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Contents

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Chapter 1

Quantum Technology: Computational Models for Quantum Device Design (12w5061)

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Introduction

Some of the world's experts on quantum engineering met at the Banff Centre in January 2012 at a workshop hosted by the Banff International

Research Station (BIRS) on recent developments and challenges in designing and building functional quantum devices. Until recently quantum physics was predominantly in the domain of physics explaining the workings on the universe at tiny scales and also at the heart of many philosophical discussion about the nature of the universe. Advances not only in physics, but also in manufacturing, computer science and mathematics now promise a wide range of applications of quantum phenomena from communication, to electronics, computing, biology and medicine. Quantum devices would function quite unlike classical devices and enable new functionalities to create faster, smaller and more energy efficient devices for everyday use.

Progress in nano-fabrication enables the realisation of ever smaller and increasingly sophisticated nanoscale devices whose behaviour is no longer classical but strongly influenced by quantum effects such as quantum tunnelling and quantum transport. These effects are unavoidable and are already an issue in conventional semiconductor technologies. Quantum effects such as coherence, interference and entanglement also hold significant potential to enable entirely new devices with novel functionalities. Recent advances in the novel area of coherent control of quantum systems, especially optimal control design, further enable unparalleled levels of control of quantum dynamics, in theory, simulations, and increasingly, laboratory experiments. This creates the real possibility of a new age of quantum engineering utilising quantum effects to devise novel technologies. Emerging applications range from quantum sensors, to spintronics and coherent electronics, to information processing with improvements in communication, imaging, metrology and medical diagnostics along the way.

However, the realisation of such devices is still a challenge. It will require the development of a comprehensive framework for quantum

engineering, including new tools for modelling quantum devices, simulations of quantum dynamics, continued improvements to existing models by incorporating experimental data and systematic system identification, control to achieve desired outcomes, and robustness analysis to identify robust device designs. The workshop aimed to advance this goal by bringing together experts in computational modelling, quantum simulations, coherent control, system identification and control engineering to identify key challenges and form new connections. BIRS, with its stimulating environment, excellent facilities and administrational support, provided the ideal location for this.

Device design requires the creation of a physical model of the device for which a control must be found to guide its dynamical behaviour. This in turn requires physically accurate simulation and also identification of quantum systems to build and verify such models. Hence, a critical task is the development of mathematical models that capture the key properties of the device sufficiently accurately. Modelling even basic properties of complex quantum systems or devices is already non-trivial, but the models required for coherent control of the dynamics of quantum devices are far more demanding. They must describe the dynamical evolution of the system and capture the influence of various possible controls such as voltages, external potentials or coherent control fields on the behaviour of the system. This means we need dynamic control system models of quantum devices.

Accurate models are particularly critical for quantum systems as most control strategies for these systems, even those involving direct or indirect feedback, are model-based. To find the best controls to accomplish various tasks from quantum state engineering to gate implementation to initialization, stabilization and decoherence suppression these strategies rely on a mathematical model of the system and the effect of controls on its behaviour. The computational methods used for simulations, control and identification are thus enabled by the model, but also limited by its restrictions. We also need highly efficient simulation tools to evaluate these models for the computationally very intensive operations required for control and design.

To a certain extent computational models for quantum devices can be derived from basic physical principles. Such models are usually not sufficiently accurate except for very simple systems as they fail to capture many details of the actual device which is subject to effects neglected in the model, imperfections, inhomogeneities, etc. To overcome such limitations, some degree of experimental system identification and model verification is needed to estimate and verify the parameters in the model. For the application of different computational techniques and efficient computation, often different models are required, which means they must be converted into each other. However, often there are no simple relations between different models, even for geometric models and much less for dynamic control models.

Overview

The workshop explored models and techniques for designing and controlling quantum devices to establish links between physical (quantum) device modelling, experimental system identification, model verification and quantum control. Having reliable mathematical and computational models and methods is crucial for quantum engineers to design realistic quantum devices. This includes tools that allow not only efficient simulation of complicated quantum devices such as semi-conductor nano-structures, superconducting devices or atom chips, but also device optimisation and dynamic control.

Models with efficient design and control methods are crucial to enable engineering complex quantum devices, whose ultimate importance lies in the applications. Quantum communication and encryption may be the most advanced area on the application side, with significant impact already on secure communication. However, similar high-impact technologies are on the horizon in electronics, metrology, imaging, biology and medicine.

Specifically the workshop evolved around the following questions:

1. What are the most promising applications in nearer future? E.g., semiconductor or superconductor devices for quantum metrology, information processing, or perhaps modelling and understanding quantum effects in biological systems, or medical applications. This further raises questions about what design tools are needed for the most promising applications. An important issue to consider is also the type of models needed for the different operations required for an application such as the relation between a geometric model of the device and the model required for dynamic control.
2. What techniques and models are needed to design, simulate and control the operation of these quantum devices? Dynamic control simulations require control system models, conventionally based on partial differential equations derived from fundamental principles, such as the control dependent Schrödinger or quantum Liouville equation. Efficient simulation algorithms for physically accurate computational models are crucial for this task. Statistical models simply describing the behaviour of an actual device may sometimes be sufficient, or at least be used to augment the differential equation approach to consider specific material properties and the complete behaviour of real devices.

3. What protocols for experimental system identification, parameter estimation and model verification are available? How efficient and reliable are these? What are their experimental requirements and how realistic are these? Do they provide the models we need? E.g. common techniques such as spectroscopy and quantum process tomography provide information about the system but do not directly provide the type of dynamic control systems models we require. How can techniques be combined, modified or extended to enable construction of the models we need?
4. What are efficient ways to solve the inverse of the modelling and simulation problem? How can we design and control efficient quantum devices, by, e.g., finding optimal device geometries, optimal dynamic voltage profiles applied to control electrodes or optimally shaped control pulses? What should be the main objectives of such optimizations? What are the practical constraints? How well can current algorithms cope with large-scale problems?
5. What computational tools are available and what new tools are required or desirable, especially with a view towards integration of device or system design, dynamic simulations of quantum evolution, experimental data analysis and design and control optimisation. What is the efficiency of simulation algorithms on different hardware platforms? What role could effective visualisation of raw simulation data and automated, though perhaps human-guided, data analysis play? How could it make the process of model analysis and verification more efficient?

These questions were discussed in the context of four areas over four days, details of which are reported below: (a) device modelling,

simulation and control of solid state qubits, (b) device modelling, simulation and control of biological and spin systems, (c) quantum system identification, (d) controlling quantum systems.

A diverse range of issues were covered, from computational modelling of physical systems, to high performance and parallel computing, to device design and nano-fabrication, to control and optimisation, to machine learning and pattern analysis, to quantum theory and experimental physics. This was reflected in the list of participants which come from mathematics, computer science, engineering and physics with relevant background in some of these areas. The discussions about identification, simulation and optimisation of quantum devices have shown that there are still many links between physics and control of quantum devices and engineering and computer science to be explored.

During the last day of the workshop the overall findings were drawn together to formulate major challenges to address to devise a roadmap towards computer-aided design systems for quantum devices, listing the required functionalities and various approaches to provide these, ensuring interoperability between the approaches as far as feasible, and dependencies between results and approaches. Furthermore initial plans for future workshops and conferences on quantum technologies were considered to track progress on addressing these issues.

Solid State Device Modelling

The presentations for the first day centred around the current state of quantum device modelling, particularly in solid-state qubit systems

(dots and donors):

- Richard Muller: Development of Few-Electron Si Quantum Dots for Use as Qubits;
- Jim Fonseca: NanoHUB and NEMO5: Science Cyberinfrastructure and Nanoelectronic Modeling Tool;
- Erik Nielsen: The QCAD Framework for Quantum Device Simulation;
- Suzey Gao: Semiclassical Poisson and Poisson-Schroedinger Solvers in QCAD;
- Rajib Rahman: High Precision Quantum Device Simulation with Atomistic Tight-Binding Technique Coupled with Semi-Classical Poisson Solver and Many-Electron Configuration Interaction Method.

The audience broke into four groups to address the following discussion topics on quantum system/device modelling. A summary of the reports is as follows.

Realism of the Model: Speed vs Accuracy and Comparison with Experiment

The central issue of theoretical modelling of quantum devices is how to describe the essential physics without compromising too much on realism and without the model becoming intractably complex. It was noted that one should first define clearly the purpose of modelling and the level of modelling (micro vs. macro) necessary. The realism required depends on the purpose of the modelling and the degree and precision of experimental knowledge of the system. For some systems, such as quantum dots, it was noted that semi-classical calculations appear adequate for most experimentalist needs at present, but fully quantum mechanical models may be required for new applications such as quantum information. At the engineering level a proper discretisation scheme or coarse graining level that represents the correct physics of the system is essential. There is always a trade-off between accuracy and complexity when choosing a discretisation or coarse graining of the microphysics involved. In some cases, such as when constructing stochastic models for real-time feedback control, improved model accuracy will come at the expense of increased feedback delay time, possibly favouring simpler models even if less accurate.

Compressed sensing was discussed as a particular tool to extract a minimal set of information pertinent to the behaviour of the system. Compressed sensing has been useful to reconstruct large but low-rank matrices given $n \ln n$ random entries using the Singular Value Threshold Algorithm. However, compressed sensing can be quite sensitive to noise (beyond a threshold) and it is less obvious how to apply this approach to reconstruct generators of the dynamics as opposed to dynamical maps or states.

Multi-Scale Models: from Microscopic Physics to High Level Device Description

For solid-state based quantum devices such as donor atoms embedded in a substrate or quantum dots one cannot hope to perform an ab-initio calculations up to and including the device scales without some form of coarse graining of the physics. The question therefore arises how to interface large scale ab-initio approaches such as density functional theory, which cover relatively small device volumes, with higher-level effective approaches such as tight-binding and Poisson-Schrödinger equation modelling.

The example of simulating coherent transport (CTAP) through a triple (donor) well structure was given: atomistic tight-binding simulations of a multi-million atom device with a range of gate biases were used to derive a higher-level description in terms of adiabatic control of an effective 3×3 Hamiltonian. Another example was the evolution of models for highly-doped phophorous in silicon nanostructures fabricated by STM, from ab-initio treatments of small supercell regions, to tight-binding to a high-level and computationally simpler self-consistent effective mass model informed by microscopic band-structure calculation.

The importance of understanding the domains of validity of a model or theory, and careful model validation by comparing with data from experiments or verifying predictions made by the model, were stressed, raising also the question of efficient strategies for validating models.

QCAD Approaches: Optimisation and Time-Dependence

For solid-state devices the focus was on how to optimise the atomistic tight-binding calculations, interfacing with gate-level simulation and going beyond single electron descriptions. For multi-electron device simulations the most expensive computations in the tight binding model are the Coulomb integrals. One approach to make the integrations more tractable that was discussed is the transformation from a plane-wave to a Gaussian basis although it was noted by one participant that this had been tried before (presumably unsuccessfully?). Other approaches such as matrix product states or density matrix renormalisation group may be useful if the system is effectively low-dimensional.

It was also discussed what other systems could be modelled with QCAD and what additional capabilities would be needed to be able to model other physical systems. Superconducting flux qubits appear to be amenable to QCAT modelling, which could be useful to model the effect of imperfect J-junction topography, for example. A comment was made about the problem that de-facto standard simulation tools, mainly finite element analysis, often do not preserve properties of the simulated continuous systems: a moving rigid body may gain or loose momentum or a cavity may exhibit fictitious eigenmodes in an electromagnetism (E-M) simulation. This loss of fidelity follows from discretisation processes which fail to preserve underlying geometric and topological structures on the continuous model. There are excellent results from the calculus of differential forms in E-M and Lagrangian mechanics to avoid such problems.

To incorporate the effect of temporally varying fields, one could compute an effective Hamiltonian in the configuration space imposing piecewise constant classical E-M fields. This was thought to be adequate for many applications. A suggestion was made to look at existing quantum dynamics simulators, e.g., Q-TIP. In this context it was also discussed how to differentiate between coherent and incoherent dynamics and control, considering signatures, e.g., related to quantum transport, external E-M fields, effective spin descriptions. There was also discussion about what a quantum system is, and it was and agreed that calling QCAD a quantum device simulator may be too broad, since there are types of quantum devices (e.g. biological systems) that it is not designed to simulate.

Open Quantum System Description

It was discussed when a given open quantum system should be modelled as Markovian, non-Markovian or a combination of the two, when the Lindblad formalism of open quantum systems was sufficient and when a stochastic model or something else was needed, and what effective descriptions of non-Markovian behavior existed in general. In principle, for any system interacting with non-Markovian environment, one may always enlarge the original system so that the interaction of the enlarged system with the environment can be treated as Markovian, where the shell added to enlarge the system is a finite-dimensional subspace of the bath degrees of freedom that encompasses the non-Markovian dynamics. In practice, finding the right shell is not trivial, however, and the resulting enlarged system may be too large to be computationally tractable.

Another approach to derive open-system models employed by some participants is singular value decomposition of the Redfield superoperator to obtain matrix product operators acting on matrix product states, and finally non-Hermitian generators for non-unitary dynamics. However, this approach is likely to work only for 1D models. An alternative used in NMR is to start with a Gaussian theory and move to lower temperatures by adding n-point correlations with increasing n, a sort of Lie closure of a cluster expansion. In biological systems it was suggested that 5-point functions are needed for an effective description.

Time-dependent density matrix renormalisation group (tDMRG) was also discussed, but it was suggested that instead of simulating the dynamics of the entire system as in tDMRG, we should find a renormalisation group approach that yields an effective theory where the effective degrees of freedom are those we actually care about. Information spreads via the system-bath interactions in the Hamiltonian and the Lieb-Robinson bound tells us how this information spreads. A renormalisation program restricting the computation to the cone of this spread could render otherwise intractable problems tractable. Objections to this approach were that general theories often yield prohibitively expensive routes to computation. On the positive side it was noted that Hamiltonian interactions of finite order yield graphs of finite order and often only low-order interactions need to be considered. For example, in NMR, the scale of the coupling and relaxation are both several Hertz, so that we only have to worry about a few concatenations of two-body interactions. It was also noted that in a linear chain or tree, there is a unique traversal of the chain making compression tractable.

No matter what approach is employed to model an open quantum system, the resulting model is likely to be high-dimensional and model reduction techniques may still be required to make the model computationally tractable. A compressed description could also help understand dynamic decouplings and the dissipative dynamics in general. While there are a number model reduction techniques out there, it is not obvious which are the most suitable and there was concern that it may be difficult to obtain accuracy criteria.

Biological and Spin Systems

The role of quantum effects and technology in biological systems has more recently attracted a lot of attention. This is likely because of the wide range of interesting applications in biology and medicine, including improving understanding of biological processes, light harvesting systems, sensors for biological and medical applications, medical imaging, targeted drug delivery, etc. Some of which have been discussed in the following presentations at the workshop:

- Martin Plenio: Quantum Coherence and Biological Systems;
- Stephan Hoyer: Quantum Control of Light-Harvesting Systems;
- Paul Rees: Mathematical Techniques for Cell Cycle Analysis;
- Jianming Cai: Diamond Based Single Molecule Magnetic Resonance Spectroscopy;
- Ilya Kuprov: Algorithms and Software for Large-Scale Quantum Spin Dynamics Simulations;
- André Bandrauk: Modelling Molecules in Intense Laser Pulses and Attosecond Simulations;
- Lloyd Hollenberg: Quantum Sensing Technology and Modelling.

Naturally there is a lot of overlap with modelling, simulating, system identification and control topics related to other areas. So details on some of the discussions have been combined with topics in other sections of this report. Nevertheless the different nature of the systems might mean that different techniques might be more effective and of course new ideas and insights from this area can also be useful for other applications areas. This exchange has inspired workshop participants to consider this area more actively, as in particular there is a range of immediate applications on the horizon, especially possibly in the form of bio-molecular systems as devices.

Simulation Tools for Quantum Biology

On the question of simulating quantum effects and technology in biology, it was mainly discussed how far any of the existing simulation tools would be useful for solid-state device simulation. In particular how far could similar approximations and simplifications used for biological systems also apply to solid state devices. Density matrix renormalisation group (DMRG) techniques were considered to be of practical use only in 1D and mainly of use for questions of ground state, though.

Structured and Non-Markovian Environments

A particular issue that arises in biological systems is the use structured and/or non-Markovian environments as a tool to design systems/devices. They might well be a resource utilised in biological systems by nature.

There are problems that cannot be solved by coherent control alone such as preparation of a system in a certain state starting with an unknown state or entropy reduction in general. Noise can help here by offering new effective pathways. Recent work on reservoir engineering has shown that environments can be designed to initialise and stabilise systems in a wide range of desired states. Furthermore, reservoir engineering can be achieved by various means including incoherent control, measurements and direct feedback. In the latter cases it is the backaction from the measurement and the feedback that effectively modifies the Lindbladians. Control by backaction is an interesting and promising idea but it was noted that in some cases such as for ensembles of spins backaction effects are very small. The focus of the discussion was mostly on classical baths but the question of nonclassical baths was raised.

Role of Coherence in Biological Systems and Quantum Biotechnology

It was noted that one can see quantum effects in room temperature NMR processes and that entanglement is pervasive throughout chemistry. Spin dynamics results from processes involving singlet-correlated electrons. Such processes may play a role in biological systems although likely at a different time scale than excitonic dynamics. For example, in radical chemistry the radicals interact as singlet pairs. Whether the pair is singlet or triplet depends on the magnetic field, making these processes magneto-sensitive both to internal and/or external fields. Such magnetic field effects appear to be used by birds to navigate while biological functions in humans appear to be insensitive to even strong magnetic fields as experienced in MRI scanners.

In the application of qubit technology for sensing in biology, quantum coherence of the probe itself needs to be maintained in the complex ambient environment. So far the nitrogen-vacancy defect centre in diamond is proving to be an ideal qubit platform for such quantum sensing (nano-magnetometry) applications in biology with a number of early proof-of-concept experiments already reported. It was noted that for sensing applications the modelling challenge is to understand the biological origins and nature of the magnetic fields being detected.

Quantum System Identification

Knowledge about a system's behaviour is required in order to use it to build practical quantum devices. This means that the model must describe the system with sufficient accuracy for the task. If no such model exists a-priori, certain parameters of the model, or even the complete model, must be derived from data about the physical system. While techniques such as spectroscopy and quantum process tomography provide information about a system, they typically do not describe it completely, especially its dynamical evolution and response to external fields. Yet, for effective control, such dynamic models of the system are required.

At the workshop common system identification techniques were discussed with a view to understanding how efficient, reliable and practically realisable these are. To start the discussion various system identification approaches and how these are used for quantum systems were presented in the following talks:

- Frank Langbein: Bayesian Learning;
- Jonathan Quinn: Optimisation;
- Steffen Glaser: Optimal Control of Uncoupled and Coupled Spins;
- Daniel Oi: Maximum Likelihood Hamiltonian and Decoherence Estimation;
- Daniel Burgarth: Quantum System Identification with Limited Resources;
- Clemens Muller: Characterizign Defects in Superconducting Phase Qubits;
- Edmond Jonckheere: Differential Topology of Adiabatic Gap.

These formed the basis for the group discussions on the topics summarised below.

Bayesian Estimation

Bayesian learning has been discussed in detail as a possible approach to system identification. The main difference to more traditional approaches is that Bayesian models explicitly encode the uncertainty in what is being learned based on the available observations. Bayesian approaches do not start with effectively arbitrary priors or pick a maximum likelihood answer, but provide a probability distribution. While this improves the prediction qualities of the model and avoids over- and underfitting problems, it does not actually provide a definite model and is quite expensive to compute. So approximations and simplifications are required.

Core questions to address are to determine to what accuracy the model actually has to be learned for the problem it is used for, and what we can actually learn about a quantum system purely from (repeatedly) measuring it. Selecting suitable parameters and models appears crucial, but also requires a way to test the appropriateness of the model and the parameters to learn. Likely a hybrid approach between a completely modelled and a completely statistical model is required to achieve a useful, but computationally feasible system identification approach.

