equivalently, the spaces of polynomials with two finite ramification points with special exponents at one of the points have a basis explicitly given via multiple orthogonal Jacobi-Piñeiro polynomials. In a similar way, multiple orthogonal Laguerre polynomials appear in the Bethe Ansatz related to the trigonometric Gaudin model and multiple orthogonal little q-Jacobi polynomials in the Bethe Ansatz related to the XXZ model.

This is a joint work with A. Varchenko.

Speaker: Alejandra Premat

Monomial Bases for Demazure Modules

Abstract: We will discuss certain monomial bases of quantum Demazure modules for the algebra Uq(affine-sln) and show how to compute them using a description of the crystal graphs by Young diagrams. We will also see that the transition matrices from these bases to the Global bases are upper triangular with ones in the diagonal.

Speaker: D. Sagaki - S. Naito

Crystal of Lakshmibai-Seshadri paths associated to a level-zero integral weight for an affine Lie algebra

Let $\lambda = \sum_{i \in I_0} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$, be an integral weight of level zero that is a sum of level-zero fundamental weights ϖ_i, $i \in I_0$, for an affine Lie algebra $\mathfrak{g}$. We study a certain crystal $\mathbb{B}(\lambda)_{cl}$, which is (modulo the null root of $\mathfrak{g}$) the crystal of all Lakshmibai-Seshadri paths of shape λ, and prove that the $\mathbb{B}(\lambda)_{cl}$ is isomorphic as a crystal to the tensor product $\bigotimes_{i \in I_0} \mathbb{B}(\varpi_i)_{cl}^{\otimes m_i}$ of the crystals $\mathbb{B}(\varpi_i)_{cl}$, $i \in I_0$. Here we note that for each $i \in I_0$, the $\mathbb{B}(\varpi_i)_{cl}$ turns out to be isomorphic as a crystal to the crystal base of the level-zero fundamental module $W(\varpi_i)$ over the quantum affine algebra $U'_q(\mathfrak{g})$.

Speaker: Anne Schilling

Title: Crystal structure on rigged configurations

Abstract: Rigged configurations label the Bethe vectors of a given spin model. According to a bijection by Kirillov and Reshetikhin (generalized by Kirillov, S., Shimozono) rigged configurations correspond to highest weight crystal paths. The natural question arises whether there exist ”unrestricted” rigged configurations corresponding to any crystal path, not necessarily highest weight. In this talk we define unrestricted rigged configurations and describe the crystal structure on this set.

Speaker: Mark Shimozono

Title: Schubert calculus on the affine Grassmannian

Abstract: We present a generalization of the Robinson-Schensted-Knuth correspondence which conjecturally realizes the Cauchy identity that gives the perfect pairing between the Schubert bases of cohomology and homology of the affine Grassmannian of type $A^{(1)}_{n-1}/A_{n-1}$. This involves two kinds of tableaux that are defined using respectively the weak and strong Bruhat orders on the affine Weyl group. When n goes to infinity the bijection converges to the usual RSK map. We state a Pieri rule for the multiplication in cohomology, which uniquely determines the basis.

We are also investigating the properties of a jeu de taquin algorithm on weak order tableaux which may lead to a rule for the structure constants for homology. These constants generalize the fusion Littlewood-Richardson coefficients that come from the tensor product of representations at a given level.

This is ongoing joint work with Thomas Lam, Luc Lapointe, and Jennifer Morse.

Speaker: Catharina Stroppel

Title: The classification of projective functors for Kac-Moody Lie algebras

We consider the Bernstein-Gelfand–Gelfand category O attached to a semisimple complex Lie algebra. Projective functors are the direct summands of the functors given by tensoring with finite dimensional representations. These functors were classified by Bernstein and Gelfand. We want to give an alternative approach to this classification using deformation theory. We will explain how this alternative proof can be generalized to the Kac Moody situation giving rise to a classification of projective functors. As an explanation we briefly mention the connection to knot and tangle invariants.

Speaker: S. Viswanath

Dynkin diagram sequences and tensor product stabilization

In this talk, we will consider sequences of Dynkin diagrams Z_k of the form $X -o-o-o- \cdots -o-o- Y$ where X and Y are two fixed Dynkin diagrams and k is the number of intermediate nodes. The classical series A_k, B_k, C_k, D_k are all of this form and we can construct many more such series of indefinite Kac-Moody algebras as well (e.g $E_n, G_n, (E-E)_n, \cdots$).

Our goal will be to show that for the Z_k, multiplicities of irreducible representations in tensor product decompositions exhibit a stabilization behavior as $k \to \infty$. This parallels the situation for the series A_k where this result is implied directly by the Littlewood-Richardson rule. We'll use Littelmann's path model to do this.

The stable values of these multiplicities can be used as structure constants to define a “stable tensor product” operation on a space $\mathcal{R}(X|Y)$ that could be called the “stable representation ring”. We'll show that this multiplication operation is indeed associative, making $\mathcal{R}(X|Y)$ a bonafide C algebra that captures tensor products in the limit $k \to \infty$.

Speaker: Milen Yakimov

General finiteness of the fusion tensor product

Kazhdan and Lusztig proved a finiteness result for the fusion tensor product for smooth modules over an affine Kac-Moody algebra which can be viewed as an analog of the fact that the product of finite dimensional modules over a simple Lie algebra is finite dimensional. In the classical situation Kostant's theorem from the late 70's provides a much more general finiteness: for any subalgebra k of a complex simple Lie algebra g which is reductive in g, the category of finite length, admissible (g,k)-modules is stable under tensoring with finite dimensional g-modules (with applications to category O, Harish-Chandra modules, etc.). We will describe a proof of an analog of this theorem for the fusion tensor product of smooth affine modules, based on an approach different from the one of Kazhdan and Lusztig.

Conclusion

The important ideas which emerged from the workshop were the relation between the Demazure modules, the level zero representations of affine Lie algebras , the Weyl modules and the path model for these representations. It is hoped that these relations should help in solving a conjecture of Kashiwara which predicts that the Kirillov–Reshetikhin modules for the quantum affine algebras admit a crystal basis. Also, it now appears very likely that the specialization to $q = 1$ of the tensor product of representations of the quantum affine algebra should be the fusion product of the representations of the classical affine algebras. While much of the work reported was on the untwisted affine algebras, it also became clear that the corresponding problems for the twisted affine algebras were also important.

The average age of participants was younger then usual and women were well represented among the speakers and participants. We consider this a success. The workshop has already stimulated research activity amongst its participants. S. Viswanath [28] and Ben Webster [29] have already posted articles following up on results presented at the conference. Several other collaborations between the participants, Hernandez and Greenstein, Hernandez and deMoura are ongoing and preprints should be available soon. On the whole we believe that the workshop was very useful and provided a good venue for interaction between the various directions of research in representation theory.

List of Participants

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Brundan, Jonathan (University of Oregon)
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Chapter 4

Workshop in Homotopical Localization and the Calculus of Functors (05w5078)

April 2–7, 2005

Organizer(s): Kristine Bauer (University of Calgary), Ralph Cohen (Stanford University), George Peschke (University of Alberta), Hal Sadofsky (University of Oregon)

Overview and Introduction to the Subject

This workshop focused on two relatively recent developments in homotopy theory: homotopical localization, and the calculus of homotopy functors. An effort was made to promote the, as of yet, sparsely explored interrelationship between these two subjects. To develop a sense of purpose and perspective, let us mention a few evolutionary highlights of algebraic topology/homotopy theory, and observe how its concerns and viewpoints progress over time (we use present day terminology throughout):

1. Early activity in the subject centered around combinatorial invariants of polyhydra, such as the Euler characteristic, Betti numbers, etc. These were adequate to classify the members of certain families of spaces, such as connected surfaces which are compact and without boundary. More generally, they provided a tool for distinguishing spaces.

2. Next followed a functorial approach to invariants for the disconnectivities in general topological spaces: homotopy groups, various species of (co-)homology theories, etc. As a 'biproduct' the homotopy invariance of the earlier invariants was obtained.

3. The next evolutionary layer came with the notion of a homotopy functor (one which preserves homotopy equivalences). This provided a unifying platform for all of the specific and geometrically motivated constructs which characterized the previous stage. In addition, it set the stage for a systematic comparison of such functors; e.g. which functors detect a homotopy theoretical property in a given space? which homotopy functor factors through another? etc.

4. With homotopy functors in the center of view, the need for tools to study such resulted in the study of functors on the category of homotopy functors.

Each step further in this development was motivated by the prospect of gaining insight in earlier steps. As history testifies, each step has been successful in this regard.

How do homotopical localization and the calculus of homotopy functors fit in? Homotopy localization of spaces or spectra generates homotopy functors with certain predictable properties. Such functors fit naturally into framework of 3 above. Building on ideas and the groundwork provided by the works of Adams [1], Bousfield [5, 6], Bousfield-Kan [9], Sullivan [23], and others a flurry of activity over the 1990's culminated in a fully developed theory which permits implementations in suitable model categories; see the works of Farjoun [13] and Hirschhorn [17].

The calculus of homotopy functors belongs to level 4. above. It aims to study a homotopy functor F by a tower of homotopy functors

$$\cdots \to T_nF \longrightarrow T_{n-1}F \to \cdots \to T_1F \to T_0F.$$

This tower is strikingly analogous with Taylor polynomial approximations of a smooth function as we'll describe below.

At this point we'd like to describe homotopical localization and the Goodwillie Calculus in more detail.

Mathematical Background

We will be working in categories where it is possible to do homotopy theory or something related to homotopy theory. The most basic example of such a category is the category $\mathcal{T}$ of topological spaces.

There are many variations on this category, some of which are considered in Goodwillie's work, and some of which have been considered in the work of other authors. One can do homotopy theory in the category of topological spaces with distinguished basepoints, $\mathcal{T}_*$ (where all functions must preserve the basepoint), topological spaces *over* some fixed base space Y, and the category of spectra, $\mathcal{S}$. We will use $\mathcal{T}_*$ in the succeeding and take this opportunity to describe three basic constructions. Let X be a space with a distinguished basepoint x_0, and I be the unit interval. The *suspension* of X is

$$\Sigma X = (I \times X)/(\{0,1\} \times X \cup I \times \{x_0\}).$$

The *based loop space* on X is

$$\Omega X = \text{Maps}((I, \{0,1\}), (X, x_0))$$

in other words, all continuous maps from the interval to X which take the endpoints of the interval to the basepoint. The *smash product* of X with Y is

$$X \wedge Y = (X \times Y)/(\{x_0 \times Y \cup X \times \{y_0\}\}).$$

Since $\mathcal{S}$ features prominently, and may not be familiar, we also describe it briefly, taking liberties with the definition for the sake of conciseness. A **spectrum** may be thought of as being a sequence of topological spaces with basepoints

$$\{X_0, X_1, \ldots\}$$

together with continuous functions (preserving basepoints)

$$s_i \Sigma X_i \to X_{i+1}$$

which are nice inclusions.

It is not important to elaborate the details of the morphisms (the functions) in $\mathcal{S}$, these being somewhat technical but to note the most germane properties of this category. There is a functor

$$S^\infty : \mathcal{T}_* \to \mathcal{S}$$

which takes a space X with distinguished basepoint to the spectrum

$$\{X, \Sigma X, \Sigma^2 X \ldots\}.$$

In $\mathcal{S}$, the function $\Sigma(\cdot)$ is invertible, and fibers of maps (equivalent to) desuspensions of cofibers.

There is also a smash product in $\mathcal{S}$ which is also denoted $\wedge$ and which is determined by wanting

$$S^\infty(X \wedge Y) = S^\infty(X) \wedge S^\infty(Y).$$

$\wedge$ in $\mathcal{S}$ plays a role similar to $\otimes$ in a category of modules.

Our interest in $\mathcal{S}$ arises because any functor

$$X \mapsto h_*(X)$$

from $\mathcal{T}_*$ to graded groups which satisfies the axioms of a homology theory is actually given by

$$X \mapsto \pi_*(S^\infty(X) \wedge H)$$

for an appropriately chosen spectrum H. So spectra represent homology theories.

Goodwillie's Calculus

We begin by considering a functor

$$F : \mathcal{T} \to \mathcal{T}$$

such that F preserves weak homotopy equivalences. For purposes of simplicity, we also assume that $F(*)$ is contractible (F is *reduced*). There is a special class of such functors which are referred to as *excisive*. An excisive functor is a functor which takes homotopy pushout squares to homotopy pullback squares. Loosely this condition can be though of as taking cofiber sequences of spaces to fiber sequences of spaces. In other words, if F is excisive, then the functor

$$X \mapsto \pi_*(F(X))$$

satisfies the axioms of a *homology theory*. (It is a consequence as discussed above that linear functors are represented by spectra; in fact an excisive reduced functor from based spaces is represented by the spectrum $F(S^0)$.)

One reason excisive functors have a special role is that in many cases homology theories are computable, so that even if we can't always identify $F(X)$ precisely, we can at least compute its homotopy groups. Goodwillie considers excisive functors to be analogous to linear functions in single variable calculus.

One way to think about the beginning of the functor calculus is to imagine searching for an algorithm which allows one to approximate an arbitrary (reduced homotopy) functor by an excisive one. In ordinary calculus, the analogy is to finding a linear approximation to an arbitrary function.

Goodwillie solves this problem in [14]. Given an arbitrary reduced homotopy functor F, Goodwillie gives an algorithm for computing a linear (excisive) functor

$$P_1 F : \mathcal{T} \to \mathcal{T}$$

which comes with a natural transformation $\eta : F \to P_1 F$ which is *initial* among natural transformations from F to linear functors. That is, given any natural transformation $\nu : F \to G$ where G is linear, ν factors through η. With the restrictions we've given, it is easy to describe the algorithm for making $P_1 F$. With the restrictions we've given, there is a natural map

$$F(X) \to \Omega F(\Sigma X).$$

The target functor (as a functor of X) is also a reduced homotopy functor, so the construction can be iterated. Then $P_1 F$ is (loosely) the limit of

$$F(X) \to \Omega F(\Sigma X) \to \Omega^2 F(\Sigma^2 X) \to \dots$$

The notion of a linear approximation to a functor turns out to be just the beginning of an analogy between Taylor polynomials and Taylor series. Goodwillie calls an excisive functor is "1-excisive." Goodwillie gives a definition of n-*excisive*: roughly speaking, a functor is n-excisive if it takes any $n+1$-cubes of spaces in which every square is a pushout to some $n+1$ cubes of spaces in which the initial corner is the pullback of the rest of the cube. From this definition it is obvious that it is easier to be $n+1$ excisive than n excisive (that is, n-excisive functors are automatically $n+1$-excisive).

For each (reduced, homotopy) functor F, there is an n-excisive approximation $P_n F$ and a natural transformation $\eta_n : F \to P_n F$ which is initial among natural transformations from F to n-excisive functors. Just as 1-excisive functors are to be thought of as analogous to linear functions, n-excisive functors should be thought of as analogous to polynomial functions of degree n. So P_nF can be though of as the degree n polynomial approximation to F. Because $n-1$-excisive implies n-excisive, the universal property of the natural transformation

$$\eta_n : F \to P_nF$$

implies there is a functor $\pi_n : P_nF \to P_{n-1}F$ so that

$$\pi_n \circ \eta_n = \eta_{n-1}.$$

There are two important structural observations to make here. First the natural transformations π_n give us a tower of functors $\{P_nF\}$ and the natural transformation η_n give compatible maps from F into this tower. One can ask what the relationship is between F and the homotopy inverse limit of this tower. In particular, one hopes that for any particular space X, F is *analytic* at X (that is, $F(X) = \lim(P_nF)(X)$).

Second, recall that linear functors are described by spectra. Polynomial functors of degree greater than 1 don't have such a simple description, but for each n, the fiber of the natural transformation $\pi_n : P_nF \to P_{n-1}F$ is completely describe by a spectrum with the nth symmetric group, Σ_n, acting on it, and techniques for determining what this spectrum actually is are described in [16]. This functor should be though of as a homogenous functor of degree n. So while excisive functors of degree n may be somewhat complicated, they are described by a finite number of extensions of functors which are themselves determined by equivariant spectra. In principle, this leads to descriptions of (analytic) functors from spaces to spaces in terms of equivariant stable data together with extension information.

This is already interesting in the case where F is the identity functor. In this case the functor is, of course, understood, but because homotopy groups are extremely difficult to compute for most topological spaces, the homotopy groups of the functor evaluated at most interesting spaces are not understood. The homogenous layers are discussed in [16] and [19], and the entire tower is discussed in [2]. This work is further developed for particular values of the space X in[3] where the homotopy groups of the spaces in the Goodwillie tower shed light on the homotopy groups of X.

Homotopical localization

Homotopical localization has its roots in algebraic localization. Serre introduced C-theory as a tool that allowed him to prove local versions of classical theorems like the Hurewicz theorem. Some years later the implicit ideas are developed in different directions by Quillen and Sullivan.

Quillen, in [22], gives a development of localization in "model categories". At its most fundamental, this gives conditions where a new category can be constructed from an old category by "inverting" some collection of morphisms which are to be thought of as equivalences (in the new category). A specific and commonly used example is to take the old category to be the category of topological spaces and the equivalences to be maps which induce isomorphisms on $H_*(-;\mathbf{Q})$. (More examples can be easily produced by substituting other coefficients for $\mathbf{Q}$.)

Sullivan, in [23] takes a different approach. He describes for a set of primes S and sufficiently nice CW complexes X a construction X_S which "inverts" primes in S. That is, if $X \to Y$ is a map which induces an isomorphism in $H_*(-;\mathbf{Z}[S^{-1}])$, then the induced map $X_S \to Y_S$ will be an equivalence.

Bousfield in [5] generalized these ideas considerably. A *homology theory* $E_*(-)$ is a homotopy invariant functor from spaces to graded abelian groups which satisfies the usual properties of singular homology except that if $*$ represents the one point space, the graded group $E_*(*)$ is not required to be concentrated in dimension 0. Given such a homology theory, Bousfield constructs a functor L_E from the category of spaces to itself which he calls E-localization, and a natural transformation, η from the identity functor to E. E-localization is determined up to homotopy by the following two properties:

1. L_EX is E-local.
2. The natural transformation evaluated at X gives a map $X \to L_EX$ which is initial (up to homotopy) among maps from X to E-local spaces.

Here by Y is *E-local*, we mean that if $E_*(A) = E_*(*)$, then $[A, Y] = *$, the one point set. So all maps from A to Y are homotopic to the constant map.

Fundamental to the construction of Bousfield's localization functors are the class of maps which are to localize to homotopy equivalences. Bousfield ([8]), Dror ([12]) and other authors study more general localizations based on collections of maps which are to become equivalences.

There is a sequence of homology theories related to cobordism known as Johnson-Wilson theories

$$E(0)_*(-) = H_*(-;\mathbf{Q}), E(1)_*(-), E(2)_*(-), \ldots$$

(here $E(1)_*(-)$ is closely related to complex K-theory). Since work of Morava as expanded by Miller, Ravenel and Wilson [20] and the celebrated Nilpotence Theorem [11, 18] localization with respect to these theories has become one of the central organizing principles of stable homotopy, and to a lesser extent, unstable homotopy. Localization with respect to the homology theory $E(n)$ is generally denoted $L_n(-)$, and this family of localizations are referred to as the chromatic localizations.

Scope of workshop

The workshop was intended to center on areas where the calculus of functors meets homotopical localizations.

Let L be a homotopical localization functor on some category. L is guaranteed to come with an important structure; a natural transformation from the identity functor to L:

$$\eta_X : X \to L(X)$$

such that

$$\eta_{LX} : LX \to L(LX)$$

is a homotopy equivalence (L is homotopy idempotent).

This is also a property satisfied by the functors in Goodwillie's Taylor Tower when interpreted suitably. Consider the category whose objects are homotopy functors from (for example) $\mathcal{T}$ to $\mathcal{T}$. Then P_n applied to this category of functors is idempotent and comes with a natural transformation from the identity functor. In Dwyer's presentation at the workshop, he described how to produce P_n as a homotopical localization.

One of the more fascinating results in these area is that of Arone and Mahowald in [3]. This paper analyzes the Goodwillie tower of the identity functor from spaces to spaces. One of the main results is that for certain spaces (at least for spheres) the layers in the Goodwillie tower for the identity functor are essentially the chromatic localizations, L_n. While the implications of this fact are far from completely understood, Michael Ching's work presented at this workshop displays these same objects (the derivatives of the identity functor) arising as the spaces in an operad.

A second place where an interaction between chromatic localizations and Goodwillie's techniques was demonstrated at this meeting was in Kuhn's report on his work. If X is a spectrum, it determines a certain infinite loop space (written $\Omega^\infty X$). Kuhn is able to use a number of techniques including Goodwillie calculus to compute $E_*(\Omega^\infty X)$ in terms of $E_*(X)$ for homology theories $E_*(-)$ related to chromatic localizations.

While initially the calculus of homotopy functors was designed for functors on spaces or spectra, the theory has in the mean time found parallel instances in a number of other categories, such as chain complexes, vector spaces and the category of open subsets of a manifold. This begs for an eventual full bodied framework for the calculus of homotopy functors on suitable model categories.

There were two main goals to this conference. First, we sought to introduce researchers in the calculus of functors or homotopical localization to each other's subject. Second, we sought to develop an overlap of these two research areas by exploring current research in both areas. Towards the first objective, Tom Goodwillie provided a series of expository lectures which laid out the foundations of the calculus of functors. A complementary series of lectures were provided by Bill Dwyer, who gave an excellent introduction to localizations and explained how to construct Goodwillie's Taylor stages as homotopy localizations within a suitable category of diagrams of spaces as mentioned above. These lectures laid the groundwork for what followed.

Outcome of the meeting

While it is unfair to categorize the contributions of the participants of this conference into such a short list of topics, it is beneficial to enumerate those topics which form current trends in the calculus of functors and homotopical localizations. What follows is a short compilation of those topics which pertain most to the intended goals of this meeting.

- Manifold Calculus: As mentioned in the introduction, the calculus of functors has applications to areas reaching beyond homotopy theory. In particular, Goodwillie's machinery can be applied to functors from the category of open sets of a manifold to the category of topological spaces. In tandem lectures, Ismar Volic and Brian Munson gave a gentle introduction to manifold calculus. The talks pertained to research in both the machinery of calculus (Munson's results address the lifting problem from the second stage of the tower to the third stage of the tower), and applications of this machinery to the study of embeddings (Volic described joint work with Pascal Lambrechts and Greg Arone related to finite type invariants of knots).

- Calculus and Operads: Recently, there has been a flurry of activity trying to understand an apparent operad structure on the layers of the Goodwillie tower of the identity functor from spaces to spaces. One of the great testimonies to the beauty of the calculus of functors is complexity of the Goodwillie tower of the identity functor, which is seemingly innocuous. In particular, this complexity is the main obstacle to obtaining a chain rule. Motivated by our instinct from the calculus of real variables, we would expect that the layers of the tower for $F \circ G$, where F and G are homotopy functors of spaces, would be the composition of the layers of F with the layers of G. However, the expected formulation fails. Rather, the identity functor plays a critical role. In his talk, Michael Ching showed that the layers of the identity functor form an operad, and conjectured a solution to the chain rule problem, relying on the left and right module structures of the layers of F and the layers of G over this operad. An alternative approach to understanding the operad structure of the layers of any homotopy functor of spaces equipped with a natural transformation $F \circ F \to F$ was suggested in the talk of Andrew Mauer. Mauer's approach relies on the formulation of the layers of such a functor in terms of the cross effects of this functor. This is also related to Dev Sinha's talk, in which he presented another formulation of the operad structure on cross effects of the identity of functors, at least for spheres. The relationship between Sinha's work and Ching's work can be seen by relating both of their operads to the Lie operad.

- Tensor calculus of homotopy functors: ad hoc special session by Tom Goodwillie with an outline of an obstacle toward a 'theory of differential forms' of homotopy functors (spaces) to (spectra).

- Relationships between calculus of functors and localizations: Nick Kuhn's work with $K(n)$ localizations and calculus, Taylor stages in the calculus of homotopy functors are homotopy localizing functors in a suitable category of diagrams of spaces: Bill Dwyer

Abstracts of Talks

M. Ching *Operads and calculus of functors*
I'll talk about some aspects of the relationship between the calculus of homotopy functors and the theory of operads. In particular, I'll describe the operad structure on the derivatives of the identity functor and try to explain how the derivatives of other functors might fit into this framework.

C. Casacuberta *Continuity of homotopy idempotent functors*
A functor L in a simplicial model category is called simplicial or continuous if it defines a map from map$(X, Y) \to$ map(LX, LY) for all X, Y, which is natural and preserves composition and identity. As shown by Farjoun and Hirschhorn, f-localizations can be constructed as continuous functors. Thus, a necessary condition for a homotopy idempotent functor to be equivalent to some f-localization is that it be equivalent to a continuous functor.

In joint work with different coauthors, we discuss continuity of homotopy functors in several model categories, with emphasis on simplicial sets, spectra, and groupoids. In the latter, remarkably, continuity is automatic.

W.G. Dwyer *Localization and Calculus I and II*
A general discussion of the idea of localization in homotopy theory. Followed in part II by specialization to the localization of diagram categories, and further specialization to the case of a particular diagram category associated to the Goodwillie tower.

E. Farjoun *Open problems and some recent progress in localization and cellularization theory*
The talk will revisit some of the progress made recently in understanding localization and co-localization functors. We shall list some interesting problems and describe related partial progress. The talks will concentrate mostly on general properties of localization with respect to a map in both algebraic homological algebra and topological categories.

T. Goodwillie *Introduction to the Calculus of Homotopy Functors, I,II, and III*
Overview of basic definitions and results (excisive and n-excisive approximations of functors, classification of homogeneous functors, chain rule); key examples; matrix notation. Followed in part II by: more about homogeneous functors, with an emphasis on results which require no information about connectivity.

A geometric view of the functor/function analogy. In this view, Top is a variety and functors Top $\rightarrow$ Spectra are global functions. I will say which categories are the tangent spaces of Top. I will discuss tangent vector fields and more generally tensor fields, in both a coordinate-free way and a coordinate-dependent way. I will show that there are two tangent connections, both of which are flat, and that their difference is the tensor field known as smash product of spectra. I will say something about higher-order jets and about differential operators. I cannot make much sense of differential forms (except 0-forms and 1-forms), but I may talk about them anyway. Applications are work in progress, but I will make sure to at least say something trivial about some nontrivial examples, and maybe something nontrivial about some trivial examples.

M. Hovey *$E(n)_* - E(n)$-comodules*
I will recap my results with Neil Strickland about the structure of the category of $E(n)_*E(n)$-comodules (e.g. the Landweber filtration theorem works there as well). I will describe why we need to know more about comodules (derived functors of product in the category of comodules form the E_2-term of a spectral sequence converging to the $E(n)$-homology of a product of spectra; this is relevant for the chromatic splitting conjecture). Then I will describe some new results I have about the honest injective $E(n)_*E(n)$-comodules. There are only $n+1$ isomorphism classes of indecomposable injectives, and most interestingly, the endomorphism ring of the k-th one is $(E(k)^\wedge)^*(E(k)^\wedge)$, where $E(k)^\wedge$ is the completion of $E(k)$ at I_k.

So in the category of $E(n)_*E(n)$-comodules, you are seeing all the $E(k)^\wedge$ operations for $0 \leq k \leq n$, and therefore seeing all the different stabilizer groups S_k for $0 \leq k \leq n$. This is a good thing, since the relation between the different stabilizer groups is basically what the chromatic splitting conjecture is about.

N. Kuhn *Periodic homology of infinite loop spaces*
If E_* is a homology theory, one can ask to what extent the E_*-homology of an infinite loop space is determined by the E_*-homology of the associated spectrum. Using a combination of the Hopkins-Smith Periodicy Theorem, as packaged in the telescopic functors of Bousfield and me, and Goodwillie calculus, I can give a quite definitive answer to this question when the homology theory is Morava K-theory. There are calculations still to be done that may inform on the Telescope conjecture.

A. Mauer-Oats *An operad from the derivatives of a monad*
McClure and Smith have a simple idea that explains how to produce an operad from a functor operad by evaluating on the unit of the smash product. The cross effects of a (reasonably good) monad F are a functor-operad of spaces. We explain the proper way to prolong a multivariate functor to spectra, and use this to produce an operad of symmetric spectra. If a certain problem of cofibrancy can be overcome, the spectra in the operad will be the derivative spectra of F.

B. Munson *The layers of the embedding tower*
I will discuss the layers of the embedding tower and their relationship to the obstructions to finding embeddings.

D. Sinha *A pairing between graphs and trees*
We give an elementary pairing between graphs and trees, which facilitates the study of the Lie operad and free Lie algebras. It arises in topology through both homology of configuration spaces and (conjecturally) in studying Hopf invariants and Whitehead products. We sketch its possible application in using the embedding calculus to define knot invariants, and hope that it might be of interest in the homotopy calculus as well.

D. Stanley *Complete invariants of t-structures*
Let R be a Noetherian ring. We give a classification of Bousfield classes on the bounded derived category of R. This also gives complete invariants of t-structures on the same category. We also show that the t-structures on the unbounded derived category of Z-modules do not form a set.

I. Volic *Embedding calculus and formality of the little cubes operad*
I will first give a brief introduction to embedding calculus and say how a certain Taylor tower can be assigned to an isotopy functor. Then I will describe joint work with Greg Arone and Pascal Lambrechts in which the central observation is that the stages of the Taylor tower in the case of Emb(M, V), the space of embeddings of a manifold in a vector space (up to immersions), have the structure of maps of certain modules over the little cubes operad. Using Kontsevich's formality of this operad, one then concludes that the cohomology spectral sequence for Emb(M, V) arising from the Taylor tower collapses rationally. In the special case of spaces of knots, this was conjectured by Vassiliev. Additionally, using the interplay between embedding and orthogonal calculus, one also deduces that the rational cohomology of Emb(M, V) only depends on the rational homotopy type of M when 2dim$(M) + 1 <$ dim(V).

M. Weiss *Stratifications and homotopy colimit decompositions*
This talk will discuss the art of converting stratifications into homotopy colimit decompositions, perhaps with applications to the theory of surface bundles. Every well behaved stratified space has a homotopy colimit decomposition indexed by a certain topological category in which all endomorphisms are invertible up to homotopy. In many cases one can do better and match the stratification with a homotopy colimit decomposition indexed by a discrete category in which all endomorphisms are invertible. The matching property means roughly that the strata correspond to the isomorphism classes of the indexing category.

List of Participants

A determined effort was made to ease the entry into these subjects by young researchers. Specifically, out of 34 participants, 3 were graduate students and a number of 5 were within the first 3 years of their postdoctoral career. We had talks from one of the graduate students and from three of the postdocs.

Arlettaz, Dominique (Universite de Lausanne)
Bauer, Kristine (University of Calgary)
Casacuberta, Carles (University of Barcelona)
Chebolu, Sunil (University of Washington)
Ching, Michael (Massachusetts Institute of Technology)
Chorny, Boris (University of Western Ontario)
Dover, Lynn (University of Alberta)
Dror-Farjoun, Emmanuel (Hebrew University of Jerusalem)
Dwyer, William (Notre Dame University)
Goodwillie, Tom (Brown University)
Gutierrez, Javier (University of Barcelona)
Hovey, Mark (Wesleyan University)
Krause, Eva (University of Alberta)
Kudryavtseva, Elena (University of Calgary/Moscow State University)
Kuhn, Nick (University of Virginia)
Lambrechts, Pascal (Louvain-la-Neuve)
Mauer-Oats, Andrew (Northwestern University)
McCarthy, Randy (University of Illinois at Urbana-Champaign)

Munson, Brian (Stanford University)
Nicas, Andrew (McMaster University)
Nikolaev, Igor (University of Calgary)
Palmieri, John (University of Washington)
Peschke, George (University of Alberta)
Prince, Tom (University of Alberta)
Ravenel, Douglas (University of Rochester)
Sadofsky, Hal (University of Oregon)
Scull, Laura (University of British Columbia)
Sinha, Dev (University of Oregon)
Stanley, Don (University of Regina)
Varadarajan, Kalathoor (University of Calgary)
Volic, Ismar (University of Virginia)
von Bergmann, Jens (University of Calgary)
Weiss, Michael (University of Aberdeen)
Zvengrowski, Peter (University of Calgary)

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Chapter 5

Complex Data Structures (05w5504)

April 9–14, 2005

Organizer(s): Jim Berger (Statistical and Applied Mathematics Institute), Nancy Reid (University of Toronto), James Stafford (University of Toronto), Mary Thompson (University of Waterloo)

Introductory Remarks

Projects, and pilot projects, within the National Program on Complex Data Structures (NPCDS) met April 9 - 14 at the Banff International Research Station. Leaders in Computer Experiments, Data Mining, Genomics and Survey Methods each organized a day of activity in their respective fields. An additional day was devoted to three pilot projects that have inaugural workshops later this year in the areas of Biomedicine, Forestry and Marine Ecology. Research presentations were incredibly varied and included topics that concerned pharmacophore identification, complex HIV proteomic data structures, communications security, studies of complex traits, social behaviour, forest fires, high throughput genomics, tracking of leatherback turtles, turbulence, and so on. Underlying such a diverse set of topics was a genuine common interest in complex data, regardless of its origin. This, in effect, bonded participants in their vision of what NPCDS can bring to the statistical sciences community in Canada. As such the event was instrumental in generating considerable enthusiasm for the Program's model. Concretely, the establishment of interdisciplinary projects with quantitative leadership was viewed as a vehicle that gives our community a greater voice in the research agenda's of other disciplines. These projects have the potential to create a culture in our discipline where training takes place in intensely interdisciplinary environments ensuring young researchers become effective collaborators in the long run. This was evident by the number of excellent presentations given by graduate students including Norberto Pantoja Galicia, Jason Loeppky, Pritam Ranjan and so on.

The Science

Data Mining

Certainly this workshop started in extremely strong fashion setting the tone for the remainder of the week. The first day's topic was data mining, a field with many connotations though organizers were able to encapsulate much of the research in this area through a focus on the rare target problem. Data mining is a new and fast-changing discipline, which aims at the discovery of unusual and unexpected patterns in large volumes of data. It came to life in response to the challenges and opportunities provided by the increasing number of large, complex, high-dimensional databases covering important areas of human activity, coming from the industrial, economical, social and biomedical sectors.

Stan Young of the National Institute of Statistical Sciences addressed the use of Gibbs sampling for pharmacophore identification, a problem where large libraries of molecules are searched for comparatively similar reactive properties. Here the binding of a small molecule to a protein is inherently a three dimensional matching problem. As crystal structures are not available for most drug targets, there is a need to be able to infer the key binding features of small molecules and their disposition in space, the pharmacophore from bioassay data. They use fingerprints of 3D features and a modification of Gibbs sampling to determine the common pharmacophore parts for a set of compounds. We use a clique detection method to map the features back onto the binding conformations. The method works for known pharmacophores. We show the basic algorithm is fast, 15 minutes for 15 molecules, and it can easily deal with a hundred compounds and tens of thousands of conformations. They demonstrated the successful use of PharmID on a multiple binding mode problem. Being able to sort out multiple pharmacophores from the same data set is potentially useful in cell-based assays where different molecules could be hitting different biological targets. Knowing the 3D pharmacophore for a biological target was a key for more efficient compound design and 3D database searching.

Stan's talk was followed up by a mesmerizing demonstration of the predatory behaviour of the Human Immunodeficiency Virus (HIV). This was in the context of George Hatzakis's (McGill University) lecture concerning modeling HIV complex clinical and proteomic data structures. Within the context of Clinical/Bio-Informatics, it is common to use numerical techniques to model and optimize clinical management of patients treated for Human Immunodeficiency Virus-1 (HIV-1). HIV infection is for the most part chronic and asymptomatic. Optimal therapy should suppress the HIV-virus, prevent the emergence of antiviral drug-resistance and control long-term side effects. In George's presentation he addressed the former 2 aspects. To achieve virus-suppression one has to longitudinally follow and understand how an HIV-patient progresses. However, clinical and laboratory follow-up information is non-stationary and characterized by transients and trends. George used Artificial Intelligence based models to follow the progression of a subset of patients from the Electronic Anti-Retroviral Therapy (EARTH) International-cohort and addressed several what-if scenarios related to morbidity and mortality. Also, to identify those patients that could mostly benefit from the new class of drugs based on the CCR5 and CXCR4 chemokine inhibitors, he analyzed the proteomic sequences of the V3 loop on over 1000 patients coming from the HOMER BC-cohort. Clustering techniques were presented.

Further presentations concerned the development of particular data mining tools as demonstrated most effectively by Antonio Ciampi and Steven Wang who spoke on soft classification trees and clustering categorical data respectively. Perhaps one of the most compelling presentations was given by Shirley Mills and Ted Normington, both of Carleton University, who are involved in various research projects in consultation with the Communications Security Establishment: Data mining in action leading to secure national borders (we hope).

Genomics and Statistical Genetics

Not to be outdone by the data miners the second day of the workshop was devoted to the genetic revolution that is taking both the medical world and our imaginations by storm. Advances in many areas of Genomics have become the most exciting story in the biological, life, and health sciences in recent years, and have captured the imagination of the public at large. One of the most interesting technological breakthroughs in genomics has been the miniaturization of classical experimentation techniques in molecular biology. This has led to the ability to conduct massively parallel experiments on the scale of the whole genome. The most widely known examples of such technology are various kinds of microarrays or DNA chips, which can now measure the expression activity of most of the predicted genes in humans. There exist similar high-throughput technologies to detect Single Nucleotide Polymorphisms (SNP chips), protein abundance (proteome chips), RNA activity, protein-protein interaction systems, and others.

For the first time in history, biologists are facing huge volumes of noisy data. The challenge of analyzing this data has been described as the biggest bottleneck in modern biology. Huge dimensionality and small sample size creates a challenge throughout an experiment, from the design, visualization and exploratory phases, to the analysis itself.

The genetics/genomics theme at the meeting was led by Dr. Brent Zanke, VP of the Ontario Cancer Research Network (OCRN) who spoke on the use of high throughput genomics to predict disease risk and treatment response. Here the coincidence of functionally relevant polymorphisms in genes that are part of a single pathophysiologic pathway may cause significant risk for an individual and collectively account for a large proportion of population at attributable risk. For instance, the activities of phase I enzymes such as the cytochrome P450s, phase II enzymes, such as glutathione-s-transferases, DNA repair enzymes, cell cycle control enzymes and apoptosis effectors. Polymorphisms in each of these enzymes that individually would confer only minor increased disease susceptibility could collectively cause significant individual risk. Many case-control studies evaluating isolated polymorphisms have failed to identify significant disease association, potential victims of underpowered study designs.

In anticipation of genome-wide disease association studies an international human variation-mapping (HapMap) project was launched in October 2002 to catalogue blocks of LD and haplotype diversity (http://genome.gov/10005336). As much as 85% of the human genome may be organised into haplotype blocks that are 10,000 bases or larger. The exact pattern of SNP variants within a given haplotype block differs among individuals, though for most less than 5 distinct haplotype clades exist. This limited haplotype diversity makes complete genotyping of individuals of Northern European or Asian descent possible with measurement of as few as 50,000-100,000 haplotyping SNPs (htSNPs) and measurement of approximately 250,000-500,000 htSNPs in individuals of African descent.

Brent and colleagues are studying haplotype diversity in patients with colon cancer and controls to detect associations with the presence of the disease and with treatment response to those with cancer receiving chemotherapy. Tests such as these will reduce health care costs and reduce the social cost of cancer. With an investment from Genome Canada our group will measure over 1 billion SNPs in 2400 individuals over the next 6 months. The statistical analysis of this data set will present new issues in multiple testing correction and multivariate analysis.

Rafal Kustra and Celia Greenwood are leading efforts to confront the new statistical issues in this context. They present an initial analysis of the first batch of data in an international effort to derive a prognostic test of colon cancer using dense maps (hundreds of thousands) of genetic markers and detailed clinical and lifestyle data. They discussed attempts in building a predictive, multivariate model using boosting and proposed a dimension reduction techniques motivated by statistical and evolutionary genetics. Their untested proposal is intended to spark discussion on dealing with huge dimensionality of genomic data in the presence of highly refined existing knowledge about genetics, knowledge which could potentially be used to construct more successful predictive models.

Rafal and Celia were followed up by Shelley Bull who addressed issues in multiple testing and effect estimation for candidate gene and genome-wide studies of complex traits. While it is well-recognized that the examination of multiple hypotheses corresponding to multiple SNPs within a candidate gene and/or to multiple genes or genetic markers across the genome can lead to inflated false positive rates and failure to replicate findings in an independent sample, the impact of multiple testing and strict type I error control on effect estimation has received less attention. To put these issues in context Shelley first considered some background concerning gene discovery and gene characterization, and the related data structures. Approaches to multiple testing adjustments in genetic linkage and association analysis, whether family-based or case-control designs, can usefully depend on the correlation structure among neighbouring genetic loci. However, multiple testing and stringent type I error control typically induce bias in the associated effect parameter estimates. They proposed a bootstrap algorithm and resampling-based estimators that yield bias-reduced estimates from the original sample in general settings.

Jenny Bryan then spoke on statistical problems in gene clustering from high-throughput data. The term ”high-throughput data” encompasses a large variety of current assays in which a response is measured across a range of condition or subjects for a large number genes (often for practically an entire genome). This certainly includes transcriptional profiling via microarrays, as well as highly parallel phenotypic studies in, for example, the yeast deletion set. A common use of such data is to cluster genes, with the hope that apparent gene clusters will have substantial overlap with biological gene groups, such as pathways, protein complexes, or regulons. Jenny cast this problem in the form of a traditional statistical inference problem and drew some practical conclusions about preferred algorithms and such matters. She used this framework effectively to generate group discussions on the general”disparate data” unification problem in gene clustering (should we create meta-datasets and then cluster? should we cluster datasets separately and then merge? should we use biclustering-type techniques?).

The final genomics speaker, Bob Nadon, addressed data analysis, software, and pedagogy in big science biology. Big science biology is generating massive data sets that provide motivation for algorithm development and potential for long-term funding. This continuing collaboration between biology and the computational sciences will be most productive if knowledge and tools are made available in formats readily accessible to applied scientists. Bob described such a project in microarray analysis that integrates software, pedagogy, and data analysis research.

Pilot projects

After two exciting days devoted to Data Mining and Genomics attendees were treated to a short day composed a simply a morning session in which nascent NPCDS projects were featured. This was extremely rewarding for both the speakers and audience, the latter be given a sense of future directions of NPCDS while the former received an abundance of helpful advice and suggestions to assist their endeavours.

The first of these speakers, John Braun, discussed forest ecology under the title Forests, Fires and Stochastic Modelling. He asserted that statisticians have an important role to play in the study of various aspects of forestry. The talk began with a description of how an upcoming NPCDS workshop would facilitate interactions between statisticians and researchers into wildfire behaviour as well as forest ecology and hydrology. This was followed up by a description of a work in progress connected with a problem of forest fire prediction given observed lightning strokes. The prediction problem is not solved; however, the talk will describe how interactions with forest fire researchers have spurred development of statistical methodology.

John's lecture was followed up by a joint effort from Chris Field and Joanna Flemming who both gave an overview of statistical methods in marine ecology. This included a general overview of the Marine Ecology Workshop to be held at Dalhousie in August, 2005 as part of the NPCDS programme. They also gave brief descriptions of example problems involving the dynamics of plankton levels in the tropical Pacific and a more detailed analysis of a problem involving tracking data of leatherback turtles, a long distance migrant.

The third presentation was given by Peter Song who discussed an array of methods he has developed for use in biomedical research. This included a personal overview of the methodological development in the of longitudinal and clustered data analysis (LCDA). Arguably, the methodology of the LCDA has provided powerful tools to practitioners for their subject-matter innovative research in past two decades or so. In his talk, he covered both Liang and Zeger's marginal models and the generalized linear mixed models. Peter used a few real world data sets as running examples to enhance discussions.

Computer Experiments for Complex Systems

The design and analysis of experiments continue to make important and far-reaching contributions to scientific investigation. Historically, experimenters have relied on physical experiments to help understand processes. The rapid growth in computing power has made the computational simulation of complex systems feasible and has helped avoid physical experimentation that might otherwise be too time consuming, costly, or even impossible to observe. The advent of such widespread computer experiments raises a host of challenging statistical issues, which this project will explore.

The fourth day of the workshop was devoted to the topic of computer experiments and was marked by a large number of student presentations which were all quite excellent. Jason L. Loeppky, a postdoctoral student at UBC, addressed issues in model calibration. Computer models are widely used in engineering and science to simulate physical phenomena. Before using a computer model, for example to optimize systems, a natural first step is often to assess whether it reliably represents the real world. Data from the computer model are compared with data from field measurements. Similarly, field data may be used to calibrate or tune unknown constants in the computer model.

Calibration is particularly problematic in the presence of systematic discrepancies between the computer model and field observations. In Jason's talk he introduced a likelihood based approach to the estimation of the calibration parameters and further showed how one could use this to test the reliability of the computer model. The approach and the test were illustrated through a series of examples, and compared to the results of a Bayesian implementation.

Zhiguang Qian, a graduate student at Georgia Institute of Technology, discussed building surrogate models based on detailed and approximate simulations, while Pritam Ranjan, a graduate student from SFU, discussed designing efficient simulations for exploring features of response surfaces. Pritnam's talk was particularly interesting as in many engineering applications, one is interested in identifying the values of the inputs in computer experiments that lead to a response above a pre-specified threshold. In his talk he introduce statistical methodology that identifies the desired contour in the input space. The proposed approach had three main components. Firstly, a stochastic model is used to approximate the global response surface. The model is used as a surrogate for the underlying computer model and provides an estimate of the contour together with a measure of uncertainty, given the current set of computer trials. Then, a strategy for choosing subsequent computer experiments that improve the estimation of the contour is outlined. Finally, he discussed how the contour is extracted and represented. The methodology is illustrated with an example from a multi-class queuing system.

Yet another graduate student, Crystal Linkletter of SFU, presented where she discussed inert column variable selection. In many situations, simulation of complex phenomena requires a large number of inputs and is computationally expensive. Identifying which inputs most impact the system can be a critical step in the scientific endeavour so that these factors can be further investigated. In computer experiments, it is common to use a Gaussian spatial process to model the output of the simulator. Crystal introduced a new, simple method for identifying active factors in computer screening experiments. The approach is Bayesian and only requires the generation of a new inert variable in the analysis. The posterior distribution of the inert factor is used as a reference distribution with which we assess the importance of the experiment factors. The methodology was demonstrated on simulated examples as well as an application in material science.

The final speaker of the day was Derek Bingham, who leads the NPCDS project in this area and supervised a number of the students who presented. In Derek's talk latin hypercube sampling was presented as a popular method for evaluating the expectation of functions in computer experiments. However, when the expectation of interest is taken with respect to a non-uniform distribution, the usual transformation to the probability space can cause relatively smooth functions to become extremely variable in areas of low probability. Consequently, the equal probability cells inherent in hypercube methods often tend to sample an insufficient proportion of the total points in these areas. Derek introduced Latin hyper-rectangle sampling to address this problem. Latin hyper-rectangle sampling is a generalization of Latin hypercube sampling that allows for non-equal cell probabilities. A number of examples were given illustrating the improvement of the proposed methodology over Latin hypercube sampling with respect to the variance of the resulting estimators. Extensions to orthogonal-array based Latin hypercube sampling, stratified Latin hypercube sampling and scrambled nets were also described.

Complex survey data

Survey data now being collected by many government, health and social science organizations have increasingly complex structures precipitating an urgent demand for new statistical methodology to further research in substantive areas. In cross-sectional studies, which are taken at one point in time, it is typical to use very complex sampling designs, involving stratification and clustering as the components of random sampling. There can also be complexities in the resulting data file due to the patterns of nonresponse. In longitudinal studies, which follow individuals or groups of individuals over time, there is additional complexity stemming from possible complex correlation structures induced by repeated measurements on the same sampling unit, by irregularly spaced data and differing numbers of repeated observations on individuals. This datatype, with all it various complexities, is increasingly common in substantive areas due to its power to infer causality, to separate individual and population trends and to detect changes in time.

The final day of the workshop was devoted to the efforts of the survey methods project within NPCDS, although due to many of the team members being drawn to the meeting of the International Statistical Institute in Australia, the session was limited to four speakers. Nevertheless this is a very active project, involving many graduate students one whom presented in this session.

The first speaker was Milorad Kovacevic of Statistics Canada who discussed survey bootstrap methods and analysis of survey data. Here a variety of approaches for estimating design-based variances of estimated model parameters were reviewed. The particular approach of bootstrapping through the rescaling of the survey weights - which he call the survey bootstrap, was presented as gaining popularity due to its portability. Namely, once bootstrap samples have been taken and bootstrap weights calculated, the user estimates the quantities of interest in exactly the same way with the full sample and with each of the bootstrap samples, and then combines these estimates to obtain variance estimates. There are situations, however, in which this direct variance estimator may be unstable. Recently, methods have been proposed for making inferences using an estimating function bootstrap in a model-based setting, which seem to provide more stable results. These methods have been adapted to produce different design-based estimating function survey bootstraps. In Milorad's presentation he covered some of these new developments. Results of a simulation study motivated by a real-life analysis were presented.

The next speaker, Brajendra Sutradhar considered generalized quasilikelihood approaches for survey based incomplete longitudinal binary data. When the response variable in a longitudinal model is subject to missing completely at random (MCAR) or missing at random (MAR), the existing 'independence' or 'working correlations' based generalized estimating equations (GEE) approaches yield consistent estimators for the effects of the covariates. These GEE based estimators may, however, be inefficient. There also exists a true correlation structure based GEE approach to deal with exponential family based longitudinal responses subject to MCAR or MAR. The existing correlation models used in such incomplete data analysis are, however, quite restricted. In Brajendra's presentation he exploited a robust correlation model based generalized quasilikelihood (GQL) approach, where the correlation model can accommodate AR(1), MA(1) and exchangeable correlation structures for longitudinal binary responses. Furthermore, for the cases when individuals are selected based on a complex survey sampling scheme rather than simple random sampling, it becomes necessary to incorporate the survey weights in the estimation approach. For this purpose, Brajendra developed a survey design based GQL (DBGQL) estimating equation approach as a generalization of the GQL approach. The DBGQL estimation approach was illustrated by analysing a real life binary longitudinal data set subject to MCAR or MAR.

The next talk, a joint effort by Roland Thomas and Irene Lu, was of particular interest as the research resulted from a collaboration borne out of NPCDS/SAMSI joint efforts in the context of the SAMSI thematic program: Latent Variable Models in the Social Sciences. The title of their talk was "Latent Regression with Social Science Data: A Comparison of Various Methods Using Simulation and Complex Survey Data Examples" The presentation focused on methods for estimating regression coefficients for the linear latent regression models frequently encountered in social science research. In the social sciences, latent variables are typically measured using batteries of questionnaire items from which latent variable scores can be predicted in numerous ways. These scores comprise fallible estimates of the underlying latent variables, and it is well known that naive methods of analysis based on these scores are likely to result in biased estimates. These biases are quantified not only for simple scoring methods, but also for methods based on Item Response Theory (IRT). The conclusion is that the use of scores, no matter how sophisticated, yields unacceptably large bias and should be avoided. An alternative approach via discrete structural equation modeling (SEM) is also evaluated. This approach, which implicitly includes the IRT model structure, is shown to provide lower levels of regression parameter bias, though its bias cannot be ignored for the smaller sample sizes. Finally, the speakers described a recent adaptation (Bollen, 1996) of the instrumental variables approach to social science data, and shows that this simple approach provides low levels of parameter bias comparable to the more computationally involved discrete SEM method. The performance of the various approaches was compared using simulation, and is also illustrated on complex survey data from Statistics Canada's Youth in Transition Survey.

The day, and indeed the workshop, ended in fine form with a presentation from yet another graduate student, this time Norberto Pantoja Galicia from the University of Waterloo who was one of the participants of the internship program that is jointly funded by NPCDS and Statistics Canada. Norberto discussed a nonparametric test for association of interval censored event times in the National Population Health Survey (NPHS). Here outcomes from the questionnaire of the NPHS a longitudinal survey conducted by Statistics Canada, offer the necessary information to explore the relationship between smoking cessation and pregnancy. A formal nonparametric test for association was presented. This test requires estimation of the joint density for interval censored event times, which takes into account complexities of the sample design.

Concluding Remarks

For NPCDS this event at BIRS was timely as the Program is currently entering the second half of its four-year funding cycle and it offered an opportunity for participants to assess what has been accomplished thus far. The general view was "a lot!": with potentially seven national projects established in a two year span the

Program has engaged the broader community in a robust way. Credit *must* be attributed to the many individual researchers who are investing time and energy into this endeavour. During the week at Banff, general meetings were held where progress, and the future of the program, was discussed openly. For example, issues concerning capacity led to consideration of an RFP for training initiatives, which is now being actively pursued. In addition, plans for the renewal of the program have been set in motion.

List of Participants

Berger, Jim (Statistical and Applied Mathematics Institute)
Bingham, Derek (Simon Fraser University)
Braun, John (University of Western Ontario)
Bryan, Jennifer (University of British Columbia)
Bull, Shelley (University of Toronto)
Chipman, Hugh (Acadia University)
Ciampi, Antonio (McGill University)
Cook, Richard (Department of Statistics and Actuarial Science, University of Waterloo)
Field, Chris (Dalhousie University)
Greenwood, Celia (University of Toronto)
Gustafson, Paul (University of British Columbia)
Hatzakis, George (McGill University)
Kourti, Theodora (McMaster University)
Kovacevic, Milorad (Statistics Canada)
Kustra, Rafal (University of Toronto)
Léger, Christian (Université de Montréal)
Linkletter, Crystal (Simon Fraser University)
Loeppky, Jason (University of British Columbia)
Lu, Irene (York University)
Mills, Shirley (Carleton University)
Mills Flemming, Joanna (Dalhousie University)
Nadon, Robert (McGill University)
Norminton, Ted (Carleton University)
Pantoja Galicia, Norberto (University of Waterloo)
Qian, Zhiguang (Georgia Tech)
Ramsay, Jim (McGill University)
Ranjan, Pritam (Simon Fraser University)
Reid, Nancy (University of Toronto)
Routledge, Rick (Simon Fraser University)
Song, Peter (University of Waterloo)
Stafford, James (University of Toronto)
Susko, Ed (Dalhousie University)
Sutradhar, Brajendra (Memorial University of Newfoundland)
Thomas, Roland (Carleton University)
Wang, Steven (York University)
Wang, Liqun (University of Manitoba)
Welch, Will (University of British Columbia)
Young, Stan (National Institute of Statistical Sciences)
Zanke, Brent (Ontario Cancer Research Network)
Zhu, Mu (University of Waterloo)

Chapter 6

Applications of torsors to Galois cohomology and Lie theory (05w5030)

April 23–28, 2005

Organizer(s): Vladimir Chernousov (University of Alberta), David Harari (École Normale Superiéure), Shrawan Kumar (University of North Carolina at Chapel Hill), Arturo Pianzola (University of Alberta)

A BRIEF INTRODUCTION

The idea of building mathematical structures out of local data has been a cornerstone of both modern Mathematics and Physics. Manifolds, distributions, simplicial complexes, vector bundles, and homogeneous spaces attest to this fact. The mathematical tools that measures the obstruction preventing us from gluing local data in a compatible way are the various cohomology theories.

In the middle of the last century the theory of algebraic varieties was establishing itself as a invaluable tool that allowed "geometric methods" to be applied to arithmetical questions. But already A. Weil had explicitly singled out that one of the most powerful classical tools, namely the construction of the quotient of a manifold by the action of a Lie group (homogeneous spaces), had no successful analogue for algebraic groups acting on varieties. (The reason being that the Zariski topology of a variety, which plays the role of the classical topology for a manifold, is too weak: there are too few open sets to trivialize actions, and these sets are too big). The answer to this riddle came from the work of Serre and of Grothendieck. The resulting theory of principal homogeneous spaces (Torsors for short), hinges around endowing schemes with the étale topology, and using various theories of "descent" to produce a coherent cohomology theory to go with it.

Several of the fundamental problems in algebra and number theory are related to the problem of classifying G-torsors and in particular of computing the Galois cohomology $H^1(k, G)$ of an algebraic group G defined over an arbitrary field k. The study of Galois cohomology is still in its early stages and many natural questions and long standing conjectures are still open. During the past two decades new insight into this theory began arising under the influence of algebraic geometry and algebraic K-theory. We note that new possibilities provided by algebraic K-theory still only begin to manifest themselves in full strength.

It has also recently become apparent that torsors can also be used to understand affine Kac-Moody Lie algebras and groups and superconformal algebras. It is possible, but at this point not known, that these methods could extend to a more general class of Lie algebras (Extended Affine Lie Algebras) around which there is today a considerable amount of interest.

Exploring the connections between these two aspects of torsors: The algebraic Geometry on one hand, and the infinite dimensional Lie theory on the other, was one of the purposes of the meeting.

SUMMARIES OF TALKS

G–forms and cohomological invariants

by E. Bayer–Fluckiger (EPFL Lausanne, Switzerland)

Let k be a field of characteristic $\neq 2$. Milnor's conjecture, recently proved by Voevodsky, provides a classification of quadratic forms over k up to isomorphism. This gives hope for progress in related questions, for instance the classification of quadratic forms invariant by a finite group.

Let G be a finite group. One of the natural examples of G–forms is given by trace forms of G–Galois algebras. If L is a G–Galois algebra, let us denote by q_L its trace form. Let q_0 be the unit G–form – if we denote by L_0 the split G–Galois algebra, then we have $q_{L_0} = q_0$. If G has odd order, then it is known that $q_L \simeq_G q_0$ (where $\simeq_G$ denotes G–compatible isometry). If the 2–Sylow subgroups of G are elementary abelian of rank r, then in a joint paper with J–P. Serre we give a complete criterion for the isomorphis of the trace forms of two G–Galois algebras in terms of an r–dimensional $\bmod\ 2$ cohomological invariant.

Let us denote by $W(k)$ the Witt ring of the field k, and let $I = I(k)$ be the ideal of even dimensional quadratic forms. Let d be the 2–cohomological dimension of k. Let L and L' be two G–Galois algebras. Then Milnor's conjecture implies that if $\phi \in I^d$, then the quadratic forms $\phi \otimes q_L$ and $\phi \otimes q_{L'}$ are isomorphic. Philippe Chabloz recently proved that these forms are actually isomorphic as G–forms. Going futher in this direction, note that Milnor's conjecture implies that if $\phi \in I^{d-1}$ and if we denote by $e_{d-1}(\phi)$ its cohomological invariant, then $\phi \otimes q_L \simeq \phi \otimes q_{L'}$ if and only if $e_{d-1}(\phi) \cup d(q_L) = e_{d-1} \cup d(q_{L'})$. This talk presented some partial generalisations of this fact. One can define a notion of *G–discriminant* for q_L, denoted by $d_G(q_L)$. It is then natural to conjecture that $\phi \otimes q_L$ and $\phi \otimes q_{L'}$ are isomorphic as G–forms if and only if $e_{d-1}(\phi) \cup d_G(q_L) = e_{d-1} \cup d_G(q_{L'})$. This is known in some cases, by the work of Chabloz, Monsurro, Parimala, Schoof and the author.

Essential dimension of homogeneous forms

by G. Berhuy (Nottingham University, UK)

The essential dimension of an algebraic structure is roughly the minimal number of independent parameters needed to describe it up to isomorphism. This notion has been defined first by Reichstein an Buhler for Galois extensions of given group G in a more geometric way, then extended to any G-torsor by Reichstein (where G is an algebraic group defined over an algebraically closed field of characteristic 0).

In this talk, we compute the essential dimension of the generic homogeneous polynomial of degree d in n variables when the $g.c.d.$ of n and d is a (possibly trivial) prime power. For this, we define a new numerical invariant attached to G-torsors in a geometric way, namely the canonical dimension. We then relate the canonical dimension of a certain GL_n/μ_d-torsor to the essential dimension of the generic homogeneous polynomial, and we use the properties of canonical dimension to compute it.

The algebraic connective K-theory

by S. Cai (UCLA, USA)

By using the Brown-Gersten-Quillen spectral sequence, we give a simple definition of the algebraic connective K-theory, the universal homology theory overriding the K-homology (chow groups) and algebraic K-theory. The definition of a homology theory (a Borel-Moore functor) is verified, and standard properties are proved. Relations with K-homology and K-theory are explored.

Groupe de Picard et groupe de Brauer des compactifications lisses d'espaces homogènes, I et II

by J-L. Colliot-Thélène (Université Paris-Sud, France)
et B. È. Kunyavskiĭ (Bar-Ilam University, Israel)

Soit k un corps de caractéristique nulle, $\overline{k}$ une clôture algébrique de k, et $g = \mathrm{Gal}(\overline{k}/k)$. Soient G un k-groupe connexe et X/k un espace homogène de G. Le *stabilisateur géométrique*, c'est-à-dire le groupe d'isotropie d'un $\overline{k}$-point de $\overline{X} = X \times_k \overline{k}$ est bien défini à $\overline{k}$-isomorphisme non unique près. On *note* $\overline{H}$ ce groupe. Supposons le groupe $\overline{H}$ connexe. Il y a alors un k-tore T naturellement associé au G-espace homogène X, tel que $\overline{T}$ soit le plus grand quotient torique $\overline{H}^{\mathrm{tor}}$ de $\overline{H}$. Soit X_c une k-compactification lisse de X. La $\overline{k}$-variété $\overline{X}_c$ est unirationnelle, le groupe de Picard $\mathrm{Pic}(\overline{X}_c)$ est un g-module continu discret **Z**-libre de type fini et le groupe $\mathrm{Br}(\overline{X}_c)$ est fini. On note $\mathrm{Br}_1(X_c)$ le noyau de l'application de restriction $\mathrm{Br}(X_c) \to \mathrm{Br}(\overline{X}_c)$. Le quotient du groupe de Brauer $\mathrm{Br}_1(X_c)$ par l'image du groupe $\mathrm{Br}(k)$ est un sous-groupe du groupe fini $H^1(g, \mathrm{Pic}(\overline{X}_c))$.

À tout g-module continu discret M et tout entier naturel i on associe le groupe

$$\mathrm{Sha}^i_\omega(k, M) = \mathrm{Ker}[H^i(g, M) \to \prod_h H^i(h, M)],$$

où h parcourt les sous-groupes fermés procycliques de g.

Théorème A *Soient k un corps de caractéristique nulle, G un k-groupe linéaire connexe, X une k-variété espace homogène de G, de stabilisateur géométrique connexe. Soit X_c une k-compactification lisse de X.*

(i) *Le g-module $\mathrm{Pic}(\overline{X}_c)$ est un g-module flasque, c'est-à-dire que pour tout sous-groupe fermé $h \subset g$, on a $H^1(h, \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}(\overline{X}_c), \mathbf{Z})) = 0$, soit encore $\mathrm{Ext}^1_h(\mathrm{Pic}(\overline{X}_c), \mathbf{Z}) = 0$.*

(ii) *Pour tout sous-groupe fermé procyclique $h \subset g$, on a $H^1(h, \mathrm{Pic}(\overline{X}_c)) = 0$.*

(iii) *Soit T le k-tore associé au G-espace homogène X, et soit $\hat{T}$ son groupe des caractères. Si G est un groupe linéaire quasitrivial, i.e. extension d'un k-tore quasitrivial par un k-groupe simplement connexe, alors le quotient du groupe $\mathrm{Br}_1(X_c)$ par l'image du groupe $\mathrm{Br}(k)$ s'injecte dans le groupe $\mathrm{Sha}^1_\omega(k, \hat{T})$, et est isomorphe à ce dernier groupe si $X(k) \neq \emptyset$ ou si k est un corps de nombres.*

Sous l'hypothèse de (iii), nous montrons comment le g-module **Z**-libre de type fini $\mathrm{Pic}(\overline{X}_c)$ est déterminé, à addition près d'un g-module de permutation, par le k-tore T – en particulier il ne dépend pas du groupe quasi-trivial G.

Ce théorème est une extension naturelle de résultats connus dans le cas où $\overline{H} = 1$ (Voskresenskiĭ 1975, Colliot-Thélène et Sansuc 1976, Borovoi et Kunyavskiĭ 2004). Ces résultats furent rappelés dans le premier exposé.

Un ingrédient important de la démonstration du théorème A est le théorème suivant, pour la démonstration duquel un ingrédient essentiel nous a été suggéré par O. Gabber.

Théorème B *Soit A un anneau de valuation discrète de corps des fractions K, de corps résiduel k de caractéristique nulle. Soit G un K-groupe quasitrivial et soit E/K un G-espace homogène de stabilisateur géométrique connexe et de tore associé trivial. Soit X un A-schéma propre, régulier, intègre, dont la fibre générique contient E comme ouvert dense. Alors il existe une composante de multiplicité 1 de la fibre spéciale de X/A qui est géométriquement intègre sur son corps de base k.*

Totaro's question on zero-cycles on G_2, F_4, and E_6 torsors

by S. Garibaldi (Emory University, Atlanta, USA)

It is a natural naive question to ask: How can one tell if a collection of polynomial equations has a common solution over a given field k? A more sophisticated version of this question asks: If a variety X has a zero-cycle of degree 1, does X necessarily have a k-point, i.e., a closed point of degree 1? Various examples show that some restrictions on the variety X are necessary for a positive answer. Several people (Veisfeiler, Serre, Colliot-Thélène) have suggested hypotheses that may be sufficient to guarantee a positive answer.

In a 2004 paper, Totaro asked whether a G-torsor X that has a zero-cycle of degree $d > 0$ will necessarily have a closed étale point of degree dividing d, where G is a connected linear algebraic group. This question is closely related to several conjectures regarding exceptional algebraic groups. Totaro gave a positive answer to his question in the following cases: G simple, split, and of type G_2, type F_4, or simply connected of type E_6. Detlev W. Hoffmann and I proved that the answer is also "yes" for all groups of type G_2 and some nonsplit groups of type F_4 and E_6. We make no restrictions on the characteristic of the base field. The key tool is a lemma regarding linkage of Pfister forms.

Twisted forms of toroidal Lie algebras

by P. Gille (Université Paris-Sud, France)
jointly with A. Pianzola (University of Alberta, Canada)

The main thrust of our project is the study of Toroidal Lie algebras via cohomological methods This leads us to the theory of reductive group schemes as developed by M. Demazure and A. Grothendieck [8]. More precisely, Algebraic Principal Homogeneous Spaces (also called Torsors for short) and their accompanying non-abelian étale cohomology, arise naturally once this new point of view is taken into consideration.

Let A be a finite dimensional algebra over a field k. An *R-form* of A is an algebra L over R for which there exists a faithfully flat and finitely presented extension S/R such that

$$L \otimes_R S \simeq_S A \otimes_k S$$

(isomorphism of S-algebras).

Since $A \otimes_k S \simeq (A \otimes_k R) \otimes_R S$, the R-algebra L is nothing but an R-form (trivialized by $\mathrm{Spec} S$ in the flat topology of $\mathrm{Spec} R$) of the R-algebra $A \otimes_k R$. Since $\mathrm{Spec} R$ is affine, the isomorphism classes of such R-algebras are parametrized by $H^1(R, \mathrm{Aut}_k\, A_R)$ (pointed set of Cěch cohomology on the flat side of $\mathrm{Spec} R$ with coefficients on $\mathrm{Aut}_k\, A_R$). The group sheaf $\mathrm{Aut}_k\, A_R$ is in fact an affine R-group scheme (because A is finite dimensional). If $\mathrm{Aut}_k\, A$ is smooth (for example if char $k = 0$), then S may be assumed to be an étale cover.

Because of connections with Extended Affine Lie Algebras (EALAs for short), the case when R is a ring of Laurent polynomial in finitely many variables is of special importance (one variable corresponding the affine Kac-Moody case). For simplicity, we will restrict our attention to this special this case.

We assume henceforth that k is of characteristic 0. Fix $n \geq 0$ and let $R_n = k[t_1^{\pm 1}, \dots t_n^{\pm 1}]$. For any positive d, define $R_{n,d} = k[t_1^{\pm\frac{1}{d}}, t_2^{\pm\frac{1}{d}}, \cdots t_n^{\pm\frac{1}{d}}]$, and let $R_{n,\infty}$ be the inductive limit of all the $R_{n,d}$.

By definition, forms are trivialized in some $fppf$ extension of the base ring. In the case of Laurent polynomials, one has very precise control over the trivializing base change.

Theorem *Let A be a finite dimensional k-algebra. Every R_n-form L of A is isotrivial (i.e. trivialized by a finite étale cover of R_n). More precisely, there exist a finite Galois extension K/k and a positive integer d such that*

$$L \otimes_{R_n} (R_{n,d} \otimes_k K) \simeq_{R_{n,d} \otimes_k K} A \otimes_k (R_{n,d} \otimes_k K).$$

Similarly, every reductive group scheme over R_n is isotrivial.

Multiloop algebras are the quintessential examples of forms. Assume k to be algebraically closed, and fix once an for all a compatible family $(\zeta_n)_{n>0}$ in $k^\times$ of primitive roots of unity (thus $\xi_{hd}^h = \xi_d$).

We begin by introducing the ingredients needed in the definition of multiloop algebras. Let $\sigma = (\sigma_1, \dots, \sigma_\ell)$ be a commuting family of finite order automorphisms of the k-algebra A. Let m_i be the order of σ_i.

For each $(i_1, ..., i_n) \in \mathbb{Z}^n$, consider the simultaneous eigenspaces

$$A_{i_1 \dots i_n} := \{x \in A : \sigma_j(x) = \xi_{m_j}^{i_j} \text{ for all } 1 \leq j \leq n\}$$

(which of course depend only on the i_j modulo the m_j). Finally, consider the rings extension $R_n \subset R_{n,\mathbf{m}} = k[t_1^{\pm 1/m_1}, \dots, t_n^{\pm 1/m_n}]$ where $\mathbf{m} = (\mathbf{m_1}, ..., \mathbf{m_n})$. The *multiloop algebra* associated to this data is the k-algebra

$$L = L(A, \sigma) = \oplus A_{i_1 \dots i_n} \otimes t^{i_1/m_1} \dots t_n^{i_n/m_n} \subset A \otimes_k R_{n,\mathbf{m}}$$

Observe that L has a natural R_n-algebra structure. One easily verifies that $L \otimes_{R_n} R_{n,\mathbf{m}} \simeq_{R_{n,\mathbf{m}}} A \otimes_k R_{n,\mathbf{m}}$, and that $R_{n,\mathbf{m}}/R$ is free of finite rank (hence $fppf$. In fact étale and even Galois). Thus L is an R_n-form of A which is trivialized by the extension $R_{n,\mathbf{m}}/R_n$.

Let $\mathfrak{g}$ be a finite dimensional split simple Lie algebra over an algebraically closed field k of characteristic zero. In nullity 1 loop algebras provide us with concrete realization of the affine Kac-Moody algebras (a result of V. Kac). We can in fact prove a much stronger assertion: In nullity 1 *every* form of $\mathfrak{g}$ is a loop algebra. This follows from the following result of Pianzola.

Theorem *Let* $\mathbf{G}$ *be a reductive group scheme over* $R_1 = k[t_1^{\pm 1}]$. *Then* $H^1(k[t_1^{\pm 1}], \mathbf{G}) = 1$.

This result ought to be thought as a the validity of "Serre Conjecture I " for $k[t_1^{\pm 1}]$ (the usual Conjecture I, which is consequence of a Theorem of Steinberg, corresponding to the generic fiber of R_1; namely the function field $k(t_1)$).

With this in mind, we now turn our attention to the case $n = 2$ where some interesting and unexpected behaviour arises. Assume now that K is a field of dimension 2. Serre's Conjecture II asserts that $H^1(K, \mathbf{G}) = 1$ whenever $\mathbf{G}$ is a semisimple algebraic of simply connected type. At the present time, this conjecture is still open. There is however one case where the conjecture is known to hold, and this is precisely the case when $K = k(t_1, t_2)$.

By analogy with the one dimensional case, it seems inevitable to raise the following.

Question. *Let* $\mathbf{G}$ *be a semisimple group scheme over* $R = k[t_1^{\pm}, t_2^{\pm 1}]$. *Assume* $\mathbf{G}$ *is of simply connected type. Is* $H^1(R, \mathbf{G})$ *trivial? . More generally, if* $\mathbf{G}/R$ *is semisimple and* $\lambda : \widetilde{\mathbf{G}} \to \mathbf{G}$ *is its universal covering with (central) kernel* μ, *is the boundary map* $H^1(R, \mathbf{G}) \to H^2(R, \mu)$ *bijective ?*

We have shown that the the boundary map $H^1(R, \mathbf{G}) \to H^2(R, \mu)$ is always surjective. Furthermore, if $\mathbf{G}$ is split, then $H^1(R, \widetilde{\mathbf{G}}) = 1$ (and therefore the answer to the above question is positive). But somehow surprisingly however, the answer in general is negative (we have constructed an explicit counterexample, but the classification of these exotic objects seems hard). The failure seems to be directly related to anisotropic kernels.

Diagrams and torsors

by J.F. Jardine (University of Western Ontario, London, Ontario, Canada)

Maps between objects X and Y in a homotopy category can be identified with path components of a category of cocycles, in great generality. This correspondence can be used to give a simple demonstration of the identification of isomorphism classes of torsors (torsors are generalizations of principal bundles) for sheaves of groups G with maps in the homotopy category of simplicial sheaves. This identification is a homotopy theoretic classification G-torsors; this classification result has been known at this level of generality since the late 1980s, but the new proof is much simpler and more conceptual.

A G-torsor can be characterized as a sheaf X admitting a G-action for which the corresponding Borel construction $EG \times_G X$ is isomorphic to a point in the homotopy category of simplicial sheaves. More generally, for arbitrary index categories I, I-torsors are defined to be diagrams of weak equivalences which have trivial homotopy colimits. Using the machinery of Quillen's Theorem B (which is one of the main foundational results of algebraic K-theory), one can show that homotopy colimit and derived pullback together define a bijection

$$[*, BI] \cong \pi_0(I - \mathbf{Tors})$$

relating morphisms from a point to BI in the homotopy category with the set of path components of the category of I-torsors. This definition of I-torsor and the homotopy classification both exist quite generally, and specialize to definitions of higher torsors and motivic torsors with corresponding homotopy classification results.

Higher torsors can be thought of as special types of diagrams which take values in simplicial sheaves, and are defined on sheaves of categories I enriched in simplicial sets. Sheaves of groupoids enriched in simplicial sets are the objects of a homotopy theory which is equivalent to the full homotopy theory of simplicial sheaves, for which the fibrant objects represent higher stacks. The homotopy classification result for higher torsors does not depend on the theory of higher stacks, and the result for the full category of sheaves of categories enriched in simplicial sets was unexpected.

A bound for canonical dimension of the (semi-)spinor groups

by N. Karpenko (Universite d'Artois, Lens, France)

In the talk we discuss the *canonical dimension* $\mathrm{cd}(G)$ of a linear algebraic group G defined over a field F which was introduced recently by Berhuy–Reichstein. The general question raised by Berhuy–Reichstein is to determine $\mathrm{cd}(G)$ for every split simple algebraic group G.

For the spinor group, representing a particularly difficult case of the above general question, one knows that $\mathrm{cd}\,(\mathrm{Spin}_{2n+1}) = \mathrm{cd}\,(\mathrm{Spin}_{2n+2})$, so that we will discuss only $\mathrm{cd}(\mathrm{Spin}_{2n+1})$ here.

Although the canonical dimension of, say, a smooth projective variety X can be expressed in terms of algebraic cycles on X, there are no general recipes for computing $\mathrm{cd}(X)$ or $\mathrm{cd}(G)$. A better situation occurs with the *canonical p-dimension* cd_p, a "p-local version" of cd, where p is a prime: a recipe for computing $\mathrm{cd}_p(G)$ of an arbitrary split simple G is obtained by Merkurjev and Karpenko. In particular, one has

$$\mathrm{cd}_2(\mathrm{Spin}_{2n}) = \frac{n(n-1)}{2} - 2^l + 1\ ,$$

where l is the minimal integer such that $2^l \geq n+1$ (and for any odd prime p, one has $\mathrm{cd}_p(\mathrm{Spin}_{2n+1}) = 0$). Since $\mathrm{cd}\,(G) \geq \mathrm{cd}_p(G)$ for any G and p, we have a *lower bound* for the canonical dimension of the spinor group, given by its canonical 2-dimension.

We establish the following *upper bound* for spinor groups:

$$\mathrm{cd}\,(\mathrm{Spin}_{2n+1}) \leq n(n-1)/2.$$

This result improves the previously known upper bound $n(n+1)/2$, established by Berhuy–Reichstein. The proof makes use of the theory of non-negative intersections, of duality between Schubert varieties, and of the Pieri formula for a variety of maximal totally isotropic subspaces.

Note that the lower bound for $\mathrm{cd}(\mathrm{Spin}_{2n+1})$, given by $\mathrm{cd}_2(\mathrm{Spin}_{2n+1})$, coincides with our upper bound if (and only if) $n+1$ is a power of 2. Therefore, for such n, we get the precise value: if $n+1$ is a power of 2, then

$$\mathrm{cd}(\mathrm{Spin}_{2n+1}) = \mathrm{cd}(\mathrm{Spin}_{2n+2}) = \frac{n(n-1)}{2}\ .$$

Our second main result is the following upper bound for the semi-spinor groups $\mathrm{Spin}^{\sim}_{2n+2}$, obtained by the similar technique: for any odd one has

$$\mathrm{cd}(\mathrm{Spin}^{\sim}_{2n+2}) \leq n(n-1)/2 + 2^k - 1,$$

where $k = v_2(n+1)$ (the 2-adic order of $n+1$).

Importance of the spinor and semi-spinor groups in this context is explained by the fact that these groups represent the only difficult cases of the following general question: let G be a split simple algebraic group, having a unique torsion prime p (a prime p is a *torsion prime* of G if and only if $\mathrm{cd}_p(G) \neq 0$); is it true that $\mathrm{cd}(G) = \mathrm{cd}_p(G)$?

Zero cycles on homogeneous varieties

by D. Krashen (IAS, Princeton)

The study of projective homogeneous varieties and their invariants has been a source of many interesting problems and has various applications in recent years. For example, Panin's description of the algebraic K-theory of homogeneous varieties has resulted in the useful index reduction formulas of Merkurjev, Panin and Wadsworth. The study of algebraic cycles and the motives of these varieties has also played an important role in quadratic form theory, and in particular, Voevodsky's proof of the Milnor conjecture. The structure of the Chow groups and motives of these varieties continues to be an active area of research with many unresolved questions.

In the talk, we introduce tools for studying the Chow group of 0-dimensional cycles on a projective variety using results from Suslin and Voevodsky's work on algebraic singular homology. This allows us to connect the study of the group of zero cycles to studying the more geometrically naive notion of R-equivalence (i.e. connecting points with rational curves) on symmetric powers of the original variety.

We apply these ideas by showing that the symmetric powers of certain homogeneous varieties may be related to spaces which parametrize commutative étale subalgebras in a central simple algebra. To make this precise, we define moduli spaces of étale subalgebras in a central simple (or Azumaya) algebra. These spaces are very interesting in their own right, as many open questions in the area of central simple algebras concern the existence and structure of certain types of subfields in a division algebra. We show that in certain cases these moduli spaces are R-trivial, and we apply this to determining the Chow group of zero cycles for certain homogeneous varieties. This allows us to show that the Chow group of zero dimensional degree zero cycles is trivial for involution varieties as well as for certain Severi-Brauer flag varieties. This was previously known to be true for involution varieties of index no more than 2 (by work of Swan, Karpenko and Merkurjev) and for Severi-Brauer varieties (by work of Panin).

On Cachazo-Douglas-Seiberg-Witten Conjecture for simple Lie algebras

by S. Kumar (University of North Carolina at Chapell Hill, USA)

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over the complex numbers. Consider the exterior algebra $R := \wedge(\mathfrak{g} \oplus \mathfrak{g})$ on two copies of $\mathfrak{g}$. Then, the algebra R is bigraded with the two copies of $\mathfrak{g}$ sitting in bidegrees (1,0) and (0,1) respectively. To distinguish, we will denote the first copy of $\mathfrak{g}$ by $\mathfrak{g}_1$ and the second copy of $\mathfrak{g}$ by $\mathfrak{g}_2$.

The diagonal adjoint action of $\mathfrak{g}$ gives rise to a $\mathfrak{g}$-algebra structure on R compatible with the bigrading. We isolate three 'standard' copies of the adjoint representation $\mathfrak{g}$ in R^2, where R^2 is the total degree 2 component of R. The $\mathfrak{g}$-module map

$$\partial : \mathfrak{g} \to \wedge^2(\mathfrak{g}), \qquad x \mapsto \partial x = \sum_i [x, e_i] \wedge f_i,$$

considered as a map to $\wedge^2(\mathfrak{g}_1)$ will be denoted by c_1, and similarly,

$$c_2 : \mathfrak{g} \to \wedge^2(\mathfrak{g}_2), \text{ and}$$
$$c_3 : \mathfrak{g} \to \mathfrak{g}_1 \otimes \mathfrak{g}_2, \quad x \mapsto \sum_i [x, e_i] \otimes f_i,$$

where $\{e_i\}_{i \le i \le N}$ is any basis of $\mathfrak{g}$ and $\{f_i\}_{1 \le i \le N}$ is the dual basis of $\mathfrak{g}$ with respect to a normalized Killing form $\langle \, , \, \rangle$ of $\mathfrak{g}$. We denote by C_i the image of c_i.

Let J be the (bigraded) ideal of R generated by the three copies C_1, C_2, C_3 of $\mathfrak{g}$ (in R^2) and define the bigraded $\mathfrak{g}$-algebra

$$A := R/J.$$

The Killing form gives rise to a $\mathfrak{g}$-invariant $S \in A^{1,1}$ given by

$$S := \sum_i e_i \otimes f_i.$$

Motivated by supersymmetric gauge theory, Cachazo-Douglas-Seiberg-Witten made the following conjecture. They proved the conjecture for classical $\mathfrak{g}$. Subsequently, Etingof-Kac proved the conjecture for $\mathfrak{g}$ of type G_2 by using the theory of abelian ideals in $\mathfrak{b}$.

Conjecture [CDSW] *(i) The subalgebra $A^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants in A is generated, as an algebra, by the element S.*

(ii) $S^h = 0$.

(iii) $S^{h-1} \neq 0$, where h is the dual Coxeter number.

The aim of this work is to give a uniform proof of the above conjecture part (i). In addition, we give a conjecture, the validity of which would imply part (ii) of the above conjecture.

To prove part (i), we first prove that the graded algebra $B^{\mathfrak{g}}$ is isomorphic with the singular cohomology of a certain (finite dimensional) projective subvariety $\mathcal{Y}_2$ of the infinite Grassmannian $\mathcal{Y}$ associated to $\mathfrak{g}$, where $B := R/\langle C_1 \oplus C_2 \rangle$. The definition of the subvariety $\mathcal{Y}_2$ is motivated from the theory of abelian ideals in the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. This isomorphism is obtained by using Garland's result on the Lie algebra cohomology of $\hat{\mathfrak{u}} := \mathfrak{g} \otimes t\mathbb{C}[t]$; Kostant's result on the 'diagonal' cohomolgy of $\hat{\mathfrak{u}}$ and its connection with abelian ideals in $\mathfrak{b}$; and a certain deformation of the singular cohomology of $\mathcal{Y}$ introduced by Belkale-Kumar.

Steenrod operations in algebraic geometry

by A. Merkurjev (UCLA, USA)

Steenrod operations in algebraic geometry were originally defined by Voevodsky in the context motivic cohomology. P.Brosnan found an elementary definition of the Steenrod operations on the Chow groups of algebraic varieties. His definition uses equivariant Chow groups of Edidin and Graham and the construction relies on embedding to a smooth scheme.

In the talk a new direct construction of the Steenrod operations modulo 2 is presented. Namely, the Steenrod operations (of homological type) of a scheme X are defined as the Segre classes of the tangent cone of X. All the standard properties of the Steenrod operations can be proven directly.

Non-commutative version of purity

by I. Panin (Steklov Institute, St. Petersburg, Russia)

Let $\mathcal{F}$ be a covariant functor from the category of commutative rings to the category of sets. We say that $\mathcal{F}$ *satisfies purity for* R if

$$\bigcap_{\mathrm{ht}\mathfrak{p}=1} \mathrm{Im}\,[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(K)] = \mathrm{Im}\,[\mathcal{F}(R) \to \mathcal{F}(K)].$$

For certain functors $\mathcal{F}(R)$ injects into $\mathcal{F}(K)$ for all regular local rings R. In this case the purity of $\mathcal{F}$ for R implies that

$$\bigcap_{\mathrm{ht}\mathfrak{p}=1} \mathcal{F}(R_{\mathfrak{p}}) = \mathcal{F}(R) \subset \mathcal{F}(K).$$

Now we switch to a specific functor. For that consider a characteristic zero field k, a reductive algebraic k-group G (connected one) and a functor $\mathcal{F}$ which takes a commutative k-algebra R to $H^1_{\text{ét}}(R, \mathrm{G})$. We make the following conjecture:

the functor $\mathcal{F}$ satisfies the purity for regular local rings containing k.

The conjecture is a kind of extension of the known conjecture of A. Grothendieck and J.-P. Serre. They conjectured the injectivity. Here a purity is conjectured. It can be viewed as a non-commutative version of Gersten's conjecture in K-theory. In the talk we discussed in certain details this conjectures for interesting examples of reductive groups like PGL_n, $\mathrm{SL}_{1,A}$, $\mathrm{O}(q)$, $\mathrm{SO}(q)$.

Algebras of prime degree over function fields of surfaces

by R. Parimala (Tata Institute, Mumbai, India)
jointly with M. Ojanguren (Lausanne University, Switzerland)

It is an open question whether division algebras of prime degree are cyclic. Over number fields, cyclicity of all central simple algebras is a classical theorem due to Hasse-Brauer-Noether. Further the index and the exponent coincide for all division algebras over a number field. Artin raised the question whether the index and the exponent coincide for central simple algebras over a C_2-field. Artin's question is answered in the affirmative for function fields of surfaces over an algebraically closed field of characteristic zero by de Jong. We explain a method of proof of cyclicity of prime degree algebras over such fields using de Jong's techniques.

Tori in quasi-split groups

by M. S. Raghunathan (Tata Institute, Mumbai, India)

In this talk a proof of the following result was outlined:

Let k be any field and G a quasi-split k-algebraic group, S a maximal k-split torus in G, $Z(S) = T$ the centraliser of S and $N(S)$ the normaliser of S. Let $W = N(S)/Z(S)$ be the Weyl group-scheme over k. Let $i : W \hookrightarrow \mathrm{Aut}(T)$ be the natural inclusion. Now k-isomorphism classes of tori of dimension l (= dimension T) are in bijective correspondence with elements of the Galois cohomology set $H^1(k, Aut(T))$. A necessary and sufficient condition that a torus B over k is realisable as a k- subtorus of G is that class $[B]$ of B in $H^1(k, Aut(T))$ be in the image of $H^1(k, W)$. This is a consequence of the following stronger assertion: let $\pi : H^1(k, N(T)) \longrightarrow H^1(k, W)$ and $i : H^1(k, N(T)) \longrightarrow H^1(k, G)$ be the natural maps. Then π maps kernel $i(= i^{-1}$(trivial class in $H^1(k, G))$ onto $H^1(k, W)$.

A key ingredient of the proof is the theorem of Steinberg that every regular semisimple k-conjugacy class in G contains a k-rational point.

Group-theoretic compactification of Bruhat-Tits buildings

by B. Rémy (Lyon 1, France)
jointly with Yves Guivarc'h (Rennes 1, France)

Let G be a simply connected semisimple algebraic group, defined over a non-archimedean local field F. We denote by G_F the locally compact group of its rational points, and we denote by X the Bruhat-Tits building of $G_{/F}$. We are interested in compactifying the vertices V_X of X by group-theoretic means, so that we eventually obtain structure results on the rational points G_F (i.e. parametrizations of remarkable classes of closed subgroups of G_F). We first prove convergence of some sequences of compact open subgroups of G_F in the Chabauty topology. This enables us to define the desired compactification of V_X. We obtain then a structure theorem showing that the Bruhat-Tits buildings of the Levi factors all lie in the boundary of the compactification. We obtain an identification theorem with the polyhedral compactification, previously defined by E. Landvogt. We finally prove two parametrization theorems extending the correspondence between maximal compact subgroups and vertices of X: one is about Zariski connected amenable subgroups, and the other is about subgroups with distal adjoint action.

Cyclic algebras over p-adic curves

by D. Saltman (Texas University, USA)

The study of the structure of division algebras goes back 150 years since they were first defined. The issue has always been how to describe their structure. The first examples of division algebras were so called cyclic algebras - defined simply using a cyclic Galois extension. Since then non-cyclic algebras have been found, but only with complicated (precisely composite) degree, where the degree of a division algebra is an integer describing its size. Thus in some ways the first question about division algebras is still unsolved, namely, whether all division algebras of prime degree are cyclic.

Another strain in the theory of division algebras is their study over special fields, where over time the "special" fields have gotten more and more general. This approach is best illustrated by the Hasse-Brauer-Noether-Albert theorem that all division algebras over global fields are cyclic. In this talk we discussed a "higher dimensional" special field, namely, the function field of a curve over a p-adic field. What we showed was that when q is a prime not equal to p, then any division algebra of degree q over such a field is cyclic.

Of equal importance to the actual result is the methods we used. The fields we are concerned with are best viewed as the function field of surfaces S over the p-adic integers. By a result of Grothendieck, such surfaces have Brauer group 0. What this means is that the division algebras over such surfaces are determined by their so called "ramification". As a consequence of this, showing that a division algebra is cyclic is equivalent to showing that one can "split" its ramification by a cyclic Galois extension of the right size. It turn out that another way to view this result is that it is a result on splitting ramification over surfaces, and as such it has had application to a much broader class of fields than treated here.

The arguments of Grothendieck and Tits on splitting fields

by B. Totaro (Cambridge University, UK)

One of the great achievements of mathematics is the 19th-century classification of the simple Lie groups by Killing and Cartan. There are four infinite families of groups and just five exceptional groups. Chevalley showed in 1958 that the same classification works for the simple algebraic groups over any algebraically closed field.

The classification of simple algebraic groups over an arbitrary field is much richer. It includes as a special case some of the fundamental problems of algebra, such as the classification of quadratic forms over an arbitrary field. Nonetheless, one can hope to answer basic questions such as: given a simple algebraic group of a given type over a field, what degree of field extension is needed to make it into the standard (Chevalley) group?

Using the idea of torsors, and the definition of the Chow ring of a classifying space, we give an improved proof of a theorem by Grothendieck which gives a strong connection between the classification of simple algebraic groups over arbitrary fields and the topology of the corresponding compact Lie groups.

As a result, we can do topological calculations and read off information about splitting fields. Tits solved these problems for many types of groups, but we are able to solve these problems in the remaining cases, notably for the groups E_8 and $\mathrm{Spin}\,(n)$. We find, for example, that any algebraic group of type E_8 over any field becomes isomorphic to the Chevalley group E_8 after a field extension of degree dividing 2880. The number 2880 is best possible. This is satisfying in that questions about E_8, the largest exceptional Lie group, have often been the hardest of all questions about Lie groups.

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Chapter 7

Densest Packings of Spheres (05w5022)

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Introduction

A family of (not necessarily infinitely many) non-overlapping congruent balls in d-dimensional space of constant curvature is called a packing of congruent balls in the given d-space that is either in the Euclidean d-space $\mathbb{E}^d$ or in the spherical d-space $\mathbb{S}^d$ or in the hyperbolic d-space $\mathbb{H}^d$. The goal of this report is to give a state of the art description of d-dimensional sphere packings. On the one hand, the research on sphere packings seems to be one of the most active areas of (discrete) geometry on the other hand, it is one of the oldest areas of mathematics ever studied. The topics discussed in separate sections of this report are the following ones:

- Hadwiger numbers of convex bodies and kissing numbers of spheres;
- Touching numbers of convex bodies;
- Newton numbers of convex bodies;
- One-sided Hadwiger and kissing numbers;
- Contact graphs of finite packings and the combinatorial Kepler problem;
- Isoperimetric problems for Voronoi cells, the strong dodecahedral conjecture and the truncated octahedral conjecture;
- The strong Kepler conjecture;
- Bounds on the density of sphere packings in higher dimensions;
- Solidity and uniform stability.

Each section outlines the state of the art of relevant research along with some of the ”most wanted” research problems. Generally speaking the material covered belongs to combinatorics, convexity and discrete geometry however, often the methods indicated cover a much broader spectrum of mathematics including computational geometry, hyperbolic geometry, the geometry of Banach spaces, coding theory, convex analysis, geometric measure theory, (geometric) rigidity, topology, linear programming and non-linear optimization.

Hadwiger numbers of convex bodies and kissing numbers of spheres

Let $\mathbf{K}$ be a convex body (i.e. a compact convex set with nonempty interior) in d-dimensional Euclidean space $\mathbb{E}^d, d \geq 2$. Then the Hadwiger number $H(\mathbf{K})$ of $\mathbf{K}$ is the largest number of non-overlapping translates of $\mathbf{K}$ that can all touch $\mathbf{K}$. An elegant observation of Hadwiger [H] is the following.

Theorem 7.0.1 *For every d-dimensional convex body $\mathbf{K}$,*

$$H(\mathbf{K}) \leq 3^d - 1,$$

where equality holds if and only if $\mathbf{K}$ is an affine d-cube.

On the other hand, in another elegant paper Swinnerton-Dyer [S] proved the following lower bound for Hadwiger numbers of convex bodies.

Theorem 7.0.2 *For every d-dimensional ($d \geq 2$)convex body $\mathbf{K}$,*

$$d^2 + d \leq H(\mathbf{K}).$$

Actually, finding a better lower bound for Hadwiger numbers of d-dimensional convex bodies is a highly challenging open problem for all $d \geq 4$. (It is not hard to see that the above theorem of Swinnerton-Dyer is sharp for dimensions 2 and 3.) The best lower bound known in dimensions $d \geq 4$ is due to Talata [85], who applying Dvoretzky's theorem on spherical sections of centrally symmetric convex bodies succeeded to show the following inequality.

Theorem 7.0.3 *There exists an absolute constant $c > 0$ such that*

$$2^{cd} \leq H(\mathbf{K})$$

holds for every positive integer d and for every d-dimensional convex body $\mathbf{K}$.

Now, if we look at convex bodies different from a Euclidean ball in dimensions larger than 2, then our understanding of their Hadwiger numbers is very limited. Namely, we know the Hadwiger numbers of the following convex bodies different from a ball. The result for tetrahedra is due to Talata [86] and the rest was proved by Larman and Zong [60].

Theorem 7.0.4 *The Hadwiger numbers of tetrahedra, octahedra and rhombic dodecahedra are all equal to 18.*

In order to gain some more insight on Hadwiger numbers it is natural to pose the following question.

Problem 7.0.5 *For what integers k with $12 \leq k \leq 26$ does there exist a 3-dimensional convex body with Hadwiger number k? What is the Hadwiger number of a $d-$dimensional simplex (resp., crosspolytope) for $d \geq 4$?*

The second main problem in this section is fondly known as the kissing number problem. The kissing number τ_d is the maximum number of nonoverlapping d-dimensional balls of equal size that can touch a congruent one in $\mathbb{E}^d$. In three dimension this question was the subject of a famous discussion between Isaac Newton and David Gregory in 1964. So, it is not surprising that the literature on the kissing number problem is "huge". Perhaps the best source of information on this problem is the book [35] of Conway and Sloane. In what follows we give a short description of the present status of this problem.

$\tau_2 = 6$ is trivial. However, determining the value of τ_3 is not a trivial issue. Actually the first complete and correct proof of $\tau_3 = 12$ was given by Schütte and van der Waerden [19] in 1953. The subsequent (two pages) often cited proof of Leech [35], which is impressively short, contrary to the common belief does contain some gaps. It can be completed though, see for example, [66]. Further more recent proofs can be found in [29], [1] and in [72]. None of these are short proofs either and one may wonder whether there exists a proof of $\tau_3 = 12$ in THE BOOK at all. (For more information on this see the very visual paper [32].) Thus, we have the following theorem.

Theorem 7.0.6 $\tau_2 = 6$ *and* $\tau_3 = 12$.

The race for finding out the kissing numbers of Euclidean balls of dimension larger than 3 was always and is even today one of the most visible research projects of mathematics. Following the chronological ordering, here are the major inputs. Coxeter [1] conjectured and Böröczky [27] proved the following theorem, where $F_d(\alpha) = \frac{2^d U}{d!\omega_d}$ is the Schläfli function with U standing for the spherical volume of a regular spherical $(d-1)$-dimensional simplex of dihedral angle 2α and with ω_d denoting the surface volume of the d-dimensional unit ball.

Theorem 7.0.7 $\tau_d \leq \frac{2F_{d-1}(\beta)}{F_d(\beta)}$, *where* $\beta = \frac{1}{2}$*arcsec* d.

It was another breakthrough when Delsarte's linear programming method (for details see for example [77]) was applied to the kissing number problem and so, when Kabatiansky and Levenshtein [59] succeeded to improve the upper bound of the previous theorem for large d as follows. The lower bound mentioned below was found by Wyner [87] several years earlier.

Theorem 7.0.8 $2^{0.2075d(1+o(1))} \leq \tau_d \leq 2^{0.401d(1+o(1))}$.

As the gap between the lower and upper bounds is exponential it was a great surprise when Levenshtein [35] and Odlyzko and Sloane [75] independently found the following exact values for τ_d.

Theorem 7.0.9 $\tau_8 = 240$ *and* $\tau_{24} = 196560$.

In addition, Bannai and Sloane [3] were able to prove the following.

Theorem 7.0.10 *There is a unique way (up to isometry) of arranging* 240 *(resp.,* 196560*) nonoverlapping unit spheres in* 8*-dimensional (resp.,* 24*-dimensional) Euclidean space such that they touch another unit sphere.*

The latest surprise came when Musin [70], [71] extending Delsarte's method found the kissing number of 4-dimensional Euclidean balls. Thus, we have

Theorem 7.0.11 $\tau_4 = 24$.

In connection with Musin's result we believe in the following conjecture.

Conjecture 7.0.12 *There is a unique way (up to isometry) of arranging* 24 *nonoverlapping unit spheres in* 4*-dimensional Euclidean space such that they touch another unit sphere.*

Using the spherical analogue of the technique developed in [11] K. Bezdek [22] gave a proof of the following theorem that one can regard as the local version of the above conjecture.

Theorem 7.0.13 *Take a unit ball* $\mathbf{B}$ *of* $\mathbb{E}^4$ *touched by 24 other (nonoverlapping) unit balls* $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_{24}$ *with centers* $C_1, C_2, \dots, C_{24}$ *such that the centers* $C_1, C_2, \dots, C_{24}$ *form the vertices of a regular 24-cell* $\{3,4,3\}$ *in* $\mathbb{E}^4$. *Then there exists an* $\epsilon > 0$ *with the following property: if the nonoverlapping unit balls* $\mathbf{B}'_1, \mathbf{B}'_2, \dots, \mathbf{B}'_{24}$ *with centers* $C'_1, C'_2, \dots, C'_{24}$ *are chosen such that* $\mathbf{B}'_1, \mathbf{B}'_2, \dots, \mathbf{B}'_{24}$ *are all tangent to* $\mathbf{B}$ *in* $\mathbb{E}^4$ *and for each* $i, 1 \leq i \leq 24$ *the Euclidean distance between* C_i *and* C'_i *is at most* ϵ*, then* $C'_1, C'_2, \dots, C'_{24}$ *form the vertices of a regular 24-cell* $\{3,4,3\}$ *in* $\mathbb{E}^4$.

There is a great list of record kissing numbers in dimensions from 32 to 128 in [74] and also, we refer the interested reader to the paper [39] of Edel, Rains and Sloane for some amazingly elementary but efficient constructions.

Touching numbers of convex bodies

The touching number $t(\mathbf{K})$ of a convex body $\mathbf{K}$ in d-dimensional Euclidean space $\mathbb{E}^d$ is the largest possible number of mutually touching translates of $\mathbf{K}$ lying in $\mathbb{E}^d$. The elegant paper [37] of Danzer and Grünbaum gives a proof of the following fundamental inequality. In fact, this inequality was phrased by Petty [76] as well as by P. S. Soltan [83] in another equivalent form saying that the cardinality of an equilateral set in any d-dimensional normed space is at most 2^d.

Theorem 7.0.14 *For an arbitrary convex body* $\mathbf{K}$ *of* $\mathbb{E}^d$

$$t(\mathbf{K}) \leq 2^d$$

with equality if and only if $\mathbf{K}$ *is an affine* d*-cube.*

In connection with the above inequality K. Bezdek and Pach [15] conjecture the following even stronger result.

Conjecture 7.0.15 *For any convex body $\mathbf{K}$ in $\mathbb{E}^d, d \geq 3$ the maximum number of pairwise tangent positively homothetic copies of $\mathbf{K}$ is not more than 2^d.*

This problem is still quite open. It seems that the only published upper bound is $3^d - 1$ in [15].

It is natural to ask for a non-trivial lower bound for $t(\mathbf{K})$. Brass [30] as an application of Dvoretzky's well-known theorem gave a partial answer for the existence of such a lower bound.

Theorem 7.0.16 *For each k there exists a $d(k)$ such that for any convex body $\mathbf{K}$ of $\mathbb{E}^d$ with $d \geq d(k)$*

$$k \leq t(\mathbf{K}).$$

It is remarkable that the natural sounding conjecture of Petty [76] stated next is still open for all $d \geq 4$.

Conjecture 7.0.17 *For each convex body $\mathbf{K}$ of $\mathbb{E}^d, d \geq 4$*

$$d + 1 \leq t(\mathbf{K}).$$

A generalization of the concept of touching numbers was introduced by K. Bezdek, M. Naszódi and B. Visy [19] as follows. The mth touching number (or the mth Petty number) $t(m, \mathbf{K})$ of a convex body $\mathbf{K}$ of $\mathbb{E}^d$ is the largest cardinality of (possibly overlapping) translates of $\mathbf{K}$ in $\mathbb{E}^d$ such that among any m translates always there are two touching ones. Note that $t(2, \mathbf{K}) = t(\mathbf{K})$. The following theorem proved by K. Bezdek, M. Naszódi and B. Visy [19] states some upper bounds for $t(m, \mathbf{K})$.

Theorem 7.0.18 *Let $t(\mathbf{K})$ be an arbitrary convex body in $\mathbb{E}^d$. Then*

$$t(m, \mathbf{K}) \leq \min\left\{(m-1)4^d, \binom{2^d + m - 1}{2^d}\right\}$$

holds for all $m \geq 2, d \geq 2$. Also, we have the inequalities

$$t(3, \mathbf{K}) \leq 2 \cdot 3^d,\ t(m, \mathbf{K}) \leq (m-1)[(m-1)3^d - (m-2)]$$

for all $m \geq 4, d \geq 2$. Moreover, if $\mathbf{B^d}$ (resp., $\mathbf{C^d}$) denotes a d-dimensional ball (resp., d-dimensional affine cube) of $\mathbb{E}^d$, then

$$t(2, \mathbf{B^d}) = d + 1,\ t(m, \mathbf{B^d}) \leq (m-1)3^d,\ t(m, \mathbf{C^d}) = (m-1)2^d$$

hold for all $m \geq 2, d \geq 2$.

We cannot resist on raising the following question (for more details see [19]).

Problem 7.0.19 *Prove or disprove that if $\mathbf{K}$ is an arbitrary convex body in $\mathbb{E}^d$ with $d \geq 2$ and $m > 2$, then*

$$(m-1)(d+1) \leq t(m, \mathbf{K}) \leq (m-1)2^d.$$

Newton numbers of convex bodies

According to L. Fejes Tóth [44] the Newton number $N(\mathbf{K})$ of a convex body $\mathbf{K}$ in $\mathbb{E}^d$ is defined as the largest number of congruent copies of $\mathbf{K}$ that can touch $\mathbf{K}$ without having interior points in common. (Note that unlike in the case of Hadwiger numbers here it is not necessary at all to use translated copies of the given convex body. In fact, often it is better to use rotated or reflected ones.) For the special case when $\mathbf{K}$ is a ball we refer the reader to Section 2 of this paper. Here we focus on the case when $\mathbf{K}$ is different from a ball. Somewhat surprisingly, in this case only planar results are known. Namely, Linhart [65] and Böröczky [26] determined the Newton numbers of regular convex polygons.

Theorem 7.0.20 *If $N(n)$ denotes the Newton number of a regular convex n-gon in $\mathbb{E}^2$, then*

$$N(3) = 12, N(4) = 8 \text{ and } N(n) = 6 \text{ for all } n \geq 5.$$

L. Fejes Tóth [42] proved the following - in some cases quite sharp - upper bound for the Newton numbers of convex domains (i.e. compact convex sets with nonempty interior) in $\mathbb{E}^2$.

Theorem 7.0.21 *A convex domain with diameter D and minimum width W cannot be touched by more than*

$$\left[(4 + 2\pi)\frac{D}{W} + 2 + \frac{W}{D}\right]$$

non-overlapping congruent copies of it.

This result was improved by Schopp [81] as follows.

Theorem 7.0.22 *The Newton number of any convex domain of constant width in $\mathbb{E}^2$ is at most 7 and the Newton number of a Reuleaux triangle is exactly 7.*

We close this section with a rather natural question, which to the best of our knowledge has not been yet studied.

Problem 7.0.23 *Prove or disprove that the Newton number of a d-dimensional ($d \geq 3$) Euclidean cube is $3^d - 1$.*

One-sided Hadwiger and kissing numbers

K. Bezdek and P. Brass [20] assigned to each convex body $\mathbf{K}$ in $\mathbb{E}^d$ a specific positive integer called the one-sided Hadwiger number $h(\mathbf{K})$ as follows: $h(\mathbf{K})$ is the largest number of non-overlapping translates of $\mathbf{K}$ that touch $\mathbf{K}$ and that all lie in a closed supporting half-space of $\mathbf{K}$. In [20], using the Brunn-Minkowski inequality, K. Bezdek and P. Brass proved the following sharp upper bound for the one-sided Hadwiger numbers of convex bodies.

Theorem 7.0.24 *If $\mathbf{K}$ is an arbitrary convex body in $\mathbb{E}^d$, then*

$$h(\mathbf{K}) \leq 2 \cdot 3^{d-1} - 1.$$

Moreover, equality is attained if and only if $\mathbf{K}$ is a d-dimensional affine cube.

The notion of one-sided Hadwiger numbers was introduced to study the (discrete) geometry of the so-called k^+-neighbour packings, which are packings of translates of a given convex body in $\mathbb{E}^d$ with the property that each packing element is touched by at least k others from the packing, where k is a given positive integer. As this area of discrete geometry has a rather large literature we refer the interested reader to [20] for a brief survey on the relevant results. Here, we emphasize the following corollary of the previous theorem proved also in [20].

Theorem 7.0.25 *If $\mathbf{K}$ is an arbitrary convex body in $\mathbb{E}^d$, then any k^+-neighbour packing by translates of $\mathbf{K}$ with $k \geq 2 \cdot 3^{d-1}$ must have a positive density in $\mathbb{E}^d$. Moreover, there is a $(2 \cdot 3^{d-1} - 1)^+$-neighbour packing by translates of a d-dimensional affine cube with density 0 in $\mathbb{E}^d$.*

It is obvious that the one-sided Hadwiger number of any circular disk in $\mathbb{E}^2$ is 4. However, the 3-dimensional analogue statement is harder to get. As it turns out the one-sided Hadwiger number of the 3-dimensional Euclidean ball is 9. One of the shortest proofs of this fact was found by A. Bezdek and K. Bezdek [10]. Since here we are studying Euclidean balls their one-sided Hadwiger numbers we simply call one-sided kissing numbers.

Theorem 7.0.26 *The one-sided kissing number of the 3-dimensional Euclidean ball is 9.*

As we have mentioned before Musin [71] has just announced a proof of the long-standing conjecture that the kissing number of the 4-dimensional Euclidean ball is 24. Based on this result K. Bezdek [22] gave a proof of the following.

Theorem 7.0.27 *The one-sided kissing number of the 4-dimensional Euclidean ball is either 18 or 19.*

The proof of the above theorem supports the following conjecture.

Conjecture 7.0.28 *The one-sided kissing number of the 4-dimensional Euclidean ball is 18.*

Contact graphs of finite packings and the combinatorial Kepler problem

Let $\mathbf{K}$ be an arbitrary convex body in $\mathbb{E}^d$. Then the contact graph of an arbitrary finite packing by non-overlapping translates of $\mathbf{K}$ in $\mathbb{E}^d$ is the (simple) graph whose vertices correspond to the packing elements and whose two vertices are connected by an edge if and only if the corresponding two packing elements touch each other. One of the most basic questions on contact graphs is to find out the maximum number of edges that a contact graph of n translates of the given convex body $\mathbf{K}$ can have in $\mathbb{E}^d$. Harborth [55] proved the following remarkable result on the contact graphs of congruent circular disk packings in $\mathbb{E}^2$.

Theorem 7.0.29 *The maximum number of touching pairs in a packing of n congruent circular disks in $\mathbb{E}^2$ is precisely*

$$\lfloor 3n - \sqrt{12n-3} \rfloor.$$

In a very recent paper [31] Brass extended the above result to the ”unit circular disk packings” of normed planes as follows.

Theorem 7.0.30 *The maximum number of touching pairs in a packing of n translates of a convex domain $\mathbf{K}$ in $\mathbb{E}^2$ is $\lfloor 3n - \sqrt{12n-3} \rfloor$, if $\mathbf{K}$ is not a parallelogram, and $\lfloor 4n - \sqrt{28n-12} \rfloor$, if $\mathbf{K}$ is a parallelogram.*

The analogue question in the hyperbolic plane has been studied by Bowen in [23]. We prefer to quote his result in the following geometric way.

Theorem 7.0.31 *Consider circle packings in the hyperbolic plane, by finitely many congruent circles, which maximize the number of touching pairs for the given number of congruent circles. Then such a packing must have all of its centers located on the vertices of a triangulation of the hyperbolic plane by congruent equilateral triangles, provided the diameter D of the circles is such that an equilateral triangle in the hyperbolic plane of side length D has each of its angles equal to $\frac{2\pi}{N}$ for some $N > 6$.*

It is not hard to see that one can extend the above result to $\mathbb{S}^2$ exactly in the way as the above phrasing suggests. However, we get a more general approach if we do the following: Take n non-overlapping unit diameter balls in a convex position in $\mathbb{E}^3$, that is assume that there exists a 3-dimensional convex polyhedron whose vertices are center points moreover, each center point belongs to the boundary of that convex polyhedron, where $n \geq 4$ is a given integer. Obviously, the shortest distance among the center points is at least one. Then count the unit distances showing up between pairs of center points but, count only those pairs that generate a unit line segment on the boundary of the given 3-dimensional convex polyhedron. Finally, maximize this number for the given n and label this maximum by $c(n)$. The following theorem was found by D. Bezdek [12] who also pointed out its interesting relation to protein folding as well as to Dürer’s unsolved geometric problem on edge-unfolding of convex polyhedra. He calls the convex polyhedra showing up in the theorem below ”higher order deltahedra” mainly because they form an extension of ”deltahedra” classified earlier by Freudenthal and van der Waerden in [47].

Theorem 7.0.32 $c(n) \leq 3n - 6$*, where equality is attained for infinitely many n namely, for those for which there exists a 3-dimensional convex polyhedron whose each face is an edge-to-edge union of some regular triangles of side length one such that the total number of generating regular triangles on the boundary of the convex polyhedron is precisely $2n - 4$ with a total number of $3n - 6$ sides of length one and with a total number of n vertices.*

Now, we are ready to phrase the **Combinatorial Kepler Problem**. As its name suggests this problem is strongly related to the Kepler Conjecture on the densest unit sphere packings in $\mathbb{E}^3$ (for more details see Section 7 of this paper).

Problem 7.0.33 *For a given n find the largest number $K(n)$ of touching pairs in a packing of n congruent balls in $\mathbb{E}^3$.*

This problem is quite open. The first part of the following theorem was proved by D. Bezdek [12] the second part by K. Bezdek [22].

Theorem 7.0.34

(i) $K(4) = 6, K(5) = 9, K(6) = 12$ *and* $K(7) = 15$.

(ii) $K(n) < 6n - 0.59n^{\frac{2}{3}}$ *for all* $n \geq 4$.

We close this section with two upper bounds for the number of touching pairs in an arbitrary finite packing of translates of a convex body, proved by K. Bezdek in [18] . In order to state these theorems in a short way we need a bit of notation. Let $\mathbf{K}$ be an arbitrary convex body in $\mathbb{E}^d, d \geq 3$. Then let $\delta(\mathbf{K})$ denote the density of a densest packing of translates of the convex body $\mathbf{K}$ in $\mathbb{E}^d, d \geq 3$. Moreover, let $\mathrm{Iq}(\mathbf{K}) = \frac{(\mathrm{Svol}_{d-1}(\mathrm{bd}\mathbf{K}))^d}{(\mathrm{Vol}_d(\mathbf{K}))^{d-1}}$ be the isoperimetric quotient of the convex body $\mathbf{K}$, where $\mathrm{Svol}_{d-1}(\mathrm{bd}\mathbf{K})$ denotes the $(d-1)$-dimensional surface volume of the boundary bd$\mathbf{K}$ of $\mathbf{K}$ and $\mathrm{Vol}_d(\mathbf{K})$ denotes the d-dimensional volume of $\mathbf{K}$. Moreover, let $\mathbf{B}$ denote the closed d-dimensional ball of radius 1 centered at the origin in $\mathbb{E}^d$. Finally, let $\mathbf{K}_0 = \frac{1}{2}(\mathbf{K} + (-\mathbf{K}))$ be the normalized (centrally symmetric) difference body assigned to $\mathbf{K}$ with $H(\mathbf{K}_0)$ (resp., $h(\mathbf{K}_0)$) standing for the Hadwiger number (resp., one-sided Hadwiger number) of $\mathbf{K}_0$.

Theorem 7.0.35 *The number of touching pairs in an arbitrary packing of $n > 1$ translates of the convex body $\mathbf{K}$ in $\mathbb{E}^d, d \geq 3$ is at most*

$$\frac{H(\mathbf{K}_0)}{2} \cdot n - \frac{1}{2^d \cdot (\delta(\mathbf{K}_0))^{\frac{(d-1)}{d}}} \cdot \Big(\frac{Iq(\mathbf{B})}{Iq(\mathbf{K}_0)}\Big)^{\frac{1}{d}} \cdot n^{\frac{(d-1)}{d}} - (H(\mathbf{K}_0) - h(\mathbf{K}_0) - 1).$$

Theorem 7.0.36 *The number of touching pairs in an arbitrary packing of $n > 1$ translates of the convex body $\mathbf{K}$ in $\mathbb{E}^d, d \geq 3$ is at most*

$$\frac{3^d - 1}{2} \cdot n - \frac{\omega_d^{\frac{1}{d}}}{2^{d+1}} \cdot n^{\frac{(d-1)}{d}},$$

where $\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ is the volume of a d-dimensional ball of radius 1 in $\mathbb{E}^d$.

Isoperimetric problems for Voronoi cells

Recall that a family of non-overlapping 3-dimensional balls of radii 1 in Euclidean 3-space, $\mathbb{E}^3$ is called a unit ball packing in $\mathbb{E}^3$. The density of the packing is the proportion of space covered by these unit balls. The sphere packing problem asks for the densest packing of unit balls in $\mathbb{E}^3$. The conjecture that the density of any unit ball packing in $\mathbb{E}^3$ is at most $\frac{\pi}{\sqrt{18}} = 0.74078\ldots$ is often attributed to Kepler that he stated in 1611. The problem of proving the Kepler conjecture appears as part of Hilbert's 18th problem [56]. Using an ingenious argument which works in any dimension, Rogers [79] obtained the upper bound $0.77963\ldots$ for the density of unit ball packings in $\mathbb{E}^3$. This bound has been improved by Lindsey [64], and Muder [68], [69] to $0.773055\ldots$. Hsiang [57], [58] proposed an elaborate line of attack (along the ideas of L. Fejes Tóth suggested 40 years earlier), but his claim that he settled Kepler's conjecture seems exaggerated. However, so far no one has found any serious gap in the approach of Hales [50], [51], [52], [53], although no one has been able to fully verify it either. This is not too surprising, given that the detailed argument is described in several papers and relies on long computer aided calculations of more than 5000 subproblems. Hales shows the following remarkable theorem.

Theorem 7.0.37 *The densest packing of unit balls in* $\mathbb{E}^3$ *has density* $\frac{\pi}{\sqrt{18}}$, *which is attained by the "cannon-ball packing".*

For several of the above mentioned papers Voronoi cells of unit ball packings play a central role. Recall that the Voronoi cell of a unit ball in a packing of unit balls in $\mathbb{E}^3$ is the set of points that are not farther away from the center of the given ball than from any other ball's center. As it is well-known, the Voronoi cells of a unit ball packing in $\mathbb{E}^3$ form a tiling of $\mathbb{E}^3$. One of the most attractive problems on Voronoi cells is the Dodecahedral Conjecture first phrased by L. Fejes Tóth in [40]. According to this the volume of any Voronoi cell in a packing of unit balls in $\mathbb{E}^3$ is at least as large as the volume of a regular dodecahedron with inradius 1. Very recently Hales and McLaughlin [54] announced a solution to this problem:

Theorem 7.0.38 *The volume of any Voronoi cell in a packing of unit balls in* $\mathbb{E}^3$ *is at least as large as the volume of a regular dodecahedron with inradius* 1.

Now, we can make a step further and take a look of the following stronger version of the Dodecahedral Conjecture called the **Strong Dodecahedral Conjecture**. It was first articulated in [16].

Conjecture 7.0.39 *The surface area of any Voronoi cell in a packing with unit balls in* $\mathbb{E}^3$ *is at least as large as* $16.6508\ldots$ *the surface area of a regular dodecahedron of inradius* 1.

It is easy to see that if true, then the above conjecture implies the Dodecahedral Conjecture. The strongest inequality known towards the Strong Dodecahedral Conjecture is due to K. Bezdek and E. Daróczy-Kiss published in [21]. In order to phrase it properly we introduce a bit of terminology. A face cone of a Voronoi cell in a packing with unit balls in $\mathbb{E}^3$ is the convex hull of the face chosen and the center of the unit ball sitting in the given Voronoi cell. The surface area density of a unit ball in a face cone is simply the spherical area of the region of the unit sphere (centered at the apex of the face cone) that belongs to the face cone divided by the Euclidean area of the face. It should be clear from these definitions that if we have an upper bound for the surface area density in face cones of Voronoi cells, then the reciprocal of this upper bound times 4π (the surface area of a unit ball) is a lower bound for the surface area of Voronoi cells. Now, we are ready to state the main theorem of [21].

Theorem 7.0.40 *The surface area density of a unit ball in any face cone of a Voronoi cell in an arbitrary packing of unit balls of* $\mathbb{E}^3$ *is at most*

$$\frac{-9\pi + 30\arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right)\right)}{5\tan\left(\frac{\pi}{5}\right)} = 0.77836\ldots,$$

and so the surface area of any Voronoi cell in a packing with unit balls in E^3 *is at least*

$$\frac{20\pi\tan\left(\frac{\pi}{5}\right)}{-9\pi + 30\arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right)\right)} = 16.1445\ldots.$$

Moreover, the above upper bound $0.77836\ldots$ *for the surface area density is best possible in the following sense. The surface area density in the face cone of any* n*-sided face with* $n = 4, 5$ *of a Voronoi cell in an arbitrary packing of unit balls of* $\mathbb{E}^3$ *is at most*

$$\frac{3(2-n)\pi + 6n\cdot\arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{n}\right)\right)}{n\tan\left(\frac{\pi}{n}\right)}$$

and equality is achieved when the face is a regular n*-gon inscribed in a circle of radius* $\frac{1}{\sqrt{3}\cdot\cos\left(\frac{\pi}{n}\right)}$ *and positioned such that it is tangent to the corresponding unit ball of the packing at its center.*

The Kelvin problem asks for the surface minimizing partition of $\mathbb{E}^3$ into cells of equal volume. According to Lhuilier's memoir [63] of 1781, the problem has been described as one of the most difficult in geometry.

The solution proposed by Kelvin is a natural generalization of the hexagonal honeycomb in $\mathbb{E}^2$. Take the Voronoi cells of the dual lattice giving the densest sphere packing. This gives truncated octahedra, the Voronoi cells of the body centered cubic lattice. A small deformation of the faces produces a minimal surface, which is Kelvin's proposed solution. Just recently Phelan and Weaire [78] produced a remarkable counter-example to the Kelvin conjecture. Their work indicates also that Kelvin's original question is even harder than it was expected. In fact, the following simplier and quite fundamental question seems to be still open. One can regard this as the isoperimetric inequality for parallelohedra and one can call the conjecture below the **Truncated Octahedral Conjecture**. (Recall that a parallelohedron is a 3-dimensional convex polyhedron that tiles $\mathbb{E}^3$ by translation.)

Conjecture 7.0.41 *The surface area of any parallelohedron of volume 1 in $\mathbb{E}^3$ is at least as large as the surface area of the truncated octahedral Voronoi cell of the body-centered cubic lattice of volume 1 in $\mathbb{E}^3$.*

The strong Kepler conjecture

In this section we propose a way to extend Kepler's conjecture to finite packings of congruent balls in 3-space of constant curvature that is in Euclidean 3-space $\mathbb{E}^3$, in spherical 3-space $\mathbb{S}^3$ and in hyperbolic 3-space $\mathbb{H}^3$. The idea goes back to the theorems of L. Fejes Tóth [41] in $\mathbb{E}^2$, J. Molnár [67] in $\mathbb{S}^2$ and K. Bezdek [13], [14] in $\mathbb{H}^2$ which in short, can be phrased as follows:

Theorem 7.0.42 *If at least two congruent circular disks are packed in a circular disk in the plane of constant curvature, then the packing density is always less than $\frac{\pi}{\sqrt{12}}$.*

The hyperbolic case of this theorem proved by K. Bezdek in [13] (see also [14]) seemed quite unexpected because there are (infinite) packings of congruent circular disks in $\mathbb{H}^2$ in which the density of a circular disk in its respective Voronoi cell is significantly larger than $\frac{\pi}{\sqrt{12}}$. Also, we note that the constant $\frac{\pi}{\sqrt{12}}$ is best possible in the above theorem. Last we have to mention that since the standard methods do not give a good definition of density in $\mathbb{H}^2$ (in fact all of them fail to work as it was observed by Böröczky [25]) and since even today we know only a rather "fancy" way of defining density in hyperbolic space (see the work of Bowen and Radin [24]) it seems important to study finite packings in bounded containers of the hyperbolic space where there is no complication with the proper definition of density. All this supports the idea of the following conjecture that we call the **Strong Kepler Conjecture**:

Conjecture 7.0.43 *The density of at least two non-overlapping congruent balls in a ball of the 3-space of constant curvature (having radius strictly less than $\frac{\pi}{2}$ in the case of $\mathbb{S}^3$) is always less than $\frac{\pi}{\sqrt{18}} = 0.74048\ldots$.*

The following theorem proved by K. Bezdek [22] supports the above conjecture.

Theorem 7.0.44 *The density of at least two non-overlapping congruent balls in a ball of the 3-space of constant curvature (having radius strictly less than $\frac{\pi}{2}$ in the case of $\mathbb{S}^3$) is always less than Rogers' upper bound for the density of packings of congruent balls in $\mathbb{E}^3$ that is less than $0.77963\ldots$.*

Bounds on the density of sphere packings in higher dimensions

Recall that a family of non-overlapping $d-$dimensional balls of radii 1 in the $d-$dimensional Euclidean space $\mathbb{E}^d$ is called a unit ball packing of $\mathbb{E}^d$. The density of the packing is the proportion of space covered by these unit balls. The sphere packing problem asks for the densest packing of unit balls in $\mathbb{E}^d$. Indubitably, of all problems concerning packing it was the sphere packing problem which attracted the most attention in the past decade. It has its roots in geometry, number theory and information theory and it is part of Hilbert's 18th problem. The reader is referred to [35] (especially the third edition, which has about 800 references covering 1988-1998) for further information, definitions and references. In what follows we report on a few selected developments some of which are fantastic recent news.

The Voronoi cell of a unit ball in a packing of unit balls in $\mathbb{E}^d$ is the set of points that are not farther away from the center of the given ball than from any other ball's center. As it is well-known the Voronoi cells of a unit ball packing in $\mathbb{E}^d$ form a tiling of $\mathbb{E}^d$. One of the most attractive results on the sphere packing problem was proved by C. A. Rogers [79] in 1958. It was rediscovered by Baranovskii [4] and extended to spherical and hyperbolic spaces by Böröczky [27]. It can be phrased as follows. Take a regular $d-$dimensional simplex of edge length 2 in $\mathbb{E}^d$ and then draw a $d-$dimensional unit ball around each vertex of the simplex. Let σ_d denote the ratio of the volume of the portion of the simplex covered by balls to the volume of the simplex. Then the volume of any Voronoi cell in a packing of unit balls in $\mathbb{E}^d$ is at least $\frac{\omega_d}{\sigma_d}$, where ω_d denotes the volume of a $d-$dimensional unit ball. This has the following immediate corollary.

Theorem 7.0.45 *The (upper) density of any unit ball packing in $\mathbb{E}^d$ is at most σ_d.*

Daniel's asymptotic formula [80] yields that

$$\sigma_d = \frac{d}{e}2^{-(0.5+o(1))d} \text{ (as } d \to \infty).$$

Then 20 years later, in 1978 Kabatjanskii and Levenshtein [59] improved this bound in the exponential order of magnitude as follows. They proved the following theorem.

Theorem 7.0.46 *The (upper) density of any unit ball packing in $\mathbb{E}^d$ is at most*

$$2^{-(0.599+o(1))d} \text{ (as } d \to \infty).$$

In fact, Rogers' bound is better than the Kabatjanskii-Levenshtein bound for $4 \leq d \leq 42$ and above that the Kabatjanskii-Levenshtein bound takes over ([35], p. 20).

There has been some very important recent progress concerning the existence of economical packings. On the one hand, improving earlier results, Ball [2] proved through a very elegant completely new variational argument the following statement. (See also [48] for a similar result of W. Schmidt on centrally symmetric convex bodies.)

Theorem 7.0.47 *For each d, there is a lattice packing of unit balls in $\mathbb{E}^d$ with density at least*

$$\frac{d-1}{2^{d-1}}\zeta(d),$$

where $\zeta(d) = \sum_{k=1}^{\infty} \frac{1}{k^d}$ is the Riemann zeta function.

On the other hand, for some small values of d, there are explicit (lattice) packings which give densities (considerably) higher than the bound just stated. The reader is referred to [35] and [73] for a comprehensive view of results of this type.

All these explicit constructions raise the well-known challenging question whether one can find a smaller upper bound than Rogers' bound for the density of unit ball packings, especially in low dimensions. The next theorem due to K. Bezdek [17] does exactly this by improving Rogers' upper bound for the density of unit ball packings in Euclidean $d-$space for all $d \geq 8$. Since this result extends also some of the results of Section 7 to higher dimensions we phrase it in details. For this we need a bit of notation. As usual, let $\text{lin}(\dots)$, $\text{aff}(\dots)$, $\text{conv}(\dots)$, $\text{Vol}_d(\dots)$, ω_d, $\text{SVol}_{d-1}(\dots)$, $\text{dist}(\dots)$, $\|\dots\|$ and $\mathbf{o}$ refer to the linear hull, the affine hull, the convex hull in $\mathbb{E}^d$, the $d-$dimensional Euclidean volume measure, the $d-$dimensional volume of a $d-$dimensional unit ball, the $(d-1)-$dimensional spherical volume measure, the distance function in $\mathbb{E}^d$, the standard Euclidean norm and to the origin in $\mathbb{E}^d$.

Let $\text{conv}\{\mathbf{o}, \mathbf{w}_1, \dots, \mathbf{w}_d\}$ be a $d-$dimensional simplex having the property that the linear hull $\text{lin}\{\mathbf{w}_j - \mathbf{w}_i | i < j \leq d\}$ is orthogonal to the vector $\mathbf{w}_i$ in $\mathbb{E}^d, d \geq 8$ for all $1 \leq i \leq d-1$ that is let

$$\text{conv}\{\mathbf{o}, \mathbf{w}_1, \dots, \mathbf{w}_d\}$$

be a $d-$dimensional orthoscheme in $\mathbb{E}^d$ moreover, let

$$\|\mathbf{w}_i\| = \sqrt{\frac{2i}{i+1}} \text{ for all } 1 \leq i \leq d.$$

It is clear that in the right triangle $\triangle\ \mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d$ with right angle at the vertex $\mathbf{w}_{d-1}$ we have the inequality $\|\mathbf{w}_d-\mathbf{w}_{d-1}\|=\sqrt{\frac{2}{d(d+1)}}<\sqrt{\frac{2}{(d-1)d}}=\|\mathbf{w}_{d-1}-\mathbf{w}_{d-2}\|$ and therefore $\angle\mathbf{w}_{d-1}\mathbf{w}_{d-2}\mathbf{w}_d<\frac{\pi}{4}$. Now, in the plane aff$\{\mathbf{w}_{d-2},\mathbf{w}_{d-1},\mathbf{w}_d\}$ of the triangle $\triangle\ \mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d$ let

$$\lhd\ \mathbf{w}_{d-2}\mathbf{w}_d\mathbf{w}_{d+1}$$

denote the circular sector of central angle $\angle\mathbf{w}_d\mathbf{w}_{d-2}\mathbf{w}_{d+1}=\frac{\pi}{4}-\angle\mathbf{w}_{d-1}\mathbf{w}_{d-2}\mathbf{w}_d$ and of center $\mathbf{w}_{d-2}$ sitting over the circular arc with endpoints $\mathbf{w}_d,\mathbf{w}_{d+1}$ and radius $\|\mathbf{w}_d-\mathbf{w}_{d-2}\|=\|\mathbf{w}_{d+1}-\mathbf{w}_{d-2}\|$ such that $\lhd\ \mathbf{w}_{d-2}\mathbf{w}_d\mathbf{w}_{d+1}$ and $\triangle\ \mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d$ are adjacent along the line segment $\mathbf{w}_{d-2}\mathbf{w}_d$ and are separated by the line of $\mathbf{w}_{d-2}\mathbf{w}_d$. Then let

$$D(\mathbf{w}_{d-2},\mathbf{w}_{d-1},\mathbf{w}_d,\mathbf{w}_{d+1})=\triangle\ \mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d\cup\lhd\ \mathbf{w}_{d-2}\mathbf{w}_d\mathbf{w}_{d+1}$$

be the convex domain generated by the triangle $\triangle\ \mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d$ with constant angle

$$\angle\mathbf{w}_{d-1}\mathbf{w}_{d-2}\mathbf{w}_{d+1}=\frac{\pi}{4}.$$

Now, let

$$W=\operatorname{conv}\left(\{\mathbf{o},\mathbf{w}_1,\dots,\mathbf{w}_{d-3}\}\cup D(\mathbf{w}_{d-2},\mathbf{w}_{d-1},\mathbf{w}_d,\mathbf{w}_{d+1})\right)$$

be the $d-$dimensional wedge (or cone) with $(d-1)-$dimensional base

$$Q_W=\operatorname{conv}\left(\{\mathbf{w}_1,\dots,\mathbf{w}_{d-3}\}\cup D(\mathbf{w}_{d-2},\mathbf{w}_{d-1},\mathbf{w}_d,\mathbf{w}_{d+1})\right)\ \text{ and apex } \mathbf{o}.$$

Finally, if $B=\{\mathbf{x}\in\mathbb{E}^d|\ \text{dist}(\mathbf{o},\mathbf{x})=\|\mathbf{x}\|\le 1\}$ denotes the $d-$dimensional unit ball centered at the origin $\mathbf{o}$ of and $S=\{\mathbf{x}\in\mathbb{E}^d|\ \text{dist}(\mathbf{o},\mathbf{x})=\|\mathbf{x}\|=1\}$ denotes the $(d-1)-$dimensional unit sphere centered at $\mathbf{o}$, then let

$$\widehat{\sigma}_d=\frac{\text{SVol}_{d-1}(W\cap S)}{\text{Vol}_{d-1}(Q_W)}=\frac{\text{Vol}_d(W\cap B)}{\text{Vol}_d(W)}$$

be the the surface density (resp., volume density) of the unit sphere S (resp., of the unit ball B) in the wedge W. For the sake of completeness we remark that as the regular $d-$dimensional simplex of edge length 2 can be dissected into $(d+1)!$ pieces each being congruent to conv$\{\mathbf{o},\mathbf{w}_1,\dots,\mathbf{w}_d\}$ therefore

$$\sigma_d=\frac{\text{Vol}_d(\text{conv}\{\mathbf{o},\mathbf{w}_1,\dots,\mathbf{w}_d\}\cap B)}{\text{Vol}_d(\text{conv}\{\mathbf{o},\mathbf{w}_1,\dots,\mathbf{w}_d\})}.$$

Now, we are ready to state the main result of [17]. Recall that the surface density of any unit sphere in its Voronoi cell in a unit sphere packing of $\mathbb{E}^d$ is defined as the ratio of the surface area of the unit sphere to the surface area of its Voronoi cell.

Theorem 7.0.48 *The surface area of any Voronoi cell in a packing of unit balls in the $d-$dimensional Euclidean space $\mathbb{E}^d, d\ge 8$ is at least $\frac{d\cdot\omega_d}{\widehat{\sigma}_d}$, that is the surface density of any unit sphere in its Voronoi cell in a unit sphere packing of $\mathbb{E}^d, d\ge 8$ is at most $\widehat{\sigma}_d$. Thus, the volume of any Voronoi cell in a packing of unit balls in $\mathbb{E}^d, d\ge 8$ is at least $\frac{\omega_d}{\widehat{\sigma}_d}$ and so, the (upper) density of any unit ball packing in $\mathbb{E}^d, d\ge 8$ is at most $\widehat{\sigma}_d<\sigma_d$.*

In fact, K. Bezdek [22] extended the above theorem to spherical space ($\mathbb{S}^d$) as well as to hyperbolic space ($\mathbb{H}^d$) in the following local sense. Consider packings of congruent balls of small radii only. Then for sufficiently small radii r of the given space $\mathbb{S}^d$ (resp., $\mathbb{H}^d$) one can define the quantity $\widehat{\sigma}_{\mathbb{S}^d}(r)=\frac{\text{Vol}_d(W\cap B)}{\text{Vol}_d(W)}$ (resp., $\widehat{\sigma}_{\mathbb{H}^d}(r)=\frac{\text{Vol}_d(W\cap B)}{\text{Vol}_d(W)}$) in a way very similar to the Euclidean case. (Here we simply omit the obvious details.) With this notation the following theorem holds.

Theorem 7.0.49 *Consider an arbitrary packing of spheres of radius r in $\mathbb{S}^d$ (resp., $\mathbb{H}^d$) with $d\ge 8$. Then there exists an $r(d)>0$ such that the (volume) density of each ball (of the given packing) in its respective Voronoi cell is at most $\widehat{\sigma}_{\mathbb{S}^d}(r)$ (resp., $\widehat{\sigma}_{\mathbb{H}^d}(r)$) provided that $r\le r(d)$.*

Further improvements on the upper bound $\widehat{\sigma}_d$ of K. Bezdek for the dimensions from 4 to 36 have been obtained very recently by Cohn and Elkies [33]. They developed an analogue for sphere packing of the linear programming bounds for error correcting codes, and used it to prove new upper bounds for the density of sphere packings, which are better than K. Bezdek's upper bounds $\widehat{\sigma}_d$ for the dimensions 4 through 36. Their method together with the best known sphere packings yield the following remarkable theorem in dimensions 8 and 24.

Theorem 7.0.50 *The density of the densest unit ball packing in $\mathbb{E}^8$ (resp., $\mathbb{E}^{24}$) is at least $0.2536\dots$ (resp., $0.00192\dots$) and is at most 0.2537... (resp., $0.00196\dots$).*

Cohn and Elkies [33] conjecture that their approach can be used to solve the sphere packing problem in $\mathbb{E}^8$ (resp., $\mathbb{E}^{24}$).

Conjecture 7.0.51 *The E_8 root lattice (resp., the Leech lattice) that produces the corresponding lower bound in the previous theorem in fact, represents the largest possible density for unit sphere packings in $\mathbb{E}^8$ (resp., $\mathbb{E}^{24}$).*

If linear programming bounds can indeed be used to prove optimality of these lattices, it would not come as a complete surprise, because for example, the kissing number problem in these dimensions was solved similarly (for more details see Section 2).

Last but not least we mention the following striking result of Cohn and Kumar [34] according to which the Leech lattice is the densest lattice packing in $\mathbb{E}^{24}$. (The densest lattices have been known up to dimension 8.)

Theorem 7.0.52 *The Leech lattice is the unique densest lattice in $\mathbb{E}^{24}$, up to scaling and isometries of $\mathbb{E}^{24}$.*

We close this section with a short summary on the recent progress of L. Fejes Tóth's [45] "sausage conjecture" that is one of the main problems of the theory of finite sphere packings. According to this conjecture if in $\mathbb{E}^d, d \geq 5$ we take $n \geq 1$ non-overlapping unit balls, then the volume of their convex hull is at least as large as the volume of the convex hull of the "sausage arrangement" of n non-overlapping unit balls under which we mean an arrangement whose centers lie on a line of $\mathbb{E}^d$ such that the unit balls of any two consecutive centers touch each other. By optimizing the methods developed by Betke, Henk and Wills [7], [8] finally, Betke and Henk [6] succeeded to prove the sausage conjecture of L. Fejes Tóth in any dimension of at least 42. Thus, we have the following natural looking but, far not trivial theorem.

Theorem 7.0.53 *The sausage conjecture holds in $\mathbb{E}^d$ for all $d \geq 42$.*

It remains a highly interesting challange to prove or disprove the sausage conjecture of L. Fejes Tóth for the dimensions between 5 and 41.

Conjecture 7.0.54 *Let $5 \leq d \leq 41$ be given. Then the volume of the convex hull of $n \geq 1$ non-overlapping unit balls in $\mathbb{E}^d$ is at least as large as the volume of the convex hull of the "sausage arrangement" of n non-overlapping unit balls which is an arrangement whose centers lie on a line of $\mathbb{E}^d$ such that the unit balls of any two consecutive centers touch each other.*

Solidity and uniform stability

The notion of solidity, introduced by L. Fejes Tóth [43] to overcome difficulties of the proper definition of density in the hyperbolic plane, has been proved very useful and stimulating. Roughly speaking a family of convex sets generating a packing is said to be solid if no proper rearrangement of any finite subset of the packing elements can provide a packing. More concretely, a circle packing in the plane of constant curvature is called solid if no finite subset of the circles can be rearranged such that the rearranged circles together with the rest of the circles form a packing not congruent to the original. An (easy) example for solid circle packings is the family of incircles of a regular tiling $\{p, 3\}$ for any $p \geq 3$. In fact, a closer look of this example led L. Fejes Tóth [46] to the following simple sounding but difficult problem: he conjectured that the incircles of a regular tiling $\{p, 3\}$ form a strongly solid packing for any $p \geq 5$, i.e. by removing any circle from the packing the remaining circles still form a solid packing. This conjecture has been verified for $p = 5$ by Böröczky [28] and Danzer [D] and for $p \geq 8$ by A. Bezdek [9]. Thus, we have the following theorem.

Theorem 7.0.55 *The incircles of a regular tiling $\{p, 3\}$ form a strongly solid packing for $p = 5$ and for any $p \geq 8$.*

The outstanding open question left is the following.

Conjecture 7.0.56 *The incircles of a regular tiling $\{p, 3\}$ form a strongly solid packing for $p = 6$ as well as for $p = 7$.*

In connection with solidity and finite stability (of circle packings) the notion of uniform stability (of sphere packings) has been introduced by A. Bezdek, K. Bezdek and R. Connelly [11]. According to this a sphere packing (in the space of constant curvature) is said to be uniformly stable if there exists an $\epsilon > 0$ such that no finite subset of the balls of the packing can be rearranged such that each ball is moved by a distance less than ϵ and the rearranged balls together with the rest of the balls form a packing not congruent to the original one. Now, suppose that $\mathcal{P}$ is a packing of (not necessarily) congruent balls in $\mathbb{E}^d$. Let G be the contact graph of $\mathcal{P}$, where the centers of the balls serve as the vertices of G and an edge is placed between two vertices when the corresponding two balls are tangent. The following basic principle can be used to show that many packings are uniformly stable.

Theorem 7.0.57 *Suppose that $\mathbb{E}^d$ can be tiled face-to-face by congruent copies of finitely many convex polytopes $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ such that the vertices and edges of that tiling form the vertex and edge system of the contact graph G of the packing $\mathcal{P}$ of some balls in $\mathbb{E}^d$. If each $\mathbf{P}_i$ is strictly locally volume expanding with respect to G, then the packing $\mathcal{P}$ is uniformly stable.*

By taking a closer look of the Delaunay tilings of some lattice sphere packings one can derive the following corollary (for more details see [11]).

Theorem 7.0.58 *The densest lattice sphere packings $A_2, A_3, D_4, D_5, E_6, E_7, E_8$ up to dimension 8 are all uniformly stable.*

Last we mention another corollary (for details see [11]), which was observed also by Bárány and Dolbilin [5] and which supports the above mentioned conjecture of L. Fejes Tóth.

Theorem 7.0.59 *Consider the triangular packing of circular disks of equal radii in $\mathbb{E}^2$ where each disk is tangent to exactly six others. Remove one disk to obtain the packing $\mathcal{P}'$. Then the packing $\mathcal{P}'$ is uniformly stable.*

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Chapter 8

Moment maps in Various Geometries (05w5072)

May 21–26, 2005

Organizer(s): Tara Holm (University of California, Berkeley), Lisa Jeffrey (University of Toronto), Yael Karshon (University of Toronto), Eugene Lerman (University of Illinois Urbana-Champaign), Eckhard Meinrenken (University of Toronto)

Background

Symplectic geometry was invented by Hamilton in the early nineteenth century, as a mathematical framework for both classical mechanics and geometrical optics. Physical states in both settings are described by points in an appropriate phase space (the space of coordinates and momenta). Hamilton's equations associate to any energy function ("Hamiltonian") on the phase space a dynamical system. Hamilton realized that his equations are invariant under a very large group of symmetries, called canonical transformations or, in modern terminology, symplectomorphisms. A symplectic manifold is a space which is locally modeled by the phase spaces considered by Hamilton. In mathematical terms, a symplectic manifold is a manifold M with a closed, non-degenerate 2-form ω. A smooth function $H \in C^\infty(M)$ defines a vector field X_H on M by Hamilton's equations,

$$\mathrm{d}H = -\omega(X_H, \cdot).$$

New techniques have transformed symplectic geometry into a deep and powerful subject of pure mathematics. One concept of symplectic geometry that has proved particularly useful in many areas of mathematics is the notion of a *moment map*. To recall the original setting for this notion, let M be a symplectic manifold, and G a Lie group acting on M by symplectomorphisms. A moment map for this action is an equivariant map $\Phi\colon M \to \mathfrak{g}^*$ with values in the dual of the Lie algebra, with the property that the infinitesimal generators of the action, corresponding to Lie algebra elements $\xi \in \mathfrak{g}$, are the Hamiltonian vector fields $X_{\langle\Phi,\xi\rangle}$. The linear momentum and angular momentum from classical mechanics may be viewed as moment maps, corresponding to translational and rotational symmetries, respectively.

In the past thirty years, tremendous progress has been made in the study of moment maps and related areas: symplectic quotients, geometric quantization, localization phenomena, and toric varieties. This has had applications to the study of moduli spaces, representation theory, special metrics, and symplectic topology.

In recent years, moment maps have been generalized in many different directions and have led to advances in geometries related to symplectic geometry. These include Poisson geometry, Kähler geometry, hyper-Kähler geometry, contact geometry, and Sasakian geometry. While some headway has been made in understanding moment maps in these fields, there remain many open questions. One of the goals of this workshop was to explore phenomena that are well understood in symplectic geometry but are not as well understood in these new settings, and to seek potential applications of this new direction of research. For this purpose we brought together experts from these fields, thus generating a fruitful exchange of ideas, which also enabled us to formulate and discuss interesting open problems.

Objectives of the workshop

Let us review some of the achievements in and applications of equivariant symplectic geometry in the past few years. We will then indicate some of the open questions that were our motivation for holding the workshop.

We first recall some terminology. Let a Lie group G act on a symplectic manifold (M, ω). As we already recalled, a *moment map* is an equivariant map $\Phi\colon M \to \mathfrak{g}^*$ to the dual of the Lie algebra such that the G action is generated by the Hamiltonian vector fields of the components of Φ. The *symplectic quotient* is $\Phi^{-1}(0)/G$. *Localization* formulas express global invariants of M in terms of local data at the fixed point set of an abelian subgroup of G. When G is a torus of half the dimension of M and M is compact, (M, ω, Φ) is a *toric manifold*.

A *contact structure* is an odd dimensional counterpart of a symplectic structure. Similarly, a *Sasakian structure* is an odd dimensional counterpart of a Kähler structure, and a *3-Sasakian structure* is an odd dimensional counterpart of a hyper-Kähler structure. The goal of the workshop was to obtain a better understanding of moment maps and their applications in these other geometries.

The development of equivariant symplectic geometry over the last two decades was greatly motivated by attempts to understand the topology of moduli spaces of stable bundles over Riemann surfaces. The symplectic and Morse theoretic approach to the problem was pioneered by Atiyah and Bott in 1983, when they produced a set of generators for the cohomology ring of the moduli space $M(n, d)$ of semi-stable rank n, degree d holomorphic vector bundles over a Riemann surface, for n and d co-prime.

Given a Hamiltonian group action of a Lie group G on a compact symplectic manifold M, with moment map $\Phi\colon M \to \mathfrak{g}^*$ such that 0 is a regular value for Φ, there is a natural map from the equivariant cohomology $H^*_G(M)$ to the cohomology of the reduced space, $H^*(\Phi^{-1}(0)/G)$, obtained as the restriction $H^*_G(M) \to H^*_G(\Phi^{-1}(0))$ followed by the isomorphism $H^*_G(\Phi^{-1}(0)) \to H^*(\Phi^{-1}(0)/G)$. Kirwan refined the Morse-theoretic methods of Atiyah and Bott to prove that this map, $\kappa\colon H^*_G(M) \to H^*(\Phi^{-1}(0)/G)$, is surjective. This enables one to compute Betti numbers of symplectic quotients $\Phi^{-1}(0)/G$. The non-abelian localization theorem of Jeffrey and Kirwan gives an explicit formula for the kernel of κ. Jeffrey and Kirwan used their version of the non-abelian localization formula, and a description of $M(n, d)$ as a finite-dimensional quotient of a so-called "extended moduli space", to obtain a mathematically rigorous proof of Witten's formulas for the intersection pairings in the cohomology of $M(n, d)$.

In 1998, Alekseev, Malkin and Meinrenken introduced quasi-Hamiltonian spaces and Lie group valued moment maps. They expressed the moduli space of flat G-connections as a quasi-Hamiltonian quotient of a product $G^2 \times \cdots \times G^2$, and were thus able to recover Witten's formulas for intersection numbers in the cohomology of moduli spaces. In the moduli space case, quasi-Hamiltonian spaces enable one to avoid the use of extended moduli spaces; more generally, quasi-Hamiltonian spaces enlarge the collection of situations to which similar techniques can be applied.

In 2002, Bott, Tolman and Weitsman proved surjectivity of Kirwan's map $\kappa : H^*_{LG}(M) \to H^*(\Phi^{-1}(0)/G)$ in the case where LG is the loop group of a compact Lie group G, M is a Banach manifold and Φ a proper moment map. As a consequence one obtains that, while Kirwan's map is not surjective for quasi-Hamiltonian spaces, its image together with *finitely many* cohomology classes generates the cohomology ring of the quotient. This work is related to Tolman and Weitsman's earlier work (1998) determining the kernel of the Kirwan map κ and thereby the structure of the cohomology ring of the symplectic quotient $H^*(\Phi^{-1}(0)/G)$ when G is a finite-dimensional Lie group.

In 2003, Xu introduced quasi-symplectic groupoids. This approach enables him to unify into a single framework various moment map theories, including ordinary symplectic moment maps and group valued moment maps.

Moment maps and symplectic quotients can be defined in other categories, such as contact or hyper-

Kähler. However, the topology of quotients in these categories remains elusive. As noted in a recent book by Ginzburg, Guillemin, and Karshon, phenomena such as Kirwan surjectivity and localization are often due to the underlying moment map and group action more than to the geometry. However, we do not yet understand these phenomena for contact or hyper-Kähler manifolds. For example, Kirwan surjectivity fails for contact structures, and it is not yet clear why or how. Surjectivity is conjectured for hyper-Kähler quotients, and known to be true for many classes of examples, but a general theorem has not been proved. An interesting example of a hyper-Kähler quotient is the moduli space of rank 2 parabolic Higgs bundles. Hausel and Thaddeus have produced generators and relations for the cohomology ring of this space. This work is analogous to the work of Jeffrey and Kirwan on the moduli space $M(n,d)$. Another usage of hyper-Kähler quotients is that they provide examples of Einstein manifolds.

In 1988 Delzant classified symplectic toric manifolds. These turn out to be symplectic quotients of $\mathbb{C}^N$. In particular, they inherit a complex structure from $\mathbb{C}^N$, making them into smooth Kähler toric varieties. The images of their moment maps are simple rational polytopes satisfying certain integrality conditions. The polytope determines the toric manifold together with its symplectic form and torus actions. This theorem of Delzant, while simple in retrospect, inspired a lot of interesting mathematics. For example, the removal of the integrality condition on simple rational polytopes leads to orbifold singularities. Symplectic toric orbifolds were classified in 1997 by Lerman and Tolman in terms of simple rational polytopes with positive integers attached to facets. Delzant's work inspired Banyaga and Molino to initiate the study of *contact* toric manifolds. The classification of contact toric manifolds has been recently completed by Lerman. Lerman used this classification to prove conjectures of Toth and Zelditch on toric integrable geodesic flows. Most, but not all, of the contact toric manifolds turn out to be Sasakian. These contact toric manifolds are classified by rational polyhedral cones.

Yet another direction inspired by Delzant's work is that of hyper-Kähler toric manifolds. These manifolds were first studied by Bielawski and Dancer, who defined them to be hyper-Kähler quotients of a flat quaternionic vector space. They obtained a formula for the Betti numbers of these manifolds in terms of the corresponding arrangements of hyperplanes. Bielawski also showed that these are all complete hyper-Kähler manifolds with torus symmetries of maximal dimension. At the same time, Bielawski obtained a classification of toric 3-Sasakian manifolds. In 2000, Konno computed the full cohomology ring of a hyper-Kähler toric manifold in terms of the hyperplane arrangement. In a later paper Konno computed the total Chern classes of these manifolds.

An important use of toric varieties, in both complex and symplectic geometry, is to provides a large "hands-on" family of examples. In particular, they have been used in searches for examples of special Kähler metrics.

A formula for the Kähler metric on a toric manifold, in terms of natural linear functions on the polytope, was obtained by Guillemin in 1994. Guillemin's work, in turn, inspired Abreu, who studied other metrics on symplectic toric manifolds. For example, Abreu obtained an explicit description of Bochner-Kähler metrics studied by Bryant. He also obtained a combinatorial formula for their scalar curvature and used it to explicitly construct Kähler metrics that are extremal in the sense of Calabi. One question that remains open is to obtain explicit formulas for Kähler-Einstein metrics on $\mathbb{CP}^2$ blown up at three generic points; such metrics are only known to exist.

Recently a great deal of progress has been made by Boyer, Galicki and their collaborators in proving the existence of Sasakian-Einstein metrics on a large class of contact manifolds. These metrics, however, are not known explicitly. One expects that an analogue of Guillemin's formula for Kähler metrics on symplectic toric manifolds to hold for the Sasakian toric manifolds. These metrics are unlikely to be Einstein (this follows from very recent work of Guillemin and Burns). However, it might be possible to construct the Sasakian-Einstein metrics explicitly in terms of polyhedral cones.

There have been a variety of other applications of moment maps to the study of special metrics. Futaki and Tian used localization to compute an invariant which provides an obstruction to the existence of constant scalar curvature metrics in a fixed Kähler class. For a toric variety, Mabuchi expressed this invariant in terms of the corresponding polytope. Claude Lebrun and Michael Singer used moment maps to explore scalar-flat Kähler metrics on ruled surfaces. "Extremal" metrics and "central" metrics are ones for which certain elementary symmetric functions of the Ricci curvature are moment maps for Killing fields. An outstanding conjecture is whether the existence of constant scalar curvature metrics, or Kähler-Einstein metrics, is equivalent to certain notions of "stability". Results in this direction have been obtained by Tian (1997) and Tian-Chen (as announced very recently). Another part of this conjecture was recently proved by Donaldson for the special case of toric manifolds in complex dimension 2. In a different direction, one can exhibit the scalar curvature as a moment map in an infinite dimensional setting. This description is due to Mabuchi and was used by Donaldson. It is analogous to Atiyah and Bott's influential work on the Yang Mills functional.

One of our motivating goals was to determine which invariants developed in symplectic geometry for understanding symplectic quotients (for example their cohomology ring) carry over to the settings of hyper-Kähler, contact, Sasakian, and 3-Sasakian geometries. In particular, we proposed to explore the question of surjectivity in contact and hyper-Kähler geometry. Additionally, we aimed to study natural metrics on such quotients and to use this to seek explicit descriptions for special metrics on Kähler and Sasakian manifolds.

At the workshop, besides an under-representation of the odd dimensional structures (contact, Sasakian, 3-Sasakian), the lectures and discussions addressed many aspects of moment maps in a wide variety of contexts: Kähler geometry and special metrics, applications to symplectic topology, approaches through Lie groupoids, algebraic geometric, several aspects of hyper-Kähler geometry, and more.

Activities of the workshop

The formal activities during the workshop included research talks, survey lectures on special topics, and two problem sessions, aimed as forums for discussion. We believe that this format has been highly successful and very stimulating. Below, we will summarize some of the new developments and open questions presented at the workshop.

Moment maps and symplectomorphism groups

Let (M, ω) be a symplectic manifold, and $\mathrm{Diff}_\omega(M)$ its group of symplectomorphisms. The group $\mathrm{Diff}_\omega(M)$ contains an important subgroup $\mathrm{Diff}_{\mathrm{Ham}}(M)$ of *Hamiltonian diffeomorphisms*, i.e., the subgroup generated by time-one flows of Hamiltonian vector fields. The topology of the groups $\mathrm{Diff}_{\mathrm{Ham}}(M)$ and $\mathrm{Diff}_\omega(M)$ has been the subject of intense research over the past few years.

Miguel Abreu (Instituto Superior Tecnico, Lisbon) (joint work with Granja and Kitchloo) reported on recent progress on the topology of $\mathrm{Diff}_\omega(M)$. The basic new input goes back to Donaldson, and uses the moment map geometry for the action of a symplectomorphism group on the space of compatible almost complex structures. In conjunction with his earlier work [1] with McDuff, employing Gromov's technique of pseudo-holomorphic curves, this approach turns out to be particularly successful for a class of 4-dimensional symplectic manifolds, including rational ruled surfaces.

Susan Tolman (University of Illinois at Urbana-Champaign) (joint work with McDuff) described exciting new results on the fundamental group of symplectomorphism groups of 4-dimensional symplectic toric varieties M, i.e., spaces carrying an effective Hamiltonian action of a torus of dimension $\frac{1}{2}\mathrm{dim} M = 2$. A well-known theorem of Delzant (see e.g. [11]) states that such spaces are completely determined (up to equivariant symplectomorphism) by the convex polytope in $\mathbb{R}^2$ given as their moment map image. Moreover, one can specify exactly which polytopes arise as moment polytopes of Delzant spaces. In their work, McDuff-Tolman discovered a relationship between the topology of the symplectomorphism group of such spaces with the shape of the moment polytope. This then leads to the following problem: Which Delzant polytopes admit a linear function so that the center of mass of the polytope depends linearly on the facet position? The solution to this problem allows them to prove that, for all but a few exceptional cases, the inclusion of the (compact) group of Kähler automorphism into the group of symplectomorphism induces an isomorphism of fundamental groups.

Victor Guillemin (M.I.T.) (joint work with Sternberg) described a very different aspect of symplectomorphism groups. He explained that for certain maps from finite dimensional manifolds into the group of symplectomorphisms, there is an intriguing notion of a moment map even if there is no Hamiltonian group action! In his beautiful talk, he motivated how this type of generalized moment map fits with Weinsteins *symplectic category* [27]. This is the "category" with objects Obj symplectic manifolds M, and morphisms $\mathrm{Mor}(M_1, M_2)$ the canonical relations, meaning, Lagrangian submanifolds of $M_1^- \times M_2$. (Here "category" is put in quotes, since composition is not always defined.) Concrete applications of this theory arise in micro-local analysis, in the study of families of Fourier integral operators.

Moment maps and Poisson geometry

Poisson manifolds are manifolds M equipped with a Poisson bracket $\{\cdot, \cdot\}$ on the algebra of smooth functions on M. Symplectic manifolds are special cases of Poisson manifolds, where the bracket is given as

$$\{f, g\} = X_f(g).$$

A Poisson structure determines a singular foliation (in the sense of Sussmann) whose leaves are symplectic manifolds.

Rui Fernandes (Instituto Superior Tecnico, Lisbon) (joint work with Crainic). The Poisson bracket descends to a canonical Lie bracket on the space of 1-forms on any Poisson manifold. In this way, the cotangent bundle T^*M acquires the structure of a *Lie algebroid*. A global object 'integrating' this Lie algebroid is a *symplectic groupoid*, i.e., a groupoid

$$S \rightrightarrows M,$$

where S carries a symplectic structure such that both groupoid maps are Poisson maps, and such that the symplectic form is compatible with the groupoid multiplication. Not all Poisson manifolds admit such a *symplectic realization*. The precise obstructions were found a few years ago by Fernandes-Crainic [10]. In his BIRS lecture, Fernandes explained how this theory extends to the presence of Poisson group actions. He showed that if M admits a symplectic realization S, then the induced action on S is Hamiltonian with a canonical moment map. (This moment map satisfies a cocycle condition, and is a coboundary if and only if the action on M admits a moment map.) Finally, Fernandez explained in which sense 'symplectic realization' commutes with 'reduction'.

Anton Alekseev (University of Geneva) (joint work with Meinrenken [3]). A *Poisson Lie group* is a Lie group K with a Poisson structure for which the product map is Poisson. This condition defines a Lie bracket on the dual of the Lie algebra $\mathfrak{k}^*$, which integrates to the so-called *dual Poisson Lie group* K^*. If K carries the zero Poisson structure, then the dual Poisson Lie group is $\mathfrak{k}^*$ with the Kirillov Poisson structure. A construction of Lu-Weinstein [23] shows that any compact Lie group K admits a canonical Poisson Lie group structure. Later, Ginzburg-Weinstein [14] proved that, in this case, the dual Poisson Lie group K^* is Poisson diffeomorphic to $\mathfrak{k}^*$. However, no explicit form of such a diffeomorphism was known. Alekseev explained that for the group $K = \mathrm{U}(n)$, there is a distinguished and very concrete Ginzburg-Weinstein diffeomorphism

$$\mathfrak{u}(n)^* \to \mathrm{U}(n)^*.$$

The proof of this result (which verifies a conjecture of Flaschka-Ratiu [13]) is based on a study of Gelfand-Zeitlin systems on $\mathfrak{u}(n)^*$ and $U(n)^*$, respectively. As a corollary, one obtains the following interesting result: There is a canonical diffeomorphism

$$\gamma\colon\ \mathrm{Herm}(n) \to \mathrm{Herm}^+(n)$$

from hermitian matrices to positive definite Hermitian matrices, with the property that the eigenvalues of the kth principal submatrix of $\gamma(A)$ are the exponentials of those of the kth principal submatrix of A.

Groupoids and generalized moment maps

Markus Pflaum (Goethe Universität, Germany) Differentiable groupoids can be interpreted as an interpolation between the notion of a manifold and the notion of a Lie group. In this survey talk, Markus Pflaum gave a general introduction to the theory of Lie groupoids (cf. [12]), and explained two major applications of this theory in symplectic geometry. The first application deals with the integrability of Poisson manifolds by symplectic groupoids (cf. Fernandes' lecture). The second application is Moerdijk's approach [24] to orbifolds via proper étale Lie groupoids, which is an important ingrediant in the work by Pflaum–Posthuma–Tang on the deformation quantization and index theory for orbifolds.

Henrique Bursztyn (University of Toronto) presented a survey lecture on generalized moment maps (cf. [9]). He explained how, quite generally, any Poisson map between Poisson manifolds defines an infinitesimal 'Lie algebroid' action, and hence may be viewed as a moment map. This includes ordinary $\mathfrak{k}^*$-valued moment maps, but also Lu's [22] non-linear moment maps taking values in a dual Poisson Lie group K^*. To include more exotic types of moment maps, one has to go beyond Poisson structures to so-called *twisted Dirac structures*. In particular, any compact Lie group carries a natural twisted Dirac structure, and the associated moment map theory defines the q-hamiltonian spaces of Alekseev-Malkin-Meinrenken [2]. Among the advantages of this approach is that the somewhat mysterious 'minimal degeneracy conditions' become very natural. Furthermore, the techniques work well also for non-compact Lie groups, as well as for complex Lie groups.

Topology of symplectic quotients

Let (M, ω) be a symplectic manifold, equipped with a Hamiltonian action of a Lie group K, with moment map Φ. A standard result of Marsden-Weinstein asserts that under suitable assumptions, the *symplectic quotient*

$$M/\!\!/G = \Phi^{-1}(0)/G$$

inherits a symplectic structure from the 2-form on M. It is a fundamental problem in symplectic geometry to understand the geometry and topology of $M/\!\!/G$ in terms of the equivariant geometry of the original space M.

Greg Landweber (University of Oregon) (joint work with Harada [18]). In this survey lecture, Landweber gave a general overview of equivariant K-theory (the generalized cohomology theory given as the Grothendieck ring of equivariant vector bundles) in the context of Hamiltonian group actions. He explained the K-theory analog of the Atiyah-Bott Lemma, which says that the K-theory analogue of the equivariant Euler class is not a zero divisor. As a result, one obtains a K-theoretic analogue of the Kirwan surjectivity theorem. As Landweber remarks, the torsion in K-theory is better behaved than that in cohomology with integer coefficients. Essentially, K-theory eliminates just enough torsion for Atiyah and Bott's arguments to work.

Liviu Mare (University of Regina) (joint with Harada, Holm and Jeffrey [17]). Classical results of Atiyah [6], Guillemin-Sternberg [15] and Kirwan [19] say that for any compact torus T, and any Hamiltonian T-space with proper moment map, the image of the moment map is a convex polyhedron, and the fibers of the moment map are connected. Atiyah-Pressley [8] proved a similar convexity result for the maximal torus $\widetilde{T}$ in the standard extension of the based loop group ΩG for a compact, simply connected Lie group. The main result presented in this lecture says that also in this case, the fibers of the moment map are connected.

Nick Proudfoot (UT Austin) ([25]) Suppose G is a reductive algebraic group, acting on a variety Q. Then the cotangent bundle T^*Q has an algebraic symplectic form, and the lifted G-action is Hamiltonian with an algebraic moment map. In his talk, Proudfoot discussed the relation between the symplectic quotient of T^*Q, with various GIT (geometric invariant theory) quotients of Q.

Kähler geometry and special metrics

A Kähler manifold is a manifold with compatible complex and symplectic reduction. The presence of a complex structure leads to stronger versions of some of the results of moment map geometry.

Reyer Sjamaar (Cornell University) (joint work with V. Guillemin [16]). For Hamiltonian torus actions on *Kähler manifolds*, Atiyah [6] had proved an important refinement of the convexity theorem: Not only is the image of the moment map a convex polytope, but in fact the moment map image of any orbit closure is convex. (Note that orbit closures need not be smooth submanifolds.) Brion generalized the result to actions of a complex reductive group. The results presented in this lecture generalize this result even further, to actions of a maximal solvable subgroup. Two interesting examples of Borel-invariant subvarieties of a Hamiltonian Kähler G-manifold are: (1) Generalized Schubert varieties (introduced by Białnicky-Birula, and (2) the co-called facial varieties. That is, for each face of the moment polytope there is a certain variety whose moment map image is the given face. (In general, there is no G-invariant subvariety with this property.)

Vestislav Apostolov (UQAM) (joint work with Calderbank, Gauduchon, and Tonnesen-Friedman [5]). In recent work, Apostolov and his coauthors introduced the notion of *Hamiltonian 2-forms on Kähler manifolds*. These are closed differential forms of bi-degree $(1,1)$, defined as solutions of a certain linear differential equation on the Kähler manifold. Hamiltonian 2-forms arise, for example, in the theory of Bochner-flat or conformally Einstein Kähler manifolds. Apostolov's lecture was concerned with the local and global classification of Hamiltonian 2-forms. As applications, he obtained new examples of so-called *orthotoric Kähler-Einstein manifolds*.

Hyper-Kähler geometry

Hiroshi Konno (University of Tokyo) gave a survey lecture on the geometry and topolgy of hyper-Kähler quotients. Examples for such quotients include: toric hyper-Kähler manifolds, hyper-Kähler polygon spaces, the moduli space of torsion free sheaves on $\mathbb{C}^2$, and Nakajima quiver varieties.

Tamas Hausel (UT Austin) explained techniques for the computation of cohomology groups of hyper-Kähler manifolds, such as moduli space of instantons, quiver varieties, representation varieties, and moduli of Higgs bundles. The techniques are: (i) global analysis to determine the space of L^2-harmonic forms (this approach is motivated by Sen's conjecture); (ii) circle-equivariant cohomology techniques (motivated by ideas of Nekrasov-Shatashvili-Moore) and (iii) calculation of zeta functions by arithmetic harmonic analysis (motivated by mirror symmetry).

Graeme Wilkin (Brown University) (joint work with Daskalopoulos and Wentworth). In their 1982 paper, Atiyah and Bott [7] used Morse theory of the Yang-Mills functional to study the topology of the moduli space of semistable vector bundles over a Riemann surface. Wilkin described a similar technique for the moduli space of rank 2 semi-stable Higgs bundles. A complication in this example is that the moduli spaces are singular, and hence the method has to refined to take the singularities into account. A main result of this approach is a proof of Kirwan hyper-Kähler surjectivity for some rank-2 Higgs bundles.

Moment maps and path integrals

Jonathan Weitsman (Santa Cruz). Quantum field theory is a source for many exciting predictions in mathematics, mostly based however on non-rigorous 'functional integral teachniques'. A prototype is Witten's formulas [28] for intersection pairings, based on path integral calculations for the Yang-Mills functional (norm square of the moment map). In his talk, Weitsman indicated that in some case, these path integral arguments can in fact be made rigorous. The main techique is a new construction of measures on Banach manifolds associated to supersymmetric quantum field theories. As examples, he discussed the Wess-Zumino-Novikov-Witten model for maps of Riemann surfaces into compact Lie groups, as well as 3-dimensional gauge theory.

Open problems

In addition to the traditional lectures, we ran two problem sessions during our week at Banff. These sessions were meant to foster discussion and to identify open problems relevant to the workshop. Each session had a moderator who solicited the open problems from the audience and transcribed them onto the board. We used a format very similar to the problem sessions run at the workshops at the American Institute of Mathematics [4]. We present here the record of the problems discussed at these sessions.

Compactification of cotangent bundles

Problem 8.0.1 (N. Kitchloo) *Let X be a compact manifold. Does the symplectic manifold (T^*X, ω) have a "natural" compactification $(Y, \widetilde{\omega})$ so that $\widetilde{\omega}|_{T^*X} = \omega$?*

Several commented that this question is a bit misleading, since T^*X has infinite volume. We may modify it to ask about the disc bundle in T^*X.

Nick Proudfoot pointed out that this is equivalent to asking whether or not X is a Lagrangian submanifold of some compact symplectic manifold.

Eugene Lerman noted that this is true for $X = S^3$, and is true more generally if X is a Riemannian manifold with a periodic geodesic flow: then we may "cut" T^*X with respect to the energy functional. For example, we may do this when $X = S^3$ or when X is a Zoll surface. Of course, if not all periods are the same, one may end up with an orbifold.

If X is a complex manifold, there is a natural S^1 action on the fibres; however, this action is not symplectic.

Allen Knutson commented about the case when X is a real algebraic variety. Then X is the real locus of $X(\mathbb{C})$, a complex algebraic variety. Let Y be a desingularization of the closure of $X(\mathbb{C})$ in projective space. Note that the singularities are all far from X. Then X still sits inside as a Lagrangian submanifold.

Eugene Lerman pointed out that we may take Y to be Thom space of T^*X or the one-point compactification. If we view this as a symplectic stratified space, X is a Lagrangian submanifold. This may not be "natural".

Markus Pflaum mentioned that a similar question was addressed in [21].

Circle actions and the Hard Lefschetz Property

Let (M, ω) be a $2n$-dimensional compact symplectic manifold. Consider the map

$$\begin{array}{rcl} L : H^i(M) & \to & H^{i+2}(M) \\ c & \mapsto & [\omega] \cup c. \end{array}$$

We say that M satisfies the *Hard Lefschetz property* if

$$L^k \colon H^{n-k}(M) \to H^{n+k}(M)$$

is an isomorphism for each $0 \leq k \leq n$.

Participants note: All compact Kähler manifolds satisfy Hard Lefschetz. Specifically, if M is a projective variety, then ω is the restriction of the Fubini-Study form on projective space, so the Kähler class is the Poincaré dual of a hyperplane section. So L is the intersection with this hyperplane section, and Hard Lefschetz holds.

Problem 8.0.2 (Y. Karshon) *Suppose that (M, ω) admits a Hamiltonian S^1 action with isolated fixed points. Does (M, ω) satisfy the Hard Lefschetz property?*

This problem has been around for at least 13-14 years; Yael isn't sure of its origin.

Reyer Sjamaar comments that his student Yi Lin has worked on a related question. Symplectic quotients often inherit nice properties from the original manifold: if the original manifold is Kähler, so is its symlectic quotient. Yi Lin has shown that symplectic quotients **do not** inherit the Hard Lefschetz property.

Nick Proudfoot asked why having an action should say anything about Hard Lefschetz. Yael Karshon replied that having a Hamiltonian action with isolated fixed points is a very strong assumption.

Reyer Sjamaar pointed out that, by a result of Susan Tolman and Jonathan Weitsman, if the S^1 action is in addition semi-free, then $H^*(M)$ is isomorphic as a ring to $H^*((P^1)^k)$. Under the isomorphism, $[\omega]$ maps to the class that is the product of Fubini-Study 2-forms and takes the first Chern class to the first Chern class. Thus the Hard Lefschetz property holds for these examples.

Nitu Kitchloo asked if it makes any difference if ω is integral. Then we may classify ω by a map to $\mathbb{C}P^\infty$. This gives a principle S^1 bundle P over M, and Hard Lefschetz is equivalent to $H^*(P)$ being a "very small" cohomology ring. This follows from the Leray-Serre spectral sequence for the cohomology of the total space.

Sue Tolman points out that an easier version of this problem is as follows.

Problem 8.0.3 (S. Tolman) *Are the Betti numbers of M unimodal? That is, do they satisfy*

$$\beta_1 \leq \beta_3 \leq \cdots \leq \beta_{half}$$

and

$$\beta_2 \leq \beta_4 \leq \cdots \leq \beta_{half}?$$

$\mathbb{Z}_2$-graded ("super") symplectic manifolds and reduction

Let M be a $\mathbb{Z}_2$-graded symplectic manifold (for a reference, see [20]). That is, M is locally a manifold, and over each open set U, we have a trivial bundle $E = V \times U$. The "functions" on U are $C^\infty(U) \otimes \Lambda^*(V)$. The "odd" variables live in "flat" directions corresponding to V ("ectoplasm has no topology!"). This is one way to define a super manifold. Extend this to global structure by patching. A **symplectic form** in this setting is anti-symmetric on the even (standard) directions and symmetric on the odd (V) directions.

Consider the case where the space of odd variable seems NOT flat. Take $M = pt$. Then we have only an odd vector space V and the "symplectic form" is a Euclidean metric (inner product). For $G \subseteq SO(V)$ acting, we can define a moment map. It seems that the "symplectic quotient" will not necessarily be a $\mathbb{Z}_2$-graded symplectic manifold in the above sense.

Now consider the "quantization," which is the space of functions on the manifold. This is the spinor representation $\mathbb{S}(V)$ of the Clifford algebra of V. Now restrict to the G-invariant part to "reduce" the "quantization."

Problem 8.0.4 (S. Wu) *What is the classical analogue of "reduction" so that quantization commutes with reduction? How may we generalize the concept of graded symplectic manifolds to include such examples?*

Problem 8.0.5 (S. Wu) *Give examples of mixed odd/even cases.*

A partial answer to this second question was given by Greg Landweber: coadjoint orbits of Lie supergroups fall into this situation.

Symplectic reduction and GIT quotients

Let M be a Kähler manifold and G a connected complex non-reductive affine algebraic group acting on M. Let K be the maximal compact, but note that $G \neq K_{\mathbb{C}}$. K acts on M by isometries.

For example, G could be the group of $n \times n$ invertible upper triangular matrices, and then we have K the compact torus $(U(1))^n$.

Problem 8.0.6 (A. Knutson) *Is there a notion of K-equivariant moment map*

$$\Phi : M \to G/K$$

so that the symplectic quotient of M by K is homeomorphic to the GIT quotient of M by G, when the GIT quotient makes sense?

Jonathan Weitsman commented that a reference might be M. Leingang's thesis, which contains a generalization of [2] to moment maps with values in symmetric spaces. However, this may be restricted to the case when G is reductive.

Allen Knutson continued that Problem 8.0.6 is perhaps most interesting when G is unipotent, and in this case, $K = 1$. So in this case, can we view the GIT quotient of M by G as a real symplectic submanifold of

M? Topologically, the stable set is a G-bundle, which topologically has a continuous section. In this case, topologically, the GIT quotient is a submanifold. Here the GIT quotient is a quotient $M^s \to M^s/G$, and since G is contractible, this fibration has a continuous section.

Reyer Sjamaar pointed out that if G is the maximal unipotent of a reductive group $\widetilde{G}$ which also acts on M, then this GIT quotient $M//G$ exists, and there is a nice choice of such a section $M^s \to M^s/G$. Namely, take the inverse image $\phi^{-1}(C)$, where C is a Weyl chamber for $\widetilde{G}$, and ϕ is a moment map for a compact real form of $\widetilde{G}$.

In a later discussion, Allen Knutson and Gideon Maschler found a natural answer at least to the question, “Is there a moment map?” The issue of existence of a quotient needs further exploration.

Ricci curvature and proper moment maps

Let M be a complete Kähler manifold equipped with a Hamiltonian isometric action of compact Lie group with compact fixed point set and moment map bounded in some direction. Generally the moment map is not proper.

Problem 8.0.7 (R. Bielawski) *If we assume that Ricci curvature is non-negative (or even zero), then is the moment map proper?*

EXAMPLE: (Nick Proudfoot) The circle S^1 acting on $\mathbb{C}^2 = \mathbb{C}_{(1)} \oplus \mathbb{C}_{(-1)}$ is a counterexample to the problem without the assumption that the moment map is bounded in some direction.
EXAMPLE: The circle S^1 acting on $\mathbb{C}^2 = \mathbb{C}_{(1)} \oplus \mathbb{C}_{(0)}$ is a counterexample to the problem without the assumption that the fixed point set is compact.

EXAMPLE: (Roger Bielawski) The statement fails without the Ricci curvature hypothesis:

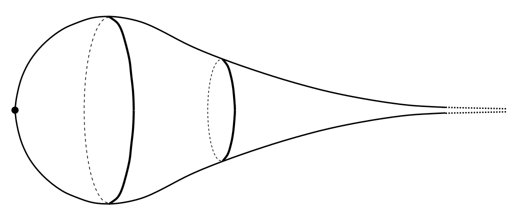

Figure 8.1: S^1-invariant complete Kähler metric on $\mathbb{C}$ with bounded moment map.

Symplectically, this is a disc. Since the volume of the manifold is finite, the moment is map bounded and so not proper as a map to $\mathbb{R}$. This can be done while making the metric complete.

Some partial results: the answer is yes (even without the curvature assumption), if the injectivity radius of M has a positive lower bound. The answer is also yes for circle actions such that the fixed point set F is connected and the cohomology class of the Kähler form restricted to F is a multiple of the first Chern class of F.

The motivation for this problem is the following. Given a real analytic compact Kähler manifold M, there exists a unique hyper-Kähler metric on a neighborhood of the manifold M in its cotangent bundle T^*M (due to Feix and independently to Kaledin). This extends the given metric, and the standard $G = S^1$-action (on the fibres) is isometric and Hamiltonian. The holomorphic symplectic form on T^*M comes from the standard symplectic form on M. The fixed point set of this action is M, the moment map is bounded below, and the Ricci curvature is zero. In general, we know very little about completeness of these metrics.

If the moment map is proper, then M must have NEF tangent bundle. Up to the Campana-Peternell conjecture in algebraic geometry, this implies that if M is projective, then M fibers over its Albanese variety with rational homogeneous fibers.

Proving the above statement would provide a necessary condition for completeness of the metric on T^*M.

Topology of the symplectomorphism group

Suppose (M, ω) is a compact symplectic manifold, and that the Chern class $c_1(M) \in H^2(M; \mathbb{Z})$ is a negative (or non-positive) multiple of $[\omega] \in H^2(M; \mathbb{R})$.

According to Sue Tolman, this implies that there are no Hamiltonian S^1 actions on M. The idea of the proof is to look at the maximum and minimum of the $\mathbb{R}$-valued moment map. The S^1 equivariant cohomology of a point consists of weights, so it makes sesnse to describe them as positive and negative. The restriction of the equivariant first Chern class c_1 to the maximum fixed point set must be negative, and at the minimum the restriction is positive. This restriction of c_1 is the sum of the isotropy weights.

Problem 8.0.8 (M. Abreu) *When c_1 is a non-positive multiple of the class of the symplectic form, is the group of Hamiltonian diffeomorphisms, $Ham(M)$, contractible?*

Problem 8.0.9 (M. Abreu) *When c_1 is a negative multiple of the class of the symplectic form, is the identity component of the group of symplectic diffeomorphisms, $Symp_0(M)$, contractible?*

The motivation here is that, under these hypotheses and according to the above argument of Sue Tolman, $Ham(M)$ has no compact subgroups. One would believe that any topology of $Ham(M)$ is related to some compact subgroup. The torus T^{2n}, with curvature $c_1 = 0$, motivates the two different statements for the problem.

Note that for surfaces Σ, we have the following cases:

- When $\Sigma = S^2$, $c_1 > 0$ and $Ham(M)$ is not contractible, in fact it is homotopy equivalent to $SO(3)$;
- When $\Sigma = T^2$, $c_1 = 0$ and $Ham(\Sigma)$ is contractible; and
- When $\Sigma = \Sigma_g$ has genus $g > 1$, then $c_1 < 0$ and $Symp_0(\Sigma)$ is contractible.

Thus, for surfaces, the statements hold.

A related problem is the following.

Problem 8.0.10 (M. Abreu) *Is the group of compactly supported symplectomorphisms of $\mathbb{R}^{2n}$ contractible?*

Smale answered this question in the affirmative for $n = 1$, and Gromov proved the result for $n = 2$.

Sasaki-Einstein metrics

Recently the physicists, Gauntlett, Martelli, Sparks, and Waldram have constructed explicit Sasakian-Einstein metrics on $S^2 \times S^3$. These even include irregular Sasakian-Einstein metrics, where the flow of the Reeb vector field has non-closed orbits. They are the first examples of such metrics and actually give counterexamples to a conjecture of Cheeger and Tian. The metrics are related to local Kähler-Einstein metrics found in the late 1980's by Page and Pope, and generalize to higher dimensions. It was then shown by Martelli and Sparks that these Sasakian-Einstein metrics are related to toric contact geometry. It turns out that for a certain choice of contact form, the characteristic foliation is regular and the base space is a Hirzebruch surface, and for another choice of contact 1-form one gets the Sasakian-Einstein metrics.

Problem 8.0.11 (C. Boyer) *Is it possible to develop a general theory of these structures?*

Boyer believes that such Sasakian-Einstein metrics should exist on the k-fold connected sums of $S^2 \times S^3$, but currently there is little hope of getting explicit metrics. One must prove existence theorems. This is the hard part as there are some real subtleties. First the regular contact structure over Hirzebruch surfaces does not give positive Ricci curvature, because generally Hirzebruch surfaces are not Fano. For quasi-regular contact structures, this can be overcome using certain branch divisors to shift the orbifold canonical divisor to be Fano. Boyer does not yet understand how this works in the irregular case though. Given this, the techniques that we have been using to prove the existence of Sasakian-Einstein metrics do not work here. The singularities of the pair $(variety, orbifold\ anticanoncal\ divisor)$ are not Kawamata log terminal.

Apostolov mentioned a recent paper [26] where Wang and Zhu prove that Kähler-Einstein metrics exist on toric Fano manifolds if and only if the Futaki invariant vanishes. Thus, the program is to generalize the Futaki type invariants to the Sasakian setting. Hopefully one can describe these Sasakian Futaki invariants as functions of the weight vector one gets by writing an arbitrary Reeb vector as a linear combination of a basis for the Lie algebra of the torus.

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Chapter 9

Critical Scaling for Polymers and Percolation (05w5025)

May 28 – June 2, 2005

Organizer(s): David Brydges (University of British Columbia), Jennifer Chayes (Microsoft Research), Gordon Slade (University of British Columbia)

Background

Equilibrium statistical mechanics is the mathematical framework created by Gibbs for predicting macroscopic properties of matter from a microscopic description. Within this framework thermodynamic functions of state such as *temperature* and *entropy* are defined in such a way as to satisfy the laws of thermodynamics. In principle the formalism determines the model specific relation called the equation of state. The first example of such a prediction is the famous ideal gas law $P = (n/V)RT$ relating pressure P to the number of atoms/volume and temperature T. This is the equation of state for an assembly of non-interacting particles. This equation of state is notable for having no singularities: in the physical domain $T > 0$ the pressure is a smooth function of temperature. However interacting systems will not in general relate thermodynamic variables in a smooth way and therefore the equation of state has singularities which reflect phase transitions. For example the density of water changes discontinuously as a function of temperature at the boiling point. In practice the determination of the complete equation of state is not realistic for systems with interactions, but the nature of the singularities, the exponents of power law divergences at these singularities, are more accessible. Thus these singularities have been the focus of research.

The basic setup for equilibrium statistics is a probability space Ω whose points $\omega \in \Omega$ are possible configurations of the physical system and a probability measure called the *Gibbs* measure which has the form

$$\frac{1}{Z}e^{-\beta H(\omega)} \times \text{Uniform Measure}$$

where β is proportional to the inverse temperature and H is the energy of the system in configuration ω. Z normalises the Gibbs measure so that it is a probability measure. Z is called the partition function (of β and parameters in H).

In the context of polymer physics we are interested in the thermodynamic properties of large molecules called *polymers* formed by atoms or smaller molecules arranged in a chain or other topologies such as trees (*branched polymers*) or rings. The subunits from which the polymer is formed are called *monomers*.

The goal of the physical chemist is to predict the properties of the polymer starting with knowledge of the forces acting between the monomers. The desired properties include the average size of the polymer as a function of the number of monomers in the polymer, when the number is large. Size is measured by diameter or, in the case of chains, end-to-end distance. This size can exhibit discontinuities when parameters such as temperature are varied. In other words a long chain molecule such as DNA can suddenly change its conformation and therefore its size. These discontinuities are called *theta phase transitions*.

The medium in which the polymer is immersed is called the *solvent*. Instead of starting with detailed information on the solvent molecules and the forces that act between them and the monomer it is common to leave out any explicit description of the solvent and suppose that the forces experienced by the monomers have been adjusted to take into account the effect of the solvent. In physics this is expressed by the words "the interactions are equivalent to a simplified model with an *effective interaction*". The idea is that a complicated model may have a large scale structure that is the same as a simpler model and we may hope to classify all models into equivalence classes labeled by these simpler models. For example, consider a polymer modeled as a simple random walk: there is a first monomer at $X_0 = 0$ and then the next one in the chain occupies a position X_1 randomly chosen near X_0 according to some probability density p and then the one after that X_2 is chosen independently according to the same density but centred on X_1, ..., i.e. we have a Markov chain $X_0, X_1, \ldots$ of random variables whose law satisfies $P(X_{i+1} = x | X_i = y) = p(x - y)$. The density $p(x - y)$ is likely to be very complicated, being determined by microscopic chemistry. However the theorem of Donsker [16] tells us that if we scale the random variables so as to see only the large scale structure by looking at the chain from far away,

$$Y_t = \lim_{T \to \infty} T^{-\nu} X_{\lfloor Tt \rfloor}, \quad \nu = \frac{1}{2},$$

then the form of p is not important. All that will matter is the matrix of second moments of p. This is called taking the *scaling limit*. No matter how we choose p the scaling limit will be a continuous random path $Y(t)$ called Brownian motion. The probability law of Brownian motion is completely determined by the matrix of second moments. All different choices of p with the same second moments give rise to the same scaling limit. One can take this a step further and show that there is a direction-dependent scaling such that the scaling limit is isotropic, i.e., standard Brownian motion. In this model we see a good example of the notion of a *critical exponent*, namely ν. The existence of the scaling limit for $\nu = 1/2$ implies that the typical random walk with n monomers will have an end-to-end distance of $O(n^{1/2})$.