Because of the computational costs and the potentially large amount of measurements needed, important questions on the Bayesian approach are how well it will scale to larger systems, with potentially thousands of random variables to be learned, and how robust the approach is to deal with real physical systems under non-ideal conditions, and what priors and approximations should be used (depending on the model). To determine the dynamics of a quantum system it is not at all clear at what times to measure, how to initialise the system and what to measure (repeatedly) to obtain a dynamical model that is sufficient for control. Partly this might be determined from a physical model of the system using utility theory and value of information approaches. Yet this is only as good as the model describes the system and in particular achieving high accuracy might be difficult.

To improve data acquisition it was suggested to use compressed sensing techniques, possibly also for sparse defect identification and to determine biases on the fly. This would largely depend on selecting physically possible compressive sensing bases that reveal sufficient

information about the system.

There was also some discussion about whether spectroscopy or Bayesian learning is more suitable. However, it seemed that Bayesian learning might well be used in spectroscopy or spectroscopy could provide data to Bayesian models to learn the system behaviour. Possibly Bayesian techniques may also be used to estimate parameters in a tight-binding model.

Selecting a specific model, e.g. for a maximum likelihood approach, usually requires an optimisation or search method which may struggle with the complicated “landscape” of the probability distribution. Various approaches toward approximating the probability distributions to enable identifying the peaks without computing the whole distribution may be used, also for the evaluation of the distribution over all possible outcomes. Gibbs sampling or swarm intelligence approaches could also be employed for evaluation and optimisation.

Role of Control in System Identification

One central issue raised was whether (open loop and feedback) control could help to estimate parameters. In particular feedback control and system identification seem to address related issues. Linear classical systems are often secretly subject to feedback, which often suppresses the impact of the actual parameters on the system's behaviour. This might work similarly for using feedback to control quantum systems and avoid that the actual parameters need to be determined, but only their impact on the system dynamics must be suppressed. Specifically to deal with idiosyncrasies of a specific system, feedback might then be better than complete identification.

However, unlike classical systems, a quantum feedback mechanism becomes part of the system that is intended to be controlled. Hence, feedback might make the system non-Markovian. While this can be avoided by considering the control and system variables together, one must be cautious, particularly if the feedback mechanism may in fact amount to a measurement. Effectively the results from an observation is fed back into some parametrised family of controls. When the measurements are weak measurements, as in some examples of quantum optics, then, to some degree, it is understood how to treat such systems. Extending and improving the understanding of such approaches and identifying realisable classes weak feedback systems might lead to bootstrapping methods for quantum devices.

These discussions further raised questions on how far a system with feedback is or remains controllable and how to deal with non-Markovian systems; including discussing examples of non-Markovian systems and how to determine that a system is Markovian. Typically we apply control to a system, make an observation and throw the system away, hoping that we can learn the dynamics from these measurements. Instead, is it possible to learn more about the system if we do not throw it away? At least for understanding the "Markovianity" of the bath, the answer seems to be positive. The control process and the partial measurement in the first iteration could push information into the environment, and the subsequent iteration could perhaps detect that the environment has remembered this information.

Another issue considered was if feedback models from control can be combined with with Lie Algebraic approaches into an overall framework as either side misses something: the interface between quantum and classical models, semi-classical modelling; Heisenberg vs. Schrödinger picture. Are the feedback approach and Lie algebra approach compatible?

Modelling for System Identification

Models provide prior knowledge for system identification, usually as a hard constraint and not a probabilistic prior. Models may nevertheless be used to provide priors to Bayesian estimates. It is critical to choose the model such that the relevant parameters can be estimated efficiently, while it also captures the essential physics and is suitable for control. While system identification without any model (or rather a very generic model) may be possible, it might be advantageous to use models more specific to the problem that encode existing knowledge about the system for faster and more accurate identification. This may in certain cases even reduce very complicated systems to low-dimensional models (in terms of identification complexity).

Currently the focus seems to be too strongly on single devices performing a simple operation and not a complete mechanism. This includes models to deal with open quantum systems robustly, especially considering environmental noise. To model more complex mechanism, hierarchical models may also be required. This hierarchy might be on a multi-scale level, but also on other aspects, such as function or physics. Certainly one will have to use different models as they are needed and relations, as far as relevant, between the different models must be clear.

Certainly not everything needs to be modelled, but it remains unclear how much we actually need to know about a system to control it effectively. However, if the observables relevant to a calculation we care about cannot separate two potential implementations, we likely do not have to care either. This, however, means system identification is not just about finding parameters for a model, but includes the at least equally important issue of determining whether a particular model describes the system sufficiently well for the control task. Furthermore, if the model is determined to be insufficient, approaches to expand it are needed.

Another question considered was the issue of how far the environment can be characterised with system identification approaches, and consequently also how it could be used as part of the function of the device instead of considering it to be a defect. This would require identifiable properties of the environment that can be included in the model. Restrictions in the model, prior knowledge and incomplete knowledge about the environment might affect the environment's characterisability.

Quantum Control

For most applications of quantum physics in engineering, control takes a central role. Modelling, simulation, system identification, etc. seem to be largely in the service of dynamic control. The only exception there is the actual device design, which forms the static companion of dynamic control. Both are crucial to obtain effective devices and a large part of the task can be automated, meaning that in particular dynamic control algorithms, alongside the actual design of a device and how it can be controlled, are central to many quantum technologies.

Consequently at the workshop many different approaches toward controlling quantum systems have been discussed based on the following talks:

- Sophie Schirmer: Closing the Modelling-Simulation-Control-Identification Loop;
- Matthew James: Quantum Feedback Networks;
- Ian Petersen: Robust Stability of Uncertain Quantum Systems;
- Giulia Gualdi: Appoximating Open Quantum System Dynamics in a Controlled and Efficient Way: A Microscopic Approach to Decoherence;
- Tomasso Calarco: Quantum Technology Taken to its (Speed) Limit;
- Barry Sanders: Efficient Quantum Algorithms for Simulating Hamiltonian Evolution on a Quantum Computer / Optimal Phase Estimation Using Particle Swarm Optimisation;
- Thomas Schulte-Herbruggen: Symmetry Principles in Quantum Systems Theory.

These lead to many group discussions on a variety of topics as summarised below.

Adiabatic Transport and Adiabatic Computing

The discussion on adiabatic transport and computing focused on clarifying the different terms.

Adiabatic passage or transport refers to adiabatic following of the system state under a perturbation of the Hamiltonian. It is an old technique that has been used for laser control of atomic and molecular systems, e.g., in the form of STIRAP for a long time and has recently found new applications in solid-state quantum devices. Adiabatic transport can be optimised in a non-adiabatic way when the target state is known.

Adiabatic computing is related to the idea of adiabatic passage. It can be used for computation is by encoding the result of a computation in the ground state of a target Hamiltonian. As a given Hamiltonian is adiabatically transformed into the target Hamiltonian, the ground state of the starting Hamiltonian evolves into the ground state of the target Hamiltonian, which is the desired solution. Control may be important for adiabatic computing to change the path in Hamiltonian space. D-wave was mentioned as an example of a commercially available prototype quantum computer constructed from superconducting elements based on the principle of adiabatic computation (using Hamiltonian homotopy of classical Ising models), but doubts were expressed whether it is indeed an adiabatic *quantum* computer or simply an adiabatic classical computer.

Validity of Rotating Wave Approximation

The validity of the rotating wave approximation (RWA) depends on the relative time scales in the problem at hand. In particular, the Rabi frequency must be much smaller than transition frequency. The RWA may therefore break down when control fields with high amplitudes are applied, which could invalidate solutions obtained by unconstrained optimisation based on RWA Hamiltonians. It was also suggested there may be a connection between this problem and the Markovian assumption in that the neglected dynamics in the environment may interact with the dynamics neglected in the rotating wave approximation.

Necessity of Measurement for Quantum Control

Open-loop coherent control does not require measurements while the control is acting on the system. However, state preparation or gate implementation using open-loop coherent control relies on knowledge of the initial state of the system. Initialisation can be accomplished either by measurement, collapsing the state of the system to an eigenstate of the observable (in the simplest case), or by relaxation dynamics that drive the system to a definite steady state. Coherent control assisted by relaxation dynamics can also be used to drive the system to a desired target state but coherent control alone cannot do this as entropy is conserved under unitary evolution. The latter process could be interpreted as measurement by the environment. Beyond this, it was agreed that measurement can be useful in certain quantum control problems depending on the control objective. Measurement, for example, enables stabilization of states, the possibility of environment-driven control and reservoir engineering.

Coherent Control, Feedback and Quantum Error Correction

Quantum error correction relies on encoding quantum information in a physical quantum system in such a way that errors that take the system state outside a code space can be detected and subsequently corrected. Typically error detection is done by performing so-called syndrome measurements, which identify the error that has occurred (amongst a discrete set of codeword errors) but leave the code space invariant. Subsequent correction is usually achieved by applying unitary gates. Error correction in encoded logic gates can also be accomplished using coherent or measurement-based feedback.

Quantum error correction can be thought of as a form of discrete-time feedback control in the sense that the syndrome measurements identify the error that has occurred (amongst a discrete set of possible errors) and the subsequent unitary operations applied are conditional on the outcome of the error syndrome measurement. However, it is different from the continuous real-time feedback control schemes commonly found in classical engineering applications, and some participants considered that error correction was only superficially like feedback control. There was consensus, however, that the implementation of the dynamical maps required to correct the errors can be posed as a coherent control problem and that control could lower the overhead due to error correction by lowering the error rate and improving gate fidelities.

On the question of quantum feedback network modelling, the question was raised about constructing a Lie algebra description of a quantum network connected by quantum wires and allowing swap operations based on the Lie algebra description of the subsystems. The argument was made in the 90s by Lidar, Zanardi et al that decoherence free subspaces constituted some sort of error correction, but, when viewed this way, it is a mode of error correction which requires exponential resources. It was thought that an analogy with classical error correction was not useful and that a feedback control formulation may be useful in the quantum case because of the inherent dynamics of quantum systems.

Deleterious Backaction Effects vs Control by Backaction

Unlike in the familiar classical case, measurements of a quantum system disturb the system and can change its state and dynamics. Similarly, even direct Hamiltonian feedback conditioned on a measurement results in additional terms that disturb the system. One explanation for this so-called backaction effect is that the system and controller become entangled. The strength of the measurement backaction is related to the measurement strength; generally the more information about the system we acquire the more the measurement will disturb the system. Similarly, the strength of the feedback backaction depends on the feedback control strength. It was suggested that this implies a trade-off between the quality of the control and the quality of knowledge about the system, and perhaps that entanglement between the controller and the system could be the quantifiable measure of the backaction.

Backaction effects are usually considered a nuisance especially from the point of view of control. However, recent work on reservoir engineering has shown that backaction effects can be a resource for control especially quantum state engineering and stabilization provided we have precise knowledge of the backaction. It was therefore suggested that perhaps it is our ignorance of the specific nature of this back action that is deleterious. It was also suggested that measurement-based quantum computation could be regarded as an example of control by backaction. It was noted that there remains considerable confusion regarding the terminology.

Relevance of Robustness and Stability

The meaning of the terms robustness and stability were debated. Stability is a concept that derives from dynamical systems. A steady state of a dynamical system is stable if the state remains close to the steady steady when subjected to small perturbations. A stronger concept is that of asymptotic stability or attractivity which provides that the system returns to the steady state after a small perturbation. For quantum systems subject to Markovian dynamics, asymptotic stability is equivalent to the existence of a unique steady state and implies global attractivity, i.e., convergence to a known equilibrium state. It was thought that this could be useful for system initialisation which could then be followed by open loop control.

Robustness can be regarded as a measure of feature persistence in a system subject to perturbations. For a system subject to control such perturbations can arise due to environmental effects or systematic errors or noise in the controls. Minimising the sensitivity of the systems to such perturbations was generally thought to be important. Robustness with respect to model uncertainty, especially uncertainties in Linblad terms, was considered to be an important issue and the question was raised whether coherent or measurement-based control can compensate for uncertainties in the Linblad terms. It was thought that this might not be possible in general, but may be possible for physically relevant problems.

A related issue of stabilisation to an identity operator was raised. It was thought that this might be possible only on a certain subspace of the whole space. It was thought that this might relate to the concept of rank-k numerical range. This also led to a discussion of dynamic decoupling for a given subspace and raised issues of error avoidance versus error correction.

Quantum Device Networks

Assuming we succeed in building individual quantum elements such as qubits, the next step is to connect them and create networks of interacting quantum systems. In this context, it was thought to be important to understand the connectivity of the interaction graph, conditions for approximate locality, the velocity of information propagation and the topology of graph.

Outcomes and Challenges

The final day's discussion focused on outcomes and directions. On the general issue of modelling, capturing essential physics in a simple way and scalability of the models were seen as the most important issues for systems from silicon-based solid-state devices to quantum biology. In particular, capturing the influence of the environment on the system was considered to be very important and identifying the best approaches (Lindblad, QSDE, etc) for modelling different types of open systems. Among the challenges are complexity and uncertainty. Beyond basic system identification, how can we quantify our confidence in model and how can we validate models? How much uncertainty can be tolerated before control becomes ineffective? The latter issue is closely related to robustness and stability of controlled systems, which were thought to be important issues that require further study. It was also noted that some powerful tools (Bayesian learning, finite element analysis, etc.) have not yet been fully exploited for quantum modelling. Another suggestion made was to focus on Heisenberg picture as this formalism is more natural for quantum stochastic differential equations and understanding the dynamics of quantum observables. Finally, it was thought to be important to progress from study of single devices to quantum networks and to develop a better, and especially more integrated, mathematical framework for quantum control.

The workshop also improved the mutual understanding of the various problems and issues involved and enabled participants to network, across different fields. This was instrumental in promoting collaborations and the formation of the EU-funded network QUAINT, and a Welsh network for Quantum Technologies (QYMRU). Some of the participants met again a the the Kavli Institute for Theoretical Physics in Santa Barbara as part of the special program on Control of Complex Quantum Systems during the first quarter of 2013. Other follow-up meetings include a Hereaus Seminar (2013) and a workshop at the Newton Institute in 2014. There are also efforts on the way to create a conference series on quantum technologies and engineering linked with a journal as a regular platform to discuss and disseminate results.

Acknowledgements

We thank the participants of the workshop for their active engagement, interesting talks and discussions, and notes from the discussions which formed the basis for this report, and BIRS for the financial support and excellent organization.

Appendix: Talk Abstracts

Modelling Molecules in Intense Laser Pulses and Attosecond Simulations

Bandrauk, André D
Université de Sherbrooke

The description and simulation of the nonlinear nonperturbative interaction of molecules with intense laser pulses requires the numerical solution of multidimensional partial differential equations, pde's such as the time-dependent Schroedinger equation, TDSE for fs-asec electron-nuclear movement beyond traditional Born-Oppenheimer approximations and for the zps time scale, TimeDependent Dirac Equations, TDDE's for relativistic effects such as the increase of spin-orbit coupling with increasing nuclear charge and pulse intensity. The complete description of the nonlinear nonperturbative response of intense molecule-laser interactions requires finally the coupling of Maxwell's (laser, photon) equations to include collective effects via propagation in a medium. The main goal of the FAZSST large scale computation project is to use advanced numerical algorithms on large memory machines to advance the development of the new Attosecond Science via numerical solutions of multidimensional TDSE and TDDE's coupled

to appropriate Maxwell equations.

Quantum System Identification with Limited Resources

Burgarth, Daniel
Imperial College

The aim of quantum system identification is to estimate the ingredients inside a black box, in which some quantum-mechanical unitary process takes place, by just looking at its input-output behavior. Here we establish a basic and general framework for quantum system identification, that allows us to classify how much knowledge about the quantum system is attainable, in principle, from a given experimental setup. Unlike usual quantum metrology, our theory applies to cases where the set of observables are not topographically complete, and where knowledge about the system can only be retrieved indirectly. When the topology of the system is known, the framework enables us to establish a general criterion for the estimability of the coupling constants in its Hamiltonian.

Diamond Based Single Molecule Magnetic Resonance Spectroscopy

Cai, Jianming
University of Ulm

We demonstrate that a single nitrogen-vacancy (NV) center in diamond can be used to construct a nanoscale single molecule spectrometer. The proposed device may find applications in single molecule spectroscopy in chemistry and biology, such as in determining protein structure or monitoring macromolecular motions and can thus provide a tool to help unravelling the microscopic mechanisms underlying biomolecular function.

Quantum Technology Taken to its (Speed) Limit

Calarco, Tomasso
University of Ulm

Quantum effects form the basis of most present-day information technologies. However, the full power of quantum coherence has not yet been tapped for everyday technological applications. The exquisite level of control of current atomic physics experiments may enable this, for instance in the field of quantum communication and quantum computing - but scalable quantum information processing requires extremely precise operations. Quantum optimal control theory allows to design the evolution of realistic systems in order to attain the best possible performance that is allowed by the laws of quantum mechanics. I will present a range of its applications to a variety of quantum technologies, and discuss its use in probing the ultimate limits to the speed of the corresponding quantum processes.

NanoHUB and NEMO5: a Science Cyberinfrastructure and a Nanoelectronic Modeling Tool

Fonseca, Jim
Purdue University

The NanoHUB is a cyberinfrastructure for the development, deployment, and use of scientific software. Students, teachers, developers, and researchers are able to collaborate in an online environment. A web browser provides the user with access to available tools that are run on high performance parallel computers. One tool that is under construction for nanoelectronic modeling is NEMO5. The core capabilities of NEMO5 lie in the atomic-resolution calculation of nanostructure properties: a multimillion-atom strain calculation, bulk electron and phonon band structures, a 1-D Schrödinger-Poisson simulation, a multiphysics simulation of a resonant tunneling diode, and quantum transport through a nanowire transistor.

Semiclassical Poisson and Poisson-Schrodinger Solvers in QCAD

Gao, Suzey
Sandia National Laboratories

We discuss the technical details of the semiclassical Poisson and the self-consistent quantum Poisson-Schrödinger solvers in the Quantum Computer Aided Design (QCAD) LDRD project at Sandia. We have developed as a first step a nonlinear Poisson solver, including Boltzmann or Fermi-Dirac statistics and incomplete ionization of dopants, which provides a good first-order description of the electrostatics in quantum devices. We have also developed a self-consistent Poisson-Schrodinger (P-S) solver to capture the unique quantum effects. This description includes the exchange-correlation potential within the local density approximation. Efficient convergence of the self-consistent solution is achieved using a predictor-corrector iteration scheme. Applications of the QCAD framework to 1-, 2-, and 3-D quantum systems demonstrate high accuracy and robustness of the solvers.

Optimal Control of Uncoupled and Coupled Spins

Glaser, Steffen
Technical University Munich

Based on principles of optimal control theory, the physical limits of quantum control can be explored and time-optimal and relaxation-optimized pulse sequences can be designed to control the dynamics of spin systems. Furthermore, robust pulse sequences, compensating experimental imperfections and taking into account experimental limitations can be optimized. Recent advances include time-optimal pulses for saturation and for maximizing contrast in magnetic resonance imaging (MRI), robust pulses that effect point-to-point transfers and/or desired unitary transformations that tolerate a large range of rf scaling, e.g. a factor of six in toroid NMR probes. As demonstrated experimentally, these pulses make it possible to perform sophisticated 2D NMR experiments on this hardware platform. The design of cooperative pulses opens new avenues for improved performance of multi-pulse experiments. Finally, the application of optimal control methods to the problem of spin decoupling yields not only significantly improved

performance but also unprecedented flexibility in the design of tailored decoupling sequences.

Approximating Open Quantum System Dynamics in a Controlled and Efficient Way: A Microscopic Approach to Decoherence

Gualdi, Giulia
University of Kassel

We demonstrate that the dynamics of an open quantum system can be calculated efficiently and with predefined error, provided that a basis exists in which the system-environment interactions are local and hence obey the Lieb-Robinson bound. We show that this assumption can generally be made. Defining a dynamical renormalization group transformation, an effective Hamiltonian for the full system plus environment is obtained that comprises only those environmental degrees of freedom that are within the effective light cone of the system. The reduced system dynamics can therefore be simulated with a computational effort that scales at most polynomially in the interaction time and the size of the effective light cone. Our results hold for generic environments consisting of either discrete or continuous degrees of freedom.