The Donsker theorem is an ultimate version of the Central Limit Theorem, but the theory of scaling limits starts where the central limit theorem ends. For example, modeling a polymer by simple random walk is rather optimistic: we know that different monomers cannot occupy the same position so we ought at least to consider that, and adopt as a better model, a *self avoiding walk*. More generally we should consider attractive and repulsive interactions. The simple random walk model is the analogue of the ideal gas mentioned in the first paragraph. Once we introduce interactions the existence of a scaling limit for some ν is still largely Terra Incognita. We think the scaling limits exist because we can prove they do in a small number of cases and because this belief now permeates theoretical physics. Almost all theoretical physicists work on models which are vast simplifications of reality. They do so because they think their models classify the large scale structures of reality. The long term health of their enterprise will be improved if we succeed in adding to the list of cases where this can be proved to be true. As with any hard problem, the struggle is spinning off new developments in mathematics and has formed a nice community of researchers with very different backgrounds.

Self-avoiding walk is the archetypical problem that embodies a combinatorial aspect of polymer physics. In this model we generally start with a simple cubic lattice $\mathbb{Z}^d$, hoping that the scaling limit will make the lattice invisible. The points in $\mathbb{Z}^d$ are called sites and they represent the possible positions of a monomer. A long chain molecule consisting of n monomers is represented as a sequence $\omega = (\omega_0, \ldots, \omega_n)$. We define the energy $H(\omega)$ of the polymer ω to be infinite if ω has a self-intersection and zero otherwise. This reduces the role of β in the Gibbs measure to cases $\beta = 0$, in which case the Gibbs measure is the standard measure on simple random walk and $\beta > 0$, in which case the Gibbs measure is the uniform measure on the subset of Ω consisting of self-avoiding walks (SAW). These are the sequences $\omega = (\omega_0, \ldots, \omega_n)$ consisting of *distinct* nearest neighbour sites. Thus the temperature appears in a trivial way in this model. The fundamental question for this model is what is the right scaling: does there exist ν such that the scaling limit exists and what is that limit? How does the end-to-end distance of self-avoiding walk grow as a function of n?

Percolation is the archetypical model for a phase transition. We again start with the simple cubic lattice $\mathbb{Z}^d$. An unordered pair $\{x, y\}$ of nearest neighbour sites is called an edge. Each edge can be either *open* or *closed*. Think of open as meaning that fluid can pass from x to y and closed as meaning that it cannot.

A configuration ω of percolation is a possible choice of open/ closed for every edge. We make this choice independently for each edge; edge $\{x, y\}$ is open with probability p and closed with probability $1 - p$. In this model the connection to Gibbs measures is not apparent but there is a connection called the $Fortuin - Kasteleyn$ representation which will not be discussed here. p plays the role of temperature as follows: the sites of the lattice fall into clusters connected by open edges. One can ask whether there is an infinitely large cluster. For $d > 1$ there is a critical probability $p_c(d)$ such that for $p > p_c$ there is an infinitely large cluster whereas for $p < p_c$ all clusters are finite. Thus the probability that the origin lies in an infinite cluster is zero for an interval $p \in [0, p_c)$ and non-zero for $p \in (p_c, 1]$.[1] The outstanding open questions concern the existence and values of critical exponents. An example of a critical exponent is the expected size of the cluster containing the origin as a function $(p - p_c)^\beta$ for $p > p_c$. In $d = 3$ dimensions the existence of ν and β has not been proved.

One may wonder what this model has to do with polymers. At the outset it was a separate subject but now both models are slowly becoming united in a larger framework of random geometry and there is a commonality of concepts and techniques.

Scaling limits in two dimensions

The 1984 paper by Belavin et al. [8] theory started two decades of progress by theoretical physicists in two dimensional statistical mechanics based on conformal field theory (CFT) and more recently also string theory and quantum gravity. In addition there are exact but non-rigorous solutions to lattice models based on the Bethe Ansatz and Yang Baxter equations. We conflate these subjects under the initials CFT.

The book [15] provides a good review: the range of statistical mechanical models for which critical exponents can be calculated (in advance of knowing if they exist!) is remarkable. Mathematicians have yet to find their Euclid for CFT and so they have to regard these calculations as conjectures. There are partial axiomatic programs, for example [18].

The methods of conformal field theory give information on correlations but less directly on random geometry. The family of stochastic processes SLE$_\kappa$ studied in the work of Lawler Schramm and Werner over the last five years describes the geometric objects within statistical mechanical models. For example, as reviewed in the lecture by Lawler and in [34], the distribution on simple random curves prescribed by SLE(8/3) is the only possible scaling limit for SAW if the scaling limit exists and is conformally invariant. SLE$_\kappa$ is effective for calculations and some scaling exponents such as ν have been verified to be equal to the values provided by Nienhuis [39] by CFT (actually by first mapping to Solid on Solid models). Thus there is still a very hard open question to show that the discrete process approaches a conformally invariant limit. Another fundamental question is to give the ”correct” parametrization of the path which would correspond to the limit of the natural discrete parametrization. This was discussed at this meeting by Lawler and in detail by Kennedy who has examined it by Monte Carlo simulations. Here is a brief summary of Kennedy's lecture.

“SLE$_{8/3}$ is believed to describe the scaling limit of the two-dimensional self-avoiding walk. However, these two processes have different natural parameterizations. SLE is parameterized by the capacity of the curve, and the length of the SAW leads to a natural parameterization in the scaling limit. One can reparameterize the SAW using its capacity. Monte Carlo simulations were presented which indicate that with this parameterization the SAW process agrees with the SLE process. A more interesting question is to find the parameterization of the SLE that would make it agree with the SAW with its natural parameterization. Lawler gave several properties such a parameterization should have - local dependence on the curve, additivity and an appropriate transformation property under conformal maps which reflects the Hausdorff dimension of the curves. A possible candidate is what probabilists call the “pth variation” where p is taken to be $1/\nu$. Monte Carlo simulations presented showed that while this parameterization makes the SLE agree with the SAW for one random variable, for another random variable there is roughly 6% discrepancy. Understanding the source of this difference is an important open problem for future simulations. Several lines of attack have been developed as a result of conversations with other participants at the conference.”

For percolation the geometrical object is the boundary of a percolation cluster for the critical model. Recalling that the critical clusters in percolation are known to be finite in dimension two one needs a method

[1] In dimension $d = 2$ and in very large dimensions it is known that there is no infinite cluster at p_c.

to constrain them to be as large as the scaling limit scale L as the scaling limit $L \to \infty$ is taken. This can be achieved by by using boundary conditions that force a percolation boundary to pass from one side a region of scale L to the other. The scaling limit of the resulting boundary curve has been identified with SLE_6. Remarkably, existence and conformality in the scaling limit was established by Smirnov, not for the whole model, but for expectations of a specific crossing probability [45].

In this conference Camia gave a report on his work with Newman in which the full scaling limit and conformality of percolation has been established. The crucial point is that this work considers the set of all interfaces as opposed to one forced by boundary conditions. This is fundamental progress because one wants to make contact with the methods of CFT which are based on random fields and one wants to identify the interfaces with contours of the random field. Here is a brief summary of the talk by Camia.

"In my talk, I discussed some aspects of the convergence of critical percolation interfaces to their continuum scaling limits, following a recent joint paper with Charles M. Newman [14]. More specifically, I looked at the "percolation exploration path," conjectured by Oded Schramm to converge to SLE(6) and used by Stanislav Smirnov and Wendelin Werner to rigorously obtain various critical exponents for percolation, and at the set of all percolation interfaces. Percolation is so far the only model for which one can go beyond a single interface and prove the scaling limit of the set of all interfaces. This gives rise to a "full" scaling limit in terms of fractal, continuous loops in the plane. Similar objects should arise when taking the full scaling limit of other models, like Ising and Potts models, and should be described by conformal loop ensembles (CLE) as described by Scott Sheffield and Wendelin Werner in their talks. Such relations, for models other than percolation, are still conjectural. In the case of percolation, the full scaling limit was first constructed in a joint paper with C.M. Newman [13].

The field is rapidly moving forward, and various talks at the meeting showed that the understanding of the continuum counterpart of various discrete models is reaching its maturity. There is hope that this will lead in the future to proofs that the beautiful continuum objects described by Sheffield and Wendelin are indeed the continuum scaling limits of discrete models, extending the results known for percolation to Ising, Potts and O(N) models."

The importance of making complete contact with CFT is illustrated by "Duplantier duality". Using CFT Duplantier noticed that, in the scaling limit, for a spin model or percolation, there must be a relation between the SLE that describes the boundary of a cluster and the SLE that describes the outer boundary of a cluster, namely Duplantier duality $\kappa \to 16/\kappa$. The continuing inspiration coming from these lines of thought is evident in this summary of his lecture.

"I presented a unified heuristic point of view on the Stochastic Loewner Evolution (SLE). It consisted in relating critical exponents for conformally invariant random paths in the plane to similar ones on a random surface with fluctuating metric. The key ingredient was the so-called Knizhnik, Polyakov, and Zamolodchikov (KPZ) relation between these exponents. The status of this relation is to be considered as true in theoretical physics, but conjectural from the point of view of rigorous mathematics. The machinery it provides, however, is strikingly efficient. Any exponent from SLE can be predicted this way. This was illustrated in several instances:

The duality $\kappa \to 16/\kappa$ which is believed to map hulls of SLEs to their external boundaries is reflected by a similar duality built in the two analytical determinations of the inverse KPZ map.

The pressure effect on an SLE path coming from a drift term of strength ρ in the Brownian source of the Loewner equation of that path (the $SLE(\kappa, \rho)$), can be analyzed through the KPZ relation in terms of an equivalent number of multiple SLEs, or of a certain equivalent number of Brownian paths.

The "shadow exponents" describing the probability that some Brownian paths screen some others from the exterior can also be calculated systematically in terms of these Quantum Gravity equivalences.

The multifractal harmonic and rotational spectra of the SLE curves have been obtained from that approach.

A challenge is now to establish the proper rigorous form of this fundamental tool coming from conformal field theory."

And here is a summary of Cardy's lecture which also underlines the need for a complete understanding of CFT.

"There has been a very fruitful interdisciplinary connection formed between the study of critical behaviour by theoretical physicists and the approach to random spatial processes of probabilists. In recent years this has been brought to the fore by the spectacular progress by mathematicians using ideas such as SLE, which have reinterpreted and put on a more systematic basis the earlier results of the physicists in conformal field theory, inspired by ideas which were born in string theory. Now this subject is reaching a point where there is a mutual flow back and forth between the SLE ideas and CFT. My talk gave a very simple example of this interdisciplinary thinking, concentrating on a small result which, however, both illuminates what should be the correct extension of SLE to many random curves, and also on the physics side relates to potentially measurable phenomena in the quantum Hall effect."

Conformal Field Theory is the study of correlation functions for a random field on a Riemann surface. The simplest example is called the (massless) Gaussian field on the complex plane $\mathbb{C}$. This may also be the fundamental example since a method called the Coulomb gas representation is used to write correlations of other CFT's in terms of the Gaussian field. One immediately discovers that the Gaussian free field ϕ is not actually a random field, but instead is a generalized random function. This means that for each test function f on the plane there is a random variable $\phi(f)$ which would be given by $\phi(f) = \int \phi(x) f(x)\, dx$ except that $\phi(x)$ does not exist because there is too much oscillation at arbitrary small scales. This would seem to be a major obstacle to making a connection between SLE and the free field based on the idea mentioned above: that scaling limits of interfaces should be contours of a conformal field. Thus we were excited by the lectures of Sheffield and Werner in which contours of the free field were defined and related to variants of SLE_κ. Here is summary of the lecture given by Werner who gave a talk based on joint work with Scott Sheffield.

"Motivated by identifying and understanding better the possible conformally invariant scaling limits of various 2-dimensional models such as O(N) models or the Ising model, we define a natural property that these continuous limits should satisfy:

We are considering random collections of disjoint simple non-nested loops in a domain D. A sample is therefore a collection of loops $(\gamma_j, j \in J)$. We assume conformal invariance so that one can define for any simply connected domain such a law P_D (in a conformally invariant way).

Suppose now that $D' \subset D$. Then, one can define two sets of loops: Those that stay in D' (for which we say that $j \in \widetilde{J}$) and those that exit D' (for which $j \in I$). One defines $\widetilde{D} = D' \setminus \cup_{j\in I} \gamma_j$. Roughly speaking the condition is that conditionally on $\widetilde{D}$, the law of $(\gamma_j, j \in \widetilde{J})$ is $P_{\widetilde{D}}$.

If this is true for all D', then we say that (γ_j) is a conformal loop ensemble.

We study various properties of these loop-ensembles. In particular, we show that

1) The outer boundaries of loop-soup clusters (related to the Brownian loop-soup introduced in joint work with Greg Lawler) are examples of such loop-ensembles. These examples are parametrized by the intensity of the loop-soup c. This works for all $c \le c_0$ where c_0 is a critical intensity.

2) In fact these are the only conformal loop-ensembles.

3) The level-lines of the Gaussian Free Field studied by Schramm and Sheffield are other examples of such conformal loop-ensembles and they coincide with the c_0 case in 1).

4) One can construct all these conformal loop-ensembles via SLE related processes, the SLE$(\kappa, \kappa - 6)$ processes and the relation between κ and c is $c = (3\kappa - 8)(6 - \kappa)/2\kappa$ where $\kappa \in (8/3, 4]$".

High Dimensions

For dimensions above two nothing is known about the scaling limits of self-avoiding walk and percolation until one gets to the *critical dimension*, which is four in the case of self-avoiding walk and six in the case of percolation. Above the critical dimension there is a method called the *Lace Expansion* which one can read about in the lecture notes of Slade [44]. The range of usefulness of this tool has expanded very greatly from the original application [12] to a variant of self-avoiding walk. In this conference Sakai reported on a first time application to the Ising model. The significance of this application is that it provides a way to prove "universality" which is the conjecture that drives much of theoretical physics: that the scaling limit is independent of details of the local interactions: refer to the discussion at the beginning of Donsker's theorem. Sakai's success should encourage us to look for ways to extend it to other spin models. Here is a summary of the lecture by Sakai.

"In this talk, I describe a lace expansion for the Ising model, which can be applied to prove Gaussian infrared asymptotics for the critical two-point function for Ising ferromagnets above four dimensions, assuming that the dimension d or the range of the spin-spin coupling is sufficiently large [42, 43]. As a consequence, the other observables also exhibit the mean-field behavior for $d > 4$ [1, 3, 4, 5]. The main point is that the proof of these results does not require the reflection positivity [17].

For reflection-positive models, it is known that the two-point function obeys a Gaussian infrared bound for $d > 2$. Although the nearest-neighbor model satisfies the reflection positivity, other finite-range models (e.g., the next-nearest-neighbor model) do not. Since local details of the models should not affect the critical behavior (i.e., universality), all these finite-range models must exhibit the same mean-field behavior for $d > 4$, no matter whether the reflection positivity holds or does not. Therefore, our approach using the lace expansion is more robust.

The lace expansion has been used for stochastic-geometrical models, such as self-avoiding walk (e.g., [12, 21]), lattice trees and lattice animals (e.g., [20]), percolation (e.g., [19]), oriented percolation (e.g., [38]) and the contact process (e.g., [41]), to prove a Gaussian infrared bound or asymptotics of the critical two-point function above the upper-critical dimension. The lace expansion gives rise to a recursion equation similar to the one for the random-walk Green's function, and this is the foundation of the Gaussian behavior for the two-point function. The lace expansion for the Ising model has just been proved for the first time [42]."

Applications of the Lace expansion to percolation are also in the news. The important development is the analysis of a cluster constrained to be very large. The idea is that in critical percolation no cluster is infinite but the slightest increase in the density of occupied bonds will cause the critical clusters to link up into an infinite cluster. In some sense one can therefore see the infinite cluster before it has appeared. Here is a summary of the lecture of van de Hofstad based on joint work with Frank den Hollander, Antal Járai and Gordon Slade.

"The incipient infinite cluster (IIC) describes the infinite cluster which is on the verge of arising in critical percolation models. Kesten [31] first constructed the incipient infinite cluster for two-dimensional percolation. Kesten's IIC describes the infinite cluster which is on the verge of appearing at the critical value, and is constructed by conditioning the origin to be connected to infinity by an appropriate limiting procedure. This construction of the IIC is different in spirit as the one suggested by Aizenman in [2], which is closely related to the scaling limit and the behaviour of all critical clusters in a large cube simultaneously, and is also studied in [22, 23] in the high-dimensional case.

We discuss Kesten's results in two dimensions, as well as the extension by Járai [27, 28] for the two-dimensional case and give some of its properties, such as its dimension and its backbone dimension, which follow from the connection to SLE_6 (see [31, 33]).

We also present the constructions of Kesten's IIC for percolation above the upper critical dimension. We will give 2 different constructions for the IIC for sufficiently spread-out percolation above six dimensions, and 3 different constructions for the sufficiently spread-out oriented percolation IIC above 4+1 dimensions. We will also discuss properties of the IIC, such as its dimension and, in the oriented case, its scaling limit. The high-dimensional results are taken from the papers [24, 26, 25].

One reason to study Kesten's IIC is that it is the natural context for a random walk on a critical cluster. Random walk on a super-critical cluster is expected to converge to Brownian motion, which is recently proved by Berger and Biskup (see the talk by Marek Biskup). Kesten [32] studied the random walk on the two-dimensional IIC, and proved that it is subdiffusive. He also proved that a random walk of n-steps on a branching process cluster scales like $n^{1/3}$, which suggests that a random walk on a critical branching random walk cluster conditioned to survive forever, has displacement of order $n^{1/6}$. Since in high-dimensions, Kesten's IIC has similar scaling properties as critical branching random walk cluster conditioned to survive forever, this suggests that a random walk on the IIC has displacement $n^{1/6}$ after n-steps. The latter problem is still open."

There are at least three further developments one should hope for. (1) The Lace expansion applies to models whose interactions are repulsive. A self-avoiding walk with a small nearest-neighbour attraction is not amenable to the Lace expansion. The best attempt so far is [48]. (2) The critical dimension models are not accessible to the Lace expansion. The Renormalisation Group is likely to be the key to progress on these models. You can see a start in this direction at [10, 11]. (3) Stochastic models such as *True Self-Avoiding Walk* are almost Terra Incognita in dimensions greater than one. A good starting point is to develop more understanding of random walk in random environments. Here is a summary of the lecture by Biskup on this topic.

"Random walk in random environment is a subject of considerable interest in probability community. One particular setting concerns simple random walk on supercritical percolation cluster. In 2003, Sidoravicius and Sznitman proved that, in dimensions four and higher, such walk scales to Brownian motion under the usual diffusive scaling of space and time. Their proof uses heavily the path-transience of simple random walk in high dimensions and, as such, it does not seem to be generalizable to include the "hard" dimensions $d = 2$ and 3. In my talk, I have described the recent result – obtained jointly with Noam Berger – that establishes the invariance principle for such walks in any dimension $d > 1$. The principal idea of our proof is to consider the embedding of the percolation cluster into R^d that makes the random walk an L^2 martingale".

Random walk in random environment gives an opening into the study of walks which have attractive self interaction because integrating over the environment constructs a self interaction. A particularly nice example is provided by the self-reinforced walk discussed in the lecture by Rolles. Here is a summary of her lecture. In her third paragraph she is referring to the fact that this model is equivalent to a random walk over an environment which has been integrated out.

"Linearly edge-reinforced random walk was introduced by Diaconis in 1986. Diaconis asked whether edge-reinforced random walk on $\mathbb{Z}^d$ is recurrent or transient. As usually, the random walk is called recurrent if almost all paths visit all vertices infinitely often. For all dimensions $d \geq 2$, this question is still open.

Recently, progress has been made in studying the edge-reinforced random walk on ladders $\mathbb{Z}\times\{1, 2, \ldots, d\}$ and, more generally, on graphs of the form $\mathbb{Z} \times T$ with a finite tree T: For large constant initial weights, recurrence was proved by Merkl and Rolles in [36] and [40]. A more detailed analysis was obtained in [37]. There, it is shown that the edge-reinforced random walk on infinite ladders has the same distribution as a random walk in a random environment given by spatially decaying random edge weights. Convergence theorems and estimates for the position of the random walker at large times are given.

A crucial tool in the analysis is a representation of the edge-reinforced random walk on finite graphs as a mixture of reversible Markov chains; see e.g. [29]. Transfer operator techniques are used to analyze the random environment."

Other models

Fortunately we are not constrained to work on the archetypal problems, which are not always making good progress. The archetypal problems arose by distilling challenges from other subjects to their simplest level. New archetypes and fresh ideas will arise from problems such as the localisation problems discussed by Whittington and den Hollander. Here is a summary of their talks.

"A random copolymer is a copolymer in which the sequence of monomers is determined by a random process and is then quenched. Suppose we have two immiscible liquids, A and B, and suppose that one type of monomer prefers to be in the A phase and the other prefers to be in the B phase. For instance, think of the two monomer types as being hydrophobic and hydrophillic and the two liquids as oil and water. Depending on the chosen parameters (e.g., temperature or relative interaction strength) the polymer can localize at the interface so as to optimize the numbers of monomers in their preferred phases, or delocalise into one of the bulk phases to optimize the entropy of the system. One has a choice as to how the configurational properties of the polymer are modeled (e.g., as directed or undirected self-avoiding walks, with possibly other appropriate restrictions). Similarly there is some choice about the details of the interaction Hamiltonian.

The aim is to establish the existence of a phase transition in the system and to study the properties of the phase transition curve. This can be done either at the thermodynamic level or at the level of path properties. Most work has focused on localization at a single infinite flat interface but there has been recent interest and progress in multi-interface and random interface problems, e.g., as models of polymers in an emulsion.

An early paper on the directed walk model is Bolthausen and den Hollander, Localization transition for a polymer near an interface, [9], and some results on the self-avoiding walk model can be found in Madras and Whittington, Localization of a random copolymer at an interface, [35]. For a recent review see Soteros and Whittington, Statistical mechanics of random copolymers, [46]. There are important recent results on the single interface problem by G. Giacomin, and work in progress on random interfaces by den Hollander and Whittington."

Jarai introduced us to self-organised criticality and the relations it has with models we already study such as the uniform spanning tree. Here is a summary of his lecture.

"In the last 15 years, a lot of attention was devoted in the physics literature to so called self-organized critical (SOC) systems. In these systems critical scaling appears with a somewhat different flavour than in the well-known examples of percolation and the Ising model. Namely, criticality is generated by a highly non-local dynamics that is a result of a separation of scales, rather than a parameter passing through a critical point. The main challenge in the area is to develop rigorous methods to study SOC. For some SOC models there is a close connection with a corresponding classical critical model, which opens up the possibility to study SOC via these connections. For example Chayes, Chayes and Newman have shown the intimate relationship between invasion percolation and critical percolation. Dhar and Majumdar have established a mapping between the Abelian sandpile model and the uniform spanning tree, which is the $q \to 0$ limit of the Potts model. In a rigorous approach to SOC, one of the first problems is to establish the existence of infinite volume limits, which can be difficult due to the non-local interaction present. In this regard, recent progress has been made for the Abelian sandpile model based on the connection with spanning trees in dimensions $d > 4$. Future challenges include extending these to lower dimensions. It is expected that in d = 2 conformal invariance plays a role, and therefore a description using SLE is to be explored".

¿From the world of Biology we have a lecture on Vesicles by Buks van Rensburg.

"Vesicles in the biological world, such as blood cells, are known to posses a number of different phases. For example, red blood cells, which are normally shaped as an indented disk (also called "discocytes") become sickled shaped in individuals with cickle cell anemia. In addition, red blood cells could be burred, pinched, pointed, indented, berry-shaped, etc., in either normal or pathological conditions depending on on factors such as dehydration and membrane properties. Vesicles appear to have a very rich phase diagram, and some of these phases can be examined by building mathematical models of a vesicle.

Perhaps the simplest approach to a vesicle would be a two or three dimensional discrete vesicle in the square or cubical lattices. These lattice vesicles include models such as square, partition and convex polygon vesicles in two dimensions, and cubical, rectangular and plane partition models of vesicles in three dimensions. Curvature and osmotic pressure terms van be build into the models by defining partition and generating functions with activities conjugate to volume, area or perimeter. From a mathematical point of view these models offer considerable challenges in combinatorics and in statistical mechanics, and tricritical scaling appears to be the appropriate physical frame for a mathematical description.

In my talk I will present some results on models of three dimensional cubical and rectangular vesicles in a perimeter-area-volume ensemble. These models have a multicritical point and the values of scaling exponents around this point is determined by the asymptotic analysis of the generating functions in both cases. For example, in the volume-area ensemble, the crossover exponent ϕ has value $2/3$ in both these models, and this transition is an inflation-deflation transition between vesicles which are cubical or square shaped".

Finally there was an introduction to one of the outstanding problems of condensed matter physics: to prove that adding a small density of randomly placed impurities to a conductor does not make it an insulator, at least in dimensions three or more. The first first fundamental issue is whether a Schroedinger operator with a random potential has absolutely continuous spectrum. Here is a summary of the lecture by Aizenman on joint work with Robert Sims and Simone Warzel.

"Spectral and dynamical properties of linear operators with extensive disorder are of interest for condensed matter physics, as well as the broad subject of mathematics of graphs and related structures.

An outstanding challenge in the field is to shed light on the existence of extended states in the presence of disorder. Such states play a basic role in conduction properties of an "electron gas" and in the spreading of vibrations in randomized systems. The converse of conduction is Anderson localization which was proven to occur at high disorder, and is particularly pronounced in low dimensions [47], in particular at d=1 where it precludes extended states even at weak levels of the disorder.

The models discussed start from a well familiar Laplacian, or the incidence matrix on a regular graph, which is then modified by incorporating the effect of disorder, represented by random terms with a homogeneous distribution and a control parameter. Specific examples are provided by:

i the discrete Schrdinger operator with a random potential, and

ii the Laplacian acting on functions supported along the edges of a graph, whose edge length are stretched by random factors.

Other models are not hard to formulate; the obstacles encountered in their analysis are similar. The challenge is to prove that if the dimension is high enough, and the disorder not too strong, then the spectral measures associate with the action of the random operators on local functions have an absolutely continuous (ac) component. Such delocalised states associated with ac spectrum are expected to occur in dimensions $d > 2$. The issues under study resemble a quantum version of percolation, in that it concerns conduction, or the spread of correlations, connectivity, etc., occurring through a local mechanism but measured at a distance. A common experience is that moderate dimensions, like $d = 3$, are out of reach of mathematics. For present case the spectral analysis has been most effective in dimension $d = 1$, where no extended states exist at any strength of the disorder, and at the opposite extreme of the tree graphs, which in a sense represent the case of infinite dimension. In fact, the latter situation is the only for which existence of delocalised states has been established; a proof due to A. Klein [30]. The talk presented a new method for establishing the persistence of ac spectra on tree graphs [6, 7].

The main result is the continuity of the Lebesgue measure of the ac spectrum as the disorder is turned on. The proof makes an essential use of the study of the fluctuations of the relevant Green functions, which are bounded through a non-linear recursion relation which these obey on a tree. Useful input is obtained from the analytical theory of the Lyapunov exponent. The new criterion for the continuity of the ac spectrum has been applied also to the case of radial quasi-periodic potentials. These allow to draw an instructive contrast between the effects of radially correlated disorder versus one which is weakly correlated among different tree branches."

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Chapter 10

Mathematical Issues in Molecular Dynamics (05w5052)

June 4–9, 2005

Organizer(s): Robert D. Skeel (Purdue University), Paul Tupper(McGill University)

Introduction

Computations for molecular systems work with a set of values Γ representing the microscopic state of the systems, e.g., atomic positions, atomic velocities, volume of the simulation box. Most such computations can be classified into one of two categories:

1. Sampling. Given the relative probability for different values of Γ, calculate the expectation of some observable $O(\Gamma)$. An example is the probability of two biomolecules being (noncovalently) bound versus unbound.

2. Dynamics. Given the relative probability for initial values $\Gamma(0)$ and equations of motion for $(\mathrm{d}/\mathrm{d}t)\Gamma(t)$, calculate the expectation of $O(\Gamma(t))$. An example is the position autocorrelation function. The equations of motion may be deterministic or stochastic.

The probability for different values of Γ is often specified by the ensemble the system is, see Section 10. It is important to note sampling computations commonly employ dynamics but the dynamics is typically unphysical.

We begin in Section 10 with a consideration of the models we use in molecular dynamics, and whether we can justify their use over other, more physically realistic, models.

A major topic, discussed in Section 10, concerns the effects of discretization errors in numerical trajectories for deterministic molecular dynamics (MD). Due to the highly chaotic nature of the dynamics, the numerical trajectory completely departs from the analytical trajectory very early in the simulation. Yet numerical experiments show that averages calculated from such erroneous trajectories are close the correct ones.

Section 10 considers problem 1, sampling, above. There are roughly two approaches: dynamical sampling, which using uses continuous time deterministic or stochastic dynamics, and Markov Chain Monte Carlo (MCMC) sampling, which uses a Markov Chain to explore state space. In this section we consider primarily the former class, which includes deterministic extended Hamiltonian approaches and stochastic Langevin dynamics. We also consider Hybrid Monte Carlo methods which combines aspects of dynamical sampling with MCMC techniques.

Section 10 considers other topics in sampling. In particular, various Monte Carlo Markov Chain (MCMC) methods are considered along with methods for speeding up their rate of convergence.

Often we do not want to just sample configurations or states of the system, but actual trajectories of the system. An example of this is when we specify an initial configuration and a final configuration and we wish to be able to sample from the set of all trajectories that run between them. Section 6 discusses this kind of situation, which is known as sampling from path space.

For many systems microscopic models are simply to restrictive in terms of the time and length scales that are attainable with current computations. Section 7 discusses coarse graining: replacing microscopic models with mesoscopic ones that preserve the same effective behaviour of the system.

Models and Justifications

The main focus of the meeting was classical molecular dynamics, whether determinsitic or stochastic. Since we believe that fundamentally molecular systems are quantum and deterministic, using classical models—particularly stochastic ones— requires some justification.

Classical vs. Quantum

At the most fundamental level the dynamics of atoms and molecules must follow the rules of *quantum* mechanics and the dynamics prescribed by Schrödinger's or Heisenberg's equations of motion. The presentation of J. Straub described the results of a careful study of the molecular dynamics of vibrational energy transfer within a protein. The predictions of classical models for the molecular dynamics, based on both direct molecular dynamics as well as the theory of "small vibrations" or normal modes of motion, were compared with those of quantum mechanical perturbation theory. It was noted that for the case of vibrational energy transfer in a protein, classical dynamics can lead to poor results for the time scales and pathways of energy transfer. More attention must be paid to this issue if the ultimate objective is to develop accurate numerical methods for realistic simulations of the molecular dynamics of biomolecular systems.

Stochastic vs. Deterministic

The use of stochastic equations of motion for (real) dynamics also requires an explanation: Let us start from the assumption that the desired molecular system can be described by classical mechanics entirely and correctly. This implies the existence of a microscopic state $\overline{\Gamma}$ and an associated Liouville operator $\overline{\mathcal{L}}$ which completely characterize the molecular system and the time evolution of distributions $\overline{\rho}(\overline{\Gamma}, t)$ in phase space in particular, i.e.,

$$\overline{\rho}_t = \overline{\mathcal{L}}\overline{\rho}.$$

Almost all numerical simulations will work with a reduced representation over a smaller phase space $\Gamma \subset \overline{\Gamma}$. Crucial is the assumption that the time evolution over such a reduced space is still Markovian and the existence of a (perhaps approximative) Liouville operator $\mathcal{L}$ is generally taken as granted. See work by Mori and Zwanzig and, more recently, by Stuart and co-workers for rigorous results. Note that we have to, in general, assume that $\mathcal{L}$ corresponds to some form of stochastic ODE with the noise process representing forces from the missing degrees of freedom. The feeling in that regard is that Langevin/Brownian dynamics is not appropriate and that a more general form of the stochastic and dissipative coupling terms is required (see dissipative particle dynamics (DPD)). Also, in practice the missing degrees of freedom are in contact with the system only on the boundaries. Although simulations are sometimes implemented this way, it is far more common to use periodic boundary conditions and it is an interesting question how to handle this. Constant energy simulations make sense really only for molecules in gas phase.

The Accuracy of Long Time Numerical Trajectories —Why MD Works?

The primary focus on the mathematical aspects of molecular dynamics was placed on the long-time accuracy of integrating Newton's or Hamilton's *classical* equations of motion. Many of the paradigms in that field of research were derived from the early work on celestial mechanics, where the assumption of classical dynamics is most certainly a good one.

In the case of classical MD, this question is posed for the difficult case of deterministic dynamics, specifically Hamiltonian dynamics. The equations for Hamiltonian dynamics are

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma = J\nabla H(\Gamma), \qquad \Gamma = \begin{bmatrix} q \\ p \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

for example, $H(q,p) = \frac{1}{2}p^{\mathsf{T}}M^{-1}p + U(q)$ where M is a diagonal matrix of masses and $U(q)$ is potential energy. Numerical evidence indicates that both (i) time averages for steady-state sampling and (ii) ensemble averages for time correlation functions are approximated well from numerical trajectories. The challenge is to explain why this is so.

One view

In the opening talk R. Skeel sketched an explanation for the success of Hamiltonian molecular dynamics for calculating time correlation functions in terms of the underlying Liouville equations for the probability density. Calculating numerical trajectories is basically the method of characteristics for approximating the probability density at any point in time. For accuracy it is not necessary to have long accurate trajectories, e.g., if two trajectories were exchanged this would not change the density. Because each time step generates a temporal discretization error, long-time accuracy is possible only if these errors are damped. This would seem to require that the Hamiltonian system have the mixing property. However, this is not easy to demonstrate: Suppose that the initial density is not very close to the stationary density but that after a long time t it reaches a density $\rho(q,p,t)$ which is very close. Now consider $\rho(q,-p,t)$. In any traditional norm this is equally close to the stationary density. However, if we started from here and did dynamics, the density would ultimately be equal to the original initial density. So no matter how close we are to the stationary density, it could evolve to be far away. So a convergence proof based on mixing seems difficult because an argument based on showing contractivity in some metric would have to use a metric that distinguishes between $\rho(q,p,t)$ and $\rho(q,-p,t)$. Perhaps, an argument that distinguishes between stable and unstable manifolds would work. Alternatively, the following is plausible: a solution $\rho(q,p,t)$, for smooth initial values $\rho(q,p,0)$, spends almost all of its time very near to $\rho(q,p)$:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|\rho(\cdot,\tau) - \rho(\cdot)\|\mathrm{d}\tau = 0$$

—for some suitably weak norm. This means that nearly all initial conditions produce mixing dynamics with the “expected” rate of convergence.

The preceding discussion is oversimplified and needs to be amended because ergodicity, and mixing, are not generic properties for Hamiltonian systems. It is only on most of phase space that the almost-always mixing property can hold.

Another view

In the closing discussion another explanation for the success of MD was given by S. Reich. To explain what makes MD work, we need to look at the spectral properties of $\mathcal{L}$. Mixing would correspond to an operator $\mathcal{L}$ with a spectrum $\sigma(\mathcal{L})$ and $\sigma_{\mathrm{ess}}(\mathcal{L}) := \sigma(\mathcal{L}) - 0$ with $\Re(\sigma_{\mathrm{ess}}(\mathcal{L})) \le \gamma < 0$. The constant γ characterizes the decay of correlation. Biomolecular systems would probably not be called mixing in the above sense as γ would be very close to zero (at least relative to simulation times) and $\sigma_{\mathrm{ess}}(\mathcal{L})$ contains both mixing (the essential spectrum) and metastable states (isolated eigenvalues).

In MD the time evolution of a density ρ could be approximated by a sum of individual trajectories and weighted Dirac delta functions, i.e.,

$$\rho_{\mathrm{num}}(\Gamma,t) = \sum_i \alpha_i \delta(\Gamma - \Gamma_i(t)).$$

An interesting question is the convergence of ρ_{num} to ρ under an appropriate norm (e.g., Wasserstein norm). It is optimistic to think that this interpretation of MD would bring “weaker” requirements with respect to trajectory accuracy. Rather it would be expected that one needs to require the convergence of numerical trajectories to shadow trajectories of a modified system for a vast majority of such trajectories. This is also supported by the fact that hyperbolic systems are mixing and almost all trajectories of hyperbolic systems can be shadowed.

One talk that supported this view was W. Hayes'. He presented the state-of-the-art on practical shadowing of real trajectories of Hamiltonian systems. The work originates in the area of celestial mechanics, where there has been great success in showing that computed trajectories are close to nearby analytic trajectories. (In a related talk, W. Newman showed how long celestial mechanics trajectories can be computed to machine precision. Can this be done for MD too?) Some, but not a lot of hope was expressed for the application of Hayes' work to MD, though results are preliminary.

Yet another view

There is another view, presented by A. Stuart, that does not require that numerical trajectories are shadowed over long time intervals by true trajectories. Begin with the observation that we are typically not interested in the statistical behaviour of the entire system, but only a one or two degree-of-freedom subsystem. The remainder of the system acts as a heat bath for this component. Often the trajectories of the one component are well approximated in distribution by those of a stochastic process, such as the solution to a stochastic differential equation. The remainder of the system acts as a source of random noise and damping. A rigorous results of this sort is given in [25]. Now, for numerically computed trajectories to give the right statistics, it is only necessary that they too are close in distribution to the trajectories of this stochastic process. This has been shown to be true for special systems through numerical experiments in [37] and [38], though as of yet there are no general theorems.

Dynamical Sampling

The problem of sampling is to generate states Γ^n, $n = 1, 2, \ldots$ from a prescribed distribution. Assuming ergodicity, one can compute them from the solution of an ODE or stochastic differential equation (SDE). Configurations generated this way are (i) tainted by the initial conditions, (ii) correlated, making it more difficult to estimate variance, and (iii) biased due to the use of a finite step size. Markov chain Monte Carlo methods, which we consider in Section 10, do not suffer the last of these drawbacks.

For a given Hamiltonian system there are a few different distributions on the state space that we might want to sample from. These different distributions are typically referred to as ensembles and correspond to different boundary conditions. The two that we discuss in this report are the NVE ensemble and the NVT ensemble. In each case it is important to note that the probability density can only be expressed in closed form up to a multiplicative constant, known as the partition function. This constant is unknown for most interesting systems.

The NVE or microcanonical ensemble describes an isolated system and corresponds to a fixed number of particles, fixed volume, and fixed energy for the system. If the fixed energy is E, the density of the distribution is given by

$$\frac{\delta(H(q,p)-E)}{\int \delta(H(u,v)-E)\mathrm{d}u\mathrm{d}v}.$$

So zero weight is given to all states where $H(q,p) \neq E$. This density is invariant under Hamiltonian dynamics. Assuming that the Hamiltonian dynamics is ergodic on the energy level set, exact Hamiltonian dynamics sample from the NVE distribution. Hence, when the differential equations of a Hamiltonian system are numerically integrated (that is, we do MD), we are attempting to compute NVE ensemble averages. Of course, many things could go wrong with this computation, and there will typically be a bias. MD is the only widely used procedure for sampling from the NVE ensemble.

Two talks specifically addressed errors in microcanonical simulation. In his talk, P. Tupper described some work attempting to justify the assumption of ergodicity in microcanonical dynamics and MD. The talk by S. Bond surveyed some results from backward error analysis and showed how (under certain assumptions) these results can be applied to compute estimates of the error in averages from molecular dynamics simulations. Results from several test problems were explored including examples from constant temperature molecular dynamics, which corresponds to the next ensemble we consider.

The NVT or canonical ensemble corresponds to a fixed number of particles, fixed volume, and fixed temperature. The density associated with this ensemble is proportional to $\exp(-\beta H(q,p))$ where $\beta = 1/kT$, k is Boltzmann's constant, and T is temperature. This ensemble is believed to be good representation of the distribution of the Hamiltonian system if it is in thermal contact with a much larger system of temperature T. A nice feature of this ensemble is that position q and momentum p are independent for separable Hamiltonians.

There are many different ways to sample from this distribution including

1. deterministic thermostats,
2. Langevin thermostats,
3. MCMC, (see Section 10) and
4. hybrid Monte Carlo (Monte Carlo using MD for proposals).

In the case of dynamical sampling, ergodicity is an important issue, and for both types of methods the rate of convergence is also very important. We consider methods 1, 2, and 4 in subsequent subsections.

An important point to note about all these methods is that unlike MD applied to unadorned Hamilton's equations, the dynamics is not *real*. In each case something has been added to the dynamics that does not correspond directly to any component of the real physical system we are attempting to model. This is done so that states generated by the trajectories sample the canonical distribution. However, it is no longer clear what the trajectories generated can be used for other than this. For example, is there any physical validity to a velocity autocorrelation function computed with Langevin dynamics? The answer may be no.

Dynamical thermostats

Nosé Dynamics. Leimkuhler, Barth and Sweet are developing extended Hamiltonian formulations for thermostating molecular dynamics. It was the observation of the physicist Nosé [33] that we can augment the energy function $H(q,p)$, by incorporating a single additional phase variable, s, together with its canonical momentum, π, in the following way:

$$H^{\text{Nose}} = H(q, \widetilde{p}/s) + \frac{\pi^2}{2\mu} + U(q) + gkT \ln s.$$

A simple integration argument convinces us that,

$$\iint \delta\left[H^{\text{Nose}}(q, sp, s, \pi) - E\right] \mathrm{d}s\mathrm{d}\pi = \text{constant} \times \exp\left(-\frac{N}{gkT} H(q,p)\right),$$

moreover,

$$\iint f(q, \widetilde{p}/s)\delta\left[H^{\text{Nose}}(q, \widetilde{p}, s, \pi) - E\right] \mathrm{d}s\mathrm{d}\pi\mathrm{d}\widetilde{p}$$
$$= \text{constant} \times \int f(q,p) \exp\left(-\frac{N}{gkT} H(q,p)\right) \mathrm{d}p,$$

so if we choose $g = N$, we are able to reduce the microcanonical density function for H^{Nose} to the canonical one for H.

Time transformation. Accurate sampling of the NVT ensemble for certain types of systems calls for large fluctuations of the thermostating variable s, including potentially very small values. When this happens, the equations of motion from Nosé's Hamiltonian are poorly scaled and standard numerical methods become unstable. For this reason, a time transformation $dt/dt' = s$ is used in the derivation of the Nosé-Hoover equations. However, the way this is traditionally done destroys the symplectic structure. The basis of the

Nose-Poincaré method of [6] is rather a *Poincaré transformation*, modifying the original Nosé Hamiltonian to:

$$
\begin{aligned}
H^{\mathrm{NP}}(q,\widetilde{p},s,\pi) &= s\left[H^{\mathrm{Nose}} - E\right] \\
&= s\left[H(q,\widetilde{p}/s) + \frac{\pi^2}{2\mu} + U(q) + gkT\ln s - E\right].
\end{aligned}
$$

It is a better alternative starting point for the development of numerical methods than Nosé-Hoover. Now the construction of numerical methods is slightly complicated by the modification of the Hamiltonian–we cannot directly use the Verlet integrator here, for example–but there are several ways to solve H^{NP} which work well.

Nosé-Poincaré Chains. An improvement on the Nosé approach is based on what are termed *Nosé-Poincaré chains*. In Nosé-Hoover chains, one thermostats the thermostating variable, introducing an additional variable and controlling it via an additional temperature equation. This process can be repeated. For example we can use a Hamiltonian like this:

$$
H^{\mathrm{NPC};1} = s_0\left[H(q,\frac{\widetilde{p}}{s_0}) + \frac{\pi_0^2}{2\mu s_1^2} + \frac{\pi_1^2}{2\mu_1} + gkT\ln s_0 + kT\ln s_1 + f(s_1) - E\right].
$$

In this formula the time-rescaling has been included. If the function f is bounded below with bounded exponential integral, then the integration argument goes through. A canonical sampling argument exists in this case. We can make these chains as complicated as we like by extending them with more variables. The design of the regularizing function and choice of thermal masses is important for good results. For the right choice of parameters, Nosé-Poincare chains have reasonably good ergodicity properties: if we use 3 or 4 augmenting variables and choose carefully the thermal masses $\mu_1, \mu_2, \ldots$,, then we can get an accurate recovery of canonical sampling from these chains. The need for a careful choice of thermostat masses is a flaw, though, and the NPC methods tend to be unstable.

RMT Chains. Better methods are possible based on a more complicated interaction between the bath and the physical variables, termed *Recursive Multiple Thermostating* (RMT). The theory of this model is considered in detail in a recent article of Leimkuhler and Sweet, and practical selection of coefficients is discussed in the work of Barth,Leimkuhler and Sweet presented at the meeting. In nonlinear models the RMT formulation is found to be more sensitive to the details of the underlying system. Heuristics have been presented for selection of thermostat masses. Some preliminary encouraging data were presented on the use of these methods for study of an alanine dipeptide model, based on an implementation of RMT in CHARMM, the popular molecular dynamics code.

In separate work, Leimkuhler and Jia have proposed a general framework of thermostating dynamics for multiscale problems. Using these ideas, we can flexibly introduce canonical sampling over particular components while preserving physically relevant multiscale structural characteristics of the application and maintaining these characteristics in the design of efficient numerical algorithms. Certain classes of problems with fast and slow variable separation were examined in detail. Moreover, a method was proposed for following the slow evolution in an nonequilibrium setting, where only the fast degrees of freedom are assumed to be in equilibrium. The sampling properties of these new formulations were tested in numerical experiments using an enhanced reversible averaging scheme and a Nosé-Poincaré chain. For some choices of model setup, a fully Hamiltonian formulation is impractical.

In more recent work of Gill, Jia, Leimkuhler and Cocks, this multiscale partial thermostating method was applied with modified RMT thermostat for a simplified version of the quasicontinuum molecular dynamics (QCMD) model of materials science. Effectively the combination of this coarse-graining strategy with advanced thermostats provides a powerful adaptive boundary condition for a localized, detailed atomistic calculation. This type of method is useful for studies of nanoindentation, and for defect nucleation (e.g., cracks). This work was also discussed at the meeting.

The summary of the state of the art in dynamical thermostats is that they have interesting properties and can work in practical situations. Their ergodic properties are verifiable in at least some numerical studies, and they provide a flexible adaptive framework for molecular simulation while benefiting from the well known reliability of the classical MD simulation framework. More work is needed to probe their robustness, to understand their ergodic properties, and to evaluate their efficiency vis a vis stochastic dynamics methods. Some of these questions were raised in discussions at the meeting.

Langevin Thermostating

The Langevin equation in the case of sampling is

$$M\frac{\mathrm{d}^2}{\mathrm{d}t^2}q = -\nabla U(q) - M\gamma\frac{\mathrm{d}}{\mathrm{d}t}q + \sqrt{2k_{\mathrm{B}}TM\gamma}\frac{\mathrm{d}}{\mathrm{d}t}W(t) \quad (1)$$

where γ is a damping constant and $W(t)$ is a vector of independent standard Wiener processes.

Like for deterministic thermostating, the Langevin equations have the canonical ensemble as an invariant distribution. Unlike for deterministic dynamics, there are plenty of situations where Langevin dynamics are proven to be ergodic [29]. (One situation where Langevin dynamics has not been proven to be ergodic is that of Dissipative Particle Dynamics (DPD). T. Shardlow explained the problems here and gave an overview of his proof of ergodicity for 1d DPD. More on DPD in section on coarse graining.)

It is, of course, necessary to have an integration scheme for Langevin dynamics. M. Tretyakov described his work with Milstein on quasi-symplectic integrators. These integrators contract volume with the same rate as the original SDEs, and become symplectic methods in the limit of zero noise ($\gamma \to 0$). However, these methods applied to non-globally Lipschitz Langevin equations are not themselves ergodic; they will eventually diverge to infinity with probability one. Tretyakov discussed how this turns out to not be a problem in practice.

An issue that might be worth considering is the relationship between the Langevin based methods, with the coupling to a "physical" heat bath through the Caldeira-Leggett-Zwanzig Hamiltonian form, and the Nosé based methods, with the extended Lagrangian.

One special limiting case of Langevin dynamics is Brownian dynamics, obtained from (1) by letting $M \to 0$ while $M\gamma$ is a constant. This gives:

$$g\frac{\mathrm{d}}{\mathrm{d}t}q = -\nabla U(q) + \sqrt{2k_{\mathrm{B}}TM\gamma}\frac{\mathrm{d}}{\mathrm{d}t}W(t).$$

Brownian dynamics samples the from canonical distribution for the configuration variable q. One issue on which there was some disagreement was under which conditions it is better to just use Brownian rather than Langevin dynamics to sample configurations. A related question is what is the best parameter γ to use in Langevin dynamics for the fastest sampling.

Comparison of Thermostats. Both deterministic thermostats (such as Nosé) and Langevin (stochastic) thermostats can be used successfully to sample configurations from the canonical distribution. Both analytic dynamics have the canonical measure as an invariant. Langevin dynamics are provably ergodic in some cases. This not true of dynamical thermostated dynamics, but it may not be important in practice. In both cases numerical methods applied to them will lead to a bias (that decreases with reduced step length) and potentially instability over long simulations. A natural question is which— if any– is better for practical problems. In particular, which allows more efficient sampling of the canonical ensemble? One issue is that generating a complete set of random numbers per time-step may be costly in some circumstances. Some pointers for measuring ergodicity and convergence are given in [2], in particular, they describe an empirical "ergodic measure," which can be used to compare algorithms and to optimize them. Another possibility is to compare the two methods when used as proposals in Hybrid Monte Carlo (see next subsection). More work is needed to carefully and systematically compare the effectiveness of the best algorithms for both deterministic and stochastic dynamics.

Hybrid Monte Carlo

Hybrid Monte Carlo (HMC) is an MCMC method that uses MD to generate proposals. We saw many interesting developments in this area. J. Izaguirre presented some work on using the shadow Hamiltonian to reduce the number of rejections. The idea is that in typical HMC you use the amount of energy drift experienced to decide whether or not to accept of MC move. However, it is hard to differentiate between energy drift and a short-term fluctuation just by looking at the energy. However, it has been shown, e.g, [9], how to compute the shadow Hamiltonian value on the fly, and this can be used to determine whether drift is really occurring. Izaguirre and his collaborators have incorporated this into HMC.

HMC in this form, like other Monte Carlo methods, only sample over configuration space. It seems unlikely that there is any way to interpret long trajectories as 'real' dynamics in even some weakened spectral sense (see 10). S. Reich and co-worker addressed this issue in taking the approach of Izaguirre even further by reconsidering the means of generating new proposals in HMC. Typically, a proposal is made by discarding all momentum information and re-sampling it from the canonical distribution. Akhamtskaya & Reich suggested (i) to implement HMC over phase space, (ii) to increase the acceptance rate by using a shadow Hamiltonian (see Izaguirre & Hampton), and (iii) to only partially re-sample momenta. It seems feasible that the associated Markov chain has spectral properties that are similar to discrete-time Langevin dynamics and/or DPD in the limit of weak thermal coupling. A main theoretical obstacle is the necessary momentum reversal after a rejected MD step. However, using a sufficiently high-order shadow Hamiltonian, rejections should almost become negligible. This is clearly an area for further research.

As with all MC methods, there is still the issue in HMC of how to optimize parameters to maximize performance. Many authors optimize on the rejection rate. What should be optimized (minimized) is the integrated autocorrelation function, which gives the variance of estimators. Rejection rate affects this, but it is not the whole story.

Other Strategies for Efficient Configurational Sampling

An abundance of effective techniques have been recently proposed. There is a need to put these methods in an abstract framework and do a theoretical comparison of efficiency.

Many of these techniques are importance sampling schemes. Importance sampling uses points drawn with biased probability density $\rho_{\mathrm{b}}(q)$ to estimate expectations with respect to the distribution with density $\rho(q)$ using a weighted average, with weights proportional to $\rho(q)/\rho_{\mathrm{b}}(q)$. However, for high-dimensional problems it is difficult to find an easily-sampled $\rho_{\mathrm{b}}(q)$ that is close enough to $\rho(q)$, and for which the density function $\rho_{\mathrm{b}}(q)$ can easily be computed. Commonly the density is known only up to a multiplicative constant, in which case one samples from $\rho_{\mathrm{b}}(q)$ using an MCMC method and uses

$$\langle O(q)\rangle = \frac{\langle O(q)\sigma(q)/\sigma_{\mathrm{b}}(q)\rangle_{\mathrm{b}}}{\langle \sigma(q)/\sigma_{\mathrm{b}}(q)\rangle_{\mathrm{b}}}$$

where σ, σ_{b} denote unnormalized densities. In practice, with finite sampling, this can be approximated by $(\sum_n O(q^n)\sigma(q^n)/\sigma_{\mathrm{b}}(q^n))/(\sum_n \sigma(q^n)/\sigma_{\mathrm{b}}(q^n))$ (which is a biased estimate).

Fast growth methods

These are importance sampling methods that use the Jarzynski identity [22]. The Hamiltonian is parameterized and changed it as a function of time while doing Markovian dynamics, which must satisfy a “balance condition.” Typically the normalizing constant of the initial distribution for the Markovian dynamics is not known and it is sampled using dynamics or an MCMC method.

The talk by R. Neal proposes the use of temperature as a parameter for enabling a sampling not hindered by potential energy barriers. Also, it permits the initialization of the trajectories using configurations drawn exactly from a given distribution (as opposed to an approximate equilibration), so the method becomes one of perfect sampling (no bias due to initial configurations).

Theoretical studies such as [34] indicate that fast growth methods as they are currently formulated offer no advantage over equilibrium sampling, e.g., umbrella sampling.

Hamiltonian importance sampling

The problem of finding an easily-sampled $\rho_{\mathrm{b}}(q)$ that is close enough to $\rho(q)$, and for which the density $\rho_{\mathrm{b}}(q)$ can easily be computed can be overcome using a $\rho_{\mathrm{b}}(q)$ that is defined in terms of Hamiltonian dynamics. For a molecular system, we sample particle positions uniformly over some (wrapped-around) region, and also sample momenta for these particles from their distribution at a high temperature. We then simulate Hamiltonian dynamics for this system, while periodically multiplying the momenta by some factor slightly less than one, which eventually cools the system to whatever temperature, T, we are interested in. This procedure defines a distribution $\rho_{\mathrm{b}}(q)$, which can be used to estimate expectations with respect to $\rho(q)$, the canonical distribution at temperature T. Crucially, for each sampled q, we can compute $\rho_{\mathrm{b}}(q)$, and hence the appropriate weight to attach to this point in the average: The density of the initial point sampled from the high-temperature distribution is easily calculated; since Hamiltonian dynamics conserves phase space volume, the density of a point found after simulating Hamiltonian dynamics for some time is the same as that of the original point; and finally, multiplying a momentum variable by a factor less than one simply increases the probability density of the resulting point by the same factor.

Replica exchange

In the discussion on the last day, R. Neal compared parallel tempering with the use of the Jarzynski equality. He said that Jarzynski's method tends to require a finer spacing of distributions than parallel tempering. But on the other hand, information in parallel tempering propagates between distributions via a random walk, which tends to take n^2 steps to move a distance of n. These two effects more-or-less cancel out, so that there is no clear advantage of one method over the other (at least in this respect—their properties differ in other ways that may be relevant in some problems). This points to a research direction of trying to modify one of these methods to obtain the advantageous property possessed by the other method.

Adaptive biased-force method

Eric Darve presented novel techniques to calculate the potential of mean force along a reaction coordinate, the so-called Adaptive Biasing Force (ABF) and an extension to Monte-Carlo simulations (MC-ABF). The goal of ABF is to improve the sampling of phase space when calculating the free energy along a reaction coordinate $\xi(r)$ or potential of mean force:

$$A = -k_B T \ln\langle\delta(\xi(r) - \xi)\rangle$$

In typical molecular systems, the molecules remain trapped in low energy basins for extensive periods of time escaping only rarely. See Figure 10.1.

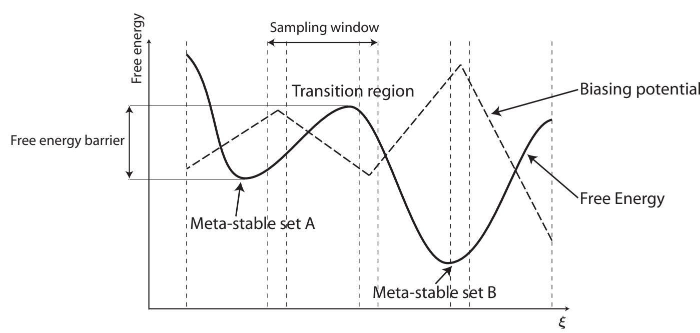


Figure 10.1: Typical free energy profile showing two basins of attraction and a free energy barrier at transition states. In the method of Umbrella Sampling, a biasing potential is used to improve the sampling of the system. The simulation is often performed inside of windows to improve the efficiency.

ABF improves the efficiency of this type of calculation by applying an external biasing force which leads to a uniform sampling along the reaction coordinate. The biasing force is obtained by applying the following equation for the derivative of the free energy:

$$\frac{dA}{d\xi} = -\left\langle \frac{d}{dt}\left(m_\xi \frac{d\xi}{dt}\right)\right\rangle_\xi, \quad m_\xi^{-1} = \sum_r \frac{1}{m_r}\left(\frac{\partial\xi}{\partial x_r}\right)^2$$

The applied force is taken approximately equal to $(dA/d\xi)\nabla\xi$. It is continuously updated using samples gathered during the simulation.

Darve presented results on dichloro-ethane, fluoro-methane with a water-hexane interface and results on the insertion of a helix inside a membrane: see Figure 10.2.

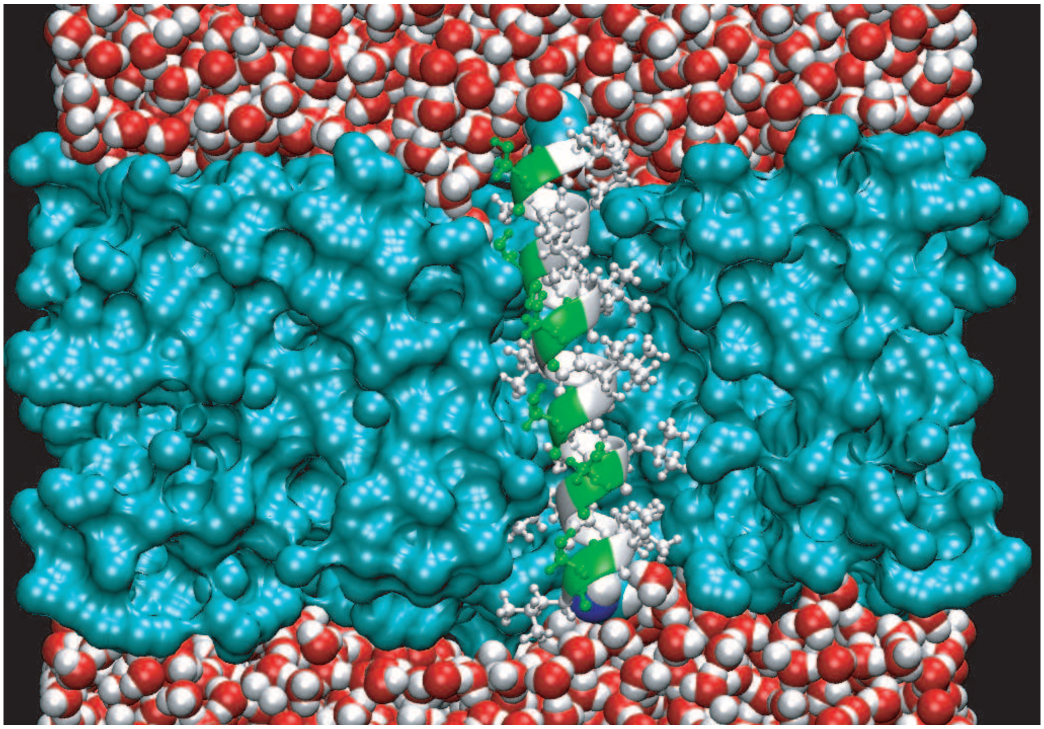
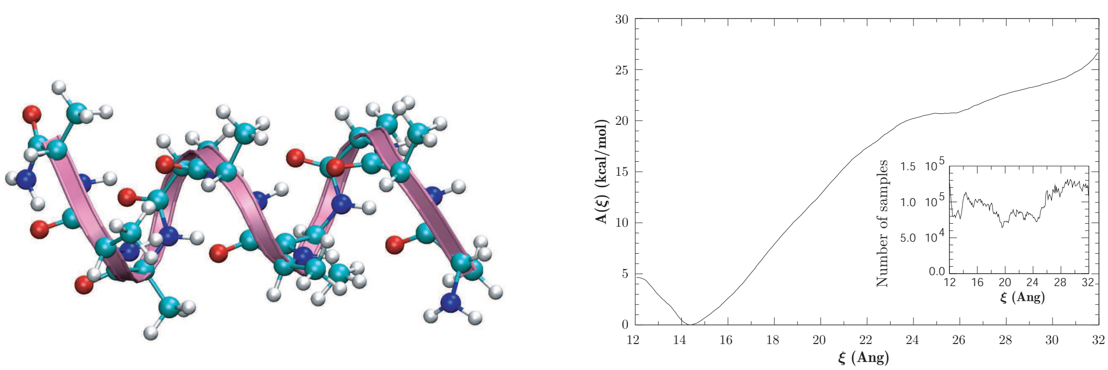


Figure 10.2: Helix inside a mimetic membrane. This system is used to understand the self-assembly of helices in cell membranes to form ion channels.



(a) Deca–L–alanine in its folded configuration

(b) Free energy profile for Deca–L–alanine calculated using the adaptive biasing force. The inset shows the number of samples as a function of ξ.

Figure 10.3: Deca-L-alanine

Christophe Chipot presented simulation results on the unfolding of deca-alanine using ABF. See Figure 10.3.

In addition to Molecular Dynamics, Darve presented a new algorithm based on ABF for Metropolis Monte-Carlo simulations, called MC-ABF. This algorithm is based on applying a bias to the transition probabilities. MC-ABF was applied to calculate the density of states for the Ising model. The Ising model is a square system of L^2 spins (up or down) whose energy is given by: $E = -\sum_{i,j} \sigma_i \sigma_j$ where the indices i and j correspond to neighboring spins. In MC-ABF, the density of states $g(E)$ is estimated using a recurrence formula:

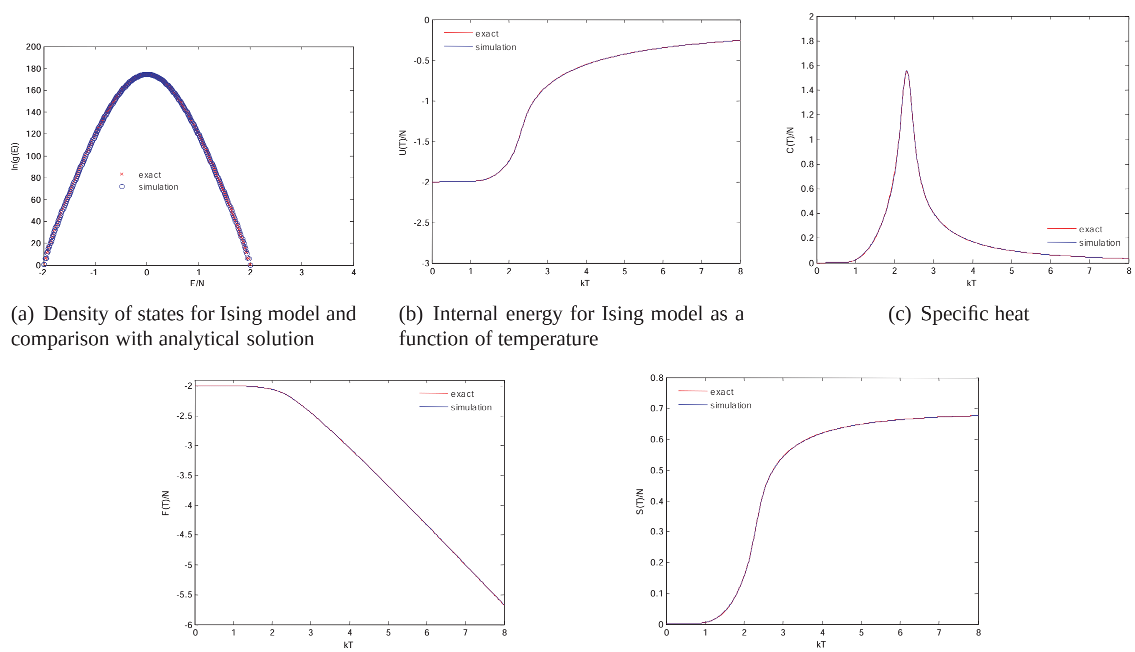
$$g(E_i) = g(E_{i-1}) \frac{\Pi^+(E_{i-1})}{\Pi^-(E_i)} \tag{2}$$

where $\Pi^+(E_{i-1})$ is the probability to transition from energy E_{i-1} to E_i and $\Pi^-(E_i)$ the probability to transition from E_i to E_{i-1}. The modified acceptance rule is then given by:

$$p_b(E_i \to E_j) = \min\left(\frac{g^{ABF}(E_i)}{g^{ABF}(E_j)}, 1\right)$$

where g^{ABF} is the current estimate of $g(E)$ computed using Equation 2. This biasing leads to a uniform sampling in energy space. In particular states at high and low energies ($2L^2$ and $-2L^2$) are visited as often as intermediate states with energy close to zero. This is despite a difference in population at those energies of about $\exp(180) \approx 10^{80}$.

Results were presented for the calculation of the density of states as well as internal energy, specific heat, Helmholtz free energy and Entropy. See Figure 10.4.



(a) Density of states for Ising model and comparison with analytical solution

(b) Internal energy for Ising model as a function of temperature

(c) Specific heat

(d) Helmholtz free energy

(e) Entropy

Figure 10.4: Ising model of size 16×16.

Transformation of Configuration Space Variables

One of the computational grand challenge problems is to develop methodology capable of sampling conformational equilibria in systems with rough energy landscapes. If met, many important problems, most notably protein folding and protein aggregation, could be significantly impacted. In his talk, Tuckerman present a new approach in which molecular dynamics is combined with a novel variable transformation designed to warp configuration space in such a way that barriers are reduced and attractive basins stretched. The essence of the method is as follows: Consider a one-dimensional potential $V(x)$ with a barrier. The canonical partition function is

$$Q = \int dp\, dx\, \exp\left\{-\beta\left[\frac{p^2}{2m} + V(x)\right]\right\}$$

If the barrier in $V(x)$ is high, then if we try to evaluate this partition function using molecular dynamics or Monte Carlo, the presence of the barrier will make crossing events between the two wells on either side of the barrier rare and, hence, hinder the sampling. However, consider a nonlinear variable transformation in the above partition function:

$$u = f(x) = x_0 + \int_{x_0}^{x} dy\, e^{-\beta V_{\rm ref}(y)}$$

where $V_{\rm ref}(x)$ is an arbitrary reference potential. When substituted into the partition function, one obtains

$$Q = \int dp\, du\, \exp\left\{-\beta\left[\frac{p^2}{2m} + (V(x(u)) - V_{\rm ref}(x(u))\right]\right\}$$

One can then use Monte Carlo or thermostated molecular dynamics with a Hamiltonian of the form $p^2/2m + V(x) - V_{\rm ref}(x)$. It is, therefore, very likely that the techniques being developed by Leimkuhler and coworkers will be applicable with this method. Note that this transformation yields exactly the same partition function and therefore, preserves all of the thermodynamic and equilibrium properties of the system. From the form of the transformed partition function, the optimal choice of the reference potential becomes clear: One should choose $V_{\rm ref}(x)$ to be equal to the true potential in the barrier region and zero outside the this region. Note that this is not equivalent to umbrella sampling or guiding potential methods, as the variable u naturally moves on the difference potential $V(x) - V_{\rm ref}(x)$ so that no re-weighting of the phase space is needed. By applying transformations of this type on the full set of backbone dihedral angles in polymers and proteins, Tuckerman was able to show that very large gains in the efficiency of sampling configuration space could be obtained for large polymer chains and small model proteins. The method is further enhanced by including adaptive transformations that remove barriers that arise “on the fly” in a simulation by neighboring solvent atoms or short-range non-bonded type interactions. Currently, Tuckerman and coworkers are implementing the method in their PINY_MD code, a code that contains a full “bio-builder” that will allow all-atom models of proteins to be treated, thereby allowing the method to be tested on realistic problems.

Multicanonical simulations

Numerical experiments have demonstrated that combining replica exchange with multicanonical Monte Carlo leads to much more effective sampling [30].

Sampling from Path Space

Many reactions in (bio)molecular systems occur on time-scales outside the range of current direct molecular dynamics simulations. In simulations of slow molecular reactions, much of the simulation time is in effect ”wasted” waiting for a large enough fluctuation to carry the system from the reactant state to the product state. Following Dellago, Chandler and co-workers [8], one can instead attempt to create only transition paths, i.e., those trajectory segments that connect reactant and product states and exclude the waiting time in the reactant well. However, even if one has “harvested” many such reactive trajectories, it is not always immediately clear where the “bottleneck” of the reaction is (the transition state), and which measure best describes the progress of the reaction (the reaction coordinate). Bayes relation for conditional probabilities can be used to extract transition states and reaction coordinates from an ensemble of transition paths (obtained, e.g., by path sampling) and an equilibrium ensemble (obtained, e.g., by umbrella sampling) [21]:

$$p(\mathrm{TP}|x) = \frac{p(x|\mathrm{TP})p(\mathrm{TP})}{p_{\rm eq}(x)} \tag{3}$$

where $p(x|\mathrm{TP})$ is the probability density of the phase-space variable x, $p_{\mathrm{eq}}(x)$ is the equilibrium density, and $p(\mathrm{TP})$ is the fraction of time spent in transition paths, averaged over long equilibrium trajectories. The conditional probability of being on a transition path, $p(\mathrm{TP}|x)$, can be expressed in terms of splitting (or commitment) probabilities, and assumes a maximum at the transition state. The Bayesian relation can be used to locate transition states, optimize reaction coordinates, and calculate rate coefficients directly from path sampling [21, 4]. Moreover, it leads to a simple transition-path sampling algorithm in which trajectories are created by shooting from phase points near a presumed dividing surface between reactant and product states [21, 4].

Another approach to path-sampling was presented by A. Stuart in his talk. He considered the problem in a more general framework: molecular dynamics is just one possible application; another is nonlinear filtering in signal processesing. He described an abstract MCMC method for sampling in such problems, based on generalizing the Metropolis-adjusted Langevin algorithms to infinite dimensions. This leads naturally to the study of stochastic reaction-diffusion equations which, in their invariant measure, sample from the required distribution [16].

Coarse graining

Classical MD simulations are typically the method of choice for studying biophysical and soft matter systems at the molecular level. Characteristically, they are limited to system sizes of approximately 10^4 atoms and to times of around 100 ns. Thus, to study systems such as polymer melts, the computationally accessible time and length scales are simply far too short for the system to be able to reach equilibrium as dynamic processes during equilibration occur under hydrodynamic conditions.

A plethora of coarse graining and multiscale modeling methods have been developed to overcome the above difficulties. The approaches range from position space coarse-graining to free energy methods and field theory. Polymer research has been probably the leading field in multiscale modeling in both practical and theoretical aspects. For example, the projection operator formalism has been used by Akkermans and Briels [1] to investigate the fluctuating forces in coarse graining, and and the so-called GENERIC approach [14, 15, 35], which has very different nature but is also based on operator projection formalism, provides an analytically rigorous method for coarse-graining. GENERIC (General Equation for Non-Equilibrium Reversible-Irreversible Coupling) is based on the idea is that there is a general form for the time-evolution of non-equilibrium systems and that it can be given in a general form

$$\frac{dx}{dt} = L(x)\frac{\delta E(x)}{\delta x} + M(x)\frac{\delta S(x)}{\delta x},$$

where x characterizes the state of the system, $L(x)$ is an antisymmetric matrix and $M(x)$ is a symmetric and positive definite matrix. They are connected to the second and the first law of thermodynamics, respectively. $E(x)$ and $S(x)$ are functionals for the total energy and entropy, respectively. It is important to to notice that although GENERIC is rigorous it is not unique. There is no unique way to coarse grain and that constitutes one of the main conceptual difficulties. Other commonly used analytical approaches rely on the Ornstein–Zernike equation [17] and the hypernetted chain closure [5, 26]. A good review of some of the recent developments is given in Ref. [23].

Dissipative Particle Dynamics (DPD)

As a conceptually simple approach, Dissipative Particle Dynamics (DPD) [10, 20, 24] has recently become popular in soft matter simulations. In DPD the pairwise interaction potentials are "soft" in contrast to the Lennard–Jones -type potentials. "Softness" means that the DPD potential has a finite value at zero particle separation, i.e., the Fermi exclusion principle is not accounted for. It is not obvious that the above approximation is reasonable but Forrest and Suter [13] showed that by explicitly averaging over fluctuations in over long times that approximation becomes justifiable.

Another motivation behind DPD is that it conserves hydrodynamics, i.e., all the interactions are pairwise conservative. In DPD the force exerted on particle i by particle j is

$$\vec{F}_{ij} = \vec{F}_{ij}^{D} + \vec{F}_{ij}^{R} + \vec{F}_{ij}^{C}, \tag{4}$$

where $\vec{F}_{ij}^D$, $\vec{F}_{ij}^R$, and $\vec{F}_{ij}^C$ are the dissipative, random and conservative forces, respectively. The different components are given as

$$\begin{aligned} \vec{F}_{ij}^D &= -\gamma\,\omega^D(r_{ij})(\vec{v}_{ij}\cdot\vec{e}_{ij})\,\vec{e}_{ij} \quad \text{and} \\ \vec{F}_{ij}^R &= \sigma\,\omega^R(r_{ij})\,\xi_{ij}\,\vec{e}_{ij}, \end{aligned} \tag{5}$$

where $\vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j$, $r_{ij} \equiv |\vec{r}_{ij}|$, $\vec{e}_{ij} \equiv \vec{r}_{ij}/r_{ij}$, and $\vec{v}_{ij} \equiv \vec{v}_i - \vec{v}_j$ for particles with positions $\vec{r}_i$ and velocities $\vec{v}_i$. The ξ_{ij} are symmetric Gaussian random variables with zero mean and unit variance. They are independent for different *pairs* of particles and different times.

The coupling of the the dissipative and random forces, $\vec{F}_{ij}^D$ and $\vec{F}_{ij}^R$, comes from the fact that the thermal heat generated by the random force must be balanced locally by dissipation. The precise relationship between these two forces is determined by the fluctuation-dissipation theorem [10] and is given as

$$\omega^D(r_{ij}) = [\omega^R(r_{ij})]^2 \quad \text{and} \quad \sigma^2 = 2\gamma k_\mathrm{B} T^*, \tag{6}$$

where T^* is the canonical temperature of the system.

The most common choice for the weight functions ω^D and ω^R is the soft-repulsive form

$$\omega^R(r_{ij}) = \begin{cases} 1 - r_{ij}/r_c & \text{for } r_{ij} < r_c; \\ 0 & \text{for } r_{ij} > r_c, \end{cases} \tag{7}$$

where r_c is the cut-off distance and $\omega^D(r_{ij})$ is given by the fluctuation-dissipation relation above. This is also the form that the conservative force takes in the standard DPD formulation.

In addition, DPD can be used as a momentum conserving thermostat. That, and issues related to integration of the DPD equations of motion are discussed in Refs. [3, 32, 39].

Systematic derivation of DPD

The standard DPD is purely phenomenological. Recently, Flekkøy et al. [11, 12] were able to formally link DPD to molecular level properties by using a Voronoi tessellation based technique. The advantage of their method is that it can be used to resolve different length scales simultaneously. The method is formally akin to the well-known renormalization group procedure extensively used in the theory of critical phenomena.

Effective interactions from the pair correlations

Inverting the radial distribution functions $g(r)$ in order to obtain pair potentials offers another starting point. That approach can be used to obtain the pair potentials for DPD simulations in a more realistic and systematic way. Using pair correlations is based on the so-called Henderson theorem [18] which stipulates that under fixed conditions two pair potentials which give rise to the same $g(r)$ cannot differ by more than a constant. This constant is determined by the condition

$$V(r \to \infty) \to 0, \tag{8}$$

where r is the interparticle distance and V is the pair potential. The Henderson theorem analogous to the Hohenberg–Kohn theorem [19], i.e., all ground state properties are determined by the electron density. What makes the application of the Henderson theorem appealing is that the radial distribution function obtained from a simulation includes effects from the many-body interactions. Furthermore, this way it is possible to define new interaction sites and to compute the radial distribution function between them, and thus to systematically obtain new coarse-grained models at different levels of description.

The Inverse Monte Carlo (IMC) procedure of Lyubartsev and Laaksonen [28] is practical implementation of Henderson's idea. In IMC, one inverts the radial distribution functions – experimental or from microscopic simulations – to obtain effective potentials for a coarse-grained model with a fewer number of degrees of freedom. This approach has been recently used to study salt solutions and lipids [27, 31]

It is important to notice that the effective potential includes corrections from many-body interactions to the well-defined potential of mean force (PMF) [17], which is defined as

$$V^{\mathrm{pmf}}(r) \equiv -k_\mathrm{B} T \ln g(r), \tag{9}$$

where r is the interparticle distance and $g(r)$ the pair correlation function. In other words, the effective potential is *not* the same as the potential of mean force in Eq. (9). Inclusion of the many-body corrections is the reason why an iterative IMC scheme is needed. The iterative IMC procedure guarantees self-consistency, i.e., the effective potentials lead to the same pair correlation behavior as the underlying MD simulations.

Coarse graining – the future

The above provides just a scratch on the surface. There are a lot of other interesting and promising approaches. In general, multiscale modeling is a very rapidly developing field and progress is partly driven by the fact that despite the increase in CPU power, it is algorithm and method development that is crucial for treating complex problems such as protein folding, polymers, or cellular membranes.

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Chapter 11

Geometric and Asymptotic Methods in Group Theory (05w5011)

June 11–16, 2005

Organizer(s): Rostislav Grigorchuk (Texas A&M University), Alexander Olshanskiy (Vanderbilt University), Akbar Rhemtulla (University of Alberta), Mark Sapir (Vanderbilt University), Dani Wise (McGill University)

The goal of the conference was to bring together specialists in geometric, probabilistic and asymptotic methods in group theory. Special attention was paid to the following topics:

1. Amenability and randomness in groups
   - Random walks and generic properties of groups
   - Poisson boundaries of groups
   - Amenable actions of groups
   - Minimal volume entropy problem for graphs
2. Actions on rooted trees, growth and self-similarity
   - Growth and diameters of Schreir graphs of groups generated by finite automata
   - R. Thompson group
   - Subgroup growth of groups
3. Groups, boundaries and geometries
   - Cubulation of groups and right angled Artin groups
   - Quasi-isometric rigidity of groups
   - Asymptotic cones of groups
   - Algebraic geometry over groups and Tarski problems
4. Lattices in Lie groups
   - Bounded generation property
   - Property τ
   - Expanders

Many informal discussions lead to creating new ideas of collaboration between specialists in different areas of group theory. In particular, it may be possible to make groups of intermediate growth and torsion groups act on cubings using the end structure of their Schreir graphs; several properties of 1-relator groups which are believed to be generic may be connected to important properties of random walks on lattices, etc.

Here is the list of participants of the conference and the abstracts of their talks.

1. **Miklos Abert** (University of Chicago)
On chains of subgroups in residually finite groups
ABSTRACT: We analyze descending chains of finite index subgroups and the corresponding permutation representations for residually finite groups. As a result we show that we can obtain an arbitrary countable set of residually finite groups as the intersection set of a conjugacy class of a suitable chain. Although the representations do not approximate the group in general, we show that in certain cases, e.g. for higher rank real lattices, a weaker form of approximation holds.

2. **Roger Alperin** (San Jose State University)
Subgroup Separability in Linear Groups
ABSTRACT: We'll survey some examples and non-examples for separability of subgroups in f.g. linear groups.

3. **Jason Behrstock** (Columbia University)
Relative hyperbolicity and the mapping class group
ABSTRACT: We will describe some recent work on the asymptotic geometry of the mapping class group. In particular, we will give a geometric proof that the mapping class group is not hyperbolic relative to any finite collection of finitely generated subgroups. We will also contrast this with a new description of a way in which the mapping class group is non-positively curved. Parts of this talk are joint work with C. Drutu and L. Mosher.

4. **Ievgen Bondarenko** (Texas A&M University)
Growth of Schreier graphs associated to groups generated by bounded automata
ABSTRACT: We describe an algorithm for calculating the growth of diameters of Schreier graphs and the orbital contracting coefficient associated with actions of groups generated by bounded automata on levels and on the boundary of the tree. As a corollary we get estimates for the polynomial growth of Schreier graphs associated with action on the boundary. This is joint work with V. Nekrashevych.

5. **Jim Cannon** (Brigham Young University)
Dead-end elements in Thompson's group
ABSTRACT: We report on work by student Ben Woodruff (BYU PhD, 2005) who uses Blake Fordham's results to sharpen the Cleary-Taback description of dead-end elements in Thompson's group and to give a characterization of the same, notes the existence of regular strings of dead-end elements, and notes that each has a ”tail” (varying lengths) of elements pointing back toward the identity consisting of elements that are almost dead-end elements. Such structures create obvious difficulties in the most obvious proposed methods to prove the nonamenability of Thompson's group. It remains, of course, to study the asymptotic density of such elements.

6. **Yair Glasner** (University Of Illinois at Chicago)
Finitely generated vs' Normal subgroups in 3-manifold groups
ABSTRACT: Let G be the fundamental group of a 3 dimensional manifold of finite volume. We define an invariant topology on G, by taking the normal subgroups and their cosets as a basis for the topology. This is a refinement of the profinite topology. We prove that every finitely generated subgroup of G is closed in this topology. As a corollary we deduce that a maximal subgroup of infinite index in G cannot be finitely generated. The main tool used in the proof is the Marden conjecture that was recently established by Ian Agol, and by Calegari-Gabai.

This is a joint work with Pete Storm and Juan Souto.

7. **Frédéric Haglund** (University of Paris-Sud)
Commensurability and separability for uniform lattices in some polygonal complexes
ABSTRACT: We consider a (word hyperbolic) Coxeter group whose Davis-Moussong complex is two-dimensional. We show (in almost every case) that a uniform lattice of this complex is commensurable with the initial Coxeter group if all of its quasi-convex subgroups are separable. The previous "if" is an "if and only if" for example when the Coxeter group is right-angled. As an application there is a single commensurability class of uniform lattices as soon as the link of a vertex in the Davis-Moussong complex is a bipartite graph - for example in Bourdon buildings.

8. **Chris Hruska** (Chicago)
Cubulating relatively hyperbolic groups
ABSTRACT: We give criteria for proving that a group acts properly discontinuously on a locally finite CAT(0) cube complex. The criteria are inspired by ideas from the theory of relative hyperbolicity but are not limited to relatively hyperbolic groups.

We also give criteria for determining when a relatively hyperbolic group acts on a finite dimensional cube complex and when such an action is cocompact, generalizing a theorem of Sageev from the word hyperbolic setting. More generally, we describe a "cusped cofinite" structure relative to the action of the parabolic subgroups. This structure is analogous to the cusped structure of a finite volume manifold with pinched negative curvature. This is joint work with Dani Wise.

9. **Tim Hsu** (San Jose State University)
Cubulating graphs of free groups with cyclic edge groups
ABSTRACT: We prove that if G is a finitely generated group that has a decomposition as a graph of free groups with cyclic edge groups, and G is "generic" (essentially, contains no Baumslag-Solitar subgroups), then G is the fundamental group of a compact CAT(0) cube complex. We also discuss generalizations of this result. This is joint work with D. Wise.

10. **Vadim Kaimanovich** (Bremen)
Amenability of self-similar groups and random walks with internal degrees of freedom
ABSTRACT: The talk is devoted to a discussion of the relationship between random walks with internal degrees of freedom and natural matrix presentations of self-similar groups which allows one to prove amenability of such groups by establishing triviality of the Poisson boundary for appropriate random walks on such groups.

11. **Martin Kassabov** (Cornell University)
Symmetric groups and Expanders
ABSTRACT: A finite graphs with large spectral gap are called expanders. These graphs have many nice properties and have many applications. It is easy to see that a random graph is an expander but constructing an explicit examples is very difficult. All known explicit constructions are based on the group theory — if an infinite group G has property T (or its variants) then the Cayley graphs of its finite quotients form an expander family.

This leads to the following question: For which infinite families of groups G_i, it is possible to find generating sets S_i which makes the Cayley graphs expanders?

The answer of the question is known only in few case. It seems that if G_i are far enough from being abelian then the answer is YES. However if one takes 'standard' generating sets the resulting Cayley graphs are not expanders (in many cases).

I will describe a recent construction which answers the above question in the case of the family of all symmetric/alternating groups. It is possible to construct explicit generating sets S_n of Alt_n, such that the Cayley graphs $C(Alt_n, S_n)$ are expanders, and the expanding constant can be estimated.

Unlike the usually constructions of expanders, the proof does not use an infinite group with property T (although such group exists) but uses the representation theory of the symmetric groups directly.

12. **Olga Kharlampovich** (McGill)
Equations in groups with free regular length function.
ABSTRACT: I will discuss the Elimination process for solving equations in groups with free regular length function (in particular, in a free group).

13. **Avinoam Mann** (Einstein Institute of Mathematics, Hebrew University, Jerusalem)
Positively finitely generated groups and their ζ-functions
ABSTRACT: A profinite group is positively finitely generated (**PFG**) if, for some k, the set of k-tuples generating it has a positive Haar measure. We denote this measure by $P(G, k)$. E.g. finitely generated pronilpotent groups are **PFG**, while free profinite groups of rank at least two are not.

Let $m_n(G)$ be the number of maximal subgroups of G of index n. A theorem of Mann-Shalev characterizes PFG groups by the property that $m_n(G)$ grows polynomially, i.e. $m_n(G) \leq n^s$, for some constant s. It follows that f.g. prosoluble groups are **PFG**, and more generally, any f.g. profinite group that does not generate the variety of all profinite groups is **PFG** (Borovik-Pyber-Shalev). This includes the profinite completions of arithmetic groups with the congruence subgroup property **CSP**.

In many cases we can interpolate the values of $P(G, k)$ to an analytic function defined in a right half-plane of the complex plane. The reciprocal of this function is termed the probabilistic ζ-function of G. It exists, e.g., when G is prosoluble, or when it is an arithmetic group with the **CSP**.

14. **Dave Witte Morris** (University of Lethbridge)
Bounded generation of special linear groups
ABSTRACT: We present the main ideas of a nice proof (due to D.Carter, G.Keller, and E.Paige) that every matrix in $\mathrm{SL}(3, \mathbf{Z})$ is a product of a bounded number of elementary matrices. The two main ingredients are the Compactness Theorem of first-order logic and calculations of Mennicke symbols. (These symbols were developed in the 1960s in order to prove the Congruence Subgroup Property.) Similar methods apply to $\mathrm{SL}(2, A)$ if $A = \mathbf{Z}[\sqrt{2}]$ (or any other ring of integers with infinitely many units).

15. **Roman Muchnik** (Chicago)
Amenability of free Grigorchuk group
ABSTRACT: I will describe how the methods developed by V. Kaimanovich can be used to prove that the Free Grigorchuk group is amenable. The main tool used by V. Kaimanovich is to obtain a Random walk with 0 entropy. I will also describe some modifications to simplify computations.

16. **Graham Niblo** (Southampton)
An eccentric characterisation of hyperbolicity
ABSTRACT: We give a new characterisation of hyperbolicity for geodesic metric spaces in terms of the geometry of balls. It is related to Papasoglu's "thin bigons" characterisation of hyperbolic graphs. This is joint work with Indira Chatterji

17. **Stephen Pride** (Invariant Ideals for Groups)
University of Glasgow
ABSTRACT: Given a group of type FP_n, David Cruickshank and I defined a table of ideals $E(i,j)$ $(0 \leq i \leq n, j$any integer$)$ in the abelianized group ring. This table is an invariant of the group. The first column $E(1,-)$ is the chain of classical Alexander ideals. I will give the definition of these tables, describe some of their properties, give examples of calculations, and raise some open questions.

18. **Michah Sageev** (Utah & Technion)
Quasi-isometries and right angled Artin groups.
ABSTRACT: We discuss some results regarding the quasi-isometric rigidity and classification of right angled Artin groups. This is joint work with Bestvina and Kleiner.

19. **Dmytro Savchuk** (Texas A&M University)
Schreier graphs related to the Thompson's group
ABSTRACT: We will explicitly describe the Schreier graphs of the Thompson group F with respect to the stabilizer of $\frac{1}{2}$ and generators x_0 and x_1 and of its unitary representation in $L_2([0,1])$ induced by standard action on the interval $[0,1]$.

The main result is that these two graphs coincides modulo finite subsets.

20. **Dan Segal** (All Souls College, Oxford)
Subgroups of finite index in profinite groups
ABSTRACT: We answer a 30-year old question of Serre by proving that *all subgroups of finite index in a finitely generated profinite group are open*. This is deduced from the main theorem: *If w is a d-locally finite word and G is a d-generator finite group then every element of the verbal subgroup $w(G)$ is a product of $f(w,d)$ w-values.* (w is called d *-locally finite* if $F_d/w(F_d)$ is finite, where F_d is the free group of rank d, and $f(w,d)$ denotes a number that depends only on w and d.) The proof is complicated and depends on CSFG. (joint work with Nikolay Nikolov)

21. **Dan Segal** (All Souls College, Oxford)
Groups with polynomial index growth
ABSTRACT: A group G has *PIG* if there exists α such that $\left|\overline{G}/\overline{G}^n\right| \leq n^\alpha$ for every finite quotient $\overline{G}$ of G and every natural number n. This holds for example if G is an arithmetic group with the congruence subgroup property, or if G is 'boundedly generated' (a product of finitely many cyclic groups), in particular if G is a soluble group of finite rank. But there also exist uncountably many finitely generated residually finite groups with PIG that are neither linear nor virtually soluble. We answer a question posed by me 20 years ago with the Theorem: *Let G be a finitely generated soluble residually finite group. Then G has* PIG *if and only if G has finite rank*. Along the way we prove that *every infinite residually finite boundedly generated group has an infinite linear image*. The proofs use some representation theory of finite soluble groups and a lot of 'quasi-commutative algebra': the study of abelian group rings with operators. (joint work with Laci Pyber)

22. **Vladimir Shpilrain** (The City College of New York)
Translation equivalence in free groups
ABSTRACT: Motivated by the work on hyperbolic equivalence of homotopy classes of closed curves on surfaces, we investigate a similar phenomenon for free groups. Namely, we study the situation where two elements g, h in a free group F have the property that for every free isometric action of F on an $\mathbb{R}$-tree X the translation lengths of g and h on X are equal or have bounded ratio. This is joint work with I.Kapovich, G.Levitt, P.Schupp.

23. **Tatiana Smirnova-Nagnibeda** (University of Geneva)
Minimizing entropy over the Outer space
ABSTRACT: We solve the minimal volume entropy problem in the class of universal covers of finite connected metric graphs.

24. **Benjamin Steinberg** (Carleton University)
The spectra of lamplighters and related groups via automata
ABSTRACT: The speaker and Silva showed that any wreath product $G \wr Z$, with G a finite Abelian group, can be realized as the group generated by a special kind of automaton called a Cayley machine. In this talk we calculate the KNS spectral measure associated to the group of a Cayley machine, and in particular to such generalized lamplighters. KNS spectral measures were introduced by Grigorchuk and Zuk for groups acting spherically transitively on rooted trees.

We also show that the KNS spectral measure associated to an automata group coincides with the spectral measure of the simple random walk on the automata group if and only if the action of the group is free in a Baire category sense sense. This is the case for wreath product groups of the above form, and so we have given an automata-theoretic calculation of their spectral measures. A different approach has been used by Dicks and Schick to calculate these spectral measures.

This is joint work with M. Kambites and P. Silva.

25. **Zoran Sunik** (Texas A&M University)
Free-by-free right-angled Artin groups
ABSTRACT: A group H is poly-free if it has a subnormal series

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = H$$

in which all factors are free. Equivalently, H is a finitely iterated semidirect product of free groups. The shortest length of a subnormal series with free factors is called the poly-free length of H.

Our main results are as follows.

All right-angled Artin groups are poly-free. The poly-free length of a right-angled Artin group $A\Gamma$ is bounded between the clique number and the chromatic number of the graph Γ that defines the group $A\Gamma$. An explicit realization of a subnormal (in fact normal) series with free factors and of length equal to the chromatic number of Γ is provided.

A complete characterization of graphs that define right-angled Artin groups of poly-free length 2 is given. Such graphs must have an independent set of vertices D such that every cycle in Γ contains at least two vertices from D.

Finally, considerations involving the Euler characteristic allow us to conclude that a right-angled Artin group $A\Gamma$ has poly-free length 2 with both factors of finite rank, i.e., $A\Gamma$ is a semidirect product of two free groups of finite rank if and only if the defining graph Γ is a tree or a complete bipartite graph.

This work is motivated by a question of Bestvina asking if all Artin groups are virtually poly-free.

26. **Balint Virag** (Toronto)
Torsion generators and slow random drives in Britain
ABSTRACT: A b-spinner graph is a sum of directed cycles of length at most b. An example is a simplified road map of Britain consisting of roundabouts and short two-way streets. Another example is any directed Cayley graph of the Grigorchuk group.

We give up-to-constant optimal upper bounds on the rate of escape of random walks on spinner graphs in terms of their growth.

For torsion groups of intermediate growth, there is no set of generators for which the corresponding random walk escapes at a positive speed. (This statement is false for any infinite nilpotent group.)

This is joint work with David Revelle.

27. **Andrzej Zuk** (CNRS, Paris VI)
Automata groups
ABSTRACT: We present recent results concerning groups generated by finite automata.

List of Participants

Abert, Miklos (University of Chicago)
Alperin, Roger (San Jose State University)
Behrstock, Jason (Columbia University)
Bondarenko, Ievgen (Texas A&M University)
Cannon, Jim (Brigham Young University)
Chatterji, Indira (Columbia University)
Glasner, Yair (University of Illinois at Chicago)
Grigorchuk, Rostislav (Texas A&M University)
Haglund, Frederic (University of Paris-Sud)
Hruska, Chris (University of Chicago)
Hsu, Tim (San Jose State University)

Kaimanovich, Vadim (International University Bremen)
Kassabov, Martin (Cornell University)
Kharlampovich, Olga (McGill University)
Kropholler, Peter (University of Glasgow)
Lauer, Joe (McGill University)
Mann, Avinoam (Hebrew University, Jerusalem)
Miasnikov, Alexei (McGill University)
Morris, Dave Witte (University of Lethbridge)
Muchnik, Roman (University of Chicago)
Niblo, Graham (University of Southampton)
Nica, Bogdan (Vanderbilt University)
Pride, Steve (University of Glasgow)
Rhemtulla, Akbar (University of Alberta)
Sageev, Michah (Technion-Israel Institute of Technology)
Sapir, Mark (Vanderbilt University)
Savchuk, Dmytro (TexasA&M)
Segal, Dan (Oxford University)
Shpilrain, Vladimir (City College of New York)
Smirnova-Nagnibeda, Tatiana (University of Geneva)
Steinberg, Benjamin (Carleton University)
Sunik, Zoran (Texas A&M University)
Virag, Balint (University of Toronto)
Wise, Dani (McGill University)
Zuk, Andrzej (Institut de Mathematiques)

Chapter 12

Combinatorial Game Theory Workshop (05w5048)

June 18–23, 2005

Organizer(s): E. Berlekamp (Berkeley), M. Mueller (University of Alberta), R. J. Nowakowski (Dalhousie University), D. Wolfe (Gustavus Adolphus College)

A main aim of the workshop was to bring together the two camps, mathematicians working in combinatorial game theory and computer scientists interested in algorithmic and Artificial intelligence.

The Workshop attracted a mix of people from both communities (17 from mathematics, 16 from computer science and 2 undergraduates) as well as a mixture of new and established researchers. The oldest was Richard Guy, turning 90 in 2006 and the youngest was in 3rd year University. There were attendees from Europe, Asia as well as North America.

The Workshop succeeded in its primary goal and more. New collaborations were struck. There was quick dissemination and evaluation of major new results and new results were developed during the Workshop. Part of the success was due to the staff and facilities at BIRS.

The facilities at BIRS were appreciated by all the participants. The main room allowed lectures to mix computer presentations with overheads and chalkboard calculations. (No prizes for guessing which community used which technology.) The coffee lounges and break away rooms allowed discussions to continue on, in comfort, until late in the night. Our thanks go to all the staff who made the stay such a wonderful experience and to the BIRS organization for hosting the workshop.

The elder statesmen of the community, Berlekamp, Conway, Fraenkel and Guy, all took active roles in the proceedings. The first three gave survey talks on various topics and all were involved in discussions throughout the days and the evenings. The younger (established) generation were represented by the likes of Demaine, Grossman, Müller and Siegel.

As befits a workshop on combinatorial games, games were invented, played and analyzed. Philosophers Football (Phutball) was much in evidence. There was a Konane tournament played over three evenings. Much effort went into attempting an analysis of Sticky Towers of Hanoi, a game invented at the Workshop by Conway, spearheaded by Conway and the youngest attendee, Alex Fink. There were many representatives of the Go community quite a few of whom had never met each other.

Collaboration is very important in this community. For example, David Wolfe presented a progress report on work of G. A. Mesdal on a class of partizan splitting games, answering questions first raised over 30 years ago. Mesdal is a joint effort of eleven co-authors from North America and New Zealand. Eight of the eleven attended the workshop and the number of co-authors had risen to twelve by the time the Workshop ended.

In the end, between the talks and the discussions, there was simply too much to absorb in such a short time. The talks, surveys and several consequent papers are slated to appear as (tentatively titled) *Games of No Chance 3* in the MSRI book series.

All the presentations were at a high standard and all had lively discussions during and after the time allotted. Some highlights were: Conway's talk on lexicodes; Berlekamp's overview *Today's View of Combinatorial Game Theory*; and Fraenkel's *What hides beyond the curtain separating Nim from non-Nim games*; Demaine's talk on Dyson Telescopes and Moving Coin Puzzles also showed the complexity in some very new and some very old puzzles.

However the highlights were the reports by:

1. *Plambeck on a breakthrough in the analysis of impartial misèr games;*

2. *Siegel on extending the analysis of loopy games;*

3. *Friedman and Landsberg on applying renormalization techniques from physics to combinatorial games—this paper was both controversial and thought provoking and lead to the most discussions, including ones on the nature of truth and of proof;*

4. *Nakamura on the use of 'cooling by 2' to determine the winner in 'races to capture' in Go. One of the goals of the Workshop was to bring together researchers from mathematics and computer science. This was one of the talks that helped bridge the gap and engendered much discussion. The 30 minutes allotted to the talk was too short, and most participants stayed an extra hour (into the dinner-time) so as to hear the details.*

All of these were very new, very important results, produced only months before the Workshop.

1: Misère Games: On page 146 of *On Numbers and Games*, in Chapter 12, "How to Lose When You Must," John Conway writes:

> *Note that in a sense, [misére] restive games are* ambivalent Nim-heaps*, which choose their size (g_0 or g) according to their company. There are many other games which exhibit behaviour of this type, and it would be very interesting to have some general theory for them.*

Questions about the analysis of misére impartial octal games were raised in [3, 6] and no good general analytical techniques have been developed apart from finding the genus sequence [3]. (See [1, 22], see also [8, 20]). In his presentation, Plambeck provided such a general theory, cast in the language of commutative semigroups.

The misère analysis of a combinatorial game often proves to be far more difficult than its normal play version. In fact it is an open question (Plambeck) if there is a misére impartial game whose analysis is simpler than the normal play version and there is no know way of analyzing misére partizan games ([15], Problem 9).

To take a typical example, the normal play of Dawsons Chess was solved as early as 1956 by Guy and Smith [16], but even today, a complete misère analysis has not been found. Guy tells the story [15]:

"[Dawsons chess] is played on a $3 \times n$ board with white pawns on the first rank and black pawns on the third. It was posed as a losing game (last-player-losing, now called misère) so that capturing was obligatory. Fortunately, (because we still don't know how to play Misère Dawsons Chess) I assumed, as a number of writers of that time and since have done, that the misère analysis required only a trivial adjustment of the normal (last-player-winning) analysis. This arises because Bouton, in his original analysis of Nim [5], had observed that only such a trivial adjustment was necessary to cover both normal and misère play..."

But even for impartial games, in which the same options are available to both players, regardless of whose turn it is to move, Grundy & Smith [14] showed that the general situation in misère play soon gets very complicated, and Conway [6], (p. 140) confirmed that the situation can only be simplified to the microscopically small extent noticed by Grundy & Smith.

In Chapter 13 of [3], the genus theory of impartial misère disjunctive sums is extended significantly from its original presentation in chapter 7 (How to Lose When You Must) of Conways On Numbers and Games [6]. But excluding the tame games that play like Nim in misère play, theres a remarkable paucity of example games that the genus theory completely resolves. For example, the section Misère Kayles ([3], pg 411) promises, "Although several tame games arise in Kayles (see Chapter 4), wild games abounding and well need all our [genus-theoretic] resources to tackle it..." However, it turns out Kayles wasn't tackled at all. It was left to the amateur William L. Sibert to settle misère Kayles using completely different methods. One finds a description of his solution at end of the updated Chapter 13 in [4], and also in [22].

When normal play is in effect, every game with nimber $G^+(G) = k$ can be thought of as the nim heap k. No information about best play of the game is lost by assuming that G is in fact precisely the nim heap of size k. Moreover, in normal play, the nimber of a sum is just the nim-sum of the nimbers of the summands. In this sense every normal play impartial game position is simply a disguised version of Nim (see [3], Chapter 4, for a full discussion).

Genera. When misère play is in effect, nimbers can still be defined but many inequivalent games are assigned the same nimber, and the outcome of a sum is not determined by nimber of the summands. These unfortunate facts lead directly to the apparent great complexity of many misère analyses. Nevertheless progress can be made. The key definition, taken directly from [6], now at the bottom of page 141: In the analysis of many games, we need even more information than is provided by either of these values [G^+ and G^-], and so we shall define a more complicated symbol that we call the G^o-value or genus. This is the symbol

$$g.g_0g_1g_2\ldots$$

where $g = G^+(G)$, $g_0 = G^-(G)$, $g_1 = G^-(G + 2)$, $g_2 = G^-(G + 2 + 2)$, . . ., where, in general, g_n is the G^--value of the sum of G with n other games all equal to [the nim-heap of size] 2.

At first sight, the genus symbol looks to be an potentially infinitely long symbol in its exponent. In practice, it can be shown that the g_is always fall into an eventual period two pattern. By convention, the symbol is written down with a finite exponent with the understanding that its final two values repeat indefinitely.

Evidently the exponent of a genus symbol of a game G is closely related to the outcome of sums of G with all multiples of misère nim heaps of size two.

The genus computations are intended to illustrate the complexities of a misère analysis when the only tools available to be applied are those described in Chapter 13 of Winning Ways.

Plambeck's breakthrough was to introduce a *quotient semigroup* structure on the set of all positions of an impartial game with fixed rules. The basic construction is the same for both normal and misère play. In normal play, it leads to the familiar Sprague-Grundy theory. In misère play, when applied to the set of all sums of positions played according to a particular game's rules, it leads to a quotient of a free commutative semigroup by the game's indistinguishability congruence. Playing a role similar to the one that *nim sequences* do for normal play, mappings from single-heap positions into a game's misère quotient semigroup succinctly and necessarily encode all relevant information about its best misère play. Plambeck showed examples of wild misère games that involve an infinity of ever-more complicated canonical forms amongst their position sums that may nevertheless possess a relatively simple, even finite misère quotient. Suppose Γ is a taking and breaking game whose rules have been fixed in advance. Let h_i be a distinct, purely formal symbol for each $i \geq 1$. We will call the set $H = \{h_1, h_2, h_3, \ldots\}$ the *heap alphabet*. A particular symbol h_i will sometimes be called a *heap of size* i. The notation H_n stands for the subset $H_n = \{h_1, \ldots, h_n\} \subseteq H$ for each $n \geq 1$. Let $\mathcal{F}_H$ be the free commutative semigroup on the heap alphabet H. The semigroups $\mathcal{F}_H$ and $\mathcal{F}_{H_n}$ include an identity Λ, which is just the empty word. There's a natural correspondence between the elements of $\mathcal{F}_H$ and the set of all position sums of a taking and breaking game Γ. In this correspondence, a finite sum of heaps of various sizes is written multiplicatively using corresponding elements of the heap alphabet H. This multiplicative notation for sums makes it convenient to take the convention that the empty position $\Lambda = 1$. It corresponds to the *endgame*—a position with no options. Fix the rules and associated *play convention* (normal or misère) of a particular taking and breaking game Γ. Let $u, v \in \mathcal{F}_H$ be game positions in Γ. We'll say that u *is indistinguishable from* v *over* $\mathcal{F}_H$, and write the relation $u \; \rho \; v$, if for every element $w \in \mathcal{F}_H$, uw and vw are either both P-positions, or are both N-positions.

Lemma 1 The relation ρ is a congruence on $\mathcal{F}_H$.

Suppose the rules and play convention of a taking and breaking game Γ are fixed, and let ρ be the indistinguishability congruence on $\mathcal{F}_H$, the free commutative semigroup of all positions in Γ. The *indistinguishability quotient* $\mathcal{Q} = \mathcal{Q}(\Gamma)$ is the commutative semigroup

$$Q = \mathcal{F}_H/\rho.$$

Notice that the indistinguishability quotient can be taken with respect to either play convention (normal or misère). The details of the indistinguishability congruence then determine the structure of the indistinguishability quotient. Since the word "indistinguishability" is quite a mouthful, $\mathcal{Q}$ is called the *quotient semigroup* of Γ. When Γ is a normal play game, its quotient semigroup $\mathcal{Q} = \mathcal{Q}(\Gamma)$ is more than just a semigroup. *A* *re-interpretation of the Sprague-Grundy theory says that these are always groups, each isomorphic to a direct product of a (possibly infinite) set of Z_2's (cyclic groups of order two).* If u is a position in $\mathcal{F}_H$ with normal play nim-heap equivalent $*k$, the members of a particular congruence class $u\rho \in \mathcal{F}_H/\rho$ will be precisely all positions that have normal-play nim-heap equivalent $*k$. The identity of $\mathcal{Q}$ is the congruence class of all positions with nim-heap equivalent $*0$. The "group multiplication" corresponds to nim addition. For misère play, the quotient structure is a *semigroup*. Surprisingly, it's often a finite object, even for a game that has an infinite number of different canonical forms occurring amongst its sums. The elements of a particular congruence class all have the same outcome. Each class can be thought of as carrying a big stamp labelled "P" (previous player wins in best play for all positions in this class) or "N" (next player wins). In normal play, there's only one equivalence class labelled "P"—these are the positions with nim heap equivalent $*0$. In misère play, for all but the trivial games with one position $*0$, or two positions $\{*0, *1\}$, there is always more than one "P" class—one corresponding to the position $*1$, and at least one more, corresponding to the position $*2 + *2$.

At the time of the presentation, Plambeck had 20 games each of whose octal description was short but whose analysis had defied his attempts. Plambeck offered varying amounts of money for their solutions. During the Workshop, Aaron Siegel solved four of them and, in conjunction with Plambeck, has solved all of the games and produced a computer program that helps with representations of the quotient semi-groups.

2: Loopy Games. Aaron Siegel reported on two parts of the work contained in his PhD thesis, this particular presentation concerned loopy games. The traditional theory of combinatorial games assumes that no position may be repeated. This restriction guarantees that arbitrary sums of games will terminate; the result is a clean, recursive, and computationally efficient theory. However, there are many interesting games that allow repetition, including Fox and Geese, Hare and Hounds, Backsliding Toads and Frogs, Phutball and Checkers. Go is a peculiar example: the ko rule forbids most repeated positions, but local repetition is extremely important when the board must be decomposed to effect a tractable analysis.

Every game that permits repeated positions faces the possibility of nonterminating play. This is typically resolved by declaring infinite plays drawn (as in Checkers and Chess), but alternative resolutions are not uncommon. For example, Hare and Hounds declares infinite plays wins for the Hare, and some dialects of Go rules forbid them altogether. The disjunctive theory, in its most general form, assumes that in sums within finite play, the game is drawn unless the same player wins on every component in which play is nonterminating. This is vacuously true for games where infinite play is drawn to begin with, and it applies equally well to games such as Hare and Hounds. Go, with its unique ko rule, does not fit so cleanly into the theory.

The general disjunctive theory was first considered by Robert Li [17], who in the mid-1970s focused on games where it is a disadvantage to move, including a variant of Hackenbush. Shortly thereafter, Conway, together with his students Clive Bach and Simon Norton, generalized and codified the theory and coined the term loopy game. Their results, including the fundamental concepts of stoppers and sides, appeared first in a 1978 paper [6] and were reprinted in Winning Ways. At roughly the same time, Shaki [21] and Fraenkel and Tassa [13] studied approximations and reductions of partizan loopy games under a slightly different set of assumptions. Despite this flurry of initial activity, there were few advances in the two decades following the first publication of Winning Ways. Moews generalization of sidling was a rare exception: Published in his 1993 thesis [18, 19],it constituted the first real advance in the disjunctive theory since the late 1970s.

Various authors have studied loopy games in other contexts. Generalizations of the Sprague-Grundy theory to impartial loopy games were introduced by Smith [23] a full decade before Li invented the partizan theory. They were studied in the 1970s by Fraenkel and Perl [11] and Conway[3], and much more recently by Fraenkel and Rahat [12]. James Flanigan, in his 1979 thesis and two subsequent papers [9, 10], analyzed conjunctive and selective sums of partizan loopy games.

Meanwhile, the greatest advance of the 1990s came from an entirely different quarter, the study of kos in Go. The interplay between local cycles and the global state of the position gives rise to a rich and fascinating temperature theory, which appears to differ from Conways disjunctive theory in striking ways. The theory was first realized by Berlekamp, following his analysis of loop free Go positions with his student David Wolfe (see [2]. Many others have since investigated the theory of kos, including Fraser, Müller, Nakamura, Spight and Takizawa. (See [25, 24, 27, 28], for some examples.)

Siegel showed how to calculate canonical forms of loopy games and gave some of their characteristics.

One of his remarkable achievements is the software package CGSuite (for the "computationally efficient theory" of finite disjunctive sums) and then and its extension to be able to calculate the canonical form of loopy games.

Siegel, Ottaway and Nowakowski showed how rich the canonical forms of small games can be when they considered 1-dimensional Phutball played on boards of length 7, 8, 9, 10, and 11.

3: Cooling and Go. The applications of combinatorial game theory to the game of Go have, so far, been focused on endgames and eyespace values. A *capturing race* is a particular kind of life and death problem in which both of the two adjacent opposing groups are fighting to capture the opponent's group each other. Skills in winning races are very important factor to the strength of Go as well as openings and endgames techniques. In order to win the complicated capturing races, techniques of counting liberties, taking away the opponent's liberties and extending own liberties in addition to wide and deep reading are necessary. Nakamura, ""On Counting Liberties in Capturing Races of Go" showed that the 'counting' required can be regarded as combinatorial game with a score. Within this framework, he showed how to analyze capturing races that have no shared liberty or have just simple shared liberties using combinatorial game values of external liberties and an evaluation formula to find out the outcome of the capturing races. Essentially, the evaluation formula is by cooling. All applications of cooling so far have been chilling (cooling by 1) but in this case, one must cool by 2 !

4: Renormalization techniques.

Friedman & Landsberg presented a new approach to combinatorial games that unveiled connections between such games and nonlinear phenomena commonly seen in nature: scaling behaviors, complex dynamics and chaos, growth and aggregation processes. Using the game of Chomp (as well as variants of the game of Nim) as prototypes, they showed that the game possesses an underlying geometric structure that grows (reminiscent of crystal growth), and showed how this growth can be analyzed using a renormalization procedure. This approach not only obtains answers to some open questions about the game of Chomp, but opens a new line of attack for understanding (at least some) combinatorial games more generally through their underlying connection to nonlinear science.

Analysis of these two-player games has generally relied upon a few beautiful analytical results or on numerical algorithms that combine heuristics with look-ahead approaches ($\alpha - \beta$ pruning). Using Chomp as a prototype, this new geometrical approach unveils unexpected parallels between combinatorial games and key ideas from physics and dynamical systems, most notably notions of scaling, renormalization, universality, and chaotic attractors. Their central finding is that underlying the game is a probabilistic geometric structure that encodes essential information about the game, and that this structure exhibits a type of scale invariance: Loosely speaking, the geometry of small winning positions and large winning positions are the same after rescaling. (This general finding also holds for at least some other combinatorial games, as was explicitly demonstrated with a variant of Nim.) This geometric insight not only provides (probabilistic) answers to some open questions about Chomp, but suggests a natural pathway toward a new class of algorithms for more general combinatorial games, and hints at deeper links between such games and nonlinear science.

Chomp is an ideal candidate for the study, since in certain respects it appears to be among the simplest in the class of hard games. Its history is marked by some significant theoretical advances but it has yet to succumb to a complete analysis in the 30 years since its introduction by Gale and Schuh. The rules of Chomp are easily explained. Play begins with an N x M array of counters. On each turn a player selects a counter and removes it along with all counters to the north and east of it. Play alternates between the two players until one player takes the last counter, thereby losing the game. (An intriguing feature of Chomp, as shown by Gale, is that although it is very easy to prove that the player who moves first can always win, under optimal play, what this opening move should be has been an open question. The methodology provides a probabilistic answer to this question.)

For simplicity, consider the case of three-row (M=3) Chomp, a subject of recent study by Zeilberger [29] and Sun [26]. Generalizations to four-row and higher Chomp are analogous. To start, note that the configuration of the counters at any stage of the game can be described (using Zeilbergers coordinates) by the position p=[x,y,z], where x specifies the number of columns of height three, y specifies the number of columns of height two, and z the number with height one. Each position p may be classified as either a winner, if a player starting from that position can always force a win, or as a loser otherwise. The set of all losers contains the information for solving the game. One may conveniently group the losing positions according to their x values by defining a loser sheet Lx to be an infinite two-dimensional matrix whose (y,z)th component is a 1 if position [x,y,z] is a loser, and a 0 otherwise. (As noted by Zeilberger, one can express Lx in terms of all preceding loser sheets Lx-1, Lx-2, , L0.) Studies by Zeilberger [29, 30] and others have detected several numerical patterns along with a few analytical features about the losing positions, and their interesting but non-obvious properties have even led to a conjecture that Chomp may be chaotic in a yet-to-be-made-precise sense. However, many of the numerical observations to date have remained largely unexplained, and disjoint from one another.

To provide broader insight into the general structure of the game, the authors departed from the usual analytic/algebraic/algorithmic approaches. Instead showing how the analysis of the game can be recast and transformed into a type of renormalization problem commonly seen in physics (and later apply this methodology to other combinatorial games besides Chomp). Analysis of the resulting renormalization problem not only explains earlier numerical observations, but provides a unified, global description of the overall structure of the game. This approach will be distinguished by its decidedly geometric flavor, and by the incorporation of probabilistic elements into the analysis, despite the fact that the combinatorial games we consider are all games of no chance which lack any inherent probabilistic components to them whatsoever.

To proceed, consider the so-called instant-winner sheets, defined as follows: A position p=[x,y,z] is called an instant winner if from that position a player can legally move to a losing position with a smaller x- value. We therefore define an instant-winner sheet Wx to be the infinite, two- dimensional matrix consisting of all instant winners with the specified x-value, i.e., the (y,z)th component of matrix Wx is a 1 if position [x,y,z] is an instant winner, and a 0 otherwise. These instant-winner sheets will prove crucial for understanding the geometric structure of the game.

Their first insight comes from numerical simulations. They numerically construct the instant winner sheets Wx for various x values using a recursive algorithm. Each sheet exhibits a nontrivial internal structure characterized by several distinct regions: a solid (filled) triangular region at the lower left, a series of horizontal bands extending to the right (towards infinity), and two other triangular regions of different densities. Most importantly, however, we observe that the set of instant-winner sheets Wx possess a remarkable scaling property: their overall geometric shape is identical up to a scaling factor! In particular, as x increases, all boundary-line slopes, densities, and shapes of the various regions are preserved from one sheet to the next (although the actual point- by-point locations of the instant winners within each sheet are different). Hence, upon rescaling, the overall geometric structure of these sheets is identical (in a probabilistic sense). The growth (with increasing x) of the instant-winner sheets is strikingly similar to certain crystal-growth and aggregation processes found in physics in each case, the structures grow through the accumulation of new points along current boundaries, and exhibit geometric invariance during this process. The loser sheets Lx can be numerically constructed in a similar manner; their characteristic geometry is revealed. It is found to consist of three (diffuse) lines: a lower line of slope mL and density of points L, an upper line of slope mU and density U, and a flat line extending to infinity. The upper and lower lines originate from a point whose height (i.e., z-value) is ax. The flat line (with density one) is only present with probability in randomly selected loser sheets. Like the instant-winner sheets, the loser sheets also exhibit this remarkable geometric scaling property: as x increases, the geometric structure of Lx grows in size, but its overall shape remains unchanged (the only caveat being that, as previously noted, the flat line seen in is sometimes absent in some of the loser sheets).

The second key finding is that there exists a well-defined, analytical recursion operator that relates one instant winner sheet to its immediate predecessor. Namely, one can write Wx+1 = R Wx, where R denotes the recursion operator. (The operator R can be decomposed as R=L(I+DM), where L is a left-shift operator, I is the identity operator, D is a diagonal element-adding operator, and M is a sheet-valued version of the standard mex operator which is often used for combinatorial games.) They point out that once a given instant-winner sheet Wx has been constructed, the corresponding loser sheer Lx can be found via Lx = M Wx.

The task is to determine an invariant geometric structure W such that if we act with the recursion operator followed by an appropriately-defined rescaling operator S, we get W back again: W = SR W (i.e., find a fixed point of the renormalization-group operator SR.) This can be done, but before doing so, even though the recursion operator R is exact and the game itself has absolutely no stochastic aspects to it, it is necessary to adopt a probabilistic framework in order to solve this recursion relation. Namely, the renormalization procedure will show that the slopes of all boundary lines and densities of all regions in the Wxs (and Lxs) are preserved not that there exists a point-by-point equivalence. In essence, bypassing consideration of the random-looking scatter of points surrounding the various lines and regions of Wx and Lx by effectively averaging over these fluctuations.

The key to implementing the renormalization analysis is to observe that the losers in Lx are constrained to lie along certain boundary lines of the Wx plot, and are conspicuously absent from the various interior regions of Wx (for all x). In other words, the interior regions of each Wx remain forbidden to the losers. Hence the geometry of Wxs must be very tightly constrained if it is to preserve these symmetries.

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Chapter 13

Rigidity, Dynamics, and Group Actions (05w5029)

July 9–14, 2005

Organizer(s): David Fisher (Lehman College - CUNY), Elon Lindenstrauss (Princeton University), Dave Witte Morris (University of Lethbridge), Ralf Spatzier (University of Michigan)

Rigidity theory has its roots in classical theorems of Selberg, Weil, Mostow, Margulis and Furstenberg. It extends into diverse areas such as complex and differential geometry, group theory and representation theory, ergodic theory, dynamics and group actions. Our conference "Rigidity, Dynamics and Group Actions" concentrated on the rapid recent progress in these areas. The study of "large" groups (such as lattices in semisimple groups or higher rank abelian groups) and their actions was the focal point of the conference, with particular attention given to the following four closely related topics:

- local and global rigidity of actions,
- low-dimensional actions of large groups,
- orbit-equivalence rigidity, and
- invariant measures for actions on homogeneous spaces.

We had many exciting talks on these and other topics on large groups. Exciting recent progress more generally in dynamics, geometry and geometric group theory was also discussed and presented in talks. Many exciting new connections between dynamics of group actions and other areas, including number theory, geometry, and operator algebras, were discussed.

The organizers have established a resource page for the workshop[1]. There are also plans to expand an existing problem list from an earlier workshop to reflect the problem session at this workshop.

Classification of Group Actions

Let $G = SL(n, R)$ and $\Gamma = SL(n, Z)$, with $n \geq 3$. More generally, we can consider any simple Lie group G of real rank at least two, and a lattice Γ in G. For any natural number ℓ, the classical theory of roots and weights determines all of the homomorphisms from G into $GL(\ell, C)$. Roughly speaking, Margulis'

[1] http://people.uleth.ca/∼dave.morris/banff-rigidity/

Superrigidity Theorem (1975) shows that roots and weights of groups closely related to G determine all of the homomorphisms from Γ into $GL(\ell, C)$.

These two results classify the linear actions of G or Γ on (complex) vector spaces. Zimmer's non-linear generalization of Margulis' superrigidity theorem opened the way to classifying "non-linear represenions" of these groups. One such non-linear variant is to study ergodic group actions of G or Γ up to orbit equivalence. For higher rank groups and their lattices, orbit equivalence is now fairly well understood, due primarily to work of Zimmer and Furman. Recent progress has focused on other types of groups, see the recent survey by Shalom [29].

A more difficult nonlinear, problem is to classify the smooth (C^∞) actions of G or Γ on compact, smooth manifolds. This work is closely connected to understanding the structure of known algebraic actions, and also to several questions in pure ergodic theory.

Very few volume-preserving actions of Γ (on a compact manifold) are known. One example is the standard action of $\Gamma = SL(n, Z)$ on the n-torus. Certain examples similar to this are called *affine algebraic actions*; they arise from purely algebraic (group-theoretic) constructions.

In 1996, Katok and Lewis produced the first examples of non-algebraic actions. However, the actions were constructed by making minor topological modifications of algebraic ones. It may be the case that every volume-preserving action is isomorphic to an algebraic action, after certain sets of measure zero are deleted.

Local rigidity [7, 9]

A smooth action ρ of Γ is said to be *locally rigid* if every "nearby" smooth action is smoothly conjugate to ρ. Building upon many authors' results of the last 15 years, Fisher and Margulis established local rigidity for all affine algebraic actions [9]. Thus, perturbing an affine algebraic action will not result in a non-algebraic action.

Fisher recently pushed through another approach to local rigidity, generalizing some of Weil's ideas for proving local rigidity of lattices in Lie groups. It is often easier to prove infinitesimal rigidity of a subgroup or action. Weil for subgroups and now Fisher for actions showed how to go from infinitesimal to local rigidity. For actions this is a highly non-trivial problem due to the difficulty of suitable inverse function theorems. This approach has many novel applications to groups not covered by any previous local rigidity results. Fisher reported on this in his talk at the workshop. He also discussed work in progress with T.J.Hitchman which would produce further applications of this result.

Dynamics and global rigidity [6]

Margulis and Qian proved a global rigidity result for actions of Γ on tori under some further assumptions. Goetze and Spatzier completely classified the much more restricted class of "Cartan" actions on arbitrary compact manifolds.

These proofs use the study of "hyperbolic" actions of higher rank abelian groups by Katok, Spatzier and others. As for lattices, all irreducible actions of this type are conjectured to be "algebraic."

The cross fertilization between these areas has been crucial. For example, local rigidity of the higher rank abelian actions led to the proof of local rigidity of projective actions of higher rank cocompact lattices. This is also closely related to work of Katok-Spatzier discussed below in (13).

More recent developments concerning global rigidity of group actions have introduced a plethora of new techniques and ideas into the field and some of these were reported on at the meeting, but are discussed below in the section on low dimensional actions. In low dimensions, the classification problem simplifies to showing that no examples exist!

Arithmetic Quotients [10]

Recent work of Lubotzky, Zimmer, and Fisher constructs a measurable map from any volume-preserving action of G or Γ to some algebraic example. Fisher and Whyte gave conditions under which the map is continuous. Under additional assumptions, the algebraic action is "close to" the original action. More recent results of Schmidt have drawn closer connections between global rigidity and the study of arithmetic quotients.

Low-Dimensional Actions of Large Groups [2, 11, 13, 19, 23]

Zimmer conjectured that Γ cannot act (faithfully) on any compact manifold M whose dimension is much smaller than the size of G. This has not been proved in complete generality even when $dim(M) = 1$, although much progress was made in a sequence of works by Witte, Ghys, Burger and Monod, Navas, and Lifschitz and Morris. More recently, there has been dramatic progress when $dim(M) = 2$ as well, assuming the action is volume-preserving, and that G/Γ is not compact. Under these assumptions, Polterovich eliminated all the surfaces of genus at least 1, by using techniques from symplectic topology. Franks and Handel were able to eliminate the other surfaces, under a stronger assumption on Γ, by using a completely different approach based on low-dimensional dynamics, including a structure theory for area preserving diffeomorphisms of surfaces. Franks discussed some of these results in his talk. In connection with this work, M. Handel explained his joint work with Franks on fixed points for actions of higher rank abelian groups on R^2 and S^2. Higher rank abelian actions have been prominent in recent years, due to the discovery of many rigidity properties. The work of Franks and Handel again shows that such actions are very special.

Vanishing of bounded cohomology groups is an obstruction to non-trivial actions on the circle. Recent work of Ghys-Gambaudo and Polterovich indicate that bounded cohomology may also be relevant to studying actions on surfaces.

One can interpret elements of the second bounded cohomology of a group Γ as quasi-morphisms. Polterovich gave a brief overview over of quasi-morphisms and how they arise for groups of Hamiltonian diffeomorphisms at the workshop. This very inspiring lecture will serve as an excellent departure point for future work in the area.

In the complex analytic setting, S. Cantat recently established a version of Zimmer's conjecture. This combines holomorphic dynamics with arguments from algebraic geometry, and is the first result of this kind for actions preserving a non-rigid geometric structure (the complex structure).

Cocycle Superrigidity [5, 8, 30]

A fundamental tool, in the analysis of actions of large groups is Zimmer's extension of Margulis superrigidity theorem for cocycles for higher rank semisimple Lie groups without compact factors. As reported by Hitchman, he and Fisher extended these cocycle superrigidity results to actions of the Kazhdan rank 1 groups and their lattices using the harmonic maps approach to superrigidity. This builds on earlier work of Korevaar-Schoen and Corlette-Zimmer and also gives new proofs of the known cases of superrigidity. This will allow Fisher and Hitchman to prove many results for actions of these groups which had so far only been available for higher rank groups.

In recent years various superrigidity results were obtained for lattices in products of groups and even simply for products of finitely generated groups by many authors, particularly Shalom and Monod. Furman reported on work with Monod in which they generalized Zimmer's superrigidity theorem for (certain) cocycle over actions of such groups, and applied it to the study of their smooth actions.

Another extension of superrigidity for cocycles was announced at the conference by Popa. His result applies to a wide class of groups, but requires that the cocycle be over an action which is Bernouilli. This is related to Popa's recent work on orbit equivalence, which is discussed in the next subsection.

Orbit Equivalence Rigidity [12, 24, 29]

Two actions on measure spaces are said to be *orbit equivalent* if there is a bi-measurable map that takes orbits to orbits. This notion is central in ergodic theory. For discrete amenable groups, essentially all actions preserving a finite measure are orbit equivalent. At the other extreme, Zimmer's superrigidity theorem implies that non-isomorphic actions of G are never orbit equivalent. The situation for actions of a lattice is more subtle and was recently resolved by Furman. Furman was inspired by the classification of lattices up to quasi-isometry in geometric group theory. Furman's work has been used by logicians to solve longstanding problems on Borel equivalence relations.

Several authors have recently proven orbit equivalence results for more general groups. Gaboriau showed that ℓ^2-Betti numbers of groups are invariant under measure equivalence. This allowed him to distinguish free groups and their products under measure equivalence. Monod and Shalom used techniques from bounded cohomology to prove measure-equivalence rigidity of products of groups acting on CAT(-1)-spaces. More recently, Gaboriau and Popa have used techniques from operator algebras in conjunction with ideas from rigidity theory to produce uncountably many non-orbit equivalent actions of the free group.

During the workshop Popa reported on his recent work on the strong rigidity of II_1-factors of rigid groups, and in particular of Bernoulli actions of groups which have relative property (T). He also sketched some of the ideas in his more recent work, which yields orbit equivalence "super-rigidity" theorems for remarkably broad classes of groups. His lecture provided a good bridge to the world of operator algebras from the more classical areas of rigidity theory.

Flows on homogeneous spaces and related topics

In the previous section we described attempts to classify actions of large groups. Another major theme of research has been the study of the properties of concrete group actions. A basic class of such actions is the following: Let G be a locally compact group (usually either a Lie group or an S-arithmetic algebraic linear group), $\Gamma < G$ a discrete subgroup, and $H < G$ some other closed subgroup G. Then one may study the action of H on G/Γ. These actions are fascinating for their own sake and arise naturally in many contexts particularly in number theory, and also in the study of the rigidity questions discussed in the other sections of this summary.

A basic question considered about these actions in the classification of H-invariant measures on G/Γ and of H-invariant closed subsets. A major landmark in this direction has been Margulis' resolution of the long-standing Oppenheim conjecture regarding values of indefinite quadratic forms by classifying closures of $SO(2,1)$-orbits in $SL(3,R)/SL(3,Z)$.

This classification result is a very special case of much more general theorems proved a few years later by Ratner [26, 27] on invariant measures and orbit closure for actions of groups generated by unipotents (such as the Lie group $SO(2,1)$).

Invariant Measures For Actions on Homogeneous Spaces and Applications to Number Theory [3, 14, 16, 18, 20, 21, 26, 28]

Ratner's work (even her work on orbit closures) is based on the classification of measures invariant under groups generated by unipotents, and the many applications of this work are too numerous to be listed here! In the workshop H. Oh explained her work with Gorodnik and Shah on equidistribution of rational points in affine spaces refining earlier work of Eskin and McMullen on the growth of the number of such points, a key ingredient of which was Ratner's theorems.

Ratner's measure classification results apply only to finite invariant measures. If one considers flows on a quotient space G/Γ of infinite volume the situation is much less is understood. O. Sarig explained his work with Ledrappier on the horocycle flow on infinite normal covers of surfaces. Amazingly, even in this infinite geometric setting it is possible to classify invariant measures. Furthermore, Sarig reported that only one of these invariant measures satisfies a generalized law of large numbers.

Another type of actions that often arise in applications is the action of multidimensional abelian subgroups which are Ad-diagonalizable over R. At first sight it seems rather unlikely that anything useful can be said about invariant measures for such actions, since the action of a single hyperbolic diffeomorphism has many invariant measures and complicated orbit closures. But in fact, for an abelian group generated by several such diffeomorphisms, it seems that the invariant measures again are scarce. In the early $60's$ Furstenberg conjectured that ergodic measures invariant under both $\times 2$ and $\times 3$ on the unit interval are either supported on periodic orbits or are Lebesgue measure. Rudolph has proven the conjecture provided the entropy of at least one transformation is positive. Katok and Spatzier studied general affine algebraic actions of higher rank abelian groups, and proved algebraicity of the measures under a positive entropy condition and other strong ergodicity assumptions. Two new measure classification methods have been introduced that do not require these ergodicity assumptions — one by Einsiedler and Katok which deals with measures with "high" entropy and a second by Lindenstrauss dealing with measures of "low" entropy. These have been combined in [3] to classify all the measures on $SL(n,R)/SL(n,Z)$ ergodic and invariant under the action of the full diagonal group with positive entropy, which gives a partial result towards Littlewood's conjecture on simultaneous diophantine approximation. Lindenstrauss used the low entropy methods, in conjunction with his work with Bourgain in number theory, to show quantum unique ergodicity for certain arithmetic surfaces.

In the workshop E. Lindenstrauss discussed his work with M. Einsiedler, P. Michel and A. Venkatesh which gives another application of the results of [3] which gives information regarding the distribution of compact orbits of on homogeneous spaces indexed by discriminant.

Another talk which deals with the same type of action was given by Tomanov who presented generalization of his previous work with Weiss regarding the classification of closed orbits, and presented an application regarding the set of values attained by a product of k linear forms in $n \geq k$ variables at integer points.

Other related topics[15, 22]

Using ideas developed to study unipotents flows, and in particular their behavior near the cusp in the space $SL(n, R)/SL(n, Z)$, Dani, Kleinbock, Margulis and others have proven many results regarding diophantine approximations. During the workshop, D. Kleinbock discussed his recent work on quantitative divergence estimates for unipotent flows and how they give precise formulas for Diophantine exponents of affine subspaces of R^n, and Weiss explained how similar techniques work in the Teichmuller space setting.

Classical ergodic theory concerns itself with ergodicity and equidistribution problems for actions of "small" groups such as the reals and integers, and, more generally, amenable groups. It was only in the 1990's that ergodic theorems for actions of semisimple groups were established by Nevo, Stein and Margulis. They proved both strong maximal inequalities and pointwise ergodic theorems for avergaes over Riemannian balls in the group bi-invariant under a maximal compact subgroup. A. Gorodnik and A. Nevo recently generalized such theorems to a more general class of increasing compact sets. As a consequence, they obtained strong maximal inqualities, mean ergodic theorems and pointwise ergodic theorems for actions of lattices in semisimple groups, as was reported by Gorodnik.

GEOMETRY [4, 17]

A common theme of rigidity in geometry is the characterization of locally symmetric metrics in simple geometric or topological terms. The prime example is the Strong Rigidity Theorem of Mostow, Margulis and Prasad. Later examples are the rank rigidity theorems by Ballmann and Burns-Spatzier, and the characterization by Besson, Courtois and Gallot of real hyperbolic space by minimal volume and the other negatively curved symmetric spaces by minimal entropy. A related topic of interest is the study of similar rigidity properties for homogeneous spaces which are not locally symmetric, see work of Connell, Eberlein and Heber.

Minimal volume is closely related to Gromov's simplicial volume. The vanishing of the latter has important consequences for the topology and geometry of the space. Thurston had shown non-vanishing of the simplicial volume for closed real hyperbolic spaces. More generally it is known for closed manifolds of negative curvature. B. Schmidt reported on his recent work with J. Lafont that the simplicial volume of closed higher rank locally symmetric spaces of nonpositive curvature and no Euclidean facfors is not 0. This is based on a non-trivial extension of a Jacobian estimate of Besson, Courtois and Gallot to the higher rank situation by C. Connell and B. Farb.

Another approach to characterize locally symmetric spaces is by symmetry: assume that the universal cover of a closed manifold has a non-discrete group of isometries. If it is also assumed that the sectional curvature is non-positive, then the metric is automatically locally symmetric, as was proved by P. Eberlein in the 80's. B. Farb reported on his beautiful work with S. Weinberger that achieves essentially the same conclusion without the curvature assumption. This work has recently been extended to other Lorentz and other pseudo-Riemannian metrics by K. Melnick. This will prove important in the context of group actions preserving such structures.

Mostow's use of quasi-isometries in establishing strong rigidity led to many outstanding problems in geometric group theory. Gromov in particular asked for the quasi-isometric classification of groups. For special groups such as lattices in semisimple groups, this was established in the early 1990's in a remarkable series of works by Casson, Chow, Drutu, Eskin, Farb, Gabai, Gromov, Jungreis, Kleiner, Koranyi-Riemann, Leeb, Pansu, Schwartz, Sullivan, and Tukia. One obtains both quasi-isometric rigidity and classification. Thus, any group quasi-isometric to such a lattice is isomorphic to one on a subgroup of finite index. There is one quasi-isometry class of cocompact lattices for each semisimple group G. Further, there is one quasi-isometry class for each commensurability class of irreducible non-cocompact lattices, except for $G = SL(2, R)$ where there is precisely one quasi-isometry class of non-cocompact lattices.

The case of nilpotent groups is still open even though Pansu showed that the associated graded group of two quasi-isomorphic nilpotent groups have to agree. Shalom recently found further invariants for quasi-isometry which distinguish some nilpotent groups with isomorphic graded group. These invariants have been further refined by R.Sauer.

The case of solvable groups however was wide open until our workshop when A. Eskin announced his recent joint work with Fisher and Whyte on Sol and other more general solvable groups. Again they establish quasi-isometric rigidity. Interestingly, the proof borrows techniques more commonly seen in ergodic theory.

Marked length spectrum rigiidity is yet another sought after characterization of a negatively curved Riemannian manifold. Much progress has been achieved in the last two decades. U. Bader in collaboration with R.Muchnik connected marked length spectrum rigidity to a natural representation of the fundamental group coming from the canonical action on the sphere at infinity.

Lattices [1, 25]

Boundaries have played a central role in rigidity theory. Yet we still do not understand boundaries completely. H. Furstenberg's lecture on problems in boundary theory will be made available as a video on the BIRS website, and is suitable for an introduction to the field for a more general audience.

The fine theory of lattices is still making major advances as exemplified by E. Breuillard's talk on his work with Gelander on the uniform Tits' alternative. Tits' famous result says that a finitely generated linear group either has a subgroup of finite index or contains a free group. This new work gives an estimate how close to the identity one can find two generators for a free group. This improves earlier work of Eskin, Mozes and Oh for free semigroups. They also obtained uniform Kazhdan L^2 constants and uniform Cayley graph Cheeger constants.

Raghunathan gave an introductory survey lecture on the congruence subgroup problem. While this question has been resolved in many cases, the general result seems to require significant new ideas and Raghunathan gave an excellent survey of known methods, their applicability and their limitations.

List of Participants

Bader, Uri (University of Chicago)
Breuillard, Emmanuel (Institut des Hautes Etudes Scientifiques, IHES)
Cantat, Serge (Université de Rennes)
Chatterjee, Pralay (Rice University)
Dani, S.G. (Tata Institute of Fundamental Research)
Day, Matthew (University of Chicago)
Eskin, Alex (University of Chicago)
Farb, Benson (University of Chicago)
Feres, Renato (Washington University)
Fisher, David (Lehman College - CUNY)
Franks, John (Northwestern University)
Furman, Alex (University of Illinois Chicago)
Furstenberg, Hillel (Hebrew University)
Gelander, Tsachik (Yale University)
Gorodnik, Alex (California Institute of Technology)
Handel, Michael (Lehman College)
Hitchman, T.J. (Rice University)
Kleinbock, Dmitry (Brandeis University)
Klingler, Bruno (University of Chicago)
Kloeckner, Benoît (cole Normale Supérieure de Lyon)
Ledrappier, Francois (University of Notre Dame)
Lindenstrauss, Elon (Princeton University)
Margulis, Gregory (Yale University)
Melnick, Karin (University of Chicago)
Morris, Dave Witte (University of Lethbridge)

Mozes, Shahar (Hebrew University)
Oh, Hee (California Institute of Technology)
Peterson, Jesse (University of California, Los Angeles)
Polterovich, Leonid (Tel Aviv University)
Popa, Sorin (University of California, Los Angeles)
Raghunathan, Madabusi S. (Tata Institute of Fundamental Research)
Ratner, Marina (University of California Berkeley)
Sarig, Omri (Penn State University)
Schmidt, Ben (University of Michigan)
Shah, Nimish (Tata Institute of Fundamental Research, Mumbai, India)
Shalom, Yehuda (Tel Aviv University)
Spatzier, Ralf (University of Michigan)
Stolovitch, Laurent (Universite Paul Sabatier)
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Zimmer, Robert (Brown University)

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Chapter 14

Multimedia and Mathematics (05w5505)

July 23–28, 2005

Organizer(s): Rabab Ward (Institute for Computing, Information and Cognitive Systems, UBC), Robert Gray (Stanford University)

Introduction

The diverse applications of multimedia technology affect the way we communicate, work and play. The Banff International Research Station (BIRS) workshop on Multimedia and Mathematics, organized by Rabab K. Ward and Robert Gray, brought university and industry personnel together from July 23-28, 2005 to share ideas about the latest advances in the different areas of multimedia and related mathematics. Forty attendees (29 men and 11 women) from Canada, UK, Australia and the USA comprised 6 graduate students and 26 faculty members from 24 universities, as well as 8 researchers from Microsoft, Apple, Hewlett-Packard, Tiz Media Foundation, and the National Science Foundation. The rich cross-fertilization brought about by this workshop provided new insights into possible solutions to the latest technical challenges.

Academics and mathematicians, as well as practitioners, engineers, and researchers working in different industries related to multimedia devices, described the approaches, advances, and constraints involved in their area of media. With a view to discovering common ground, they explored the mathematical modeling, analysis, and representation of information relevant to their respective fields. Models used in individual media as well as in multimedia systems were examined. Under this broad umbrella, the following topics were discussed: algorithms, architecture and hardware, software, joint processing and coordination of multi-model signals and data, coding, compression, storage, retrieval, statistical learning, recognition, classification, segmentation, communication, networking, multi-model devices and systems, multimedia forensics and security, human movements and mobile devices.

The main types of signals discussed in the workshop were those involved in text, audio, speech, music, images, and video, as well as sensor data such as environmental measurements from sensor networks and biological data from medical devices. The role of multimedia in hip-hop culture was also investigated as a means of promoting mathematics among under-represented minorities.

Among the many topics discussed, three important areas received special attention: (1) data protection; (2) coding; and (3) reduction in the computational load of multimedia devices and processors. Multimedia networking and security are intertwined topics because the growth of multimedia products raises concerns for content producers about how best to protect their information. At the same time, there is a need to make better use of bandwidth in a network hardware infrastructure whose standards are fixed. This need for greater bandwidth efficiency is reflected in the number of presentations in the coding area. Reducing computational complexity received much attention, since future multimedia communication will be based on wireless devices with person-to-person connections.

Peer-to-peer video streaming and wireless multimedia represent a paradigm shift. Traditionally, most video content has originated from only a few places (mainly broadcasters) for mass distribution to consumers. Now, however, consumers equipped with digital cameras, camcorders, and camera phones, have multiple ways of generating, acquiring, and managing their own video content. Video now comes from a multitude of sources, and not a lot of computing power can be crammed into these mobile imaging devices without draining their batteries and using up their limited data storage. Along with the limited bandwidth of wireless devices, this limitation requires that the video signals be compressed. However, functions such as motion estimation and compensation, which are integral to video compression, are very computer-intensive. Today, compression and other video encoding are done by broadcasters. For mobiles, however, we need to shift the computationally demanding components such as motion estimation and compensation to the desktop machine or the mobile. Mobiles, therefore, need to have simpler, less power-hungry encoders, and we need to reduce encoder complexity in ways that wont affect compression efficiency. We arent really there yet.

In the following two sections, we summarize the topics explored at the workshop. For convenience of presentation, we classify the topics discussed into categories, ”Theory and Modeling” and ”Progress in Specific Application Areas”, even though almost every presentation discussed theory as well as applications.

Presentations and Discussion

Theory and Modeling

Of the 25 presentations, 13 can be roughly categorized under Theory and Modeling. In most of these talks, different applications to multimedia applications were also discussed and demonstrated. The following four talks could fall under the general topics of modeling images, image rendering, humancomputer interaction and a unified algebraic approach to time and signal models: Photographic Image Representation with Multiscale Gradients and Applications, e.g., to Denoising Taking Multi-View Imaging to a New Dimension: From Harry Nyquist to Image-Based Rendering A New Framework for Modeling and Recognizing Human Movement and Actions Deterministic and Stochastic, Time and Space Signal Models: An Algebraic Approach The area of image and video coding received much attention, as mentioned earlier. The following four talks were given in this area: A Signal Processor’s Approach to Modeling the Human Visual System, and Applications, e.g., to Coding Vector Quantizers for Reduced Bit-Rate Coding of Correlated Sources Analytical Modeling of Matching Pursuit Time Domain Lapped Transform and Its Applications to Coding and Error Resilience Transmission Additional presentations in this area are discussed in the following section on Progress in Specific Application Areas. There was one presentation on information representation of networks, entitled Information Representation for Network Systems. Image segmentation remains an active area of research, with many applications ranging from video retrieval to biomedical imaging. The following two presentations addressed image segmentation: Mathematical and Perceptual Models for Image Segmentation Deformable Models for Image Analysis: From ’Snakes’ to ’Organisms’ Two presentations that addressed reduction in computational load were Dimension Reduction for Classification and Anomaly Detection Multi-scale Displacement Estimation and Registration for 2-D and 3-D Datasets. We will now briefly describe the digests of the above talks, highlighting recent developments, scientific progress and some of the open problems.

Eero Simonelli talked about photographic image representation with multiscale gradients. He described recent empirical investigation and modeling of the joint statistical properties of a multiscale representation based on derivative operators. In particular, he discussed the use of Gaussian Scale Mixtures (product of a scalar random variable and a Gaussian vector) to model the statistics of clusters of wavelet coefficients at adjacent positions, scales and orientations. When applied to the problem of denoising, these models provide a natural generalization of both standard linear (Wiener) and thresholding estimators, and lead to substantial increases in performance. He also described how to extend this model to include local geometry in the form of phase and orientation information.

Tsuhan Chen talked about the recent convergence of image processing, computer vision, and computer graphics resulting in multi-view image processing. A picture may be worth a thousand words, but a single picture is not able to render the whole scene; it merely renders the scene as seen from a particular viewpoint. In 1991, Adelson and Bergen proposed the concept of the plenoptic function, a seven-dimensional function that represents all the light rays in a dynamic scene. Since then, research on sampling, storing, interpolating, and reconstructing the plenoptic function has been emerging at both academic and industrial research institutions. This area of research is commonly referred to as image-based rendering, or, more familiar to the signal-processing community, multi-view image processing. Recent convergence of image processing, computer vision and computer graphics has resulted in significant progress in multi-view image processing. Now widely used in applications ranging from special effects (remember the movie ”The Matrix”?) to virtual teleconferencing, multi-view image processing has become a critical tool for creating visually exciting content. With multi-view image processing, real-world scenes can be captured and rendered directly from images captured by cameras, eliminating the need for computationally expensive modeling of 3D geometry or surface reflectance, as is often done in traditional computer graphics. Dr. Chen also discussed recent developments in image-based rendering. While studying the mechanism for sampling multi-view data, he revealed the connections between image-based rendering, multidimensional multirate signal processing, and the Sampling Theorem discovered by Harry Nyquist 80 years ago!

Ling Guan described a new framework for modeling and recognizing human movement and actions. Humancomputer interaction (HCI) study is a key research area in many scientific disciplines. Dr. Guan started the talk with an overview of concepts, history and recent developments in HCI: face, speech, gesture, human emotion and human actions, with emphasis on emotion and action recognition. He then focussed on a fundamental, but under-investigated research area in HCI: modeling and recognizing human movement and actions. Inspired by the movement notation systems used in dance and the paradigm of the phonemes used in continuous speech recognition, he described a Continuous Human Movement Recognition (CHMR) framework. The framework is based on a novel paradigm, the alphabet of dynemes, the smallest contrastive dynamic units of human movement. A Differential Evolution-Monte Carlo particle filter is introduced, which has demonstrated highly effective and robust characteristics in tracking basic human movement skills. Using multiple hidden Markov models, the recognition process attempts to infer the human movement skill that could have produced the observed sequence of dynemes. Recent anthropometric data shows that the famous ”average sized human” model in Leonardo da Vinci’s drawing of the human figure is a fallacy, and that there is no one who is average in 10 dimensions. Incorporating the highly accurate biometric features into the CHMR framework, Dr. Guan was able to demonstrate the effectiveness of the framework in biometrics, biomedical analysis, and recognition of human skills. He proposed and forecasted that this framework will form the enabling technology for biometric authentication systems for a broad range of applications such as security/surveillance, biomedicine/physiotherapy, special effects in motion picture production, digital asset management, battlefield surveillance, coaching/training/judging in sports and performing arts, to name a few.

Jose Moura presented a new algebraic approach for deterministic and stochastic, time and space models. We are all familiar with (infinite) ”time” signal processing: time shifts, filters and convolution, signals, Fourier and z-transforms, spectrum, fast algorithms. Images, of course, are not ”time” but ”space” objects. Also, they are ”finite” objects, i.e., defined over a finite indexing set. What is the natural concept of space shift, of space filter and convolution, spectral analysis, or ”z”-transform, as well as many other related concepts? To address these questions, Dr. Moura went beyond linear algebra to present an algebraic approach where time (signal) and space (image) processing are instantiations of the same mathematical structure. The basic building block is the signal model - a triplet (A, M, f) of an algebra A of filters, a module M of signals, and a generalization of the z-transform as a bijective linear mapping f from a vector space into the module of signals. The shift is naturally interpreted as a generator of the algebra of filters, boundary conditions connect finite with infinite indexing sets, the trigonometric transforms (e.g., DCTs) are appropriate Fourier transforms, and the C-transform is the z-transform. More than a mathematical curiosity, the algebraic approach provides the appropriate structure to extend signal and image processing beyond uniform to other grids (e.g., hexagonal or quincunx), or develop fast algorithms from a few basic principles, from which we can also derive new fast algorithms for existing and new transforms. Connections with other image models, in particular, with Gauss Markov fields and pinned Markov diffusions, were discussed. This talk overviewed Moura’s recent work with Markus Pueschel on the algebraic theory of signal and image processing.

Sheila Hemami presented a signal processing approach to model the human visual system. Current image and video compression algorithms (e.g., JPEG-2000, H.264) provide very high efficiency compression and excellent quality at relatively high bit rates. These algorithms operate by treating images and video as traditional ”signals,” employing efficient transformations, correlation-based models, and entropy coding. Human visual system characteristics have been successfully applied to high-rate signal-based compression, where stimuli such as compression-induced distortions are below the visibility threshold; i.e., humans cannot see them. Operation of such signal-based compression algorithms at low rates, in which compression-induced distortions are clearly visible, has to date operated based on visual system rules-of-thumb and has produced moderate success for images, while little has been done for video. Dr. Hemami presented recent results on characterizing the human visual system in a manner that allows for immediate incorporation into imaging and video applications, such as compression and quality measurement, at not only high rates/low distortions but also at low rates/high distortions. Results were presented in two distinct areas: vision-based results that explain how humans perceive stimuli, and engineering-motivated results that allow us to incorporate our characterizations into practical algorithms.

Russ Mersereau discussed coding of correlated sources. It is well known that vectors derived from consecutive segments of most real-world signals are strongly correlated. This inter-vector correlation is not exploited in a standard VQ system. Many techniques proposed to exploit this correlation render the VQ sub-optimal or require buffering, and thus introduce encoding delay. Dr. Mersereau presented two alternative methods. The first approach, cache VQ, uses a cache memory to reduce the bit rate and the encoding time, at the cost of a slight, but controllable, increase in the coding error. The second approach, recently developed by Krishnan, Barnwell, and Anderson at Georgia Tech, overcomes cache VQ's limitations. Their approach, called dynamic codebook reordering, dramatically reduces the entropy in the representation of the VQ symbols, which can then be exploited for lossless compression. Dynamic codebook reordering can significantly reduce the bit rate for strongly correlated sources without introducing any additional distortion, coding delay, or sub-optimality when compared to a standard VQ.

Shahram Shirani presented an analytical approach that models the operation of the matching pursuit algorithm on uniformly distributed signals. Matching pursuit is a greedy algorithm that decomposes a signal into a redundant dictionary of basis functions. It has recently found applications in many areas, including image and video processing. The proposed model expresses the relationship between the bit rate and matching pursuit coder parameters such as dictionary size, quantization step size, distortion and dimension of the signal. This relationship can be used to optimize the dictionary size and quantization step size for minimum bit rate. The model is verified through experimental results, and the accuracy of the model is validated for different system parameters.

Jie Liang reviewed the theory and applications of time domain lapped transform, including the design of fast transform, its application in wavelet-based image and video coding, and error resilient design for multiple description coding.

Thrasos Pappas discussed problems arising in the segmentation of images of natural scenes. One of the challenges of this problem is that the statistical characteristics of perceptually uniform regions are spatially varying due to effects of lighting, perspective, scale changes, etc. A second challenge is the extraction of perceptually relevant information. Dr. Pappas first considered the problem of segmenting images of objects with smooth surfaces. The images are modeled as smooth spatially varying functions with sharp discontinuities at the segment boundaries, plus white Gaussian noise. Dr. Pappas discussed an adaptive clustering algorithm for segmentation, which is a generalization of the K-means clustering algorithm to include spatial constraints and to account for local intensity variations in the image. The spatial constraints are modeled through the use of Gibbs/Markov random fields, while the local intensity variations are accounted for in an iterative procedure involving averaging over a sliding window whose size decreases as the algorithm progresses. Dr. Pappas also considered a hierarchical implementation that results in better performance and computational efficiency, then discussed an adaptive perceptual colortexture segmentation algorithm that is based on low-level features for color and texture. It combines knowledge of human perception with an understanding of signal characteristics in order to segment natural scenes into perceptually/semantically uniform regions, and is based on two types of spatially adaptive low-level features. The first describes the local color composition in terms of spatially adaptive dominant colors, and the second describes the spatial characteristics of the gray-scale component of the texture. Key segmentation parameters are determined on the basis of subjective tests. The resulting segmentations convey semantic information that can be used for content-based retrieval.

Another presentation on image segmentation was given by Ghassan Hamarneh. Dr. Hamarneh started by giving a short overview on image segmentation and registration. He then focussed on deformable models ('snakes' and others) for image segmentation and mentioned issues related to incorporating prior knowledge. He then presented his work on 'deformable organisms', an artificial-life framework for image analysis incorporating high-level, intelligent, intuitive control of shape deformations. Various application examples were presented throughout the talk.

Dimension reduction for classification was discussed by Alfred Hero. There has been intense interest in analysis of massively complex data sets with thousands of dimensions. Dimension reduction methods are critical components of any analysis method due to the requirements of computation and noise reduction. Dr. Hero presented new variational methods of dimension reduction that explicitly target classification, anomaly detection, or other tasks.

Nick Kinsbury discussed the problems in motion estimation and registration of images and 3-dimensional objects. His talk considered the problems of displacement (or motion) estimation between pairs of 2-D images or 3-D datasets, especially for the case of non-rigid deformation as encountered in many medical imaging applications. He showed how the use of multi-scale directionally selective octave-band filters with analytic (complex) impulse responses can greatly reduce the computational load associated with displacement estimation by employing phase-based methods. In particular, he extended the techniques of Hemmendorf for use with dual-tree complex wavelets (DT CWT) and in an iterative scenario, such that the usual approximations associated with phase-based approaches are minimized. These methods rely on the shift-invariant and directional properties of the DT CWT, and are inherently resilient to shifts in the mean level and contrast of the two datasets and to noise, because of the band-limited nature of the signals and the use of phase shifts to estimate displacements. They are computationally efficient because a coarse-to-fine, multi-scale approach is used, and they are well-suited to displacement fields that can be represented by locally-affine models with smoothly varying parameters. The algorithm can also be designed largely to ignore data in areas where the two datasets do not match (e.g., where a tumour is present in one dataset but not in the other). Dr. Kinsbury believes that the computational advantages of this method will be particularly helpful for 3-D registration tasks.

Progress in Specific Application Areas

The areas of forensics and security, video coding, automated speech recognition, automated music retrieval, video for mobile devices and network coding for the Internet and wireless networks were discussed. A presentation of a different kind but which received much discussion was that of using multimedia and hip-hop culture to promote math among under-represented minorities. There were three presentations on forensics and security, entitled Multimedia Forensics for Traitors Tracing Secure Signal Processing Emerging Paradigms in Sensor Network Security Dr. Adrian Dumitras of Apple Inc. and Dr. Amir Said of Hewlett Packard talked about the state of the art in video coding. The titles of their presentations were Optimization Methods for State-of-the-Art Video Encoders The Need for Better Models for Coding Sparse Multimedia Representations Workshop attendees also discussed recent developments and open problems in the area of speech and music. The following three talks addressed this field: Computer Speech Recognition: Building Mathematical Models Mimicking the Human System Managing Spoken Documents A Personal History of Music Information Retrieval Panos Nasiopoulos and Kostas Plataniotis gave a joint presentation regarding consumer-grade mobile devices. The titles of these presentations were Digital Video for Mobile Devices A Unified Framework for the Consumer-Grade Image Pipeline Philip Chow of Microsoft gave the following talk on the newly emerging theory and applications of network coding: Network Coding for the Internet and Wireless Networks

Ray Liu presented first on the art of multimedia security. The recent growth of networked multimedia systems has increased the need for techniques that protect the digital rights of multimedia. Traditional protection alone (such as encryption, authentication and time stamping) is not sufficient for protecting data after it is delivered to an authorized user or after it has traveled outside a closed system. To address the post-delivery protection and introduce user accountability, a class of technologies known as digital fingerprinting is emerging. Due to the global nature of the Internet, ensuring the appropriate use of media content is no longer a traditional security issue with a single threat or adversary. Rather, new threats are posed by coalitions of users who can combine their contents to undermine the fingerprints. These attacks, known as collusion attacks, provide a cost-effective method for removing an identifying fingerprint, and thus pose a strong threat to protecting the digital rights of multimedia. To mitigate the serious threat posed by collusion, theories and algorithms are being investigated and developed for constructing forensic fingerprints that can resist collusion, identify colluders, and corroborate their guilt. Therefore, multimedia forensics has become an emerging field built upon the synergies between signal processing theory, cryptology, coding theory, communication theory, information theory, game theory, and the psychology of human visual/auditory perception. Dr. Liu provided the audience with a broad overview of the recent advances in multimedia forensics, with a focus on multimedia fingerprinting for traitor tracing. He then talked about tracing traitors using collusion-resistant fingerprinting for multimedia that jointly considers the encoding, embedding, and detection of fingerprints. A general formulation of fingerprint coding and modulation with a unified framework covering orthogonal fingerprints, coded fingerprints, and group fingerprints was discussed. Finally, traitor-within-traitor dynamics and behavior was modeled and analyzed. As a result of this work, optimal strategies for traitors and for detectors can now be developed.

Ton Kalker talked about secure signal processing. He observed that (professional) multimedia signals are increasingly made available only in protected format. Typically, the security wrappers can only be removed by the targeted devices or applications (e.g., the DRM agent in a rendering device). This poses serious problems for intermediate processing applications that do no have access to the appropriate cryptographic keys (for liability reasons, security reasons or otherwise) and/or that do not have sufficient computational resources. In his talk, Dr. Kalker discussed options for processing of protected signals in their protected format, both by adopting the cryptographic methods (e.g., homomorphic encryption) or by adapting the signal processing methods (scalable coding).

Deepa Kundur talked about the emerging paradigms in Sensor Network Security. She provided an overview of the field of sensor network security and highlighted particular challenges in symmetric key distribution, secure aggregation, secure routing, and actuation security. Through examination of these problems, fundamental compromises among the degree of protection, complexity and network performance were highlighted, leading to a discussion of appropriate primitives and paradigms for securing sensor networks. The talk concluded with a discussion of the principal issues for protecting emerging optical free space sensor networks and multimedia sensor networks.

Adriana Dumitras discussed optimization of video encoders. Much work has been done on identifying the best methods to optimize video encoders. These efforts have focused on removing spatial, temporal and perceptual redundancies from a video source, with the objective of representing the data efficiently. However, so far there is no unique ”best method” to optimize a video encoder. Instead, various methods exist that address (usually distinctly) different aspects of the optimization problem and different applications. This diversity is motivated and enabled by the tremendous flexibility allowed in the encoder design by video coding standards, the development of unoptimized video encoding tools as part of the non-normative verification or experimental models in the standards' developments, and the powerful competition in the video industry. Dr. Dumitras presented a taxonomy and an overview of the methods that enable video encoder optimization by tradeoffs at the algorithmic, software and hardware implementation levels.

Amir Siad talked about the need for better models for multimedia coding. A main objective in multimedia signal processing is to numerically eliminate redundancy and create sparse representations. However, for compression an effective representation needs to be effectively entropy coded. There is a need to have good combined models for both the signal and how its information is distributed, in the sense of what and where the most important components are. Simple recursive set-partitioning methods were shown to be very effective in coding sparse data, both in terms of compression and computational complexity, but their use still has not been extended to more complicated media types. Dr. Said discussed the challenges and possibilities for improving performance using more sophisticated data models.

Li Deng of Microsoft discussed computer speech recognition and how to build mathematical models that mimic the human system. The main goal of computer speech recognition/understanding is to automatically convert naturally uttered human speech into its corresponding text (and then into its meaning). While amazing success, both technologically and commercially, has been achieved in the past by straightforward mathematical methods (e.g., hidden Markov modeling, maximum likelihood and discriminative learning, dynamic programming, etc.), solving the remaining problems leading to its ultimate success appears to require a deep understanding of human speech recognition mechanisms. Dr. Deng analyzed various human sub-systems, including linguistic-concept generator, motor-control, articulation, vocal tract acoustic propagation, ears, auditory pathways, and auditory cortex, working in synergy to accomplish the remarkable task of highly robust, low-error speech recognition/perception and understanding. How can the essence of such human information processing power be abstracted in building a computer system with similar (or better) performance? How can we build mathematical models to enable the development of advanced machine-learning algorithms and techniques that will run efficiently in a computer? Is it possible to explore and exploit some special power of the computing machines inherently lacking in the human system so as to achieve super-human speech recognition? These are some of the issues Dr. Deng addressed in the talk.

Mari Ostendorf talked about the management of spoken documents. As storage costs drop and bandwidth increases, there has been a rapid growth of information available via the Web or in online archives, raising problems of finding and interpreting collections of documents. Significant recent progress has been made in text retrieval, analysis, summarization and translation, but much of this work has focused on written language. Increasingly, speech and video signals are also availableincluding TV and radio broadcasts, congressional records, oral histories, voice mail, call center recordings, etc.which can be thought of as spoken documents'. Because it takes longer to listen to audio than to read text, spoken documents are clearly a prime candidate for automatic indexing, information extraction, and other such technologies. In her talk, Dr. Ostendorf provided an overview of the speech processing technology underlying spoken document management, including mathematical frameworks for both word and metadata recognition, and for integrating video and language cues. In addition, she discussed issues that arise in text processing when moving from written to spoken language and implications for statistical models of language.

George Tzanetakis gave a very lively presentation about music retrieval, complete with beautiful and varied pieces of music. Music Information Retrieval (MIR) is an emerging research area that explores how large digital collections of music can be effectively analyzed for searching and browsing. It combines ideas from many different fields, including Signal Processing, Machine Learning, Music Cognition, and Human-Computer Interaction. Dr. Tzanetakis gave a historic overview of MIR, with specific emphasis on topics he had more personal experience with, such as audio-feature extraction, automatic musical genre classification, rhythm analysis, query-by-humming, and sensor-enhanced musical instruments. He concluded the talk by making predictions about the future of MIR and how it will radically transform the way music is produced, distributed and consumed.

Panos Nasiopoulos talked about digital video and mobile devices. Mobile wireless technologies and digital video broadcast technologies are gradually converging within efforts from 3GPP and DVB 2.0 to complete this merging in the upcoming generations of mobile technologies. In order to support this convergence, existing video technologies need to be upgraded to ensure the reliability and quality of the delivered content. This calls for highly efficient video codecs in addition to reliable error resilience techniques that overcome the bandwidth constraints and highly error-prone conditions of wireless networks.

Kostas Plataniotis talked about a unified framework for consumer grade image pipeline. A new modeling and processing approach suitable for consumer-grade image processing was presented. Using vector modeling principles, nonlinear image operators and adaptive filtering concepts, single-sensor camera image processing problems are treated from a global viewpoint yielding new classes of processing solutions. The following varied applications of the framework were covered: spectral interpolation (demosaicking), spatial interpolation of the acquired (mosaic-like) single-sensor gray-scale images as well as demosaicked full-color images, demosaicked image post-processing and color image enhancement, camera image denoising and sharpening, camera image compression, spatio-temporal video demosaicking, and camera image indexing and rights management. Results obtained using the framework were provided. The list of the topics covered, while certainly not exhaustive, provided a good indication of the usefulness and often necessity of the proposed framework in consumer grade image processing. Open research problems and other potential applications of the framework were also discussed.

Melanie Louisa West gave a multimedia presentation about using multimedia and hop-hop culture to teach math to under-represented minorities. There is widespread agreement among educators that a strong need exists for programs to increase math and science competency among under-represented minority students. Lack of interest and motivation are known contributing factors for this lack of representation. Ms. West proposed combining multimedia with elements of hip-hop culture to promote interest in math among under-represented minorities. In today's society, hip-hop music has captured the minds of urban youth. Music sales, fashion trends, and advertisement strategies reflect this. Consequently, Ms. West believes that incorporating hip-hop into math instruction for under-represented minorities holds great promise for success. The elements of rhyme, rhythm, and repetition make raphip-hop's linguistic componentan excellent creative vehicle for presenting concepts that require memorization. Math, in particular, lends itself to rap because the creative use of natural language provides a platform for transferring the conceptualization of math into real life experiences through story-telling. By combining the learning experience with an activity that is already an integral part of a person's life, Ms. West believes that this will not only increase interest in learning, but will also maximize information retention. This coupled with the incorporation of multimedia elements that will be widely accessible (on public display for peers and or publish for a general audience) will motivate the individual (or group) to do their very best. An innovative aspect of this proposed approach is that it combines teaching students at the elementary school level with multimedia content created by students at the high school level. This accomplishes two goals; it makes it easier to motivate the younger students, and at the same time, provides a great vehicle for exposing the older students to multimedia.

Summary and Highlights

The workshop topic provided a timely cross-disciplinary bridge between the relatively new area of multimedia and the well-established discipline of mathematics. For many researchers in a specific area of multimedia, the workshop provided an excellent opportunity to broaden their perspective. The workshops high-quality presentations made clear the surprisingly similar mathematical approaches applied to speech, audio, image, and video-processing research.

The presentations and informal discussions enabled participants to examine the variety of approaches in different media areasan invaluable opportunity made possible by the mixed formalinformal style of the workshop. For example, the group discussion resulting from Amir Said 's presentation confirmed that coding can only be optimized if we have good models. Another example is that of Professor Pappas' presentation on image segmentation, which generated heated debate by questioning the need for an intermediate step, given that the final task is semantic image understanding/classification. The researchers with speech recognition/understanding expertise have found that integrated pattern-recognition approaches that avoid the step of speech segmentation always provide better results than modular processing approaches that involve explicit segmentation. The discussions on such disparities provided much needed information that will hopefully generate new interest in cross-media research and exploration. Li Deng, the General Chair of the 2006 IEEE Workshop on Multimedia Signal Processing, was one of the attendees. He decided, together with the Technical Committee, to continue such discussions and explorations with a special panel at the upcoming workshop on "Differences and Similarities of Image/Video and Speech/Audio Processing Techniques." Professor Pappas has accepted their request to organize the panel. We believe that this will have a significant impact on the future of multimedia research, an initiative inspired by this BIRS workshop.

We hope BIRS will continue sponsoring cross-disciplinary workshops such as the one we organized. Cross-disciplinary research sharing similar mathematical approaches stands to benefit the most from such workshops. The different branches of media processing research make it impossible to gain expertise in every sub-area, and this BIRS workshop helped immeasurably to foster an awareness of new trends in the various sub-disciplines. This is particularly important to some industrial researchers whose work has a relatively short-term scope. Most researchers in multimedia cannot afford the time-consuming process of mastering the subtleties of all the multimedia processing techniques. The BIRS workshop provided an ideal opportunity to make close connections among them and to deepen our understanding of problem areas.

The workshop succeeded in its aim to bring mathematicians, engineers, and scientists to interact and get exposed to each others' ideas and advances in these disciplines. As different multimedia technologies have evolved and continue to evolve at a very rapid rate, the exact definition of multimedia remains illusive, even though multimedia technologies are now being widely deployed in industries in a multitude of applications. All of these applications affect the way we live, communicate, interact with each other, work, and play.

The cross-fertilization among the different disciplines, academics and practitioners, engineers and mathematicians encouraged by the workshop was very useful in exposing the different communities to a new range of challenging and timely technical advances, the underlying mathematical problems and applications, and implementation challenges.

List of Participants

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Chapter 15

Mathematical Epidemiology (05w5003)

August 20–25, 2005

Organizer(s): Herb Hethcote (University of Iowa), Simon Levin (Princeton University), Pauline van den Driessche (University of Victoria)

Background

Population growth and spread, global climate change, and the emergence and reemergence of novel and deadly forms of infectious diseases have increased the need for sound quantitative methods to guide disease intervention practice [1, 15, 18]. In the 20th century, influenza was pandemic several times and new diseases such as Lyme disease, Legionnaire's disease, toxic-shock syndrome, hepatitis C, hepatitis E, and hantavirus were encountered. The human immunodeficiency virus (HIV), which is the etiologic agent for acquired immunodeficiency syndrome (AIDS), was identified in 1981 and now causes over 3 million deaths per year in the world. Drug and antibiotic resistance have become serious issues for diseases such as tuberculosis, malaria, and gonorrhea. Prions have been identified as the infectious agents for bovine spongiform encephalopathy (BSE or mad-cow disease), Creutzfeldt-Jacob disease, kuru, and scrapie in sheep. Changing patterns of social behavior and travel present new classes of disease transmission problems. For example, West Nile virus has spread to North America. Biological terrorism with diseases such as smallpox or plague has become a new threat. In the 21st century, we have already encountered severe acute respiratory syndrome (SARS) and will undoubtedly face more new infectious disease challenges.

The epidemiological modeling of infectious disease transmission has a long history in mathematical biology, but in recent years it has had an increasing influence on the theory and practice of disease management and control [15]. Mathematical modeling of the spread of infectious diseases has become part of epidemiology policy decision making in several countries, including the United Kingdom, Netherlands, Canada, and the United States. Epidemiological modeling studies of diseases such as gonorrhea, HIV/AIDS, BSE, foot and mouth disease, measles, rubella, and pertussis have had an impact on public health policy in these countries. Thus modeling approaches have become very important for decision-making about infectious disease intervention programs. Recent approaches include deterministic models, computer simulations, Markov Chain Monte Carlo models, small world and other network models, stochastic simulation models, and microsimulations of individuals in a community. These techniques are often implemented computationally and use data on disease incidence and population demographics. Sometimes the epidemiology, immunology, and evolution of a disease must all be considered. For example, some recent research has studied the rational design of influenza vaccines by considering the effects on the immunology of influenza immunity in individuals of the yearly epidemics of influenza A variants, the vaccine composition each year, and the yearly evolutionary drift of influenza A virus variants.

One barrier to effective modeling of infectious diseases and intervention policies has been the lack of communication between the modelers and the policy makers. A primary objective of the BIRS Mathematical Epidemiology workshop was to encourage communication among internationally-recognized applied mathematicians, statisticians, and epidemiologists. To promote communication, 50 minute lectures were followed by 30 minute discussion periods on specific diseases, epidemiological problems, public health policies, comparisons of disease intervention strategies, recent advances, open questions, new approaches, and future directions for research. The formal lectures and discussions in the mornings and evenings were supplemented by more informal discussions and special sessions in the afternoons. The topics included Compound Matrices, Incidence Functions, Modelling Rubella Vaccination, and Wildlife Diseases.

Partcipants in this BIRS workshop (August 20-25, 2005) presented the latest results on the theory and applications of mathematical modeling of infectious disease epidemiology and control. Mathematicians, statisticians, and epidemiologists presented successful examples of mathematical modeling studies. They also described current epidemiological problems and questions about strategies for vaccination and other prevention methods that could be studied using mathematical modeling approaches. The variety of approaches included not only deterministic and stochastic modeling, but also network and agent-based modeling. Some talks emphasized new methods in dynamical modeling of infectious diseases, while others considered new applications of modeling approaches and new methods for parameter estimation from data. The participants included many young scientists including assistant professors, postdoctoral fellows, and graduate students.

Both mathematical modelers and public health policy decision makers will ultimately benefit from this workshop on modeling as a decision making tool for the epidemiology and control of infectious diseases. Epidemiologists and public health policy makers have much to learn about successful and potential applications of modeling approaches to understanding disease transmission and using interventions to reduce disease incidence. It was the consensus of the participants that workshops should be organized by modelers for public health officials, in which they would work on epidemiology modeling and computer simulations of infectious disease transmission and control. Applied mathematicians and statisticians learned about new and challenging problems in modeling the spread and prevention of diseases. This workshop may lead to new projects and collaborations involving the applications of modeling approaches to problems in understanding infectious disease transmission and intervention strategies.

Challenges in modeling influenza and antigenic variation

Viggo Andreasen opened the conference with a detailed presentation of the biology of influenza viruses, and of mathematical models used to understand influenza's evolutionary ecology. As Andreasen discussed, influenza's biology is complicated, even though the disease is caused by a relatively small RNA virus. The virus consists of eight separate RNA segments encoding a total of 10 genes. In temperate regions of the globe, the virus causes regular, annual epidemics. In the years 1918, 1957, and 1968, however, the virus caused major pandemics worldwide. As Andreasen described, pandemics are associated with antigenic "shifts" – that is, reassortment of entire viral RNA segments between avian and human forms of the virus. In non-pandemic years, by contrast, annual epidemics are possible because of antigenic "drift" – that is, the gradual accumulation of point mutations in hemagglutinin, the gene encoding the primary surface antigen of influenza. The phylogeny of drifting influenza viruses is very unusual when compared to all other known RNA viruses (such as HIV).

Influenza drift is the result of selection for novel antigenic viral strains, selected because of "cross-reactivity" between related strains. The processes of influenza transmission and drift can be modeled by generalizing the standard SIR framework in one of several ways:

1) Assume a fixed set, K, of n distinct strains. Instead of using only three classes of individuals ($S(t)$ $I(t)$ and $R(t)$), we introduce more classes (order 2^n) of individuals, which indicate the set of strains, $J \subset K$, from which an individual has previously recovered, or the strain with which an individual is currently infected. The resulting model requires a very large number of ODEs, but it has been thoroughly analyzed in the case of $n = 4$ strains. 2) Assume of a one-dimensional continuum of strains, and index susceptible individuals according to the strain of their most recent infection. This results in a PDE version of the SIR model, and the steady-state evolutionary rate depends upon the kernel of cross-immunity between strains. The phylogenetic structure of viral strains cannot be studied in such a model. 3) Employ individual-based stochastic simulations that keep track of the full infection history of each individual, transmission events between individuals, and mutational events to viral strains.

Although individual-based simulations have successfully reproduced the empirical patterns of influenza drift evolution [13, 26], such simulations are very complicated and do not easily reveal the underlying principles that govern the structure of influenza drift. Dr. Andreasen introduced new work based on an earlier framework he has developed with colleagues [19, 2]. Andreasen and Sasaki have recently analyzed simplified 2-strain “annualized” models which attempt to determine the conditions under which a mutant viral lineage will co-exist with its parent strain (phylogenetic branching), and under what conditions a mutant viral lineage will extinguish earlier lineages (phylogenetic pruning). This modeling approach helps to identify generic principles that govern the structure of drifting influenza viruses.

In the second session, Junling Ma presented another model intended to help gain analytic understanding of the complex process of influenza drift. Ma’s approach synthesized a variety of data about drift to support the development of a simple modeling framework that captures key aspects of influenza drift. He presented an argument beginning from a Poisson process of random mutations arising and showed that a few simple assumptions allow construction of a novel modeling framework similar to the earlier models of Andreasen et al. [3, 19, 2].

Junling Ma thereby provided a rationale for the “linear” strain-evolution framework of Andreasen, Levin and others. He showed that his model leads naturally to cycles of about one year, with explicit evolution, tying in to an earlier theory by Dushoff and others [9] that strong annual cycles in influenza arise from resonance between a natural tendency to cycle and exogenous seasonal forcing. This earlier theory was developed in part at a 2003 BIRS workshop in honor of Lee Segel.

The discussion ranged over a broad set of existing challenges in influenza modeling, including: how to how to bridge scales from cellular interactions, to individual outcomes, to population-level patterns of disease incidence and viral evolution; and how to guide choice of vaccine strains and policies of vaccine allocation. We expect that collaborations started here will lead to significant progress on these important questions.

Most of the diseases discussed at the workshop exhibit antigenic variation: the ability of the disease organism to change its surface in order to evade the immune system. Discussions focused on models incorporating multiple strains for subtypes of a virus circulating in a host population. For example, the virus responsible for Dengue Hemorrhagic Fever may appear in one of four subtypes, which has complicated the development of an effective vaccine. These subtypes co-circulate in the host population and the course of infection within a host depends on the previous history of infections with other subtypes. Subsequent infections are hypothesized to increase one’s viral load, and increase one’s infectiousness. This effect is called antibody dependent enhancement (ADE). Even the simplest assumptions of this phenomenon lead to large complex models, which possess very interesting dynamics, both from the mathematical and epidemiological perspective. The complexity of these models is necessary to resolve questions of outbreak patterns and development of effective vaccines and vaccination strategies.

Lora Billings presented a dynamical system model of co-circulating subtypes in diseases such as dengue, with both autonomous and seasonally driven outbreaks [2]. She showed that for sufficiently small ADE, the number of infectives of each subtype synchronizes, with outbreaks occurring in phase. When the ADE increases past a threshold, the system becomes chaotic, and outbreaks from differing subtypes become desynchronized. However, windows of synchronization can persist. This drives down the number of susceptibles, and can threaten persistence of the virus. She concluded that increased number of subtypes and ADE effect may provide a competitive advantage to a virus, but there are limits.

The current state of the art in both the epidemiological data and the analysis of the models falls short of answering the questions of vaccine strategies, but desynchronized outbreaks of different subtypes can be partially understood. There are several clear directions for research: one, the further development of analytical techniques for these types of dynamical systems, particularly, bifurcation analysis and integration methods, and two, further study of immunological mechanisms and within-host modeling of immune response to both understand the details of susceptibility and immunity and to properly model the spread of subtypes within the host population. The challenges surrounding within-host aspects of antigenic variations, such as drug resistance and its consequences for treatment and vaccination programs, were not discussed at the meeting and should be addressed in a future workshop.

New approaches: Network modeling

Mark Newman's talk on Disease Dynamics on Contact Networks provided an introduction to the use of network approaches in epidemiology, together with a number of examples of their application to real-world problems. The spread of an infection on a network can be mapped onto a percolation problem, where the probability of there being a connection between two nodes is given by the transmissibility of infection. This quantity is simply the probability of transmission between an infective and a susceptible, over the entire infectious period of the infective. With this mapping, the well-developed machinery of percolation theory (most notably, generating function methods) can be brought to bear on the problem. Epidemic thresholds, probabilities of disease invasion and epidemic sizes can then be calculated. A number of examples of networks were presented, together with a discussion of the problems and issues that accompany attempts to capture the structure of real-world networks. The central point is that a network consists of both nodes (individuals) and edges (connections between individuals). The statistics of sampling individuals from a population is a well-studied problem, but appropriate techniques for sampling edges are less well understood. Many techniques, such as contact tracing, may introduce biases into the sample of the network obtained. The network structure is highly dependent on the infection setting, as witnessed by the impact of increased long-range travel: the spread of Black Death in medieval Europe involved mainly local spread of infection whereas the SARS outbreak rapidly jumped between countries and even continents.

For many rapidly spreading infections, the contact network can be treated as being fixed, but such an assumption would be quite inappropriate for sexually transmitted infections. In many settings, transmission is enhanced by superspreaders: individuals who give rise to many more secondary infections than the average person. The percolation analysis highlights this phenomenon, with the basic reproductive number depending not only on the average number of contacts made (i.e. the mean of the degree distribution) but also on the second moment of the degree distribution. This result echoes the familiar "mean + variance over mean" result from mathematical epidemiology. A hospital-based network model was presented, depicting hospital wards, patients and caregivers. Fitting the model to data on an outbreak of Mycoplasma pneumonia suggested that the probability of transmission between patients and caregivers was highly asymmetric, with a much higher transmission probability from caregivers to patients than from patients to caregivers. As a consequence, the model makes a strong prediction regarding control: each caregiver should be limited to one ward, and caregivers should be given antibiotics. These recommendations are in stark contrast with conventional public health wisdom, which states that patients should be confined to wards and patients should be treated. In the resulting discussion, it was pointed out that the standard policy might be more concerned with mitigating the effects of infection (i.e. preventing patient deaths) rather than preventing transmission.

The spread of SARS in a city such as Vancouver was studied using a simulated contact network, based on demographic data. Properties of the network were discussed, together with epidemiological questions (such as the probability of invasion) that can be addressed using the percolation approach. Interestingly, despite all of the structure that was included in the network, it appeared, in many ways, to behave very similarly to a random graph model. Sexual partnership networks have a quite different structure to the social networks that govern the spread of respiratory infections. The dynamic structure of the network, as sexual partnerships are formed and break up, is an important feature, as is the degree to which partnerships overlap (concurrency). If the infectious period of sexually transmitted infections is short, then most transmission events must be associated with partnerships that are either concurrent or that closely follow other partnerships. "Gap dynamics" are, therefore, an important determinant of transmission, in addition to concurrency. Survey data that examines partnership dynamics, including concurrency and gap dynamics, were presented. There was considerable discussion of biases in such data. The talk concluded with the question of whether network models are really appropriate for sexually transmitted infections, despite their long history of use in this area.

A lively discussion followed. Questions of different network structures were raised. Bipartite graphs have been used in some instances, such as the EpiSims model for spread of smallpox in Portland, Oregon, that describes people and places, such as offices, schools and stores. In such models, places can be considered as being infected, so that people visiting those places can acquire infection. Vector borne diseases may be more appropriately described using random graphs, if it is assumed that the vector (e.g. mosquito) does not distinguish strongly between different people. On the other hand, such networks may exhibit aggregation if the vector shows preference for biting certain classes of people.

The usefulness of the basic reproductive number concept in network settings was questioned. In reply, it was pointed out that different network structures (and hence R_0) may explain the different patterns of spread of HIV in different settings. Control measures can also be explored using the analytic approach. The difficulties in applying network approaches to the real-world were a recurring theme in the talk and discussion. Important issues remain regarding how we can gain insights into the structure of networks on which infection spreads. There are only a small number of instances (such as SARS, for which intensive contact tracing was carried out, or the hospital study presented, whose small scale enabled a complete description of the network to be obtained) in which detailed network data is available. In other settings, we only have a sample of the network or a sample of the individuals involved in the network.

Mercedes Pascual spoke about her joint work with Juan Aparicio on translating from networks to populations using modified mean-field models of disease dynamics. Such models ignore network structure and assume homogeneous mixing. At the opposite extreme, high-dimensional models that are both individual-based and stochastic incorporate the distributed nature of transmission. In between, moment approximations have been proposed that incorporate the effect of correlations on the dynamics of mean quantities of interest. As an alternative closer to traditional epidemiological models, she presented results on 'modified mean-field equations' for disease dynamics, in which only mean quantities are followed and the effect of heterogeneous mixing is incorporated implicitly. She illustrated the idea of formulating these equations from the basic reproductive number of the disease (R_0), and illustrated the approach with SIR dynamics in random and small world networks. She asked how much detail is needed on the transmission network to predict the population course of disease dynamics. She derived an expression for R_0 in small networks and showed that in spite of high levels of clustering, the resulting system of differential equations are able to capture the initial transients and the long-term equilibrium of the more complex network simulations. Pascual argued, however, that modified mean field equations will be most useful when the network is not known, and therefore, when the analytical expression for R_0 is not know. Thus, she addressed how much information is needed on the network to parameterize the model using only the initial transients (i.e. the beginning of an epidemic). From initial data on incidence vs. time, she estimated R_0 and used it as a parameter in the modified mean field equations. This exercise showed that no information on the network is required to parameterize the system and predict the course of the disease. Limitations of the approach were discussed.

A second method relies on power-law relationships between global and local densities. Pascual specifically investigated the previously proposed empirical parameterization of heterogeneous mixing in which the bilinear incidence rate kSI is replaced by a nonlinear term kS^qI^p [25, 21], for the case of stochastic SIRS dynamics on different contact networks, from a regular lattice to a random structure via small world configurations. She showed that, for two distinct dynamical regimes involving a stable equilibrium and a noisy endemic steady-state, the modified mean field model approximates successfully the long term dynamics and short term transients of decaying cycles. A regime of coherent cycles in the small world regime is not well-approximated by this simple model. Pascual argued that future work should couple aspects of the two proposed approximations to better capture the effects of heterogeneous mixing.

Pascual asked whether the demographic noise introduced by finite populations in individual-based models must be kept. That is, do we need the noise even when network structure is only implicitly incorporated? She presented some recent results on the dynamics of a stochastic SIR models for infectious diseases with immigration. In particular, she derived the power spectra of both infective and susceptible numbers and gave conditions under which large and sustained cyclic stochastic fluctuations are expected. This analytical result formalizes the well-known observation that demographic noise sustains persistent oscillations when the corresponding deterministic system approaches an equilibrium with decaying cycles [4, 22]. These results show that the dominant period of the deterministic and stochastic system do not necessarily coincide. More importantly, they suggest a complementary explanation for the major dynamical transitions observed in epidemics of childhood infectious diseases after vaccination, from regular to irregular cycles [11, 5]. Seasonal forcing does not appear to change the basic character of the power spectra, other than adding an annual peak. They also show that childhood diseases fall in regions of parameter space prone to high noise amplification, an observation that raises interesting evolutionary questions. Discussion of the interplay of seasonality, stochasticity and nonlinear disease dynamics clearly shows that this is an important area in need of further study.

Modeling emerging/reemerging diseases such as HIV, SARS and West Nile Virus

The presentation by Brandy Rapatski and James Yorke [23] dealt with the epidemiology of HIV. There have been only a few attempts in the literature to estimate the probability of HIV transmission per sexual contact. A number of years ago J. Jacquez and J. Koopman at the University of Michigan analyzed a data set dealing with gay men in San Francisco that were part of a hepatitis B vaccine trial for which multiple blood samples were taken during the early years of the HIV epidemic. From analysis of the data from 1978-1984, before the introduction of antiretroviral therapy, Jacques and Koopman concluded that the highest probability of transmission occurred during the first few months after infection, a period called primary infection. Rapatski and Yorke reanalyzed the same data with a model that incorporated three stages of disease, primary infection, asymptomatic infection (lasting on average 7 years), and symptomatic infection (lasting on average 3 years). Using data on the fraction of gay men that were HIV positive vs time during the years 1978-1984, they concluded that to sustain the rapid increase in the number of infected gay men into the later years of the San Francisco epidemic that the probability of transmission must be highest during the third stage, the symptomatic stage of disease, rather than during primary infection. Their conclusion, given the data, seemed very surprising given the adoption by the field that HIV is mainly spread during primary infection. This talk, which was supported by rigorous modeling and data analysis, presented an important change in the view of HIV spread. Much discussion followed both about the methods used and the conclusion, but no one identified any flaws. In fact, all approximations seemed to be conservative and adding more realistic features to the model only appeared to increase the probability of transmission in the third stage.