Quantum technology: qubits in quantum computing to sensing in biology

Hollenberg, Lloyd
Centre for quantum Computation and Communication Technology, University of Melbourne, Australia

The fundamental building blocks of a quantum computer are qubits - individually controlled two-level quantum systems. As the march towards quantum computing continues apace, the quantum control of qubit systems has been achieved in a range of physical platforms - electronic, photonic and atomic - together with a deep understanding of their environmental interactions. In this talk we will discuss the use of a qubit system as a nanoscale probe, where the qubit is deliberately exposed to the worst possible, but most interesting, environment - room temperature biology. The nitrogen-vacancy (NV) defect centre in diamond represents an ideal single spin qubit for use in biology as a nanoscale magnetometer probe. It possesses a broad absorption band from 512-560 nm, sustained fluorescence from 630-750 nm, is chemically inert and bio-compatible and most importantly has relatively long quantum coherence at room temperature. These defect centres have been used as highly stable fluorescence beacons to track the position and diffusion of diamond nanocrystals in vitro and in vivo. Recent experimental demonstrations of nanomagnetometry using these single spin systems create opportunities for new applications in biology. We explore the viability of diamond-based nanomagnetometry bio-applications by performing the full suite of quantum control and measurement protocols on NV-nanodiamonds in a living human HeLa cell, and demonstrate the enhancements that the quantum properties of the NV qubit enable for orientation tracking in the intra-cellular context.

Quantum Control of Light-Harvesting Systems

Hoyer, Stephan
University of California, Berkeley

I will review the evidence for the relevance of quantum coherence to photosynthetic energy transfer, and discuss our theoretical progress towards controlling these dynamics using shaped ultra-fast laser pulses. I will also present a scheme for experimental validation in a pump-probe spectroscopy setup.

Quantum Feedback Networks

James, Matthew
Australian National University

Background: Systems and control theory is concerned with ensuring that systems behave in a desired way. At the most basic level, the system should operate in a stable manner, and beyond this, systems and control theory endeavours to assess, and where possible, optimise, performance. For example, in the 18th century James Watt designed a governor system to ensure that his steam engines maintained a constant speed, (and as is well known, the steam engine powered the industrial revolution). The steam engine governor is a classic example of a feedback control system. A feedback system is a network: an interconnection of the system being controlled (e.g. steam engine or atom) and the system facilitating the control (e.g. governor or another atom). Networks are ubiquitous in science and technology, e.g. electronic circuits, the internet, etc, etc.

Over many years, many methods have been developed for modeling and design. Models and design techniques have been developed for particular purposes, at different levels of abstraction and detail. For example, detailed transport equations could be used to describe the flow of electrons and holes in a transistor within a standard operational amplifier chip, or a simple model could be used when the opamp is included in a negative feedback circuit. Widely used abstractions include Gaussian and white noise models (e.g. for the movement of particles in a fluid, or to describe sensor noise) and Boolean algebra (used to describe classical logic). Methodologies that have been developed at these levels of abstraction include probability and stochastic processes, information theory, and computational complexity.

At the present time, quantum technology is developing rapidly, spurred on by the significant potential of key application areas, notably, quantum computation, information, and metrology. The qbit is now omnipresent, and fundamental to the conceptual and experimental developments in quantum technology in recent decades. The qbit, of course, is a quantum generalization of the classical bit from classical logic, and indeed quantum generalizations of information theory and computation have been developed. Gaussian states are very common, particularly in quantum optics, and a theory of quantum stochastic processes has been available now for several decades. However, this later theory has yet to be fully exploited in quantum technology.

This talk will discuss ideas concerning quantum feedback networks (QFN). These ideas go back to Yurke-Denker in the 1980's, Wiseman-Milburn, Gardiner and Carmichael in the 1990's, and Yanagisawa and others in the 2000's. QFNs consist of components modeled as open quantum systems, interconnected by free fields and/or by direct physical couplings (it is worth thinking about how electronic amplifiers are cascaded together, and how a steam engine is connected to a governor). An important feature of the developing theory of QFNs are the simple symbolic rules for describing interconnections, based on underlying quantum stochastic models. The talk will discuss this theory, as well as some recent applications in quantum optics and quantum information.

Differential Topology of Adiabatic Gap

Jonckheere, Edmond
USC

In this talk, we provide a new interpretation of the adiabatic theorem in quantum computation with special attention to the gap problem and the crossing avoidance of the various energy levels. It is indeed argued that the mere plotting of the various energy levels versus the time spent in the adiabatic process hides some deeper issues as to how the energy levels are intertwined. Central in this new interpretation is the numerical range of a matrix constructed with the initial and terminal Hamiltonians. The fundamental fact is that the various energy levels encountered along the time through the adiabatic process are the various critical value curves of the numerical range viewed at an angle that parameterizes the time spent in the process. The boundary of the numerical range represents the ground state, while that curve closest to the boundary is the first excitation level. Generically, the boundary curve is smooth, but all other energy level curves are highly singular with cusps. With this interpretation,

the known scaling of the gap is easily recovered and new results are derived from topological consideration on the numerical range. But probably most importantly, this approach explains how the various energy levels are intertwined by a two-stage unraveling process: (i) by going from the classical energy level plots to the highly singular critical value curves in the numerical range, and (ii) by lifting the critical value curves to (smooth) Legendrian knots in a contact space. The smooth curves in the contact space are possibly knotted, possibly linked, which provides the ultimate explanation of the non-crossing phenomena observed "down" at the most elementary level of the classical energy level plots.

Algorithms and Software for Large-Scale Quantum Spin Dynamics Simulations

Hogben, HJ[1], Edwards, LJ[2], Krzystyniak, M[2], Charnock, GTP[2], Hore1, PJ[1], Kuprov, Ilya[2]
[1] Chemistry Department, University of Oxford, South Parks Road, Oxford, UK.
[2] Oxford e-Research Centre, University of Oxford, 7 Keble Road, Oxford, UK.

We introduce a software library incorporating our recent research into efficient time-domain simulation algorithms for large spin systems. Liouville space simulations (including symmetry, relaxation and chemical kinetics) of most liquid state NMR experiments on 40+ spin systems can now be performed without effort on a desktop workstation. Much progress has also been made with improving the efficiency of ESR, solid state NMR and Spin Chemistry simulations. The functionality available at the time of writing includes:

- Low-dimensional matrix representations for spin operators in large spin systems that enable the simulation of magnetic resonance experiments on systems previously considered too big for any practical simulations.
- Generalized symmetry module (any number of groups of equivalent spins of any quantum number).
- Krylov subspace based time propagation routines that avoid matrix exponentiation.
- Generalized rotation module and a Lebedev powder integrator.
- Generalized relaxation theory module, supporting all types of magnetic resonance spectroscopy (NMR, ESR, DNP, Spin Chemistry, etc.). Anisotropic rotational diffusion tensors are supported in full generality.
- Optimal Control waveform design module using BFGS-GRAPE algorithm with exact gradients. Optimization of broadband pulses, selective pulses and universal rotations is implemented in both Cartesian and phase-amplitude coordinates.
- Functions for multi-grid parallel soft pulses and (algebraic) decoupling.
- Functions for the simulation of magnetochemical experiments.
- Functions for the simulation of common NMR experiments (COSY, DQF-COSY, NOESY, HSQC, HMQC, HETCOR, etc.), basic ESR experiments (ESEEM, ENDOR, etc.) as well as building blocks for the writing of user-specified experiment simulations.

Spinach is an open-source Matlab library available at http://spindynamics.org.

Bayesian Learning and Optimisation

Langbein, Frank C, Quinn, Jonathan A
Cardiff University

Machine learning aims to construct a model for an unknown or only partially known environment. This is directly related to identifying quantum systems. We give an overview of various machine learning techniques, the underlying models and efficient algorithmic techniques, especially focused on Bayesian learning techniques. In the second part we overview core search and optimisation methods, useful for model fitting, parameter identification and control.

Development of Few-Electron Si Quantum Dots for Use as Qubits

Muller, Richard P, Bielejec, ES, Bishop, N, Ezra Bussman, E, Carroll, MS, Gao, S, Landahl, A, Lilly, MP, Lu, TM, Nielsen, E, Pluym, T, Rahman, R, Shirkhorshidian, A, Stalford, HL, Tracy, LA, Witzel, WM, Young, RW
Sandia National Laboratories, Albuquerque, NM

This talk will describe a modeler's perspective on existing experimental efforts to create a Si qubit, the challenges to overcome, and the opportunities for modeling to have an impact on the experimental direction. The talk will review options we?ve considered for the materials stack and the gate layout, and will discuss how issues like spin- and charge disorder, valley splitting, and gate design impact the development of few-electron quantum dots. I will conclude with observations on how computer simulation can help accelerate this development. This work was supported by the Laboratory Directed Research and Development program at Sandia National Laboratories.

Characterizing Defects in Superconducting Phase Qubits

Müller, Clemens
Université de Sherbrooke

Spectroscopy of superconducting phase qubits often shows characteristic anti-crossings, indicating the presence of additional coherent and strongly coupled quantum systems. The origin of these two-level defects is not yet fully understood, although a variety of models have been proposed. I will show a summary of our joint theory-experiment efforts to determine the parameters of possible microscopic models of the defects in order to determine their physical nature. I will also talk about the possibility of applying Hamiltonian tomography on this system and present the specific challenges we face there.

The QCAD Framework for Quantum Device Simulation

Nielsen, Erik
Sandia National Laboratories

The Quantum Computer Aided Design (QCAD) LDRD project at Sandia National Labs is developing an integrated tool designed for the simulation of few-electron quantum devices. We discuss the software framework and capabilities of this tool, and compare these with existing tools. We emphasize the close relationship with our experimental effort, and discuss how QCAD has been utilized for rapid design guidance. This work was supported by the Laboratory Directed Research and Development program at Sandia National Laboratories.

Maximum Likelihood Hamiltonian and Decoherence Estimation

Oi, Daniel
Strathclyde University

Advances in quantum systems engineering and control are leading to the ability to fabricate and manipulate large scale devices operating in the coherent regime. However, precision operation requires knowledge of the behaviour of the system, both in response to control fields and to environmental noise and decoherence. The traditional approach to system behaviour, process tomography, presupposes the ability to prepare complete sets of initial states and to be able to measure in many different bases, abilities which may not be initially present without control. A method is required to bootstrap the characterisation process utilising a more restricted set of preparation and measurement primitives. Even for small systems, signal complexity can overwhelm conventional analysis. We have applied maximum likelihood methods to extract model parameters and invert system dynamics. Scaling up and increasing efficiency remain as challenges, these may be possible using adaptive experiment design and compressive sensing methods.

Robust Stability of Uncertain Quantum Systems

Petersen, Ian
UNSW

This talk considers the problem of robust stability for a class of uncertain quantum systems subject to unknown perturbations in the system Hamiltonian. Some general stability results are given for different classes of perturbations to the system Hamiltonian. Then, the special case of a nominal linear quantum system is considered with either quadratic or non-quadratic perturbations to the system Hamiltonian. In this case, robust stability conditions are given in terms of strict bounded real conditions.

Quantum Coherence and Biological Systems

Plenio, Martin
University of Ulm

The interplay between coherence and the vibrational environment is of key interest for the quantum dynamics of bio-molecular systems. In this talk I will present some fundamental ideas in this regard, then I will discuss briefly methods to simulate the dynamics in the relevant regime of intermediate strength coupling between system and environment. Finally, if time allows I may discuss some aspects concerning measurement techniques in these systems.

High Precision Quantum Device Simulation with Atomistic Tight-Binding Technique Coupled with Semi-Classical Poisson Solver and Many-Electron Configuration Interaction Method

Rahman, Rajib
Sandia National Laboratories

Due to the rapidly shrinking dimensions of semiconductor devices, modeling techniques need to incorporate the underlying atomistic nature of the materials the devices are built from. Widely used continuum models very often fail to capture the subtle quantum properties that govern the operations of modern devices. The semi-empirical tight-binding method offers a promising way to simulate these devices as it treats interactions on an atomic scale and is also scalable to realistic device sizes of millions of atoms. In general, the tight-binding approach captures various features of materials and devices such as hetero-structure properties, inhomogeneous strain distributions, lattice miscuts, surface roughness, full bandstructure, interfaces, electro-magnetic fields - all under a unified framework.

However, tight-binding being essentially a single electron theory misses out important many electron interactions that some quantum devices rely on. Here, we describe two methods to augment the tight-binding method with many electron interactions. The first is based on coupling the tight-binding Hamiltonian with a semi-classical Poisson solution of the device with realistic gate geometries and voltages to describe mean field interactions in terms of the electron density. The second is based on coupling tight-binding to an exact many-electron configuration interaction method, which captures Coulomb, exchange, and correlation effects with accurately. The complete suite of these tools constitutes a powerful package for high precision device simulation.

As applications of these techniques, we show simulations of two important quantum information processing systems in silicon. Simulations of a single donor spin qubit in silicon in a field effect transistor show the subtle electric field tuning of the donor spin and explain experimental measurements. Computations of the many electron levels of a double quantum dot in silicon along with nearby defects reveal complex interplay of valley physics and many electron effects, which are manifested in the nature of the voltage tuned exchange curve of the double quantum dot.

Mathematical Techniques for Cell Cycle Analysis

Rees, Paul
Swansea University

The aim of systems biology is to understand the complex interactions which occur between the components of a biological system. By identifying the relevant components within the complex system a host mathematical tools can be used to attempt to identify the nature of the mechanisms between them. Here we take inspiration from a traditional systems engineering approach which relates the output of a system to the input by a transfer function. This requires a significant simplification of the complex biological system however we will demonstrate that appropriately designed experiments couple with simple mathematic models of the transfer function can accurately elucidate biological function. In casting the problem in this manner we are able to use systems engineering theory to describe and analyze the biological system. We outline this technique by giving examples for the assessment of drug treatments on cancer cell populations and the study of uptake of nanoparticles and the toxic effect on cells.

Efficient Quantum Algorithms for Simulating Hamiltonian Evolution on a Quantum Computer / Optimal Phase Estimation Using Particle Swarm Optimisation

Sanders, Barry

I present efficient quantum algorithms for simulating Hamiltonian evolution on a quantum computer both for an oracle setting for the Hamiltonian and also without the oracle model for the specific case that Hamiltonians are sums of non-commuting tensor products of Pauli operators. For the time-dependent Hamiltonian case, we establish sufficient smoothness criteria to deliver bounded-error quantum simulations of state evolution efficiently. Technically we employ Lie-Trotter-Suzuki approximations for ordered-operator exponentials with strict error bounds for these expansions. For the specific case of Hamiltonians that are sums of non-commuting tensor products of Pauli operators, our efficient classical algorithm delivers circuit designs for efficient quantum simulation, and I show applications to Kitaev's toric-code and honeycomb Hamiltonian systems.

Closing the Modelling-Simulation-Control-Identification Loop

Schirmer, Sophie

Swansea University

When we talk about quantum devices there is an implied assumption of functionality, i.e., the device is supposed to perform certain tasks as required by an application. Achieving such functionality requires careful device design as well as control to manipulate the behaviour of the system. Effective device and control design is a non-trivial task that requires understanding the intrinsic dynamics of the system as well as its interaction with the control apparatus, sensors and actuators, as well as the effect of the environment. Generally, this means that we require a model not only of an isolated system but its interaction with a controller and environment. In simple cases such models can be constructed directly from first principles, e.g., for a simple quantum system we may be able to write down a Hamiltonian for the intrinsic dynamics of the system and maybe the effect of certain types of coherent control fields. For more complex systems from nano-electronics to biomolecules, however, this approach usually

does not suffice, and it is crucial to incorporate data from observations of the actual behaviour of the system to construct effective models. In this talk we will discuss some general requirements for control system models as well as some techniques for constructing and adaptively refining such models.

Symmetry Principles in Quantum Systems Theory

Schulte-Herbruggen, Thomas

Technical University Munich

Elucidating quantum optimal control in terms of symmetry principles has triggered us in a number of recent advances to be elucidated in survey:

(i) it leads to a new and handy controllability criterion,
(ii) it guides the design of universal quantum hardware,
(iii) it governs which quantum system can simulate another one given,
(iv) with little modification it specifies the limit between time-optimal control and relaxation-optimised control of open systems, and
(v) it provides a pattern that may help to understand new coherent pathways in noise-assisted energy transfer in light-harvesting biomolecules.

How principles turn into practice is illustrated in a plethora of examples showing practical applications in solid-state devices and circuit-qed. The algorithmic tools are presented in a unified programming framework.

Participants

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Bandrauk, Andre (Universite de Sherbrooke)

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Nielsen, Erik (Sandia National Laboratories)
Oi, Daniel (Strathclyde University)
Petersen, Ian (UNSW)
Plenio, Martin (University of Ulm)
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Chapter 2

Interactive Information Theory (12w5119)

January 15 - 20, 2012

Organizer(s): Natasha Devroye (University of Illinois at Chicago), Ashish Khisti (University of Toronto), Ian F. Blake (University of British Columbia)

The BIRS workshop 12w5119 was a great success despite it being the coldest week of the year, with temperatures between -20 and -40 celsius the entire week. The workshop topic was "Interactive Information Theory." It is important to note that this title and topic may be interpreted in a variety of ways, i.e. there is no standard definition of what constitutes "interactive" information theory. The goal of our workshop was to bring together researchers from different areas, each with their own interpretation of this topic, to interact.

Overview of the Field

Information theory, while a mathematical theory, has had a powerful impact on practical communication systems. While many ideas in information theory such as random code constructions, iterative decoding and maximum entropy principles were originally developed as solutions to well defined mathematical problems, interestingly, today's practical engineering solutions that approach fundamental limits, are closely related to these ideas. In the 60-odd years of information theory, the dominant focus and most successes have emerged in "one-way" problems in which information flows in a single direction. When information flows in two-directions, not only does establishing the fundamental limits of communication become more challenging due to the interactive or two-way nature of the problems, but the need for new "directional" information metrics themselves arise. As a result, due to their difficulty, interactive communication problems have received less attention and are less well understood than more classical one way problems. This is unfortunate, as interactive problems come closer to modeling true communication than do one-way models: in a human conversational dialogue, we naturally adapt our speed, tone, and content to account for the other party's reactions in a fully interactive fashion rather than speak in monologues. Communicating two-way data by treating it as two one-way data flows ignores the ability of a communication system to adapt to the received messages, to have the terminals *interact*. An interactive model and theory for communication is not only more relevant and general, but also of immediate use in current and future applications such as distributed data storage in data centers and two-way video-conferencing, tele-presence and tele-medicine applications. In the past years, significant progress in network coding, on interactive function computation and compression, interference management, particularly in two-way relaying scenarios, and a renewed interest in feedback and directed information and its applications, have given information theory new insight into interactive problems; various disconnected pieces of the interactive communication puzzle have emerged in different areas, what remains to be done is to piece them together. The time was ripe for a meeting bringing together active, young, and established experts in interactive communication to exchange recent results and determine promising avenues to tackle the fundamental problem of interactive communications.

While the term “interactive information theory” encompasses many concepts, the key ingredient which sets our topic apart from more classical information theory is the concept of “interaction”, in which communication nodes adaptively change their behavior (encoding, decoding, compression, etc.). Specifically, nodes are said to interact if their next actions (which may for example be what message to send or how to encode them) are based on what they have received from the other terminal nodes, forming a type of “closed-loop” information exchange process. State of the art results in both source and channel coding surprisingly involve very simple interactive protocols, often involving only one or two rounds of communication. Part of the challenge lies in the fact that it is difficult to find computable (single-letter) rate expressions for more sophisticated interactive schemes or show their optimality.

Recent Developments and Open Problems

Given the breadth of this topic, we formed four rough themes for the workshop (which by no means exhaust the definition of interactive information theory – notably missing is the interplay of control, interaction and information theory as discussed by several of the speakers), each addressed on a different day.