Zhien Ma spoke about the work of his group on modeling the SARS outbreak in China during November 2002 to June 2003 [29, 28]. A compartmental model is proposed that mimics the SARS control strategies implemented by the Chinese government after the middle of April 2003: the division of the whole population into two parallel blocks corresponding to the so-called free environment and the isolated environment and the partition of these blocks further into the compartments of susceptible, exposed, infective, suspected SARS, diagnosed, removed and health care workers. A novel approach was introduced to calculate the transfer rate from the free environment to the isolated environment. This approach incorporated undiagnosed suspected SARS individuals that were put into isolation because fast SARS tests were not available. Methods were developed for parameter identification using the daily reported data from the Ministry of Health of China. Simulations based on these parameters agree with the accurate data well, thus providing additional validation of the model. Finally some parameters were varied to assess the effectiveness of different control measures: these new parameters correspond to the situation when the quarantine measures in the free-environment were prematurely relaxed (thus the observation that the second outbreak with the maximal number of daily SARS patients is much higher than the first outbreak) or when the quarantine time of SARS patients is postponed (noting the delayed peak time but with much higher number of SARS patients at the peak). The basic reproductive number and the basic adequate contact rate were also calculated.

Interestingly, the modeling work was carried out in 12 days by Zhien Ma and his group in May of 2003, before the SARS infections had subsided in China. Yet, their results came very close to predicting the real SARS case data in China that accumulated almost a month later. This demonstrates the need for modelers to consider approaches to real-time modeling and prediction on an ongoing outbreak, as opposed to the traditional prediction of future outbreaks or retrospective analysis, which are abundant in the literature.

Zhilan Feng presented work done together with John Glasser (CDC) in which they investigated potential public health response strategies for an emergent infectious disease. They constructed a general compartmental ODE model incorporating the possibility of infectiousness during clinically distinguishable stages, during which patients could be quarantined or isolated with varying efficiencies. They tested their model by application to SARS data in Hong Kong. Analysis of this model with increasingly accurate and complete information indicates that recommended public health interventions may change during the course of an epidemic. This led into a more general discussion of how mathematicians can best help public health decision-makers who are planning for or responding to epidemics.

Spatial aspects are important in infectious disease transmission, but are often taken into account implicitly in models. Throughout the workshop, the spatio-temporal component of disease spread was often alluded to, for example during discussions on network models, but seldom discussed explicitly. Of the various modeling techniques at hand to address spatial aspects, the one that involves integro-differential equations is at the same time the most accurate and the hardest to use. Shigui Ruan gave a presentation entitled Nonlocal Epidemic Models, in which he presented models employing this approach. He first introduced a host-vector model for a disease without immunity, with the specificity that the current density of infectious vectors is related to the number of infectious hosts at earlier times. This results in an integro-differential equation model, in which a diffusion term is used to model the spatial spread in a region. Examples of these host-vector diseases include West Nile Virus and malaria.

Ruan showed how, for the general model, the stability of the steady states can be studied using the contracting convex sets technique. When the spatial variable is one-dimensional and the delay kernel assumes some special form, the existence of traveling wave solutions is established using the linear chain trick and the geometric singular perturbation method. In a second part, Shigui used a multi-compartment model to describe the nonlocal spread of SARS, discussing in particular the effect of global travel on the transmission of the disease.

Recent advances in modeling disease transmission and vaccination

David Greenhalgh spoke on estimation of R_0 and evaluation of vaccination programs from age-structured serological data. There were questions on whether or not age structured bootstrap samples were used in these kinds of studies. It is likely that when the infection process is independent of the age and the age specific samples are good, then the age specific bootstrap method is applicable. In rubella, children may be infected by the adults and vice-versa, so the samples are age-dependent. However, many people have the opinion that it is not easy to validate the model. One reason for this could be changes in the behaviour of the individuals who are vaccinated. There was also discussion on general difficulties on validating the given mathematical model which predicts the proportions of newborns to be vaccinated.

John Glasser gave a talk entitled “Mathematical Epidemiology of Varicella and Herpes Zoster”. The United States has recently begun to recommend children be vaccinated against varicella (chickenpox); however, there is a complex process by which the varicella-zoster virus reactivates resulting in herpes zoster (shingles). Previous work has considered this reactivation, but this work includes the effect of boosting of immunity to herpes zoster due to either the periodic reactivation of the virus within a person or contact with a varicella-infected person.

Previous studies had cast doubt on the varicella vaccination policy of the United States because of a predicted temporary increase in herpes zoster infections in adults who are no longer boosted by exposures to children with chickenpox. Other considerations, including possible evolutionary changes in the virus caused by vaccination, might provide further evidence for or against the policy.

Chris Bauch spoke on the behavior-incidence dynamics in childhood disease vaccination. The interplay between disease prevalence, population behavior and vaccine coverage is explored in a game theoretical setting for the case of pertussis in England and Wales during the 1970’s. A model that considers imitation dynamics is able to give a good fit to the time-series data of pertussis vaccine uptake. The model is able to recover the oscillatory dynamics characteristic of some childhood diseases. The model also predicts that the probability and amplitude in oscillations increase with the intensity of imitation behavior in the population or with increases in disease prevalence. It is suggested that game theoretical approaches could aid in predicting the population behavior towards vaccination and therefore facilitate public health decision making.

In this session, we also briefly discussed issues related to parameter estimation, uncertainty and sensitivity analysis. The capability of the model to uniquely identify model parameters needs to be addressed. Parameter estimation can be achieved using maximum likelihood methods, least squares fitting, etc. Uncertainty in parameter estimates can be quantified under different assumptions in the data (e.g., heterogeneity in variance, correlated errors, etc.). Sampling techniques (e.g., latin hypercube, simple random sampling) are useful to explore the parameter space and assess the uncertainty of epidemiological quantities of interest. Sensitivity analysis (e.g., partial derivatives, partial rank correlation coefficients) of parameters on the model solution of interest are useful not only in determining the sensitivity on parameters, but also in constructing asymptotic variance-covariance matrices from which parameter variance and correlation information can be obtained.

Modeling wildlife diseases

Linda Allen spoke about modeling wildlife diseases including hantavirus infections in rodents and chytrid infection in amphibians. Hantavirus pulmonary syndrome is an emerging zoonotic disease that is carried by wild rodents. The mortality rate in humans is as high as 37%. Humans are usually exposed to the virus through geographically isolated outbreaks. Two new mathematical models for hantavirus infection in rodents were presented. The models were based on a male/female SEIR epidemic model. The first model was a system of ordinary differential equations (ODEs) while the second model was a system of stochastic differential equations (SDEs). The SDE model can be derived directly from the ODE model assuming variation with respect to the birth, death, and infection process [17]. These new models capture some of the realistic dynamics of the male/female rodent hantavirus interaction: higher seroprevalence in males and variability in seroprevalence levels.

Two diseases associated with recent amphibian declines are ranavirus infection and chytridiomycosis. Chytridiomycosis is a disease caused by the fungal pathogen *Batrachochytrium dendrobatidis*. Both pathogens causing these diseases are found throughout the world. In this presentation, models for amphibian populations infected by the fungal pathogen were discussed [12]. The amphibian host population is structured according to two developmental stages, juveniles and adults. The juvenile stage is a post-metamorphic, nonreproductive stage, whereas the adult stage is reproductive. Each developmental stage is further subdivided according to disease status, either susceptible or infected. There is no recovery from disease. Each year is divided into a fixed number of periods. The first period represents a time of births. Amphibians are generally explosive breeders, resulting in a large increase in population density during the breeding season. During the remaining time periods there are no births, only survival within a stage, transition to another stage or transmission of infection. Conditions were derived for population extinction. High transmission rates can destabilize the disease-free equilibrium and low survival probabilities can lead to population extinction.

There are several reasons for studying wildlife diseases [14, 8, 27, 16].

1) If the wildlife species is of conservation interest and there are concerns about the impact of the disease on the survival of the populations, for example, rabies in Ethiopian wolves.

2) If there is increasing worry about the possibility of either transmission from wildlife to humans or to domestic animal species. In this case we often think of wildlife as the reservoir species.

3) Wildlife diseases pose a threat to global diversity. Control of wildlife diseases is important for the preservation of our natural world.

Emerging diseases often occur because of anthropogenic changes to the environment or human encroachment. These changes result in increased contact with wildlife species which allows disease to jump between species. Wildlife diseases are often associated with diseases in humans (zoonotic disease) and domestic animals. A few examples of wildlife disease that are transmitted to humans include hantavirus pulmonary syndrome (transmitted by wild rodents such as rats and mice), influenza in birds, and plague from prairie dogs and rats. Rabies cases in humans are often due to bites by infected bats. The annual number of human deaths worldwide caused by rabies is estimated to be between 40,000 and as high as 70,000. An estimated 10 million people receive post-exposure treatments each year after being exposed to rabies suspect animals.

Vector-transmitted diseases affecting wildlife and humans include West Nile Virus and Lyme disease. Canine distemper virus is a spillover infection from domesticated dogs that has resulted in extinction of black footed ferret and African wild dog populations. A few emerging diseases are known to only impact wildlife, such as chytridmycosis, a fungal infection in amphibians, and chronic wasting disease (transmissible spongiform encephalopathy) in deer and elk.

The main differences between modelling wild life and human disease identified in the discussion are that

i) Wildlife populations do not remain constant over time; indeed, they can be highly variable due to environmental factors or the landscape. This can have a profound impact on the dynamics of the disease.

ii) Multiple species interactions are often involved. For example a reservoir for infection does not have to consist of one species, but can be made up of a number of species which interact (at least) via the pathogen and allow the disease to persist. There are many diseases which infect multiple species and we often observe “apparent competition” between these species via the pathogen.

iii) In many cases, wildlife population dynamics are believed to be controlled by pathogens. For example, red grouse and Trichostrongylus tenuis (although we can also think of examples where diseases have had a profound effect on human populations, e.g. HIV in Africa).

iv) Data can be more easily obtained from animal systems. In particular, it is often possible to do experiments on wildlife populations, or individual animals without the ethical issues involved with human disease systems.

The workshop concluded with remarks by the organizers and suggestions for follow-up activities. The organizers and participants thank BIRS and the funding agencies for their support for this excellent workshop. The following references were suggested by participants, but this list is not comprehensive.

List of Participants

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Chapter 16

Topology (05w5067)

August 27–September 1, 2005

Organizer(s): Ian Hambleton (McMaster University), Michael Hopkins (Massachusetts Institute of Technology), Matthias Kreck (University of Heidelberg), Ronald Stern (University of California Irvine).

Introduction

The idea of the conference was to bring together distinguished senior and some of the best junior mathematicians representing a broad variety of subjects in topology. Topology has become - like many other areas of mathematics - a field which is a collection of many areas all having the size to justify a conference of its own. Most conferences nowadays are of the latter type. If one looks back, important progress often was based on a combination of methods and ideas from these subareas and also from some neighbouring areas. To mention a few examples:

- The Donaldson and Seiberg-Witten theory addresses problems in topology of 4-manifolds and uses methods ranging from partial differential equations, differential geometry, index theory, algebraic geometry to algebraic topology.
- Novikov conjecture and related conjectures where high-dimensional manifold theory, in particular surgery, index theory, algebraic K-theory and geometric group theory are centrally involved.
- The attempts to create elliptic cohomology use methods ranging from stable homotopy theory, algebraic geometry, index theory to theoretical physics (conformal field theory).

Almost no mathematician is able to be familiar with all these subjects and even to follow the main results and fundamental ideas is very hard. A conference like this provides an unusual opportunity to hear some of the most important and fundamental developements and - even more important - to discuss ideas with experts from other areas.

The conference was attended by forty participants. When the organizers selected the participants, they had the difficulty that to cover all these areas with leading experts left rather limited room for young people. And so they had to give up some very prominent names. The result seemed to us a good mixture of leaders and excellent young people some of which are already leaders themselves. One indication of success: we heard during the conference that *four* of our main speakers had been asked to give talks at the next ICM 2006 in Spain.

To give enough time to discussions between individuals and in groups, we limited the talks to five per day and 45 minutes each. All three of the outstanding developements mentioned above were represented in the talks. We had asked the speakers to address a broad audience and most of them succeeded very well. Our impression was that our goal was fully achieved. We have asked the participants to send us comments and we quote from them after the problem list.

We were uncertain about a problem session and finally decided against one. But on Wednesday evening a group of about twenty people met in the lounge and spontaneously a problem discussion came up. More precisely, we asked everybody to formulate her/his favourite problem. Since the list looks very nice, we gave those who did not participate in this round the chance to add their favourite problems afterwards. The problems are attached after the summaries of talks.

Many participants asked for another conference of this type. We like the idea, and are planning to apply again for 2007.

Program

Sunday, August 28, 2005
8:45-9:00 Introduction and Welcome to BIRS by BIRS Station Manager, Max Bell 159
9:00-9:45 Bruce Kleiner, Univ. of Michigan: Geometrization and uniformization of metric spaces
10:00-10:45 Arthur Bartels, Univ. Münster: The Farrell-Jones Conjecture for groups acting on trees
11:15-12:00 Denis Auroux, MIT: Fiber sums of Lefschetz fibrations
16:00-16:45 Jacob Lurie, Harvard: Elliptic Cohomology and Derived Algebraic Geometry
17:00-17:45 Jongil Park, Seoul National University: Rational blow-downs and smooth 4-manifolds with one basic class

Monday, August 29, 2005
9:00-9:45 Weimen Chen, University of Massachusetts: Pseudo holomorphic curves and finite group actions in dimension 4
10:00-10:45 Shmuel Weinberger, University of Chicago: A Sullivan conjecture for equivariant structure sets
11:15-12:00 Martin Bridson, Imperial College London: Limit Groups: non-positive curvature, logic, and group theory
16:00-16:45 Thomas Mark, Southeastern Louisiana University: Ozsvath-Szabo invariants of fiber sums
17:00-17:45 William Dwyer, Notre Dame: Duality in algebra and topology

Tuesday, August 30, 2005
9:00-9:45 Oleg Viro, Uppsala University: Virtual links, their relatives and Khovanov homology
10:00-10:45 Jesper Grodal, University of Chicago: p-compact groups and their classification
11:15-12:00 Peter Ozsvath, Columbia University: Heegard Floer homology of links
16:00-16:45 Jacob Rasmussen, Princeton University: Differentials on Khovanov-Rozansky homology
17:00-17:45 Yongbin Ruan, University of Wisconsin: Twisted K-theory on orbifolds and its stringy product

Wednesday, August 31, 2005
7:00-9:00 Wolfgang Lück, Münster: L^2-invariants and their applications
10:00-10:45 Stefano Vidussi, Univ. of California, Riverside: Taubes conjecture and twisted Alexander invariants
11:15-12:00 Karen Vogtmann, Cornell University: Tethers and homology stability
16:00-16:45 Andras Stipsicz, Alfrd Rnyi Institute of Mathematics: Contact Ozsvath–Szabo invariants and tight structures on 3-manifolds
17:00-17:45 Walter Neumann, Columbia University: Graph manifolds and singularities

Summaries of Talks

Bruce Kleiner: Uniformization and Geometrization of metric spaces

I discussed the problem of parameterizing metric spaces by nice model spaces. More precisely, the goal was to find conditions on a metric space Z which guarantee that there is quasisymmetric homeomorphism $f : X \to Z$, where the model space X is either optimal in some way, or at least canonical. This recoginition problem is motivated by a long development in Geometric Mapping Theory and by rigidity questions in Geometric Group Theory.

Arthur Bartels: The Farrell-Jones Conjecture for groups acting on trees

In my talk the Farrell-Jones Conjecture in algebraic K-theory was discussed. This conjecture proposes a computation of the algebraic K-theory of group rings RG as equivariant homology groups. If the conjecture holds for a group G, then $K_*(RG)$ is in some sense computable in terms of $K_*(RV)$, where V runs over the family of virtually cyclic subgroups of G. For torsion free groups the conjecture implies the vanishing of the Whitehead group.

The result presented in this talk is joint work with Wolfgang Lueck and Holger Reich and assertes that the conjecture holds for groups G that act properly, cocompactly and simplicially on a tree. The proof uses controlled algebra and the (negatively curved) geometry of the tree.

As a corollary of this result and of work on Nilgroups of virtually cyclic groups by Kuku and Tang, Grunewald one obtains rational vanishing results for Waldhausen's Nilgroups appearing in his work on amalgamated free products and HNN-extensions.

Denis Auroux: Fiber sums of Lefschetz fibrations

It is a key problem in 4-manifold topology to understand which smooth compact oriented 4-manifolds carry a symplectic structure (i.e., a non-degenerate closed 2-form). Symplectic 4-manifolds are much more general than complex projective surfaces, but are still a very special class of 4-manifolds. One way to approach symplectic 4-manifolds is via Lefschetz fibrations.

A Lefschetz fibration is a map $f : M^4 \to S^2$ with isolated non-degenerate critical points, near which f behaves like a complex Morse function. Hence, the generic fiber is a smooth closed oriented surface, and the singular fibers present ordinary double point singularities only, obtained by pinching a simple closed loop (the ”vanishing cycle”) on the regular fiber. A theorem of Gompf states that (almost) every Lefschetz fibration carries a symplectic structure; conversely, Donaldson has shown that, after blowing up a finite set of ”base points”, every compact symplectic 4-manifold can be presented as a Lefschetz fibration (with a distinguished set of -1-sections).

The topology of a Lefschetz fibration is encoded by its monodromy, which is a morphism from a free group (the fundamental group of the complement of a finite set in S^2) to the mapping class group of the fiber, mapping the standard generators to Dehn twists. Choosing a set of generating loops, we can express the monodromy by a ”factorization” of the identity element as a product of positive Dehn twists in the mapping class group. Moreover, the various factorizations corresponding to a same Lefschetz fibration are equivalent up to two operations: global conjugation, and Hurwitz moves. There is therefore a one to one correspondence between isomorphism classes of Lefschetz fibrations, and Hurwitz and conjugation equivalence classes of factorizations in the mapping class group.

The classification of Lefschetz fibrations is well-understood in genus 0 and 1 (classical results of Moishezon and Livne), and in genus 2 in the absence of reducible singular fibers (Siebert and Tian). However, many ”exotic” examples have been constructed in higher genus, and the classification there is not understood at all.

A simpler question is that of classification up to stabilization by fiber sums. The main result that one can get is the following. For any genus, there exists a Lefschetz fibration f_g^0 such that, given any two genus g Lefschetz fibrations $f_1 : M_1 \to S^2$ and $f_2 : M_2 \to S^2$ such that (1) M_1 and M_2 have same Euler characteristic and signature, (2) f_1 and f_2 have the same numbers of singular fibers of each type, (3) f_1 and f_2 each admit a section with the same self-intersection, for all large enough n, after fiber summing with n copies of f_g^0 the Lefschetz fibrations f_1 and f_2 become isomorphic.

Using Donaldson's theorem, a corollary is the following "symplectic Wall's theorem": given two compact symplectic 4-manifolds with $[\omega]$ integral and the same values of $(c_1^2, c_2, c_1.\omega, \omega^2)$, they become symplectomorphic after performing on each of them a certain number of blow-ups and fiber sums with some f_g^0.

The proof is almost purely group-theoretic, and involves a study of factorizations in the mapping class group of a surface with one boundary component.

Jacob Lurie: Elliptic Cohomology and Derived Algebraic Geometry

Let E be an elliptic curve over a commutative ring R. If certain mild hypotheses are satisfied by E, then Landweber's exact functor theorem ensures the existence of an essentially unique (elliptic) cohomology theory A such that $A(*) \simeq R$ and $A(\mathbf{C}P^\infty)$ is the ring of functions on the formal completion of the elliptic curve E. In particular, these conditions are satisfied whenever E is classified by an étale map Spec

$$R \xrightarrow{\phi} \mathcal{M},$$

where $\mathcal{M}$ denotes the moduli stack of elliptic curves; let A_ϕ be the associated cohomology theory.

The assignment

$$\phi \mapsto A_\phi$$

may be viewed as a presheaf of cohomology theories on the moduli stack of elliptic curves. The work of Goerss, Hopkins, and Miller implies that this presheaf of cohomology theories can be refined (in an essentially unique way) to a presheaf of E_∞-ring spectra $\mathcal{O}$ on the moduli stack of elliptic curves. It then makes sense to take the (right-derived functor of) global sections, giving an E_∞-ring spectrum tmf

$$[\Delta^{-1}] = R\Gamma(\mathcal{M}, \mathcal{O}).$$

A more refined approach (which includes the "point at ∞" on $\mathcal{M}$) yields a spectrum tmf, the spectrum of *topological modular forms*, so named for the existence of a ring homomorphism from $\pi_* tmf$ to the ring of integral modular forms, which is an isomorphism after inverting 6. The spectrum tmf may be regarded as a universal elliptic cohomology theory, and is a suitable target for "elliptic" invariants such as the Witten genus.

It is natural to think of the presheaf $\mathcal{O}$ as a kind of structure sheaf on the moduli stack $\mathcal{M}$ of elliptic curves. This can be made precise using the language of *derived algebraic geometry*: a generalization of algebraic geometry in which E_∞-ring spectra are allowed to play the role of commutative rings. The pair $(\mathcal{M}, \mathcal{O})$ may naturally be viewed as a Deligne-Mumford stack in the world of derived algebraic geometry, which is a kind of "derived version" of the classical moduli stack of elliptic curves. One may then ask if $(\mathcal{M}, \mathcal{O})$ has some moduli-theoretic significance in derived algebraic geometry; our main result is an affirmative answer to this question.

Given an E_∞-ring spectrum R, there is a natural notion of an *elliptic curve* over R in derived algebraic geometry (which specializes to the usual notion of elliptic curve when R is an ordinary commutative ring). Any elliptic curve E has a formal completion $\hat{E}$; we define an *orientation* of E to be an equivalence Spf $R^{\mathbf{C}P^\infty} \simeq \hat{E}$ of formal groups over R. The main result then asserts that there is a natural homotopy equivalence

$$\{ \text{ Oriented Elliptic Curves } E \to \text{Spec } R\} \Leftrightarrow \text{Map}(\text{Spec } R, (\mathcal{M}, \mathcal{O}));$$

in other words, $(\mathcal{M}, \mathcal{O})$ classifies *oriented* elliptic curves in derived algebraic geometry.

This result, and the accompanying ideas, can be used to shed light on virtually all aspects of the theory of elliptic cohomology.

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Jongil Park: Rational blow-downs and smooth 4-manifold

It has been known that most simply connected smooth 4-manifolds with b_2^+ odd and large enough admit infinitely many distinct smooth structures due to the gauge theory, in particular, Seiberg-Witten theory. But we still do not know which smooth 4-manifolds with b_2^+ small have more than one smooth structure. Though it is not known yet whether the most fundamental 4-manifolds such as $S^4, \mathbf{C}P^2$ and $S^2 \times S^2$ admit more than one smooth structure, it has been some progress in last couple of decades.

In the case when $b_2^+ = 1$, S. Donaldson first proved that a Dolgachev surface is not diffeomorphic to $\mathbf{C}P^2 \sharp 9 \overline{\mathbf{C}P}^2$ ([D]) and D. Kotschick proved in the late 1980's that the Barlow surface is not diffeomorphic to $\mathbf{C}P^2 \sharp 8 \overline{\mathbf{C}P}^2$ ([K]). Recently, I constructed a new simply connected symplectic 4-manifold with $b_2^+ = 1$ and $b_2^- = 7$ ([P1]), and then R. Fintushel, R. Stern, A. Stipsicz and Z. Szabó found many new exotic smooth 4-manifolds with $b_2^+ = 1$ using rational blow-downs and knot surgeries in double node neighborhoods ([FS2], [PSS], [SS1]). So it has been proved up to now that rational surfaces $\mathbf{C}P^2 \sharp n \overline{\mathbf{C}P}^2$ with $n \geq 5$ admit infinitely many distinct smooth structures. Second, in the case when $b_2^+ = 3$, it was also known in the mid 1990's that the $K3$ surface $E(2)$ and the topological 4-manifold $3\mathbf{C}P^2 \sharp n \overline{\mathbf{C}P}^2$ with $n \geq 14$ admit infinitely many distinct smooth structures. And later, the same statement with $n \geq 10$ was also proved. Recently, Stipsicz and Szabó constructed infinitely many distinct smooth structures on $3\mathbf{C}P^2 \sharp 9 \overline{\mathbf{C}P}^2$ ([SS2]), and then I proved that the topological 4-manifold $3\mathbf{C}P^2 \sharp 8 \overline{\mathbf{C}P}^2$ also admit infinitely many distinct smooth structures ([P2]).

In this talk I would like to survey these recent developments.

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Weimen Chen: Pseudo holomorphic curves and finite group actions in dimension 4

As for the abstract of my talk, the main point is to propose to study a class of smooth finite group actions on 4-manifolds, the so-called symplectic symmetries. (These are smooth finite group actions which preserve some symplectic structure of the 4-manifold.) The hope is that the symplectic symmetries will form an interesting and large enough class of smooth finite group actions to study, which on one hand are more tractable than the general smooth actions while on the other hand are more flexible than the holomorphic actions.

From the technical point of view, the equivariant Seiberg-Witten-Taubes theory allows one in principle to detect the fixed-point set structure of a symplectic symmetry by looking at the induced action in a neighborhood of a 2-dimensional, pseudoholomorphic subset. Such information is crucial in studying finite group actions. A key issue is how the regularity of the pseudoholomorphic subset is related to the fixed-point data of the symplectic symmetry. More generally, one can consider the orbifold version of the Seiberg-Witten-Taubes-Gromov theory, which may find applications beyond finite group actions on 4-manifolds.

Shmuel Weinberger: A Sullivan Conjecture for Equivariant Structure Sets

This talk discussed the problem of classifying topologically tame G-manifolds up to equivariant homeomorphism within an equivariant homotopy type. After a quick review of classical surgery and the obstacles one faces in finding an equivariant variant of it, I discussed two key ideas: stratfied surgery, which suffices formally to solve the problem in the isovariant setting, ixed set and then the second idea is purely homotopical and uses categorical ideas (such as the homotopy fixed set and the Goodwillie calculus) to relate isovariance versus equivariance to spaces of Poincare embeddings, and ultimately to ordinary embeddings. Unfortunately, there was not enough time to discuss examples. This talk was based on a combination of results that were joint with Cappell, Klein, and Yan in various combinations.

Martin Bridson: Limit groups: non-positive curvature, logic and group theory

I am interested in exploring the universe of finitely presented groups. In this lecture, I want to focus on the region immediately adjacent and ask what natural class of groups best approximate free groups? Having identified the right class (and we shall see that there really is ”a right class”), I want to set about the task of proving that groups in this class enjoy many of the non-trivial properties of free groups. The property that I am particularly interested in is the one that first got me thinking about this area: some years ago, Howie, Miller, Short and I proved that a subgroup of a direct product of n free groups is of type F_n if and only if has a subgroup of finite index that is itself a direct product of (at most n) free groups. Interest in extending this result became more interesting when work of Delzant and Gromov showed that understanding the subdirect products of surface groups is important in addressign the question of which finitely presented groups are fundamental groups of compact Kahler manifolds. The theorem of BHMS extends from free groups to surface groups, but the proof is rather mysterious and one would like a more coherent explanation of why this type of splitting theorem works.

For this and other reasons I want to approximate free groups. In this talk, we looked at Gromov-Hausdorff limits of free groups, limits coming from representations of finitely generated groups into free groups (which in turn comes from looking at algebraic geometry over groups), we also looked at ”fully residually free groups” (groups whose balls of arbitrary radius can be injected into a free groups), and we looked at groups whose first order logic is that of a free group (existential and/or universal theory). Remarkably, all approaches lead to the class of ”limit groups” with the subclass of ”elementarily free groups”, these being the groups that have the same universal theory as a non-abelian free group. The hardest parts of this classification are due to Zlil Sela.

I described the beautifully simple structure theory of the groups in this class, the simple classifying spaces, with their metrics of negative and non-positive curvature and graph-of-groups decompositions. I finished by quickly mentioning some of the results that one can prove about this class. The basic message is that the programme of extending from free groups to limit groups non-trivial theorems is working. The most striking example is the splitting theorem for subdirect products of limits groups (proved by Howie and I). Other examples include recent work with my students Wilton and Tweedale in which we prove that elementarily free groups are measure equivalent to free groups. Further examples, proved by Hoiwe and I, include the fact that a non-trivial, finitely generated normal subgroup of a limit group must be of finite index, and having finitely generated H_1 is equivalent to being finitely generated.

Thomas Mark: Ozsváth-Szabó invariants of fiber sums (joint work with Stanislav Jabuka)

Ozsváth-Szabó 4-manifold invariants associate to a closed $Spin_c$ 4-manifold (X, σ) having $b^+(X) \geq 2$ a function $\Phi_{X,\sigma} : \mathbf{A}(X) \to \mathbb{Z}$, where $\mathbf{A}(X)$ is the graded algebra $\Lambda^*(H_1(X)/tors) \otimes \mathbb{Z}[U]$. Here $\mathbf{A}(X)$ is graded such that elements of $H_1(X)$ carry degree 1, while U is of degree 2. The function $\Phi_{X,\sigma}$ is nonzero only on homogeneous elements of degree $d(\sigma) = \frac{1}{4}(c_1^2(\sigma) - 2e(X) - 3\sigma(X))$, where $e(X)$ is the Euler characteristic and $\sigma(X)$ is the signature. Furthermore, there are at most finitely many $Spin_c$ structures σ for which $\Phi_{X,\sigma}$ is nontrivial.

Our goal here is to understand the behavior of these invariants under fiber sum of 4-manifolds. Recall that if $f_i : \Sigma \hookrightarrow X_i$ $(i = 1, 2)$ are embeddings of a closed oriented surface Σ in 4-manifolds X_i such that each embedding has trivial normal bundle, the *fiber sum* $Z = X_1 \#_\Sigma X_2$ of X_1 and X_2 along Σ is defined by removing a neighborhood of $f_i(\Sigma)$ from each of X_1 and X_2 and gluing the resulting manifolds (which have boundary diffeomorphic to $\Sigma \times S^1$) along their boundaries using f_i to identify the Σ factors and conjugation in the S^1 factor. (In general, the resulting manifold depends on the embeddings f_i.) We assume throughout that $[\Sigma] \in H_2(X_i;\mathbb{Z})$ is a primitive nontorsion element.

To simplify the statment of the results, we make the assumption that X_1 and X_2 have (strong) *simple type*, which is to say that the only $Spin_c$ structures σ_i for which $\Phi_{X_i,\sigma_i} \neq 0$ have $d(\sigma_i) = 0$. Furthermore, we will consider only the sum of invariants corresponding to $Spin_c$ structures differing by elements of $H^2(Z;\mathbb{Z})$ dual to *rim tori*: these are tori of the form $\gamma \times S^1$ in $\Sigma \times S^1 \subset X_1 \#_\Sigma X_2$, where γ is a circle on Σ. Specifically, if $\mathbf{R} \subset H^2(Z;\mathbb{Z})$ is the subspace spanned by the Poincaré duals of rim tori, we set $\Phi^{rim}_{Z,\sigma} = \sum_{r\in\mathbf{R}} \Phi_{Z,\sigma+r}$. Implicit in the results below is the fact that the fiber sum of two manifolds of simple type is again of simple type.

Theorem 16.0.60 *Assume that the genus of Σ is $g \geq 2$, and suppose σ_i are $Spin_c$ structures on X_i such that $\langle c_1(\sigma_i), [\Sigma]\rangle = 2k$, with $|k| = g-1$. Let $\sigma \in Spin^c(Z)$ satisfy $\sigma|_{X_i\setminus\Sigma\times D^2} = \sigma_i|_{X_i\setminus\Sigma\times D^2}$ for $i = 1, 2$. Then*

$$\sum_{n\in\mathbb{Z}} \Phi^{rim}_{Z,\sigma+nPD[\Sigma]} T^n = \left(\sum_{n_1\in\mathbb{Z}} \Phi_{X_1,\sigma_1+n_1PD[\Sigma]} T^{n_1}\right)\left(\sum_{n_2\in\mathbb{Z}} \Phi_{X_1,\sigma_1+n_2PD[\Sigma]} T^{n_2}\right)$$

as polynomials in the formal variable T. If $0 < |k| < g-1$, we have

$$\sum_{n\in\mathbb{Z}} \Phi^{rim}_{Z,\sigma+nPD[\Sigma]} T^n = 0.$$

Theorem 16.0.61 *Suppose the genus of Σ is 1, and σ_i are $Spin_c$ structures as above with $\langle c_1(\sigma_i), [\Sigma]\rangle = 0$. Then for any glued $Spin_c$ structure $\sigma \in Spin^c(Z)$ as above, we have*

$$\sum_{n\in\mathbb{Z}} \Phi^{rim}_{Z,\sigma+nPD[\Sigma]} T^n = (T^{1/2} - T^{-1/2})^2 \left(\sum_{n_1\in\mathbb{Z}} \Phi_{X_1,\sigma_1+n_1PD[\Sigma]} T^{n_1}\right)\left(\sum_{n_2\in\mathbb{Z}} \Phi_{X_1,\sigma_1+n_2PD[\Sigma]} T^{n_2}\right).$$

These formulae can be used, for example, to compute the Ozsváth-Szabó invariants of elliptic surfaces: the result is in accord with the conjecture that the Ozsváth-Szabó and Seiberg-Witten invariants are identical. We should note, however, that Theorem 1 admits a generalization for manifolds that are not of simple type, for which an analogue in Seiberg-Witten theory is not known.

William Dwyer: Duality in Algebra and Topology

The talk, which represents joint work with John Greenlees and Srikanth Iyengar, discusses the idea of interpreting properties of ordinary commutative rings so that they can be extended to the more general rings that come up in homotopy theory. Among the rings that arise are Eilenberg-MacLane ring spectra, the cochains on a space with coefficients in a commutative ring spectrum, or the chains on a loop space with similar coefficients. It is something of a surprise that differential graded algebras or ring spectra can appear naturally even in purely algebraic settings. One line of reasoning leads to a new homological formula for the injective hull of the residue class field of a local ring; essentially the same formula in another setting gives, for any prime p, the p-summand of the Brown-Comenetz dual of the sphere spectrum. A homotopical interpretation of the notion of Gorenstein ring gives a common way of understanding Gorenstein rings, Poincare duality spaces, and the formal component of Gross-Hopkins duality. The main theme here is that it is interesting to take ring spectra seriously and to try to manipulate them as if they were ordinary rings.

Oleg Viro: Virtual links, their relatives and Khovanov homology

We extend Khovanov homology to links in the projective space. Unexpectedly, full fledged Khovanov homology with integer coefficients are defined only for non-zero homologous links. For zero-homologous links any construction over $\mathbb{Z}$ fails, provided it is based on $1+1$ TQFT.

More generally, integer Khovanov homology extends to the case of link in an oriented 3-manifold fibered over a surface with fiber $\mathbb{R}$, if the projection of the link realizes $w_1(\text{the surface})$.

The construction requires a study of several new kinds of virtual links: twisted virtual links (generalizing the usual ones), blunted Gauss diagrams, checkerboard virtual links, etc. Most of them admit not only combinatorial 1-dimensional, but also 3-dimensional interpretation.

Jesper Grodal: p-compact groups and their classification

In this talk I'll announce and explain a proof of the classification of 2-compact groups, joint with K. Andersen, hence completing the classification of p-compact groups at all primes p. A p-compact group, as introduced by Dwyer-Wilkerson, is a homotopy theoretic version of a compact Lie group, but with all its structure concentrated at a single prime p. Our classification states that there is a 1-1-correspondence between connected 2-compact groups and root data over the 2-adic integers (which will be defined in the talk). As a consequence we get the conjecture that every connected 2-compact group is isomorphic to a product of the 2-completion of a compact Lie group and copies of the exotic 2-compact group $DI(4)$, constructed by Dwyer-Wilkerson. The major new input in the proof over the proof at odd primes (due to Andersen-Grodal-Møller-Viruel) is a thorough analysis of the concept of a root datum for 2-compact groups and its relationship with the maximal torus normalizer. With these tools in place we are able to produce a proof which to a large extent avoids case-by-case considerations.

Peter Oszvath: Heegaard Floer homology for links

I will describe recent joint work with Zoltan Szabo, in which we define an invariant for links, generalizing an earlier construction for knots. The filtered Euler characteristic of this theory is closely related to the multi-variable Alexander polynomial.

Jacob Rasmussen: Differentials on Khovanov-Rozansky homology

I discussed a conjecture (joint with Nathan Dunfield and Sergei Gukov) which describes how the knot Floer homology should be related to the $sl(N)$ knot homologies defined by Khovanov and Rozansky. For each $N > 0$, their construction assigns to a knot K a sequence of bigraded homology groups $H_N(K)$ whose graded Euler characteristic is the $sl(N)$ knot polynomial of K. Work of Gornik suggests that these homology groups should be equipped with a family of differentials $d_n(0 < n < N)$. For each such n, $H_N(K)$ is itself the underlying group of a chain complex with differential d_n. The homology of this chain complex is expected to be $H_n(K)$. This suggests that we should be able to take a limit of the H_N's to obtain a triply graded homology theory with graded Euler characteristic the HOMFLY polynomial of K. The conjecture suggests that this homology should be equipped with anticommuting differentials d_n, not only for $n > 0$ (which would be provided by Gornik's construction) but also for $n \leq 0$ as well. In particular, the homology with respect to d_0 is expected to give the knot Floer homology. In the actual talk, I sketched the construction of Gornik's differentials, formulated the conjecture, and finally, ended by describing a simple class of "thin" knots for which at least part of the conjecture can be seen to hold. (For such knots, the $sl(N)$ homology is determined by the HOMFLY polynomial and signature.) It can be shown that two-bridge knots are thin.

Yongbin Ruan: Twisted K-theory on orbifolds and its stringy product

Wolfgang Lueck: L^2-invariants and their applications

The purpose of this talk is to present recent developments about L2-invariants and their applications to problems in other areas such as topology, group theory, K-theory, geometry and global analysis. It addresses non-experts. We begin with a list of theorems which a priori have nothing to do with L^2-invariants but whose proofs uses L^2-methods. We develope the basic definitions of L^2-Betti numbers and basic tools. Then we mention some important theorems about L^2-Betti numbers and explain in some cases how the theorems in the first list are proved using L^2-methods. Finally we discuss open problems about L^2-invariants.

Stefano Vidussi: Taubes' conjecture and twisted Alexander invariants

It is well-known that the Seiberg-Witten invariants of a 4–manifold provide obstructions to the existence of a symplectic structure. When the 4–manifold is of the form $S^1 \times N$, these obstructions can be described in terms of the Alexander polynomial of N. C. Taubes formulated the conjecture that, if $S^1 \times N$ is symplectic, then N fibers over the circle. P. Kronheimer studied the case where N is obtained as 0–surgery along a knot $K \subset S^3$ and showed that the aforementioned constraints on the Alexander polynomial Δ_N give evidence to Taubes' conjecture, i.e. Δ_N must be monic and its degree must coincide with the genus of the knot. Still, these conditions are short of characterizing fibered knots. In this talk we discuss how to extend these ideas to the case of a general 3–manifold and how these conclusions can be strengthened by taking into account the twisted Alexander polynomials associated to an epimorphism of $\pi_1(N)$ into a finite group. This way we get new evidence to Taubes' conjecture and, practically, new obstructions to the existence of symplectic structures on $S^1 \times N$, even in the case of 0–surgery along a knot. As an application! o! f these results we show that if N is the 0–surgery along the pretzel knot $(5, -3, 5)$, a case that cannot be decided with the use of the Alexander polynomial, $S^1 \times N$ is not symplectic: this answers a question of Kronheimer. In a similar way, we show that Taubes' conjecture holds for knots up to 12 crossings. (*Joint work with Stefan Friedl of Rice University*)

Karen Vogtmann: Tethers and homology stabillity

I defined what it means for a sequence G_n of groups to have homology stability and pointed out some important consequences of homology stability (Quillen's finite generation of K-groups and the Madsen-Weiss computation of the stable homology of mapping class groups). I then described the method introduced by Quillen in the 1970's for proving homology stability, by looking at the equivariant homology spectral sequence of the group G_n acting on a highly-connected complex X_n, with simplex stabilizers G_{n-k-1}. I then showed how to find a suitable complex for $G_n = Aut(F_n)$, giving an action which makes the spectral sequence argument work in the simplest possibly way. This complex involves finding "enveloping spheres" for coconnected sphere systems in a 3-manifold with fundamental group F_n. The complex can alternatively be described by "tethering" the spheres to the basepoint, from both sides. This idea of tethering turns out to be useful in other contexts giving, for instance, a simplified proofs of homology stability for braid groups (first proved by Arnold in 1970), for mapping class groups of orientable surfaces (Harer 1980's), and symmetric automorphism groups of free groups.

Andras Stipsicz: Contact Ozsvath–Szabo invariants and tight structures on 3-manifolds

Recall that an oriented 2-plane field ξ on an oriented 3-manifold Y is a *contact structure* if ξ can be given as the kernel of a 1-form α satisfying $\alpha \wedge d\alpha > 0$. A contact structure is *overtwisted* if there is an embedded 2-disk D in Y such that ξ is tangent to D along ∂D; otherwise ξ is *tight*. It turns out that overtwisted structures are determined by the homotopy type of Y, while the tight structures capture important geometric information of the underlying 3-manifold.

Contact structures can be constructed by performing surgeries along *legendrian* links, that is, along links for which the tangent vectors are in ξ. The tightness of (Y, ξ) can be detected by computing its contact Ozsváth-Szabó invariant $c(Y, \xi)$, which is an element of the Heegaard-Floer homology group $\widehat{HF}(-Y)$. It is known that $c(Y, \xi)$ is zero if (Y, ξ) is overtwisted and is nonzero if (Y, ξ) is the boundary of a Stein domain.

We have studied the existence and classification problem of tight contact structure on a special class of 3-manifolds, called *small Seifert fibered* 3-manifolds. Y is small Seifert fibered if it admits a Seifert fibration over S^2 with 3 singular fibers. As an application of Donaldson's famous diagonalizability theorem for definite 4-manifolds, we find a tight contact structure which is not the boundary of any symplectic 4-manifold.

Walter Neumann: Graph manifolds and singularities

The topology of a complex singularity is determined by its 3-manifold link. The topologies are known but until recently it was rarely possible to give explicit analytic descriptions for any but the simplest topology. The "splice singularities" of Jonathan Wahl and the author do this for many rational homology spheres. The talk will describe a nice characterization of these singularities that we have (finally) proved.

List of problems

Problem 1 *(Adem) A finite group G acts freely on a finite complex X with the homotopy type of a product of k spheres if and only if every elementary abelian subgroup in G is of rank at most k.*

Problem 2 *(Akbulut) Formulate and prove a Resolution Theorem for polynomial maps. This is the only missing issue to topologically characterizing real algebraic sets, i.e. to determine when a given space is a real algebraic set.*

Problem 3 *(Bartels) Borel conjecture. Let M and N be closed aspherical manifolds of dimension ≥ 5 that are homotopy equivalent. Then there is a homeomorphism $f : M - - > N$ that is homotop to the given homotopy equivalence.*

Problem 4 *(Boden) The smooth Poincaré Conjecture in dimensions three and four.*

Problem 5 *(Bridson) Construct counterexamples to the Andrew's Curtis Conjecture: Let $F = F_n$ be the free group of a finite rank n with a fixed set $X = \{x_1, \ldots, x_n\}$ of free generators. Is the normal closure of a set $Y = \{y_1, \ldots, y_n\}$ equals F if and only if Y is Andrews-Curtis equivalent to X, which means one can get from X to Y by a sequence of Nielsen transformations together with conjugations by elements of F?*

Problem 6 *(Collin) If a non-trivial Dehn surgery on a knot K in S^3 has cyclic fundamental group, must K be fibered?*

Problem 7 *(Edwards) The Hilbert-Smith Conjecture: If G is a compact subgroup of the homeomorphism group of a topological mannifold, then G is a Lie group.*

Problem 8 *(Grodal) Find a topological proof of the classification of finite simple groups.*

Problem 9 *(Hambleton) Formulate a local to global principal for smooth manifolds.*

Problem 10 *(Kirby) Is a slice knot a ribbon knot?*

Problem 11 *(Kreck) Is a random smooth manifold asymmetric, i.e. has no non-trivial finite group action?*

Problem 12 *(Lueck) The Atiyah Conjecture: Denote by $N(G)$ the group von Neumann algebra associated to G viewed as a ring (not taking the topology into account). For a $N(G)-$module M let $\dim_{N(G)}(M) \in [0.\infty]$ be its dimension. Let $\frac{1}{FIN(G)} \subset \mathbf{Q}$ be the additive abelian subgroup of $\mathbf{Q}$ generated by the inverses $|H|^{-1}$ of the orders $|H|$ of finite subgroups H of G. Notice that $\frac{1}{FIN(G)}\mathbf{Z}$ agrees with $\mathbf{Z}$ if and only if G is torsion-free. Consider a ring A with $\mathbf{Z} \subset A \subset C$. The Atiyah Conjecture for A and G says that for each finitely presented $AG-$ module M we have $\dim_{N(G)}(N(G) \otimes_{AG} M) \in \frac{1}{FIN(G)}\mathbf{Z}$.*

Problem 13 *(Lurie) Let G be a group acting on a set X. Suppose that the action of G is simply 3-transitive on X (that is, given any two triples (x,y,z) and (x',y',z') of distinct points in X, there is a unique g in G such that (gx,gy,gz) = (x',y',z')). Suppose furthermore that every element g in G which exchanges two distinct points (so that (gx,gy) = (y,x)) has order 2. Does there exist a commutative field k such that the action of G on X can be identified with $PGL_2(k)$ acting on the projective line over k?*

Problem 14 *(Mark) Does every simply connected symplectic 4-manifold X satisfy $c_1^2(X) \leq 9\chi_h(X)$? Here $\chi_h(X) = \frac{1}{4}(sign(X)+e(X))$ where $sign(X)$ is the signature of the intersection form and $e(X)$ is the Euler characteristic.*

Problem 15 *(Mrowka-Ozsváth) Find a proof of the existence of uncountably many exotic smooth structures on $\mathbf{R}^4$ without using instantons, possibly using Seiberg-Witten or Heegaard Floer homology.*

Problem 16 *(Mrowka We have learned starting with the work of Furuta that subtle information can be obtained from refining the Seiberg-Witten invariants from homology classes in the suitable configuration space to a stable homotopy class of map. To what extent can a similar story be told for the Donaldson invariants and the Gromov invariants?*

Problem 17 *(Neumann) Lehmer Conjecture: Let $M_1(P)$ denote the Mahler measure for a univariate integer polynomial $P(x)$. Suppose that is $P(x)$ not a product of cyclotomic polynomials. Lehmer conjectured that $M_1(P) \geq M_1(1 - x + x^3 - x^4 + x^5 - x^6 + x^- x^9 + x^1 0)$. Here $M_1(P) = \exp[\int_0^1 \ln|P(e^{2\pi it})|dt]$.*

Problem 18 *(Park) Does there exist an exotic smooth structure on the complex projective plane $\mathbf{C}P$?*

Problem 19 *(Pederson) The Arf/Kervaire Invariant One Problem: Do there exist framed manifolds with Kervaire invariant one?*

Problem 20 *(Ranicki) Extend the algebraic surgery model for high-dimensional topological manifolds to dimensions 3 and 4. While at it, use the model to obtain combinatorial formulae for the Pontrjagin classes!*

Problem 21 *(Reich) Farrell-Jones conjecture. For a torsion free group Γ the so-called assembly map A : $H_n(B\Gamma; \mathcal{K}^{-\infty}(\mathbf{Z})) \to K_n(\mathbf{Z}\Gamma)$ is an isomorphism for all $n \in \mathbf{Z}$.*

Problem 22 *(Stern) Is every topological $n-$manifold, $n \geq 5$, a simplicial complex?*

Problem 23 *(Stolz) What is the geometric interpretation of elliptic cohomology and what is its relationship to conformal field theory*

Problem 24 *(Teichner) The $A - B$ slice problem.If $B^4 = A \cup B$ is a decomposition of the 4-ball into two smooth submanifolds, such that the intersection with S^3 is a thickening of the Hopf link, determine which side (A or B) is strong. The definition of strong must be invariant under Bing doubling (and thus the obvious homological definition does not work). If there is such a definition then the topological surgery and s-cobordism theorems are false (for free fundamental groups) in dimension 4.*

Problem 25 *(Vidussi) Does there exist a closed smooth 4-dimensional manifold with only finitely many exotic smooth structures?*

Problem 26 *(Vogtman) Using Kontsevich's identification of the homology of the Lie algebra l_∞ with the cohomology of Out(F_r), Morita defined a sequence of 4k-dimensional classes μ_k in the unstable rational homology of Out(F_{2k+2}). Are theses Morita classes trivial in $H^*(\mathrm{OutF_g})$?*

Problem 27 *(Wahl) Get a hold on diffeomorphisms of $3-$manifolds.*

Problem 28 *(Weinberger) What does a random manifold mean? See problem of Kreck. The main point is that most manifolds we consider, e.g. have group actions, are not random. For example a random graph with valence less than or equal to three has no symmetries.*

Comments by some participants

Adem:

I enjoyed the meeting at Banff, I'm glad to hear that you will reapply.

Auroux:

Thanks for putting together such a great conference! I think it was a great idea to have such a broad topology conference. It's definitely useful and can help keep the topology community united. The talks were great, and almost all speakers made a very good effort to keep things elementary.

One suggestion, though: at this meeting, some 3-/4-manifold specialists were confused during homotopy theory talks, and vice-versa. It may be useful in the future to have a series of remedial talks on the first day, planned once the main topics become clear – for this meeting, it would have been useful to have maybe a 90-minute crash-course on homotopy theory for low-dimensional topologists (introducing ring spectra, p-completions, and other monsters, giving concrete examples to make them less scary) and a 90-minute crash-course on low-dimensional topology for homotopy theorists (maybe brief overviews of SW and Ozsvath-Szabo theories ?)

Bartels:

I enjoyed the meeting very much. Most talks were very good and speakers made an (successful) effort to adress the general audiance. Given the number of talks on 4-dimensional manifolds I think it would have been a good idea to have one survey talk on 4-dimensional manifolds to set the stage for the specialliced talks. The talk of Bridson presented a class of groups that seems to be interesting to study in relation with the Farrell-Jones conjecture.

Bridson:

I think taht the idea of sustaining communication between the broad community of "topologists" is a fruitful one, and that this meeting provides an excellent example of the benefits. For the most part, speakers made a real effort to communicate to the whole audience and as a result I have a much better idea of what is happening in adjacent subfields of topology, and who I should ask which questions to. This was a meeting quite different to the highly specialised ones that happen with such great regularity these days. I think that it has played a valuable role, and I hope that it may be repeated on a regular (bi-annual?) basis.

Chen:

Thanks for organizing such a wonderful workshop. I particularly likes this format of having a diverse range of topics.

Dwyer:

I really enjoyed the meeting, and especially the chance to hear something of what's going on across the board in topology.

Grodal:

I think the conference went great!

Kirby:

The conference went very well, thanks to the organizers and thanks to the speakers who with few exceptions did an excellent job of making their specialty accessible to everyone else. This is not easy, and is particularly hard when the audience covers all of topology. But it is vital that we have such conferences and such talks or else topology will just break up into its subareas which no longer interact.

Kleiner:

Thanks very much for organizing the conference and for the invitation to participate. I enjoyed the conference overall. The only way it could have been improved, from my own standpoint, would have been if a few of the lectures were pitched to a more general audience, closer to a colloquium style. However, I suspect that most of the other participants were better versed in homotopy theory and the fine points of surgery, so my comments simply reflect the fact that I'm more of a geometer/geometric group theorist than a topologist.

Lueck:

In my opinion this conference shall be in the format as this year, very broad and not specialized. There are enough special conferences and I like to get an impression to hear from leading representatives what happens in other fields.

In my opinion this is a very good meeting. I have no complaints about the organization or the facilities, they are excellent.

Mark:

I found the Banff workshop to be very informative and a broadening experience. Conferences such as this one, involving a range of mathematicians in various subdisciplines, are too rare. The workshop opened my eyes to problems and techniques in topology of which I was previously unaware, which is extremely valuable.

Thanks for your efforts with organization, and I hope the application for the next workshop goes well.

Mrowka:

I very much enjoyed the meeting and think that more of the same would be great for topology.

Park:

As it usually happens in any conference which puts several areas together, I hardly catch a theme of topics without introducing the contents of topics enough. So, although I am sometimes bored, what do you think that one hour talk is better than 45 minutes talk for speakers and audiences? Except this, I really like this type of conference!

Ranicki:

Thanks again to the organizers for inviting me to a most enjoyable conference. The only negative comment I have is that the organizers did not have the imagination to follow the Oberwolfach tradition (possibly initiated by Matthias himself) of distributing the abstracts of all the talks proposed, and there was no opportunity of presenting posters (e.g. in the room set aside for BIRS across the corridor from the lecture room). Also, the speakers should have been asked to provide reading lists for their talks, so that members of the audience could follow up the talks if so inclined. Thanks again, and good luck with your 2007 proposal

Rasmussen:

This is my second time at BIRS, and my impression of the place has not changed very much from the last visit. I think it is simply the best conference venue for encouraging collaborative work and interaction that I have been to. The setup (breakfast room, everyone staying in the same place, meals together) is great for encouraging interaction between people who might not otherwise get together. I had a lot of fun going to talks from other areas, but I can't say that I got ideas useful for my own reseach from them, or that I was in a position to make meaningful suggestions about them. Despite this criticism, I should say that I really had a great and productive time this week. Thanks to you and the other organizers for putting this thing together.

Stipsicz

It was a great conference, I enjoyed it a lot,

Vidussi:

Some comments on the conference. I definitely enjoyed the idea of having a meeting with people that work in different areas of topology. It is very difficult and time consuming to keep track of the developments of various areas only by reading papers. A conference's talk, instead, gives an easier access to main results and ideas, and allows interaction with a specialist. If there is an improvement that I can suggest, this would be to stress out in advance that the talks are meant for a "general" audience. (You pointed that out at the beginning of the conference but some - including possibly myself - did not fully comply with this.) Personally, my interest in some of the topics discussed at the conference grew; for example, I am currently reading a review paper of W. Lueck on L^2 invariants, and trying to understand if this may have applications in my research.

Third, the schedule and number of talks was perfect, and Banff is a great place for a conference.

Finally, I am very grateful to the organizers for inviting me and giving me the opportunity to give a talk.

Vogtman:

I thought it was great, I learned a lot about what's happening in the rest of topology. Thanks!!!

List of Participants

Adem, Alejandro (University of British Columbia)
Akbulut, Selman (Michigan State University)
Auroux, Denis (Massachusetts Institute of Technology)
Baldridge, Scott (Louisiana State University)
Bartels, Arthur (Universität Münster)
Bauer, Kristine (University of Calgary)
Boden, Hans (McMaster University)

Bridson, Martin R. (Imperial College London)
Chen, Weimin (University of Massachusetts at Amherst)
Collin, Olivier (Université du Québec Montréal (UQAM))
Dwyer, William (Notre Dame University)
Edwards, Bob (University of California Los Angeles)
Grodal, Jesper (University of Copenhagen)
Hambleton, Ian (McMaster University)
Kirby, Robion (University of California - Berkeley)
Kleiner, Bruce (University of Michigan)
Kreck, Matthias (University of Heidelberg)
Lueck, Wolfgang (Universitat Munster)
Lurie, Jacob (Harvard University)
Mark, Thomas (Southeast Louisiana University)
Matic, Gordana (University of Georgia)
Mrowka, Tom (Massachusetts Institute of Technology)
Neumann, Walter (Columbia University)
Ozsvath, Peter (Columbia University)
Park, Jongil (Seoul National University)
Pedersen, Erik (SUNY Binghamton)
Ranicki, Andrew (University of Edinburgh)
Rasmussen, Jacob (Princeton University)
Reich, Holger (Universität Münster)
Ruan, Yongbin (University of Wisconsin-Madison)
Stern, Ronald (University of California Irvine)
Stipsicz, Andras (Hungarian Academy of Sciences)
Stolz, Stephan (University of Notre Dame)
Teichner, Peter (University of California)
Vidussi, Stefano (Kansas State University)
Viro, Oleg (Uppsala universitet)
Vogtmann, Karen (Cornell University)
Wahl, Nathalie (University of Chicago)
Weinberger, Shmuel (University of Chicago)

Chapter 17

Analytic and Algebraic Methods in Complex and CR Geometry (05w5086)

September 4–8, 2005

Organizer(s): John Bland (University of Toronto), John D'Angelo (University of Illinois), Laszlo Lempert (Purdue University), Joseph J. Kohn (Princeton University), Yum-Tong Siu (Harvard University)

This workshop focused on both complex analysis and algebraic geometry. Its primary purpose was to foster interactions among researchers in these areas. This report will describe analytic, algebraic, and geometric perspectives and how they blend.

Both the lectures and the informal conversations held in the workshop developed these connections. It is natural to place (most of) the discussions into one or more of three categories: those with a flavor from Partial Differential Equations, those motivated by CR Geometry, and those concerning Algebraic Geometry. Nearly all the lectures made at least some connections among these areas.

We begin by discussing the Cauchy-Riemann operator $\overline{\partial}$ and its impact on complex analysis. The study of $\overline{\partial}$ as a partial differential operator leads to the basic questions of existence and regularity. These basic questions from partial differential equations naturally lead to theorems relating the geometry of the boundary of a domain to the behavior of $\overline{\partial}$ on the domain. Since the 1960's so-called L^2 methods and their applications have played a major role. We recall some of these developments.

Many important developments in complex analysis in the twentieth century arose from the solution of the *Levi Problem* identifying domains of holomorphy with pseudoconvex domains. Pseudoconvexity is a local geometric property of the boundary, whereas the notion of domain of holomorphy belongs to the function theory on the domain itself. The solution of the Levi Problem includes an existence and regularity result for $\overline{\partial}$. A domain Ω in $\mathbf{C}^n$ is a domain of holomorphy if and only if the following statement holds: For each nonnegative integer q and each smooth $(p, q+1)$ form α on Ω such that $\overline{\partial}\alpha = 0$ on Ω, there is a smooth (p, q) form u on Ω such that $\overline{\partial}u = \alpha$.

The so-called $\overline{\partial}$-Neumann problem extends the above idea by considering the Cauchy-Riemann equations on $\Omega \cup b\Omega$. Suppose that the boundary $b\Omega$ is smooth and consider differential forms with L^2 coefficients on the closed domain. Given a $\overline{\partial}$-closed form α, orthogonal to the harmonic space, the $\overline{\partial}$-Neumann problem constructs the N-operator and the solution $\overline{\partial}^* N\alpha$ to the equation $\overline{\partial}u = \alpha$. Spencer first posed this problem in the 1950's in order to extend Hodge Theory to manifolds with boundary, but many analytic difficulties arose before Kohn solved the problem in 1962 using the method of L^2 estimates.

Local regularity holds when $\overline{\partial}^* N\alpha$ must be smooth wherever α is smooth; local regularity follows from subelliptic estimates, which imply that N is a *pseudo-local* (but not a pseudodifferential) operator. *Global regularity* for the $\overline{\partial}$-Neumann problem holds when $\overline{\partial}^* N\alpha$ is smooth everywhere on the closed domain assuming that α is itself everywhere smooth. Several years after solving the $\overline{\partial}$-Neumann problem, Kohn established a global regularity result using weighted L^2 techniques. A smooth solution to $\overline{\partial}u = \alpha$ exists when α is everywhere smooth on the closed smoothly bounded domain. For a long time it was not known however whether the $\overline{\partial}$-Neumann solution was always smooth. When subelliptic estimates (described below) hold, of course, the $\overline{\partial}$-Neumann solution is smooth. In 1996 Christ proved that global regularity of the $\overline{\partial}$-Neumann solution fails for some worm domains. Boas and Straube showed that global regularity for the $\overline{\partial}$-Neumann solution holds for domains with a defining function that is plurisubharmonic on the boundary. They also verified global regularity when the set of points of infinite type satisfies certain topological conditions, but the problem of global regularity is not yet completely understood. A related open problem concerns finding necessary and sufficient conditions for compactness estimates.

Results about global regularity often produce geometric applications. The smooth extension to the boundary of biholomorphic mappings between certain weakly pseudoconvex smoothly bounded domains provides a striking example. Siu's work on the nonexistence of smooth Levi-flat hypersurfaces in the complex projective plane $\mathbf{P^2}$ gives a second example. In the workshop Ohsawa spoke further about the use of L^2 methods to study Levi flat objects. Siu has also applied techniques of L^2 estimates to establish the invariance of plurigenera first for the case of general type and later when the manifold is not necessarily of general type. Thus L^2 estimates for $\overline{\partial}$ have provided a deep link between analysis and algebraic geometry.

Perhaps the major advance at this workshop was Siu's talk on the famous question of the finite generation of the canonical ring of a compact algebraic manifold X of complex dimension n of general type. Siu described the techniques he introduced from L^2 estimates for $\bar{\partial}$ to handle the obstacles of this problem.

He introduced the infinite sum Φ over all m of the m-th root of the sum of the absolute-value squares of elements of a basis of m-canonical sections. By adapting Skoda's L^2 estimates of $\bar{\partial}$ for the generation of ideals, he first reduced the problem to proving that Φ and one of its finite partial sums are each dominated by a constant multiple of the other. His method involves as intermediate steps the proofs of the rationality of the vanishing orders of Φ and the finiteness of the number of irreducible components of the super level sets of the Lelong number of $\sqrt{-1}\partial\bar{\partial}\log\Phi$. For such proofs he used algebraic geometric techniques which are adapted from and motivated by the following two analytic techniques of the complex Monge-Ampère equation for $\left(\sqrt{-1}\partial\bar{\partial}\log\Phi\right)^n$:

(i) an observation of Demailly that $\frac{1}{\Phi}$ is equivalent to the metric $e^{-\varphi}$ of the canonical line bundle K_X of X with φ maximum among all plurisubharmonic φ subject to the normalization of the supremum of $\varphi - \psi$ being 0 for some fixed background metric $e^{-\psi}$ of K_X, and

(ii) a result of Bedford and Taylor that the complex Monge-Ampère equation is the Euler-Lagrange equation for maximizing a function among plurisubharmonic functions.

Notice that two analytic techniques, developed in the study of the complex Monge-Ampère equation, have algebraic applications here. First, Fefferman's work (Annals 1976) on the asymptotic order of the solution of the complex Monge-Ampère equation on a strongly pseudoconvex domain motivates the algebraic geometric technique to prove the rationality of vanishing orders of Φ. Second, Yau's regularity results (Comm. Pure and Applied Math. 1978) for the complex Monge-Ampere equation when the right-hand side has complex analytic singularities motivates the algebraic-geometric techniques for proving the finiteness of the number of irreducible components of the super level sets of the Lelong number of $\sqrt{-1}\partial\bar{\partial}\log\Phi$. By incorporating the techniques developed for the Fujita conjecture type problems and the techniques of Shokurov's nonvanishing theorem, Siu's method translated the analytic techniques to the algebraic geometric settings so that when either some vanishing order of Φ is irrational or there are infinite number of super level sets of the Lelong number of $\sqrt{-1}\partial\bar{\partial}\log\Phi$, some new pluricanonical sections can be produced by L^2 estimates of $\bar{\partial}$ to give a contradiction to the definition of Φ.

We return to the $\overline{\partial}$-Neumann problem on a smoothly bounded domain Ω. The geometry of the boundary enters because of the $\overline{\partial}$-Neumann boundary condition. For a $(0,1)$ form ϕ this condition is the same as saying that the $(1,0)$ vector dual to ϕ is tangent to $b\Omega$. This condition therefore leads to the notion of a CR manifold. CR manifolds are real manifolds whose tangent spaces behave like those of real submanifolds in complex manifolds. The special case of a real hypersurface in complex Euclidean space arises of course as the (smooth) boundary of a domain. The $\overline{\partial}$-Neumann problem therefore provides a deep link between the CR geometry of the boundary of Ω and the function theory on Ω.

The ideas in the proofs of existence and regularity results for $\overline{\partial}$ have led to the development of CR geometry, the calculus of pseudo-differential operators, and subelliptic multiplier ideal sheaves. All three of these topics have evolved considerably, and each played a major role in the workshop.

We next discuss subellipticity and related ideas. After much preliminary work, in 1978 Kohn introduced subelliptic multipliers as a technique for proving subelliptic estimates in the $\overline{\partial}$-Neumann problem. Subelliptic estimates imply local regularity results for the $\overline{\partial}$ operator. In the 1970's Skoda introduced the use of L^2 methods in algebraic geometry. Currently the algebraic geometry community has become actively involved in the study and use of multiplier ideal sheaves. The work of Siu, Nadel, Demailly and others have demonstrated convincingly the power of such analytic methods in algebraic geometric problems. Nearly all participants in the workshop have used either analytic or algebraic aspects of these ideas in their work, and if not, have worked on closely connected problems.

The solution of the $\overline{\partial}$-Neumann problem on strongly pseudoconvex domains can be understood by thinking of the determinant of the Levi form $\det(\lambda)$ as a subelliptic multiplier; for any 1-form ϕ in the domain of $\overline{\partial}^*$, one can control the Sobolev $\frac{1}{2}$-norm of $\det(\lambda)\phi$ in terms of the usual Dirichlet form. When this determinant vanishes things become quite difficult. Kohn posed the problem of determining necessary and sufficient conditions for subelliptic estimates for $\overline{\partial}$. D'Angelo introduced a finite type condition that, through deep work of Catlin, turned out to be necessary and sufficient for subelliptic estimates on $(0,1)$ forms on pseudoconvex domains. A similar result holds for forms of higher degree. Catlin's proof does not use subelliptic multipliers; instead he constructs bounded plurisubharmonic functions with large Hessians. The precise relationship between the two approaches to subelliptic estimates is not yet understood. Because they apply in the smooth category, Catlin's techniques have significant unrealized potential in subelliptic multiplier theory.

All these ideas are closely related to singularity theory. D'Angelo has discussed a precise analogy: strongly pseudoconvex points correspond to the maximal ideal in the ring of germs of holomorphic functions at a point, and finite type corresponds to ideals primary to the maximal ideal. Thus the problem of subelliptic estimates helped establish a basic connection between hard analysis (PDE estimates) and singularity theory. Developing this connection was one of the reasons for holding this workshop.

The workshop itself succeeded in forging new connections on precisely this topic. For example Lazarsfeld spoke about the *type* of a punctual ideal, a concept invented in algebra for several reasons, and independently in analysis for the purpose of understanding the relationship between finite type and subelliptic estimates. The type of a punctual ideal in the ring of germs of holomorphic functions is finite if and only if the ideal is primary to the maximal ideal, and it provides an interesting numerical measurement (always a rational number) of the singularity. The lecture of Lazarsfeld showed how ideas in algebra such as the integral closure of an ideal, normalized blow-ups, and the Briancon-Skoda Theorem impact the study of the type of a punctual ideal. The theory of finite type shows how to reduce the type of an ideal in the ring of germs of smooth functions to the types of a family of punctual ideals. Closely related to these ideas is an algebraic version of Kohn's theory of subelliptic multipliers in the (simpler) holomorphic setting, a topic which was discussed by many of the participants in the informal discussion held throughout the workshop. Lazarsfeld also gave a simple treatment and extension of a result of McNeal-Nemethi showing how a supremum over all holomorphic arcs can be replaced by a maximum over a finite list of well-chosen holomorphic arcs, thus rendering evident the rationality of the type. This material illustrates well the sort of connections forged by the workshop.

Hwang spoke about the relationship between the Arnold multiplicity and the usual notion of multiplicity connected with orders of vanishing. The Arnold multiplicity is a local invariant of an effective divisor on a complex manifold. It is the infimum of the set of m for which a certain integral is finite; if f is a local equation for the divisor, the Arnold multiplicity is the infimum of the set of m for which $|f|^{\frac{-2}{m}}$ is locally integrable. Hwang established a decisive estimate for the Arnold multiplicity when the base manifold is the quotient of a complex semi-simple Lie Group by a maximal parabolic subgroup. To do so he proved a product theorem concerning the behavior of the Arnold multiplicity for divisors on the product of two manifolds. Again we observe a powerful connection between analysis and algebraic geometry. Hwang discussed upper-semi continuity properties of these multiplicities, making a nice connection with other issues. For example, semi-continuity fails for the type of a family of punctual ideals depending nicely on a parameter, and this result has impacted subelliptic estimates. On the other hand, inequalities relating the type to the co-length, which behaves better under change of parameter, play a role in work on finite type.

D'Angelo spoke on a monotonicity result for holomorphic volumes. At first glance this result is not obviously related to the theme we have discussed so far; on the other hand volumes involve integrals of squared norms of Jacobians, and the results are thereby connected with both complex geometry and L^2 ideas. Note that the determinant of the complex Hessian of the squared norm of a holomorphic mapping is precisely equal to the sum of the squared moduli of all possible Jacobians of the components of the mapping. The ideas are thus connected with properties of the integral of the determinant of the Levi form. The monotonicity result leads to a corollary with a nice algebraic-geometric flavor. Let p be a proper polynomial mapping between balls, of degree d. Then the volume of the image of the ball under p is at most $\frac{\pi^n d^n}{n!}$, with equality if and only if the mapping is homogeneous. For balls and eggs one proves the monotonicity result for volumes of holomorphic images by carefully studying the L^2 norms of monomials. A result for more general pseudoconvex domains can be proved using Stokes's theorem, in case the map has some regularity at the boundary. Again we see how L^2 methods are closely related to complex geometry.

Several other talks in the meeting nicely illustrated L^2 methods. McNeal spoke about a generalization (due to McNeal-Varolin) of the celebrated Ohsawa-Takegoshi Theorem. Suppose first that D is a pseudoconvex domain in $\mathbf{C}^n$ and that H is a complex hyperplane. Let f be holomorphic on $H \cap D$, and in L^2 with respect to some weight. Ohsawa-Takegoshi proved that f can be extended to a function F holomorphic in D whose L^2 norm with respect to the same weight is controlled by the L^2-norm of f. McNeal-Varolin showed how to gain strength in this estimate by manipulating the weights. During the talk Siu observed a parallel with these ideas and his use of the L^2 extension result in order to establish the invariance of plurigenera.

A natural problem in complex analysis asks to express a nonnegative Hermitian symmetric polynomial as a squared norm of a holomorphic mapping, or more generally as a quotient of squared norms of holomorphic mappings. D'Angelo has asked, as a complex variable analogue of Hilbert's 17th problem, for a characterization of quotients of squared norms of holomorphic polynomial mappings. Work of Catlin-D'Angelo relating isometric imbedding of holomorphic bundles to squared norms and quotients of squared norms of holomorphic mappings provides a general framework for such questions. Their result assumes a nondegeneracy condition analogous to strong pseudoconvexity; the degenerate case is quite subtle, because the class of quotients of squared norms is not closed under limits. In his talk at the workshop Varolin announced a complete solution to this question. His proof uses L^2 techniques and a form of the resolution of singularities. Furthermore the setting applies for many bundles, and even the proof in the (simplest) case of powers of the tautological bundle over projective space requires proving the theorem for more general spaces. Varolin's condition states that the real Hermitian polynomial R, which can always be written as $||F||^2 - ||G||^2$ for holomorphic mappings F and G, is a quotient of squared norms if and only if the function

$$\frac{||F||^2 + ||G||^2}{||F||^2 - ||G||^2}$$

is bounded. The proof involves the Bergman kernel function in a rather general setting. As in the above work on isometric embedding, the Bergman kernel function appears as an approximate generating function for tensor powers of a metric.

Varolin also discussed other positivity conditions and Siu mentioned the connection with a famous paper of Calabi on isometric imbedding from the early 1950's. Various forms of a non-linear version of the Cauchy-Schwarz inequality play a key role in all the work on isometric embedding. The condition that a bundle metric satisfies the non-linear Cauchy-Schwarz inequality involves curvature, but it is distinct from the usual curvature conditions. It could therefore could play a role in developing new connections between analysis and complex geometry.

Next we turn to some connections between PDE and CR geometry. Perhaps the most basic example of a CR manifold is the unit sphere. Because the unit ball is biholomorphically equivalent with the Siegel generalized upper half-plane, its boundary (the sphere) is CR equivalent with the Heisenberg group. This connection between several complex variables and harmonic analysis has been especially fruitful in studying the strongly pseudoconvex case, but new ideas are needed in general.

Several talks considered issues centering around differential and pseudo-differential operators on CR manifolds, typically motivated by the Heisenberg group. Melrose began the workshop with a general and abstract treatment of a calculus of pseudo-differential operators that takes into account the anisotropic behavior of the tangent spaces on strongly pseudoconvex boundaries. The anisotropic behavior there has one *parabolic direction*. He showed that operators in a very general class behave properly under composition. Precise descriptions of the kernels of these operators of course epitomizes the theme of the workshop; the relationship between the geometry of the boundary of a domain and analysis on the domain. His general results apply in some cases admitting multiple parabolic directions and also apply to other applied boundary problems.

A central problem of local CR geometry is the embeddability question. Is an abstract CR manifold (of hypersurface type) locally CR-embeddable in $\mathbf{C}^{\mathbf{N}}$? Kuranishi solved the problem for strongly pseudoconvex CR manifolds of dimension at least nine using L^2-estimates. Akahori and later Webster proved the result in dimension seven. Akahori used L^2-methods, whereas Webster used integral formulas for solving the $\overline{\partial}$-equation. Catlin has generalized these results when appropriate finite-type conditions replace strong pseudoconvexity. It has been long known that the result fails in three dimensions, but the case of dimension five remains open.

In the problem session Greiner proposed an approach to prove local embeddability for CR manifolds of dimensions at least five. This approach relies on Greiner's program of constructing fundamental solutions explicitly. Previous approaches construct the embedding by an iterative procedure. In each step one solves an approximate $\overline{\partial}_b$ equation for $(0,1)$-forms on an embedded CR manifold. The solution is obtained by solving a precise $\overline{\partial}_b$ equation on $(0,2)$ forms. Use of this secondary equation requires the dimension to exceed five. Greiner's approach, by contrast, constructs an embedding in one step, by finding CR functions with prescribed differential at one point. To do so he solves a $\overline{\partial}_b$ equation on (0,1) forms, using explicit kernels. To make this approach work, one needs to extend Greiner's explicit results on fundamental solutions from one PDE to systems of PDE.

Studying which three-dimensional CR manifolds can be embedded is a challenging part of the general problem. Various partial results have opened new avenues for investigating the relationships between function theory for pseudoconcave manifolds and CR deformation theory for the boundary.

Epstein discussed the embedding problem for abstract three-dimensional CR manifolds. He related this question to the Dirac operator $\overline{\partial}+\overline{\partial}^*$. He considered the collection of embeddable CR structures near a given embeddable one, and gave a necessary and sufficient condition; namely, that the restriction of the Szegö projection be Fredholm. Epstein began by describing an extension of the $\overline{\partial}$-Neumann problem to a class of $Spin_C$ manifolds. He used it to study the relative index between two generalized Szegö projectors on a contact manifold. For example, suppose that a three-dimensional contact manifold bounds two strongly pseudoconvex complex surfaces. Then the relative index can be expressed in terms of the differences of their Euler characteristics, their signatures, and the dimensions of their cohomology groups $H^{0,1}$. In certain cases it follows that the relative index assumes only finitely many values among embeddable deformations close to a given embeddable structure. In these cases the set of embeddable CR-structures is closed in the C^∞-topology.

The talks by Greiner and Tie considered sub-Riemannian geometry, motivated again by the Heisenberg group. Greiner's talk provided many explicit relationships between CR geometry and geodesics. He considered second order partial differential operators given as sums of squares of vector fields; these operators arise for example as the Kohn Laplacian in the case of three-dimensional CR manifolds, and information about them is therefore useful for complex analysis. Greiner built explicit formulas for fundamental solutions from geometric invariants. A new phenomenon in this sub-Riemannian geometry is the notion of the "characteristic submanifold" attached to every point p: the locus of points connected to p by an infinite number of geodesics.

Tie's talk also evolved from generalizing some of the basic ideas from CR geometry. For example, we have seen that the anisotropic behavior of the CR geometry of a strongly pseudoconvex manifolds leads to harmonic analysis on the Heisenberg group, which has a nilpotent Lie algebra. For certain 3-dimensional CR manifolds of finite type, Lie algebras of higher step arise. Tie discussed a specific example of step 3 and its relations to Hamilton's equations and the Heisenberg group.

Polarization techniques play a key role whenever real-analytic functions arise, e. g., as defining equations of domains or as metrics on holomorphic line bundles. The ability to vary z and $\overline{z}$ separately lies at the foundation of complex analysis. Segre introduced the varieties which have been used extensively by Webster and others in diverse problems. More recently Baouendi-Ebenfelt-Rothschild developed an iterative procedure to generate additional Segre sets. These ideas have had many uses. In particular Ebenfelt and Rothschild proved a CR transversality result for generic real-analytic CR submanifolds of finite commutator type. The result says that the germ of a finite holomorphic mapping between two such manifolds is necessarily CR transverse. In other words, in codimension d, one obtains a result guaranteeing that a certain derivative mapping has rank d. The codimension one version of this result is a version of the Hopf lemma. The technique of Segre sets also provides a characterization of finite commutator type, due to Baouendi-Ebenfelt-Rothschild; for a generic CR manifold M of codimension d, the Segre set $S_{2d}(p)$ contains an open neighborhood of p if and only if M is of finite commutator type at p. Thus an issue about iterated commutators of vector fields (a part of *complex* control theory) has a description in terms of Segre sets and polarization.

Segre sets also arose in the talk of Christ, revealing quite an interesting connection. Christ considered L^p estimates for generalized Radon transforms. Generalized Radon transforms are defined by integration over families of submanifolds of an ambient space and associated with a certain geometric structure. A basic and fascinating problem here is to relate the analysis to the underlying geometry. Part of Christ's talk considered this idea as a problem in continuum combinatorics. The relationship between geometry and analysis described here meshed especially well with the several talks on pseudodifferential operators and sub-Riemannian geometry.

The recent striking work of Kohn on hypoellipticity despite loss of derivatives was mentioned in the original proposal for this workshop. One talk directly considered this topic. Tartakoff discussed his work with Derridj and Bove showing that Kohn's example of a C^∞ hypoelliptic operator P_k is also locally analytic hypoelliptic. The proof yields a simplification of Kohn's proof. The second order operator P_k has the simple expression

$$P_k = LL^* + (\overline{z}^k L)^* (\overline{z}^k L),$$

where L is a Lewy operator of the form

$$L = \frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t}.$$

Tartakoff also provided a generalization $P_{k,m}$ which is hypoelliptic in both senses but loses $\frac{k-1}{m}$ derivatives. The techniques of proof involve complicated estimations which evoke earlier work by Tartakoff and Treves on global analytic hypoellipticity for operators such as the $\overline{\partial}$-Neumann operator.

A major advance (1981) in CR geometry was the Baouendi-Treves approximation Theorem: A CR function on a CR submanifold of $\mathbf{C^n}$ can be locally uniformly approximated by entire holomorphic functions. The proof uses convolution with a complex Gaussian kernel. A CR function is of course a solution to the homogeneous tangential Cauchy-Riemann equations. Boggess spoke about global and semi-global versions of the Baouendi-Treves result. In particular Boggess and Dwilewicz proved such a result for real hypersurfaces in $\mathbf{C^n}$ that are graphs over a linear space of codimension one.

An important idea in CR geometry concerns the tangential version of the inhomogeneous Cauchy-Riemann equations. As in the case of holomorphic functions, one obtains information about the solutions of the homogeneous equation by studying the inhomogeneous equation as a system of PDE. In the smooth category many such results have been worked out. Shaw spoke about estimates for the tangential Cauchy-Riemann equations on CR manifolds with minimal smoothness. The main point is to prove Hölder and L^p regularity for the tangential Cauchy-Riemann equations on CR manifolds of class C^2. One application of these estimates is to prove the embedding theorem of Boutet de Monvel for strongly pseudoconvex CR manifolds of real dimension at least five and of class C^2.

Stolovitch considered a basic question about CR singularities. Consider a real-analytic $(n+r)$-dimensional submanifold of $\mathbf{C^n}$ having a CR-singularity at the origin. Let us restrict to quadrics for which one can define generalized Bishop invariants. Such a quadric intersects the complex linear manifold $z_{p+1} = \cdots = z_n = 0$ along some real linear set $\mathcal{L}$. Stolovitch discussed what happens to this intersection under perturbation of the quadric. In some cases, if such a submanifold is formally equivalent to its associated quadric, then it is holomorphically equivalent to it.

We next discuss some of the connections with algebraic geometry and complex differential geometry.

Mabuchi considered three notions of stability; K-stability, Chow-Mumford stability, and Hilbert-Mumford stability, and clarified their asymptotic relationships. He showed that asymptotic Chow-Mumford-Veronese stability coincides with asymptotic Hilbert-Mumford stability and that K-stability implies asymptotic Chow-Mumford-Veronese stability. For a polarized projective algebraic manifold with vanishing Futaki character, Mabuchi showed that asymptotic Chow-Mumford stability relative to an algebraic torus implies K-semi-stability.

de Oliveira considered symmetric differentials and the hyperbolicity of hypersurfaces in $\mathbf{P}^3$ with appropriate nodal singularities. The existence of symmetric differentials on an algebraic surface X has a strong impact on the algebraic and transcendental hyperbolicity of X. Unfortunately, smooth hypersurfaces in $\mathbf{P}^3$ have no symmetric differentials. It turns out that there are smooth families whose general member is a smooth hypersurface of degree $d \geq 6$ in $\mathbf{P}^3$, but whose special member which is singular has many symmetric differentials. By using a resolution of singularities in his argument, he showed that the special singular member with appropriate nodal singularities has sufficient independent symmetric differentials to make it quasi-algebraically hyperbolic. This situation exhibits jumping of the cotangent plurigenera along a family.

Miyaoka provided some new examples of stable and semistable Higgs bundles. Higgs bundles arise from representations of the fundamental group of complex or algebraic manifolds, and are part of the active subject of noncommutative Hodge theory. They have played an important role in gauge theory and the geometrization of mathematical physics

Yeung discussed integrality and arithmeticity of lattices in quotients of the ball. The main result is that a co-compact lattice in a complex two ball is integral. He also discussed related geometric and arithmetic problems. Although arithmetic geometry was not the primary focus of this meeting, Yeung's results indicate intriguing connections between algebraic, analytic, and arithmetic geometry.

Nearly every good conference has at least one excellent talk that, at first glance, seems a bit removed from the other talks. Often such talks profoundly impact future developments in the subject, because they provide fresh ideas. Larusson gave such a talk at this meeting, on the subject of model categories and homotopical algebra, a subject invented by Quillen. Model categories provide an abstract setting for developing analogues of the homotopy theory of topological spaces for various other sorts of objects, and they have found important applications not only within homotopy theory itself but also in algebra and algebraic geometry. Recently they have appeared in complex analysis and provided a natural conceptual framework for the Oka Principle. Of course the Oka Principle intimately connects the Cauchy-Riemann equations with topology; one expects, on a Stein manifold, to be able to do with holomorphic functions what one can do with continuous functions.

Many important parts of complex analysis were not explicitly mentioned at the workshop, but the subject remains finely woven, and many such topics made at least a spiritual appearance. We mention in particular the possibilities associated with extending the ideas of the workshop to infinite dimensional holomorphy, an area thriving due to deep work of Lempert.

The workshop ended with a discussion of open problems in complex analysis and algebraic geometry and their connections.

There is no doubt that the workshop forged significant connections between complex analysis and algebraic geometry. The lectures, discussions, and the session on open problems enabled a diverse group of mathematicians with common interests to see first-hand how techniques from other parts of mathematics can be used in their own research specialties. Furthermore the amenities of the BIRS helped create a lively and stimulating environment. Research in both complex analyis and algebraic geometry has advanced as a result of this meeting.

List of Participants

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Siu, Yum-Tong (Harvard University)
Stolovitch, Laurent (CNRS-Université Paul Sabatier)
Straube, Emil (Texas A&M University)
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Varolin, Dror (Stony Brook University)
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Chapter 18

Interactions between Noncommutative Algebra and Algebraic Geometry (05w5035)

September 10–15, 2005

Organizer(s): Michael Artin (Massachusetts Institute of Technology), Colin Ingalls (University of New Brunswick), Zinovy Reichstein (University of British Columbia), Lance Small (University of California, San Diego), James J. Zhang (University of Washington)

abstract

This is the final report by the organizers of the workshop on *Interactions between noncommutative algebra and algebraic geometry*, held at the Banff International Research Station September 10 - 15, 2005. The workshop was attended by 36 mathematicians from eight different countries (Australia, Canada, China, France, Great Britain, Israel, Norway, and the United States).

This report is subdivided into three parts. In the first part we introduce the subject matter of the workshop and briefly discuss its history, from the beginning of the 20th century to the present. In the second part we describe some of the currently active research topics which involve the use of algebro-geometric methods in noncommutative algebra or conversely, topics in algebraic geometry (and related mathematical physics), where noncommutative algebra plays an important role. These topics formed the core of the scientific content of the workshop. The third part consists of summaries of lectures given at the workshop.

Introduction

Noncommutative phenomena are perhaps as old as mathematics itself; they manifest themselves in the simplest mathematical objects, such as permutations or matrices. Noncommutative algebra developed into a separate subject in the early 20th century. The initial steps, taken by Dickson and Wedderburn, among others, were motivated by attempts to better understand ”hypercomplex numbers”, such as the quaternions, discovered by W. Hamilton in 1843. Subsequent steps, due to E. Artin, R. Brauer, H. Hasse, E. Noether, etc., came in the context of abstract algebra, which was a rapidly developing subject in the 1920s and 30s. The next phase, lasting roughly from the 1930s to the early 1980s and led by A. Albert, S. Amitsur, N. Jacobson, I. Kaplansky, A. Goldie, I. Herstein, among others, focused on developing the structure theory for various types of noncommutative rings.

To illustrate how quickly one encounters open problems in noncommutative algebra and to give the reader a bit of a flavor of the subject, consider the Weyl algebra $A = \mathbf{C}\{x_1, x_2\}$, given by two generators, x_1 and x_2, and one relation,

$$x_1x_2 - x_2x_1 = 1\,.$$

If we replace the 1 on the right hand side by any other complex number $\alpha \neq 0$, we will get an isomorphic algebra; however, if we set $\alpha = 0$ then the resulting algebra will become the commutative polynomial ring $P = \mathbf{C}[x_1, x_2]$ in two variables. Thus we can think of A as a noncommutative deformation of P; the two have some properties in common, but A is considerably more complicated. One property they have in common is that in both cases we can obtain a skew field by formally inverting all non-zero elements. Recall that a skew field (otherwise known as a division algebra) satisfies all the axioms of a field (i.e., one can perform the four arithmetic operations, and all the usual associative, distributive, etc. rules hold) except that multiplication is not required to be commutative. Of course, if we formally invert all non-zero elements in the polynomial ring $P = \mathbf{C}[x_1, x_2]$, the skew field we obtain will be the rational function $\mathbf{C}(x_1, x_2)$ in two variables. If we do the same thing to A, we will obtain a noncommutative skew field D, called the Weyl skew field. There are many proper skew subfields $S \subset D$ isomorphic to D itself. Given one such S, we can view D as an S-vector space in two different ways, with scalar multiplication on the left or on the right. It is thus natural to ask whether or not these two vector spaces have the same dimension. This seemingly simple question is, in fact, a long-standing open problem. A more general question along the same lines is whether or not the left and the right dimensions of A over B are the same, where A is an arbitrary skew field and B is a skew subfield. This question was posed by E. Artin and settled in the negative by P.M. Cohn [25] and A.H. Schofield [44]. Returning to the Weyl skew field D, note that much is unknown here. If the left and right dimensions of D over S turn out to be the same, what invariants distinguish D from the examples constructed by Cohn and Schofield? On the other hand, if the right and left dimensions are different (for some S), is there an effective way to compute them or to determine whether or not they are the same for a given S?

In last 20 years the subject of noncommutative algebra has been rapidly developing in several different directions. Once common theme has been the increasing penetration of algebro-geometric methods into the subject and conversely the increasing use of noncommutative ring theory within algebraic geometry and related mathematical physics. One important outgrowth of this interaction is an entirely new research area, called noncommutative algebraic geometry. Recall that one of the foundational steps in the early development of (commutative) algebraic geometry was the realization that every commutative ring R can be thought of the the ring of regular functions on a suitably defined space, namely, $X = \operatorname{Spec}(R)$. This dichotomy is of fundamental importance in commutative ring theory: by passing from R to X, one can often translate purely algebraic questions into problems about the geometry of X, in a setting where both geometric tools and geometric intuition are available. Noncommutative algebraic geometry is motivated by an attempt understand noncommutative rings in a similar manner. Curiously, this turned out to be somewhat easier to do with graded rings, using methods derived from projective (rather than affine) algebraic geometry. The reason is that affine algebraic geometry tends to rely on techniques like localisation that are rarely available in the noncommutative setting, whereas the more global and categorical approaches to projective geometry can and have been generalized. Another goal of noncommutative algebraic geometry is to build up and study "noncommutative algebraic varieties" or "noncommutative schemes". In addition to clarifying the structure of noncommutative rings they are of independent interests, and may have interesting and unexpected applications (e.g., in mathematical physics). At this point we are rather far from fully realizing these goals. However, the methods developed in noncommutative projective geometry have already found a number of applications; in particular, they have been used to solve several outstanding open problems in noncommutative ring theory [5, 6, 4, 7, 8].

The purpose of the workshop was to discuss various aspects of the interaction between noncommutative ring theory and algebraic geometry, including the latest developments in noncommutative algebraic geometry. In particular, the following topics were discussed.

Areas of recent activity

We will now outline several areas of interaction between algebraic geometry and noncommutative algebra, where there have been interesting new developments in recent years. Most of these developments were discussed during the workshop.

Foundation of noncommutative algebraic geometry

One important question in the field concerns the foundations of noncommutative algebraic geometry. For example, what is “the right” axiomatic definition of a noncommutative space? The approach usually taken in (commutative) algebraic or differential geometry is to first define what a space of the desired type should look like locally, in a sufficiently small open neighbourhood of each point, then specify what kind of transition functions are allowed to ”glue” these local charts together. For example, in a differentiable manifold, a sufficiently small neighbourhood of every point looks like an open ball in $\mathbf{R}^n$, with differentiable transition functions between these local charts. A scheme looks like $\mathrm{Spec}(R)$ in a neighbourhood of each point, with regular transition functions between the charts. As we pointed out above, it is not possible (or at least has not been possible so far) to mimic this approach for noncommutative spaces, because it ultimately relies on the assumption that one can easily pass to a smaller open subset of a given point. In the commutative setting this is done through the technique of localization (i.e., inverting certain elements in a ring), which is usually not available in the noncommutative setting. The successful approaches so far have taken the global point of view from the very beginning. Here is a partial list of papers addressing this subject. M. Artin [3], M. Artin and J.J. Zhang [9], M. Kontsevich and A. Rosenberg [34], Y.I. Manin [37], A. Rosenberg [39, 40, 41], M. Van den Bergh [52], F. Van Oystaeyen and A. Verschoren [54], A.B. Verevkin [55, 56], V. Ginzburg [31], W. Crawley-Boevey, P. Etingof and V. Ginzburg [27]. One purpose of having many different approaches to noncommutative spaces is to understand them from different points of view. The foundations of noncommutative projective geometry that were established by A.B. Verevkin [55, 56] and M. Artin and J.J. Zhang [9] have been largely accepted but this is just the beginning of this theory, and much foundational work remains to be done.

Finite-dimensional division algebras of transcendence degree 2.

Division algebras (or skew fields) that are finite over their centres have been studied since the beginning of the 20th century. These algebras play an important role in algebraic geometry, the theory of algebraic groups, algebraic number theory and algebraic K-theory. Some of the most exciting recent developments in this field have to do with algebras defined over function fields of surfaces. Recall that every finite-dimensional central simple algebra A/K can be written in the form $A = \mathrm{M}_n(D)$, where D is a division algebra with centre K. The index d of A is the degree of D, i.e., $\sqrt{\dim_K(D)}$. The exponent of A is the smallest positive integer e such that $A^{\otimes e}$ is a matrix algebra over K. It is known that $e \leq d$ and that e and d have the same prime divisors. If K is the function field of a surface it has been long conjectured that $e = d$; this is sometimes called the period-index problem. Special cases of this conjecture were proved by M. Artin and J. Tate [2] in the 1980s, but a full solution was obtained only a few years ago by J.A. de Jong [28]. Similar results in the context of arithmetic surfaces were proved earlier by D.J. Saltman [42, 43] (who spoke on this topic at the workshop), and subsequently strengthened by M. Lieblich [36].

Another important open problem in the theory of central simple algebras is the Albert conjecture. Recall that a cyclic algebra of degree n over a field K, containing a primitive nth root of unity ζ_n, is a K-algebra given by two generators, x and y and three relations,

$$x^n \in K\,, \quad y^n \in K\,, \quad \text{and} \quad xy = \zeta_n yx\,.$$

Albert’s conjecture asserts that every division algebra of prime degree $n = p$ is of this form. This conjecture (which might or might not been stated by Albert), has motivated much of the research in the theory of central simple algebras, going as far back as perhaps the 1930s.