Focus area 1: Interactive source coding and computing

Source coding, or data compression, involves the process of encoding information using fewer bits. The lowest number of compressed bits needed on average to represent an information source X, modeled as a random variable which generates a sequence of symbols $\{X_i\}$ according to the distribution $p(x)$, is given by that source's entropy $H(X)$. From this compressed version, the source may be reconstructed in either a lossy or a lossless manner. Work on interactive source coding considers variants of the basic problem in which a source which generates symbols (X_i, Y_i) according to a joint distribution $p(x, y)$. Symbols X_i are known to node N_x and symbols Y_i are known to node N_y. If one of the nodes, say N_y, wishes to noiselessly reconstruct X given one compressed version of it over a noiseless channel we obtain the celebrated “one-way” Slepian-Wolf result [1] which states that the minimal rate at which X may be compressed such that perfect reconstruction is possible at node N_y is $H(X|Y)$. If both nodes wish to reconstruct each others' messages, or if multiple rounds of messages may be exchanged in order for one node to reconstruct the others' source, we obtain a fully “two-way” problem. Both lossless extensions of Slepian Wolf's result [1] to this two-way source coding problem [2], which may take place over numerous interactive - and half-duplex - rounds led by the work of Orlitsky [3, 4, 5, 6], as well as lossy source coding with [7] or without [8, 9] a “ common helper” - again using K rounds of interactive messages, have been obtained. In a related vein, motivated by sensor networks and database privacy, recent work has focused on functional compression [10], where the goal is to separately compress possibly correlated sources in order to compute an arbitrary function of these sources given the compressed data. Of particular interest are the techniques used in proving these results, and the results themselves which outline when and how interactive source coding is beneficial. For example, Ma and Ishwar's very recent result indicates that *interaction* is strictly beneficial [9] in reducing the amount of bits needed to be communicated in certain scenarios. The recent work of Braverman and Rao [11, 12, 13, 14] have looked at similar problems from a theoretical computer-science perspective. However, in general only bounds on various interactive source-coding schemes exist under various constraints exist; open problems abound as we are only starting to understand the role of interaction in communicating sources.

Focus area 2: Interactive channel coding and networking

Source coding's so-called dual is channel coding, in which determining and attaining the channel capacity is the goal.

A one-way channel is characterized by the set of conditional probability distributions $p(y|x)$ between channel input x and channel output y. The channel capacity corresponds to the maximal rate (in bits/channel use) at which a message may be reliably, i.e. with probability of error $\to 0$ as the number of channel uses increases, be communicated between the source and the destination, and is given by the maximal mutual information $I(X;Y)$ between the channel inputs and outputs. The simplest extension of this point-to-point channel to a two-way scenario was first proposed by Shannon in 1961 [15], where he introduced the “two-way channel” $A \leftrightarrow B$ in which A and B wish to exchange messages over a common channel characterized by the set of conditional distributions $p(y_A, y_B|x_A, x_B)$. While capacity is known for specific channels, it remains unknown in general. This is surprising given the model's simplicity, and highlights our lack of understanding on how to communicate in an interactive fashion. The inherent difficulty lies in the fact that each source is also a destination in a two-way channel, and as such its channel input at channel use n may depend on its message w as well as all past received channel outputs y_1^{n-1}, i.e. $x_n(w, y_1^{n-1})$. While the capacity region of a two-way channel may be expressed in terms of *directed-information,* $I(\mathbf{X} \to \mathbf{Y})$ between channel inputs and outputs, a directional version of the classical mutual information $I(X;Y)$ expression taking into account the causal, channel output-dependent structure of the channel inputs $x_{1n} = f(w_1, y_1^{n-1})$, and $x_{2n} = g(w_2, y_2^{n-1})$ [15, 16, 17, 18, 19], this is often deemed unsatisfactory, as it is, with our current techniques, prohibitively complex to evaluate. Nonetheless, general inner and outer bounds [15, 18, 20, 21, 22, 23, 24] for the two-way channel exist. By bringing together active contributors to this area we intend to exchange recent results and determine possible directions in which to attack this problem. Two-way communications have very recently been extended to network scenarios [25, 26]; two-way channel coding problems over networks highlight the interplay between classical two-way problems and those seen in networks, including how to relay information, combine information from various paths, and deal with interference between multiple streams. For example, the two-way relay channel is the logical extension of the classical relay channel [20, 27, 28] - a channel in which a transmitter send a message to a receiver with the help of a designated third "relay" node - to allow for two messages to be exchanged between the two nodes with the help of a relay [29, 30, 31]. A large body of work has emerged (see for example [32, 33] and references therein). The capacity region is in general still unknown, but exact and performance guaranteeing (finite-gap, generalized degrees of freedom) capacity results have been derived under different channel conditions. Much remains to be done to fundamentally understand the interplay of two-way techniques with interference networks which extend upon the simple two-way and two-way relay channels. Interaction may also be used to denote channels in which nodes cooperate, or channels in which feedback in present: in both cases nodes are able to adapt their inputs based on other inputs/outputs in the network.

Focus area 3: Adaptive hypothesis testing

Yet another area of interactive information theory involves the extension of hypothesis testing – a statistical method of making decisions using (noisy) data – to dynamic, adaptive settings. Connections between information theory and hypothesis testing, or decision theory, have long been made (see [35] and references therein); recent developments however have focussed on *sequential* hypothesis testing scenarios where some degree of control or interaction is available. For example, rather than being *given* a set of measurements from which we will make a decision about the underlying hypothesis, one could ask how one should best select observations (or control the sensing) in a multi-hypothesis test, as very recently done in [36, 37]; this is interactive in the sense that one controls and dynamically may change where one senses (or what types of observations are received), and information theory and its metrics (e.g. Kullback-Leilber divergence) are often useful in obtaining bounds on the performance. In a similar spirit, one may consider information theoretic limits of "active learning" schemes as in [38, 39]. While sequential hypothesis testing has been studied, its active or interactive counterpart has only recently been considered and much remains to be theoretically understood.

Focus area 4: Interactive information security

Security is a core requirement in any networked infrastructure system. The purpose of this focus topic was to understand fundamental limits of secret-key generation between two parties using interactive communication. Interaction between the legitimate parties is a fundamental requirement in secure communication. It provides advantage to the legitimate terminals by virtue of their active participation in the protocol. An adversary, despite having significantly better resources (computation power, communication channel etc) cannot decode the share secret-key if he/she does not participate in the protocol. While the foundation of interactive secret-key generation was laid out almost 20 years ago [40], in recent years a significant progress has been made on this area, due to the applications to wireless systems.

Presentation Highlights

The workshop was structured as follows: each of four days had a different "theme", which was opened by one (or sometimes two) one hour tutorial-style lectures in the morning, followed by a series of 30 minute talks on related topics. For brevity, we only highlight several of the talks; and simply list the title and speaker for all others.

Day 1: Interactive source coding and computing

Tutorial talk: MARK BRAVERMAN, PRINCETON UNIVERSITY, *Tutorial on information complexity in interactive computing.* Mark opened up the workshop by discussing interactive computing from a computer-science perspective. In particular, he discussed several new extensions of information-theoretic notions to the two-way communication setting, which he used to prove a direct sum theorem for randomized communication complexity, showing that implementing k copies of a functionality requires substantially more communication than just one copy. More generally, he showed that information cost $I(f)$ can be defined as a natural fundamental property of a functionality f, measuring the amount of information that the parties need to exchange in order to compute f. He described several new tight connections between $I(f)$, direct sum theorems, interactive compression schemes, and amortized communication complexity.

This tutorial was especially interesting in the context of this workshop as Mark is a computer scientist (one of the few at the workshop) and hence provided a different perspective on problems similar to those considered by information theorists. It became evident that translating results (and even vocabulary and notions) between the information and computer science theory communities, enabling the communities to exchange results more readily, could advance the state of the art in interactive communication complexity.

Tutorial talk: BABAK HASSIBI, CALTECH, *Control over Lossy Networks: The Interplay Between Coding and Control.* Babak followed up Mark's tutorial with another excellent one on tree codes for interactive communications and relationships with control theory. In particular, he asks how (and whether) one can design interactive codes in order to stabilize a plant in a closed-loop control system, where the controller and the plant are not necessarily co-located and must communicate over *noisy* channels. He developed a universal and efficient method for stabilizing plants driven by bounded noise over erasure channels; numerous open questions remain, including how to deal with unbounded noise, with other types of channels (beyond erasure), or how to optimize performance rather than only stabilizing the plant.

PRAKASH ISHWAR, BOSTON UNIVERSITY, *The Infinite-Message Limit of Interactive Source Coding.* Prakash considered distributed block source coding problems, where it is known that multiround interaction can improve the communication efficiency of distributed computation. One may then ask: what is the ultimate limit of this efficiency when the number of messages (in the multiround interaction) is unbounded – an asymptotic that has received relatively little attention in the literature. He outlined recent efforts to tackle this question for distributed computation in two-terminal and collocated networks.

NAN MA, UNIVERSITY OF CALIFORNIA - BERKELEY, *The benefit of interaction in lossy source coding.* Nan discussed the sum-rate-distortion function for a two-way lossy source coding problem in which two terminals send multiple messages back and forth with the goal of reproducing each other's sources. He constructed an example which shows that two messages can strictly improve the one-message (Wyner-Ziv) rate-distortion function, and that the multiplicative gain in terms of the ratio of the one-message rate to the two message sum-rate can be arbitrarily large, while simultaneously the ratio of the backward rate to the forward rate in the two message sum-rate can be arbitrarily small.

JIN MENG, UNIVERSITY OF WATERLOO, *Interactive Encoding and Decoding: Concept, Coding Theorems and Algorithm Design.*

ANUP RAO, UNIVERSITY OF WASHINGTON, *Towards Coding for Maximum Errors in Interactive Communication.*

Day 2: Interactive channel coding and networking

Tutorial talk: YOUNG-HAN KIM, UNIVERSITY OF CALIFORNIA - SAN DIEGO, *On the role of interaction in network information theory.* Young-Han opened the second day, on interactive *channel coding* (in contrast to the previous day's focus on *source coding*), by providing a high-level overview of interaction in channel coding problems. He discussed several channels with feedback: the posterior-matching scheme for the point-to-point feedback channel, the Cover-Leung coding scheme for the multiple access channel with feedback, Dueck's example for the broadcast channel with feedback. Interactive coding schemes for the two-way channel (and in particular the Shannon-Blackwell Binary Multiplying channel) were discussed before outlining another form of interaction – relaying in general networks.

YOSSI STEINBERG, TECHNION, *The broadcast channel with action dependent states.*

BOBAK NAZER, BOSTON UNIVERSITY, *Computation over Feedback Channels.*

DANIELA TUNINETTI, UNIVERSITY OF ILLINOIS - CHICAGO, *Cooperation in interference channels.* Daniela gave an overview of interaction in one of the classical multi-user information theoretic channels – the interference channel (not discussed in Young-Han's tutorial). In the cooperative interference channel, the two transmitting nodes are able to obtain causal, noisy versions of the others' transmission, allowing them to cooperate, or interact, in transmitting their messages to two independent destinations. She outlined recent capacity bounds for this challenging channel, emphasizing the role and potential benefits of cooperation.

HOLGER BOCHE, TECHNISCHE UNIVERSITÄT MÜNCHEN, *Conferencing Encoders for Compound and Arbitrarily Varying Multiple-Access Channel.* Continuing in the line of cooperation, Holger first discussed two coding theorems for the compound multiple-access channel with an arbitrary number of channel states, where conferencing – a type of cooperation or interaction – between the transmitters is possible. Next, he discussed the capacity region of arbitrarily varying multiple-access channels with conferencing encoders for both deterministic and random coding. Unlike compound multiple-access channels, arbitrarily varying multiple-access channels may exhibit a discontinuous increase of the capacity region when conferencing is enabled. Applications to wireless networks with cooperating base-stations were discussed.

TOBIAS OECHTERING, KTH, *Transmit strategies for the bidirectional broadcast channel & latest results.* Tobias discussed the bi-directional relay channel where two users exchange messages with the help of a relay. Here, two nodes transmit their messages to a relay in phase 1. In phase 2, the relay then forwards these two messages to nodes, allowing them to exchange messages. He discussed optimal transmit strategies for this second broadcast phase for Gaussian channels where the nodes (possibly) have multiple antennas.

PETAR POPOVSKI, AALBORG UNIVERSITY, *Protocol Coding for Two-Way Relay Communication.* Petar introduced a new type of coding which is backwards compatible in the sense that it encodes information through changing the actions taken in existing communication protocols (rather than more physical layer changes). He asks how much information may be extracted on top of the usual information by this protocol coding, or resource re-ordering. In particular, for the two-way relay communication channel he showed interesting links to interactive schemes initially suggested by Schalkwijk for the binary multiplier channel.

BESMA SMIDA, PURDUE UNIVERSITY - CALUMET, *On the utility of Feedback in Two-way Networks.* Besma spoke about developing a fundamental and practical – from a wireless communications perspective – understanding of the value of feedback in two-way, interactive networks. In general, feedback has been studied from a one-way perspective, meaning data travels in one direction, and feedback - often assumed to be perfect - in the other. Besma proposed a new unified framework which captures the key tradeoffs particular to two-way networks and presence of different types of feedback including quantized channel state information, Automatic Repeat reQuest, or extensions and combinations thereof; many open questions were discussed.

RAMJI VENKATARAMAN, YALE UNIVERSITY, *Interactive Codes for Synchronization from Insertions and Deletions.* Ramji discussed efficient codes for synchronization from insertions and deletions. As an example, he considered remotely located users who independently edit copies of a large file (e.g. video or text), where the editing may involve deleting certain parts of the file, and inserting new data in other parts. The users then want to synchronize their versions with minimal exchange of information, in terms of both the communication rate and the number of interactive rounds of communication. He focussed on the case where the number of edits is small compared to the file-size, and described an interactive synchronization algorithm which is computationally efficient and has near-optimal communication rate.

JEAN-FRANCOIS CHAMBERLAND, TEXAS A&M UNIVERSITY, *Challenges and Potential Approaches in Combining Channels with Memory, Block Codes and Queues.*

Day 3: Adaptive hypothesis testing and energy efficient green communications

Tutorial talk: ROB NOWAK, UNIVERSITY OF WISCONSIN - MADISON, *Interactive Information Gathering and Statistical Learning.* Rob opened up the third day with yet another aspect of interactive information theory: the notions of adaptive and non-adaptive information, in the context of statistical learning and inference. He considered a collection of models denoted by X and a collection of measurement actions (e.g., samples, probes, queries, experiments, etc.) denoted by Y. A particular model x in X best describes the problem at hand and is measured as follows. Each measurement action, y in Y, generates an observation $y(x)$ that is a function of the unknown model. The goal is to identify x from a set of measurements $y_1(x), ..., y_n(x)$, where y_i in Y, $i = 1, ..., n$. If the measurement actions $y_1, ..., y_n$ are chosen deterministically or randomly without knowledge of x, then the measurement process is non-adaptive. However, If y_i is selected in a way that depends on the previous measurements $y_1(x), ..., y_{i-1}(x)$, then the process is adaptive. The advantage of adaptive information is that it can sequentially focus measurements or sensing actions to distinguish the elements of X that are most consistent with previously collected data, and this can lead to significantly more reliable decisions. The key question of interest is identifying situations in which adaptive information is significantly more effective than non-adaptive information, which depends on the interrelationship between the model and measurement spaces X and Y. He covered the general problem, connections to channel coding and compressed sensing, and considered two illustrative examples from machine learning.

MATT MALLOY, UNIVERSITY OF WISCONSIN - MADISON, *Sequential testing in high dimensions.*

TE SUN HAN, NATIONAL INSTITUTE OF INFORMATION AND COMMUNICATIONS TECHNOLOGY, *Trade-off of data compression and hypothesis testing.*

SENNUR ULUKUS, UNIVERSITY OF MARYLAND, *Interacting with Nature: Information Theory of Energy Harvesting Communications.*

KAYA TUTUNCUOGLU, PENN STATE, *Interactive Transmission Policies for Energy Harvesting Wireless Systems.*

Day 4: Interactive information security

Tutorial talk: HIMANSHU TIYAGI, UNIVERSITY OF MARYLAND, *Function Computation, Secrecy Generation and Common Randomness.* This talk provided a comprehensive overview on the role of common randomness in secure function computation and secret-key generation. The setup presented included multiple source terminals, each observing a correlated source sequence. The terminals interactively exchanged messages over a public discussion channel and ultimately generated a common secret-key. The quantity of interest was the maximum secret-key rate that can be generated in this setup. The general problem remains open, but the state of the art was surveyed.

PRAKASH NARAYAN, UNIVERSITY OF MARYLAND, *Multiple-access channels, feedback and secrecy generation.* This talk presented a secret-key generation problem when the legitimate terminals do not have access to correlated sources, but instead can communicate over a noisy communication link. An adversary can also eavesdrop this communication. The problem was partially solved when there are two legitimate terminals. An open problem involving three terminals and a multiple access channel for communication was described.

FRANS WILLEMS, TU EINDHOVEN *Authentication based on secret generation.* This talk focused on an application of biometric encryption. A user who provides an enrolment biometric to the system wishes to authenticate by providing another biometric reading. The successive readings from the same user can be noisy. The system is required to authenticate legitimate user and reject a false user. An information theoretic formulation was presented and the conditions under which successful decoding is possible were derived.

AYLIN YENER, PENN STATE, *Secrecy: Benefits of Interaction.* A two-way wiretap channel with an external eavesdropper was presented and conditions under which secure communication is possible were derived. The two-way nature is particularly useful since the eavesdropper observes a super-position of two information signals and cannot separate the two individual signals from the sum.

ASHISH KHISTI, UNIVERSITY OF TORONTO, *Secret-Key Generation over Fading Channels.* This talk provided a review of secret-key generation over wireless fading channels. Fundamental limits of secret-key generation was of particular focus. The difference in the limits of various wireless systems was discussed and connections to the information theoretic formulations from earlier works were elaborated.

PULKIT GROVER, STANFORD UNIVERSITY, *Interactive communication in circuits: understanding Shannon's "magic trick".*

STARK DRAPER, UNIVERSITY OF WISCONSIN - MADISON, *Reliability in streaming data systems with feedback.*

ADITYA MAHAJAN, MCGILL UNIVERSITY, *The stochastic control approach to real-time communication: an overview.*

ANDREW ECKFORD, YORK UNIVERSITY, *Models and Capacities of Molecular Communication.* Andrew spoke about a very different channel model: a diffusion-mediated molecular communication channel. This very recent channel model's capacity is surprisingly difficult to obtain, not only because the communication medium unfamiliar to communication engineers; but the mathematical details of the communication environment are complicated. Andrew discussed mathematical models for molecular communication, which are both information-theoretically useful and physically meaningful; discussed the difficulties of dealing exactly with these models; and presented some simplified scenarios in which capacity can be evaluated.

ERSEN EKREM, UNIVERSITY OF MARYLAND, *The Vector Gaussian CEO Problem.*

Scientific Progress Made and Outcomes of the Meeting

The central outcome of the meeting was bringing together various experts in theoretical computer science, information theory, wireless communications, and statistical signal processing who had tackled different aspects of the broad "interactive information theory" theme. Each person was given a speaking slot so they were able to expose their work to others, and there was plenty of time for discussion at the end of each talk, over meals and during coffee breaks. This group was particularly interactive, and many participants came away with useful comments on their work, and exposure to results in other areas which would be useful tools. I believe many collaborations have been initiated by this workshop on a unique topic (the first of its kind); it will take time for us to see the results of bringing together this particular and diverse group of researchers. In the meantime, we let the participants themselves attest to the scientific progress made and the outcomes of the meeting directly:

> "I was very fortunate to attend the BIRS workshop on interactive information theory. As a computer scientist I don't usually get to interact with information theorists and electrical engineering researchers. This workshop provided me with such an opportunity. I have been working on applications of information theory in the field of computational complexity – a subarea of computer science. The workshop has really impacted my research agenda in two ways. It exposed me to new ideas from information theory, which allowed me and my students to make progress on the problems we have been working on. At the same time, it made me aware of the problems and challenges in information theory to which the techniques we have been developing can be applied. I hope BIRS will hold another workshop in this area in the future years."
>
> *– Mark Braverman, Princeton University and University of Toronto*

> The workshop was very beneficial in getting people from related fields together. This is valuable to our community as it created a better awareness of the specific problems people are currently addressing, and available mathematical methods to address these problems. It also gave me an opportunity to meet and exchange with new researchers in my area. This too is of great value. I think that the BIRS workshop will have a definite impact on the research in our area (interactive information theory). It may take a few months or a couple of years for the seeds planted during the workshop to come to fruition.
>
> *– Jean-Francois Chamberland, Texas A&M University*

The wonderful BIRS Workshop on Interactive Information Theory (Jan 16-20, 2012) afforded several intellectual benefits. I met and learned of the research of several computer scientists who work on areas related to my own but who normally do not attend the same meetings as I do. The tutorial lectures organized by Profs. Devroye, Khisti and Blake, as part of the Workshop program, were very useful. Several new and exciting ideas were discussed. My student, Himanshu Tyagi, gave a well-received one-hour tutorial which, I think, will have a beneficial impact on his future career in academia.

– Prakash Narayan, University of Maryland

This has been perhaps one of the three best scientific gatherings I have ever been to. The small number of selected participants and the focused topic provided opportunity for intensive discussions. After this workshop there is a potential for at least 3-4 new collaborations, which is excellent and rare in other conferences.

– Petar Popovski, Aalborg University

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Bibliography

[1] D. Slepian and J. Wolf, Noiseless coding of correlated information sources, *IEEE Trans. on Information Theory* **19(4)** (1973), 471–480.

[2] A. El Gamal and A. Orlitsky, Interactive data compression, *25th Annual Symposium on Foundations of Computer Science* (1984).

[3] A. Orlitsky, Worst-case interactive communication. I. Two messages are almost optimal, *IEEE Trans. on Information Theory* **36(5)** (1990), 1111–1126.

[4] A. Orlitsky, Worst-case interactive communication. II. Two messages are not optimal, *IEEE Trans. on Information Theory* **37(4)** (1991), 995–1005.

[5] A. Orlitsky, Interactive communication: Balanced distributions, correlated files, and average-case complexity, *32nd annual Symposium on the Foundations of Computer Science* (1991).

[6] A. Orlitsky and K. Viswanathan, Practical protocols for interactive communication, *IEEE International Symposium on Information Theory* (2001).

[7] H. Permuter, Y. Steinberg and T. Weissman, Two-way source coding with a common helper, *http://arxiv4.library.cornell.edu/abs/0811.4773* (2008).

[8] A. Kaspi, Two-way source coding with a fidelity criterion, *IEEE Trans. on Information Theory* **31(6)** (1985), 735–740.

[9] N. Ma and P. Ishwar, Interaction Strictly Improves the Wyner-Ziv Rate-distortion function, *http://arxiv.org/abs/1001.2781* (2010).

[10] V. Doshi, D. Shah, M. Médard and M. Effros, Functional Compression through Graph Coloring, *IEEE Trans. on Information Theory* **56(8)** (2010), 3901–3917.

[11] M. Braverman, Interactive information complexity, *44th annual ACM symposium on Theory of Computing* (2012).

[12] M. Braverman and A. Rao, Towards coding for maximum errors in interactive communication, *43rd annual ACM symposium on Theory of computing* (2011).

[13] M. Braverman and A. Rao, Information equals amortized communication, *IEEE 52nd Annual Symposium on Foundations of Computer Science* (2011).

[14] B. Barak, M. Braverman, X. Chen and A. Rao, How to compress interactive communication, *42nd annual ACM symposium on Theory of Computing* (2010).

[15] C. E. Shannon, Two-way communications channels, *4th Berkeley Symp. Math. Stat. Prob.* (1961).

[16] H. Marko, The Bidirectional Communication Theory- A Generalization of Information Theory, *IEEE Trans. on Communications* **21(12)** (1973), 1345–1351.

[17] J.L. Massey, Causality, Feedback and Directed Information, *International Symposium on Information Theory and its Applications* (1990).

[18] G. Kramer, Directed information for channels with feedback, *Ph.D. dissertation, Swiss Federal Institute of Technology Zurich*, (1998).

[19] S. Tatikonda and S. Mitter, The capacity of channels with feedback, *IEEE Trans. on Information Theory* **55(1)** (2009), 323–349.

[20] E.C. van der Meulen, Three-terminal communication channels, *Advanced Applied Probability* **3** (1971), 120–154.

[21] F. Jelinek, Coding for and decomposition of two-way channels, *IEEE Trans. on Information Theory* **10(1)** (1964), 5–17.

[22] G. Kramer, Capacity Results for the Discrete Memoryless Network, *IEEE Trans. on Information Theory* **49(1)** (2003), 4–21.

[23] Z. Zhang, T. Berger and J.P.M. Schalkwijk, New outer bounds to capacity regions of two-way channels, *IEEE Trans. on Information Theory* **32** (1986), 383–386.

[24] A.P. Hekstra and F.M.J. Willems, Dependence balance bounds for single output two-way channels, *IEEE Trans. on Information Theory* **35(1)** (1989), 44–53.

[25] Z. Cheng and N. Devroye, Multi-user Two-way Deterministic Modulo 2 Adder Channels - When Adaptation Is Useless, *IEEE International Symposium on Information Theory* (2012).

[26] Z. Cheng and N. Devroye, Two-way Networks: when Adaptation is Useless, *http://arxiv.org/abs/1206.6145* (2012).

[27] T. M. Cover and A. El Gamal, Capacity Theorems for the Relay Channel, *IEEE Trans. on Information Theory* **25(5)** (1979), 572–584.

[28] G. Kramer, M. Gastpar and P. Gupta, Cooperative strategies and capacity theorems for relay networks *IEEE Trans. on Information Theory* **51(9)** (2005), 3037–3063.

[29] Y. Wu, P.A. Chou and S.-Y. Kung, Information exchange in wireless networks with network coding and physical-layer broadcast, *Conference on Information Sciences and Systems* (2005).

[30] B. Rankov and A. Wittneben, Achievable Rate Regions for the Two-way Relay Channel, *IEEE International Symposium on Information Theory* (2006).

[31] P. Popovski and H. Yomo, Physical Network Coding in Two-Way Wireless Relay Channels, *IEEE International Conference on Communications* (2007).

[32] S.J. Kim and N. Devroye and P. Mitran and V. Tarokh, Achievable rate regions and performance comparison of half duplex bi-directional relaying protocols, *IEEE Trans. on Information Theory* **57(10)** (2011), 6405–6418.

[33] B. Nazer and M. Gastpar, Reliable Physical Layer Network Coding, *Proceedings of the IEEE* **99(10)** (2011), 1–23.

[34] C. Tian, J. Chen, S. Diggavi and S. Shamai, Optimality and Approximate Optimality of Source-Channel Separation in Networks, *http://arxiv.org/pdf/1004.2648* (2010).

[35] R. Blahut, Hypothesis testing and information theory, *IEEE Trans. on Information Theory* **20(4)** (1974), 405–417.

[36] S. Nitinawarat , G. Atia and V. Veeravalli, Controlled Sensing for Multihypothesis Testing, *http://arxiv.org/abs/1205.0858* (2012).

[37] M. Naghshvar and T. Javidi, Active Sequential Hypothesis Testing, *http://arxiv.org/abs/1203.4626* (2012).

[38] R.M. Castro and R.D. Nowak, Minimax bounds for active learning, *IEEE Trans. on Information Theory* **54(5)** (2008), 2339–2353.

[39] M. Raginsky and A. Rakhlin, Lower Bounds for Passive and Active Learning, *Conference on Neural Information Processing Systems* (2011).

[40] U. Maurer, Secret Key Agreement by Public Discussion From Common Information, *IEEE Trans. on Information Theory* **39** (1993), 733—742.

Chapter 3

Emergent behaviour in multi-particle systems with non-local interactions (12w5041)

January 22 - 27, 2012

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Introduction

Collective group behaviour is a fascinating natural phenomenon that is observed at all levels of the animal kingdom, from beautiful bacterial colonies, insect swarms, fish schools and flocks of birds, to complex human population patterns. The emergence of very complex behaviour is often a consequence of individuals following very simple rules, without any external coordination. In recent years, many models of group behaviour have been proposed that involve nonlocal interactions between the species [1, 2, 3, 4]. Related models also arise in a number of other applications such as granular media [5, 6, 7, 8], self-assembly of nanoparticles [9, 10], and molecular dynamics simulations of matter [11].

Due to their nonlocal nature, these systems can exhibit complex and novel phenomena that pose challenging questions and motivate the development of new mathematical techniques. They typically lead to coherent and synchronized structures apparently produced without the active role of a leader in the grouping, phenomena denominated self-organization [12, 13], and it has been reported even for some microorganisms such as myxobacteria [14].

Most of these models are based on discrete models [1, 2, 3, 15] incorporating certain effects that we might call the "first principles" of swarming. These first principles are based on modelling the "sociological behavior" of animals with very simple rules such as the social tendency to produce grouping (attraction/aggregation), the inherent minimal space they need to move without problems and feel comfortably inside the group (repulsion/collisional avoidance) and the mimetic adaptation or synchronization to a group (orientation/alignment). They model animals as simple particles following certain microscopic rules determined by their position and velocity inside the group and by the local density of animals. These rules incorporate the "sociological" or "behavioral" component in the modelling of the animals movement. Even if these minimal models contain very basic rules, the patterns observed in their simulation and their complex asymptotic behavior is already very challenging from the mathematical viewpoint. These 3-zone models are classical in fish modelling [16, 17].

The source of this tendency to aggregation can also be related to other factors rather than sociological as survival fitness of grouping against predators, collaborative effort in food finding, etc. Moreover, we can incorporate other interaction mechanisms between animals as produced by certain chemicals, pheromone trails for ants, the interest of the group to stay close to their roost, physics of swimming/flying, etc. Although the minimal models based on “first principles” are quite rich in complexity, it is interesting to incorporate more effects to render them more realistic, see [13, 18, 19, 20] for instance.

There have been several micro and macroscopic models during the recent years that have attracted a lot the attention of mathematicians as the nonlocal models [21, 22, 23, 24] including Morse potentials, [25, 26, 27] for self-propelled particles with attraction and repulsion effects, and the simple model of alignment in [28]. For instance, the authors in [26] classify the different “zoology” of patterns: translational invariant flocks, rotating single and double mills, rings and clumps; for different parameter values. On the other hand, in the simpler alignment models [28], we get generically a flocking behavior. Much more elaborated models starting from these basic bricks are capable of simulating the collective behavior in case of analyzing systems with a large number of agents N. Control of large agent systems are important not only for the somehow bucolic example of the animal behavior but also for pure control engineering for robots and devices with the aim of unmanned vehicle operation and their coordination, see [29, 30] and the references therein.

When the number of agents is large, the use of continuum models for the evolution of a density of individuals becomes essential. Some continuum models were derived phenomenologically [21, 31, 32] including attraction-repulsion mechanisms through a mean force and spatial diffusion to deal with the anti-crowding tendency. Other continuum models are based on hydrodynamic descriptions [33, 34] derived by means of studying the fluctuations or the mean-field particle limits. Hyperbolic systems have also been proposed [35, 36, 37]. The essence of the kinetic modelling is that it does connect the microscopic world, expressed in terms of particle models, to the macroscopic one, written in terms of continuum mechanics systems. A very recent trend of research has been launched in this direction in the last few years, see for instance [38, 39, 34, 40, 41, 42, 43, 44] for different kinetic models in swarming. Introducing noise in these models can lead to phase transitions, a line of research which is wide open [1, 45, 46, 47].

Finally, variational approaches have been very fruitful to attack steady states and their stability for first order models of swarming. A very classical model in this field is the Patlak-Keller-Segel model for chemotactic cell movement [48]. Lots of exciting developments have happened in this direction in the last years [49, 50, 51, 52] and these variational tools have had nice implications in the theory of first order models [53, 54]. Fluid mechanics techniques have also been adapted to the aggregation equation to deeply analyse qualitative properties [55, 56, 57].

The common feature of these models is that they all lead to some non-locality in the equations, either in the form of a large system of ODE's with global coupling, or as a PDE with non-local kernels (integral terms). The analysis, asymptotic behavior, numerical simulation, pattern formation and their stability in many of these models still remain unexplored research territory. The development of these models has in part been motivated by increased use of computers which allows for easy experimentation. In many of these models, novel and exciting phenomena have been observed numerically. However, the fundamental understanding of observed patterns and their dynamics has been lagging. The time is ripe for development of better analytical tools which would allow to gain a better insight of these models.

Presentation highlights

Several important themes were identified during the workshop. The presentations are loosely classified according to one of the topics below.

Aggregation models

One of the simplest models of interacting particles that yields very complex dynamics is

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{j\neq i}^{N} F(|x_i - x_j|)\frac{x_i - x_j}{|x_i - x_j|}. \tag{3.1}$$

where $F(r)$ models the interaction force between the particles [58]. In the hydrodynamic limit $N \to \infty$, the model yields an intergro-differential equation

$$\rho_t + \nabla_x \cdot (\rho v) = 0; \quad v(x) = \int_{\mathbb{R}^d} F\left(|x-y|\right)\frac{x-y}{|x-y|}\rho(y)dy. \tag{3.2}$$

The study of aggregation models has been a very active area of research over the past decade; there are by literaly hunderds of papers on this model and its variations; see for example [59, 24, 21, 31] and references therein. Many of the participants talked about the aggregation model and its variations.

Balague discussed recent results on solutions that concentrate uniformly on a sphere, under the power law attractive-repulsive forces. Sharp conditions that establish stability under radial perturbations were given.

Bernoff studied equilibrium configurations of swarming biological organisms subject to exogenous and pairwise endogenous forces. Under certain conditions on the interaction force, the equilibria results in a compactly-supported density. In two-dimensions he showed that the Morse Potential and other "pointy" potentials can generically lead to inverse square-root singularities in the density at the boundary of the swarm support.

Bertozzi gave an introduction to the interesting phenomena of swarming and to the open problems in this area. She reviewed numerical and analytical results for both kinematic and dynamic aggregation equations. She discussed how models are constructed and the emergence of phenomenological behavior for different types of models including flocking, milling, and other patterns. She then presented some results on well-posedness of aggregation equations including a sharp condition on blowup from smooth initial data.

Fellner highlighed the differences between discrete stochastic and continuum versions of the model.

Fetecau discussed the dynamics and equilibria for swarms in the case where the repulsion is Newtonian and attraction is a power law. In many special cases, dynamics and equilibria can be explicitly described. The equilibria have biologically relevant features, such as finite densities and compact support with sharp boundaries. In a related talk, Huang presented some recent results on the asymptotics of the steady states for some of the limiting cases of the power attraction.

Pavlovski and Kolokolnikov presented recent results asymptotics of complex patterns in two and three dimensions. They discussed the patterns that consist of small multiple spots and of a thin ring. These patterns can be understood in terms of stability and perturbations of "lower-dimensional" patterns. Asymptotic methods provide a powerful tool to describe the stability, shape and precise dimensions of these complex patterns.

Laurent's lecture was on the dynamics of aggregation patchs in the case of purely Newtonian kernel. Numerical simulations as well as some exact solutions show that the time evolving domain on which the patch is supported typically collapses on a complex skeleton of codimension one. Reversing the time, any bounded compactly supported solution converges toward a spreading circular patch. A rate of convergence which is sharp in 2D was derived.

Raoul discussed the impact of the singularity of the interaction potential at the origin on the solutions. Some results in one dimension were presented and open questions for higher dimensions were posed.

Bedrossian talked about global existence and finite time blow-up for the critical Patlak-Keller-Segel (PKS) Models with inhomogeneous diffusion. The L^1-critical parabolic-elliptic PKS system is a classical model of chemotactic aggregation in micro-organisms well-known to have critical mass phenomena. In this talk, he studied this critical mass phenomenon in the context of PKS models with spatially varying diffusivity of the chemo-attractant in three dimensions and higher. The critical mass is identified to depend only on the local value of the diffusivity and finite time blow-up results show it to be sharp under certain conditions. The methods also provide new blow-up results for homogeneous problems, showing that there exist blow-up solutions with arbitrarily large (positive) initial free energy.

Yao gave a talk on the asymptotics of the blow-up behaviour and radial solutions for the PKS model. Numerically, three types of blow-up behaviour were identified: self-similar with no mass concentrated at the core, imploding shock solution and near-self-similar blow-up with a fixed amount of mass concentrated at the core. She also presented some theoretical results concerning the asymptotic behavior of radial solutions when there is global existence.

Second order models

Models of self-propelled particles typically take acceleration as well as self-propulsion of particles into account. An example of such a model is [26],

$$\frac{d}{dt}v_i = \left(\alpha - \beta\left|v_i\right|^2\right)v_i + \sum_{j\neq i} F(|x_i - x_j|)\frac{x_i - x_j}{|x_i - x_j|}; \qquad \frac{d}{dt}x_i = v_i$$

where $F(r)$ is models the interaction force between the particles and the term $\left(\alpha - \beta |v_i|^2\right) v_i$ is the self-propulsion force. These models typically lead to complex dynamics including swarms, mills and double mills [26, 33, 34, 27]. A related model is the Cucker-Smale equations modelling the flocking of birds [28]; in its simplest form it is

$$\frac{d}{dt}v_i = \frac{\lambda}{N}\sum_{j=1}^{N} \frac{1}{\left(1+|x_i-x_j|\right)^{\beta}}\left(v_j - v_i\right); \quad \frac{d}{dt}x_i = v_i.$$

Many presentations at the conference discussed recent results for 2nd order models.

Carrillo presented an overview of 2nd order models for swarming. He gave several examples of the derivation by means of kinetic theory arguments of kinetic equations for swarming. One example is the self-propelled particles model. Starting from the particle model, one can construct solutions to a Vlasov-like kinetic equation for the single particle probability distribution function using distances between measures. Another example is the continuous kinetic version of flocking by Cucker and Smale. The large-time behavior of the distribution in phase space is subsequently studied by means of particle approximations and a stability property in distances between measures. A continuous analogue of the theorems of Cucker-Smale will be shown to hold for the solutions on the kinetic model.

Forgoston considered self-propelling agents in the presence of both noise and delay. Delay in the swarm induces a bifurcation that depends on the size of the coupling amplitude. There are several spatio-temporal scales of these swarm structures. The interplay of coupling strength, time delay, noise intensity, and choice of initial conditions can affect the swarm in complicated ways.

Stephan Martin presented the usual self-propelled particle system, but where the morse force was replaced by a quasi-morse potential, having similar properties as the Morse force, but more amenable to analysis. He then explicitly computed the stationary states in the form of rotating flocks. Simulations were also performed which agreed with the explicit analytical solution, and illustrate the parameter dependencies.

Lega presented results of molecular dynamics simulations of disks moving in a two-dimensional box and interacting through special collisions [60]. Because this work was motivated by the existence of complex behaviors in colonies of bacteria, the particles also reorient themselves at random times, thereby simulating bacterial tumbles and inputing energy into the system. She showed that at low packing fractions clusters dynamically form and break up and that, as the packing fraction increases, groups of increasingly larger size are observed, in which the particles move coherently. Such behaviors are markedly different from those observed in systems of particles interacting through elastic collisions.

Frouvelle's talk was on the variation of the Viscek model which describes the alignment and self-organisation in large systems of self-propelled particles. He considered a time-continuous version of this model, in the spirit of the one proposed by P. Degond and S. Motsch, but where the rate of alignment is proportional to the mean speed of the neighboring particles. In the hydrodynamic limit, this model undergoes a phase transition phenomenon between a disordered and an ordered phase, when the local density crosses a threshold value. The two different regimes lead to two distinct macroscopic limits, namely a nonlinear diffusion equation for the density, and a first-order non-conservative hydrodynamic system of evolution equations for the local density and orientation.

Panferov discussed phase transitions in models of Vlasov-McKean type. These equations provide a mean-field description of a system of interacting particles through a pairwise potential V. If the Fourier transform of V has a negative minimum, the system has a critical threshold for the diffusion constant beyond which the trivial uniform steady state becomes unstable and the system experiences a phase transition. He showed that a large class of interactions, when the size of the domain is sufficiently large, the transition is always discontinuous and is characterized by coexistence of several stable states in a certain interval of parameter space.The transition is also shown to occur at a value of the diffusion constant strictly greater than the critical threshold. He also presented the results of a numerical study on the character of phase transition in Vicsek like models of flocking, in which a similar discontinuous transition is observed.