In a recent preprint, M. Ojanguren and R. Parimala [38] use and further develop the ideas of M. Artin, D.J. Saltman and J.A. de Jong, to prove Albert’s conjecture for division algebras of prime degree over the function field K of a complex surface. The details of this argument are still being checked by the experts. If the proof holds up, it is believed that a similar method can be used to show that the abelian closure of K has cohomological dimension 1. (Here K is the function field of a complex surface, as above.) Note that the inequality $\mathrm{cd}(K_{ab}) \leq 1$ is currently only known in a few cases; in particular, for K = a number field, or a p-adic field by class field theory and for $K = \mathbf{C}((X))((Y))$ by [26, Theorem 2.2]. For a related conjecture of Bogomolov, see [11, Conjecture 2].

Birational classification of noncommutative surfaces.

Division algebras that are infinite dimensional over their centres appear naturally in noncommutative algebra and noncommutative algebraic geometry. Similarly to the commutative situation, the classification of division algebras of transcendence degree 2 would be equivalent to the birational classification of integral noncommutative projective surfaces. Hence it is important to work out the classification of division algebras of transcendence degree 2. M. Artin proposed a conjectured list of division algebras of transcendence degree 2 in [3]. All algebras on this list are known to be of transcendence degree 2; the conjecture is that there are no others. If Artin's conjecture is proved, it will have many strong consequences in noncommutative ring theory.

Quantum projective spaces.

Quantum $\mathbf{P}^2$s have been classified by M. Artin, W. Schelter, J. Tate and M. Van den Bergh [4, 7, 8]. They are well understood. So it is natural to ask if we can classify quantum $\mathbf{P}^3$s, or more generally quantum $\mathbf{P}^n$s for all $n \geq 3$. Quantum $\mathbf{P}^n$s are fundamental objects in noncommutative algebraic geometry. Many interesting noncommutative spaces can be embedded into some quantum $\mathbf{P}^n$. On the other hand, it is not clear if every noncommutative space can be embedded into quantum $\mathbf{P}^n$s. This problem is not solved even for quantum or noncommutative K3 surfaces and the quantum Calabi-Yau 3-folds. The reason for this is that quantum $\mathbf{P}^n$s are not fully understood.

The complete classification of quantum $\mathbf{P}^n$s is an extremely difficult project. An algebraic approach to constructing quantum $\mathbf{P}^n$s is to form the noncommutative scheme $\mathrm{Proj}\ A$ where A is a noetherian Artin-Schelter regular connected graded algebra of global dimension $n+1$. Therefore the algebraic form of the above mentioned question is the classification of noetherian, Artin-Schelter regular, connected graded algebras. Researchers have been studying many special classes of noetherian Artin-Schelter regular algebras of global dimension 4. One well-studied example is the Sklyanin algebra of dimension 4, introduced by Sklyanin [50, 51]. Artin-Schelter regular algebras of dimension four have been extensively studied by many researchers (S.P. Smith, J.T. Stafford. T. Levasseur, L. Le Bruyn, M. Van den Bergh, J. Tate, M. Vancliff, B. Shelton, K. Van Rompay, L. Willaert, T. Cassidy, D. Stephenson, D.-M. Lu, J. Palmieri, Q.S. Wu and others). in recent years. This gives us hope that a complete classification of quantum $\mathbf{P}^3$'s may be in sight. Note that quantum $\mathbf{P}^3$'s will provide new examples of division algebras of transcendence degree 3. These division algebras are likely to play an important role in noncommutative projective geometry.

Combinatorial noncommutative algebra The study of finitely generated algebras like the Weyl algebra, enveloping algebras of finite dimensional algebras, Sklyanin algebras is greatly aided by the use of combinatorial techniques that go back to Shirshov, Golod and Shafarevich, Gelfand and Kirillov, and others.

The first real issue is to determine when a finitely generated algebra (or a module over it) is actually finite dimensional. In fact, Golod and Shafarevich found a criterion for infinite dimensionality of algebras involving generators and relations that led to an example of a finitely generated nil algebra (an algebra in which every element is nilpotent) that is infinite dimensional. This settled the Kurosh problem for algebras by showing that not every finitely generated algebra that is algebraic over its base field is finite dimensional. This example also gives immediately a counterexample to the Burnside problem for groups.

Until very recently, the Golod -Shafarevich example was, in some sense, the only such example. These rings all had exponential growth. This past year, Tom Lenagan and Agata Smoktunowicz produced examples of finitely generated nil algebras with polynomial growth.

Let $A = k[V]$ be a finitely generated algebra, where V is a finite dimensional generating subspace of the algebra over the field k, and let $d(n)$ be $\dim(V^n)$, where V^n is the subspace generated by all products of n or fewer elements of V. The Gelfand-Kirillov dimension GK(A) of A, is defined as

$$\mathrm{GK}(A) = \limsup_{n\to\infty}\ \log_n(d(n))\,.$$

This definition is independent of the choice of the generating set, V. For example, the GK dimension of the commutative polynomial ring in n variables is n; a free algebra has infinite GK dimension; the GK dimension of a finite dimensional algebra is zero; a finitely generated polynomial identity(PI) algebra has finite GK dimension; given any real number, γ, greater than or equal to two, there is a finitely generated PI whose GK dimension is γ. Remarkably, Victor Markov has shown that any finitely generated subalgebra of matrices over a commutative algebra has integral GK dimension.

In general, it 's not known when finitely generated algebras have integral GK dimension even if they're noetherian. An important recent positive step is Smoktunowicz's result that a finitely generated graded integral domain cannot have GK dimension properly between two and three. The conjecture is that finitely generated graded domains all have integral GK dimension.

Noncommutative Iwasawa algebras

Noncommutative Iwasawa algebras form a large and interesting class of complete semilocal noetherian algebras, constructed as completed group algebras of compact $p-$adic analytic groups. Thus, let p be a prime integer, let $\mathbf{Z}_p$ be the $p-$adic integers, and let G be a compact $p-$adic analytic group, so (equivalently - see [29]) G is a closed subgroup of $GL_d(\mathbf{Z}_p)$ for some $d \geq 1$. Then the *Iwasawa algebra of* G is

$$\Lambda_G \quad := \quad \lim_{\longleftarrow} \mathbf{Z}_p[G/N],$$

where the inverse limit is taken over the open normal subgroups N of G, (which have finite index in G by the compactness hypothesis). Closely related to Λ_G is its epimorphic image Ω_G, defined as

$$\Omega_G \quad = \quad \lim_{\longleftarrow} \mathbf{F}_p[G/N],$$

where $\mathbf{F}_p$ is the field of p elements.

These definitions, and the fundamental properties of these rings, were given in M. Lazard's monumental 1965 paper [35]. In particular, Lazard proved that G contains an open normal subgroup U, nowadays termed a *uniform* subgroup, whose Iwasawa algebra has a particularly smooth form. Thus, for U uniform, Ω_U is the $J-$adic completion of the ordinary group algebra $\mathbf{F}_pU$ by its augmentation ideal J. So Ω_U is filtered by the powers of $J\Omega_U$, and the associated graded algebra is a (commutative) polynomial $\mathbf{F}_p-$algebra. It follows by standard filtered-graded technology that Ω_U is a complete noetherian Auslander-regular scalar local domain. Similar remarks apply to Λ_U, and - thanks to the fact that Ω_G [resp. Λ_G] is a crossed product of Ω_U [resp. Λ_U] by the finite group G/U, similar conclusions can be drawn regarding Ω_G and Λ_G.

In the twenty years from 1970 Iwasawa algebras were little studied. Interest in them has been revived by developments in number theory over the past fifteen years, see for example [24]. Building on the filtered algebra and crossed product techniques outlined above, it's now known when Iwasawa algebras are prime, semiprime, domains, and when they have finite global dimension. Bounds have been found for their Krull dimension, and information obtained about their centres. Details about these results - and much else besides - can be found in the survey article [1].

The emerging picture is of a class of rings which in some ways look similar to the classical commutative Iwasawa algebras, (which are rings of formal power series in finitely many commuting variables over the $p-$adic integers), but which in other respects are very different from their commutative counterparts. And while some progress has been made in understanding these rings, many aspects of their structure and representation theory remain mysterious. A large number of open questions are discussed in [1].

Cluster algebras and cluster categories.

Cluster algebras were invented by Fomin and Zelevinsky [32, 33] in 2000 as a tool to approach Lusztig's theory of canonical bases in quantum groups and total positivity in algebraic groups. Since then, cluster algebras have become the center of a rapidly developing theory, which has turned out to be closely related to a large spectrum of other subjects, notably Lie theory, Poisson geometry, Teichmüller theory, integrable system, algebraic combinatorics and polyhedra, and quiver representations. Recent work by many authors has shown that this last link is best understood using the cluster category, which is a triangulated category associated with every Dynkin diagram. A partial list of papers are [10] by Assem, Brüstle, Schiffler and Todorov, [12, 13, 14, 15, 16, 17, 18] by Buan, Marsh, Reineke, Reiten and Todorov, [19, 20] by Caldero, Chapoton and Schiffler, [21, 22] by Caldero and Keller, [30] by Geiss, Leclerc and J. Schröer. The combinatorics of clusters is shown to be tightly related to tilting objects in cluster categories. There have been many new questions motivated by the study of cluster algebras [57] and cluster categories and it is expected that there will be more activities in this direction. Derived categories or triangulated categories have been used more and more in many areas. The recent development of the cluster category is a good example of such.

In the workshop Keller gave a talk on some recent developments and present the cluster multiplication theorem, obtained in his joint work with Caldero [21, 22], which directly links the multiplication of the cluster algebra to the triangles in the cluster category using a Hall algebra approach. Reiten gave a talk based on her recent work with Iyama about algebras of global dimension 3 where its bounded derived category of the finite length modules is Calabi-Yau of dimension 3. This derived category has connections with cluster algebras and the noncommutative crepant resolutions of Van den Bergh.

Noncommutative stacks. The noncommutative phenomena of algebraic stacks (i.e., Artin stacks and Deligne-Mumford stacks) has been observed for many years. Using some ideas from Connes' noncommutative geometry, Chan and Ingalls recently defined a noncommutative coordinate ring associated to a Deligne-Mumford stack with a finite flat scheme cover [23]. This has been extended to the case of Artin stacks by Behrend. There are many moduli problems suggesting that noncommutative algebras are the correct algebraic structure which describe the underlying geometric spaces. The noncommutative crepant resolutions of Van den Bergh [53] is a good example. Most of noncommutative algebras appearing with stacks are finite over their centres.

Summaries of selected talks

Speaker: Jacques Alev (Universit'e de Reims)

Title: Poisson trace group of certain quotient varieties.

Summary: Let V be a symplectic space of dimension $2n$, G a finite group of symplectomorphisms of V, $X = V/G$ the quotient variety, A_n the Weyl algebra of index n and A_n^G the invariant algebra which can be seen as "noncommutative functions" over X, hence as a quantization of X. A standard theme is to compare all possible algebro-geometric invariants of the (usually singular) Poisson variety X and of the algebra A_n^G: Poisson (co)homology of X, Hochschild (co)homology of A_n^G, desingularizations of X, etc. Alev presented his computation of $\dim HP_0(X)$ in certain cases and compared it to $\dim HH_0(A_n^G)$.

Speaker: Daniel Chan (University of New South Wales)

Title: Minimal resolutions of canonical orders and McKay correspondence.

Summary: Recently, the Mori program was adapted to orders over surfaces. In particular, there are noncommutative generalisations of discrepancy, canonical singularities and resolutions of singularities. Chan reviewed some of these concepts and showed how minimal resolutions of canonical orders can be written down explicitly. We also discussed McKay correspondence for these canonical orders. This talk was based on joint work with Colin Ingalls and Paul Hacking.

Speaker: William Crawley-Boevey (University of Leeds)

Title: Noncommutative Poisson structures

Summary: This talk described a notion of Poisson structures on noncommutative rings which seems to be better than the straightforward generalization of Poisson brackets to such rings. The speaker also discussed some open problems in this area.

Speaker: Victor Ginzburg (University of Chicago)

Title: Double derivations and cyclic homology.

Summary: Ginzburg described a new construction of cyclic homology of an associative algebra A that does not involve Connes' differential. His approach is based on the complex ΩA, of noncommutative differential forms on A, and is similar in spirit to the de Rham approach to equivariant cohomology. The cyclic homology is defined as the cohomology of the total complex $((\Omega A)[t], d + t \cdot i)$, arising from two anti-commuting differentials, d and ii, on ΩA of degrees $+1$ and -1, respectively. The differential d, that replaces the Connes differential B, is the Karoubi-de Rham differential. The differential i that replaces the Hochschild differential b, is a map analogous to contraction with a vector field. This new map has no commutative counterpart.

Speaker: Ken Goodearl (University of California Santa Barbara)

Title: Quantum matrices and matrix Poisson varieties.

Summary: Goodearl discussed the relations among prime and primitive ideals of the generic quantized coordinate ring $A = \mathcal{O}_q(M_n(\mathbf{C}))$, Poisson prime and Poisson primitive ideals of the classical coordinate ring $R = \mathcal{O}(M_n(\mathbf{C}))$, and symplectic leaves in the Poisson variety $M_n(\mathbf{C})$. The Poisson algebra R is the "semiclassical limit" of A, and so it is conjectured that there should be a bijection between the primitive spectrum of A and the Poisson primitive spectrum of R, hence also a bijection with the space of symplectic leaves in $M_n(\mathbf{C})$. All of these bijections should be equivariant with respect to natural actions of the torus H of pairs of invertible diagonal matrices. Consequently, the H-invariant prime ideals of A should naturally match up with the H-orbits of symplectic leaves in $M_n(\mathbf{C})$. Specifically: Each H-invariant prime of A is conjectured to be generated by a set of quantum minors, and these quantum minors should match minors defining the closure of a corresponding H-orbit of symplectic leaves in $M_n(\mathbf{C})$. In recent joint work with K.A. Brown and M. Yakimov, Goodearl determined these orbits of symplectic leaves, and described sets of minors defining their closures. These results lead to precise conjectures concerning generating sets for H-invariant prime ideals in A, which were discussed in the talk.

Speaker: Birge Huisgen-Zimmermann (University of California Santa Barbara)

Title: Top-stable degenerations of finite dimensional representations

Summary: Given a finite dimensional representation M of a finite dimensional algebra A, two hierarchies of degenerations of M are analyzed: the poset of those degenerations of M which share the top M/JM with M – here J denotes the radical of the algebra – and the sub-poset of those which share with M the full radical layering $\left(J^l M/J^{l+1}M\right)_{l\geq 0}$. In particular, the speaker addressed the existence of proper top-stable or layer-stable degenerations – more generally, the sizes of the corresponding posets including bounds on the lengths of saturated chains – as well as structure and classification. Here are two sample theorems to indicate the level of detail one can draw from the proposed geometric setting. The most transparent case is that of a squarefree top T. In this situation, two numerical invariants (with quite natural intuitive interpretations) govern the size of the poset of top-stable degenerations of M, namely:

• The difference $t - s$, where t is the number of simple summands in the top of M and s the number of indecomposable summands of M, and

• the difference $m = \dim_K Hom_A(P, JM) - \dim_K Hom_A(M, JM)$, where P is a projective cover of M.

Theorem A. Top-stable degenerations. Suppose $T = M/JM$ is a direct sum of t pairwise non-isomorphic simple A-modules and P a projective cover of T. Write M in the form $M = P/C$ with $C \subseteq JP$.

(1) The lengths of chains of proper top-stable degenerations of M are bounded above by $m + t - s$.

(2) Existence: M has a proper top-stable degeneration if and only if $m + t - s > 0$, if and only if either M fails to be a direct sum of local modules, or else C fails to be invariant under homomorphisms $P \to JP$.

(3) Unique existence: M has a unique proper top-stable degeneration if and only if M is a direct sum of local modules and $m = 1$. If $m = 0$ and $t - s = 1$, M has precisely two distinct proper top-stable degenerations. For all values $m + t - s \geq 2$, there are infinitely many top-stable degenerations in general.

(4) Bases: W.l.o.g., A is a path algebra modulo relations, and P a direct summand of A. If $M' = P/C'$ is a top-stable degeneration of M, then M and M' share a basis consisting of paths in the underlying quiver. That is, there exists a set B of paths such that $\{q + C \mid q \in B\}$ is a basis for M and $\{q + C' \mid q \in B\}$ is a basis for M'.

(5) The maximal top-stable degenerations of M always possess a fine moduli space, classifying them up to isomorphism. It is a projective variety of dimension at most $\max\{0, m + (t - s) - 1\}$.

(6) The case $m = 0$: M has only finitely many top-stable degenerations, and the degeneration order coincides with the Ext-order.

As for proper layer-stable degenerations in the case of squarefree top: If M is a direct sum of local modules, there are none. Otherwise, "huge" hierarchies of layer-stable degenerations may arise.

As a by-product, the theory provides a method for computing the top-stable degenerations from quiver and relations of A and a presentation of M. Hence, there is a rich supply of examples. Huisgen-Zimmermann displayed three examples of particular interest and described the conjectural classification in the general situation in terms of these specific instances.

The lecture ended with a sample of the theory for the more involved situation of an arbitrary top:

Theorem B. Suppose $M/JM \cong S_1^{t_1} \oplus \cdots \oplus S_n^{t_n}$, where $S_1, \ldots, S_n$ are the isomorphism types of the simple left A-modules (corresponding to a full set of primitive idempotents e_i of A).

Then M has no proper layer-stable degenerations if and only if

(a) M is a direct sum of local modules, say $M = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{t_i} M_{ij}$, where $M_{ij} = Ae_i/C_{ij}$.

(b) $\dim Hom_A(P, JM) = \dim Hom_A(M, JM)$, and

(c) For each i, the C_{ij} are linearly ordered with respect to inclusion.

Speaker: Tom Lenagan (University of Edinburgh)

Title: Prime ideals and the automorphism group of quantum matrices.

Summary: This talk was based in joint work in progress, in collaboration with Stéphane Launois. In work with Launois and Rigal, the speaker has recently shown that the algebra of quantum matrices is a UFD in the generic case (q is not a root of unity), in the sense that each height one prime is principal, generated by a normal element. The present work starts by establishing a criterion to decide when the algebra of quantum matrices is primitive. This is linked to the description of the height one primes since each height one prime is either invariant under the action of the natural torus that acts on quantum matrices, or it is in the so-called 0-stratum. The algebra of quantum matrices is primitive precisely when there is no height one prime in the 0-stratum, and, in this case, there are only a finite number of height one primes, each one invariant under the torus action. For example, the algebra of 2x3 quantum matrices is primitive. Next, the speaker considered the automorphism group of quantum matrices by studying the action of this group on the prime spectrum, and, in particular on the height one primes. The situation is much more complicated in the non-primitive case, where there are infinitely many height one primes, than in the primitive case, where there are only finitely many primes. In the nonsquare case, Lenagan described the automorphism group. In the square case the situation is not yet fully resolved, but there are partial results.

Speaker: Valery Lunts (Indiana University)

Title: Motivic measures and zeta functions.

Summary: A "motivic measure" is a ring homomorphism $K[V] \to A$ from the Grothendieck ring of varieties $K[V]$ to an arbitrary ring A. Lunts considered two interesting motivic measures. The first one is related to stable birational geometry of varieties and the second – to derived categories of coherent sheaves. He also discussed a counterexample to a conjecture of Kapranov on the rationality of motivic zeta function. This lecture was based on joint work with Michael Larsen.

Speaker: Daniel Rogalski (University of California San Diego)

Title: Birationally commutative surfaces are naive blow-ups

Summary: The aim of the work presented in this lecture is to classify a wide class of graded rings of GK-dimension 3 in terms of geometry. We say that a connected graded domain A is a birationally commutative if its graded ring of fractions looks like $K[t, t^(- 1); \sigma]$ where K is a commutative field. The main theorem states that if A is such a domain which is noetherian, generated in degree 1, and with $GKdim\ A = 3$, then A can be described as a naive blow-up of some twisted homogeneous coordinate ring of a surface. This is an analog of the commutative result that all surfaces in a given birational class are related by blowing-up. This talk was based on joint work with Toby Stafford.

Speaker: David Saltman (University of Texas at Austin)

Title: Brauer groups of function fields of surfaces.

Summary: The goal is to take a second look at the Brauer group of function fields of surfaces. One aspect is to generalize, in a way, a result proved for p-adic curves. Let $K = F(S)$ be the function field of a regular surface (not necessarily over a field but excellent and Noetherian). Let $\alpha \in Br(K)$ be a Brauer group element of order a prime q unequal to any residue characteristics. Assume K has a primitive q root of one. Then we state a geometric obstruction on the ramification locus of α to its being represented by a division algebra of degree q. Absent this obstruction, we show that there is a cyclic extension of degree q which splits all the ramification of α. In another direction, we recall and redevelop the H^3 obstruction to ramification data coming from a Brauer group element. We want further properties of this obstruction, the ultimate goal being to make it computable. Along the way, we consider the case where $S = Spec(R)$ and R is a regular local domain (of dimension 2 etc.) with henselization R^h. We also consider the function field $K = q(R)$, and the relationship between $Br(K)$ and $Br(K^h)$ for $K^h = q(R^h)$.

Speaker: Paul Smith (University of Washington)

Title: Noncommutative covers of weighted projective varieties.

Summary: Let A be a commutative graded ring generated by a finite number of elements of positive degrees and let $X_{nc} = Proj_{nc}A$ be the Artin-Zhang Proj, and $X = Proj(A)$ the usual commutative weighted projective variety. There is a map $f : X_{nc} - - > X$ in the sense of noncommutative geometry. Moreover, X_{nc} is a quotient stack with coarse moduli space X, and f ”is” the natural map of stacks. We study X_{nc} and the map f from the point of view of noncommutative geometry. Often f is a birational isomorphism, and often X can be singular while X_{nc} is smooth so functions as a sort of noncommutative resolution of X. Locally X_{nc} is covered by affine spaces that have coordinate rings that are skew group rings for finite cyclic groups over commutative rings. We describe a sheaf B of noncommutative algebras on X such that $Mod(X_{nc}) = Qcoh(B)$. The case where A is the polynomial ring on two generators of weights 4 and 6 was used to illustrate some of the ideas. This is an important example because then X_{nc} is the compactified moduli stack for pointed elliptic curves. We give an easy proof (in the spirit of noncommutative geometry) of Mumford's result that the Picard group of the uncompactified moduli stack is $Z/12$.

Speaker: Michaela Vancliff (University of Texas at Arlington)

Title: Using an Algebro-Geometric Method to Construct Clifford Quantum $\mathbf{P}^3$s with a Predetermined Finite Point Scheme.

Summary: The classification of generic quantum $\mathbf{P}^3$s (generic regular algebras of global dimension four) has been hindered by the lack of sufficiently generic examples of quantum $\mathbf{P}^3$s on which to formulate and test conjectures. Candidates for generic quantum $\mathbf{P}^3$s are regular algebras of global dimension four that have a finite point scheme and a one-dimensional line scheme, but such algebras are rare in the literature. One possibility for constructing such an algebra is to build it by deforming a regular Clifford algebra of global dimension four that has a finite point scheme.

Speaker: Nikolaus Vonessen (University of Montana)

Title: Group actions on central simple algebras

Summary: Suppose an algebraic group G acts on a central simple algebra A of degree n (and characteristic 0). The goal is to be able to answer the following questions a about the action:

(a) Is A^G a simple algebra, and if so, what is its degree? Its center?

(b) Does A have a G-invariant maximal subfield?

(c) Can the G-action on the center $Z(A)$ be extended to a splitting field L, and if so, what is the minimal possible value of $\operatorname{trdeg}_{Z(A)} L$?

It turns out that under mild assumptions on A and the action, one can obtain much information along these lines by using techniques from birational invariant theory (i.e., the study of group actions on algebraic varieties, up to equivariant birational isomorphisms). The talk illustrated the results with the example of the natural action of GL_m on the universal division algebra $UD(m, n)$ generated by m generic $n \times n$-matrices. In this case the invariants form a division subalgebra of degree n if and only if assuming $n \geq 3$ and $2 \leq m \leq n^2 - 2$. Related methods also make it possible to give an asymptotic estimate of the dimension of the space of SL_m-invariant homogeneous central polynomials $p(X_1, \ldots, X_m)$ for $n \times n$-matrices. This talk was based on joint work with Zinovy Reichstein.

Speaker: Amnon Yekutieli (Ben Gurion University)

Title: Deformation quantization in algebraic geometry.

Summary: The goal is to study deformation quantization of the structure sheaf O_X of a smooth algebraic variety X in characteristic 0. The universal deformation formula of Kontsevich gives rise to an L_∞ quasi-isomorphism between the pullbacks of the DG Lie algebras $T_{poly,X}$ and $D_{poly,X}$ to the bundle of formal coordinate systems of X. Using simplicial sections one obtains an induced twisted L_∞ quasi-isomorphism between the mixed resolutions $Mix(T_{poly,X})$ and $Mix(D_{poly,X})$. If certain cohomologies vanish (e.g. if X is D-affine) it follows that there is a canonical function from the set of gauge equivalence classes of formal Poisson structures on X to the set of gauge equivalence classes of deformation quantizations of O_X. This is the quantization map. When X is affine the quantization map is in fact bijective. This is an algebro-geometric analogue of Kontsevich's celebrated result.

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Chapter 19

Time-frequency analysis and nonstationary filtering (05w5026)

September 24–29, 2005

Organizer(s): Hans Feichtinger (University of Vienna), Karlheinz Gröchenig (Institute of Biomathematics and Biometry, Munich), Michael Lamoureux (University of Calgary), Gary Margrave (University of Calgary)

Objectives

The primary objective of this workshop was to bring together both theoretical researchers and the more applied practitioners in time-frequency analysis for a constructive exchange of ideas. There are many very advanced concepts in the recent theoretical publications in this field but most of these have had little impact to date upon applications to real world signals. The organizers invited some of the top theoreticians in time-frequency analysis to interact with mathematical physicists and engineers, particularly such as those in geophysics and communications engineering where nonstationary filtering is a fundamental tool. The workshop provided a format with time for formal presentations as well as unstructured time for interaction and collaboration.

This workshop served as a the capstone for the special semester "Modern Methods of Time-Frequency Analysis which was held at the Erwin Schrodinger Institute in Vienna during spring 2005. The ESI session brought together a wide spectrum of scientists from Europe, while the following BIRS workshop involved these top researchers with the North American contingent. The ESI special semester was organized by Feichtinger, Gröchenig, and Benedetto; two of whom are also organizers for this proposal. More information on the ESI workshop is available at: **http://www.univie.ac.at/NuHAG/ESI05/index.html**

A secondary objective is to encourage long-term collaboration between the theoreticians and the applied researchers. While the former often have a deeper understanding of the potential of time-frequency analysis, the latter have access to physical data and are in touch with practical necessities such as computational limitations.

Overview of the Field

Time-frequency analysis finds its roots in Fourier analysis, where a signal in time can be analyzed in the frequency domain as a sum of sines and cosines. Originally developed by Fourier to solve an open problem in heat flow on a plate, the techniques of Fourier analysis have had wide application in such diverse areas as partial differential equations and mathematical physics, signal processing and electrical engineering, geometry and Sturm-Liouville problems, probability theory and Brownian motion, to name just a few.

Ironically, Fourier's new ideas and techniques were radical enough that they forced a reassessment of the Calculus, and ushered in a new mathematical era of analysis that could properly deal with infinite sums, convergence issues, unusual functions that are continuous but differentiable nowhere, and other important difficulties. So while his work lead directly to the theoretical areas of Fourier and Harmonic analysis, indirectly Fourier's ideas are responsible for the development of the flourishing areas of real and complex analysis, measure theory, functional analysis, and more.

The development of Fourier analysis and exploitation of the factorization of translation-invariant linear systems (convolution integrals) by the Fourier transform has lead to a rich field with many practical applications, particularly in mathematical physics (PDEs) and engineering (signal processing, vibration analysis, and systems theory). However, there is growing recognition that the ever more complex applications absolutely necessitate the inclusion of nonstationary systems in analysis and filtering techniques. Extensions of Fouriers concepts to the nonstationary setting are numerous and include: the Gabor transform, the wavelet transform, the Wigner transform, pseudodifferential operators, Fourier integral operators, and more. While most of these extensions have origins within quantum theory, it is now true that applications abound in many other fields such as geophysics and engineering.

The essential idea in all these extensions is that rather than analyzing signals only in the time domain, or only in the frequency domain, we instead can make a joint representation of the signal in a time-frequency domain. A musical score is a excellent analogy of this analysis: an entire piece of music (signal) can be represented as a collection of notes (frequencies) that are played at particular instances of time. In this domain, it is elementary to identify particular notes, modify them, remove them, or even rearrange them. In real applications such as medical imaging or cellular communications, these time-frequency components may be identified as noise to be removed, features to be identified and enhanced, or encodings of complex data messages to be transmitted and received. Modifying the signal is a filtering operations; since the effects of the filter changes with time and frequency, this is called a nonstationary filter.

Essential in all applications is choosing a suitable time-frequency representation of a signal, whether that be through a short time Fourier transform, a Gabor transform, a wavelet transform, or via a pseudodifferential operator. Then choosing an appropriate operator to modify the signal, be that a Gabor multiplier, a pseudodifferential operator, or some other form of time varying linear operator. Questions of achievability, stability, and computational speed are all critical issues.

Theoretical work includes identifying appropriate function spaces that represent signals well in the time-frequency domain (the modulation spaces), identifying mapping properties of operators on these spaces, questions about choices of bases and frames for such spaces, and many of the analogous results that are common in general Banach space theory and its applications.

Recent Developments and Open Problems

This is an exciting moment in time-frequency analysis as the theory is evolving rapidly while new applications are also constantly emerging. Similar to the trend from linear to nonlinear problems, the move from stationary to nonstationary leads to a richer solution set but at the expense of greater mathematical and computational complexity. Stationary filtering has been an important signal processing tool in industry for many years but today we have an emerging understanding of nonstationary filtering that promises to have a immense impact on signal processing as well as the associated modelling of the real world. The rapid increase of available computing power makes the implementation of complex nonstationary filters possible today where they were only concepts a short while ago.

Examples of recent concrete applications in nonstationary filter theory include the development of Gabor deconvolution and Gabor wavefield extrapolation for seismic imaging, nonstationary filtering in cell phone networks, nonstationary noise reduction, modelling of spatially variable quantum systems, coherent state techniques, and filtering and analysis in commercial music production. In addition, any physical system that can be modeled as a variable coefficient partial differential equation can be re-expressed as an equivalent nonstationary filter problem.

Many of the open problems are deep questions in the analysis of functions, including such things as optimal choices of base functions for frames, linear independence of time-frequency translates of base functions, properties of modulation spaces and linear operators on these spaces, and the representation of linear operators or nonstationary filters via pseudodifferential operators and Gabor multipliers.

Presentation Highlights

The talks concentrated on four or five general areas, both theoretical and applied. Research questions about frames, whether in the Gabor, wavelet, shearlet or other exotic domains, were addressed by Bodmann, Fournasier, Heil, Jorgensen, Kutyniok, Larson, and Torresani. The Gabor transform, and representing nonstationary filters through Gabor multipliers and pseudodifferential operators, were addressed by Ali, Balazs, Feichtinger, Lamoureux, Okoudjou, and Strohmer. Localization operators were addressed by Groëchnig, Oliaro and Toft. Applications considered included seismic and medical imaging, signal processing, deconvolution, and psychoacoustics; talks on applications were given by Balazs, Casazza, de Hoop, Fishman, Gibson, Hermann, Hlawatsch, Klauder, Margrave, Mitchell, Pfander, Sacchi, Shen and Stolk.

Details of individual talks are given below.

The Talks

SPEAKER: Syed T. Ali
TITLE: A Suggestion for a Vectorial Gabor Transform
ABSTRACT: Using some recent results on coherent states over matrix and C*-algebraic domains, a possible candidate for a vectorial Gabor transform will be presented. Such a transform is expected to have applications to signals with additional (internal) degrees of freedom. Some interesting holomorphic properties of such transforms will be discussed.

SPEAKER: Peter Balazs
TITLE: Gabor Multipliers with Application to Psychoacoustics
LINK-Preprint: http://www.kfs.oeaw.ac.at/xxl/dissertation/dissertation.pdf
ABSTRACT: In this talk the basic ideas of the PhD thesis 'Regular and Irregular Gabor Multipliers with Application To Psychoacoustic Masking' will be presented. The concept of frame multiplier will be introduced. Frame multipliers are a generalization of Gabor multipliers to frames without further structure. Basic results, like the dependency of the operator on the symbol, are proved. Furthermore irregular Gabor frames are investigated. In particular some results on irregular Gabor multipliers are proved like the continuous dependency of Gabor multipliers on the symbol, the lattice and the windows. Finally a concept is presented how to implement a masking filter, which approximates the simultaneous and temporal masking known in psychoacoustics. As the linear frequency scale (in Hz) is not very well adapted to human perception, another is chosen (Bark), this filtering can be seen as an irregular Gabor multiplier with adaptable mask.

SPEAKER: Bernhard Bodmann
TITLE: Frame paths and error bounds for sigma-delta quantization
ABSTRACT: We study the performance of finite frames for the encoding of vectors by applying first-order sigma-delta quantization to the frame coefficients. Our discussion is restricted to uniform tight frames, given by N vectors in a d-dimensional Hilbert space, and mostly concerns the use of quantizers that assume only integer multiples of a step-size δ (mid-tread). We prove upper and lower bounds for the maximal reconstruction error in terms of geometric quantities of a path interpolating the sequence of frame vectors. We calculate these bounds for various known frame families and introduce the so-called d-circles and semicircles frames. The latter give a slight improvement in the upper bound over the harmonic frames. The maximal error for all of these families is asymptotically of the order $\delta d^{3/2}/N$, with numerical constants that are comparable to that of coordinatewise application of the sigma-delta algorithm.

SPEAKER: Pete Casazza
TITLE: Pure Mathematics, Applied Mathematics and Engineering: A common thread
LINK-Preprint: http://www.math.missouri.edu/˜pete/

ABSTRACT: We will see that the famous 1959 Kadison-Singer Problem is equivalent to fundamental unsolved problems in a dozen areas of research in both mathematics and engineering, including problems in signal reconstruction, internet coding, reconstruction from sparse representations and much more. This gives all these areas of research common ground on which to interact and helps to explain why each area has volumes of literature on their respective problems without a satisfactory resolution.

SPEAKER: Maarten de Hoop
TITLE: Analysis of 'wave-equation' imaging of reflection seismic data with curvelets
ABSTRACT: In this presentation we discuss how techniques from dyadic parabolic decomposition of phase space and curvelet frames can be exploited in representing and analyzing the process of 'wave-equation' seismic imaging. The approach aids in the fundamental understanding of the notion of scale in the data and how it is coupled through imaging to scale in - and regularity of - the medium. Furthermore, the use of curvelets admits a rigorous treatment of the concept of controlled illumination.

SPEAKER: Hans Feichtinger
TITLE: What do we know about Gabor multipliers?

SPEAKER: Lou Fishman
TITLE: Phase Space and Path Integral Methods in Seismic Wave Propagation Modeling and Imaging
ABSTRACT: Seismic wave propagation modeling and imaging are complicated by the large-scale and rapidly-varying environments encountered in the earth. Traditionally, these classical problems have been addressed by

1. direct approximations on the wave field (e.g., asymptotic ray theory, Gaussian beams, spectral representations),
2. the application of approximate wave equations (e.g., formal one-way wave equations), and
3. the direct application of computational PD methods (e.g., finite differences, finite elements, spectral methods).

This talk will survey the application to seismic wave propagation modeling and imaging of what is loosely referred to as "phase space and path integral methods." These methods were originally developed in the quantum physics and theoretical Pd communities, and include the Feynman path integral constructions for the Sc hrdingier equation, and the theories of pseudodifferential and Fourier integral operators, for example. For fixed-frequency modeling, the primary aims of this approach are

1. to incorporate well-posed, one-way methods into the inherently two-way global formulations,
2. to exploit the correspondences between the classical wave propagation problem, quantum physics, and modern mathematical asymptotes (micro local analysis), and
3. to effectively extend Fourier-analysis-based constructions to inhomogeneous environments.

It will be seen that the explicit, exact, well-posed, one-way reformulation of "elliptic wave propagation" problems (e.g., the scalar Helmholtz equation) in phase space provides an explicit mathematical framework for wave-equation-based seismic migration, both unifying the diverse approximations (e.g., wide-angle parabolic modeling, generalized phase screens, generalized phase shift plus interpolation (GOSSIP)), and systematically extending the physically based GPSPI algorithm. These developments result in improved seismic imaging algorithms.

SPEAKER: Massimo Fornasier
TITLE: Frame adaptive methods for signal processing and operator equations
ABSTRACT: We illustrate several adaptive algorithms for the solutions of bi-infinite singular linear systems. Such algorithms are realized from exact iterative schemes (e.g., Richardson, steepest-descent methods, matching pursuit) by finite dimensional approximations of each iteration, performed with a greedy approach.

We show that these algorithms are convergent and optimal with respect to certain sparseness classes of vectors as soon as the system matrix has sufficient off-diagonal decay. Bi-infinite linear systems of this type typically arise in the solution of functional operator equations (e.g., integral and differential equations) by discretization with respect to frames, i.e., stable, complete, and redundant expansions. We present applications in signal processing/transmission and PDE.

SPEAKER: Peter Gibson
TITLE: Gabor deconvolution of one-dimensional seismic data
ABSTRACT: The last several years have seen a new technique for deconvolution based on the Gabor transform incorporated into industrial seismic image processing, as a replacement for so-called Wiener deconvolution coupled with certain corrections. The Gabor methods are nonstationary, and are thus much better suited to the extraction of reflectivity, of which the data is a nonstationary combination. The real nonstationarity stems from the relationship of the reflectivity to the Green's function of the standard model for a layered earth. In this sense Gabor deconvolution can be viewed as a technique for solving a nonlinear inverse problem while simultaneously removing the effects of a non-Dirac source signal. In this talk we will describe in detail the theory and implementation of Gabor deconvolution as it is applied to actual seismic data.

SPEAKER: Karlheinz Gr ochenig
TITLE: Mapping properties of localization operators
LINK-Author: http://ibb.gsf.de/homepage/karlheinz.groechenig/
ABSTRACT: We will discuss the mapping properties of localization operators, which are a version of nonstationary filters. Planned topics:

1. Boundedness of localization operators on modulation spaces,
2. What happens when the symbols are rough?
3. Composition of localization operators
4. The range of a localization operator.

SPEAKER: Christopher Heil
TITLE: The Homogeneous Approximation Property for Wavelet Frames
LINK-Author: www.math.gatech.edu/˜heil
ABSTRACT: The Homogeneous Approximation Property is a key property of Gabor systems which leads to necessary conditions for Gabor frames in terms of the Beurling density of the associated sequence of time-frequency shifts of the generator. We show that, with some restrictions, wavelet frames and wavelet Schauder bases also satisfy an analogue of the Homogeneous Approximation Property with respect to the affine group, and that this leads to necessary conditions for existence in terms of an affine Beurling density. However, in stark contrast to the Gabor case, we show that the density depends on the generator, and there is no Nyquist density. This is joint work with Gitta Kutyniok.

SPEAKER: Felix Herrmann
TITLE: Non-linear seismic data regularization and separation with directional curvelet frames
LINK-Preprint: zoozoo.eos.ubc.ca/ felix/BIRS_prep.pdf
ABSTRACT: In this paper, directional frames known as curvelets are applied to solve two important tasks in seismic data processing namely data interpolation and primary-multiple separation. We show that by extending the Fast Discrete Curvelet Transform (from CurveLab at www.curvelet.org) to include non-uniform Fourier Transforms (from NFFT www.math.mu-luebeck.de/potts/nfft/) a new directional frame is defined which is particular suitable to solve non-parametric seismic data interpolation problems. We show that minimizing the ℓ^1-norm as part of inverting the frame synthesis operator gives the sparsest set of curvelet coefficients that explain the unstructured data. Hitting this set with the regularly sampled synthesis operator gives the interpolated result. This approach is a practical application of recent ideas on robust uncertainty principles.

The second topic involves using curvelets to separate two signal components – the primaries and multiples with the multiples constituting those events that include a bounce at the surface. The aim is to separate the multiples from the primaries in the presence of noise and given an inaccurate prediction for the multiples. The main distinction of this signal separation problem is that the two signal components are sparse in the same frame as opposed to the signal components in Morphological Component Analysis. We show that we arrive at an viable alternative to match filtering approaches by formulating this signal-separation problem in terms of a weighted ℓ^1 optimization problem with the weights defined by the predicted multiples.

SPEAKER: Franz Hlawatsch
TITLE Linear Methods for Time-Frequency Filtering (joint work with Gerald Matz)
ABSTRACT: Time-frequency (TF) filters are linear time-varying (LTV) systems whose filter characteristics (gain/attenuation, pass/stop) are specified in the TF domain. Such a TF filter specification is convenient and intuitive in many applications. In this talk, we discuss various linear TF filter methods that can be grouped into the following two broad classes.

- Explicit filter design: The LTV filter is calculated such that a TF representation ("symbol") of the filter is equal to, or optimally approximates, a given TF weight function. A variation of this principle using an orthogonal projector side constraint results in "time-frequency projection filters" with very sharp time-frequency pass/stop selectivity.
- Implicit filter design: The LTV filter is implemented as an analysis-weighting-synthesis scheme based on a linear TF representation (an example of such a TF filter is provided by a Gabor multiplier). Thus, the filter is designed implicitly during the filtering process.

We also explain the connections of TF filters with the theory of underspread operators and TF transfer functions. The performance and application of the TF filters presented is demonstrated through numerical simulation.

SPEAKER: Palle Jorgensen
TITLE: Computation of wavelet coefficients in generalized multiresolution systems
LINK-Preprint: http://arxiv.org/abs/math.CA/0405301
ABSTRACT: We consider wavelets in $L^2(R^d)$ which have generalized multiresolutions. This means that the initial resolution subspace V_0 in $L^2(R^d)$ is not singly generated. As a result, the representation of the integer lattice Z^d restricted to V_0 has a nontrivial multiplicity function. We show how the corresponding analysis and synthesis for these wavelets can be understood in terms of unitary-matrix-valued functions on a torus acting on a certain vector bundle. Specifically, we show how the wavelet functions on R^d can be constructed directly from the generalized wavelet filters.

SPEAKER: John Klauder
TITLE: Signal Transmission in Passive Media
ABSTRACT: Under rather general assumptions, and in a relatively simple and straightforward manner, it is shown that the characteristics of signals which travel through homogeneous, as well as inhomogeneous, passive media have the principal features usually associated with the phenomena of precursors, as generally follows from more elaborate studies. The simplicity of the present arguments permit analytic studies to be made for a greater variety of media than is normally the case.

SPEAKER: Gitta Kutyniok
TITLE: Shearlets: Sparse Directional Representations of Images within a Multiresolution Analysis Structure
LINK-Preprint: http://www.math.uni-giessen.de/Numerik/gittak/publications.html
ABSTRACT: In this talk we describe a new class of multidimensional representation systems, called shearlets. They are obtained by applying the actions of dilation, shear transformation and translation to a fixed function, and exhibit the geometric and mathematical properties, e.g., directionality, elongated shapes, scales, oscillations, recently advocated by many authors for sparse image processing applications. In contrast to other approaches these systems can be studied within the framework of a generalized multiresolution analy-

sis, which leads to a recursive algorithm for the implementation of these systems, that generalizes the classical cascade algorithm. This is joint work with Demetrio Labate.

SPEAKER: Michael Lamoureux
TITLE: The Rotation Algebra in Time-Frequency Analysis
ABSTRACT: The translation and modulation operators that appear in the Gabor transform generate a representation of a well-studied family of operators on Hilbert space, known as the rotation algebras. These algerbas arise naturally in physics in the study of Bloch electrons, and mathematically are noncommutative generalizations of a two torus. We will present some of the basic properties of this field of algebras and their connection with Gabor theory.

SPEAKER: David Larson
TITLE: Frames and Operator Theory
ABSTRACT: A few years ago the speaker and his collaborators developed an operator-interpolation approach to wavelets and frames using the local commutant (i.e. commutant at a point) of a unitary system. This is really an abstract application of the theory of operator algebras to wavelet and frame theory. The concrete applications of operator-interpolation to wavelet theory include results obtained using specially constructed families of wavelet sets. Our methods include the construction of certain groups of measure preserving transformations, and groups and algebras of operators, with special algebraic properties. Other results include applications of a theory of projection decompositions of positive operators, and a theory of operator-valued frames. We will discuss some unpublished and partially published results, and some brand new results, that are due to this speaker and his former and current students, and other collaborators.

Note this talk was cancelled due to travel delays.

SPEAKER: Gary Margrave
TITLE: A stable, explicit nonstationary filter for wavefield extrapolation
ABSTRACT: We present a new approach to the design of stable and accurate wavefield extrapolation operators needed for explicit depth migration. We split the theoretical operator into two component operators, one a forward operator that controls the phase accuracy and the other an inverse operator, designed as a Wiener filter that stabilizes the first operator. Both component operators are designed to have a specific fixed length and the final operator is formed as the convolution of the components. We utilize this operator design method to build an explicit, wavefield extrapolation method based on the migration of individual source records. Two other features of our method are the use of dual operator tables, with high and low levels of evanescent filtering, and frequency-dependent spatial down sampling. Both of these features improve the accuracy and efficiency of the overall method. Empirical testing shows that our method has a similar performance to the time-migration method called phase shift, meaning it scales as NlogN. We illustrate the method with tests on the Marmousi synthetic dataset. We call our method FOCI which is an acronym for forward operator conjugate inverse.

SPEAKER: Ross Mitchell
TITLE: Time/Frequency Applications in Medical Imaging
ABSTRACT: Medical imaging research at the Hotchkiss Brain Institute, University of Calgary, is focused on the application of mathematics, computer science, physics and engineering to help understand, diagnose, treat and monitor neurological disease. Several multi-disciplinary research teams, consisting of both basic scientists and clinicians, have been deployed within Foothills Medical Center. This seminar will provide an overview of the Fourier-based medical imaging modalities of computerized tomography and magnetic resonance imaging. It will then cover several neurological applications of time/frequency analysis. In particular, our team is using time/frequency techniques to investigate signals and images from patients suffering from stroke, brain cancer, multiple sclerosis, and epilepsy. We believe that time/frequency techniques have tremendous potential to advance the science of medical imaging, and improve outcomes for patients.

Note: this presentation will be targeted towards a non-medical audience. Nevertheless, it may contain some graphic images.

SPEAKER: Kasso Okoudjou
TITLE: On some Fourier multipliers for modulation spaces
LINK-Author: http://www.math.cornell.edu/~kasso
ABSTRACT: In this talk, I will use some time-frequency analysis techniques to study the continuity propeprties of a class of Fourier multipliers on the modulation spaces. It must be pointed out that, in general, these Fourier multipliers are known to be unbounded on Lebesgue spaces.

SPEAKER: Alessandro Oliaro
TITLE: Continuity of localization operators in L^p spaces
ABSTRACT: We study some properties of two-wavelet localization operators, i.e., operators which depend on a symbol and two different windows. In the case when the symbol F belongs to $L^p(\mathbb{R}^{2n})$, we give results on $L^q(\mathbb{R}^n)$ boundedness, non-boundedness and compactness of the corresponding operator.

SPEAKER: Goetz Pfander
TITLE: Sampling of operators and channel measurements

SPEAKER: Mauricio Sacchi
TITLE: On the Regularization of the Local Radon Transform - Applications to Seismic Coherent Noise Atenuation

SPEAKER: Zuowei Shen
TITLE: Deconvolution: A wavelet frame approach.

SPEAKER: Chris Stolk
TITLE: Combining finite elements and geometric wave propagation in 1-D
ABSTRACT: We consider the initial value problem for a strictly hyperbolic partial differential equation on the circle. We transform the equation to an operator valued ODE $du/dt = R(t)u$, where $R(t)$ is bounded. The transformation involves application of differential operators, solving an elliptic differential equation, and applying a coordinate transformation involving the characteristics, which can be done at cost $O(N)$. The resulting ODE is solved using a multiscale time-stepping method, which results in an $O(N)$ algorithm for the original hyperbolic equation.

SPEAKER: Thomas Strohmer
TITLE: Capacity of time-varying channels and pseudodifferential operators

SPEAKER: Joachim Toft
TITLE: Schatten properties for pseudo-differential operators and localization operators on modulation spaces of Hilbert type
ABSTRACT: Schatten-von Neumann (SN) classes are spaces of linear and continuous operators between Hilbert spaces. The largest SN class consists of of continuous operators, and all other SN classes are subsets of compact operators, where in particular the smallest SN-class is the set of trace-class operators. Consequently, by using such classes, it is possible to make a detailed study on compactness. In general it is hard to decide if an explicit operator belongs to a certain SN class or not. One is therefore forced to search embedding properties between SN classes and well-known spaces. In the past, such embeddings have been established between SN classes in context of pseudo-differential operators (psdo) acting on L^2, and Besov, Sobolev or modulation spaces. In the talk we present a non-trivial generalization of embedding between SN classes in psdo and modulation spaces, where the L^2 here above is replaced by general modulation spaces of Hilbert type. This generalization is obtained by a combination of careful use of time-frequence methods and Hilbert space techniques.

SPEAKER: Bruno Torresani
TITLE: Identifying sparse hybrid time-frequency models

ABSTRACT: Several signal families may be adequately modeled as sparse expansions with respect to unions of time-frequency bases or frames. We shall focus on probabilistic models involving several layers of randomness (sparse subset of the dictionary, coefficients of the expansion,...) and the corresponding estimation algorithms . A couple of two-steps estimation procedures will be studied and compared. Thesretical estimates as well as numericalm results will be presented.

Outcome of the Meeting

The best outcome of this meeting was to get the theoreticians and the applied researchers talking together. Many of the theoretical people have not been aware of the details nor the great extent to which applied researchers have been making use of time-frequency ideas in concrete applications. In fact, researchers have created commercial software and hardware in imaging (medical, seismic, etc) and telecommunications (cellphones) that take advantage of these techniques, and have developed a whole vocabulary that is distinct from the theoretical work. The applied researchers have been similarly unaware of details of extensive theoretical work that has been done on the mathematics of time-frequency analysis which will directly benefit the applications. In particular in optimal choices of frames, deconvolution work in wavelet bases, properties of localization operations which are particularly suited for rapid numerical computations, design and implementation of Gabor multipliers and other time-frequency filters all depend on good theoretical work.

It has been particularly useful to bring together the European and the North American researchers. The Vienna school is outstanding in its theoretical work and is developing a strong applied connections. The organizers at Calgary are very pleased to be learning about this theoretical work and begin applying it to their large project in the mathematics of seismic imaging. We expect further collaborations to develop from these new connections.

List of Participants

Ali, S. Twareque (Concordia University and Institut des Sciences Mathématiques)
Balazs, Peter (Austrian Academy of Science)
Benedetto, John (University of Maryland)
Bodmann, Bernhard (University of Waterloo)
Casazza, Peter (University of Missouri)
de Hoop, Maarten (Purdue University)
Drabycz, Sylvia (University of Calgary)
Feichtinger, Hans (University of Vienna)
Fishman, Lou (University of Calgary/MDF International)
Fornasier, Massimo (University of Vienna and University of Rome La Sapienza)
Gibson, Peter (York University)
Groechenig, Karlheinz (Institute of Biomathematics and Biometry)
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Chapter 20

Challenges in Linear and Polynomial Algebra in Symbolic Computation Software (05w5039)

October 1, 2005 – October 6, 2005

Organizer(s): Wolfram Decker (University of the Saarland), Keith Geddes (University of Waterloo), Erich Kaltofen (North Carolina State University), Stephen M. Watt (University of Western Ontario)

Summary

Overview of area covered

The subject of the workshop was innovation in algorithms and software addressing key bottlenecks in symbolic mathematical computation software. By symbolic mathematical computation software we mean software like Maple (represented by several participants including Jürgen Gerhard from Maplesoft), Mathematica, Macaulay 2 (represented by Michael Stillman), Magma (represented by Allan Steele), MuPAD, NTL, SINGULAR (represented by Gert-Martin Greuel) etc., whose purpose it is to aid a mathematician, scientist, engineer, or educator to solve mathematical problems on a computer. The specific area of focus for this workshop was challenges arising from linear and polynomial algebra at the core of these systems.

Symbolic computation software implements many sophisticated algorithms on polynomials, matrices, combinatorial structures and other mathematical objects in a multitude of different dense, sparse, or implicit (black box) representations. Several of the algorithms are well-known: Buchberger's Gröbner basis reduction algorithm in all its flavors, lattice bases reduction algorithms (LLL, PSLQ) [addressed by M. van Hoeij's presentation], Wiedemann's sparse linear system solver for scalars from a finite field [addressed by P. Giorgi's and J.-G. Dumas's presentations], polynomial factorization algorithms [addressed by M. van Hoeij's presentation], algorithms for solving in closed form differential and difference equations [addressed by E. Hubert's presentation], sparse interpolation algorithms [addressed by W.-s. Lee's presentation], and many more. These algorithms form the backbone of any symbolic computation software, and their improvement is the continuous effort of researchers.

In addition, several categories of algorithms for new basic problems are the subject of vigorous current investigation: diophantine linear system solution, algorithms for approximate data, e.g., floating point scalars, such as approximate polynomial greatest common divisors [addressed by L. Zhi's and H. Kai's presentations], factorization and non-linear system solving via homotopical deformation [addressed by A. Sommese's presentation], manipulation of polynomials over non-commutative domains, and more.

We estimate that the company-based systems Maple and Mathematica together are licensed to over five million users. We note that the Research & Development divisions in these companies are quite small. One objective subject of the workshop was how academia and industry can provide the users an ever-increasing speedup in the known algorithmic solutions on platforms designed with modern computer science principles. This entails the discovery of completely new algorithms, such as the ones in the new problem categories mentioned above, the change of existent algorithms for efficient computer implementation [addressed by A. Steele's presentation], and the computer science of meshing the individually implemented algorithms into a large symbolic computing environment [see the section on the two software discussions].

Overview of achieved objectives

Our workshop brought together algorithm designers and symbolic computation software builders from industry and academia. Our first objective was to review the status of the problems in the core area whose solution has the greatest impact in systems for symbolic mathematical computation. Our second objective was to design an approach that can achieve fast transfer of new mathematical algorithmic advances and new computer science concepts into the available software. We invited for discussion those who make the new mathematics for the discipline and those who make the computer programs, in particular those who are engaged in both activities.

The software builder is faced with a mammoth task: the involved mathematical analysis in current algorithms can be highly sophisticated, using deep mathematical ideas. We give as an example the computation of sparse resultant formulas via exterior algebras and Chow forms or F.-O. Schreyer's presentation.

The underlying system for programming these algorithms is highly complex, combining techniques from reusable object-oriented design with entirely original data structures and standards. For example, the LinBox group, which is developing a symbolic linear algebra library in analogy to numerical libraries such as LinPack and MatLab, had to revise the basic generic archetype for a black box matrix three times, thus requiring a re-programming of the entire library. The revisions were necessitated when new concepts such as non-native garbage collection and BLAS (basic linear algebra subprograms) were introduced. J.-G. Dumas presentation addressed several of those issues. In general, our experience is that efficient delivery of effective symbolic computation software requires ongoing and often original computer science research.

Clearly, given the proliferation of algorithmic ideas and the complexity of a modern computer environment, innovative design principles and linkages are required to bring the new breakthroughs quickly into the software that the users, including our own community, need.

This workshop provided a forum for focused discussion among the experts in industry and academia, and among algorithm designers and algorithm implementors. The goal was to understand a framework which will foster the evolution of new algorithmic ideas into usable software in a timely fashion. The pressures on being able to faster compute more are great. In some cases, the difference can be the proof or disproval of a mathematical conjecture [addressed in part in D. Lazard's talk on the Solotareff problem]. In others, the yield can be a better FFT (fast Fourier transform) algorithm.

Titles and abstracts of presentations

SCHEDULE

	Sunday	**Monday**	**Tuesday**	**Wednesday**	**Thursday**
Session chairs	M. Dewar	A. Storjohann	C. Brown	M. Stillman	
9:00-9:45	J. Demmel	P. Giorgi	L. Zhi	E. Hubert‡	SW disc.† II
9:45-10:30	E. Schost	F. Rouillier	H. Kai	F.-O. Schreyer	
11:00-11:45	M. van Hoeij	J.-G. Dumas	W.-s. Lee	G.-M. Greuel	
Session chairs	T. Lange	F. Winkler		J. Gerhard	
14:30-15:15	von zur Gathen	D. Lazard	Hike at Lake Luise/	A. Steel	
15:45-16:30	A. Sommese	SW disc.† I	Moraine Lake	M. Monagan	

†Software group discussion

‡Hubert's talk was recorded

Speaker: **James Demmel** (University of California at Berkeley)
Title: *Toward accurate polynomial evaluation in rounded arithmetic*
Abstract: Given a multivariate real (or complex) polynomial p and a domain $\mathcal{D}$, we would like to decide whether an algorithm exists to evaluate $p(x)$ accurately for all $x \in \mathcal{D}$ using rounded real (or complex) arithmetic. Here "accurately" means with relative error less than 1, i.e., with some correct leading digits. The answer depends on the model of rounded arithmetic: We assume that for any arithmetic operator $op(a, b)$, for example $a + b$ or $a \cdot b$, its computed value is $op(a, b) \cdot (1 + \delta)$, where $|\delta|$ is bounded by some constant ϵ where $0 < \epsilon \ll 1$, but δ is otherwise arbitrary. This model is the traditional one used to analyze the accuracy of floating point algorithms. Our ultimate goal is to establish a decision procedure that, for any p and $\mathcal{D}$, either exhibits an accurate algorithm or proves that none exists. In contrast to the case where numbers are stored and manipulated as finite bit strings (e.g., as floating point numbers or rational numbers) we show that some polynomials p are impossible to evaluate accurately. The existence of an accurate algorithm will depend not just on p and $\mathcal{D}$, but on which arithmetic operators and which constants are are available and whether branching is permitted. Toward this goal, we present necessary conditions on p for it to be accurately evaluable on open real or complex domains $\mathcal{D}$. We also give sufficient conditions, and describe progress toward a complete decision procedure. We do present a complete decision procedure for homogeneous polynomials p with integer coefficients, $\mathcal{D} = \mathbb{C}^n$, and using only the arithmetic operations $+$, $-$ and $\cdot$. Reference: [1].

Speaker: **Jean-Guillaume Dumas** (Université de Grenoble, France)
Title: *LinBox-1.0*
Abstract: Three major threads have come together to form the linear algebra library LinBox. The first is the use of modular algorithms when solving integer or rational matrix problems. The second thread and original motive for LinBox is the implementation of black box algorithms for sparse/structured matrices. Finally, it has proven valuable to introduce elimination techniques that exploit the floating point BLAS libraries even when our domains are finite fields. The latter is useful for dense problems and for block iterative methods. Black box techniques are enabling exact linear algebra computations of a scale well beyond anything previously possible. The development of new and interesting algorithms has proceeded apace for the past two decades. It is time for the dissemination of these algorithms in an easily used software library so that the mathematical community may readily take advantage of their power. LinBox is that library. In this talk, we sketch the current range of capabilities, describe the design and give several examples of use. Reference:

http://www.linalg.org

Speaker: **Joachim von zur Gathen** (B-IT, University of Bonn, Germany)
Title: *High-performance computer algebra*
Abstract: There are two scenarios for putting the asymptotically fast algorithms of computer algebra to work: in software and in hardware. The first is exemplified by polynomial arithmetic, in particular factorization, on sequential and parallel machines. The size of cutting edge problems is measured in megabits. The second one deals with a few hundred bits and yields fast cryptographic coprocessors at the size of current key lengths. Reference: [4].

Speaker: **Pascal Giorgi** (University of Waterloo)
Title: *Integer Linear System Solving*
Abstract: Recent implementations of algorithms for integer linear system solving can compute solutions of systems with around $2,000$ equations over word size numbers in about a minute. These performances are achieved for dense matrices using the highly optimized BLAS library. Currently we are exploiting the same approach to provide practical implementations for large sparse systems. In our talk we describe our prototype implementation of an experimental algorithm for sparse solving which reduces much of the computation to level 2 and 3 BLAS and seems to improve the bit complexity from n^3 to $n^{2.5}$. Reference: [3].

Speaker: **Gert-Martin Greuel** (University of Kaiserslautern Germany)
Title: *Computing equisingularity strata of plane curve sigularities*
Abstract: Equisingular families of plane curve singularities, starting from Zariski's pioneering 'Studies in Equisingularity I–III' have been of constant interest ever since. The theory was basically topologically motivated and so far it was only considered in characteristic 0. We develop a new theory for equisingularity in any

characteristic which gives even new insight in characteristic 0. Moreover, it is algorithmic and the algorithms for computing equisingularity strata have been implemented in Singular.

Speaker: **Mark van Hoeij** (Florida State University)
Title: *Complexity results for factoring univariate polynomials over the rationals and bivariate polynomials over finite fields*
Abstract: In this talk, a polynomial time complexity bound is given for the algorithm in “Factoring polynomials and the knapsack problem” [6]. A complexity result is also given for factoring bivariate polynomials over finite fields. Specifically, to solve the combinatorial problem, it suffices to Hensel lift to accuracy $min(p, n) \cdot (n - 1) + 1$ where p is the characteristic of the finite field and n is the total degree.

Speaker: **Evelyne Hubert** (INRIA Sophia Antipolis)
Title: *Rational and Replacement Invariants of a Group Action*
Abstract: Group actions are ubiquitous in mathematics. They arise in diverse areas of applications, from classical mechanics to computer vision. A classical but central problem is to compute a generating set of invariants. The proposed presentation is based on a joint article with I. Kogan, North Carolina State Universtity, and is part of a bigger project for differential systems invariant under a Lie group that was started with E. Mansfield, University of Kent at Canterburry.

We consider a rational group action on the affine space and propose a construction of a finite set of rational invariants and a simple algorithm to rewrite any rational invariant in terms of those generators.

The rewriting of any rational invariant in terms of the computed generating set becomes a trivial replacement. For the general case we introduce a finite set of replacement invariants that are algebraic functions of the rational invariants. They are the algebraic analogues of the normalized invariants in Cartan’s moving frame construction. The construction generalizes to the computation of a fundamental set of differential invariants.

Speaker: **Hiroshi Kai** (Ehime University)
Title: *Reliable rational interpolation by symbolic-numeric computation*
Abstract: A rational interpolation is computed by simultaneous linear equations numerically. But, if the linear equations are solved by fixed precision floating point arithmetic, there appear a pathological feature such as undesired pole and zero. An algorithm is presented to eliminate the feature and then give a reliable rational interpolation with the help from stabilization theory and computer assisted proof. Reference: [7].

Speaker: **Daniel Lazard** (INRIA France)
Title: *New challenges in polynomial computation and real algebraic geometry: Example of Solotareff approximation problem*
Abstract: Most of the computations related to polynomial equations and inegalities are done either by numeric computation, either by using Gröbner bases, Collin’s cylindrical decomposition or triangular systems. With the progress of all these methods, the main algorithmic challenge becomes to select well specified classes of problems which may be solved by using appropriately several of these methods.

Examples of such an approach may be found in global optimization or parametric systems (see Rouillier’s talk).

We illustrate this with Solotareff approximation problem (Kaltofen’s challenge 2) for which CAD fails in degree 6, while a complete solution in degrees up to 10 may be obtained by mixing theoretical considerations on quantifier elimination and with well choosen operations of localization and projection done through Gröbner bases. Reference [10].

Speaker: **Wen-shin Lee** (University of Antwerp, Belgium)
Title: *Sparse Polynomial Interpolation and Representation*
Abstract: As polynomials are one of the fundamental objects in symbolic computation, being able to represent and manipulate them efficiently can have dramatic effects on the cost of many algorithms.

This talk focuses on sparse polynomials. I discuss black box sparse interpolation and explore sparse representations of polynomials. The interplay between these problems and recent development [5] are also addressed.

Speaker: **Michael Monagan** (Simon Fraser University)
The talk was on sparse rational interpolation.

Speaker: **Fabrice Rouillier** (INRIA France)
The talk was on parametric system solving.

Speaker: **Éric Schost** (Ecole Polytechnique France)
Title: *Point counting in genus 2, and some of the problems it raises*
Abstract: Computing the number of points in the Jacobian of a hyperelliptic curve is a basic question for hyperelliptic cryptosystem design. For curves of genus 2 over prime fields, present solutions rely on a variety of tasks: polynomial system solving, root finding, computation with algebraic numbers, ...

This talk (given from a computer algebraist point-of-view) aims at describing problems met when trying to reach ”cryptographic size”, some solutions, and how they meet, or can motivate, research in symbolic computation. This is joint work with Pierrick Gaudry.

Speaker: **Frank-Olaf Schreyer** (Universität des Saarlandes, Germany)
Title: *Computing the higher direct image complex of coherent sheaves*
Abstract: The higher direct image complex of a coherent sheaf (or finite complex of coherent sheaves) under a projective morphism is a fundamental construction that can be defined via a Cech complex or an injective resolution, both inherently infinite constructions. Using exterior algebras and relative versions of theorems of Beilinson and Bernstein-Gel'fand-Gel'fand, we give an alternate description in finite terms.

Using this description we can give explicit descriptions of the loci in the base spaces of flat families of sheaves in which some cohomological conditions are satisfied—for example, the loci where vector bundles on projective space split in a certain way, or the loci where a projective morphism has higher dimensional fibers.

Our approach is so explicit that it yields an algorithm suited for computer algebra systems.

Speaker: **Andrew Sommese** (University of Notre Dame)
Title: *Exceptional Sets and Fiber Products*
Abstract: Regard the solution set of a polynomial system $f(x\colon y) = 0$ with algebraic parameters as a family $X \to Y$ of algebraic sets. A symbolic/numeric algorithm based on fiber products is given to compute the subsets of X consisting of points where the fiber dimension of X is greater than it is for generic values of the parameters. A discussion of motivating problems from engineering is given.

Speaker: **Allan Steel** (University of Syndey)
Title: *Linear and Polynomial Algebra in Magma: A Detailed Overview*
Abstract: I give a detailed overview of the many structures and algorithms in the Magma Computer Algebra system for computing in Linear and Polynomial Algebra. The key challenges and successes are highlighted, particularly in the goal of practical implementations of asymptotically-fast algorithms.

Speaker: **Lihong Zhi** (Key Lab of Mathematics Mechanization, AMSS Beijing China)
Title: *Structured Low Rank Approximation of a Sylvester Matrix*
Abstract: The task of determining the approximate greatest common divisor (GCD) of polynomials with inexact coefficients can be formulated as computing for a given Sylvester matrix a new Sylvester matrix of lower rank whose entries are near the corresponding entries of that input matrix. We solve the approximate GCD problem by new methods: one is based on structured total least norm algorithm, another is based on structured total least squares algorithm, in our case for matrices with Sylvester structure. We present iterative algorithms that compute a minimum approximate GCD and that can certify an approximate ϵ-GCD when a tolerance ϵ is given on input. Each single iteration is carried out with a number of floating point operations that is of cubic order in the input degrees. In the univariate GCD case, we explore the displacement structure and reduce the complexity of each single iteration to be of only quadratic with respect to the degrees of the input polynomials. We also demonstrate the practical performance of our algorithms on a diverse set of univariate and multivariate pairs of polynomials. This is joint work with Erich Kaltofen, Bingyu Li and Zhengfeng Yang [11, 9, 8].

Summary of the two discussions on software

Both discussions were moderated by Stephen M. Watt.

The first discussion on Monday afternoon covered two topics, one given by Gert-Martin Greuel on the *Oberwolfach References on Mathematical Software (ORMS)* project

http://orms.mfo.de

and one by James Demmel on plans for the next release of LAPACK

http://www.netlib.org

and ScaLAPACK

http://www.netlib.org/scalapack

, including arbitrary precision versions, [joint work with Jack Dongarra et al.]. In particular, arbitrary precision, was discussed. One approach is to use F90 operator overloading so that one can produce fixed precision versions of any precision, calling someone else's arbitrary precision package. A web site to enter opinions was

http://icl.cs.utk.edu/lapack-forum/survey/

, which now has the survey's results.

The second discussion on Thursday morning addressed problems in transferring algorithms into systems. The use of generic algorithm techniques either by templates in C++ or by types in Aldor was promoted. The philosophical difference between opensource free software and commercial products was noted. For purpose of comparing implementations, the creation of a standard repository for tests and specific versions of software was deemed to be useful. E. Kaltofen pointed out that many symbolic computation problems require parallel computation like those done in ScaLAPACK. He suggested that more parallel symbolic computation algorithms and implementations should be developed in the next five years.

Assessment

This workshop provided a unique opportunity for leading researchers and developing younger investigators to exchange ideas on current challenges in several important areas of computer algebra. The areas of concentration of the workshop were:

- Linear algebra, both for exact methods (Dumas's and Giorgi's talk) and numerical methods (Demmel's presentation in the first discussion on software).
- Polynomial algebra. Polynomial factorization was covered by three speakers (von zur Gathen, van Hoeij and Steel), sparse polynomial interpolation by Monagan and problems in commutative algebra and polynomial systems by Greuel, Lazard, Roullier and Schreyer.
- Applications of symbolic computation to cryptography were presented by Schost.
- Hybrid symbolic-numeric algorithms were a focus, covered by Kai, Lee, Sommese and Zhi.
- Differential equations were addressed by Hubert, the talk which we chose to record.

We feel the workshop was valuable for several reasons: First, many speakers chose to discuss new ongoing work. Second, Demmel's numerical computation point-of-view made it apparent that numerical methods must be an integral part of symbolic computation software. One of the questions Demmel raised, that of the difference of structured vs. unstructured condition numbers in the case of the Sylvester matrices has subsequently been addressed [8]. Third, there was participation from the software industry, namely Gerhard from Waterloo Maplesoft and Dewar from the Numerical Algorithms Group (NAG).

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List of Participants

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Chapter 21

Progress in algebraic geometry inspired by physics (05w5081)

Oct 08 – Oct 13, 2005

Organizer(s): Jim Bryan (University of British Columbia), Michael Thaddeus (Columbia University), Ravi Vakil (Stanford University)

This is a report of the workshop *Progress in algebraic geometry inspired by physics*, held at the Banff International Research Station, October 8–13, 2005.

This meeting was a great success and a stimulating beginning to the 2005–06 academic year. Some 37 participants attended from top institutions in Canada, the USA, Europe, Korea, Hong Kong, and Japan. As anticipated in its proposal, the workshop covered many of the topics where theoretical physics has most greatly influenced algebraic geometry in recent years.

Gromov-Witten theory, for example, which originated as a quantum field theory governing the propagation of loops or strings on a Ricci-flat space-time, has become a mathematical theory of the enumerative geometry of algebraic curves on projective varieties. It was discussed in many of the lectures, such as those of Conan Leung and Jim Bryan.

A related topic from physics was discussed in two related lectures by Lothar Göttsche and Hiraku Nakajima: the Nekrasov partition function. This partition function can be regarded, thoroughly physically, as a partition function in an $N = 2$ supersymmetric quantum field theory, but also has mathematical interpretations both in terms of Gromov-Witten invariants and in terms of their analogues and forerunners, the Donaldson invariants of real 4-dimensional manifolds.

Mirror symmetry provides another example of an explicitly physical topic discussed at the meeting. Mirror symmetry began as a duality between quantum field theories, and was reinterpreted in physics as the "T-duality" of Strominger-Yau-Zaslow, in which string theory on a torus of large radius is dual to that on a torus of small radius. Mirror symmetry has received many mathematical interpretations:

- in terms of duality of polytopes by Victor Batyrev and Lev Borisov,
- the "homological mirror symmetry" of Maxim Kontsevich involving derived categories of sheaves and related to the Fourier-Mukai transform, and
- the torus fibrations inspired by Strominger, S.-T. Yau, and Zaslow.

These were represented in the conference by the lectures of Batyrev, Hori and Mark Gross, respectively.

The principal topic of Kentaro Hori's lecture was, however, different, and perhaps more surprising to most participants at the meeting. As one of the few card-carrying physicists present, Hori was able to inform the mathematicians that physics is able to shed light on matrix factorizations of polynomials — certainly a new and intriguing direction that we are likely to hear more of in the future.

But there were also topics of a more purely mathematical nature. Izzet Coskun and Bumsig Kim gave talks on moduli spaces of curves, for example, a more "classical" topic as it goes back in some sense to the nineteenth century. In its modern incarnation, interest dates to the late 1960's, long before the resurgence of physicists' interest in algebraic geometry. Yet it is also clearly a subject that has been revivified and reanimated by the indirect influence of physics. Coskun's talk made this clear: stable curves can be better understood using stable maps, which were only introduced by Kontsevich thanks to the motivation of physics.

Another "classical" topic which kept cropping up was that of K3 surfaces, which were discussed in (at least) the lectures of Leung, Gross, and Bertram. There was no explicit reference to physics in any of these talks, but the indirect influence was clear: these K3 surfaces are (rather elaborate) toy models of Calabi-Yau threefolds, proposed by string theorists to constitute the missing dimensions of space-time.

Some other recurring themes, though less classical, were also purely mathematical. Derived categories of coherent sheaves made an appearance several times, in the lectures of Aaron Bertram, Kentaro Hori, and Alistair Craw, for example. These certainly play a role in physics, as is evident in the work of Michael Douglas, and this was a motivation for Bertram's construction, but the elementary transformations that he described in holomorphic symplectic geometry had a purely mathematical elegance. Craw explained how the study of derived categories could be led in another direction — towards combinatorics — by applying them to the theory of toric varieties.

Another contemporary mathematical theory that was often invoked at the meeting was that of orbifolds or Deligne-Mumford stacks. These are now understood to have a quantum cohomology theory analogous to that of smooth varieties (work due to a number of researchers), and their Gromov-Witten theory, K-theory, and Hochshild cohomology were discussed by Charles Cadman, Takashi Kimura, and Andrei Caldararu respectively.

The concept of topological quantum field theory or TQFT should not be overlooked either. This is not really part of physics; it is more a mathematical formalism, put forward in around 1990 by Michael Atiyah and Graeme Segal, inspired by such physicists as Edward Witten and Robbert Dijkgraaf. But it simplifies and systematizes many calculations in algebraic geometry inspired by physics, any time we want to calculate some invariant on a moduli space by degenerating or cutting up the space on which it is based into smaller constituents (for example, by cutting up a Riemann surface into pairs of pants, interpretable as thrice-punctured spheres). It was discussed, for the enumerative geometry of spaces of admissible covers, in an attractive lecture by Renzo Cavalieri, and alluded to in the talks by Leung and Bryan as well.

There was much informal discussion of all of these topics, and more, at the meeting. The number of formal lectures was intentionally kept small — only sixteen — to provide ample time for informal discussions. However, each of the sixteen speakers was given a full 75 minutes to speak, which ensured an in-depth treatment in each lecture. The topics discussed in the lectures are briefly summarized below.

Conan Leung first reviewed the conjectural Yau-Zaslow formula, which expresses the generating function on the number of curves in a K3 surface as a quasi-modular form.

Then he explained his recent joint work with Junho Lee on the proof of this formula for the index 2 case, generalizing previous work with Jim Bryan for the index 1 case.

The technique employed was the gluing formula for Gromov-Witten invariants.

Lothar Göttsche spoke on his recent work on instanton counting, Donaldson invariants, and line bundles on moduli spaces. (This is joint work with Hiraku Nakajima and Kota Yoshioka.) They computed the Donaldson invariants of a rational surface in terms of the aforementioned Nekrasov partition function, which can be viewed as a generating function for the Donaldson invariants of the affine plane. For a line bundle L on the rational surface X, they computed the holomorphic Euler characteristic

$$\chi(M_X^H(c_1, c_2), \mathcal{O}(\mu(L)))$$

of associated line bundles on the moduli space of H-stable rank 2 bundles on X. Using the Nekrasov conjecture, this yielded explicit generating functions for the Donaldson invariants and the holomorphic Euler characteristics in terms of modular forms and elliptic functions.

Reporting on joint work with Bernd Siebert, **Mark Gross** described a "nonlinear Mumford construction," by which he menat the following. Mumford's construction produces explicit degenerations of abelian varieties, starting with data of a polyhedral decomposition of a real torus and a (multi-valued) convex piecewise linear function on the torus. This can be generalized by replacing the torus with a more general integral affine manifold with singularities. From these data, one can easily produce the central fiber of the degeneration, so the challenge is to smooth this fiber.

Gross showed how Kontsevich and Soibelman's approach translates naturally into this setting, producing explicit smoothings of K3 surfaces. Tropical rational curves emerged naturally out of his construction.

Aaron Bertram spoke about new moduli associated to a K3 surface, studied in joint work with Daniele Arcara. For a K3 surface S whose Picard group is generated by a divisor class C of self-intersection $2g-2$, he considered the "old" moduli space M of stable coherent sheaves on S with invariants $ch_0 = 0$, $ch_1 = H$, $ch_2 = g-1$ agreeing with those of the push-forward of a sheaf on C of rank 1 and degree $2g-2$. This is a smooth holomorphic-symplectic manifold.

The object of Bertram's talk was to exhibit a sequence of moduli spaces

$$M \leftrightarrow M' \leftrightarrow M'' \leftrightarrow \cdots$$

that are linked by Mukai flops over projective bundles over products of Hilbert schemes of points on S. These new moduli spaces are not (at least in any manifest way) moduli spaces of coherent sheaves on S, but rather are moduli space of stable objects in the derived category of coherent sheaves on S under a family of stability conditions motivated by physics. Bertram argued that this sequence of flops was the natural generalization of Thaddeus flips to K3 surfaces.

Kentaro Hori reported on his work on matrix factorizations and complexes of vector bundles. Physics shows the equivalence of certain aspects of matrix factorizations of, say, a degree 5 polynomial in 5 variables

$$G(x_1, \ldots, x_5),$$

and complexes of coherent sheaves of the quintic hypersurface $G(x_1, \ldots, x_5) = 0$ in complex projective 4-space. Recently D. Orlov proved the equivalence of the category of matrix factorizations of G and the bounded derived category of coherent sheaves on the quintic.

In his talk, Hori described these equivalences and argued that they are the "right ones" for physics. He suggested that a proper understanding of the physics may have many applications, for example, to stability or to homological mirror symmetry.

Victor Batyrev also spoke about mirror symmetry for Calabi-Yau threefolds, but discussed a subtle feature not previously studied: their integral cohomology. For Calabi-Yau varieties X and Y of dimension d that are mirror to each other, mirror symmetry predicts that the Hodge numbers of X and Y are related by the equality

$$h^{p,q}(X) = h^{d-p,q}(Y).$$

Batyrev's main interest was to understand the relationship between the torsion in their integral cohomology rings. For $d = 3$, he observed that the torsion in H^2 and H^3 must be exchanged by mirror symmetry. His verification of this statement for Calabi-Yau complete intersections in toric varieties reduced to an explicit computation of the fundamental group and the Brauer group.

Izzet Coskun gave a lecture about the cones of ample and effective divisors on Kontsevich moduli spaces. The cones of ample and effective divisors are among the most important invariants associated to any variety. But the study of these cones for moduli spaces is especially important. For example, in a celebrated series of papers in the 1980's, Harris, Mumford, and Eisenbud were able to prove that the moduli space of stable curves is of general type in genus greater than 23 by studying these cones.

In recent work with Joe Harris and Jason Starr, Coskun reduced the computation of the ample cone of the Kontsevich space of (genus zero) stable maps to projective space to a standard conjecture about curves. They also determined the stable effective cone of the Kontsevich moduli spaces. He described these results in his talk and discussed applications to the theory of rational connectivity and the divisor theory of the moduli spaces of pointed stable curves. For example, similar techniques have allowed him to determine the effective cone of the moduli space of pointed (genus zero) stable curves, modulo permutations.

Hiraku Nakajima discussed his joint work with Kota Yoshioka on instanton counting. This refers to the computation of Nekrasov's deformed partition functions of $N = 2$ supersymmetric Yang-Mills theories by integrating in the equivariant cohomology or Grothendieck groups of instanton moduli spaces over four-dimensional Euclidean space, which are quiver varieties associated with the Jordan quiver. These partition functions are analogues of the Donaldson invariants, and equal to the Gromov-Wiotten invariants of certain noncompact Calabi-Yau threefolds. Nakajima reviewed the recent results on these functions.

Alastair Craw reported on work about quivers and exceptional collections for projective toric manifolds. He described how certain collections of line bundles on a projective toric manifold can be used to reconstruct that manifold as a moduli space of quiver representations. To put it another way, he introduced new quiver gauge theory constructions of projective toric manifolds. His condition on the line bundles was remarkably weak, and in particular holds for nice "full strong exceptional collections" (if they exist) that describe the derived category of coherent sheaves. Indeed, Craw's program leads to new examples of such collections. (This was joint work with Greg Smith.)

Harry Tamvakis spoke about the Gromov-Witten invariants of isotropic Grassmannians. He has studied them in joint work with Anders Buch and Andrew Kresch. For a homogeneous space which is the quotient of a classical Lie group by a maximal parabolic subgroup, Tamvakis explained a series of results which show that the three-point genus-zero Gromov-Witten invariants can be equated with, and hence derived from, classical triple intersection numbers on related homogeneous spaces. He applied this principle to prove structure theorems for the small quantum cohomology of these homogeneous spaces, which give new results in the case of a Grassmannian parametrizing non-maximal isotropic subspaces of a vector space equipped with a symplectic or orthogonal form. Buch was also a participant in the workshop, and explained many technical aspects of this work informally in the evenings.

In his lecture on "Hurwitz-Hodge integrals and the crepant resolution conjecture," **Jim Bryan** stated the following. A well-known principle from physics asserts that string theory on an orbifold is equivalent to string theory on any crepant resolution of its coarse moduli space. In mathematics, this can be stated as saying that the Gromov-Witten potentials for the orbifold and the crepant resolution contain equivalent information: that is, one can be transformed to another by an appropriate change of variables. Bryan illustrated this in some examples, showing how it leads to interesting new formulas for integrals of Hodge classes over Hurwitz schemes. The lecture touched on important work joint with Rahul Pandharipande, Andrei Okounkov, Tom Graber, Dagan Karp, and others.

Bumsig Kim spoke about the moduli space of rational plane curves with a unique irreducible singular point. He showed that this moduli space can be decomposed as a union of irreducible smooth rational varieties of varying dimensions. He showed how to compute the degree of the largest component with fixed tangent line at the singular point. He was reporting on joint work with Dosang Joe and Hyungju Park.

Andrei Caldararu gave a stimulating lecture entitled "Towards computing the Hochschild cohomology ring of orbifolds," in which he attempted to explain the ingredients that should go into proving the generalization of Kontsevich's Theorem for complex manifolds to orbifolds. More explicitly, he went over the proof of Kontsevich's Theorem and pointed out what changes have to be made when dealing with orbifolds. For example, the inertial orbifold appears in a natural way in the course of the argument.

Takashi Kimura described the latest results from his long-standing collaboration with Tyler Jarvis. They apply to the setting of a global quotient, that is, a smooth projective variety equipped with the action of a finite group G. To these data, they have associated a G-equivariant Frobenius algebra, which they call the "stringy K-theory," whose G-coinvariants yield the orbifold K-theory of the quotient. They then introduced a stringy Chern character, which is a ring isomorphism from stringy K-theory to its cohomological counterpart. It contains "corrections" to the ordinary Chern character. The proof of the isomorphism follows from a new, simple reformulation of the relevant obstruction bundle, which does not involve stable maps. Hence their work significantly simplifies earlier work in simpler situations.

Renzo Cavalieri spoke about his doctoral work which gave the intersection numbers on moduli spaces of admisible covers the structure of a topological quantum field theory. More precisely, he explained how to construct a two-level weighted topological quantum field theory whose structure coefficents are equivariant intersection numbers on moduli spaces of admissible covers. Such a structure is parallel (and related, albeit somewhat mysteriously) to the local Gromov-Witten theory of curves of Jim Bryan and Rahul Pandharipande.

Cavalieri described the explicit computation of the theory using techniques of localization on moduli spaces of admissible covers of a parametrized projective line. The Frobenius algebras he obtained were one parameter deformations of the class algebra of the symmetric group. In certain special cases he could produce explicit closed formulas for such deformations in terms of the representation theory of the symmetric group.

Charles Cadman also described the work of his doctoral thesis, which uses high technology from the theory of stable maps to Deligne-Mumford stacks to solve a thoroughly classical problem, namely the enumeration of rational plane curves with tangency conditions to a fixed cubic. His key idea was to consider what he calls the "stack of nth roots" associated to a scheme X with a Cartier divisor D: that is, the stack whose objects are morphisms to X together with sections of an nth root of the pullback of the line bundle $\mathcal{O}(D)$, whose nth powers correspond to the natural section of $\mathcal{O}(D)$. This is a Deligne-Mumford stack whose coarse moduli space is X, and (for smooth X and D) stable maps to this stack correspond to maps to X with tangencies of order n along D. Recursions solving the enumerative problem can then be obtained, following Kontsevich, by applying the Witten-Dijkgraaf-Verlinde-Verlinde equations in the quantum cohomology of the stack of nth roots.

List of Participants

Abramovich, Dan (Boston University)
Batyrev, Victor (University of Tubingen)
Bertram, Aaron (University of Utah)
Bryan, Jim (University of British Columbia)
Buch, Anders (Aarhus University)
Cadman, Charles (University of British Columbia)
Caldararu, Andrei (University of Wisconsin, Madison)
Cavalieri, Renzo (University of Michigan)
Chen, Linda (Ohio State University)
Ciocan-Fontanine, Ionut (University of Minnesota)
Coskun, Izzet (Massachusetts Institute of Technology)
Craw, Alastair (Stony Brook University)
Gholampour, Amin (University of British Columbia)
Gottsche, Lothar (International Centre for Theoretical Physics)
Gross, Mark (University of California at San Diego)
Hori, Kentaro (University of Toronto)
Jarvis, Tyler (Brigham Young University)
Kim, Bumsig (Korea Institute for Advanced Study)
Kimura, Takashi (Boston University)
Lee, Yuan-Pin (University of Utah)
Leung, Nai Chung (Conan) (The Chinese University of Hong Kong)
Li, Jun (Stanford University)
Mare, Augustin-Liviu (University of Regina)
Mizerski, Maciej (University of British Columbia)
Mustata, Andrei (University of Illinois, Urbana-Champaign)
Nakajima, Hiraku (Kyoto University)
Payne, Sam (Clay Institute/Stanford University)
Purbhoo, Kevin (University of British Columbia)
Roth, Michael (Queen's University)
Shapiro, Jacob (University of British Columbia)
Song, Yinan (University of British Columbia)
Tamvakis, Harry (University of Maryland)
Thaddeus, Michael (Columbia University)
Tseng, Hsian-hua (University of Wisconsin)
Vakil, Ravi (Stanford University)
Watts, Jordan (University of Calgary)

Chapter 22

Therapeutic Efficacy in Population Veterinary Medicine (05w5201)

Oct 19 – Oct 22, 2005

Organizer(s): Fahima Nekka (Université de Montréal)

Report

- General

This workshop has been organized by the MITACS BIO5 team around the general theme of therapeutic efficacy in population veterinary medicine at Banff International Research Station. It has brought researchers working in applied mathematics, veterinary sciences, behavioural sciences as well as in microbiology and nutrition. Additional to academic researchers, speakers and participants from other public sectors attended the workshop: Agriculture and Agri-Food Canada and the Public Health Agency of Canada. Representatives of Pfizer Animal Health and Elanco Animal Health were present. The representatives of Scheringer-Plough Animal Health and Avantis, who are among the sponsors of the workshop, were not able to attend but asked for a follow up on the workshop outcomes. The conferences covered different aspects relating to animal collective therapy, in particular in swine and poultry, in terms of determinants and outcomes, spanning the areas of: animal behaviour, quantification of feeding behaviour and its relationship with pharmacokinetics, pharmacodynamics and antibiotic resistance, risk assessment in terms of antibiotic use and genetic determinants for antibiotic resistance and its different transfer modes, resistance to infection diseases, zoonotical borne viruses, identification of contamination sources, characterization of microbial hazards and manure, impact on the environment. A complete portrait of animal behaviour in the context of therapeutic efficacy has been drawn. A whole overview of the Canadian Integrated Program for Antimicrobial Resistance Surveillance (CIPARS/PICRA) has been given to explain the national program of antimicrobial use in food animals and surveillance system for antimicrobial resistance arising from food animal production. An update of PK/PD analysis in antibiotics was very useful to highlight the role of the prudent use of antibiotics in preserving their effectiveness and to clarify the objectives of the seed project. A general idea of mathematical approaches used to handle biological complexity has been given with emphasis on the need for collaborative efforts between mathematical sciences and experimental research. The keynote speakers have given their own ideas of possible collaborations with the MITACS team, in terms of their research interests/expertise and in complement to the current MITACS project. Very interesting discussions took place, always balanced between the different areas of research. Presence of industrial researchers from Pfizer Animal health in particular, allowed gaining a clear idea of the pharmaceutical industry expectations and practices.

- Financial support of the workshop
  - MITACS

  - AFMnet
  - MSRI
  - CRM
  - Pfizer Animal Health
  - Scheringer-Plough Animal Health
  - Avantis

- Organizing Committee

  - Chair, Fahima Nekka, Université de Montréal
  - Jérôme del Castillo, Université de Montral
  - Renée Bergeron, Université Laval
  - Jacques Bélair, Université de Montréal
  - Jun Li, Université de Montréal
  - Don Schaffner, Rutgers University
  - Heidi Shraft, Lakehead University/AFMnet
  - Claude Miville, FPPQ
  - Jeff Lucas, MITACS

- Minutes from the discussion about collaborations and perspectives, October 22nd

Were present at this last day meeting researchers from MITACS team (F. Nekka, J. del Castillo, R. Bergeron, J. Bélair); from FPPQ (C. Miville); from AFMnet (A. Paulson, M. McLaughlin, H. Schraft, L. Truelstrup, H. Eberl, J. France); from Pfizer Animal Health (Bruce Groves). The objective was to identify potential collaborations to be added to the current MITACS project or to make it a joint MITACS-AFMnet project, to identify additional funding sources and to address the involvement of pharmaceutical companies.
1. Discussion on research avenues

The following questions have been suggested as being important to be addressed: o Is the veterinary use of antibiotics (AB) appropriate? o How could we improve AB use to make it safer and more efficient: this is the main aim of the seed project which is centred around the judicious use of antibiotics. o How does the risk of using the labelled dose compare with the risk of using an unapproved one? o It would be interesting to model withdrawal time according to dosage. o Another avenue is to compare the efficacy of alternatives to AB versus AB efficacy. o The risk of using the approved dose must be weighed against the risk of using an off-label dose. o One concern about risk assessment is that it may lead to a ban on AB use.

Three main areas of research have been identified for the full project o Impact of feeding behaviour on dosage efficacy o Alternatives to AB and their assessment o Risk analysis, including assessment, policy making and risk communication. Qualitative risk rating systems show major limitations (for example, passing from high-dimensional information to low-dimensional evaluation causes loss of information). Use of more/new mathematical methods, including Rapid Risk Rating Technique, for quantitative human health impact of continued animal use of antibiotics. ! With MITACS funding, we have to make sure that new mathematics are being developed. In the seed project, from the mathematical point of view, we have used dynamical systems (represented by the multi-compartmental approach defined by systems of ODE) with stochastic input. We analyzed the statistical properties in terms of stability and conservation of the dynamical system. This approach is new in pharmacokinetics. Use of this approach has to be widespread in biological problems and include other sources of stochasticity and analyse their impacts since the generally- used assumption of determinism are questionable when considering the randomness involved in biological reality. We have also introduced competition mechanisms in collective behaviour which accounts for dynamical interactions between individuals (the interaction between individuals is incorporated in the evolution of the group). The approach we have used to model competition situations has to be put within the framework of hierarchical nonlinear models used for repeated measurement data.
2. Stakeholders and potential funding sources

- Veterinary Drugs Directorate (VDD) Divisions
- Health Canada
- Public Health Agency of Canada (PHAC)
- Canadian Integrated Program for Antimicrobial Resistance Surveillance (CIPARS)
- Canadian Food Inspection Agency (CFIA)
- Agriculture and Agri-Food Canada (AAC)
- Pork producers associations (Alberta Pork, Sask Pork, Manitoba Pork, Ontario Pork, etc.).
- Fonds québécois de la recherche sur la nature et les technologies (FQRNT)
- Conseil des recherches en pêche et en agroalimentaire du Québec (CORPAQ)
- Pharmaceutical industry
- Nutrition companies and integrators
- CIHR NSERC
- NSERC strategic
- AFMnet
- CVMA (Canadian Veterinary Medical Association)
- CAHI (Canadian Animal Health Institute)

3. Discussion regarding the involvement of pharmaceutical companies in the project

- The pharmaceutical industry is product oriented. Mathematics are used in drug development, but not much in resistance studies.
- Pharmaceutical companies work with approved products, and research using off-label dosage may place them in an awkward position. They report to Health Canada and must demonstrate that they comply with their guidelines.
- Pharmaceutical companies may not be interested in funding a project that may benefit their competitors, for instance, research that may eventually lead to approval of the off- label dosage of a non-proprietary drug. However, they may want to fund projects that would evaluate the risk of using the approved dose of a medication.
- Perhaps suppliers of CTC (the antibiotic used by MITACS team in the seed project) would be willing to get involved in our project, given that it may lead to the approval of a higher dosage for their product.
- Much research has been done on newer antibiotics. In fact, new antibiotics are in general derivatives of known families of drugs. Old antibiotics appear to be used the most and yet have not been documented as much.