Agueh discussed the adding several forces to Cucker-Smale model to make it more realistic. Namely, the basic C-S model leads to unconditional flocking of all the birds in the swarm. Agueh presented generalizations of the C-S model to include scenarios where a typical bird is subject to a friction force driving it to fly at optimal speed, a repulsive short-range force to avoid collisions, an attractive "flocking" force which takes into account a cone of vision of the bird, and a boundary force to bring the bird back inside the swarm if it is on the edge flying outward. Unlike toe original C-S model, which has the feature that all birs flock to a swarm, simulation of the modified model show that the breakup of a swarm does occur.

A related talk by Seung-Yeal Ha was about asymptotic formation of multi-clusters for the Cucker-Smale and Kuramoto models, and the related phenomenon of synchronization. He derived sufficient conditions for the multi-cluster formation to the particle and kinetic Cucker-Samle and Kuramoto models.

Alethea Barbaro discussed the C-S model with friction and noise. She analysed the existence of phase transitions as one increased the noise or friction coefficients. She found three steady-state solutions for small noise, while only one steady state was found for larger values of noise.

Motsch talked about a model of flocking with asymmetric interactions which aims at improving the C-S model. The C-S model relies on a simple rule: the closer two individuals are, the more they tend to align with each other. In the new model proposed, the strength of the interaction is also weighted by the density: the more an agent is surrounded, the less he will be influenced. As a consequence, interactions between agents are no longer symmetric. It was found that that the dynamics converges to a flock provided that the interaction function decays slowly enough.

Jesus Rosado discussed the well-posedness of the kinetic version of the Cucker-Smale model for flocking. He showed that the unconditional flocking result that Cucker and Smale showed for the particle model also holds in the new framework. He also duscussed some extensions of the kinetic model.

Applications in biology

De Vries and Eftimie described two related models on how various communication mechanisms between species can lead to the formation of biological aggregations. Eftimie presented a nonlocal hyperbolic model in one space dimension which incorporates social interactions among the species with orientation. She computed the speeds at which the organisms travel through the media, as a function of their nonlocal interactions. She discussed the role of communication mechanisms and social interactions on the choice of movement direction of travelling groups. De Vries talk extended the hyperbolic model to a particle-based setting using a system of ODE's instead of a hyperbolic PDE. Many rich dynamics that appear in the hyperbolic model, such such as travelling pulses, zigzag pulses, breathers, and feathers, are also observed in the individual-based models.

Doron Levy discussed recent results on modeling phototaxis in order to understand the functionality of the cell and how the motion of individual cells is translated into emerging patterns on macroscopic scales.

Birnir introduced a dynamic energy budget theory to model the to model the physiology of animals and how it influences their interactions with the environment. This theory was motivated by the study of Icelandic capelin, and helps to explain how changes in physiology can trigger entirely different group behavior influencing migration patterns over large distances.

Einarsson discussed two applications: the role of noise in modelling schooling fish, and modelling biofilm growth using cellular automata. Both models are non-deterministic, but give rise to complex structures. The model reproduces biofilm development in the form of flat biofilms, ripples, streamers, towers, mushroom growth etc.

Rodriguez discussed traveling wave solutions for a Reaction-Diffusion system for crime patterns. This system of equations can be divided into three regimes, which lead to one, two, or three steady-states solutions. There is also an invasion phenomenon of crime hotspots via traveling wave solutions in one dimension.

Applications in engineering and science

In engineering context, multi-particle systems model inter-agent interactions. Typically, these agents can be robots, vehicles, soldiers etc, that need to interact collectively to perform a desgnated task. There is a strong interplay between the mathematical models on one hand and engineering applications on the other. Several speakers gave a talk from a more applied perspective. The diversity of topics illustrate the vitality of the subject.

Gazi talked about the stability of swarms with second order agent dynamics. The inter-agent interactions in the individual based swarm model are provided with artificial potential functions. In this context, he discussed aggregation, social foraging, and formation control.

Lindsay presented some recent results on a singularity formation problem in a nonlinear fourth order PDE modelling a Micro-Elcetro Mechanical Systems (MEMS) Capacitor. The singularity is observed to form in multiple locations within the domain with these locations exhibiting an analyzable dependence on the model parameters and the geometry of the domain. He outlined an asymptotic method which can predict the location(s) where singular solutions form based on the geometry of the domain and the parameters of the system. The theory was demonstrated on several examples.

Ward used asymptotic methods to compute the mean first passage time (MFPT) for a Brownian particle in a three-dimensional domain that contains N small non-overlapping absorbing windows on its boundary. This problem has wide applications in cellular biology where it may be used as an effective first order rate constant to describe, for example, the nuclear export of messenger RNA molecules through nuclear pores. Using detailed analytical properties of a surface Green's function, a three-term asymptotic approximation for the MFPT for the unit sphere was computed. The third term in this expansion depends explicitly on the spatial arrangement of the absorbing windows on the boundary of the sphere. The MFPT is minimized for particular trap configurations that minimize a certain discrete variational problem, which is closely related to the well-known optimization problem of determining the minimum energy configuration for N repelling Coulomb charges on the unit sphere.

Putkaradze gave a lecture titled "Molecular monolayers as interacting rolling balls: crystals, liquid and vapor". Molecular monolayers, especially water monolayers, are playing a crucial role in modern science and technology. He considered simplified models of monolayer dynamics, consisting of rolling self-interacting particles on a plane with an off-set center of mass and a non-isotropic inertia tensor. He further assumed that the physical properties to be the similar to water molecules. The standard tools of statistical mechanics do not apply: for example the system exhibits two temperatures – translational and rotational– for some degrees of freedom, and no temperature can be defined for other degrees of freedom. In spite of apparent simplicity, the behavior of the system is surprisingly rich. Many phenomena were investaged. As a first step towards continuous theory, he presented a Vlasov-like kinetic theory for a gas of rolling balls.

Stochastic models

Most systems in nature have a random component to them. The presence of noise is often one of the driving forces that can completely alter the behaviour of the system. Many presentations at the workshop dealt with the effect on noise on the model.

D'Orsogna's talk was about stochastic nucleation and growth of particle clusters. These model the binding of individual components to form composite structures and is an ubiquitous phenomenon within the sciences. Mean field descriptions lead to well known Becker Doering equations. In cellular biology, however, nucleation events often take place in confined spaces, with a finite number of components, so that discreteness and stochastic effects must be taken into account. She considered a fully stochastic master equation, solved via Monte-Carlo simulations and via analytical insight. This resulted in striking differences between the mean cluster sizes obtained from our discrete, stochastic treatment and those predicted by mean field treatments. Further applications to first passage time results and prion unfolding and clustering dynamics were considered.

Erban discussed three different stochastic methods for spatio-temporal modelling in cellular and molecular biology. The connections between these models and the deterministic models (based on reaction-diffusion-advection partial differential equations) were also presented. He also discussed a hybrid modelling of chomtaxis where an individual-based model of cells is coupled with PDEs for extracellular chemical signals.

Liébana presented a kinetic theory two-species coagulation. He we derived a kinetic theory that approximately describes the process dynamics and determine its asymptotic behavior. Analytical results and direct numerical simulations of the stochastic process both corroborate its predictions and check its limitations.

Haskovec presented two individual based models where social phenomena emerge purely from random behaviour of the agents, without introducing any deterministic "social force" that would push the system towards its organized phase. Instead, organization on the global level results merely from reducing the individual noise level in response to local organization, which is induced by stochastic fluctuations. The first model describes the recently experimentally observed collective motion of locust nymphs marching in a ring-shaped arena and is written in terms of coupled velocity jump processes. The second model was inspired by observations of aggregative behaviour of cockroach nymphs in homogeneous environments and is based on randomly moving particles with individual diffusivities depending on the perceived average population density in their neighbourhood. He showed that both models have regimes leading to global self-organization of the group (synchronization and aggregation). He derived the mean-field limits for both models, leading to PDEs with nonlocal nonlinearities.

Wennberg considered two models of biological swarm behavior. In these models, pairs of particles interact to adjust their velocities one to each other. In the first process, called 'BDG', they join their average velocity up to some noise. In the second process, called 'CL', one of the two particles tries to join the other one's velocity. He established the master equations and BBGKY hierarchies of these two processes. The resulting kinetic hierarchy for the CL process does not satisfy chaos propagation. Numerical simulations indicate the same behavior for the BDG model.

Outcome of the meeting

The BIRS workshop was a timely opportunity to gather foremost experts on the subject who presented many recent results on multi-particle systems. Several senior participants also gave an overview of the "state of the art" of the field. Mathematically, the two main approaches is to study multi-particle systems using the theory of dynamical systems; or by taking the continuum limit, which typically results in a PDE system that involves integral terms. Due to nonlocal nature, these systems often lead to novel phenomena that have motivated the development of new mathematical tools and pose new problems that were further explored by many participants of this workshop.

As a companion to this workshop, a special issue of Physica D dedicated to multi-particle systems is currently in production, and we expect the publication to appear in early 2013. Many of the participants from this workshop, as well some researchers who did not attend have contributed papers to this special issue. The vitality of this area of research can be demonstrated based on the interesting results obtained in the last year since this conference took place, for instance [61, 62, 63, 64, 65, 66, 67, 68, 69]. Even some of them are direct result of the interactions produced during this meeting.

The diversity of the topics involved and the backgrounds of the participants attest to the vitality of this exciting area which is currently undergoing an explosive development.

Bibliography

[1] T. Vicsek, A. Czirok, E. Ben-Jacob, I. Cohen, O. Shochet, Novel type of phase transition in a system of self-driven particles, Phys. Rev. Lett. 75 (6) (1995) 1226Ð1229.

[2] S. Camazine, J.-L. Deneubourg, N. R. Franks, J. Sneyd, G. Theraulaz, E. Bonabeau, Selforganization in biological systems, Princeton Studies in Complexity, Princeton University Press, Princeton, NJ, 2003, reprint of the 2001 original.

[3] I. D. Couzin, J. Krause, Self-organization and collective behavior of vertebrates, Adv. Study Behav. 32 (2003) 1Ð67.

[4] Y. L. R. Lukeman, L. Edelstein-Keshet, Inferring individual rules from collective behavior, Proc. Natl. Acad. Sci. U.S.A. 107 (28) (2010) 12576Ð12580.

[5] D. Benedetto, E. Caglioti, M. Pulvirenti, A kinetic equation for granular media, RAIRO Modńel. Math. Anal. Numńer. 31 (5) (1997) 615Ð641.

[6] G. Toscani, One-dimensional kinetic models of granular SSows, M2AN Math. Model. Numer. Anal. 34 (6) (2000) 1277Ð1291.

[7] J. A. Carrillo, R. J. McCann, C. Villani, Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates, Rev. Mat. Iberoamericana 19 (3) (2003) 971Ð1018. URL http://projecteuclid.org/getRecord?id=euclid.rmi/1077293812

[8] J. A. Carrillo, R. J. McCann, C. Villani, Contractions in the 2-Wasserstein length space and thermalization of granular media, Arch. Ration. Mech. Anal. 179 (2) (2006) 217Ð263. URL http://dx.doi.org/10.1007/s00205-005-0386-1

[9] D. D. Holm, V. Putkaradze, Aggregation of Þnite-size particles with variable mobility, Phys Rev Lett. 95:226106.

[10] D. D. Holm, V. Putkaradze, Formation of clumps and patches in selfaggregation of Þnite-size particles, Physica D. 220 (2) (2006) 183Ð196.

[11] J. Haile, Molecular Dynamics Simulation: Elementary Methods, John Wiley and Sons, Inc., New York, 1992.

[12] J. K. Parrish, L. E. Keshet, Complexity, pattern, and evolutionary trade-o?s in animal aggregation, Science 284 (1999) 99Ð101.

[13] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale, V. Zdravkovic, Interaction ruling animal collective behaviour depends on topological rather than metric distance: evidence from a Þeld study, Proc. Natl. Acad. Sci. 105 (2008) 1232Ð1237.

[14] A. Koch, D. White, The social lifestyle of myxobacteria, Bioessays 20 (20) (1998) 1030Ð1038.

[15] I. D. Couzin, J. Krause, R. James, G. Ruxton, N. R. Franks, Collective memory and spatial sorting in animal groups, J. Theor. Biol. 218 (2002) 1Ð11.

[16] I. Aoki, A simulation study on the schooling mechanism in Þsh, Bull. Japan Soc. Sci. Fish 48 (1982) 1081Ð1088.

[17] A. Huth, C. Wissel, The simulation of Þsh schools in comparison with experimental data, Ecol. Model. 75/76 (1994) 135Ð145.

[18] A. Cavagna, A. Cimarelli, I. Giardina, G. Parisi, R. Santagati, F. Stefanini, R. Tavarone, From empirical data to inter-individual interactions: unveiling the rules of collective animal behavior, Math. Models Methods Appl. Sci. DOI: 10.1142/S0218202510004660.

[19] C. K. Hemelrijk, H. Hildenbrandt, Self-organized shape and frontal density of Þsh schools, Ethology 114 (2008) 245Ð254.

[20] A. B. T. Barbaro, K. Taylor, P. F. Trethewey, L. Youse?, B. Birnir, Discrete and continuous models of the dynamics of pelagic Þsh: application to the capelin, Math. Comput. Simulation 79 (12) (2009) 3397Ð3414.

[21] C. M. Topaz, A. L. Bertozzi, Swarming patterns in a two-dimensional kinematic model for biological groups, SIAM J. Appl. Math. 65 (2004) 152Ð174.

[22] T. Kolokonikov, H. Sun, D. Uminsky, A. Bertozzi, Stability of ring patterns arising from 2d particle interactions, Physical Review E 84 (1) (2011) 015203.

[23] M. Bodnar, J. J. L. Velasquez, Derivation of macroscopic equations for individual cell-based models: a formal approach, Math. Meth. Appl. Sci. (M3AS) 28 (2005) 1757Ð1779.

[24] A. J. Berno?, C. M. Topaz, A primer of swarm equilibria, SIAM Journal on Applied Dynamical Systems 10 (1) (2011) 212Ð250.

[25] A. Mogilner, L. Edelstein-Keshet, L. Bent, A. Spiros, Mutual interactions, potentials, and individual distance in a social aggregation, J. Math. Biol. 47 (2003) 353Ð389.

[26] M. R. DÕOrsogna, Y.-L. Chuang, A. L. Bertozzi, L. S. Chayes, Self-propelled particles with soft-core interactions: patterns, stability and collapse, Phys. Rev. Lett. 96 (10) (2006) 104302.

[27] H. Levine, W.-J. Rappel, I. Cohen, Self-organization in systems of self-propelled particles, Phys. Rev. E 63 (1) (2000) 017101.

[28] F. Cucker, S. Smale, Emergent behavior in SSocks, IEEE Trans. Automat. Control 52 (5) (2007) 852Ð862.

[29] Y.-L. Chuang, Y. Huang, M. R. DÕOrsogna, D. Marthaler, A. L. Bertozzi, Multi-vehicle SSocking: scalability of cooperative control algorithms using pairwise potentials, IEEE International Conference on Robotics and Automation (2007) 2292Ð2299.

[30] L. Perea, G. Gńomez, P. Elosegui, Extension of the cuckerÐsmale control law to space SSight formations, AIAA Journal of Guidance, Control, and Dynamics 32 (2009) 527Ð537.

[31] C. M. Topaz, A. L. Bertozzi, M. A. Lewis, A nonlocal continuum model for biological aggregation, Bull. Math. Bio. 68 (2006) 1601Ð1623.

[32] M. Burger, V. Capasso, D. Morale, On an aggregation model with long and short range interactions, Nonlinear Analysis: Real World Applications 8 (2007) 939Ð958.

[33] Y.-L. Chuang, M. R. DÕOrsogna, D. Marthaler, A. L. Bertozzi, L. S. Chayes, State transitions and the continuum limit for a 2D interacting, self-propelled particle system, Phys. D 232 (1) (2007) 33 - 47.

[34] J. A. Carrillo, M. R. DÕOrsogna, V. Panferov, Double milling in self-propelled swarms from kinetic theory, Kinet. Relat. Models 2 (2) (2009) 363Ð378.

[35] R. Eftimie, G. de Vries, M. A. Lewis, Complex spatial group patterns result from di?erent animal communication mechanisms, Proc. Natl. Acad. Sci. 104 (17) (2007) 6974Ð6979.

[36] R. Eftimie, G. de Vries, M. Lewis, Weakly nonlinear analysis of a hyperbolic model for animal group formation, J. Math. Biol. 59 (2009) 37Ð74.

[37] R. C. Fetecau, R. Eftimie, An investigation of a nonlocal hyperbolic model for self-organization of biological groups, J. Math. Biol. 61 (4) (2009) 545Ð579.

[38] P. Degond, S. Motsch, Continuum limit of self-driven particles with orientation interaction, Math. Models Methods Appl. Sci. 18 (suppl.) (2008) 1193Ð1215.

[39] S.-Y. Ha, E. Tadmor, From particle to kinetic and hydrodynamic descriptions of SSocking, Kinet. Relat. Models 1 (3) (2008) 415Ð435.

[40] J. A. Caœnizo, J. A. Carrillo, J. Rosado, A well-posedness theory in measures for some kinetic models of collective motion, Math. Mod. Meth. Appl. Sci. 21 (2011) 515Ð539.

[41] J. A. Carrillo, M. Fornasier, J. Rosado, G. Toscani, Asymptotic SSocking dynamics for the kinetic Cucker-Smale model, SIAM J. Math. Anal. 42 (1) (2010) 218Ð236.

[42] S.-Y. Ha, J.-G. Liu, A simple proof of the Cucker-Smale SSocking dynamics and mean-Þeld limit, Commun. Math. Sci. 7 (2) (2009) 297Ð325. URL http://projecteuclid.org/getRecord?id=euclid.cms/1243443982

[43] J. A. Carrillo, M. Fornasier, G. Toscani, F. Vecil, Particle, kinetic, and hydrodynamic models of swarming, in: Mathematical modeling of collective behavior in socio-economic and life sciences, Model. Simul. Sci. Eng. Technol., Birkhňauser Boston Inc., Boston, MA, 2010, pp. 297Ð336.

[44] J. A. Carrillo, A. Klar, S. Martin, S. Tiwari, Self-propelled interacting particle systems with roosting force, Math. Mod. Meth. Appl. Sci 20 (2010) 1533Ð1552.

[45] A. Czirńok, A.-L. Barabńasi, T. Vicsek, Collective motion of self-propelled particles: kinetic phase transition in one dimension, Physical Review Letters 82 (1) (1999) 209Ð212.

[46] H. Chatńe, F. Ginelli, G. Grńegoire, Comment on Òphase transitions in systems of self-propelled agents and related network modelsÓ, Phys. Rev. Lett. 99 (2007) 229601.

[47] M. Aldana, V. Dossetti, C. Huepe, V. M. Kenke, H. Larralde, Phase transitions in systems of selfpropelled agents and related network models, Phys. Rev. Lett. 98 (9) (2007) 095702.

[48] D. Horstmann, From 1970 until present: the keller-segel model in chemotaxis and its consequences, Jahresberichte der DMV 105 (3) (2003) 103165.

[49] M. Burger, M. Di Francesco, Large time behavior of nonlocal aggregation models with nonlinear di?usion, Netw. Heterog. Media 3 (4) (2008) 749Ð785.

[50] A. Blanchet, J. A. Carrillo, N. Masmoudi, InÞnite time aggregation for the critical patlak-keller-segel model in R^2, Comm. Pure Appl. Math. 61 (10) (2008) 1449Ð1481.

[51] A. Blanchet, V. Calvez, J. A. Carrillo, Convergence of the mass-transport steepest descent scheme for the subcritical Patlak-Keller-Segel model, SIAM J. Numer. Anal. 46 (2) (2008) 691Ð721. URL http://dx.doi.org/10.1137/070683337

[52] R. C. Fetecau, Y. Huang, T. Kolokolnikov, Swarm dynamics and equilibria for a nonlocal aggregation model, Nonlinearity 24 (10) (2011) 2681Ð2716.