Invitations have been sent to US researchers. Unfortunately, due to delays in reply of the first invited researchers, tentative to reach other persons were not successful for different reasons (other meetings on similar subjects at the same period in Europe in particular).

List of Participants

Bélair, Jacques (Université de Montréal)
Bergeron, Rene (Université Laval)
Bernier, Dave (Université de Montréal)
del Castillo, Jérôme (Université de Montréal)
Eberl, Hermann J (University of Guelph)
France, James (AFMnet / University of Guelph)
Gonyou, Harold (Prairie Swine Centre)
Groves, Bruce (Pfizer Animal Health Canada)
Hayes, Tony (University of Guelph)
Lafrance, Judith (Université Laval)
Letellier, Ann (Université de Montréal)
Lucas, Jeff (MITACS)
McLaughlin, Murray (AFMnet)
Miville, Claude (Fédération des producteurs de porcs du Québec)
Nekka, Fahima (Université de Montréal)
Paulson, Allan (AFMnet / Dalhousie University)
Petit-Jette, Caroline-Emmanuelle (Université de Montréal)
Pomar, Candido (Agriculture & Agri Food Canada)
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Sanche, Steven (Université de Montréal)
Schraft, Heidi (Lakehead University)
Theede, Alan (Pfizer Animal Health Canada)
Truelstrup, Lisbeth (AFMnet / Dalhousie University)
Yang, Laurence T. (St. Francis Xavier University)

Chapter 23

Probabilistic Combinatorics: Recent Progress and New Frontiers (05w5054)

October 29 – November 3, 2005

Organizer(s): Noga Alon (Tel Aviv University), Bruce Reed (McGill University, CNRS), Benny Sudakov (Princeton University), Van Vu (University of California, San Diego)

Overview

Probabilistic Combinatorics is an interface between Probability and Discrete Mathematics. Initiated by P. Erdős over fifty years ago, it has now become one of the fastest developing areas in all mathematics, with fascinating applications to many other important areas, such as Theoretical Computer Science and Statistical Physics. Roughly speaking, Probabilistic Combinatorics comprises three main topics, for each of which we give a short description. Naturally, there are considerable overlaps between these topics.

The first topic is the application of probability to solve combinatorial problems, and conversely the application of combinatorial methods to prove results in probability theory. Typical examples of the former are the “existence” proofs of Erdős. In general, one wants to show the existence of certain objects by generating an appropriate probabilistic space and proving that the desired object has positive measure in this space. The last twenty years or so have witnessed significant progress in this approach. The development of new and powerful techniques, such as the semi-random method and various sharp concentration inequalities, has enabled researchers to attack many famous open problems, considered intractable not so long ago, with considerable success. Furthermore, many new ideas discovered in this process have turned out to be useful for problems from different areas. For instance, the recent Galvin-Kahn result on Gibb’s measures has its roots in an earlier graph colouring result of Kahn. For an example of combinatorics being used in the field of probability, one can look at some recent work of Louigi Addario-Berry and Bruce Reed, which uses combinatorial techniques to bound the point at which a random walk first returns to zero.

The second topic is the study of random combinatorial structures, such as random graphs. The typical question here is to show that at a given density, a random graph has a desired property with very high probability. The study of random graphs has recently received a major boost from industry. It has been discovered that various important real-life graphs (such as the Internet) can be modeled as a random graph of a special type. If one can analyze these graphs, then one can make predictions about the evolution of the real-life networks.

The third topic is the study of randomized algorithms. Here the main question is either to design randomized algorithms for a certain goal or to analyze natural algorithms given special inputs. While this topic can also be considered as a topic in Computer Science, it has turned out quite recently that it also has much to do with Statistical Physics. For instance, there is a natural algorithm (motivated by problems from statistical physics) for generating a random colouring of a graph. A tantalizing question is to know when this algorithm runs in polynomial time, and a proper bound would have amazing consequences in Physics.

The focus of the workshop lay specifically in the above three main research topics of Probabilistic Combinatorics. One aim of the workshop was simply to foster interaction and collaboration between researchers in these fields, and to discuss recent progress and communicate new results and ideas. To mention an example, the following conjecture of Louigi Addario-Berry (see [1]), communicated during an open problem session, was solved at the workshop by Jacques Verstraete using the technique of combinatorial nullstellensatz:

Theorem 23.0.62 *Given a graph $G = (V, E)$ and, for every $v \in V$, a list $D_v \subseteq \{0, 1, \ldots, d(v)\}$ satisfying $|D_v| > \lceil d(v)/2 \rceil$, there is a spanning subgraph $H \subseteq G$ such that for all v, $d_H(v) \in D_v$.*

Additionally, this forum was an opportunity to make state-of-the-art probabilistic techniques available to a broader audience, in particular graduate students.

With the rapid development in recent years of probabilistic techniques and their applications to various mathematical disciplines, the workshop was a key opportunity to bring together researchers representing the entire spectrum of Probabilistic Combinatorics, so as to consolidate our knowledge at present and set new horizons for future discoveries.

In the remainder of the report we describe in detail some of the advances presented at the workshop.

The Erdős-Rényi Random Graph

Joel Spencer - *Connectedness of $G(n, p)$*

I gave a talk on The Probability of Connectedness, the result being an asymptotic formula for the probability that the random graph Gn, p is connected, for the entire range of p. The key to it is a new analysis of breadth first search over the random graph Gn, p. This is an idea I have been working on for a year or so but it really came together during the workshop. I have given talks on this general topic before, most recently at the CMS Annual Meeting in Waterloo in June, but at this workshop the ideas were clearer than before.

The asymptotic probability of $G(n, p)$ being connected is $A_1 A_2$, with

$$A_1 = A_1(n, p) = (1 - (1 - p)^n)^{n-1}$$

$$A_2 = A_2(n,p) \sim \begin{cases} 1 & \text{for } p \gg n^{-1} \\ 1 - (c+1)e^{-c} & \text{for } p \sim cn^{-1} \\ \frac{1}{2}\epsilon^2 & \text{for } p \sim \epsilon n^{-1} \text{ and } n^{-1/2} \ll \epsilon = o(1) \\ \text{complicated} & \text{for } p \sim cn^{-3/2} \\ n^{-1} & \text{for } 0 < p \ll n^{-3/2} \end{cases}$$

(Note that the probability that there are no isolated vertices if the events of being isolated were independent would be $(1 - (1 - p)^{n-1})^n$ which is quite close.)

When $p \ll n^{-3/2}$ it is simpler to write that the probability of $G(n, p)$ being connected is roughly the probability that $G(n, p)$ is precisely a tree, which is $n^{n-2} p^{n-1} (1 - p)^{m-(n-1)}$ with $m = \binom{n}{2}$.

When $p \sim cn^{-3/2}$ let B be the probability $G(n, p)$ is precisely a tree. Then $G(n, p)$ is a tree plus l edges with probability $B c_l c^{3l/2}$ where the c_l are the "Wright constants". Convergence occurs and the probability that $G(n, p)$ is a tree is $B \sum_{i=0}^{\infty} c_l c^{3l/2}$.

The arrangements were excellent, giving myself and the others plenty of time to "prove and conjecture."

Louigi Addario-Berry - *The Diameter of the Minimum Weight Spanning Tree*

Given a connected graph $G = (V, E)$, $E = \{e_1, \ldots, e_{|E|}\}$, together with edge weights $W = \{w(e) | e \in E\}$, a minimum weight spanning tree of G is a spanning tree $T = (V, E')$ that minimizes

$$\sum_{e \in E'} w(e).$$

If the edge weights are distinct then this tree is unique; in this case we denote it by MWST(G). Minimum spanning trees are at the heart of many combinatorial optimization problems. In particular, they are easy to compute, and may be used to approximate hard problems such as the minimum weight traveling salesman tour. As a consequence, much attention has been given to studying their structure, especially in random settings and under various models of randomness. For instance, Frieze determined the weight of a the MWST of a complete graph whose edges have been weighted by independent and identically distributed (i.i.d.) $[0, 1]$-random variables. This result has been reproved and generalized by Frieze and McDiarmid [8] and Aldous [2]. Under the same model, Aldous derived the degree distribution of the MWST. Both these results rely on local properties of minimum spanning trees. We are interested in their global structure.

The *distance* between vertices x and y in a graph H is the length of the shortest path from x to y. The *diameter* diam(H) of a connected graph H is the greatest distance between any two vertices in H. We are interested in the diameters of the minimum weight spanning trees of a clique K_n on n vertices whose edges have been assigned i.i.d. real weights. We use $w(e)$ to denote the weight of e. In Banff we presented our proof of the following theorem, answering a question of Frieze and McDiarmid [9].

Theorem 23.0.63 *Let $K_n = (V, E)$ be the complete graph on n vertices, and let $\{X_e | e \in E\}$ be independent identically distributed edge-weights. Then conditional upon the event that for all $e \neq f$, $X_e \neq X_f$, it is the case that the expected value of the diameter of MWST(K_n) is $\Theta(n^{1/3})$.*

Benny Sudakov - *Embedding Nearly-Spanning Bounded Degree Trees*

In this talk we describe a sufficient condition for a sparse graph G to contain a copy of every nearly-spanning tree T of bounded maximum degree, in terms of the expansion properties of G. The restriction on the degree of T comes naturally from the fact that we consider graphs of constant degree. Two important examples where our condition applies are random graphs and graphs with a large spectral gap.

The problem of existence of large trees with specified shape in random graphs has a long history starting with conjecture of Erdős that a random graph $G(n, c/n)$ almost surely contains a path of length at least $(1 - \alpha(c))n$, where $\alpha(c)$ is a constant smaller than one for all $c > 1$ and $\lim_{c\to\infty} \alpha(c) = 0$. The question of existence of large trees of bounded degree other than paths in sparse random graphs was studied by de la Vega. He proved that for sufficiently large c one can almost surely embed in $G(n, c/n)$ any tree with maximum degree at most d that occupies a small constant proportion of the random graph. Our first result improves the result of Fernandez de la Vega and generalizes several results on the existence of long paths. It shows that the sparse random graph contains almost surely every nearly-spanning tree of bounded degree, i.e., tree of size $(1 - \epsilon)n$.

For a graph G let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of its adjacency matrix. The quantity $\lambda(G) = \max_{i\geq 2} |\lambda_i|$ is called the *second eigenvalue* of G. A graph $G = (V, E)$ is called an (n, D, λ)-*graph* if it is D-regular, has n vertices and the second eigenvalue of G is at most λ. It is well known that if λ is much smaller than the degree D, then G has strong expansion properties, so the ratio D/λ could serve as some kind of measure of expansion of G. Our second result shows that an (n, D, λ)-graph G with large enough spectral gap D/λ contains a copy of every nearly-spanning tree with bounded degree. This extends a result of Friedman and Pippenger [7].

Regular Graphs

Nicholas Wormald - *Large Independent Sets in Regular Graphs of Large Girth*

An *independent set* I of a graph G is a subset of the vertices of G such that no two vertices of I are joined by an edge. The *independence number* of G is the cardinality of a maximum independent set, and is denoted by $\alpha(G)$. The *girth* of G is the length of its shortest cycle.

In 1991, Shearer gave the best known lower bounds on $\alpha(G)$ for G with given maximum degree and large girth. For instance, if G is 3-regular with n vertices, Shearer's results imply that $\alpha(G) \geq \frac{125}{302}n$ provided the girth is sufficiently large, and he gave other results for graphs of maximum degree d in terms of $f(d)$ where the function f is defined iteratively.

It is known that looking at graphs with maximum degree d for such problems is equivalent to looking at d-regular graphs. In 1995, the speaker analyzed two greedy algorithms which give rise to large independent sets in random regular graphs, one simple and one more sophisticated. With Joe Lauer, we recently studied the simple greedy algorithm, applied to large girth graphs, and established a result for all regular graphs of large girth, that coincides with the corresponding result for random graphs. We use a "nibble"-type approach but require none of the sophistication of the usual nibble method arguments, using only linearity of expectation. We obtained the following result.

Theorem 23.0.64 *For all $d \geq 3$, the independence number of a graph with n vertices, maximum degree d and girth g is at least*

$$\big(1 - \varepsilon(g)\big)\frac{n}{2}\left(1 - (d-1)^{-2/(d-2)}\right),$$

where $\varepsilon(g) \to 0$ as $g \to \infty$.

This improves Shearer's result for all $d \geq 7$.

More recently, with Mohammad Salavatipour, we have analyzed the more sophisticated greedy algorithm mentioned above. The results are stronger but are given in terms of the solutions of differential equations which have only been solved numerically. With Carlos Hoppen we have examined algorithms for finding large induced forests in graphs with bounded degree and large girth. It is believed that, in all cases, the constants obtained for regular graphs of large girth coincide with those already known for random regular graphs.

It was known that, given such a bound for regular graphs of arbitrarily large girth, the same bound carries over to an asymptotic bound for random regular graphs. The current work indicates that for many problems with results on random regular graphs obtained by analyzing greedy algorithms the results can be "explained" in this way, despite the fact that they were first proved directly in the random case. It is not known to what extent this is a general phenomenon. In particular, it is not known if all 4-regular graphs with sufficiently large girth are 3-colourable.

Angelika Steger - *A Probabilistic Counting Lemma for Sparse Regular Graphs*

This is joint work with S. Gerke and M. Marciniszyn.

Over the last decades Szemerédi's regularity lemma [18] has proven to be a very powerful tool in modern graph theory. Unfortunately, in its original setting it only gives nontrivial results for dense graphs, that is graphs with $\Theta(n^2)$ edges. In 1996 Kohayakawa [14] and independently Rödl introduced a variant which holds for sparse graphs, provided they satisfy some additional structural conditions (which essentially mean that the graph does not contain regions that are too dense). However, using this sparse regularity lemma to prove e.g. extremal and Ramsey type results similar to the known results in the dense case requires as an additional step: the existence of appropriate embedding or counting lemmas. For the sparse case this missing step has been formulated as a conjecture by Kohayakawa, Łuczak and Rödl [15]. For a graph H, let $\mathcal{G}(H, n, m)$ be the family of graphs on vertex set $V = \bigcup_{x \in V(H)} V_x$, where the sets V_x are pairwise disjoint sets of vertices of size n, and edge set $E = \bigcup_{\{x,y\} \in E(H)} E_{xy}$, where $E_{xy} \subseteq V_x \times V_y$ and $|E_{xy}| = m$. Let $\mathcal{G}(H, n, m, \varepsilon) \subseteq \mathcal{G}(H, n, m)$ denote the set of graphs in $\mathcal{G}(H, n, m)$ satisfying that each $(V_x \cup V_y, E_{xy})$ is an (ε)-regular graph.

Conjecture 23.0.65 (KŁR Conjecture [15]) *Let H be a fixed graph and define*

$$\mathcal{F}(H, n, m) = \{G \in \mathcal{G}(H, n, m) : H \textit{ is not a subgraph of } G\}.$$

For any $\beta > 0$, there exist constants $\varepsilon_0 > 0$, $C > 0$, $n_0 > 0$ such that for all $m \geq Cn^{2-1/d_2(H)}$, $n \geq n_0$, and $0 < \varepsilon \leq \varepsilon_0$,

$$|\mathcal{F}(H, n, m) \cap \mathcal{G}(H, n, m, \varepsilon)| \leq \beta^m \binom{n^2}{m}^{|E(H)|},$$

where $d_2(H) = \max\left\{\frac{|E(F)|-1}{|V(F)|-2} : F \subseteq H, |V(F)| \geq 3\right\}$.

One of the key difficulties in the proof of the KŁR Conjecture is the fact that for $m = o(n^2)$ the size of a neighbourhood of a vertex is on average $o(n)$. The definition of regularity, however, only deals with linear sized subsets and thus regularity seems not to be inherited by subgraphs induced on the neighbourhoods of some vertices. In a joint paper [10] with Gerke, Kohayakawa, and Rödl we were recently able to prove that nevertheless in the sparse case a hereditary version holds as well, at least in the probabilistic setting. This result readily implies much shorter and more elegant proofs of the results known so far, namely the case of cycles C_k for all $k \geq 3$ and for $H = K_4$ and K_5. In this talk we show that in fact a much stronger property holds. Namely, small sets not only inherit with high probability the regularity property, but they also satisfy with high probability all properties that regular tuples satisfy with high probability. This allows us to show that the KŁR Conjecture holds for all complete graphs for slightly larger number of edges than the conjectured value. In return, we can show the existence of many copies instead of just one copy. That is, we get a so-called counting lemma.

Theorem 23.0.66 ([11]) *For all $\ell \geq 3$, $\delta > 0$, and $\beta > 0$, there exist constants $n_0 \in \mathbb{N}$, $C > 0$, and $\varepsilon > 0$ such that*

$$|\mathcal{F}(K_\ell, n, m, \delta) \cap \mathcal{G}(K_\ell, n, m, \varepsilon)| \leq \beta^m \cdot \binom{n^2}{m}^{\binom{\ell}{2}}$$

provided that $m \geq Cn^{2-1/(\ell-1)}$, $n \geq n_0$, and $0 < \varepsilon \leq \varepsilon_0$ and where $\mathcal{F}(K_\ell, n, m, \delta)$ denotes the family of graphs in $\mathcal{G}(K_\ell, n, m)$ that contain less than $(1-\delta)n^{|V(H)|}(\frac{m}{n^2})^{|E(H)|}$ copies of H.

Graph Colouring

Andrew King - *Advances Towards Reed's Conjecture*

My current research includes several problems: partial results towards Reed's Conjecture, probabilistic colouring work to similar ends, and the reconciliation of probabilistic models via rapidly-mixing Markov chains.

Reed's Conjecture states that for any graph G, $\chi(G) \leq \lceil (1/2)(\Delta(G) + 1 + \omega(G)) \rceil$ [19]. Generally speaking, there are two ways to work towards this result. The first involves proving it outright for certain classes of graphs, and the second involves proving that it is not far from the truth. That is, $\chi(G) \leq \lceil (1/2 + o(1))(\Delta(G) + 1 + \omega(G)) \rceil$, meaning that $\chi(G) \leq \lceil (1/2 + f(\Delta(G)))(\Delta(G) + 1 + \omega(G)) \rceil$ where f tends to 0 as Δ tends to infinity. There are partial results of this flavour, and I am working towards broadening this body of work as well as finding ways to colour graphs with few colours in polynomial time.

Since the workshop, Bruce Reed and I have proved that Reed's Conjecture holds for quasi-line graphs, improving upon a result of Chudnovsky and Ovetsky [3]. Furthermore, for these graphs a colouring using at most $\lceil (1/2)(\Delta(G) + 1 + \omega(G)) \rceil$ colours can be found in polynomial time.

Pseudorandom Graphs

Yoshiharu Kohayakawa - *Turán's Theorem for Pseudorandom Graphs*

This is joint work with V. Rödl (Emory University), M. Schacht (Humboldt-Universität zu Berlin), P. Sissokho (Illinois State University), and J. Skokan (Universidade de São Paulo).

The *generalized Turán number ex*(G, H) of two graphs G and H is the maximal number of edges in a subgraph of G not containing H. If G is the complete graph K_n on n vertices, then, by the Erdős–Stone–Simonovits theorem, we have $\mathrm{ex}(K_n, H) = \Big(1 - 1/(\chi(H) - 1) + o(1)\Big)\binom{n}{2}$, where $o(1) \to 0$ as $n \to \infty$.

We give an analogous result for triangle-free graphs H and pseudorandom graphs G. Our concept of pseudorandomness is inspired by the *jumbled* graphs introduced by A. Thomason. We say that a graph G is (q, α)-*bijumbled* if

$$\Big| e_G(X, Y) - q|X||Y| \Big| \leq \alpha \sqrt{|X||Y|}$$

for every pair of sets X, $Y \subset V(G)$, where $e_G(X, Y)$ denotes the number of pairs $(x, y) \in X \times Y$ with $xy \in E(G)$.

For simplicity, here we only state a consequence of our main result: for any triangle-free graph H with maximum degree Δ and for any $\delta > 0$, there exists $\gamma > 0$ such that any large enough n-vertex, $(q, \gamma q^{\Delta+1/2} n)$-bijumbled graph G satisfies

$$\text{ex}(G,H) \leq \left(1 - \frac{1}{\chi(H)-1} + \delta\right) |E(G)|.$$

Jan Vondrák - *2-Colourability of Randomly Perturbed Hypergraphs*

This is joint work with Benny Sudakov.

In the classical Erdős-Rényi model, a random graph is generated by starting from an empty graph and then adding a certain number of random edges. More recently, Bohman, Frieze and Martin considered a generalized model where one starts with a fixed graph $G = (V, E)$ and then inserts a collection R of additional random edges. We denote the resulting random graph by $G + R$. The initial graph G can be regarded as given by an adversary, while the random perturbation R represents noise or uncertainty, independent of the initial choice. This scenario is analogous to the *smoothed analysis* of algorithms proposed by Spielman and Teng, where an algorithm is assumed to run on the worst-case input, modified by a small random perturbation.

In subsequent work, Krivelevich, Sudakov and Tetali [16] considered random formulas obtained by adding random k-clauses (disjunctions of k literals) to a fixed k-SAT formula. They proved that for any formula with at least $n^{k-\epsilon}$ k-clauses, adding $\omega(n^{k\epsilon})$ random clauses of size k makes the formula almost surely unsatisfiable. This is tight, since there is a k-SAT formula with $n^{k-\epsilon}$ clauses which almost surely remains satisfiable after adding $o(n^{k\epsilon})$ random clauses. A related question, which was raised in this paper, is to find a threshold for non-2-colourability of a random hypergraph obtained by adding random edges to a large hypergraph of a given density.

While 2-colourability of graphs is well understood, being equivalent to non-existence of odd cycles, for k-uniform hypergraphs with $k \geq 3$ it is already NP-complete to decide whether a 2-colouring exists. Consequently, there is no efficient characterization of 2-colourable hypergraphs. The problem of 2-colourability of random k-uniform hypergraphs for $k \geq 3$ was first studied by Alon and Spencer. Recently, the threshold for 2-colourability has been determined very precisely. Achlioptas and Moore proved that the number of edges for which a random k-uniform hypergraph becomes almost surely non-2-colourable is $(2^{k-1}\ln 2 - O(1))n$. Interestingly, the threshold for non-2-colourability is roughly one half of the threshold for k-SAT. Achlioptas and Peres proved that a formula with m random k-clauses becomes almost surely unsatisfiable for $m = (2^k \ln 2 - O(k))n$. The two problems seem to be intimately related and it is natural to ask what is their relationship in the case of a random perturbation of a fixed instance.

The proof of Krivelevich et al. (for randomly perturbed k-SAT) also yields that for any k-uniform hypergraph H with $n^{k-\epsilon}$ edges, adding $\omega(n^{k\epsilon})$ random edges destroys 2-colourability almost surely. Nonetheless, it turns out that this is not the right answer. It is enough to use substantially fewer random edges to destroy 2-colourability: roughly a square root of the number of random clauses necessary to destroy satisfiability. Our main result is that for any k-uniform hypergraph with $\Omega(n^{k-\epsilon})$ edges, adding $\omega(n^{k\epsilon/2})$ random edges makes it almost surely non-2-colourable. This is almost tight in the sense that adding $o(n^{k\epsilon/2})$ random edges is not sufficient in general.

First Order Graph Properties

Oleg Pikhurko - *First Order Graph Properties*

Graph properties expressible in first order logic were studied. The vocabulary consists of variables, connectives ($\vee$, $\wedge$ and $\neg$), quantifiers ($\exists$ and $\forall$), and two binary relations: the equality and the graph adjacency ($=$ and $\sim$ respectively). The variables denote vertices only so we are not allowed to quantify over sets or relations. The notation $G \models A$ means that a graph G is a model for a *sentence* A (a first order formula without free variables); in other words, A is true for the graph G.

A first order sentence A *defines* G if G is the unique (up to an isomorphism) finite model for A. The *quantifier depth* (or simply *depth*) $D(A)$ is the largest number of nested quantifiers in A. This parameter is closely related to the complexity of checking whether $G \models A$. Let $D(G)$ be the smallest quantifier depth of a first order formula defining G.

In a sense, a defining formula A can be viewed as the canonical form for G (except that A is not unique): in order to check whether $G \cong H$ it suffices to check whether $H \models A$. Unfortunately this approach does not seem to lead to better isomorphism algorithms, but this notion, being on the borderline of combinatorics, logic and computer science, is interesting on its own and might yield unforeseen applications.

Recently, various results on the values of $D(G)$ for order-n graphs appeared. The paper of Pikhurko, Veith and Verbitsky studied the maximum of $D(G)$ (the 'worst' case). The 'best' case is considered by Pikhurko, Spencer, and Verbitsky, while Kim, Pikhurko, Spencer and Verbitsky obtained various results for the random graph $G(n, p)$.

Pikhurko presented new results for random sparse structures obtained jointly with Bohman, Frieze, Łuczak, Smyth, Spencer, and Verbitsky. Specifically, it was proved that almost surely

- $D(G) = \Theta(\frac{\ln n}{\ln \ln n})$, where G is the giant component of a random graph $G(n, \frac{c}{n})$ with constant $c > 1$;
- $D(T) = (1 + o(1))\frac{\ln n}{\ln \ln n}$ where T is a random tree of order n.

These results rely on computing the maximum of $D(T)$ for a tree T of order n and maximum degree l, so this problem was studied as well.

Combinatorial Games

Thomas Bohman - *Making and Breaking the Giant Component*

I presented the following results at the workshop. We consider a game that can be viewed as a random graph process. The game has two players and begins with the empty graph on a set of n vertices. During each turn a pair of random edges is generated and one of the players chooses one of these edges to be an edge in the graph. Thus the players guide the evolution of the graph as the game is played. One player controls the even rounds with the goal of creating a so-called giant component as quickly as possible. The other player controls the odd rounds and has the goal of keeping the giant from forming for as long as possible. We show that the product rule is an asymptotically optimal strategy for both players. (The product rule chooses between two edges by comparing the products of the sizes of the components joined. For example, the player who is trying to create a giant component would choose the edge that maximized the product of the sizes of the components joined.)

Geometric Problems

Imre Bárány - *On the Randomized Integer Convex Hull*

This is joint work with J. Matoušek.

Assume $K \subset R^d$ is a convex body. Its integer convex hull is, by definition, the convex hull of $K \cap Z^d$ where Z^d is the usual integer lattice. Notation: $I(K) = \text{conv}(K \cap Z^d)$. The integer convex hull is of central interest in integer programming. Define the lattice $L_{\rho,t} = \rho(Z^d + t)$ where $t \in [0, 1)^D$ and $\rho \in SO(d)$, which is an isometric copy of Z^d. The set of lattices $\mathcal{L} = \{L_{\rho,t}\}$ is a probability space with probability measure equal to the product of the Lebesgue measure on $[0, 1)^d$ and the Haar measure on $SO(d)$. The randomized integer convex hull is $I_L(K) = \text{conv}(K \cap L)$, where L is a random element of $\mathcal{L}$. $I_L(K)$ is a polytope.

Motivated by integer programming, we estimate the expected number of vertices of $I_L(K)$, and also the expected missed volume, that is, the expectation of $\text{vol}(K \setminus I_L(K))$. One of our results says that the expected number of vertices of $I_L(K)$ is of order $(\text{vol}(K))^{(d-1)/(d+1)}$ when K is smooth, and is of order $(\log \text{vol}(K))^{d-1}$ when K is a polytope. The expected missed volume problem leads to the following question which is a distant relative of Buffon's needle problem. Given a convex body $K \subset R^d$, what is the probability that a randomly chosen congruent copy of K is lattice point free? We show that this probability (1) is always smaller than $c_1/\text{vol}(K)$ for c_1 constant, and (2) is larger than $c_2/\text{vol}(K)$ for c_2 constant if the width of K is small enough. The constants depend only on dimension.

Ross M. Richardson - *Random Inscribing Polytopes*

This is joint work with Van Vu and Lei Wu.

Let K be a compact convex body in $\mathbb{R}^d$. Choose n points uniformly in K. The convex hull of these n points is referred to as a *random polytope*. The study of random polytopes is the study of certain key functionals of these polytopes; the volume of the random polytope and the number of $i-$dimensional faces are the most studied. There has been much recent progress in their characterization, and a broad range of techniques have arisen out of the intersection of geometry, probability, and combinatorics. A comprehensive survey by I. Bárány will soon appear in the volume *Stochastic Geometry*.

Now restrict K to have smooth boundary and everywhere positive Gaussian curvature. We define a new model of random polytopes where we now choose points on the boundary ∂K according to some positive continuous distribution. The convex hull of n points chosen in this manner is referred to as the *random inscribing polytope*.

Our work focuses on determining the distribution of the volume functional, which we denote by Z. We prove a concentration result of the following form:

$$P\left(|Z - EZ| \geq \sqrt{\lambda V}\right) \leq 2\exp(-\lambda/4) + \exp(-c\epsilon n),$$

where here $\epsilon \geq \alpha \ln n/n$, $V = \Theta(\epsilon^{(d+3)/(d-1)})$ and c, α are constants. We can use this result to show that the k^{th} moment M_k satisfies

$$M_k = O(V^{k/2}).$$

We can also prove better bounds, though with more complicated error terms.

In contrast to the integral geometric methods typically employed to study random polytopes, we rely on the notion of $\epsilon-$nets and VC-dimension to control the relevant geometry. Our concentration result employs a special instance of a more general martingale concentration theorem due to Kim and Vu. In particular we provide a quantitative notion of the volume added with the addition of a new point to the random polytope and show how this implies sharp concentration via the aforementioned tools.

We also provide a lower bound on the variance of the volume functional as well as showing the volume satisfies a central limit theorem.

Random Matrices

Van H. Vu - *Singularity of Random Matrices*

The study of random matrices is an important area of mathematics, with strong connections to various other fields. One of the main objects in this area is matrices whose entries are i.i.d. random variables. We focus on the basic model in which M_n is an n by n matrix whose entries are i.i.d. variables with Bernoulli distribution (taking values -1 and 1 with probability $1/2$).

A famous problem is to estimate the probability that M_n is singular. Let us denote by p_n this probability. Since M_n is singular if it has two identical rows, it is trivial that $p_n \geq (1/2 + o(1))^n$. A notorious conjecture in the field is that this bound is sharp:

Conjecture 23.0.67 $p_n = (1/2 + o(1))^n$.

The first result concerning singularity was obtained by Komlós in 1967, who proved $p_n = o(1)$. Later, he improved the bound to $O(n^{-1/2})$. A significant progress was made in 1995, when Kahn, Komlós and Szemerédi proved that $p_n \leq .999^n$ (see [13] and the references therein).

Recently, T. Tao and I made progress by further improving the upper bound to $(3/4 + o(1))^n$ [20]. We discovered a surprising connection between problems on random matrices and additive combinatorics. In particular, the proof of the new bound uses various ingredients from additive combinatorics (in particular, Freiman's theorem).

The details are somewhat technical, but my feeling is that the optimal bound $(1/2 + o(1))^n$ might be within sight. In fact, I believe that any improvement upon the constant $3/4$ could perhaps lead to the solution of the conjecture. Furthermore, our techniques can be used for other discrete distributions as well and in certain cases we can obtain sharp results.

A closely related question is to estimate the probability that a random symmetric matrix is singular. Let Q_n be the random symmetric n by n matrix whose upper diagonal entries are i.i.d. Bernoulli random variables. Weiss conjectured in the 1980s that Q_n is almost surely non-singular. Recently, Costello, Tao and I confirmed this conjecture. Our proof again makes a detour to additive combinatorics, with the main lemma being a quadratic version of the classical Littlewood-Offord-Erdős problem [5].

There have been several further developments in the research of random matrices reported at BIRS:

(1) The singularity problem: Costello, Tao and I generalized the singularity result for random matrices with arbitrary distribution. It seems that for any (discrete) random matrix with independent entries with distributions not concentrated on one value, the probability that the matrix is singular is exponentially small.

(2) Rank of random graphs: Costello reported a result showing that the threshold for singularity of (the adjacency matrix of) a random graph is $(\log n)/n$. (It is clear that below $(\log n)/n$, the graph has isolated vertices which correspond to all zero row; the main part is to handle the other side of the threshold.) We have extended this result to the following: For any $p > (\log n)/2n$, the corank of $G(n,p)$ equals the number of isolated vertices. As a corollary, it follows that the giant component has full rank.

(3) Richardson and Wu reported a result showing central limit theorems for random inscribing polytopes. Bárány and I extended these results for random polytopes spanned by points sampled from the Gaussian distribution.

Sequential Growth Models

Graham Brightwell - *Classical Sequential Growth Models*

Graham Brightwell gave a talk entitled "Classical Sequential Growth Models", including a discussion of joint work with Nicholas Georgiou.

Classical sequential growth models were introduced by Rideout and Sorkin in 2000; they are of particular interest as they are the only models satisfying some natural-looking conditions for discrete random models of space-time.

A particular classical sequential growth model is defined by a sequence $\mathbf{t} = (t_0, t_1, \ldots)$ of non-negative constants. The process starts with the partial order P_0 with one element labeled 0. At stage $n = 1, 2, \ldots$, the element n is added to P_{n-1} and placed above all elements in D_n, where D_n is a random subset of $\{0, 1, \ldots, n-1\}$, the probability that D_n is equal to a set D being proportional to $t_{|D|}$. The transitive closure is taken to form the partial order P_n.

One can either stop after stage n and study the finite partial order, or continue to get a partial order on the set of non-negative integers.

Special cases include random forests ($t_0 = t_1 = 1$, $t_i = 0$ for $i \geq 2$), and random binary orders (t_2 is the highest non-zero entry). Although random binary orders are very sparse, it is nevertheless the case that, a.s., in the infinite partial order, every element is incomparable with finitely many others. In a recent paper, Georgiou proves that, for any $\varepsilon > 0$, most elements r are incomparable with at most $r^{2+\varepsilon}$ other elements.

A random graph order, also known as a transitive percolation process, is defined by taking a random graph $G(n,p)$ on the vertex set $\{0, \ldots, n-1\}$, and putting i below j if there is a path $i = i_1, \ldots, i_k = j$ in the graph with $i_1 < \ldots < i_k$. This is equivalent to a classical sequential growth model with $t_n = t^n$, $t = p/(1-p)$.

In a later paper, Rideout and Sorkin provide computational evidence that suitably normalized sequences of random graph orders have a "continuum limit". Brightwell and Georgiou use results about the structure of random graph orders to confirm that this is indeed the case, and showed that the continuum limit is always a *semiorder*, i.e., a partial order representable by unit intervals in the line, one below another if it lies entirely to the left. Alternatively, a semiorder is a partial order containing no induced copy of either of the two specific partial orders $\mathbf{1} + \mathbf{3}$ and $\mathbf{2} + \mathbf{2}$.

It might be hoped that sequences of classical sequential growth models can have more interesting continuum limits, in particular ones that bear a closer resemblance to 4-dimensional Minkowski space-time. However, Brightwell and Georgiou show that classical sequential growth models are all "almost" semiorders, so that any continuum limit must also be very close to being a semiorder.

To be more precise, Brightwell and Georgiou show that, for any sequence $\{P_n\}_{n=0}^{\infty}$, where P_n is a classical sequential growth model stopped at stage n, the proportion of 4-element subsets isomorphic to either $\mathbf{1} + \mathbf{3}$ or $\mathbf{2} + \mathbf{2}$ tends to 0 as n tends to infinity.

Markov Chain Mixing Times

Prasad Tetali - *Analysis of Markov Chain Mixing Times*

Prasad Tetali gave a brief update on some recent progress in the analysis of Markov chain mixing times. The update included the status of several long-standing open problems, as well as recent theoretical developments in the topic.

The update on the theoretical development focused on isoperimetric and functional approaches to bounding mixing times. It is well known that the spectral gap of a Markov chain can be estimated in terms of conductance, facilitating isoperimetric bounds on mixing time. Observing that small sets often have large conductance, Lovász and Kannan refined this result by bounding the total variation mixing time for reversible chains in terms of the "average conductance" taken over sets of various sizes. Morris and Peres introduced the idea of evolving sets and strengthened the Lovász-Kannan result by extending the results to bound the L^∞ mixing time. Side-stepping conductance (and using a more direct functional approach, along the lines of the works on manifolds by Coulhon, Grigor'yan, and Pittet), Goel, Montenegro, and Tetali recently introduced the notion of "spectral profile" to bound L^∞ mixing time. Standard Cheeger-type inequalities show that the spectral profile bounds imply the conductance bounds. Furthermore, the known estimates on mixing times using Logarithmic Sobolev inequalities and Nash inequalities can also be derived easily with the spectral profile approach.

The strength of the above isoperimetric and spectral profile techniques has further been demonstrated in card-shuffling: A recent breakthrough result of Ben Morris provides an upper bound of d^{44} on the mixing time of the so-called Thorp shuffle on a card-deck of size 2^d, resolving a long-standing conjecture. The result of Morris has already been improved to d^{29} using the new technique of spectral profile. Morris used coupling and evolving sets techniques to prove his result, while a recent survey-style article by Montenegro and Tetali illustrates the derivation of the d^{29} mixing time for the Thorp shuffle using each technique – spectral profile as well as the evolving sets.

Tetali's report also mentioned that progress has been slow on other problems, most notably (random) sampling of contingency tables, which are of interest in statistics. The same is true for acyclic orientations, matroid bases, and Euler tours, all of which are of interest to combinatorialists. The need for new techniques in facilitating a tighter analysis of additional Markov chains such as triangulations of regular polygons and card-shuffling on general graphs has also been made clear.

List of Participants

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Chapter 24

Number Theory Inspired by Cryptography (05w5021)

Nov 05 – Nov 10, 2005

Organizer(s): David Boyd (University of British Columbia), Carl Pomerance (Dartmouth College), Igor Shparlinski (Macquarie University), Hugh Williams (University of Calgary)

Introduction

The **objective** of this workshop was to bring together most active and productive researchers, especially those with expertise in computational number theory and who are willing to share their expertise and also open to working on new topics.

Developments in both number theory and cryptography are vast and quick. However, often lack of contacts and communication between cryptographers and number theorists is an obstacle in achieving significant advances on both sides. We hope that our workshop has been a step towards bridging the gap and will foster new links between both areas.

The program of the workshop contained a number of formal talks. All talks were typically 45 minutes long (some were 30 minutes long) with substantial breaks to allow plenty of time for questions and discussion. Such discussions were of great use for both the speaker and the audience.

Besides the formal program with scheduled talks there was plenty of time for informal discussions which suit more exchange of ideas which are still in the "mid-air" and cannot be put on paper, but which could eventually become very fruitful.

Speakers who presented their research all got very valuable feedback, plus some ideas for further work. In the same time the people in the audience learned some new things. This also continued through the informal conversations and gatherings after the end of the official daily program. Several attendees came from small universities where they are the only computational/algorithmic number theorist, so any chance of personal interaction with others in this area is vital for them. This meeting has already led to many new concrete results with probably many other effects which will resonate in the future.

Although most of the people knew each other by name, not everybody met personally. So the meeting has contributed to establishing a stronger and more diverse research network, which is always valuable.

We believe that the workshop has provided a significant learning experience and exposure to current ideas and trends to younger researchers the early stages of their careers.

Overview of the Field

It is common knowledge that most of the constructions of public key cryptography, and many of the constructions of private key cryptography, are based on number theory. There is however a constantly extending flow in the opposite direction, from cryptography to number theory. Namely many problems and results of intrinsic interest for number theory have been motivated by possible cryptographic applications. These include, but are not limited to smooth numbers, elliptic and hyperelliptic curves, lattices, exponential sums, polynomials over finite fields and many others.

The aforementioned topics do not co-exist independently and separately but weave through each other and lead to very exciting, and often completely unexpected, directions of theoretic research with a great potential for practical applications in the area of information technology.

A very impressive example of such interleaving between various areas is given by recent activities stimulated by the polynomial time primality testing algorithm of by M. Agarwal, N. Saxena and N. Kayal. Follow up works around this algorithm have lead to such important results as:

- an effective version of the Bombieri-Vinogradov theorem;
- a new algorithm for constructing irreducible polynomials in finite fields;
- new lower bounds on the size of finitely generated groups in function fields over a finite field;

and will probably lead to a number of other mathematically rich explorations. Certainly these and many other topics have been discussed at the workshop as well.

Recent Developments and Open Problems

The following topics have recently been actively studied in the literature and certainly also got a lot of attention during the workshop:

- **Studying the distribution of smooth numbers.** This is related to several important algorithms, such as the integer factorization, primality testing and discrete logarithm problems. Less know applications involve attacks on padded RSA signature scheme and on the ElGamal cryptosystem. The topic has been addressed in a talk of Jonathan Sorenson.
- **Studying the structure of the groups of points on elliptic curves.** This is important for better understanding of the present and future potential of elliptic curve cryptography. Same questions for class groups of hyperelliptic curves are of great interest as well. Certainly finding new classes of cryptographically strong (or identifying new types of cryptographically weak) curves is of great importance and interest. The topic has been addressed in talks of Isabelle Dechene, Florian Luca, Kumar Murty, Takakazu Satoh, Renate Scheidler and Edlyn Teske.
- **Fast calculations on elliptic curves and other algebraic structures.** This is important for better understanding of the present and future potential of elliptic curve cryptography. Same questions for class groups of hyperelliptic curves are of great interest as well. Certainly finding new classes of cryptographically strong (or identifying new types of cryptographically weak) curves is of great importance and interest. The topic has been addressed in talks of Tanja Lange.
- **Studying the structure of class groups of algebraic number fields.** In particular, quadratic fields provide very interesting structures when an analogue of the Diffie–Hellman protocol can be executed. This area definitely requires more attention from both mathematicians and practical cryptographers. The topic has been addressed in a talk of Allison Pacelli.
- **Hash functions based on hard number theoretic problems.** Traditionally hash functions are based on various Boolean operations and whose design reminds art more than anything else. Such functions are usually very fast but have no proofs of security behind them, which sometimes leads to such dramatic events of the recent collapse of MD5. Thus since recently hash functions which are based on various algebraic structures have received a lot of attention. Such functions are usually much slower

but admit at least conditional security proofs. The topic has been addressed in talks of Qi Cheng and Kristin Lauter

- **New subexponential attacks on the discrete logarithm problem on elliptic and hyperelliptic curves** Although in generic settings still there are no viable approaches to designing a subexponential algorithm for these problems, in many special cases such attacks exist. One of the very recent approaches was discussed in a talk of Gerhard Frey and Nicolas Theriault.

- **Studying smooth and other special values occurred among group orders of various groups on cryptographic interest.** It is known that "smooth" group orders must be avoided, however the area is lacking rigorous results confirming that this can be achieved. Group orders of elliptic curves over a finite field of q elements $\mathbf{F}_q$ which divide $q^k - 1$ for some "small" k are of great interest too. They lead to elliptic curves which are not suitable for standard Diffie-Hellman protocol but instead are of great values for Weil and Tate pairing based cryptography. The topic has been addressed in talks of Florian Luca and Edlyn Teske.

- **Studying the distribution of various types of polynomials over finite fields.** In particular, this involves obtaining sharp bounds on the number of smooth polynomials is important for the discrete logarithm problem. Certainly, the results of at least the same level of precision as for the integers are expected. Moreover, for polynomials over finite fields, the celebrated Weil result provided a rigorous version of the Riemann Hypothesis and thus one can actually anticipate stronger results. The topic has been addressed in talks of Omran Ahmadi and Qi Cheng.

- **Computational challenges arising in algorithmic number theory and cryptography.** There is an on-going quest for developing new and making already algorithms faster, more portable and better adjusted to already existing hardware and software. Parallelesation is a new trend in this area as well. Recently there have been remarkable achievements in several benchmark problems, such integer factorization, primality testing, computing the number of points on elliptic curves and computing discrete logarithms. The topic has been addressed in talks of Dan Bernstein, Pedro Berrizbeitia, Francois Morain, Oliver Schirokauer and Samuel Wagstaff.

- **Bounds of new exponential sums involving functions of cryptographic interest.** Such bounds may lead to a proving expected pseudorandomness properties of various cryptographic primitives, which can be reformulated in terms of statistical distance, accepted in cryptology. Very often such exponential sums appear as eigenvalues of certain transformations of cryptographic interest and thus obtaining sharp upper bounds on their magnitude becomes the of primal importance. The topic has been addressed in talks of Kristin Lauter and Kumar Murty.

- **Extending the scope on applications of computational number theory.** Finding new surprising areas of applications is always a welcome task. One of such new areas has been outlined in a talk of Denis Charles.

- **Studying multidimensional geometric lattices associated to cryptographic constructions.** Typically it is expected that such lattices behave as a "random" lattices and thus this argument is used to justify the success of the LLL algorithm when applied to such lattices. The underlying philosophy is: *"the vector which we want to find is much shorter than it is usually expected for a lattice of this volume, thus it is very unlikely that there is another nonparallel vector of similar length, thus LLL should find the desired vector"*. Rigorous justification of this principle typically leads to new interesting number theoretic questions and studying system of equations in finite rings and fields. Although there has not been any specialised talk on this topic its main underlying motif could be seen through many workshop talks.

Scientific Progress Made

Most of the participants notice in their emails that this was a very useful workshop with an atmosphere very conducive to advancing research. Also the program was varied, stimulating and interesting, the best part of the meeting was the time we had available for discussions. The meeting happened just a few months before the submission deadline for ANTS-7 (Algorithmic Number Theory Symposium, Berlin, July 2006), which is a major event in the area of computational number theory and cryptography. Many participants got new ideas during this workshop which advanced and improved their follow-up submission to ANTS, for example, see [6, 7, 8, 10].

Besides general discussions which have created a very stimulating environment and generated many new productive ideas (as well as helped to clarify and weed out less viable approaches), the following concrete results have been achieved (this account is based on post-workshop email exachange with the participants).

- William Banks, John Friedlander, Florian Luca and Igor Shparlinski discussed and found an improvement to their joint work with F. Pappalardi [2], which has since then been accepted for Acta Arithmetica. The paper studies the distribution of values of the Carmichael function which apprears in context in computational number theory and cryptography. During these discussions an idea to use our result to denominators of Bernoulli numbers was born and the authors hope to explore this idea in the future.

- William Banks, John Friedlander and Florian Luca worked and made some significant progress on their project on numbers $n \leq x$ without a divisor in a fixed arithmetic progression. This complement a series of results (of various authors) about integers with a divisor in a given interval. The fact that all three co-authors were together for the first time in a long while helped to achieve a breakthrough in that work. In particular, during the meeting one of the very difficult issues in that paper was sorted out. That paper has been finished since then and submitted a couple of months ago. This paper, as many other papers initiated directly or indirectly by this meeting, contains the corresponding acknowledgment of the BIRS hospitality and support.

- Ian Blake and Kristin Lauter had several discussions about hash functions based on elliptic curves and found some interesting possibilities for further collaboration on this topic.

- Motivated by several workshop talks on hash functions based on advanced mathematical structures, Ian Blake and Igor Shparlinski started a joint project investigating the VHS ("Very Smooth Hash") proposed in 2005 by S. Contini, A. K. Lenstra and R. Steinfeld, and which was frequently mentioned at the workshop. Since then Igor Shparlinski visited Ian Blake in Toronto where they continued to work on the VHS. The preliminary version of their results is now available [3].

- Florian Luca and Allison Pacelli started a couple of projects about divisibilities of class numbers of function fields and algebraic number fields. Since then Florian Luca got a visiting position at Williams College for the next academic year (to work with Allison Pacelli). In turn, Allison Pacelli got an AWM travelling grant to visit Florian Luca in Montreal and Mexico. They have worked on these projects and already have an almost final preprint of about 15 pages which they hope to finish soon.

- Florian Luca and Igor Shparlinski discussed the problem of estimating the square free part of linear recurrence sequences. This is also related to estimating the number of quadratic fields generated by square roots of elements of linear recurrence sequences, which would be an analogue of some results of [5]. This project is now in progress and hopefully will be finalised in 2006.

- John Friedlander and Florian Luca discussed a conjecture from [1] related to some combinatorial number theory problem. Florian Luca made some initial progress on their conjecture in Banff and since then he settled this conjecture and submitted the paper.

- Andreas Stein and Hugh Williams had very useful discussions concerning a new method of determining rapidly large scalar multiples of divisors in the Jacobian of a hyperelliptic curve. It was particulary interesting because of possible applications of this method to the problem of fast exponentiation of ideals in real quadratic number fields, a problem of interest in implementing certain cryptographic key exchange protocols. The problems are similar, but are by no means the same; nevertheless, Hugh Williams was able after some time to apply Andreas Stein's idea to an old problem in this area.

- Tanja Lange and Igor Shparlinski finished [9] where several new bounds of exponential sums are given. These bounds imply the uniformity of distribution of some sequences of points on elliptic curves (and in particular can be of interest for pseudorandom number generation and for elliptic curve cryptography). It has already been accepted for the J. of Mathematical Cryptology.

- Gerhard Frey and Tanja Lange finished the paper [8] which has been accepted for presentation at (and publication in the proceedings of) ANTS 2006. The results of this paper have very strong cryptographic motivation and can be used to accelerate several cryptographic protocols.

- After several conversations at the meeting Allison Pacelli and Andreas Stein started a joint research project. Allison Pacelli has been invited by Andreas Stein to Wyoming to give a talk.

- For primality proving for numbers of a certain type, the computation of $\alpha^n \pmod{\nu}$ is required, where α is an element of a ring of integers of a number field or simply of the number field, n is the number to be tested and ν an ideal of the ring of integers, lying over n. The computation is interesting for other problems too. During the Banff meeting, in conversations with Alice Silverberg, Pedro Berrizbeitia learned about some progress that has been done in that area (for example, in the context of the XTR cryptosystem) and, as a consequence, he has looked again at problem that he had considered some years ago, which is to look at that precise equation for a rather specific ideal ν in a cyclotomic field. Pedro Berrizbeitia is hopping to conclude his work and to present it in the meeting in the Fields Institute, from October 31 to November 3 2006.

- Pedro Berrizbeitia and Hugh Williams had very useful discussions concerning the problem of very fast primality testing for numbers that are of cryptographic utility in fast cryptographic signature verification. They have started a joint project on pseudosquares, pseudocubes, and pseudorth-powers. Hugh Williams now has a PhD student doing his thesis on this.

- During the Banff meeting Pedro Berrizbeitia and Florian Luca, exchanged some ideas, as a consequence of this, Florian Luca, will be visiting Pedro Berrizbeitia at the University Simon Bolivar, at Caracas, Venezuela from June 24 to July 9, 2006 to teach a minicourse. Pedro Berrizbeitia and Florian Luca also hope to be able to collaborate on some specific mathematical problem during this visit, and beyond.

- Collaborative efforts between Michael Jacobson, Renate Scheidler and Andreas Stein on cryptosystems based on real hyperelliptic curves has resulted in a paper (currently in preparation) be submitted to a new journal called “Advances in Mathematics of Communications”.

- Collaboration between Michael Jacobson, Yoonjin Lee, Renate Scheidler and Hugh Williams on a function field generalization the CUFFQI algorithm of Shanks for enumerating non-isomorphic cubic fields using infrastructure of real quadratic fieldshas resulted in a paper (currently in preparation) be submitted to Mathematics of Computation in the near future.

- Takakazu Satoh has discovered a gap in his arguments during his talk (which was unscheduled and given on the first day) but he could fix it during the conference. It is quite certain that it would take several weeks if he was not attending the workshop.

- Denis Charles and Kristin Lauter had several very productive discussions with Francois Morain about their computing modular polynomials algorithm [4]. Francois Morain asked about some details and pointed out a variation on that algorithm. He intended to implement this algorithm to test its performance against the approach his student has been using.

- Denis Charles and Kristin Lauter also had useful conversations with Kumar Murty.

- Denis Charles had several illuminating discussions with Florian Lucan and Edelyn Teske regarding embedding degrees of elliptic curves over finite fields.

- Kristin Lauter had several discussions with Gerhard Frey which were very useful to her in advancing another project which is now finished and is to appear in ANTS this year [7].

- Kristin Lauter and Oliver Schirokauer have started a joint project on attacking the ECDLP (Elliptic Curve Discrete Logarithm Problem).
- Alf van der Poorten has learned from conversations with Pedro Berrizbeita about recent work of Pedro colleague Tom Berry, which is of immediate relevance to Alf van der Poorten's current research activity.
- During a lecture of Renate Scheidler, Alf van der Poorten has discovered that her work had strong interaction with his and thus he was able to give her useful information and insights.
- Gary Walsh posed an interesting problem to Alf van der Poorten to which he hope to be able to make a contribution.
- Francois Morain finished the writing of a joint paper with P. Gaudry, "Fast algorithms for computing the eigenvalue in the Schoof-Elkies-Atkin algorithm", which will appear in the Proceedings of ISSAC'06.
- After a series of discussions, Pedro Berrizbeitia invited Igor Shparlinski to give a mini-course on exponential sums at the University Simon Bolivar, at Caracas, Venezuela in 2007 and establish a research program in this direction.

List of Participants

Ahmadi, Omran (University of Toronto)
Banks, William (University of Missouri - Columbia)
Bauer, Mark (University of Calgary)
Bennett, Michael (University of British Columbia)
Bernstein, Dan (University of Illinois)
Berrizbeitia, Pedro (Universidad Simon Bolivar)
Blake, Ian (University of Toronto)
Bleichenbacher, Daniel (Lucent Technologies)
Boyd, David (UBC)
Bruin, Nils (Simon Fraser University)
Charles, Denis (Microsoft Research)
Cheng, Qi (University of Oklahoma)
Déchène, Isabelle (University of Waterloo)
Frey, Gerhard (Institut fr Experimentelle Mathematik, Universität Gesamthochschule Essen)
Friedlander, John (University of Toronto)
Jacobson, Michael (University of Calgary)
Lange, Tanja (Technische Universiteit Eindhoven)
Lauter, Kristin (Microsoft Research)
Lee, Yoonjin (Simon Fraser University)
Luca, Florian (Universidad Nacional Autonoma de Mexico)
Morain, Francois (Ecole Polytechnique (Paris))
Murty, Kumar (University of Toronto)
Pacelli, Allison (Williams College)
Satoh, Takakazu (Tokyo Institute of Technology)
Scheidler, Renate (University of Calgary)
Schirokauer, Oliver (Oberlin College)
Shparlinski, Igor (Macquarie University)
Silverberg, Alice (University of California at Irvine)
Sorenson, Jonathan (Butler University)
Stein, Andreas (University of Wyoming)
Teske, Edlyn (University of Waterloo)

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[9] T. Lange and I. E. Shparlinski, Distribution of some sequences of points on elliptic curves', *J. Math. Cryptology*, (to appear).

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Chapter 25

Flavors of Groups (05w5105)

November 17 – November 22, 2005

Organizer(s): Mladen Bestvina (University of Utah), Jeff Brock (Brown University), Jon Carlson (University of Georgia), Persi Diaconis (Stanford University), Hugo Rossi (Mathematical Sciences Research Institute)

Overview of the Field

This workshop brought together researchers working on algebraic, analytic, combinatoric, geometric and topological aspects of group theory in order to exchange techniques and ideas in preparation for the full year emphasis at MSRI in the academic year 2007-2008 on all these aspects of groups. The four particular topics present (representing the four individual semester programs at MSRI) were:

Geometric Group Theory

This is a relatively young field, with older and deeper roots in the study of groups from combinatorial and topological perspectives. In the mid 1980s, spurred by ideas of Cannon and Gromov, group theorists began to pay attention to the geometric structures which cell compexes can carry. This attention shed light on the earlier combinatorial and topological investigations, and stimulated innovative ideas which have been developing at a rapid pace: Gromov hyperbolicity, Bestvina-Brady Morse theory, splittings and actions on trees, rapid decay and the Baum-Connes conjecture.

Kleinian Groups

The study and application of recent advances in the classification of hyperbolic 3-manifolds (the solution of the tameness and ending lamination conjectures of Marden and Thurston) can lead to a better understanding of the geometry of closed hyperbolic 3-manifolds. This work also touches on Teichmuller theory, and questions concerning billiards and flows on Moduli space. Many of these avenues are potentially very fruitful for further research and synthesis between, up to now, largely disparate fields.

Combinatorial Representation Theory

There is a productive interplay between combinatorics, geometry, finite groups, Lie theory and hyperplane arrangements in the applications to representation theory. Examples are: (1) the use of symmetric functions and Hecke algebras in the modular representation theory of finite groups of Lie type, (2) the use of braid groups and finite dimensional algebras in the study of categories of highest weight modules and (3) the use of tableaux, crystals, and the path model in the study of representations of algebras with triangular decomposition.

Representation Theory of Finite Groups

Current research centers on many open questions, particularly regarding representations over the integers or rings of positive characteristic. Brauer developed block theory to better understand such representations, and in the last few years there have been many exciting new conjectures concerning correspondence of characters and derived equivalences of blocks. Topics such as p-local groups, group actions on finite complexes and homotpy representations blend algebra and topology in novel and productive ways.

With four talks a day there was plenty of time for informal discussion and interaction among the various areas of interest. On Sunday evening there was a meeting of the MSRI program organizers, initiating integrated planning for the MSRI intensive year. In all, there were 32 participants, of whom 7 were women. The participants (speakers are asterisked), with their affiliations were:

List of Participants

Barcelo, Helene (Arizona State University)
Benson, David (University of Aberdeen)
Brock, Jeffrey (Brown University)
Bromberg, Kenneth (University of Utah)
Canary, Richard (University of Michigan)
Cannon, Jim (Brigham Young University)
Carlson, Jon (University of Georgia)
Charney, Ruth (Brandeis University)
Diaconis, Persi (Stanford University)
Erdmann, Karin (University of Oxford)
Grodal, Jesper (Chicago/Copenhagen)
Hamenstaedt, Ursula (UniversitŁt Bonn)
Kamnitzer, Joel (American Institute of Mathematics / Massachusetts Institute of Technology)
Kerckhoff, Steven (Stanford University)
Kleshchev, Alexander (University of Oregon)
Malle, Gunter (Universitaet Kaiserslautern)
Masur, Howard (University of Illinois)
McCammond, Jon (University of California–Santa Barbara)
Mosher, Lee (Rutgers University-Newark)
Pettet, Alexandra (University of Chicago)
Postnikov, Alex (MIT)
Reid, Alan (University of Texas at Austin)
Rickard, Jeremy (University of Bristol)
Robinson, Geoffrey (University of Aberdeen)
Rossi, Hugo (Mathematical Sciences Research Institute)
Sapir, Mark (Vanderbilt University)
Souto, Juan (University of Chicago)
Thiem, Nat (Stanford University)
Valette, Alain (Universite de Neuchatel)
Vinroot, Christopher Ryan (University of Arizona)
Vogtmann, Karen (Cornell University)

Chapter 26

Regulators II (05w5032)

December 10 – December 15, 2005

Organizer(s): James Lewis (University of Alberta), Victor Snaith (University of Sheffield)

History of the subject

A regulator is a generalization of the logarithm. Dirichlet used the logarithm to define a map from the multiplicative group of a ring of algebraic integers to a real vector space. Then Dirichlet proved the celebrated analytic class number formula which relates all the important number theoretic invariants of the number field to the covolume of the Dirichlet regulator. Since the 1960's Dirichlet's fundamental discovery has been found potentially to occur elsewhere in number theory, in algebraic geometry, in class field theory, in algebraic K-theory, in the theory of algebraic cycles and motives, and in Hodge theory. Regulators come in many different forms, according to the context.

For instance, the Borel regulator is the higher-dimensional analogue of the Dirichlet regulator, considered as a map on algebraic K-theory in dimension one. On the other hand, in Riemann surface theory, the regulators might involve abelian integrals and Jacobians, extending the ideas of the 19th century analytic number theorists and geometers. Generally speaking, in its current incarnation, a regulator is a map from the algebraic K-theory of an algebraic variety to a suitable cohomology theory such as étale cohomology or Deligne cohomology.

The subject of regulators is a highly intricate field that gives and takes from a number of core fields in mathematics, such as algebraic and analytic geometry, and arithmetic geometry, Hodge theory, mathematical physics, algebraic and analytic number theory, algebraic K-theory, and so on. For instance, one of the simplest examples of a regulator complex projective geometry is that of the cycle class map from the so-called group of analytic subvarieties of a given dimension to standard singular cohomology. The celebrated Hodge conjecture is a statement about the image of this cycle class map.

Purpose of this meeting

A meeting of this type allowed the various groups of experts viewing the subject of regulators either arithmetically, topologically (as in Voevodsky's work, or as well as in terms of Lawson's homology), or transcendentally (i.e. Hodge theory) to compare notes. For this reason the topic of Regulators was particular ripe for a conference at that time. In May 1998 there was an Oberwolfach meeting on regulators (organised by Bloch, Kolster, Schneider and Snaith) which resulted in some of the advancements mentioned above. The Oberwolfach workshop was generally regarded as a real success and the current organisers felt that this meeting was an appropriate sequel.

Relevancy, objectives and recent developments in the subject

The recent work by Voevodsky in constructing the "motivic cohomology" theory as suggested by Grothendieck, and the resulting homological machinery associated to his approach, and its subsequent use to solve the long-standing Milnor conjecture, resulted in his winning the Fields medal in 2002. This provides an infusion of new and powerful ideas in the study of regulators. Indeed Voevodsky proved that his definition of motivic cohomology agrees with two other versions already used in regulator theory. Thus one can arguably make the case that regulators are maps (sometimes called "realizations") from Voevodsky motivic cohomology, albeit still hard to compute, to the "more computable" cohomology theories (Deligne, étale, absolute Hodge, etc.). It is often the case that regulator maps can have highly nontrivial kernels and images, which leads to higher order invariants associated motivic cohomology groups. This is generally the case if one works with varieties over the complex numbers, or even function fields of transcendence degree 1 over the rational numbers.

A case at point is the conjectured "Bloch-Beilinson" filtration, and the resulting "higher regulators" that are associated to this filtration. Working over number fields, one expects a rather different situation when it comes to the kernels of regulator maps. Another case at point is the Bloch-Beilinson conjecture on the injectivity (modulo torsion) of the Abel-Jacobi map for smooth varieties over number fields. What is the status of the conjectures related to the images and kernels of regulators for varietes over number fields, as well as over the complex numbers?

There are the camps of arithmetists, "K-theory/motivic topologists" and transcendental algebraic geometers who study these problems from different angles. It is often the case that real progress in one camp is not fully understood in the other camp. Two examples of related problems that involve the various camps are the celebrated Hodge and Tate conjectures. A consequence of some fruitful interactions between the arithmetists and transcendental geometers on regulators on algebraic varieties has led to the fascinating development of "arithmetical variations of Hodge structures" (P. Griffiths, M. Green, S. Saito, M. Saito, et al). A case at point is the highly successful NATO Advanced Study Institute on the Arithmetic and Geometry of Algebraic Cycles, in Banff (1998), where these issues among the various camps became transparent. This led to a sequel 3 week conference on The Arithmetic, Geometry and Topology of Algebraic Cycles, held in Morelia Mexico, in the summer of 2003. At that time, V. Voevodsky's recent proof of the Milnor conjecture, as well as the Bloch-Kato conjecture, was being discussed. This was a major milestone, which eventually led to Voevodsky receiving the Fields medal.

Organizational details and the Banff setting

Except for a scheduled free afternoon, noon departure on the final day, and an extra lecture at night, all lectures were planned during the day (a total of 5 daytime 1 hour talks), so as to encourage research collaboration at night. It is fair to say that this conference was an enormous success. The atmosphere was "electric", with a lot of interaction between speakers and audience, as well as fruitful discussions during coffee breaks and at nights. The quiet scenic Banff backdrop provided the perfect setting for research. Many of the participants at this workshop are familiar with the European counterpart in Oberwolfach Germany. The general concensus is that BIRS facility is superior, not only in the capacity of offering better computerized facilities, with printer and electronic library, but with a nicer scenic backdrop and a bustling town within walking distance. The support staff at BIRS performed their duties very professionally.

Scientific merit of the talks

The talks can be broken down into a number of distinct areas under the umbrella of "regulators".

(i) Motives. As K-theory is central to the subject of regulators, it permits an interpretation of everything in terms of motives. Several outstanding talks in this direction were presented by S. Bloch, H. Esnault and M. Hanamura.

(ii) Topological. A natural cohomology theory associated to "equivariant" Chow groups is the notion of Bredon cohomology. This video taped lecture was presented by Paulo Lima-Filho. Bredon cohomology is considered more sensitive than ordinary singular cohomology. Recently, this has led to a development of a "Bredon" version of Deligne cohomology for real varieties.

(iii) Polylogarithms. Based on joint work with A. Goncharov and A. Levin, and connected to the work of D. Zagier, H. Gangl presented recent developments on the subject of multiple polylogarithms associated to algebraic cycles.

(iv) Transendental methods. Using the techniques of Hodge theory, were several talks on the following. M. Asakura presented his results for elliptic surfaces, in support of a conjecture of Beilinson that generalizes the classical Hodge conjecture. P. Brosnan, in his joint work with G. Pearstein, presented results on the asymptotic nature of a variational height pairing, in terms of degenerating Hodge structures. J. Lewis presented a normal function interpretation of a candidate Bloch-Beilinson filtration on higher Chow groups. From a different perspective, there was the talk given by K. Kimura. H. Gillet presented his results towards a sheaf theoretic construction of arithmetic Chow groups.

(v) Number theory. That part of the subject of regulators connected to number theory, p-adic methods and L-functions, connections to the Borel regulator and Stark's conjecture was presented by R. de Jeu, W. Raskind, Z. Wojtkowiak, V. Maillot and V. Snaith.

(vi) Arithmetic methods. Those methods in the subject of regulators dealing with ell-adic cohomology, rigidity, varieties over finite fields, were presented by T. Geisser, A. Langer, A. Rosenschon, and S. Saito.

Summary

The subject of Regulators is a highly evolved and intrintic subject, involving some of the finest minds in the world of mathematics, including many Fields medalists. It is a subject that is expanding at an accelerated rate, and has attracted and inspired a new generation of promising young researchers.

By any reasonable measure, this conference, being a sequel to an Oberwolfach conference on Regulators held in 1998, was an outstanding success. There is certainly a desire and need for another sequel to this conference, most likely entitled, "Regulators III", to be held sometime and place in the not too distant future.

List of Participants

Asakura, Masanori (Kyushu University)
Bloch, Spencer (University of Chicago)
Brosnan, Patrick (University of British Columbia)
Buckingham, Paul (University of Sheffield)
Choo, Zacky (University of Sheffield)
Cisneros-Molina, Jose Luis (Mathematics Institute Cuernavaca, UNAM)
Colwell, Jason (University of California, San Diego)
de Jeu, Rob (Vrije Universiteit)
del Angel, P. Luis (Center of Investigations in Mathematics)
Elizondo, E. Javier (Universidad Nacional Autonoma de Mexico)
Esnault, Helene (Universitaet Duisburg-Essen)
Gangl, Herbert (Max Planck Institute for Mathematics)
Geisser, Thomas (University of Southern California)

Gillet, Heri (University of Illinois at Chicago)
Hanamura, Masaki (Tohoku University)
Joshua, Roy (Ohio State University)
Kerr, Matt (University of Durham)
Kimura, Kenichiro (University of Tsukuba)
Kolster, Manfred (McMaster University)
Langer, Andreas (University of Exeter (UK))
Lewis, James (University of Alberta)
Lima-Filho, Paulo (Texas A&M University)
Maillot, Vincent (Institut de Mathematiques de Jussieu)
Mueller-Stach, Stefan (University of Mainz)
Raskind, Wayne (University of Southern California)
Roessler, Damian (CNRS, Institut de Mathematiques de Jussieu)
Rosenschon, Andreas (SUNY at Buffalo)
Saito, Shuji (University of Tokyo)
Snaith, Victor (University of Sheffield)
Spiess, Michael (Univ. Bielefeld)
Sreekantan, Ramesh (Tata Institute of Fundamental Research)
Weiss, Al (Department of Mathematical&Statistical Sciences, University of Alberta)
Wojtkowiak, Zdzislaw (University of Nice)
Yee, Wai Ling (University of Alberta)
Zigmond, Robin (University of Durham, UK)

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The following Conference Proceedings bests illustrate the current developments in this subject.

• Beilinson's Conjectures on Special Values of L-Functions, Edited by M. Rapoport, N. Schappacher and P. Schneider, Perspectives in Mathematics, Vol. 4, 1988.

• The Arithmetic and Geometry of Algebraic Cycles, Proceedings of the NATO Advanced Study Institute, June 7-19, 1998 in Banff, Alberta. Edited by B. Gordon, J. D. Lewis, S. Müller-Stach, S. Saito, and N. Yui. NATO Science Series, Vol. 548, Kluwer Academic Publishers, 2000.

• The Arithmetic and Geometry of Algebraic Cycles, Proceedings of CRM Summer School, June 7-19, 1998 in Banff, Alberta. Edited by B. Gordon, J. D. Lewis, S. Müller-Stach, S. Saito, and N. Yui. CRM Proceedings and Lecture Notes, American Mathematical Society, Vol. 24, 2000.

Two-day Workshop Reports

Chapter 27

Second Northwest Functional Analysis Symposium (05w2089)

March 17–19, 2005

Organizer(s): Douglas R. Farenick (University of Regina), Marcelo Laca (University of Victoria), Michael Lamoureux (University of Calgary), Volker Runde (University of Alberta)

Functional analysis grew out of attempts—in the first half of the twentieth century—to find a conceptual framework for a wide range of analytic phenomena concerning algebraic systems of functions, such as existence and uniqueness of solutions to differential and integral equations. The discipline is well established at universities in Western Canada, with two large groups working at the Universities of Alberta and Victoria, respectively, and smaller groups elsewhere (Calgary and Regina, for instance).

Research in functional analysis in Western Canada is carried out in the following areas:

- Abstract harmonic analysis (Tony Lau and Volker Runde, both at Alberta);
- Banach space theory and geometric functional analysis (Sasha Litvak, Nicole Tomczak-Jaegermann, and Vaclav Zizler, all at Alberta);
- Operator algebras (Marcelo Laca, John Phillips, and Ian Putnam, all at Victoria, Berndt Brenken and Mike Lamoureux, at Calgary, and Martin Argerami and Juliana Erlijman, both at Regina);
- Operator theory (Doug Farenick at Regina, Ahmed Sourour at Victoria, and Vladimir Troitsky at Alberta).

The aims of the workshop were twofold: firstly, to enable researchers from a large geographical area to stay in touch with developments in the general field, but outside their respective areas of specialization, and secondly, to provide a forum for young researchers—junior faculty, postdocs, and graduate students—to present their results to a wider audience. For the second reason, five of the 14 talks at the workshop were given by graduate students, and four by postdocs.

Besides researchers in functional analysis from Western Canada, there were also participants whose research was not really in functional analysis, but in an area sufficiently close. For instance, Karoly Bezdek (Calgary) spoke about a topic in convex geometry, which has connections with geometric functional analysis, Bahram Rangipour (Victoria) presented results in non-commutative geometry, a discipline with many connections to operator algebras, and Alex Brudnyi was dealing with Lipschitz functions between metric spaces. As there turned out to be sufficient space at BIRS in the end, we were also able to invite people from Manitoba, and Ebrahim Samei (Manitoba) presented his results on hyper-Tauberian Banach algebras from his (soon to be defended) PhD thesis.

As indicated by the title of the workshop, there was a First Northwest Functional Analysis Symposium. It was held at BIRS in 2003 and organized by Tony Lau, Mike Lamoureux, Ian Putnam, Nicole Tomczak-Jaegermann. At the present workshop, the possibility of a third meeting in the series was discussed, and the general attitude was positive. A successor meeting next year would probably be premature, but two years should generated enough new results and sufficient turnover in the postdoc and graduate student population to justify a Third Northwest Functional Analysis Symposium.

List of Participants

Argerami, Martin (University of Regina)
Bezdek, Karoly (University of Calgary)
Binding, Paul (University of Calgary)
Bisztriczky, Ted (University of Calgary)
Brenken, Berndt (University of Calgary)
Brudnyi, Alex (University of Calgary)
Desaulniers, Shawn (University of Alberta)
Erlijman, Juliana (University of Regina)
Farenick, Doug (University of Regina)
Goncalves, Daniel (University of Victoria)
Husain, Ali-Amir (University of Alberta)
Laca, Marcelo (University of Victoria)
Laflamme, Claude (University of Calgary)
Lamoureux, Michael (University of Calgary)
Lau, Anthony To-Ming (University of Alberta)
Litvak, Alexander (University of Alberta)
Manjeghani, S. Mahmoud (University of Regina)
Nikolaev, Igor (University of Calgary)
Papish, Vlad (University of Victoria)
Phillips, John (University of Victoria)
Pivovarov, Peter (University of Alberta)
Pollock, Dan (University of Victoria)
Putnam, Ian (University of Victoria)
Rangipour, Bahram (University of Victoria)
Reznikoff, Sarah (Reed College)
Runde, Volker (University of Alberta)
Samei, Ebrahim (University of Manitoba)
Sourour, Ahmed Ramzi (University of Victoria)
Spurny, Jiri (University of Alberta)
Starling, Charles (University of Victoria)
Tcaciuc, Adi (University of Alberta)
Tomczak-Jaegermann, Nicole (University of Alberta)
Troitsky, Vladimir (University of Alberta)
Uygul, Faruk (University of Alberta)
Whittaker, Michael (University of Victoria)
Zhang, Yong (University of Manitoba)

Chapter 28

BIRS 2005 Math Fair Workshop (05w2608)

April 21–23, 2005

Organizer(s): Ted Lewis (University of Alberta), Andy Liu (University of Alberta)

This was the third BIRS Math Fair, previous ones being held in the spring of 2003 and 2004. The focus of these workshops was Mathematics Education, and the participants were teachers and educators elementary schools, junior high schools, colleges and universities, and also people from other institutions and organizations that have a deep interest in Mathematics Education.

As with the two previous workshops, the purpose of this workshop was to help teachers learn how to run a successful math fair, to exchange information about math fairs, and to put the members of this diverse group in contact with each other. The deeper purpose is to change the mathematical culture in the classroom, and we believe that this is beginning to happen. For the most part, the math fairs have been held in Alberta. The BIRS math fair workshops have helped in spreading the word about the success of our type of math fair (which is radically different from a traditional science fair) and now such math fairs have been held several provinces in Canada, in some states in the US, in Sweden, and reports have been received that a math fair based on our principles has been held in Africa.

As just one example of the local effect of this year's BIRS math fair workshop, the Edmonton Catholic School Board is involving a large number of schools in presenting math fairs in the 2005/2006 school year. Schools in other districts are doing similar things, and teachers have reported evidence that the math fair has changed classroom attitudes to the extent that students' success rates in mathematics have dramatically increased.

List of Participants

Borowiecki, Helen (St. Vincent School)
Boychuk, Halia (Bluequills College)
Campbell, Cathy (Talmud Torah School)
Cannon, Jane (University College of the Fraser Valley)
Carlson, Scott (Strathmore High School)
Chasse, Danielle (Our Lady of the Prairies)
Conroy, Heather (St. Basil School)
Dammann, Randy ()
Danchuk, Jody (Olympic Heights Elementary)
Estabrooks, Manny (Red Deer College)
Ferris, Con (Red Deer College)
Friesen, Sharon (University of Calgary)

Gelasco, Lisa (Annunciation)
Geretschlaeger, Robert (BRG Kepler, Graz/Austria)
Girvan, Doug (Red Deer College)
Gordon, Christie (Olympic Heights Elementary)
Griffiths, Christine (Muriel Martin School)
Hansen, Margaret (Leo Nickerson School)
Hohn, Tiina (Grant MacEwan College)
Holloway, Tom (University of Alberta)
Hubbard, Barb (Keenooshayo School)
Isaac, Vince (Annunciation)
Jones, Daryl (St Mary School)
Keanie, Marlene (Keenooshayo School)
Kuntz, Lisa (Wellington Jr High School)
Lagu, Indy (Mount Royal College)
Lee, Jennifer (Edmonton Catholic Schools)
Lewis, Ted (University of Alberta)
Liu, Andy (University of Alberta)
Livingstone, Kate (Olympic Heights School)
Mitchell, Shirley (PIMS)
Pouliot, Leanne (St. John Bosco)
Rice, Karla (St. Paul School)
Smart, Brenda (Keenooshayo School)
Sun, Wen-Hsien (Chiu Chang Mathematics Education Foundation)
Thomas, Elise (Olympic Heights School)
Thompson, Tanya (Ontario Schools)
Trask, Vanessa (Wellington School)
Yarovenko, Boyan (St. Martin School)
Yarovenko, Halia (St. Vincent School)

Chapter 29

Dark Side of Extra Dimensions (05w2041)

May 12–14, 2005

Organizer(s): Valeri P. Frolov (University of Alberta)

The idea that the spacetime may have more than four dimensions is very old. Starting with works of Kaluza [1] and Klein [2], higher dimensional models were used to unify gravity with other fields. In more recent time, it was demonstrated that the string theory, which is often called a theory of everything, requires higher dimensions for its consistency. Models with the spacetime with large extra dimensions were recently proposed in order to solve the hierarchy problem, that is to explain why the gravitational coupling constant is much smaller than the coupling constants of other physical interactions. In such models, our 4-dimensional spacetime is described by a 4-dimensional brane (submanifold) embedded into a higher dimensional (bulk) space. Particles and fields (except gravity) propagate within the brane, while the gravity can propagate in the bulk space. These new concepts of higher dimensional physics have a number of interesting applications in modern cosmology and theory of gravity. At the same time they require developments of the theoretical and mathematical tools to address many new important questions. At the ”Dark Side” workshop new results and open questions in this fast developing field were discussed.

One of the most important questions is to analyze how the gravitational theory is modified in the presence of extra dimensions. In the study of the Einstein equations in the 4-dimensional spacetime several powerful mathematical tools were developed, based on the spacetime symmetry, algebraical structure of spacetime, internal symmetry and solution generation technique, global analysis, and so on. At our workshop there was discussion and concrete proposal, how to develop some of these methods to higher dimensional spacetime.

Many exact solutions of the Einstein equations in 4-dimensional case were obtained by algebraic methods based on the Petrov classification. At the workshop it was proposed and discussed the generalization of the Petrov classification to higher dimensional case. It was demonstrated that the robust classification into Petrov classes can be done in arbitrary number of dimensions [1]. At the same time, the number of different degenerate subclasses within each of the Petrov class depends on the number of spacetime dimensions. To classify these subclasses in higher dimensions is much more sophisticated problem than in 4-dimensional case.

Another problem which was discussed at the workshop is an existence and properties of different ”black objects” in higher dimensions. These objects are generalization of 4-dimensional black hole solutions. According to the definition, a black hole is an (asymptotically flat) spacetime with non-trivial causal structure. Black hole boundary is an event horizon, a 3-dimensional surface which separates a spacetime region which can be ”seen” from infinity from an ”invisible” region. Under physically reasonable conditions, in 4 dimensions the horizon has the topology of $S^2 \times R^1$. Moreover, ”uniqueness theorems” were proved, which guarantee that for given value of global parameters (mass, angular momentum, and charge) the stationary solutions describing black holes are unique. Recently it was demonstrated that the uniqueness theorems are not valid if the number of spacetime dimensions is greater than 4 [4]. Higher dimensions open room for a variety of dark objects, which are natural generalizations of 4-dimensional black holes. Main difference between these dark objects is the topology of their horizons.

One of the problems which was discussed at the workshop is stability of higher dimensional dark objects and possible transitions between them. Gregory and Laflamme [6] described a particular mechanism of the instability of higher dimensional string, but what is a final state of a decaying dark string is still an open question. At the workshop there were presented results of the numerical simulations of decaying dark strings [7]. Unfortunately these results do not allow one to resolve dynamics and final state of this process. There exist evidences in favor that the black-string–black-hoole transitions may be similar to critical phenomena and the very transition from a black string to a black hole phase may have similarity with the critical gravitational collapse phenomena [8]. Another important connected problem which was discussed at the workshop is possible instability of rapidly rotating black holes and black rings [5]. This area (stability of higher dimensional dark objects and possible transitions between them) is developing very fast and for its progress developed mathematical tools are required.

Another subject which was in the focus of the workshop was study of exact solutions of higher dimensional Einstein equations. Two new families of solutions were presented and discussed at the workshop. One of them is a generalization of Mayers-Perry metrics for higher dimensional black holes to the case when there a non-vanishing cosmological constant [9]. Another new set of solutions describes the gravitational field of spinning relativistic objects (gyratons) in a spacetime with arbitrary number of dimensions [10]. An interesting property of the latter solutions, that the non-linear system of Einstein equations is effectively reduces to two linear set of equations in a flat spacetime. By solving these linear equations, one can generate a solutions of the non-linear problem.

One of the reasons why the higher dimensional theories becomes so popular recently is a possibility that in the presence of extra dimensions one can expect creation of mini black holes in future collider and cosmic ray experiments. At the workshop there was given a detailed overview of the corresponding results and were formulated concrete physical problems which are to be solved for better understanding of such processes [11].

To summarize, the workshop gave very nice view of the state of art in the higher dimensions physics and mathematics of dark objects. It has very enthusiastic support and many of participants proposed to organize again a workshop on a similar subject in future.

List of Participants

Ahmadi, Morteza (University of Lethbridge)
Barrabes, Claude (University of Tours, France)
Choptuik, Matt (University of British Columbia)
Coley, Alan (Dalhousie University)
Das, Saurya (University of Lethbridge)
Fabbri, Alessandro (Universitá di Bologna, Italy)
Frolov, Valeri (University of Alberta)
Frolov, Andrei (Stanford University)
Gegenberg, Jack (University of New Brunswick)
Gergely, Laszlo (University of Szeged)
Grimard, Lee (University of Alberta)
Hobill, David (University of Calgary)
Husain, Viqar (University of New Brunswick)
Jacobson, Ted (University of Maryland)
Kang, Gungwon (Korea Institute for Advanced Study)
Kim, Sang Pyo (Kunsan National University)
Kol, Barak (Hebrew University)
Kunstatter, Gabor (University of Winnipeg)
Kunzle, Hans-Peter (University of Alberta)
Lake, Kayll (Queen's University)

Landsberg, Greg (Brown University)
Lee, Hyun Kyu (Hanyang University)
Mann, Robert (University of Waterloo)
Myers, Robert (McGill University)
Pavluchenko, Sergey (University of Alberta)
Peet, Amanda (University of Toronto)
Schleich, Kristin (University of British Columbia)
Seahra, Sanjeev (University of Portsmouth)
Solodukhin, Sergey (International University Bremen, Germany)
Sorkin, Evgeny (Racah Institute of Physics)
Witt, Don (University of British Columbia)
Witten, Louis (University of Cincinnati)
Woolgar, Eric (University of Alberta)
Zelnikov, Andrei (University of Alberta)

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[9] G.W. Gibbons, H. Lu, D. N. Page, and C.N. Pope, Phys.Rev.Lett.93:171102,2004 e-Print Archive: hep-th/0409155.

[10] V.P. Frolov, D.V. Fursaev, Phys.Rev.D71:104034,2005; e-Print Archive: hep-th/0504027; V. P. Frolov, W.r Israel, and A. Zelnikov, e-Print Archive: hep-th/0506001, 2005.

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Chapter 30

Convex and Abstract Polytopes (05w2037)

May 19–21, 2005

Organizer(s): Ted Bisztriczky (University of Calgary), Egon Schulte (Northeastern University), Asia Ivić Weiss (York University)

The rapid development of polytope theory in the past thirty years has resulted in a rich theory featuring an attractive interplay of several mathematical disciplines. The 2-day Workshop was evidence that polytope theory is very much alive and is the unifying theme of a lot of research activity.

The Workshop provided a much desired opportunity to share recent developments and emerging directions on geometric, combinatorial, and abstract aspects of polytope theory. We had twenty-nine official participants (among them seven women, two graduate students, and many junior faculty), plus a number of graduate student participants not officially registered. With few exceptions, the participants came from North-America. It is noteworthy that the last major meeting on convex and abstract polytopes was the NATO Advanced Study Institute on "Polytopes - Abstract, Convex and Computational" in 1993 at Scarborough, Ontario.

The Workshop focused on two overlapping directions of research,

- the classical theory of convex polytopes (see [2, 4, 5]), and
- the more recent theory of abstract polytopes (see [1, 3]).

The program featured three invited 50-minute lectures and ten 20-minute talks. For convex polytopes, there was an attractive mix of talks about the combinatorial theory (concerning the numbers of faces of different dimensions, the relations among various facial structures, and generalizations such as matroids, oriented matroids, and posets), and the metrical theory (the convex-geometric study of volumes, surface areas, mixed volumes, angles, and projections and sections). One of the major themes to crystallize during the Workshop was the necessity and importance of constructing new classes of polytopes. For abstract polytopes, most talks focused on polytopes with various degrees of combinatorial or geometric symmetry (regular, chiral, or equivelar polytopes, and their geometric realization theory), as well as the structure of their symmetry groups or automorphism groups (reflection groups, Coxeter groups, and C-groups, and their representation theory).

The 2-day Workshop at BIRS was followed by a *Polytopes Day in Calgary* at the University of Calgary on Sunday, May 22, 2005, with two invited 50-minute lectures and five 20-minute talks, as well as two state of the art discussions (problem sessions), one on convex polytopes and one on abstract polytopes.

Both Workshops were very favorably received by the participants and were viewed as a success. In particular, they prompted collaboration among participants with several papers as outcome.

List of Participants

Bayer, Margaret (University of Kansas, Lawrence)
Bezdek, Karoly (University of Calgary)
Bisztriczky, Ted (University of Calgary)
Bracho, Javier (National University of Mexico)
Burgiel, Heidi (Bridgewater State College)
Csikos, Balazs (Eotvos University, Institute of Mathematics)
Dawson, Robert J.M. (Saint Mary's University)
Dinh, Thi (University of Calgary)
Edmonds, Allan (Indiana University)
Erdahl, Robert (Queens University)
Finbow-Singh, Wendy (Acadia University)
Gavrilova, Marina (University of Calgary)
Hartley, Michael (University of Nottingham (Malaysia))
Heppes, Aladar (Renyi Institute)
Herman, Allen (University of Regina)
Hubard, Isabel (York University)
Johnson, Norman W. (Wheaton College)
Lawrence, Jim (George Mason University)
Lee, Carl (University of Kentucky)
Ling, Joseph (University of Calgary)
Martini, Horst (Technische Universitaet Chemnitz)
Monson, Barry (University of New Brunswick)
Pellicer Covarrubias, Daniel (National University of Mexico)
Rybnikov, Konstantin (University of Massachusetts (Lowell))
Schmidt, Laura (University of Wisconsin-Stout)
Schulte, Egon (Northeastern University)
Soltan, Valeriu (George Mason University)
Weiss, Asia (York University)
Williams, Gordon (Moravian College)

Bibliography

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[2] G.Kalai and G.Ziegler, *Polytopes – Combinatorics and Computation*, Birkhauser, 2000.

[3] P.McMullen and E.Schulte, *Abstract Regular Polytopes*, Cambridge University Press, 2002.

[4] J.Richter-Gebert, *Realization Spaces of Polytopes*, Springer-Verlag, 1996.

[5] G.Ziegler, *Lectures on Polytopes*, Springer-Verlag, 1994.

Chapter 31

Computer Science Chairs Meeting (05w2602)

June 9 – 11, 2005

Organizer(s): Gord McCalla (University of Saskatchewan), Ken Barker (University of Calgary)

The Department Heads/Chairs of Computer Science meet annually to share ideas, discuss problems facing the community, and set directions that are in the best of interest of all Computer Science Departments across the country. Each year the venue changes to encourage those from various regions to attend the meeting even if they are challenged financially. The 2005 meeting was hosted by the University of Calgary and BIRS generously offered to host the meeting as a part of their series in Banff.

Unlike other BIRS workshops the purpose of this meeting was primarily administrative rather than being focused on addressing a particular research question. Thus, this report is intended to provide a very brief indication of the kinds of discussions undertaken at the meeting. As a result of the administrative nature of the meeting, some of the discussions were also somewhat confidential and we are unable to report details of the actual discussions undertaken.

The meeting consisted of a wide range of topics including:

- A Survey of the various Departments
- Research Challenges facing the discipline
- Graduate student funding and education
- NSERC grants and funding issues
- Software Engineering
- Computer Science Department Accreditation
- Establishing Awards for top ranked students
- Development of committees to initialize various initiatives

The details of each of these discussions are not included here but if more detail about the meetings is required interested reader can contact Gord McCalla or Ken Barker.

List of Participants

Banzhaf, Wolfgang (Memorial University of Newfoundland)
Barker, Ken (University of Calgary)
Bassett, Paul (the Canadian Information Processing Society)
Bate, John (University of Manitoba)
Bauer, Michael (University of Western Ontario)
Bhavsar, Virendra (University of New Brunswick)
Boutilier, Craig (University of Toronto)
Butz, Cory (University of Regina)
Cercone, Nicholas (Dalhousie University)
Chau, Siu-Cheung (Wilfred Laurier University)
Chiasson, Julien (Universit de Moncton)
Cribb, Peter (York University)
Delgrande, James (Simon Fraser University)
Farmer, William (McMaster University)
Greer, Jim (University of Saskatchewan)
Hackborn, Bill (Augustana University of Alberta)
Haque, Waqar (University of Northern British Columbia)
Holzmann, Wolf (University of Lethbridge)
Howe, Douglas (Carleton University)
Hughes, David (Brock University)
Hurley, Richard T. (Trent University)
Lam, Clement (Concordia University)
Lavoie, Luc (Universit de Sherbrooke)
Lethbridge, Timothy (University of Ottawa)
Lingras, Pawan (Saint Mary's University)
Martin, Pat (Queens University)
McCalla, Gord (University of Saskatchewan)
Meunier, Jean (Universit de Montral)
Mineau, Guy (Universit Laval)
Mukopadhyay, Asish (University of Windsor)
Obaid, Abdel (University of Quebec at Montreal)
Radue, Jon— (Brock University)
Sutcliffe, Rick (Trinity Western University)
Therien, Denis (McGill University)
Villemure, Serge (NSERC)
Wong, Johnny (University of Waterloo)
Zastre, Michael (University of Victoria)

Chapter 32

Cascade Topology Seminar Meeting Spring 2005 (05w2612)

July 14 – 16, 2005

Organizer(s): George Peschke (University of Alberta), Laura Scull (UBC)

This workshop was a meeting of the Cascade Topology Seminar. This is a semi-annual gathering of the region's topologists which began in 1987, overseen by Steve Bleiler (Portland State University) and Dale Rolfsen (UBC). It is designed to foster contacts between workers, as well as graduate students, in similar fields across the region. It provides a venue for local topologists to showcase their own work, and also an opportunity to bring in speakers from outside the region, helping local topologists to keep abreast of recent developments.

In all these respects, the BIRS meeting of the Cascade Topology Seminar was a great success. This meeting had 25 participants from various schools in western Canada and the Pacific Northwestern US; a few participants also came from the east (Ontario and the midwestern US). The group included established researchers, early career mathematicians and quite a few graduate students from various schools.

There were 6 one-hour lectures given during the meeting. True to the spirit of the Seminar, the speakers included a mix of established mathematicians (Ralph Cohen, Stanford U; Tony Elmendorf, Purdue U at Calumet; and John Palmieri, U Washington) and early career topologists (Ryan Budney, U Oregon; Keir Lockridge, U Washington; and Jens von Bergmann, U Calgary). The talks ranged from pure homotopy theory (stable homotopy and A_∞algebras) to more geometric topology (spaces of graphs and knots) and symplectic geometry (contact homology). Details on the titles and abstracts for individual talks can be found at

http://www.pims.math.ca/birs/workshops/2005/05w2612/Schedule05w2612.pdf

In addition to displaying the range of recent work being done by the region's topologists, the workshop was also a valuable opportunity for personal contact between the members of our various departments. In the times provided for informal discussion, current research projects were discussed, and recent advances such as the new book by Dave Morris (University of Lethbridge) were advertised. PIMS and NSF funding was extended to a number of graduate students to attend the event, and they had the opportunity to meet both each other and the more senior topologists. In addition, plans for upcoming area topology events such as expected visitors to the area and the special Topolgy sessions at the AMS meeting in Oregon and the CMS winter meeting in Victoria were discussed. Overall, the opportunity for the region's topologists to meet and discuss items of mutual interest face-to-face contributed to the sense of community which is so valuable for its members' research.

The BIRS setting provided a beautiful and congenial environment for this workshop, and the organizers wish to thank PIMS for giving us this opportunity.