[53] J. A. Carrillo, M. Di Francesco, A. Figalli, T. Laurent, D. Slepcev, Global-in-time weak measure solutions and Þnite-time aggregation for nonlocal interaction equations, Duke Math. J. 156 (2011) 229Ð271.

[54] J. A. Carrillo, M. Di Francesco, A. Figalli, T. Laurent, D. Slepcev, ConÞnement in nonlocal interaction equations, Nonlinear Anal. 75 (2) (2012) 550Ð558. URL http://dx.doi.org/10.1016/j.na.2011.08.057

[55] A. L. Bertozzi, J. A. Carrillo, T. Laurent, Blow-up in multidimensional aggregation equations with mildly singular interaction kernels, Nonlinearity 22 (3) (2009) 683Ð710.

[56] A. L. Bertozzi, T. Laurent, J. Rosado, L^p theory for the multidimensional aggregation equation, Comm. Pur. Appl. Math. 64 (1) (2011) 45Ð83.

[57] A. L. Bertozzi, T. Garnett, John B. Laurent, Characterization of radially symmetric Þnite time blowup in multidimensional aggregation equations (preprint).

[58] A. Mogilner, L. Edelstein-Keshet, A non-local model for a swarm, J. Math. Biol. 38 (1999) 534Ð570.

[59] A. J. Leverentz, C. M. Topaz, A. J. Berno?, Asymptotic dynamics of attractive-repulsive swarms, SIAM J. Appl. Dyn. Syst. 8 (3) (2009) 880Ð908.

[60] J. Lega, Collective behaviors in two-dimensional systems of interacting particles, SIAM J. Appl. Dyn. Sys. 10 (2011) 1213Ð1231.

[61] D. Balagué, J. A. Carrillo, T. Laurent, G. Raoul, Nonlocal interactions by repulsive-attractive potentials: radial ins/stability, to appear in Physica D .

[62] D. Balagué, J. A. Carrillo, T. Laurent, G. Raoul, Dimensionality of local minimizers of the interaction energy, to appear in ARMA .

[63] D. Balagué, J. A. Carrillo, Y. Yao, ConÞnement for repulsive-attractive kernels, Preprint .

[64] J. von Brecht, D. Uminsky, On soccer balls and linearized inverse statistical mechanics, to appear in J. Nonlinear Sci. .

[65] R. C. Fetecau, Y. Huang, Equilibria of biological aggregations with nonlocal repulsive-attractive interactions, preprint .

[66] A. Bertozzi, J. von Brecht, H. Sun, T. Kolokolnikov, D. Uminsky, Ring patterns and their bifurcations in a nonlocal model of biological swarms, Preprint .

[67] H. Sun, D. Uminsky, A. L. Bertozzi, Stability and clustering of self-similar solutions of aggregation equations, J. Math. Phys. 53 (2012) 115610.

[68] J. von Brecht, D. Uminsky, T. Kolokolnikov, A. Bertozzi, Predicting pattern formation in particle interactions, Math. Mod. Meth. Appl. Sci. 22 (2012) 1140002.

[69] T. Kolokolnikov, Y. Huang, M. Pavlovski, Singular patterns for an aggregation model with a conÞning potential, to appear in Physica D. .

Chapter 4

Neostability theory (12w5045)

January 29 - February 3, 2012

Organizer(s): Bradd Hart (MacMaster University), Ehud Hrushovski (Hebrew University at Jerusalem), Alf Onshuus (Universidad de los Andes), Anand Pillay (University of Leeds), Thomas Scanlon (University of Berkeley), Frank Wagner (Université Lyon I)

Overview of the Field

In recent years many unexpected results have been proven using methods from model theoretic stability theory in unstable contexts and this subject has taken center stage in model theory.

Let us start with a brief chronological review of the subject.

Starting in the late sixties and for all of the seventies and eighties, stability theory played a central role in model theory. Inaugurated with Morley's celebrated proof of his theorem on theories categorical in an uncountable cardinality, the theory reached a high degree of sophistication with Shelah's classification theory and was then developed by Shelah, Lascar, Poizat and others into what has arguably been the deepest and most applicable of the branches of model theory.

Stability theory reached an apex with the geometric stability theory of, especially, Hrushovski and Zilber. In the late 1970s Zilber introduced the group configuration in his work on totally categorical theories and then Hrushovski generalized the group configuration theorem well beyond these logically perfect theories. In so doing, it was revealed that structures of algebraic or algebraic geometric origin explain the complexity of some very general theories in which there is no apparent geometry. This approach of analyzing a stable structure according to the geometry of the types was incredibly fruitful with results such as the trichotomy theorem for Zariski geometries by Hrushovski and Zilber and the latter applications to diophantine number theory (including the Mordell-Lang Conjecture for fields of positive characteristic) by Hrushovski and later by Scanlon.

From the late nineties, especially with the work of Kim and Pillay, people have tried to generalize the results from stability theory to a more general class of structures, notably simple theories, those theories for which the basic properties of forking independence from stable theories continue to hold. Although many results have been obtained, it appears that it was the robustness of the types (automorphism invariant classes) that one has in stable theories which allowed for many of the most striking of the previously mentioned results and this robustness is lost when generalizing from stability to simplicity.

Since the early 2000's, developments turned the attention of many model theorists back into stability, or, really, back to the fundamental ideas and objects of study of stability theory, such as definable types. These developments prompted us to organize the first BIRS meeting on the subject in 2009, a meeting which directed many of the multiple approaches into defining certain research fields which are now centers of intense research and productivity, something that was reinforced with this last meeting. We will now mention what has happened with the main subfields.

First, the work Haskell, Hrushovski and Macpherson on *stably dominated types* in algebraically closed valued fields (ACVFs) combined very well with what Shelah's work on *generically stable* types in dependent theories. The combination and shared results of these two areas has developed into a central area of research in model theory with at least three well established approaches. There has been ongoing research on the foundational aspects of generically stable types from the first order point of view (for example ongoing work by Pillay and Tanovic, and Onshuus and Usvyatsov); a very important branch is the theory of invariant and generically stable Keisler measures (Hrushovski, Pillay and Simon, among others); and finally, the topology of generically stable types and applications to geometry (Hurshovski and Loeser). This is an area which promises to have deep consequences both in model theory and other areas of mathematics, which makes the study of the foundations all the more important. For instance, with this last mentioned project of Hrushovski and Loeser on stably dominated types, the theory of stably dominated types has been shown to be perfectly apt for the analysis of p-adic analytic spaces.

Secondly, the theory of non forking in dependent theories has manifested itself in disparate areas of mathematics. For example, the idea that non forking could be developed relative to any good notion of smallness (in the sense of ideals or measure zero sets) together with a generalization of the generically presented group theorem underlies Hrushovski's recent breakthrough in additive combinatorics which is now known as the "non-abelian Freiman theorem". Towsner then used these techniques to give a model theoretic proof of Szemerédi's theorem on the existence of arithmetic progressions in sufficiently dense sets of integers. On the other hand, deep ideas from combinatorics, such as Szemerédi's graph regularity theorem, have been consciously imported into general model theory. We expect this synergy to result in further spectacular results.

The foundational studies of dependent theories has continued, with many ideas and results developing and converging after the 2009 meeting. Shelah now has a "counting of types" characterization of dependent theories, while work towards understanding the different properties of non forking in dependent theories has continued with work by Chernikov, Kaplan, Simon and Usvyatsov. This research have prompted the establishment of NTP$_2$ theories as a very adequate generalization of simple and dependent theories, in the sense that it generalizes both and recent work has shown that in many cases, particularly with the behavior of forking, any result that works in both dependent and simple theories usually works also in NTP$_2$ theories.

Finally, work in stongly dependent theories and theories with finite VC (Vapnik-Chervonenkis) dimension (a notion that has deep implications in the complexity of algorithms and learning techniques) has continued, and recent preprints by Kaplan, Onshuus and Usvyatsov, Kaplan and Simon, and Aschenbrenner, Dolich, Haskell, Macpherson and Starchenko, give evidence that the combination of the notions coming from Shelah's work (dp-rank, burden and weight) and those coming from the study of the VC-rank may combine to produce some interesting results.

Recent Developments and Open Problems

The study of dependent theories has continued its fruitfulness, the most recent important development has been the proof of Chernikov and Simon ([5]) of uniform definability of types over finite sets in dependent theories. This result implies that types over finite sets have a very stable-like behaviour and, by work of Laskowski, is closely connected to long-standing problems in learning theory . It also prompts a very important question as to whether or not this result is true locally (so whether a family of dependent types over finite sets have uniform definability), which would imply that one can look for dependent-like types in any given theory and have a very important tool to work with. Another important development in dependent theory came from the paper "On non-forking spectra" by Chernikov, Kaplan, and Shelah ([1]) where they show enough examples to understand in a much better way the possible characterizations of dependent theories by bounded number of types. They show that a bounded (2^{2^κ}) number of non-forking extensions (for types over a model M of size κ) does not characterize dependent theories, although whether or not the bound $\mathrm{ded}(M)^{\aleph_0}$ works is still open. They also prove that within NTP$_2$ theories the bound 2^{2^κ} does characterize dependence, which also enriches the study of NTP$_2$ theories.

Another important result of NTP$_2$ theories was Chernikov and Kaplan's proof of Kim's Lemma ([6]). Not only is this result a very important tool when studying non forking behavior which is now valid in both simple and dependent theories, but it gives very strong evidence towards the idea that things which are true for dependent and simple theories should not only work in NTP$_2$, but that proving the fact in this context will give what in some sense is 'the right proof'.

On a quite different direction, where instead of generalizing we restrict dependent theories to a very interesting subclass, a lot has happened in the study of dp-rank and VC-density. There have been many different developments in VC-minimal and dp-minimal theories, the applications of the Aschenbrenner, Dolich, Haskell, Macpherson, Starchenko ([9] and [8]) results keep appearing, and the understanding of dp-rank and theories of finite dp-rank is one of the fastest growing areas in model theory at the time. Some of the more imporant results are the additivity of the dp-rank ([11]), all the study of dp-minimal theories by Simon and Goodrick ([4] and [3]) and the proof of uniform definability of types over finite sets in dp-minimal theories by Guingona ([7]) which was later generalized to dependent theories in the paper mentioned above. The main open question in this area is whether or not the dp-rank bounds the VC-density, and whether or not one has a definable version of the Helly number theorem for families of sets defined by dependent formulas.

Presentation Highlights

- **Mathias Aschenbrenner** opened the meeting with a talk on VC-density that was both an excellent introduction into the subject, developed initially by Vapnik and Chervonenkis in the context of computational learning theory but with a clear model-theoretic content, and his results with Dolich, Haskell, MacPherson and Starchenko. While the notion of VC dimension is by now well-established in model theory, they also study VC density. The latter not only can serve as explanation of polynomial bounds on the computational complexity of geometric arrangements, but can also be used to define strengthenings of the NIP condition.

- **Pierre Simon** spoke about honest definitions in NIP theories taking up one of the topics from Aschenbrenner's talk and showed that they exist in any NIP theory, relating their existence to earlier results about expanding an NIP structure by a predicate. One of the most remarkable results he discussed was his solution with Chernikov of the conjecture on uniform definability of types over finite sets in NIP theories. As Laskowski had demonstrated earlier, the aptly, though stultifyingly technically, named UDTFS property is closely related to some very difficult and long-standing problems on the existence of compression schemes in learning theory. The Chernikov-Simon theorem represents a major breakthrough.

- Theories without the tree property of the second kind (NTP_2) are a common generalisation of simple and NIP theories, and the more audacious would propose that this is the correct class in which to develop the most general form of stability theory. The talk by **Artem Chernikov** was one step in this direction; he showed that in an NTP_2 theory the ideal of forking formulas is S_1 in the terminology of Hrushovski. This is extremely encouraging, since the recent generalization by Hrushovski of the independence theorem and the stabilizer theorem use the existence of a suitable S_1 ideal.

- **Kobi Peterzil** reminded us that contrary to popular opinion, o-minimal theories need not eliminate imaginaries, if there is no underlying one-dimensional group, and showed that imaginaries of dimension one can be eliminated.

- Indiscernible sets and indiscernible sequences have been a mainstay of stability theory from the very beginning, appearing in essential ways in Morley's proof of his categoricity theorem. Shelah had used indiscernibles with respect to more complicated indexing structures, especially with regard to trees, to study dividing lines in the classification theory of unstable theories. With the lectures of **Joon Kim** and **Lynn Scow** we learned about the connections between generalised indiscernibles and structural Ramsey theory and further applications of these indiscernibles to the problem of distinguishing SOP_2 from SOP_3.

- Dependent (or NIP) theories and simple theories form two of the most important classes of theories generalizing stability. Combinatorially, NIP theories are characterized by the absence of a formula $\phi(x;y)$ and parameters $\{b_\sigma : \sigma \subseteq \mathbb{N}\}$ and $\{a_i : i \in \mathbb{N}\}$ for which $\models \phi(a_i;b_\sigma) \iff i \in \sigma$ which simple theories are those where no formula $\phi(x;y)$ has the tree property: there are parameters $\{b_\rho : \rho \in \omega^{<\omega}\}$ and $\{a_\sigma : \sigma \in \omega^\omega\}$ and a natural number k so that $\rho \subseteq \sigma \Longrightarrow \models \phi(a_\sigma;b_\rho)$ and for any $\rho \in \omega^{<\omega}$ the set $\{\phi(x;b_{\rho\frown\langle j\rangle}) : j \in \omega\}$ is k-inconsistent. The class of theories with the tree property may be further decomposed into those of the first kind and those of the second kind. As we learned in lectures by **Hans Adler** and **Artem Chernikov**, the class of NTP_2 theories, those without the tree property of the second kind, generalizing simple and dependent theories, enjoy strong stable-like properties.

- In the late 1960s, Keisler proposed measuring the complexity of a theory through the class of ultrafilters with respect to which ultrapowers of the given theory are reasonably saturated. On the face of it, since the definition of this relation is highly set theoretic, one would expect that the resulting partial order to be chaotic. However, the known results, mostly proven by Shelah, suggest that the Keisler order is linear and that breaks occur at meaningful points in the classification theoretic hierarchy (finite cover property, stable, stronger order property-3, *et cetera*). After the burst of initial results and a few scattered theorem proven later (the most recent of which was shown a decade and half ago), only very recently have there been any significant breakthroughs. **Maryanthe Malliaris** reported on several new results including an unconditional identification of a new class in the Keisler order.

- Several speakers addressed problems around the structure of groups whose theories satisfy various model theoretic hypotheses. **Martin Hils** explained his proof, with **Martin Bays** and **Misha Gavrilovich** that in an ω-stable theory with the definable multiplicity property, the property that $\frac{1}{n}p$ has Morley degree 1 for all $n < \omega$ is definable for definable families of Morley degree 1 subsets of divisible abelian groups. This in particular implies that for an irreducible subvariety X of a semiabelian variety over an algebraically closed field, the number of irreducible components of $[n]^{-1}(X)$ is bounded uniformly in n, and moreover that the bound is uniform in families X_t, thus closing a gap in the construction of the bad field by Baudisch, Blossier, Martin Pizarro, and Wagner. **Dugald Macpherson** described his joint work with **Katrin Tent** on the structure of pseudofinite groups with NIP or supersimple theories. **Krysztof Krupiński** and **Jakub Gismatullin** spoke about their study of definable cocycles as a method to produce interesting examples of groups G for which $G^{\circ\circ}$, the smallest type definable group of bounded index, differs from $G^{\circ\circ\circ}$, the smallest invariant (in a saturated model) group of bounded index. **Itay Kaplan** exhibited the subtlety implicit in attempted generalizations of the Baldwin-Saxl theorem, to wit that in a dependent theory given a uniformly definable family of subgroups of a definable group there is a fixed bound N for which every finite intersection of instances of such subgroups reduces to an intersection of at most N such, to families of type definable subgroups.

- **John Baldwin** spoke about the use of set theory in model theory, paying special attention to questions of absoluteness for key notions in abstract elementary classes (AECs). Many dividing lines in first-order classification theory are defined by set theoretic conditions, for example, stability may be defined by bounds on the cardinalities of type spaces, but they are known to be absolute because they also admit arithmetic characterizations. This tends to fail for other logics. **Monica VanDieren** continued the study of AECs with a lecture on non-splitting. In the first-order context, independence relations were based on non-forking rather than non-splitting, but for AECs forking does not really make sense. This handicap forced VanDieren to develop a better theory of non-splitting which yields new information even for first-order theories.

- From the point of view of first-order logic, a fixed finite model does not admit an interesting model theory since the entire structure may be described by a single sentence. However, if one studies families of finite models as one does with the theories of pseudofinite models or works with a weaker logic, then the methods of infinite model theory apply to the finite. **Cameron Hill** developed the theory of thorn-independence in finite variable logic transposing an independence theory which on the face of it works only for infinite structures to the finite. With his lecture not only did he complete the academic exercise of generalization but he showed that rosiness for finite variable logic admits a computational characterization.

- **John Goodrick** spoke about his joint work with **Byunghan Kim** and **Alexei Kolesnikov** on homology groups of first-order theories. He showed how questions about amalgamations of types are encoded by these groups.

- **Assaf Hasson** reported on his proof with **Alf Onshuus** that Henson's graphs, i.e., the generic countable K_n-free graphs, are symmetrically indivisible: For any finite partition one of the parts contains an isomorphic symmetrically embedded substructure (i.e., any automorphism of the substructure extends to an automorphism of the whole).

- A long-standing conjecture of Kueker asserts that for a countable theory T, if every uncountable model of T is $\aleph_0$-saturated, then T must be categorical in some infinite cardinal. **Predrag Tanović** outlined a proof of an important case of Kueker's conjecture: for dependent theories in which $\mathrm{dcl}(\varnothing)$ is infinite.
- Ultraimaginaries, i.e., classes modulo an invariant equivalence relation, arise in non-simple theories as canonical bases, but also appear in the simple context, for instance as non-orthogonality classes of regular types. However, Ben Yaacov had shown that the usual independence theory cannot be extended to include ultraimaginaries. **Frank Wagner** defined a reasonably behaved subclass, 'tame' ultraimaginaries (which in particular suffice for supersimple theories), proved certain basic properties and a feeble elimination result, and demonstrated how the use of ultraimaginaries can explain certain phenomena in finite rank theories and be used for a generalization to infinite rank.

This workshop was one of the last ones in which not all of the lectures were automatically recorded. Two lectures, those of Simon and Peterzil, were webcast. These recordings have already formed the basis of remote seminars. If some of the other excellent lectures were available, they, too, would have been studied by researchers and students who were unable to attend the meeting.

A particular feature of the conference was the moderated question session on Tuesday afternoon. After initial hesitation, many open questions were posed and discussed, and then assembled in a file on the conference website. The evenings were left free of talks for discussion in small groups, and long breaks between the talks enabled prolonged discussions at the end of each presentation.

Scientific Progress Made

Many people have commented to have made concrete progress of their research during the meeting.

In particular, following discussions at the meeting, Krupinski, Tanovic, and Wagner, made progress on a 40 year old conjecture of Podewski, namely that a minimal field (so that any definable set is finite or cofinite) of any characteristic is algebraically closed. Chernikov and Hills were able to prove that certain theories of valued difference fields are NTP_2. Ben Yaacov and Chernikiv also reported that after proving an "amalgamation of independent types" theorem for NTP_2 theories, they were able to prove that in NTP_2 theories equality of Lascar types over an amalgamation base is type-definable (more precisely, if a and b have the same Lascar type then they have Lascar distance at most 3.)

Several other groups of people (Berenstein-Dolich-Vassiliev, Tanovic Gismatulin-Krupinski, Macpherson-Steinhorn), are already writing papers based on results achieved during the meeting, although given the general feedback we have had we expect several more to come after the research started during the meeting matures.

Outcome of the Meeting

As expected, this meeting offered a very rich array of results. Three years ago the main achievement of the meeting was making researchers aware of the possibilities in the field. Since much progress has been made in settling the main areas and achieving the first results, all this came together to make this a meeting as fruitful as the first one, but in a more concrete way. For example, the TP_2-NTP_2 dichotomy which had appeared as a curiosity at the 2009 meeting has by now revealed itself to be a major fault line and classification theory of dependent theories is taking on the robust character of stable classification theory.