List of Participants

BEYL, F. Rudolf (Portland State University)
Bleiler, Steve (Portland State University)
Budney, Ryan (University of Oregon)
Cohen, Ralph (Stanford University)
Devinatz, Ethan (University of Washington)
Dolan, Peter (University of Oregon)
Dover, Lynn (University of Alberta)
Elmendorf, Anthony (Purdue University Calumet)
Grguric, Izak (University of British Columbia)
Krause, Eva (University of Alberta)
Lam, Kee Yuen (University of British Columbia)
Lockridge, Keir (University of Washington)
Morris, Dave (University of Lethbridge)
Nicas, Andrew (McMaster University)
Palmieri, John (University of Washington)
Peschke, George (University of Alberta)
Prince, Tom (University of Alberta)
Rodriguez Ordonez, Hugo (University of Oregon)
Ruan, Haibo (University of Alberta)
Scull, Laura (University of British Columbia)
Tomoda, Satoshi (University of Calgary)
Varadarajan, Kalathoor (University of Calgary)
von Bergmann, Jens (University of Calgary)
Yurasovskaya, Ekaterina (University of British Columbia)
Zvengrowski, Peter (University of Calgary)

Chapter 33

Connecting Women in Mathematics Across Canada II (05w2010)

Jul 21 – Jul 23, 2005

Organizer(s): Malgorzata Dubiel (Simon Fraser University), Rachel Kuske (University of British Columbia), Barbara Keyfitz (Fields Institute for the Mathematical Sciences), Judith J McDonald (Washington State University), Leah Keshet (University of British Columbia), Ortrud Oellermann, (University of Winnipeg), Gerda de Vries (University of Alberta), Mateja Sajna (University of Ottawa)

Conference Activities

The participation in the conference was by invitation: the applicants had to submit a statement of interest, a title and abstract of a talk about their work and/or research interests, and a letter of support from their supervisor. Twenty six women graduate students in mathematics from universities across Canada were selected to attend. They spend two intensive and exciting days, attending talks and presentations, and sharing experiences with ten women faculty members, speakers and mentors at the conference. The graduate students each gave a 20 min presentation or a poster on their work. The mentors coordinating these sessions insured that the women presented their work in a friendly, supportive environment and interacted with their peers and senior women in the frbara Keyfitz, Director of the Fields Institute, and Neeza Thandi, Actuary for Liberty Mutual, gave the two plenary talks. They spoke about their work, research they are involved in, and their careers.

The program included two panel discussions: Launching a Career in Mathematics, and Changing Environments in Mathematics and Academia. Both were followed by small group discussions involving students and mentors. These discussions focused on giving participants the opportunity to discuss the hurdles they have faced or may face in their studies and future careers, and how to overcome them.

For more information and the schedule of the workshop, see
http://www.cms.math.ca/bulletins/2005/cwimac05.html?nomenu=1

Assessment of Benefits

Connecting Women in Mathematics Across Canada Program has been successful in many ways. The workshop provided a venue for covering important topics relevant to pursuing a mathematical career. It brought together different viewpoints on options for career paths and different routes to reach career goals.

Young mathematicians received advice on practical issues such as applications, reviewing of files, networking, and interviewing. They also had a chance to voice problems they may be experiencing, get some new perspectives on their successes and concerns, and to make network connections, which they can use now and in the future as their careers develop.

Timely issues such as the changing the culture of science, discussed in an environment with a mix of viewpoints of the experienced and junior researchers, was very enlightening and motivating for all involved. In particular it has been a pleasure to see a growing number of women participating in our workshops that continue to come to CMS meetings and are getting tenure track positions at universities across Canada.

Future Plans

This workshop and the previous CWiMAC workshop have been the basis for an upcoming series of workshops to be held at BIRS in the coming years. These workshops will focus on examining recent advances and barriers for increasing diversity in mathematics, seeking ways to get the larger community involved.

In September 2006, BIRS workshop 06w5504 will bring together women and men mathematicians from Canada, US and Mexico to examine what the institutes and professional organizations are doing now to support women, and what other initiatives can be undertaken. They will develop recommendations for future collaboration and for activities in support of women in mathematics. In December 2006, Fields Institute will sponsor the third workshop for women graduate students.

In Summer 2007 we will reconvene for a short workshop to review the progress on the initiatives developed in 2006, and also to increase international connections. We will also collaborate with another BIRS workshop on Women in Engineering.

List of Participants

Beltaos, Elaine (University of Alberta)
Burgess, Andrea (Memorial University of Newfoundland)
Cooper, Sandy (Washington State University)
Dawes, Adriana (University of Washington (Friday Harbor Labs))
Dewar, Megan (Government of Canada)
Dubiel, Malgorzata (Simon Fraser University)
Eftimie, Raluca (University of Alberta)
Farnesi, Claudia (Concordia University)
Farzamirad, Meymanat (University of Alberta)
Keshet, Leah (University of British Columbia)
Keyfitz, Barbara Lee (Fields Institute and University of Houston)
Kuske, Rachel (University of British Columbia)
Lamkin, DeAnne (Washington State University)
Legendre, Eveline (University of Montreal)
Leite, Maria (University of Houston)
Li, Qun (McGill University)
Lisawadi, Spranee (University of Regina)
Lushi, Enkeleida (Simon Fraser University)
Masuda, Ariane (Carleton University)
McDonald, Jessica (University of Waterloo)
McDonald, Judi (Washington State University)
Meagher, Karen (University of Ottawa)
Mishna, Marni (Simon Fraser University)
Oellermann, Ortrud (University of Winnipeg)

Pandey, Pooja (University of New Brunswick)
Pronk, Dorette (Dalhousie University)
Ring, Amy (Washington State University)
Romaniuk, Yulia (University of Alberta)
Ruan, Haibo (University of Alberta)
Sanscartier, Manon (University of Saskatchewan)
Scull, Laura (University of British Columbia)
St-Hilaire, Marie-Odette (Universit de Montral)
Thandi, Neeza (Liberty Mutual Group)
Verdian-Rizi, Maryam (Simon Fraser University)
Yu, Na (University of British Columbia)

Chapter 34

West Coast Operator Algebras Seminar 2005 (05w2610)

Sep 15 – Sep 17, 2005

Organizer(s): Anthony To-Ming Lau (University of Alberta), Volker Runde (University of Alberta)

The theory of operator algebras originated with the work of F. J. Murray and J. von Neumann in the 1930s and 1940s. It has been an active—and still expanding—area of research ever since, which has manifold interactions with other areas of mathematics such as mathematical physics, algebraic topology, differential geometry, and even (quantum) computing.

The area is very well represented throughtout the North American West Coast. Berkeley—with Arveson, Jones, Rieffel, Voiculescu—and UCLA—with Effros, Ozawa, Popa, Shlyakhtenko—are probably the best known centers of high level research in operator algebras. In Canada, a strong group—Laca, Phillips, Putnam—exists at the university of Victoria. There are many more, albeit smaller, groups working on operator algebras throughout Western Canada and the Western United States.

The series of conferences now known as the West Coast Operator Algebras Seminar (WCOAS) started with a meeting at UCLA in 1991, and has been held almost every year since. It was held in Canada for the first time in 1996 (UNBC), then again in 1999 (Victoria), and finally twice at BIRS (2003 and 2005). In the years since its inception, the WCOAS has become a remarkably successful forum for the interaction of researchers that are spread out over a vast geographical area and otherwise have little opportunity to exchange ideas. In particular, it is of considerable value to graduate students and young researchers in the area.

The 2005 meeting in the series was the second one at BIRS. It had 32 participants, four of whom were graduate students and three postdocs. With two exceptions, all participants were affiliated with universities in Western Canada or in the Western United States. The two exceptions were George Elliot of Toronto and Hiroki Matui of Chiba (Japan) and currently visiting at Victoria.

The program consisted of twelve talks altogether. Three talks were one hour long:

- J. Phillips, *A survey of the analytic approach to spectral flow with some applications*;
- E. G. Effros, *On the free analogues of Hopf algebras associated with the Faà di Bruno algebra, and the Connes–Kreimer theory*;
- D. Blecher, *Dual operator algebras and non-commutative* H^∞.

Further talks of half hour length were given by D. R. Farenick, R. Floricel, K. Goodearl, A. Kumijan, H. Matui, I. Nikolaev, N. C. Phillips, D. Sherman and A. Sourour.

The talks were all of considerable mathematical quality and covered a wide range of topics, showing once again how diverse and lively the area of operator algebras has become. Even though the tight timeframe of a 2-day workshop did not leave as much time for interaction as may have been desirable, the workshop certainly accomplished its goal of bringing researchers together and providing a platform for the exchange of new mathematical ideas.

The next WCOAS will be—in all likelihood—be held at the University of Hawaii in early 2007.

List of Participants

Argerami, Martin (University of Regina)
Ashley, Dawn (University of Oregon)
Blackadar, Bruce (University of Nevada at Reno)
Blecher, David (University of Houston)
Brenken, Berndt (University of Calgary)
Deaconu, Valentin (University of Nevada at Reno)
Della Roca, Giulio (California State University at Long Beach)
Effros, Edward (University of California, Los Angeles)
Elliott, George (University of Toronto)
Erlijman, Juliana (University of Regina)
Farenick, Douglas (University of Regina)
Floricel, Remus (University of California at Berkeley)
Goodearl, Kenneth (University of California, Santa Barbara)
Ivanescu, Cristian (University of Northern British Columbia)
Kaliszewski, Steve (Arizona State University)
Kumjian, Alex (University of Nevada, Reno)
Laca, Marcelo (University of Victoria)
Lamoureux, Michael (University of Calgary)
Lau, Anthony To-Ming (University of Alberta)
Matui, Hiroki (Chiba University)
Nikolaev, Igor (University of Calgary)
Phillips, N. Christopher (University of Oregon)
Phillips, John (University of Victoria)
Pollock, Dan (University of Victoria)
Putnam, Ian (University of Victoria)
Quigg, John (Arizona State University)
Runde, Volker (University of Alberta)
Sherman, David (University of California at Santa Barbara)
Sourour, Ahmed Ramzi (University of Victoria)
Uygul, Faruk (University of Alberta)
Whittaker, Michael (University of Victoria)

Chapter 35

Alberta Postsecondary Curriculum Conference II (05w2613)

Sep 29 - Oct 01, 2005

Organizer(s): Jack Macki (University of Alberta)

Decision 1: To form the group ACUPMS: Alberta Committee on Undergraduate Programs in the Mathematical Sciences, with initial secretariat consisting of Manny Estabrooks (Red Deer College), Dave McLaughlin (Grant McEwan), Jack Macki (PIMS), Joan Stelmach (U of Calgary), and Pamini Thangarajah (Mt. Royal). (Other names suggested: Math for the Millennium, Pi, and Alberta Advanced Curriculum Study Group–this last has the great sounding acronym AACSG).

Decision 2: A new curriculum in Analysis will be prepared by a group: Gary DeYoung (Kings College), Bill Freed and Andreas Guelzow (Concordia), Bill Hackborn (Augustana), Tom Holloway (U of Alberta), Dave McLaughlin (Grant McEwan), Viena Stastna (U of Calgary), and Peter Zizler (Mt. Royal), chair Jack Macki (PIMS). It will be Jack Mackis responsibility to prepare a detailed syllabus for each of these two sequences.

Decision 3. We will set up a website for the ACUPMS. It will run on a server based at an Alberta school, and there will be a link to it from the PIMS website under Education.

Decision 4. A group will investigate e-learning: Manny Estabrooks, Andreas Guelzow, Len Bos (U of Calgary), Darius Holland (U of Calgary), Malcolm Roberts and Tom Holloway (U of Alberta). The group will be examining, among other items, the quality and feasibility of: Webworks (U of Calgary), MACSYMA (now called MAXIMA–open source), Maple online, eGrade.

Decision 5. Form a visiting committee from PIMS. This committee could consist of college and university mathematicians and non-academics with a scientific background. The mandate would be to 1. Visit, by invitation, college math departments and talk over issues – funding, failure rates (pressure to pass more students), grade inflation, admission standards.
2. If requested by the department, ask to meet with university administrators and hear their concerns.
3. Meet with representatives of client departments and faculties who send their students to study math with the department.
4. Take some time to discuss their findings among themselves, and provide a formal report.

From Thursday evening until Saturday noon, the meeting was intense and the participants hardworking and looking for solutions rather than simply criticizing. Peter Zwengrowski of the U of Calgary provided a nice break in the intensity by describing his course Mathematical Explorations, aimed primarily at Arts and

Elementary Education students. To begin, Peter asked for information on other courses aimed at these students. Mt. Royal has The Beauty of Mathematics, Kings College has Modern Applications of Mathematics and Foundations of Mathematics, Concordia has Math Motifs.

A BRIEF SUMMARY OF DISCUSSIONS

1. The Social and Political Context: Colleges are expanding, university enrollment is increasing, and these huge numbers of students are arriving with high expectations (like Garrison Keillors Lake Wobegon, where every child is above average). Among the students are the sharks (e.g., highly aggressive pre-med students). Many (too many) of entry level math courses are taught by sessionals. The financial pressures on the higher administrationat least as they see itare such that they want a high flow-through, which is administratese for pass them. Students are not accustomed to covering topics at the rapid pace of university courses. Do we need some sort of accreditation procedure to ensure introductory courses are being taught by qualified instructors. Do we need remedial courses?

A sample of thoughts presented:
We may want to consider allowing students to write the final exam in a math course many times (e.g. three times), during any exam period within say two years of their taking the course. This is a common practice in Europe.

We should make every effort make our core compatible with the B.C. core curriculum.

We will need to distinguish carefully between curriculum and pedagogy. This proposal is only about curriculum (so far).

How does computation enter at each stage of our analysis sequence?

Applications–which are relevant, how do we integrate them?

Evaluation can use a variety of techniques.

2. e-Learning, Blended Learning
Con Ferris at Red Deer College has been using eGrade for five years. Red Deer has a committee (Manny Estabrooks is on it) which is evaluating Maple On-Line and other possibilities. Andreas Guelzow is very enthusiastic about open-source MAXIMA. At the U of A, Maple is used for Engineering labs in year 1. Statistics courses use a range of tools, including on-line exams. Viena Stastna reported that on-line lab quizzes for a linear algebra course was not a success (actually, she said it was a mess). Gary DeYoung has just started a project at Kings College, using LaTeX. Len Bos is running a major project at the U of Calgary (WebWorks?) with a $100,000 grant. Joan Stelmach (U of Calgary) piloted WebWorks with a discussion board. She was amazed at the time students would spend trying to get a correct answer, rather than studying and analyzing the source of their difficulty. ePlus is better because it has hints that help avoid this problem. Peter Zwengrowski (U of Calgary) reported that they stopped using WebWorks for testing and grading in a four section ode course–it was just too much hassle. Upside pointed out by several: eLearning allows students to learn on their schedule. Some students thrive with it. Downside: Students dont learn to organize their homework as a written presentation; they do not learn to be neat and organized. Consensus: Thorough and long-term evaluation of eLearning is needed.

List of Participants

Akbary, Amir (University of Lethbridge)
Allegretto, Walter (University of Alberta)
Bailey, Jim (College of the Rockies)

Cliff, Gerald (University of Alberta)
De Young, Gary (Kings College)
Dinh, Thi (University of Calgary)
Dmitrasinovic-Vidovic, Gordana (Mount Royal College)
Estabrooks, Manny (Red Deer College)
Freed, Bill (Concordia University College)
Ganta, Reddy (Grande prairie Regional College)
Girvan, Doug (Red Deer College)
Guelzow, Andreas J. (Concordia University College of Alberta)
Hackborn, Bill (University of Alberta, Augustana campus)
Hohn, Tiina (Grant MacEwan College)
Holloway, Thomas (none)
Holzmann, Wolf (University of Lethbridge)
Kaip, Thomas (Grande Prairie Regional College)
Kharaghani, Hadi (University of Lethbridge)
Kudryavtseva, Elena (University of Calgary/Moscow State University)
LaHaye, Roberta (Mount Royal)
Ling, Joseph (University of Calgary)
Macki, Jack (University of Alberta)
McLaughlin, David (Grant MacEwan College)
Pivovarov, Peter (University of Alberta)
Roberts, Malcolm (University of Alberta)
Stastna, Viena (University of Calgary)
Stellmach, Joan (University of Calgary)
Svishchuk, Mariya (Mount Royal College)
Thangarajah, Pamini (Mount Royal College)
Timourian, James (University of Alberta)
Tomoda, Satoshi (Mount Royal College)
Zvengrowski, Peter (University of Calgary)

Focused Research Group Reports

Chapter 36

Analysis, Computations, and Experiments (05frg060)

March 12, 2005 - Mar 26, 2005

Organizer(s): Huaxiong Huang (York University), Robert M. Miura (New Jersey Institute of Technology), Demetrius Papageorgiou (New Jersey Institute of Technology), Michael Siegel (New Jersey Institute of Technology)

Introduction

This Focussed Research Group brought together a critical mass of researchers to work on fundamental problems that involve the breakup of liquid jets and on fluid and fluid jet problems that are motivated by industrial applications. Recent theoretical advances in the understanding of the breakup of single fluid jets are ripe to translate the control of breakup of jets. We also used concrete mathematical models to investigate utilizing liquid jet phenomena in the manufacture of micro- and nano-scale structures. Significant work remains to be done in the modeling and analysis of jets with more complicated geometries (e.g., compound jets) and involving complex fluids, which are typically found in industrial applications.

The FRG included applied mathematicians involved in modelling and asymptotic analysis in fundamental problems (Papageorgiou, Siegel, Howell, Young) as well as more applied problems motivated by industrial applications (Huang, Miura, Wylie), and a physicist with expertise in modelling and numerical simulation (Zhang). Many of the program participants are internationally known for their contributions to interfacial fluid dynamics.

Microdroplet Formation in a Patterned Hele-Shaw Cell

Parallel submicroliter polymerase chain reactions (pcr) have been utilized for DNA diagnostic applications [14]. A lattice of wetting (hydrophilic) patches is patterned on the interior faces of two (hydrophobic) glass plates of a Hele-Shaw cell and the patterns are aligned. A liquid first fills up the cell, and then a second, immiscible fluid is used to displace the excess liquid between the wetting patches to form multiple microdroplet liquid bridges between the plates. The droplets of liquid have a thickness which is usually much smaller than the characteristic lengths of the plates.

Preliminary studies that focused on the steady configuration indicate that the dynamic aspects of the filling process may be important. For example, droplets would not form if the filling speed is too fast. Furthermore, the viscous forces between the displacing and the droplet fluids may be important in the filling process. Motivated by these important issues, during the BIRS FRG, we modelled the dynamic filling process as a pressure-driven, two-dimensional Hele-Shaw flow.

We started a preliminary investigation of solving the model equations numerically using moving boundary methods. The standard boundary integral method has been used to simulate drop dynamics due to electrowetting in a Hele-Shaw cell [6]. However, this method cannot handle topological changes of the interface, such as during droplet formation as the interface is pushed through a wetting patch. Consequently, level set methods are used to accurately capture droplet formation with little artificial manipulation of the interface. The problem also has been reformulated using a phase field approach where the sharp interface is replaced by a thin layer characterized by an order parameter.

Influence of Surfactant on Contact Line Stability for Coating Flows

The coating of a surface is a process of obvious industrial importance and provides strength to the surface or achieves some desired physical properties [12]. We consider the two-dimensional coating flow of a moving substrate in contact with a liquid bath (e.g., see Figure 1). Experiments show that at sufficiently high coating speed, there is an instability of the fluid-substrate contact line, whereby a filament of air is ejected downstream into the liquid bath. This instability, which has been referred to as 'tip-streaming', is detrimental to the coating process.

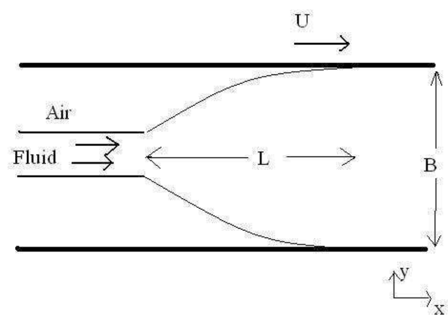

Figure 36.1: Geometry for the coating problem. The fluid coats a substrate moving to the right with speed U.

At the BIRS FRG, we investigated the role of surfactant on tip-streaming and air entrainment during coating flows. The presence of surfactant has been shown to be important during tip-streaming for the related problem of a bubble in an imposed extensional flow, see Figure 36.4. Surfactant transport at a contact line between a liquid and a moving solid substrate is a fundamental problem that has received scant attention.

Unfortunately, mathematical modelling is complicated by the presence of the contact line. It is well known that imposing a dynamic contact angle other than π gives rise to a discontinuous velocity field at the contact line. This is accompanied by a nonintegrable stress singularity at that point, which is physically unrealistic. To avoid this difficulty, we can assume that the interface is tangential to the solid at the attachment point, i.e., the contact angle is π. This has the advantage that, for single fluid systems, there exist local solutions which are devoid of nonintegrable stress singularities.

In our analysis, we therefore assume that the (microscopic) contact angle equals π. Material points on the free surface are prescribed to have speed U, the speed of the solid, and the surface velocity is continuous at the contact point. The interface then rolls onto the solid, similar to the rolling motion of a tank tread.

Influence of Soluble Surfactant on the Breakup of Two-Fluid Viscous Jets

Bubble and drop breakup is a fundamental process in fluid dynamics. At this FRG workshop, our investigation was to determine the influence of surfactant on the breakup of an extended bubble immersed in a much more viscous fluid.

Earlier studies [7] have shown that insoluble surfactant can dramatically retard the pinch-off of the interface. Instead, the interface develops a thin, quasi-stable cylindrical thread connected to nearly spherical regions (i.e., a dumbbell shape). The local surfactant concentration in the thread is large, owing to the relatively small surface area. We therefore expect that in the soluble case, there will be considerable surfactant transport from the interface to the bulk, which will have a significant effect on the pinching dynamics.

A simple model was developed at the workshop to examine the transport of soluble surfactant for a cylindrical interface separating an inviscid inner fluid from a viscous surrounding fluid. The interface location

$r = R(t)$ and velocity $\dot{R}$ are prescribed functions of time. After a series of transformations, the bulk surfactant concentration was found to satisfy an initial/boundary value problem for the heat equation. This was furthered transformed into a one-dimensional integral equation for the bulk surfactant concentration. The solution of this equation was left for future work.

Drawing of Microstructured Optical Fibres

Microstructured optical fibres, consisting of a lattice of air holes in a glass fibre, have many desirable optical properties and offer exciting possibilities for novel applications. The first step in their manufacture is the production of a preform, a few centimetres in diameter, containing the desired distribution of holes. This may be achieved, for example, by sintering together glass capillary tubes. The preform then is heated and drawn down to a typical diameter of $100\,\mu\text{m}$. A drawn microstructured fibre is shown schematically in figure 36.2, which is not to scale and has fewer holes than in practice (say 200, not necessarily circular).

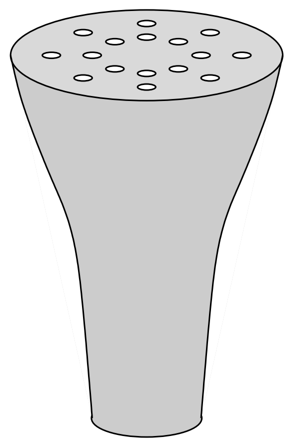

Figure 36.2: Schematic of a microstructured optical fibre.

Much empirical progress has been made in constructing fibres with increasingly complex microstructure. However, attempts to model the process mathematically have been limited to an axisymmetric fibre containing a single circular hole [4, 5, 16], which discards some of its most important characteristic features. To improve the flexibility and reliability of the process, several effects contributing to the evolution of the hole require study, including the shrinking of the fibre cross-section during drawing and the flow exerted on each hole by the other neighbouring holes. Surface tension may cause the holes to shrink, potentially closing altogether, thus pressurising the holes may need to be considered mathematically.

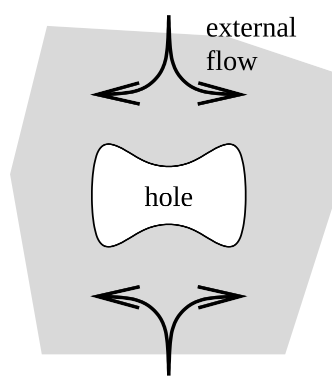

Figure 36.3: Schematic of a single hole in an external flow.

At the FRG workshop, significant progress was made on formulating the problem in a mathematically tractable way, yet retaining some physical reality. Using perturbation methods, we transform the slender three-dimensional geometry to a sequence of weakly-coupled two-dimensional problems for each fibre cross-section. This was achieved previously for a simply-connected fibre [2] and for a single bubble in an infinite fluid [8]. Further analysis is required to apply these methods to fibres containing many holes which may be pressurised.

A numerical method was devised by Kropinski [11], who developed an integral-equation approach that is spectrally accurate. However, it is necessary to discretise each fibre cross-section in this way, which is no more efficient than a boundary-integral formulation of the full three-dimensional Stokes flow problem.

We have begun to formulate an alternative approach to the problem, similar to Crowdy's [1] model for two-dimensional elliptical pores. Each hole is considered in isolation, subject to an external flow, as indicated in figure 36.3. For linear and certain nonlinear external flows [13], exact solutions for the interface evolution may be found in the form of a time-dependent conformal map from the unit disc. We propose to describe the interface using a truncated polynomial conformal map.

This approach would allow each hole, and the fibre as a whole, to be characterised by a small number of scalar coefficients in a conformal map. The asymptotic analysis of the slender geometry then would allow us to construct a system of partial differential equations for all these parameters as functions of time and axial position (as in [2]). Work has started on analysing the hole-scale and fibre-scale problems depicted in figures 36.3 and 36.2. We anticipate that a working numerical code will be completed and written up within one year.

Thermal Instability in (Viscous) Glass Threads

Viscosity of glass varies rapidly with temperature. In the drawing of glass threads, heat transfer will play an important role in the dynamics. A thread which cools too quickly will become viscous and require large forces to stretch it, so it is natural to heat the thread as it is being pulled. An important factor in the design of glass pulling devices is that they easily achieve stable and robust operating conditions.

The group has considered a thread that is heated while being pulled with a constant force, following a model proposed in [9]. Physically relevant simplifications then lead to a set of coupled nonlinear hyperbolic equations. Analytical solutions to the steady state equations for both uniformly and non-uniformly heated threads are obtained. We show the surprising result that steady states exist in which an increase in the pulling force actually causes a decrease in the exit speed of the thread at the end of the device. This situation can occur if the viscosity varies very abruptly with temperature and the heating rate is large enough. Assuming that the viscosity varies exponentially with temperature, if the heating is uniform, then such behavior does not occur because changes in the viscosity are not fast enough. However, if the heating is non-uniform, then the device can exhibit this behaviour. By considering an initial value problem, we show that these types of solutions are unstable, and if one operates the device in this parameter regime, the thread will pinch.

We also show devices with fixed pulling speed can exhibit hysteretic behavior that leads to rapid changes in the pulling force as the pulling speed is slowly varied.

Pulling Glass Microelectrodes

From an applied point of the view, the group studied a glass fibre drawing problem related to the pulling of glass microelectrodes. Glass microelectrodes play an essential role in cell electrophysiology, where they are used to inject electric current and dyes into cells and measure membrane electrical potentials. Laboratories using these microelectrodes usually make them using commercially available glass tubes and pullers that use coil heaters to soften the glass during pulling. In [9], a detailed mathematical model was developed to predict the stretching and breakup of the glass tube using a vertical puller. The model is highly nonlinear and was solved numerically. Useful insights were given, e.g., the effect of heater temperature on the formation of electrodes.

During the BIRS FRG, we simplified this model so that an analytical solution can be obtained for a simple case. It is desirable to identify the main factors that have direct influence on the electrode shape, which is of critical importance. We concluded that the source of radiation energy from the coil heater can be approximated by a piecewise constant function. This simplifies the model and allows an analytical solution under a constant pulling force, a feature of more advanced horizontal pullers. Even for the vertical puller, a semi-analytical solution can be obtained.

For an arbitrary heater strength variation, the simplified model allows the implementation of a more efficient Lagrangian-based numerical method. After the BIRS workshop, we have carried out detailed parameter studies on the break-up of the glass tubes to form glass microelectrodes. A paper [10] has been submitted for publication.

One problem that we have not investigated is the detailed breakup mechanism. In the current work, we used a phenomenological breaking stress formula. The breakup of the viscous thread here is different from the surface tension induced instability. Instead, the breakup is most likely caused by spontaneous fracture due to surface damage during the extension process.

Core-Annular Flows

Core-annular flows are two-fluid flows in circular tubes and consist of a core flow occupying the central region of the vessel surrounded by a lubricating annular fluid. The ability of the annular fluid to 'lubricate' the core fluid has potential applications in the oil and food industries, e.g., a highly viscous fluid can be made to flow efficiently with a given pressure gradient due to the slippage that the annular fluid provides. In applications, an interfacial instability can cause a breakup of the core fluid to produce drops or slugs of the higher viscosity fluid suspended in the lower viscosity one, or an emulsion at high flow rates.

This problem involves modelling and mathematical analysis based on the Navier-Stokes equations in a three-dimensional axisymmetric geometry with a free boundary separating the two fluids. Some attempts have been made to attack this formidable problem with direct simulations. When the annular fluid layer is thin compared with the core fluid radius, rational asymptotic expansions lead to an evolution equation for the interface, which includes long wave instability and nonlinearity. Of particular interest is the behaviour of solutions with long wave periods and which become chaotic via a Feigenbaum period doubling cascade.

The group considered the problem when the core fluid has a small radius compared to the pipe radius and has a viscosity that is small compared to that of the surrounding fluid. This fits nicely with the holey fiber work considered by the group, since it has a finite geometry due to the presence of the walls. Pressure-driven flow also is different and the two problems are complementary. The group considered the problem asymptotically in the case of a highly viscous annulus and an inviscid core. A nonlinear evolution equation was derived and was studied for nonlinear features, e.g., travelling waves. The equations need to be solved numerically, which should suggest some more analysis, in order to produce a publication. All these aspects are currently being investigated.

Surface-Tension-Driven Breakup of an Air Bubble in a Viscous Liquid

If you invert a nearly-full jar of maple syrup, you will see an air bubble form and rise upwards. If the air bubble becomes sufficiently elongated during the rise, it will break up into smaller bubbles. Recently, it has been shown that this phenomenon exhibits exceptional breakup dynamics [3], i.e., one which retains the effects of boundary and initial conditions to the final point of breakup. Previous examples of surface-tension driven breakup have shown that the interface shape collapses onto a single, unique form after appropriate dynamic rescaling of the coordinate axes. Such scale-invariant dynamics is obtained when the behavior is governed solely by the proximity of the breakup, with no dependence on boundary and initial conditions. The memory-preserving breakup dynamics was identified as a result of surface-tension driven breakup with an essentially static interior, with evidence provided from experiments, simulations, and theory [3]. Recent numerical simulations [15] of a surface-tension driven breakup of a cylindrical hollow inside a viscous jet provided further confirmation of this unusual property associated with static-interior breakup.

A long-wavelength model for the time-evolution of the bubble surface [3] and static-interior breakup process was derived to describe the breakup dynamics. Three common breakup scenarios are analysed: the detachment of a large bubble from a nozzle, the breakup of an infinitely long cylinder (see [15]), and the breakup of a finite-sized bubble. Exact expressions for the bubble shape and interior pressure are derived for the simpler limiting situations of infinite cylinder breakup and nozzle detachment. Our analytical results show that the shape at breakup retains an imprint of boundary and initial conditions. They also show that the long-wavelength dynamics associated with a static-interior breakup cannot give rise to new minima in the bubble shape.

As bubble breakup is approached, the solution of the long-wavelength equation for surface evolution approaches the same form regardless of initial and boundary conditions. Since the collapse does not distort the neck shape, this shape retains an imprint of initial and boundary conditions, as noted in [3].

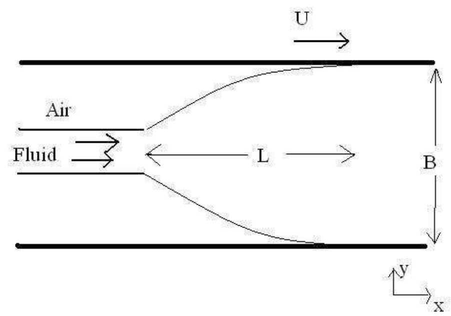

Figure 36.4: Geometry for the coating problem. The fluid coats a substrate moving to the right with speed U.

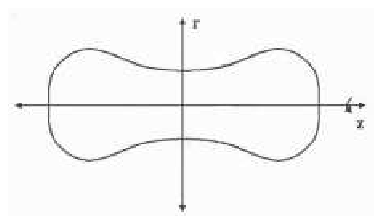

Figure 36.5: A bubble immersed in a viscous exterior liquid.

For an air bubble, the interior flow speed is always significantly larger than the exterior flow speed, and therefore, the breakup corresponds to surface-tension driven breakup with a static interior. Near breakup, the bubble neck simply collapses inward at a uniform rate, see Figure 36.5, in contrast to other situations where the interior flow is significant and the breakup dynamics evolves towards a scale-invariant form. The analysis shows, in the long-wavelength limit, that the static-interior breakup has the unusual property that all unstable modes grow at the same rate, i.e., there is no fastest growing mode. As a consequence, the breakup dynamics is highly sensitive to details of the initial shape.

In the long-wavelength limit, an initial shape with a minimum, however small, breaks up into two bubbles. An initial shape which is everywhere convex, however extended, rounds into a sphere.

List of Participants

Howell, Peter (University of Oxford)
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Miura,Robert M. (New Jersey Institute of Technology)
Papageorgiou, Demetrius (New Jersey Institute of Technology)
Siegel, Michael (New Jersey Institute of Technology)
Wylie, Jonathan (City University of Hong Kong)
Young, Yuan-Nan (New Jersey Institute of Technology)
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Chapter 37

Local index theorem in noncommutative geometry (05frg603)

April 16 – April 30, 2005

Organizer(s): John Phillips (University of Victoria, Canada), Alan Carey (Australian National University), Nigel Higson (Penn State)
Adam Rennie (University of Newcastle, Australia), Nurulla Azamov (Flinders University, Australia)

There have been two new proofs of the Connes-Moscovici local index theorem [2] produced by some of the organisers and described in [H] and [3] [4]. The latter proof applies also to the situation where the standard spectral triple [1] consisting of a C^* algebra $\mathcal{A}$ acting on Hilbert space H and an unbounded self adjoint operator D with $[D,a]$ bounded for all a in a dense subalgebra of $\mathcal{A}$ is replaced by a semifinite spectral triple. The latter means D is affiliated to a semifinite von Neumann algebra $\mathcal{N}$ which contains $\mathcal{A}$ and the resolvent of D is in the compact operators in $\mathcal{N}$.

The two week period was divided into three parts due to the fact that Alan Carey and Nigel Higson were only able to come for one week each and only overlapped by three days. The organisers agreed that the main focus of the fortnight would be on examples and applications.

For the first three to four days the emphasis was on checking the details of a preprint by Pask and Rennie in which the semifinite local index theorem was applied to certain graph C^*-algebras. The algebras studied admit a natural action of the circle group and were constrained by the requirement that the algebra should admit a trace. There were problems with the construction of a suitable trace and so considerable effort went into understanding whether the trace was continuous in an appropriate sense. After much effort, these problems were satisfactorily resolved.

After Carey and Azamov arrived talks were organised on a preprint of Azamov, Dodds and Sukochev in which the Krein spectral shift function was constructed in the semifinite von Neumann algebra setting. The question of whether it is related to spectral flow was raised. For a certain path of unbounded operators equality of the two was verified in the case of finite von Neumann algebras. It was conjectured that in general they are not directly related but that there might be a way to use spectral flow to 'subtract' discontinuities in the spectral shift function. Azamov promised to report back on the outcome of this idea after returning to Adelaide.

Phillips contributed a number of missing results and proofs to a manuscript in preparation in which an overview of the analytic approach to spectral flow in semifinite ven Neumann algebras is given. The principle objective was to outline the analytic definition of spectral flow when one was in the situation of paths of operators in a von Neumann algebra with non-trivial center. A secondary objective was to answer some natural questions which had arisen in the 10 years since Phillips original paper on this subject. The ms also contains many examples and the details of these were discussed. The ms is now nearing completion and will be the first publication arising from the BIRS interaction.

Upon Higson's arrival there were informal lectures organised on KK theory. The object was to understand how to use KK theory to understand extensions of the Pask-Rennie preprint to other settings. Higson was able to clarify some of the constructions in KK theory that might be relevant in applications of the local index theorem to graph algebras. Lectures were given by Rennie on the Cuntz algebra and $SU_q(2)$ as graph algebras and conjectural applications of the semifinite local index theorem to them.

There were a number of small group research sessions investigating various questions related to these potential applications. There were also informal discussions of applications of the local index theorem in other settings such as subelliptic operators.

After the departure of Carey and Higson, Azamov pursued the relation of the spectral shift function to spectral flow, while Phillips and Rennie made significant progress on the Cuntz algebra example and some related problems. The progress centred around understanding the KK pairing being computed by the spectral flow formula in the Cuntz algebra example. The $SU_q(2)$ example was examined again in light of the progress on the Cuntz example, but little headway was made.

List of Participants

Azamov, Nurulla (Flinders University)
Carey, Alan (Australian National University)
Higson, Nigel (Pennsylvania State University)
Phillips, John (University of Victoria)
Rennie, Adam (University of Newcastle)

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Chapter 38

Influenza Dynamics: Models and Data (05frg084)

May 28 – June 9, 2005

Organizer(s): Chris Bauch (University of Guelph), Jonathan Dushoff (Princeton University), David Earn (McMaster University), Junling Ma (McMaster University), Christina Mills (Harvard University), Joshua Plotkin (Harvard University)

The recent workshop at BIRS offered us a fantastic opportunity for collaboration and focused, productive research. The workshop exceeded our expectations in terms of the breadth of the academic subjects we explored, and the collaborations we established.

A subset of our group has been collaborating for several years. We have used mathematical models to study the spread and evolution of influenza viruses. The purpose of this workshop was to attempt reconciliation of our models with empirical data on influenza epidemics; and to form a collaboration with Christina Mills and Marc Lipsitch from the Harvard School of Public Health. We have progress to report on both of these goals. Perhaps most important is the strong collaboration we have formed with the Mills/Lipsitch group, resulting in two completed manuscripts already. The substance of these studies, as well as others that we initiated at Banff, are described below:

During our workshop at Banff, we completed a manuscript (MS #1) that uses empirical data from the infamous 1918 "Spanish Flu" pandemic and highlights theoretical puzzle about influenza persistence. The most basic, longstanding mathematical model of disease transmission divides the population into three classes (Susceptibles, Infectious, and Recovered/Immune individuals) and describes flow between these classes with a system of three ordinary differential equations. Given this standard model of disease, and given the empirical influenza epidemic curve and infection rates observed in the United States in 1918, we have estimated that a very large proportion of the population was infected (and thereafter immune) to the Spanish Flu of 1918. According to these estimates, only a very small proportion of the population remained susceptible to influenza after the pandemic – too small to support the initiation of another epidemic the following season. But the empirical data indicate that another influenza epidemic did indeed occur in 1919, which raises a theoretical puzzle. Our manuscript describes this enigma and offers several hypotheses for its resolution: the virus may have evolved to such an extent in 1918 that could re-infect individuals in 1919; or the virus could have persisted in 1919 due to heterogeneities in the host population and "pockets" of remaining susceptibles; or (perhaps most intriguing) the virus may have evolved a greater ability to spread, allowing it to persist despite the small number of susceptible hosts to support it. Our manuscript does not attempt to resolve this enigma, but rather to describe how the puzzle arises from the combination of standard mathematical models and empirical data from the 1918 influenza pandemic.

We have also drafted a second manuscript (MS #2) that analyzes the effects of spatial aggregation of data on the estimation of critical epidemiological parameters, such as the initial rate of disease spread, used in mathematical models. Measures of disease transmissibility are often estimated using data aggregated at a large spatial scale (e.g. city, state, country). Using 1918 influenza pandemic death data gathered at multiple spatial scales, we have shown that aggregation in the context of asynchronous epidemics of variable size tends to bias transmissibility estimates downward.

We also have begun a systematic analysis of methods used to estimate the initial rate of disease spread (a parameter called R_0) on the basis of epidemiological data. Data available is typically either a time-series of infected individuals, a time-series of mortality events, and/or data on the probability distribution of the disease's "serial interval" – that is the duration from infection to the end of infectiousness. Aside from several standard curve-fitting methods, we developed a novel technique for estimating the rate of disease spread, based on "serial interval" data. We are planning to write a detailed, more theoretical paper (MS #3) in which we simulate standard stochastic models of disease spread, and then apply a variety of techniques to estimate the parameter R_0 used in those simulations. We expect that estimates of R_0 may, unfortunately, depend upon which estimation techniques are employed. We plan to investigate and present these dependencies, thereby informing the broader community of scientists and public health officials who seek to infer underlying disease parameters from epidemiological data.

Finally, in light of the three manuscripts discussed above, we are planning a fourth paper (MS #4) focused on the empirical data from the 1918 influenza pandemic in Philadelphia, which killed a staggering 12,162 people within two months. Our initial analyses of these data indicate that the epidemic time-series does not conform to the standard mathematical model of disease transmission, except during the initial few weeks of expential growth. Instead, the Philadelphia data show a depression in the incidence rates after the first several weeks – which may suggest that behavioral changes or quarantine regulations had an important effect on curbing Philadelphia's epidemic. We intend to analyze the Philadelphia epidemic curve in detail, using methods described above, and to correlate our analysis with historical documents on the timing and extent of quarantine measures implemented in Philadelphia during the 1918 epidemic.

List of Participants

Bauch, Chris (University of Guelph)
Dushoff, Jonathan (Princeton University)
Earn, David (McMaster University)
Ma, Junling (McMaster University)
Mills, Christina (Harvard University)
Plotkin, Joshua (Harvard University)

Chapter 39

Hyperplane Arrangements: Cohomology and Rational Homotopy (05frg090)

Jun 11 – Jun 18, 2005

Organizer(s): Graham Denham (University of Western Ontario), Alexandru Suciu (Northeastern University)

The main focus was on the varieties of jumping loci for cohomology with coefficients in rank one local systems, and the related resonance varieties, arising from complex hyperplane arrangements. These varieties have emerged as central objects of study, providing deep and varied information about the topology of the complement of an arrangement.

Briefly, the jumping loci of a space M are the sets of representations A of the fundamental group of M for which the rank of $H^p(M, A)$ is larger than a fixed integer. The representation variety of a complex hyperplane complement is a complex torus; thus the jumping loci here are just certain subvarieties of the torus. The resonance varieties are tangent to the jumping loci at the trivial representation, and have proven to be somewhat easier to understand.

Given the multifaceted nature of the topic, the meeting brought together people with a variety of backgrounds, including commutative algebra, topology, discrete geometry, and singularity theory.

The seven participants spent the week alternating between group discussion – mutual tutorials in recent developments – and interrelated collaborations in groups of two or three.

Falk and Yuzvinsky continued their work on multinets, combining ideas from [**?**] with recent work of Falk on resonance varieties. The result seems to be a complete, combinatorial characterization of these varieties in H^1 in terms of the existence of special pencils of curves, generalizing the classical Hesse pencil.

From [**?**] and recent work of Alexander Varchenko on the Bethe anzatz [**?**], there arose some intuition that resonance varieties and the critical set of a function

$$\Phi_\lambda = \prod_{i=1}^{n} \alpha_i^{\lambda_i}$$

may be related, where each α_i is a linear form (defining a complex hyperplane) and each $\lambda_i \in \mathbb{C}^*$. We may think of λ as a point in the torus: i.e., as representation of the fundamental group of the complement. Then, generically, the cohomology of the complement with respect to the local system λ vanishes, except in middle dimension, where it has some rank β. Generically, the function Φ_λ has β isolated critical points. The papers [**?**, **?**] relate the two explicitly. Cohen, Denham, and Falk continued their joint work with Varchenko on the case of non-generic λ. Roughly, the critical set of Φ_λ may be positive dimensional, in a way that corresponds somewhat explicitly to nonvanishing cohomology. This project began in Fall 2004 when the authors were together at MSRI.

Srikanth Iyengar was able to join the group. He was new to arrangement theory and had not met the other participants before. He was able to learn quite a bit about the area. At the same time, he was able to contribute considerable technique in commutative algebra of a flavour motivated by rational homotopy theory. This was extremely useful for Denham and Suciu, in view of their recently completed project [**?**]. This relationship that began at BIRS continues fruitfully: Denham is due to visit Iyengar in March 2006, and these discussions helped motivate Denham's subsequent project with Suciu [**?**] (which began at BIRS).

In somewhat more detail, the rational homotopy theory of hyperplane complements has been shown to sit in a position where commutative algebra and rational homotopy theory overlap, in the sense of the "looking glass dictionary" of Avramov and Halperin in the 80's. The reasons for this aren't quite clear, but the precision of what is generally a somewhat vague correspondence seems to be quite profitable.

Schenck was able to continue discussions with Yuzvinsky and Denham that also began at MSRI.

Several of the research themes developed in Banff were discussed and pursued at a PIMS workshop in Vancouver in August. The idea of writing a book on arrangements to succeed [**?**] was discussed at length (on a hike up to the Stanley Glacier.) This project has been pursued, and the book is now underway.

On behalf of the focussed research group, I would like to thank PIMS for a very productive and enjoyable meeting.

List of Participants

Cohen, Daniel (Louisiana State University)
Denham, Graham (University of Western Ontario)
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Iyengar, Srikanth (University of Nebraska)
Schenck, Hal (Texas A&M University)
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Yuzvinsky, Sergey (University of Oregon)

Chapter 40

Topological Methods for Aperiodic Tilings (05frg069)

Jul 16 – Jul 30, 2005

Organizer(s): Johannes Kellendonk (Lyon I), Ian F. Putnam (Victoria), Lorenzo Sadun (Texas, Austin)

In the 1960's and 1970's, mathematicians discovered geometric patterns which displayed a high degree of regularity, and yet were not periodic [7]. The subject also gained enormous importance with the discovery of physical materials (quasicrystals) with pure point x-ray diffraction spectrum, which indicates a highly ordered atomic structure, and yet symmetry patterns in that spectrum which could not be produced by periodic atomic structures [21, 8]. Since that time, the subject has grown substantially. In doing so, it has drawn on a highly diverse collection of mathematical ideas.

One very productive idea is to regard a tiling as producing a dynamical system [14, 17, 16, 15, 20]. First, all translates of the tiling are considered, Then a metric is placed on such tilings. This arises from natural ideas in symbolic dynamics for discrete patterns, but these must be adapted to handle the geometry of Euclidean space. The translation action of the Euclidean space extends to the completion of this metric space. Under fairly mild assumptions, the space obtained, called the hull of the tiling, is compact and so provides a natural setting for using techniques from dynamics. Eventually, it has been realized that this space actually contains a great deal of interesting and computable (from standard topological techniques) information on the original tiling.

The standard assumption through much of the literature is that of 'finite local complexity' or FLC: for a fixed R, the number of different patterns in the tiling of diameter less than R, is finite modulo translation [13]. Moreover, it has been known for a long time that, under FLC the hull is locally the product of a totally disconnected set and $\mathbb{R}^d$, where d is the dimension of the tiling [16, 1, 19]. Moreover, it can been presented as inverse limit of fairly simple cell-complexes. The first natural generalization of FLC is to relax the condition 'modulo translation' to allow more general groups of isometries. This leads to G-FLC, where G is the appropriate group. This has already appeared in the work of many authors (for example, see [4, 18]. However, there are a number of interesting examples where this hypothesis fails, but this can happen in several ways. Several of the participants, Natalie Priebe-Frank, Sadun and Kellendonk, in particular, had been considering such examples, and through the course of the two weeks a unifying view of the metric was achieved. In some cases, the approximation by cell complexes seemed possible. If successful, this could lead to extending computations of cohomology invariants for new classes of tilings. Under study by a fairly large part of the FRG, including Priebe-Frank, Sadun, Kellendonk, Putnam, Hunton, Barge and Diamond, progress was made in understanding them within a global framework. Papers on this subject should be forthcoming shortly. Some other examples of non-FLC tilings were presented by Bellissard, arising from mathematical models of amorphous materials. Here, it seems that new ideas are needed to provide a better understanding of the hull.

There was a general theme for the FRG of trying to understand the nature of the cohomology of the tiling and its physical interpretations. This cohomology is closely linked witht the K-theory of the C^*-algebras associated with the tiling, first constructed by Bellissard and investigated by Forrest, Hunton, Kellendonk, Putnam and others [2, 5, 10, 11, 1, 6]. There were some interesting new interpretations made of how parts of this K-theory could arise from lower dimensional phenomena in the tiling. At a physical level, these could lead to measuring defects in physical materials. Several discussions elaborated links with groupoid cohomology and other interpretations of the cohomology.

A lot of progress was also made in computational methods and results. The use of spectral sequences for these calculations was studied intensely. Recently, tilings were discovered with a non-trivial torsion component in the cohomology. This rather surprising phenomena was investigated and discussed by Gähler, Hunton and Kellendonk. A great deal of progress was made on the calculation of several specific tilings of interest. Most notable was the pinwheel. But there were other examples, where full rotational symmetry was considered. This was the first time sufficient expertise and time had been brought to bear on these computations. Kalugin presented some very novel approaches to the understanding of matching rules from a topological view, leading to new methods for cohomology computation [9]. Recent work of Kellendonk and Putnam on their notion of pattern equivariant cohomology was presented [12]. The group spent some time developing this as an alternate view of cohomology for hulls, and indeed as a view of the hull itself, which seems very useful.

In the special case of one dimensional tilings, Barge and Diamond have a number of quite strong invariants. Moreover, a number of rather precise statements of the hull can be made. These were discussed, especially with an idea to trying to extend this program to higher dimensions.

One of the most popular features of the two weeks were the tutorials. It should be stressed that the common interest was in topological aspects of aperiodic materials, but the participants came from a remarkably wide range of backgrounds: mathematical physics, algebraic topology, operator algebras, dynamical systems, discrete geometry, ... Each day, long tutorials were presented, essentially aimed at novices, of technical tools from these different areas. For example, the use of spectral sequences for these cohomological calculations is crucial, yet only an expert in algebraic topology has this in his tool kit. All participants really gained a lot from some exceptionally revealing presentations.

A large number of other related topics were covered in various presentations: the Aubrey-Mather theory for quasi-crystals, relations with translation surfaces and orbit equivalence for Cantor minimal systems to name a few.

List of Participants

Barge, Marcy (Montana State University)
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Diamond, Beverly (College of Charleston)
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Giordano, Thierry (University of Ottawa)
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Research in Teams Reports

Chapter 41

Speciality of Malcev Algebras (05rit020)

April 30 – May 14, 2005

Organizer(s): Murray R. Bremner (University of Saskatchewan, Canada), Irvin R. Hentzel (Iowa State University, U.S.A.), Luiz A. Peresi (Universidade de São Paulo, Brasil)

Lie algebras and the PBW theorem

The Poincaré-Birkhoff-Witt (PBW) theorem (Jacobson [2]) implies that any Lie algebra is isomorphic to a subalgebra of the commutator algebra of some associative algebra. This result is established by constructing an associative universal enveloping algebra $U(L)$ for an arbitrary Lie algebra L, together with an injective Lie algebra homomorphism from L to the commutator algebra $U(L)^-$.

The speciality problem for Malcev algebras

The first step beyond Lie algebras leads to Malcev algebras. A Malcev algebra is a vector space M with a bilinear product satisfying anticommutativity and the identity

$$[[w,y],[x,z]] = [[[w,x],y],z] + [[[x,y],z],w] + [[[y,z],w],x] + [[[z,w],x],y].$$

The commutator in any alternative algebra satisfies these identities, and so every Lie algebra is a Malcev algebra. The speciality problem for Malcev algebras asks if any Malcev algebra is isomorphic to a subalgebra of the commutator algebra of some alternative algebra. This problem has been open for 50 years since it was first posed in Malcev's paper on analytic loops [4] (where these algebras were called "Moufang-Lie algebras"; they were given their present name by Sagle [6]).

Enveloping algebras for Malcev algebras

A solution to a different formulation of the speciality problem for Malcev algebras has recently been provided by Pérez-Izquierdo and Shestakov [5]. They generalize the PBW theorem to Malcev algebras in the following sense: for every Malcev algebra M they construct a universal nonassociative enveloping algebra $U(M)$ and an injective Malcev algebra homomorphism from M to the commutator algebra $U(M)^-$ such that the image of M lies in the generalized alternative nucleus of $U(M)$. The algebra $U(M)$ is in general not alternative nor even power-associative, but it inherits many of the good properties of universal enveloping algebras of Lie algebras, such as the universal mapping property, a PBW-type basis, and a (nonassociative) Hopf algebra structure. Furthermore, if M is a Lie algebra, then $U(M)$ is isomorphic to the familiar (associative) universal enveloping algebra of M.

The results we obtained at BIRS

The three of us met in Saskatoon on Friday, April 29, 2005 and drove together to Banff, arriving at BIRS in time for dinner on Saturday, April 30, 2005. On the way from Saskatoon to Banff, we agreed to start by reading and discussing the paper by Pérez-Izquierdo and Shestakov [5]. After doing that, we decided to start with a specific non-Lie Malcev algebra and use the techniques of [5] to compute explicitly the structure constants of the enveloping algebra. In the early paper by Sagle [6] there is an example of a 4-dimensional solvable non-Lie Malcev algebra M (Example 3.1, page 433). ¿From the results of Filippov [1] and Kuzmin [3] it follows that in dimension ≤ 4, this is the only (up to isomorphism) non-Lie Malcev algebra, and that it is solvable and special. We decided that our goals for our stay at BIRS would be:

1. To explicitly construct the enveloping algebra $U(M)$ with PBW-type basis and structure constants.
2. To study the polynomial identities satisfied by the nonassociative algebra $U(M)$.
3. To determine the quotient $A(M)$ of $U(M)$ by the alternator ideal, thereby obtaining an alternative enveloping algebra for M.
4. To determine a finite-dimensional quotient of $A(M)$ containing M in its commutator algebra.

To achieve these goals, we computed (using Maple and Pascal) how to express an arbitrary product of basis monomials of $U(M)$ as a linear combination of basis monomials. To do this we required various reduction algorithms to perform arguments by induction; the essential ideas behind these algorithms appear in the proof of Proposition 2.2 of Pérez-Izquierdo and Shestakov [5]. The techniques we developed at BIRS will allow us to continue this research in the following directions:

1. To solve the same problems for the 5-dimensional non-Lie Malcev algebras (Kuzmin [3]).
2. To do the same for the 7-dimensional simple non-Lie Malcev algebra (Sagle [6], Example 3.2, pages 433–435), and use this to obtain a new construction of the octonions.
3. To do the same for the free Malcev algebra, and use this to search for Malcev s-identities (identities which are satisfied by special Malcev algebras but not by all Malcev algebras).

Our time at BIRS was very productive; we expect to get at least one publication (possibly two or three) from the methods we developed during our “Research in Teams” program.

List of Participants

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Chapter 42

Random Matrices, multi-orthogonal Polynomials and Riemann-Hilbert Problems (05rit094)

April 30 – May 14, 2005

Organizer(s): John Harnad (Concordia University and Centre de recherches mathématiques, Université de Montréal)

The objective of this collaborative project was to further advance the computation of large N asymptotics in multimatrix models by extending the previously known methods used in 1-matrix models [8, 5] and applying them to the Riemannn-Hilbert formulation of multi-orthogonal polynomials developed in [2, 9].

The specific objectives were;

1. To relate the "dual" formulations of the Riemann-Hilbert problem characterizing biorthogonal polynomials obtained by the different members of this group [1, 2, 3, 9].

2. To extend the asymptotic analysis, based on the Riemannn-Hilbert method, and variational equations, to obtain rigorous large N asymptotics for the partition function in 2-matrix models [6, 7], the equilibrium distributions for the eigenvalues, and correlation functions in terms of asymptotics of the associated biorthogonal polynomials.

Considerable progress was already made on item 1 between the planning of this meeting and the actual event. The relation between the two different approaches to the Riemann-Hilbert problem for biorthogonal polynomials was in fact completely determined by M. Bertola and J. Harnad, in colllaboration with A. Its, in the months prior to the meeting, and these results were communicated to the other members of the group at the beginning of the meeting. The full details are currently being written up in final form, but a preliminary version is now available in the preprint [4].

The essential difference between the two approaches was that, whereas the large argument asymptotics in the formulation ref. [9] were fairly simple, involving only exponentials and power law dependence on the arguments, the jump discontinuities across the integration contours on which the biorthogonality is defined involves transcendental nonconstant dependence. In the approach of [2] however, the jump discontinuities are piecewise constant, but the large argument asymptotics involve fractional powers of the arguments and have sectorial behaviour, with Stokes matrices relating the different sectors. Moreover, only the "dual" fundamental systems were given an explicit integral representation in [2], with the asymptotics of the "direct" systems determined through the invariant bilinear pairing. The new approach, described in [4], gives an integral representation also for the "direct" fundamental systems, and these integral representations are used to deduce the sectorial large argument asymptotics and jump discontinuities explicitly, as well as the differential equations satisfied, without recourse to either the "folding" methods used previously, or further algebraic manipulations based on infinite recursions. Moreover, the integral representation in [4] is shown to factorize into a product of: 1) an explicitly known matrix factor, which is constant in the arguments of the system, though not in the degrees of the biorthogonal polynomials; 2) the integral representation of [9]; 3) a matrix factor that is independent of the polynomial degree N, consisting of the Wronskian matrix of an associated higher degree constant coefficient equation whose coefficients are determined by the polynomial potential. This factorization relation shows how the transcendental jump matrix in the system of [9] is transformed into a constant one, while at the same time introducing the N-independent sectorial behaviour that is required.

Since this point was already resolved by the start of the meeting, the remaining time could be devoted to addressing the set of problems listed under item 2. In fact, considerable progress in this direction had also already been made prior to the meeting, so the actual time spent at BIRS could, in part, be devoted to communicating this further progress, and to planning out the future steps needed for fully resolving these problems. The progress regarding large N asymptotics was made, partly on a heuristic, and partly a rigorous basis, by B. Eynard [6, 7].

He explained to the others in the group:

1. How the three different versions of the 2-matrix models, the "Normal" model, the "symmetry broken" normal model, and the "formal" model are related. The first of these, which is the one studied in [1, 2, 3] is based on integration on homology classes of contours; the second is based on grouping together multiple integrals by partitions of N in which the parts indicate the number of factors in the multiple integrals along a given contour and the third, the "formal" model, is based on a combinatorial definition of the partition function involving the multiplication of the weights of Feynmann graphs associated with a perturbative development about a Gaussian measure and evaluation of the integrands at the critical point contributions via gaussian integration.

2. How the existence of an "equilibrium" spectral curve may be deduced from a suitable definition of the free energy $\mathcal{F}_0$, which coincides with $\frac{1}{N^2}$ times the logarithm of the partition function in the "formal" model. This definition can be given on any "spectral curve" of the general form deduced from the "loop equation" [6] (which follows from the reparametrization invariance of the partition function),
$$E(x,y) = -(V_1'(x) - y)(V_2'(y) - x) + P(x,y),$$
where $V_1(x)$ and $V_2(y)$ are the polynomial potentials, of degrees $d_1 + 1$ and $d_2 + 1$, respectively, defining the biorthogonality measure and $P(x,y)$ is a polynomial of degree $\leq d_1 - 1$ in x and $\leq d_2 - 1$ in y. The free energy is given by residue formulæ involving the meromorphic differential ydx, which determine $\mathcal{F}_0$ as a functional on the moduli space of algebraic curves of the above the form. Its real part may be shown to be a convex function. The extrema are therefore well-defined, and the variational equations for these imply the vanishing of the real parts of the cycles of the abelian integral $\int ydx$ on the curve around any cycles.

3. Explicit forms - partly conjectural, partly proved, expressing the asymptotic forms of the fundamental systems of refs. [2, 4] in terms of ratios of Riemann theta functions on the equilibrium curve. Since these formulæ were deduced assuming the applicability of saddle point and WKB techniques which require more rigorous justification, the Riemann-Hilbert method is required to complete the analysis.

During the remainder of the two week period of the meeting, preliminary calculations were undertaken with a view to determining the branch cut structure for the Riemann surface of the spectral curve in the case when the potentials are even quartic polynomials, and the genus of the curve is 0. The purpose was to determine a contour that is homologically equivalent to the contour of integration on which the various transformations may be applied to reduce the Riemann-Hilbert problem to factors that differ from the identity only by terms that are exponentially decreasing. This is accomplished through the introduction of a generalization of the g-function, as done for the 1-matrix case in ref. [5]. The definition of this g-function seems now to be clear: it is the multivalued function defined by the abelian integral $\int ydx$ on the spectral curve.

These preliminary calculations for the genus 0 case appear to lead to the correct cut structure that should arise. It also appears, from these preliminary discussions and calculations, that the sequence of gauge transformations and deformations of the contours along which the jump discontinuities arise can be correctly defined by virtue of the analyticity and asymptotic properties of the g-function. Moreover, the presence of sectorial asymptotics in the Riemann-Hilbert problem appears to be consistent with the large N asymptotic properties of the g function, making this approach to the large N asymptotics of the biorthogonal polynomials very likely the correct one for implemention. In addition, these considerations potentially reveal a connection between (i) a physically motivated existence and uniqueness theorem for the equilibrium spectral curve, and (ii) nonlinear steepest descent analysis of the associated Riemann-Hilbert problem.

Much further work will be needed, but the preliminary results, and the general method of approach laid out at this BIRS meeting, seem to give very good promise for further development of an ongoing program that should lead to the resolution of the main unresolved questions on the large N asymptotics of 2-matrix models and biorthogonal polynomials.

List of Participants

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Chapter 43

Affinizations of Extended Affine Lie Algebras (05rit024)

May 22 – June 4, 2005

Organizer(s): Bruce Allison (University of Alberta), Stephen Berman (University of Saskatchewan), Arturo Pianzola (University of Alberta)

Extended affine Lie algebras, or EALA's for short, were introduced by Hoegh-Krohn and Torresani in 1990 [9] as natural generalizations of finite dimensional simple Lie algebras and affine Kac-Moody Lie algebras. Many of the basic facts about these algebras were proved in [1]. By definition EALA's are complex Lie algebras that possess Cartan subalgebras and invariant forms, and hence they possess root systems which turn out to be extended affine root systems. Roots of length 0 are called isotropic roots, and they generate a lattice whose rank is referred to as the nullity of the EALA. As has been shown in [3], EALA's of nullity 0 and 1 precisely coincide with finite dimensional simple algebras and affine Kac-Moody algebras respectively. Therefore there has been a lot of interest and activity in the last decade on the study of EALA's of higher rank.

An EALA $\mathcal{L}$ possesses an ideal $\mathcal{L}_c$, called the core of $\mathcal{L}$, which is defined to be the subalgebra of $\mathcal{L}$ generated by the root spaces of $\mathcal{L}$ corresponding to nonisotropic roots. ($\mathcal{L}_c$ is the derived algebra of $\mathcal{L}$ in nullity 0 and 1.) The quotient algebra $\mathcal{L}_{cc} := \mathcal{L}_c/Z(\mathcal{L}_c)$, is called the centreless core of $\mathcal{L}$. Y. Yoshii [14] has recently given an internal characterization of the Lie algebras, called centreless Lie tori, that arise as the centreless core of an EALA. Furthermore, the structure of an EALA is to a large extent governed by the structure of its centreless core. In fact, E. Neher [11] has recently announced a procedure that, given a centreless Lie torus $\mathcal{K}$, describes all EALA's with centreless core $\mathcal{K}$. For this reason, an important equivalence relation for EALA's is isomorphism of their centreless cores.

Many centreless Lie tori, and consequently EALA's, can be constructed using various "matrix" constructions, from coordinate algebras such as the noncommutative quantum tori that generalize Laurent polynomials in several variables. This is a combination of the work of number of authors in the last few years beginning with the paper of Berman, Gao and Krylyuk in [7].

Another approach to the construction of EALA's makes use of loop algebras and affinizations of Lie algebras relative to finite order automorphisms. If $\mathcal{G}$ is a Lie algebra and σ is an automorphism of $\mathcal{G}$ of period m, the loop algebra of $\mathcal{G}$ relative to σ is the algebra $L(\mathcal{G},\sigma)$ of fixed points of the automorphism $x \otimes f(z) \mapsto \sigma(x) \otimes f(\zeta_m^{-1} z)$ of the untwisted loop algebra $\mathcal{G} \otimes S$, where ζ_m is a primitive m$^{\text{th}}$ root of unit and S is the ring of Laurent polynomials in the variable z. Further, if $\mathcal{G}$ possesses a nondegenerate invariant symmetric bilinear form that is preserved by σ, one defines the affinization of $\mathcal{G}$ relative to σ to be the Lie algebra $\mathrm{Aff}(\mathcal{G},\sigma)$ obtained from $L(\mathcal{G},\sigma)$ by first forming a 1-dimensional central extension (with cocycle defined as usual using the invariant form) and then adding the 1-dimensional algebra spanned by the degree derivation $z\frac{d}{dz}$.

In his pioneering work on loop algebras in 1969, V. Kac showed that if $\mathcal{G}$ is finite dimensional simple and

σ is a finite period automorphism of $\mathcal{G}$ then $\mathrm{Aff}(\mathcal{G}, \sigma)$ is an affine Kac-Moody Lie algebra and all such Lie algebras arise in this way. In the language of EALA's this reads as follows: If $\mathcal{G}$ is a EALA of nullity zero then $\mathrm{Aff}(\mathcal{G}, \sigma)$ is a EALA of nullity one and moreover, all such algebras arise in this way. When phrased this way it becomes quite natural to ask what happens in the case of EALA's of higher nullity. In our work at BIRS on this problem, we focused on the case of nullity 2 and we worked at the level of centreless cores.

It is remarkable fact, which follows from a theorem announced recently by Neher in [12] along with the classification theorems for centreless cores of type A [7, 8, 13], that with the exception of one well-understood family, all centreless cores of EALA's are finitely generated as modules over their centroids. (The exceptional family consists of Lie algebras of the form $\mathrm{sl}_{\ell+1}(\mathbb{C}_{\boldsymbol{q}})$, where $\mathbb{C}_{\boldsymbol{q}}$ is the quantum torus determined by a quantum matrix $\boldsymbol{q}$ with at least one entry that is not a root of unity.) For this reason, we concentrated in our work on centreless cores with this additional finiteness property. While at BIRS we were able to complete the proofs of a number of results on this topic.

We showed that every centreless core of an EALA of nullity 2 that is finitely generated over its centroid is isomorphic to a Lie algebra of the form

$$L(\mathcal{G}_{\mathrm{cc}}, \sigma), \tag{1}$$

where $\mathcal{G}$ is an affine Kac-Moody Lie algebras and σ is a diagram automorphism of $\mathcal{G}$. Conversely, we showed that any Lie algebra of the form (1) is isomorphic either to a centreless core of an EALA of nullity 2 (finitely generated over its centroid) or to a Lie algebra of the form $[\mathbb{C}_{\boldsymbol{q}}, \mathbb{C}_{\boldsymbol{q}}]$, where $\boldsymbol{q} = \left(\begin{smallmatrix} 1 & \zeta \\ \zeta^{-1} & 1 \end{smallmatrix}\right)$ and ζ is a root of unity.

The class of Lie algebras of the form (1) is interesting in its own right. We were able to characterize algebras in this class in a number of different ways, including as $\mathbb{Z}^2$-graded-central-simple Lie algebras whose central grading group has finite index in $\mathbb{Z}^2$. We also gave a complete classification of the algebras in this class up to isomorphism. That is, we precisely determined when two algebras of the form (1) are isomorphic.

Precise statements and detailed proofs of the results just mentioned will appear elsewhere. Our proofs make use of techniques and results that we developed recently in a series of papers on EALA's and loop algebras including [4], [5], [6] and our paper [2] with John Faulkner.

The Banff International Research Station provided an ideal place for the three of us to get together for two weeks of uninterrupted research. We wish to thank BIRS very much for this opportunity.

List of Participants

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Chapter 44

Hamiltonian systems with symmetry (05rit606)

August 21 – August 26, 2005

Organizer(s): G. W. Patrick (University of Saskatchewan)

The *conservative, or Hamiltonian, dynamical systems* are finite or infinite dimensional dynamical systems that model physical phenomena on time scales over which dissipation is not dominant. In the idealized models, dissipation is absent, multiple time scales are usually present, and the long time dynamics is very delicate. The systems often exhibit temporal chaos. The subject has existed for centuries as celestial mechanics, but the current application domains (such as underwater vehicle dynamics [5][8][9], molecular dynamics and spectra [6][7][13], fluids and plasmas [1][4][10], and foundational physical field theories [2][3]) far exceed its origins.

When symmetry is present in these systems, one can seek solutions which are generated by one parameter subgroups of the symmetry group (i.e. of the form $\exp(\xi_e t)p_e$ where ξ_e is a Lie algebra element called the generator and $p_e \in P$). These are the *relative equilibria*, and they correspond to equilibria in the reduced spaces. The physical form of these solutions depends on the symmetry. For the three dimensional rotation group $SO(3)$, they will be uniformly rotation solutions, such as the circular orbit of a satellite. In the case of a neutrally buoyant underwater vehicle with coincident centers of mass and buoyancy, the symmetry is the Euclidean group $SE(3) = SO(3) \ltimes \mathbb{R}^3$, and the relative equilibria correspond to screw motions. When the underwater vehicle has an additional axial material symmetry, the system symmetry is $SE(3) \times SO(2)$, and the relative equilibria correspond to screw-spinning motions. When the symmetry group is not compact, such as the Euclidean symmetry group, establishing the stability of relative equilibria is delicate. Generally, in the noncompact case, a common criterion—formal stability—is insufficient to establish stability, and a more restrictive criterion—T_2 stability—must be used [12]. There is a gap between T_2 and formal stability. Stability inside the gap has been established using KAM theory, for certain relative equilibria of the system of an underwater vehicle [11] with coincident centers of mass and buoyancy. The T_2 stability theory and the KAM-desingularization technique, both due to workshop participants, are the state of the art in the area.

The workshop participants met to hammer out the details of an article that they are writing which considers an example where the gap actually occurs: the relative equilibria consisting of the falling, spinning motion of an axially symmetric underwater ellipsoid with non-coincident centers of mass and buoyancy, where the symmetry group is $SO(2) \ltimes \mathbb{R}^3 \times SO(2)$. The target audience for this work consists not just of mathematicians, but possibly also engineers and physicists, so it is important to find an exposition in the most basic language. Also this is important because the target audience must be brought to accepting that there is a subtle impact of this work on some highly regarded liturature, such as [5, 8, 9].

The stability problem, it was determined, can in the case of the symmetry in question, be generally addressed using a widely known, venerable, technique which reduces an abelian symmetry by eliminating "cyclic coordinates". After this reduction, one is reduced to a parameterized, two degree of freedom system. There is an additional circular symmetry at the parameter corresponding to the relative equilibrium in question, and the stability is a symmetry-breaking phenomenon from a completely integrable system. The general understanding of what typically occurs for any system that has the same symmetry group was established. Suitable coordinates are required to prove KAM stability inside the gap. These should be accessible to and recognizable by the target audience, and such coordinates were developed over the course of the workshop. A draft article was completed which the workshop participants anticipate will be rapidly completed and disseminated.

Another reason for considering the falling, spinning relative equilibria, is that it has nontrivial isotropy. A considerable amount of work on the isotropy problem has already been completed, but that was deliberately excluded from the theory developed in [12]. The workshop provided an opportunity to work again on this project. One of the base problems is to ensure that the presence of isotropy is fully taken advantage of in the stability theory. This was not so clear because isotropy implies singularities in the reduced spaces, at which methods which rely on a smooth structure are inapplicable, This problem was resolved over the course of the workshop: the isotropy is usually compact, in which case a purely topological proof was found that shows the presence the singularities in the reduced spaces will not affect the stability issue.

In this workshop, a team of three participants interacted very intensely. Progress was made on the isotropy project, and priorities sorted and possibilities found for further collaboration. A stalled project (the axisymmetric stability project) was reinvigorated, and work which would have taken many months, if it could have been completed at all, was largely completed in one week, with a far superior outcome. Partly this was due to an extensive preparation for the workshop, following an initial consultation a year earlier at the Bernoulli Institute of EPFL Switzerland, but it was also due to the excellent BIRS environment. The team is scattered across Canada and the UK, but the BIRS Research in Teams program enabled it to meet, concentrate on, and in large part resolve, a difficult problem, and to prepare the way for future collaborations.

List of Participants

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Chapter 45

Cohomogeneity Three Actions on Spheres (05rit047)

August 21 – September 3, 2005

Organizer(s): Jill McGowan (Howard University), Catherine Searle (IMATE-UNAM, Cuernavaca)

During our stay at BIRS, Dr. McGowan and I were able to modify our original project of classifying cohomogeneity 3 actions on spheres to the following problem: calculate the diameters and q-extents of spherical quotients of irreducible polar actions of cohomgeneities 3 and higher. First let us make the following definition: we call a cohomogeneity k action *classical polar*, when it is a polar action of cohomogeneity k corresponding to a symmetric space G/H where either G or H is a product of classical groups only. Those actions which admit products with classical groups and exceptional groups will be called *exceptional polar*. Note that there is a 1-1 correspondance between polar actions and symmetric spaces [D]. We are interested in this problem given that we have found in joint work with W. Dunbar and S. Greenwald [DGMS], and our own, [MS], when we allow for disconnected groups G to act isometrically on spheres by cohomogeneity 1, 2 or 3 (in the case where the action is classical polar) we obtain the following lower bounds for the diameter:

$$\min(\text{diam}(S^n(1)/G) = \begin{cases} \frac{\pi}{12} & \text{for cohomogeneity 1} \\ \frac{\alpha}{2} & \text{for cohomogeneity 2} \\ \beta & \text{for cohomogeneity 3} \end{cases}$$

where $\alpha = arccos(\frac{tan(\frac{3\pi}{10})}{\sqrt{3}})$, and $\beta = arccos(1/\sqrt{40 + 12\sqrt{2} - 8\sqrt{5} - 12\sqrt{10}})$.

We note that for these three cohomogeneities the diameter is strictly increasing as the cohomogeneity increases. The conjecture we are then currently trying to verify is: let G be an irreducible polar action of cohomogeneity k on S^n, then the diameter of S^n/G increases to $\frac{\pi}{2}$ as $k \to \infty$. That is, as the cohomogeneity of an irreducible action becomes large, the action "becomes" reducible. We would also like to understand what is going on in terms of the q-extents for these spaces.

We have been able to confirm this conjecture for the classical polar actions of cohomogeneities 3 and higher. The list includes the following groups:

Table 1: Classical Polar Actions of Cohomogeneity $k-1$

Nr.	G	dim(S^m)	Corresponding Symmetric Space
1	$SO(k) \times SO(n)$	$kn-1$	$SO(k+n)/(SO(k) \times SO(n)), k \geq n$
2	$S(U(k) \times U(n))$	$2kn-1$	$SU(k+n)/S(U(k) \times U(n)), k \geq n$
3	$Sp(k) \times Sp(n)$	$4kn-1$	$Sp(k+n)/(Sp(k) \times Sp(n)), k \geq n$
4	$U(2(k))$	$k(k-1)-1$	$SO(4(k))/U(2(k))$
5	$U(2(k)+1)$	$k(k-1)-1$	$SO(4(k)+2)/U(2(k)+1)$
6	$SO(k)$	$\frac{1}{2}(k-1)(k+2)-1$	$SU(k)/SO(k)$
7	$Sp(k)$	$(k-1)(2k+1)-1$	$SU(2(k)/Sp(k)$
8	$SO(2(k))$	$\frac{1}{2}2(k)(2k-1)$	$(SO(2k) \times SO(2k))/SO(2k)$
9	$SO(2k+1)$	$k(2k+1)$	$(SO(2k+1) \times SO(2k+1))/SO(2k+1)$
10	$U(k)$	k^2-1	$(U(k) \times U(k))/U(k)$
11	$Sp(k)$	$2k^2-k-1$	$(Sp(k) \times Sp(k))/Sp(k)$
12	$SU(k)$	k^2-2	$(SU(k) \times SU(k))/SU(k)$

Of the remaining groups, for those whose corresponding symmetric space is of the type $(G \times G)/G$, namely numbers 1, 6, 8 and 10 of Table 2, the result also holds true. During our stay at BIRS we were also working on the remaining groups listed in the following table.

Table 2: Exceptional Polar Actions of Cohomogeneities Greater than '2

Nr.	G	dim(S^m)	Corresponding Symmetric Space	Cohomogeneity
1	F_4	51	$(F_4 \times F_4)/F_4$	3
2	$SU(6) \times SU(2)$	39	$E_6/(SU(6) \times SU(2))$	3
3	$SO(12) \times SU(2)$	63	$E_7/(SO(12) \times SU(2))$	3
4	$E_7 \times SU(2)$	111	$E_8/(E_7 \times SU(2))$	3
5	$Sp(3) \times SU(2)$	27	$F_4/(Sp(3) \times SU(2))$	3
6	E_6	77	$(E_6 \times E_6)/E_6$	5
7	$Sp(4)$	41	$E_6/Sp(4)$	5
8	E_7	132	$(E_7 \times E_7)/E_7$	6
9	$SU(8)$	69	$E_7/SU(8)$	6
10	E_8	247	$(E_8 \times E_8)/E_8$	7
11	$SO(16)$	127	$E_8/SO(16)$	7

Since these groups do not admit "easy" matrix expressions, we are using a technique of Hsiang outlined in his book "Cohomology Theory of Topological Transformation Groups" [H] in order to calculate the principal isotropy subgroups of these actions. Once we have computed these subgroups, we then need to find their normalizers so that we may use the technique of G-manifold reductions (cf. [GS]) to compute the quotient space. That is, we must calculate the *core group* ${}_cG = N(H)/H$, where H is the principal isotropy subgroup, and also find the *core* of the corresponding sphere, ${}_cS^n$. Since ${}_cM/{}_cG \simeq M/G$, we may then compute the quotient space.

During our stay, we were able to calculate the connected component of the principal isotropy subgroup for number 2 and we made a fair amount of progress for numbers 3 and 4 (we are completing these calculations now). The only other action with non-trivial principal isotropy is number 9. For the rest of the groups in Table 2, we must use a different technique altogether, which can be found in Straume [S], namely extend the action to a larger dimensional group which will have non-trivial principal isotropy.

We also plan to see how much of Straume's paper can be extended for polar actions of cohomogeneity 3 and higher.

We would also like to add that while we modified our original proposal for our stay at BIRS, we have by no means abandoned the idea of classifying spherical actions of cohomogeneity 3 and higher. Upon conclusion of this current project, we hope to be able to tackle not only the classification problem, but also to understand how the diameters of spherical quotients of non-polar actions behave in terms of our conjecture.

In conclusion, we would like to add that we feel that our stay at BIRS was incredibly productive for us. This is the first time we have had an entire 2 weeks in which to just concentrate on our research. We are both very happy to have been provided with this opportunity.

List of Participants

McGowan, Jill (Howard University)
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Chapter 46

Symmetries of extremal conformal mappings (05rit091)

Aug 25 – Sep 03, 2005

Organizer(s): Oliver Roth (University of Würzburg), Eric Schippers (University of Manitoba)

Overview of the Field

A central problem in geometric function theory is to describe the class of conformal mappings of the disc. Two of the main reasons for the importance of this class are the Riemann mapping theorem, and the fact that it provides a model for the universal Teichmueller space. Solving extremal problems over the class (or developing methods for doing so) is one method of describing it, since the extremal function attaining the maximum must lie on the boundary.

In two approaches to solving extremal problems, the variational method of Schiffer and the extremal metric method of Teichmueller, the extremal functions are solutions of a differential equation given by a quadratic differential. This partly determines the extremal function, but it is still not known how in general to determine the function completely. In some cases further symmetries of the extremal function can be identified from the functional, which appear in different forms in the Schiffer and Teichmueller approaches.

Recent Developments, Open Problems and Scientific Progress Made

Work of Prokhorov [1] and recent work of Roth [3] and Schippers [6] indicates that a completely new approach to the above central problem *based on optimal control* might be possible.

Roth [3] has shown that Schiffer's method of boundary variation is equivalent to Pontryagin's maximum principle when applied to the Löwner differential equation. His work has been extended by Schippers [6], who exhibited a set of invariants under the Löwner flow. Moreover, a Lie–theoretic interpretation of the adjoint vector in Pontryagin's maximum principle in terms of the associated quadratic differential is obtained. This approach makes it clear that the main open problem (how does the extremal problem determine the extremal function) is closely linked with the question of uniqueness in Löwner's differential equation: In general there are many ways to generate a conformal map by means of the Löwner differential equation using different control functions for the same conformal map.

The relation between the control function and the generated conformal map in Löwner's differential equation is known to be a notoriously difficult one and there are many open questions and problems (see

[2] for a very recent solution to one of these problems). In the course of our discussion at BIRS this relation between the control and the conformal map in Löwner's equation also turned out to be crucial for constructing feedback controls in order to be able to apply the optimal control machinery to the study of extremal problems for conformal maps. We have been able to design such a control feedback problem under the assumption that every (extremal) conformal map can be generated via Löwner differential equation by a "canonical" control function. This led to a natural conjecture how this canonical Löwner equation will look like. We are currently working on a proof of this conjecture.

List of Participants

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Summer School Reports

Chapter 47

PIMS Summer School: BREAD Summer School in Development Economics (05ss100)

June 25 – July 1, 2005

Organizer(s): Siwan Anderson (University of British Columbia), Esther Duflo (Massachusetts Institute of Technology), Sendhil Mullainathan (Harvard)

The first summer school in analytical development economics organized by the fellows of BREAD (Bureau for Research and Economic Analysis of Development) was hosted by the Banff International Research Station between June 25 and July 1, 2005. BREAD is a non-profit organization dedicated to encouraging research and scholarship in development economics (http://www.cid.harvard.edu/bread/). Its members include both leading senior researchers in development economics and younger researchers working on issues of central importance for development. An important aim of BREAD is to foster academic interaction between researchers from different institutions and at different stages of their career, to promote the use of mathematical tools in the analysis of the development process. The BREAD summer school was an extremely important element of this process.

The problems addressed in the field of development economics are some of the most pressing facing researchers in economics today. The methods needed to analyze these problems in a rigorous manner have become increasingly technical. These methods range from mathematical tools developed in contract theory and positive political economy on the one hand to methods of statistical inference developed for evaluation methods ranging from randomized field experiments to estimation of dynamic structural models on the other.

A range of methodological approaches characterizes development economics. The BREAD Summer School is aimed at exposing students, in the formative periods of their research careers, to theoretical and econometric techniques outside that which they are exposed to in their home institutions. This expands and enriches their research capabilities and helps them break into new areas that may not have previously been on their research horizons. Deepening and broadening technical skills is an integral objective of the BREAD Summer School and is a key element of the formation of the students involved.

Over the course of five days, students attended three hours of lectures each morning and three hours of lectures each afternoon. Each three-hour lecture was given by a different BREAD fellow. Lectures covered the most up to date theoretical modelling techniques and empirical methods applied to the central topics in micro-economic development.

Specific lecturers included:

- Empirical Methods by Sendhil Mullainathan (Harvard University)
- Randomized Experiments by Esther Duflo (Massachusetts Institute of Technology)

- Land and Credit Markets by Abhijit Banerjee (Massachusetts Institute of Technology)
- Markets and Firms by Robin Burgess (London School of Economics)
- Education by Michael Kremer (Harvard University)
- Health and Nutrition by Duncan Thomas (University of California at Los Angeles)
- Technology Adoption and Technological Change by Andrew Foster (Brown University)
- Theories of Inequality by Dilip Mookherjee (Boston University) Political Economy
- Corruption by Rohini Pande (Yale University)

Students also presented their own work in progress and got feedback from faculty. Faculty held office hours where students go the opportunity to discus their research on a one-on-one basis.

Students feedback where extremely positive. They were particularly interested by the emphasis on methods and tools. The summer school has contributed to reinforce students' technical skills, and will hopefully contribute to make them ready to apply these skills in their own research.

List of Participants

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Chapter 48

2005 Summer IMO Training Camp (05ss006)

Jun 28 – Jul 09, 2005

Organizer(s): Bill Sands (University of Calgary)

The 2005 IMO Training Camp started on Saturday June 25 with the arrival in Calgary of four of the six student Team members and two of the three adult Team members. Two of the Team members, Peng Shi and Yufei Zhao, were permitted to show up at the Camp a few days late so that they could attend the awards ceremony in Washington for the USAMO, in which they had tied for third place. One of the adult Team members, Adrian Tang, is a graduate student at the University of Calgary and was already here. Also arriving on June 25 were two students from Edmonton and one from Vancouver, chosen, along with three students from the Calgary area, to participate in the Calgary portion of the Camp.

The participants were:

Team members: Lin Fei, Elyot Grant, Richard (Yang) Peng, David (Dong Uk) Rhee, Peng Shi, Yufei Zhao;

Adult trainers: Felix Recio (Leader), Dorette Pronk (Deputy Leader), Adrian Tang (Deputy Observer), Elena Braverman, Andy Liu, Paul Ottaway;

"Local" (Alberta and BC) students: Graham Hill and Brian Yu from Edmonton; Allen Zhang from Vancouver; Sarah Sun from Okotoks; and Zheng Guo and Yiyi Yang from Calgary.

Everyone was housed in Cascade Hall, an apartment-style Residence on campus, two students to a room. Meals were catered by the Students Union, at set times in a certain room.

Training began in the morning of June 26, with lectures and problem sets. The three adult Team members were assisted by Paul Ottaway, a graduate student at Dalhousie, who had just finished attending a math meeting at Banff and had volunteered to assist in part of the Camp. Another Calgary student, Boris Braverman, was invited to take part in this training during the day. In the afternoon, all seven Team members present at the Camp went with me to a nearby mall to purchase the Team uniforms (pants).

Graham Wright arrived in Calgary in the early afternoon of Monday June 27, to prepare for the Media Event to take place the next day. That evening he treated the adult Team members and me to supper at a nearby restaurant.

In the morning of Tuesday June 28 we were joined by three more (much younger) "local" students for the day: Jaclyn Chang and Hunter Spink of Calgary, and Mariya Sardarli of Edmonton, along with their parents. After some "fun" math activity in the morning, the "Media Event" took place from noon till 2:00 PM. Peng Shi and Yufei Zhao arrived in Calgary just in time to take part in this event and be introduced to the media and the guests along with the rest of the Team. The rest of the afternoon was spent in further "fun" training activity, and at 4:00 the Calgary portion of the Camp ended with the departure of all the local students. The

Team members and trainers were then driven to the BIRS facility in Banff for the remainder of the Training Camp.

BIRS of course was again an unmatchable setting for the Camp, with the facilities, the food, and the scenery all superb. (There were, however, noticeably fewer elk than in 2003.) Training continued, with an increased emphasis on mock contests.

Besides the concentrated training that took place at BIRS, the Team was taken on some excursions. On July 2 we drove to the Columbia Icefields with a stop at Lake Louise on the way back. That evening we went to see “War of the Worlds” at the Banff movie theatre. In the evening of July 5 most of us attended a concert at a Banff church. Then on July 6 we all walked up the Sulphur Mountain trail. The weather turned rainy once we were at the top of the mountain, and we eventually decided to take the gondola down rather than risk the slippery hike down the trail. Once at the bottom we visited the hot springs, where only two Team members were prepared to go into the pool.

On July 3 Paul Ottaway left the Camp to return to Halifax. On July 6 both Elena Braverman and Andy Liu arrived at the Camp to help out with the rest of the training, and Elena and her family took part in the excursion that day to Sulphur Mountain. On July 7 Felix left the Camp to fly to Toronto, where the next day he continued on to Merida, Mexico (the site of the IMO) to help prepare the contest. The rest of the Team stayed at Banff to continue training under the supervision of Dorette, and with the help of Elena and Andy.

The Team left Banff on July 9 and returned to Calgary. That afternoon everyone took in the Calgary Stampede. The next morning the Team left for Toronto, where they stayed in a hotel near the airport, and in the morning of July 11 they flew to Mexico.

Many thanks to:

- The staff and management at BIRS, especially **Brenda Shakotko**, the BIRS Station Manager, who made our stay there so memorable; also, **Gemai Chen**, the Calgary representative of PIMS, and **Nassif Ghoussoub**, the BIRS head, were both very supportive of the idea that the IMO Camp should be at BIRS.
- **Paul Ottaway** of the Department of Mathematics and Statistics of Dalhousie University, **Elena Braverman** of the Department of Mathematics and Statistics of the University of Calgary, and **Andy Liu** of the Department of Mathematics of the University of Alberta, who were Trainers during the IMO Camp.
- **Betty Teare**, Budgets and Administration Manager of the Department of Mathematics and Statistics of the University of Calgary, who helped to arrange the site of the Media Event, booked the food, and took the pictures at the Media Event.
- **Grady Semmens**, of Media Relations at the University of Calgary, for assistance in setting up and running our successful and enjoyable Media Event in the Learning Commons on campus.
- University of Calgary graduate student **Garth Boucher**, who drove the van taking the Team to Banff on June 28 and bringing them back on July 9.
- **Anthony Fink**, who helped meet the IMO Team at Calgary airport on June 25 and drove them to the University.
- **Tong Yu** of Edmonton, who drove Edmonton students Brian Yu (his son) and (Team member) David Rhee to Calgary on June 25.
- Former IMO Team member (and now University of Calgary student) **Alex Fink**, who helped out during the Calgary part of the IMO Camp.
- **Hugh Williams** of the Department of Mathematics and Statistics of the University of Calgary, who met Graham Wright at the Calgary airport on June 27 and drove him to the university.
- **Yanmei Fei**, a secretary in the Department of Mathematics and Statistics of the University of Calgary, who drove Graham Wright and Vancouver student Allen Zhang to the Calgary airport on June 28.

List of Participants

Braverman, Elena (University of Calgary)
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Chapter 49

Computing the Continuous Discretely: Integer-point enumeration in polyhedra (05ss027)

Aug 06 - Aug 20, 2005

Organizer(s): Matthias Beck (San Francisco State University), Sinai Robins (Temple University)

This is the final report of the MSRI/Banff Summer Graduate School at the Banff International Research Station, August 6–20, 2005. The theme was integer-point enumeration in rational polyhedra, which also goes by the name of Ehrhart Theory.

The Location

We found the Banff International Research Station nothing short of ideal for an MSRI graduate summer school. Lodging, meals, and lecture halls are all on the same site, away from the distractions, say, of a city. This setting results in an automatic networking among the students. The rapport among the students was wonderful to witness and had, needless to say, much positive influence on the mathematics that was discussed among them. The beautiful scenic setting of the Banff Center added excitement among the participants and fostered further interaction during free time. The staff on site was always helpful, and the computer support was very good. Brenda Shakotko, the BIRS Station Manager, deserves our sincere thanks for making sure that every little detail of the summer school was running smoothly.

The Schedule

We settled on an 11-day schedule for the summer school, starting with lectures on Sunday morning, having the middle weekend (Saturday and Sunday) off, and ending with the Friday afternoon session in the following week. Each morning consisted of two hour lectures separated by a half-hour coffee break. The afternoon started with two hour TA sessions, also separated by a coffee break. We had two excellent TA's, Kristin Camenga and Kevin Woods, who alternated from one day to the next. The afternoon sessions were early enough (starting at 1 p.m.) to allow ample time for the students to interact in smaller groups on their own time. This schedule worked very well for us. We got through material equivalent to one semester of a second-year graduate course. One has to keep in mind, though, that the students had a complete manuscript of the lecture notes, so that we could leave certain details to them (often in form of afternoon exercises). The middle weekend was intentionally left free, to allow the participants to explore the area around Banff. A few students volunteered to organize large group activities; we ended up having groups that went on a day hike, water rafting, a horse ride, a trip to Lake Louise and a nearby glacier, and a day bike ride. It is safe to say that there was an activity for every participant.

The Program

We were pleased that we were able to cover all 12 chapters of our book in progress *Computing the continuous discretely*, to be published in the Springer UTM series. The participants seemed excited about the material that was covered. While it is natural that not everyone can follow every detail, especially in the more advanced topics, the students showed 100% participation until the very last day. Similarly, the afternoon problem sessions were always lively, and all of them were attended by all the students. We were very grateful for the active involvement of our students and the TA's, all of whom gave us invaluable feedback on our manuscript.

The Participants

There were 30 participating graduate students from 22 universities in Canada, Mexico, and the US. Among them were 10 women and 8 under-represented minorities.

We conclude with some of the students' comments. They are quoted from a short survey that we took at the end of the first week.

"I think the lectures are great and I love their casual/informal style. "

"Too many mosquitos, but the time seems to fly during the lectures and the problem sessions. I like having a break every hour as well."

"This is wonderful material and both of you are giving nice lectures. Keep up the great work! The book is well-written, generally enjoyable to read. I would say the organization is a bit unorthodox. That makes the reading more interesting."

"I also very much enjoy the conversations and interacting with fellow math people."

"The location is great and the material is fun. Awesome summer school."

Acknowledgements

We are grateful for the generous support for the summer school from the Mathematical Sciences Research Institute, the Pacific Institute of Mathematics, and the Banff International Research Station. We believe that the MSRI Graduate Summer Schools are a wonderful institution, and that the Banff International Research Station is a perfect location for them.

List of Participants

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