This meeting had the effect of solidifying the body of work in neostability theory as a coherent project to discern robust divisions within the class of all theories and to develop stability theoretic methods in an appropriate level of generality.

Participants

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Bibliography

[1] A. Chernikov, I. Kaplan, and S. Shelah, On non-forking spectra, `arXiv:1205.3101`, preprint, 2012.

[2] M. Malliaris and S. Shelah, Constructing regular ultrafilters from a model theoretic point of view, `arXiv:1204.1481`, preprint, 2012.

[3] J. Goodrick, A monotonicity theorem for DP-minimal densely ordered groups. *J. Symbolic Logic* **75** (2010), no. 1, 221–238.

[4] P. Simon, On dp-minimal ordered structures. *J. Symbolic Logic* **76** (2011), no. 2, 448–460.

[5] A. Chernikov and P. Simon, Externally definable sets and dependent pairs II (arXiv 1202.2650).

[6] A. Chernikov and I. Kaplan, Forking and dividing in NTP2 theories, *Journal of Symbolic Logic* **77**, Issue 1 (2012), 1-20.

[7] V. Guingona, On uniform definability of types over finite sets, *Journal of Symbolic Logic* **77**, Issue 2 (2012), 499-514.

[8] M. Aschenbrenner, A. Dolich, D. Haskell, D. Macpherson and S. Starchenko, Vapnik-Chervonenkis density in some theories without the independence property, II (arXiv 1109.5437).

[9] M. Aschenbrenner, A. Dolich, D. Haskell, D. Macpherson and S. Starchenko, Vapnik-Chervonenkis density in some theories without the independence property, I (arXiv 1109.5438).

[10] M. Bays, M. Gavrilovich and M. Hils, Some definability results in abstract Kummer theory (arXiv:1112.6012).

[11] I. Kaplan, A. Onshuus and A. Usvyatsov, Additivity of the dp-rank (arXiv:1109.1601).

[12] K. Krupinski, P. Tanovic and F. Wagner, On Podewski's Conjecture (arXiv:1201.5709).

[13] F. Wagner, Plus ultra (arXiv:1201.1114).

Chapter 5

Models of Sparse Random Graphs and Network Algorithms (12w5004)

February 5 -10, 2012

Organizer(s): Nicolas Broutin (Inria, France), Luc Devroye (McGill, Canada), Gábor Lugosi (Pompeu Fabra and ICREA, Spain)

Overview of the Field

There is no doubt now that the current trend that every electronic device should be connected in one way or another (usually many) implies a greater need for efficient networks. These networks should be connected, exhibit small-world behavior and their topology should be robust to local modifications like device movement; all this should be achieved using minimal and distributed processes. The workshop we held at BIRS between Feb 5–10 2012 was aimed at a deeper understanding of some of the major network models.

The topic of the workshop was mainly *sparse geometric graphs* and their use as models for wireless, bluetooth and ad-hoc networks, but some more general models of sparse networks have also been discussed. We focused in particular on quantitative indicators of the quality of the network, and of performance of the main communications algorithms. We first describe the important aspects of the subject area: the models, the quantities of interest and the relationships with neighboring fields.

The community working in the field aims at designing and understanding models of networks. A good analysis is crucial in that it permits to improve the design and to adapt it to the needs. The models of interest arise naturally either from concrete applications and the constraints imposed by physics (random geometric graphs) or as essential objects in more fundamental questions (Erdős–Rényi random graphs, Achlioptas processes). The parameters that are crucial to the quality of a model are

- CONNECTIVITY: in general, it should be possible to go from any point to any other; but it it is sometimes acceptable if a vast majority of the nodes are interconnected (there is a giant connected component).
- MAGNITUDE OF DISTANCES: the diameter should be rather short to ensure that it is (at least theoretically) possible to move quickly from any point to any other; again, constraining the diameter is sometimes too strong a requirement and one settles for short typical distances.
- SPARSITY: it is rather easy to design networks that are very connected, and with short distances, just take a complete graph. Such a topology is however unacceptable for reasons of cost and/or scalability, and one would like the graph to be as sparse as possible.

- NAVIGABILITY: for sparse connected graph with short distances, one is only certain that short paths exist, but they may be difficult to find using local information only. To be useful in practice, one would like that the network be navigable, i.e., that short paths are easily found by some distributed algorithm using local information.
- DIFFUSIVITY: finally, one would like broadcast algorithms to perform very well. The analysis here is usually important since one would like good information to spread quickly, but also to be able to stop efficiently the propagation of viruses.

We will finish this quick overview of the field by defining the two main classical models of interest, Erdős-Rényi random graphs, and random geometric graphs. Most of the interesting and more complex models build on either of these. We will then discuss the recent developments and open questions in Section 5.

ERDŐS–RÉNYI RANDOM GRAPHS. The model introduced by Erdős–Rényi [25, 26] is one of the most studied models of random networks. In spite of its simplicity it exhibits very interesting phenomena. It is usually the model one keeps in mind to understand more complex networks. A random graph $G(n, p)$ is a graph (V, E) on n labelled vertices $V = \{1, 2, \dots, n\}$ where the edge set E consists of a random sample of the $\binom{n}{2}$ possible edges where each one is present with probability p, independently of the others. The structure of typical graphs varies greatly depending on the value of p with respect to n. With the parametrization $p = c/n$, where c is a fixed constant, one sees that the largest component of $G(n, p)$ has size at most $O(\log n)$ when $c < 1$, $\Theta(n^{2/3})$ when $c = 1$ or $\Theta(n)$ when $c > 1$. This last phase is most interesting since a large part of the graph is then interconnected. The graph only becomes completely connected when p is of the order of $\log(n)/n$ so that the average degree is of order $\log n$.

RANDOM GEOMETRIC GRAPHS. The model dates back to Gilbert [28] who introduced an underlying spatial topology for the network. One is given a connected domain, $[0, 1]^d$ for simplicity, and draws a set of n uniformly random points $\mathcal{X} = \{X_1, \dots, X_n\}$. The graph $G(n, r)$ is the graph with vertex set $\mathcal{X}$ where two nodes are tied by an edge if the Euclidean distance between them is at most r. Again, the structure of a typical graph depends highly on the respective values of r and n. When $r = (\lambda/n)^{1/d}$ for some fixed constant λ, the average degree is of order λ. As in the $G(n, p)$ model, the is a phase transition for the connectivity: there is a critical value λ_c such that the largest connected component has linear size if and only if $\lambda > \lambda_c$. A typical graph is connected only when r is of order $(\log(n)/n)^{1/d}$ so that, again, the average degree is about $\log n$.

Recent Developments and Open Problems

Sparse connected graphs

As we mentioned above, the typical graphs only get connected when the average degree is logarithmic in the size. This is usually not acceptable since this raises the question of scalability: a typical node of the network cannot have a number of links that grow with the size of the graph.

SPARSE RANDOM GEOMETRIC GRAPH MODELS. The natural model for a spatial network is the *random geometric graph* $G(n, r)$, where n points are randomly distributed in space, say the unit square $[0, 1]^2$, and edges are added between any pair of vertices whose Euclidean distance does not exceed some value r [33, 32]. Above the threshold $r^\star(n)$ for connectivity, the average degree is of order $\Omega(\log n)$ so that graph is too dense for pratical reasons. This is why a number of related models have been proposed, that built a *sparse connected overlay* of the geometric graph. An example is provided by the *irrigation graph* introduced by Dubhashi, Häggström, Johansson, Panconesi, and Sozio [23]. In this model, each node chooses independently a number $c(n)$ of its geometric neighbours (at distance at most $r(n)$) with which it establishes a connection. One would like to study how small one can pick $c(n)$ and still obtain a nice enough network. More precisely, the properties that are crucial for applications are related to the quantitative measures of the *connectivity* and the ability to design efficient *communications algorithms*, both highly dependent upon the network model.

The problem of distances and navigability

The good connectivity of the graph is only a minimal requirement for the network. There should obviously be a trade-off between connectivity and sparsity of the network. This trade-off must also take into account the fact that the network should be usable: it should be relatively fast and easy to find short routes between vertices.

DISTANCES AND SMALL-WORLD PHENOMENON. One important property concerns the magnitude of distances: indeed the communication time between two nodes is lower bounded by the shortest distance between them.Branching arguments show that the scale of distances is at least of order $1/r(n)$; so one would want the distances to have a magnitude approaching this lower bound. In other words, one would like the graph to exhibit *small-world* behavior. Many non-geometric models exhibit such properties, and for these precise asymptotics have been proved for quantities such as the typical distance or the diameter (maximum shortest pairwise distance) [36, 10, 34, 16]. However, for geometric models, the known results about distances mostly characterize the growth rate and more precise results are still lacking.

NAVIGABILITY, CONDUCTANCE AND BROADCASTING. For the navigability, one would like that the short paths, if they exists, are easily found in a random way. Two main parameters of interest here are *mixing time* and the *cover time*. The mixing time is the length of the transition period before the behavior stabilizes to the stationary distribution. It is related to the spectral properties of the graph and to its conductance, which measures the uniform expansion quality of the network. The cover time measures the number of steps required for the simple random walk to visit every node. In some sense, it is the simplest (of course inefficient in practice) toy model for broadcast in a graph where a *unique messenger* must inform every user in person; the practical version —in which every informed person becomes a messenger— is considered below. For random geometric graphs, the cover time has been addressed by [17] in dimension three and up; the most important case of dimension two is still unknown.

In practice, one of the broadcast strategies that has received a lot of attention is based on *rumor spreading* and relates to the propagation of epidemics in a population [21]. There are three versions, *push* and *pull* and *push–pull*. Some piece of information is originally in the hands of some user of the network. The algorithm consists in spreading the information in the network by pushing (each user randomly chooses a neighbor to transmit information to) or pulling (each user asks information from a randomly selected neighbor) or both. The performance of the algorithm has been studied under various assumptions on the underlying graph. Only recently [15] have used a more general approach that characterizes the broadcast time in terms of the graph conductance. In some sense, it relates the cover time of the branching random walk to the spectral properties of a single random walk and opens a very promising route towards more general characterizations of the performance of broadcast algorithms.

Diffusions in random networks

To of the main questions about propagations in networks are about diffusion of new technology (bootstrap percolation) and diffusion of a rumor.

BOOTSTRAP PERCOLATION Some processes on random networks are of great interest. The first one, bootstrap percolation, has been used to model the adoption of new technology by a population. The process takes place on any (connected) graph. One is given a set of vertices that is initially infected (has the new technology). The infection then propagates deterministically: any node that with at least k infected neighbors becomes infected. These new infected vertices can, in turn, contributed to the propagation of the infection. One says that we have percolation if the entire graph gets infected. The most important question consists in determining the proportion p_c of nodes that have to be infected at random in the first place to ensure percolation. In order to understand the influence of locality on the process, random geometric graphs is a natural model to study bootstrap percolation. Recently, some first bounds on the critical threshold p_c have been obtained by Bradonjić and Saniee [14].

RUMOR SPREADING USING CONDUCTANCE. A first step towards more general results about broadcasting algorithms consists in characterizing the time for rumor spreading in terms of an important parameter of the connectivity of the graph that would be computed for specific examples. The first result in this direction is due to Chierichetti, Lattanzi, and Panconesi [15] who provide almost tight bounds for the broadcast time using a *push–pull* strategy in terms of the graph conductance. We will discuss ways to nail down the correct asymptotics and discuss what other important graph invariant might be more suitable to express the asymptotic bounds in a useful way.

Fundamental questions of universality

Aside from the parameters of practical interest, one would like to understand better the models themselves and the common behaviors that they exhibit.

UNIVERSALITY OF RESCALED COMPONENT SIZES. A first step towards the understanding of the metric structure consists in understanding the component sizes. Aldous [3] has shown that the component sizes of critical Erdős–Rényi random graphs were asymptotically following a very specific coagulation process, the multiplicative coalescent. It seems that this process also describes the evolution of the asymptotic sequence of component sizes in many other random graphs processes, like inhomogeneous random graphs [8], the Bohman–Frieze process [7] as well as other Achlioptas processes (where one is given rules to choose an edge from a list of random available ones). The question of the universality of the multiplicative coalescent for natural models of random graphs is one of the fundamental and difficult questions.

TOWARDS UNIVERSALITY FOR GRAPH PARAMETERS. One of the more challenging questions concerns the universality of the behavior of graph models. Rather than analyzing the models one by one, one would gain a lot of insight by adopting a more abstract point of view. The main obstacle here is to define a suitable measure of similarity between graphs. The question of metrics on the space of graphs and continuity of the parameters in the induced topology is crucial here. The question has been very successfully answered for dense graphs with the notion of graph limits by [30] (see also [20]). For sparse graphs, the question is more subtle and many natural metrics (like the Gromov–Hausdorff metric) do not yield good topologies on the spaces of sparse graph that are not “critical” (only supercritical (branching) graphs are crucial in practice for their expansion properties.) The recent survey by [12] provides possible approaches. The question of convergence of geometric networks is also the main theme of Aldous’ talk at the ICM [2].

Presentation Highlights

The talks covered the spectrum of topics we intended to discuss. In particular, there were talks about properties of random geometric graphs, the structure and distance in random sparse graphs, some applications of sparse graphs, as well as some presentations about related more fundamental questions.

Random geometric graphs

Nicolas Fraiman — *Connectivity of Bluetooth graphs*

We study the connectivity of random Bluetooth graphs, these are obtained as âĂIJirrigation subgraphsâĂİ of the well-known random geometric graph model. There are two parameters that control the model: the radius r that determines the visible neighbors of each node and the number of edges c that each node is allowed to have. The randomness comes from the distribution of nodes in space and the choices of each vertex. We characterize the connectivity threshold (in c) for values of r close the critical value for connectivity in the underlying random geometric graph. This is joint work with Nicolas Broutin, Luc Devroye and Gabor Lugosi [13].

Tobias Muller — *Colouring random geometric graphs*

If we pick points $X_1, \dots, X_n$ at random from d-dimensional space (i.i.d. according to some probability measure) and fix a $r > 0$, then we obtain a random geometric graph by joining points by an edge whenever their distance is $< r$. I will talk about some results on the chromatic number and the clique number of this model [31].

Matthew Penrose — *Connectivity of G(n,r,p)*

Consider a graph on n vertices placed uniformly independently at random in the unit square, in which any two vertices distant at most r apart are connected by an edge with probability p. This generalizes both the classical random graph and the random geometric graph. We discuss the chances of its being disconnected without having any isolated vertices, when n is large, for various choices of the other parameters.

Joseph Yukich — *Probabilistic Analysis of Some Geometric Networks*
We survey some techniques for establishing general limit theorems in stochastic geometry (laws of large numbers, variance asymptotics, and central limit theorems). We show how the general theorems may be applied to deduce the limit theory for various functionals of random geometric graphs, including, for example, network connectivity functionals, clique count, total edge length, and component count. The talk is based on joint work with M. Penrose, T. Schreiber, and Y. Baryshnikov.

Structure and distances in sparse graphs

Shankar Bhamidi — *Limited choice and randomness in evolution of networks*
The last few years have seen an explosion in network models describing the evolution of real world networks. In the context of math probability, one aspect which has seen an intense focus is the interplay between randomness and limited choice in the evolution of networks, ranging from the description of the emergence of the giant component, the new phenomenon of "explosive percolation" and power of two choices. I will describe on going work in understanding such dynamic network models, their connections to classical constructs such as the standard multiplicative coalescent and local weak convergence of random trees.

Justin Salez — *Joint distribution of distances in large random regular networks.*
We study the array of point-to-point distances in large random regular graphs equipped with exponential edge-weights. The asymptotic marginal distribution of a single entry is now well-understood, thanks to the work of Bhamidi, van der Hofstad and Hooghiemstra (2010). In this talk, we will show that the whole array, suitably re-centered, converges in the weak sense to a rather simple infinite random array. Our proof consists in analyzing the invasion of the network by several mutually exclusive flows emanating from different sources and propagating simultaneously at unit rate along the edges. The result applies to both the random regular multi-graph produced by the configuration model and the uniform regular simple graph.

Applications of sparse graphs

David Aldous — *Some thoughts on data compression and entropy for sparse graphs with vertex-names*
After an informal review of classic Shannon theory of entropy and data compression for random sequences, I will speculate on analogs for sparse graphs with vertex-names.

Marc Lelarge — *A new approach to the orientation of random hypergraphs*
A h-uniform hypergraph $H = (V, E)$ is called (ℓ, k)-orientable if there exists an assignment of each hyperedge $e \in E$ to exactly ℓ of its vertices $v \in e$ such that no vertex is assigned more than k hyperedges. Let $H_{n,m,h}$ be a hypergraph, drawn uniformly at random from the set of all h-uniform hypergraphs with n vertices and m edges. In this paper, we determine the threshold of the existence of a (ℓ, k)-orientation of $H_{n,m,h}$ for $k \geq 1$ and $h > \ell \geq 1$, extending recent results motivated by applications such as cuckoo hashing or load balancing with guaranteed maximum load. Our proof combines the local weak convergence of sparse graphs and a careful analysis of a Gibbs measure on spanning subgraphs with degree constraints. It allows us to deal with a much broader class than the uniform hypergraphs.

Pat Morin — *Maximum interference in the highway and related models.*
Given a set D of n disks, the interference of a point p is defined as the number of disks of D that contain p. The interference of D is the maximum interference over all centers of disks in D. In this talk, we discuss upper and lower bounds on maximum interference in 1 dimension, 2 dimensions, in the worst case, and in probabilistic settings.

Fundamental questions

Charles Bordenave — *How does a uniformly sampled Markov chain behave ?*

This is joint work with P. Caputo and D. Chafai. In this talk, we will consider various probability distributions on the set of stochastic matrices with n states and on the set of Laplacian/Kirchhoff matrices on n states. They will arise naturally from the conductance model on n states with i.i.d conductances. With the help of random matrix theory, we will study the spectrum of these processes. An emphasis will be put on the case of the simple random walk on a sparse directed Erdős-Rényi graph.

Csaba Toth — *Convex partitions.*

A convex partition is a planar straight-line graph where every bounded face is convex and the complement of the outer face is also convex. Two results are presented in this talk: (1) For every n points in the plane, there is a convex partition G such that the total edge length of G is at most $O(\log n/\log\log n)$ times that of a Euclidean minimum spanning tree (EMST) for the n points, and this bound is the best possible. (2) If G is a convex partition and the outer face has $O(1)$ edges, then G contains a monotone path of at least $\Omega(\log n/\log\log n)$ edges, and this bound is the best possible. (Joint work with Adrian Dumitrescu.)

Some open questions

One of the main objectives of the workshop was to foster new collaborations between the participants. To this aim, we organized an open problem session the day of arrival and scheduled informal working sessions every afternoon. Some of the proposed questions are famous and notoriously difficult, and were brought up to get the opinions of participants with a fresh eye and a different background, others have spawn from the participants current interests. In any case, the set of open questions which have been proposed has great scientific value, and we reproduce them here:

LENGTH OF THE GREEDY TOUR (Presented by C. Bordenave). Throw n uniform points on the unit square, and consider the total length L_n of the greedily constructed traveling salesman path. More precisely, start from a random point, then at any stage move to the closest yet non-visited point. Let L_n denote the length of this path. It is possible to show that L_n is of order $\sqrt{n}$ [6]. Prove that $L_n/\sqrt{n}$ converges to a constant. Many problems of this kind, when the path constructed is in some sense optimal, have been solved using sub-additive arguments (see, e.g., [35]). Here the main argument collapse since the path is constructed in a greedy way.

GREEDY TOUR IN THE PLANE (Presented by C. Bordenave). In an other version of the previous problem, one starts from a homogeneous Poisson point process $\mathcal{P}$ in the plane, conditioned to have a point at the origin. Starting from the origin, proceed to a walk which always visits the nearest point of $\mathcal{P}$ not yet visited. Does this walk eventually visit every point of $\mathcal{P}$?

GREEDY ALGORITHM UNDER POISSON RAIN. Consider a homogeneous Poisson point process on $\mathbb{R}^2 \times [0, \infty)$. The first two coordinates are interpreted as space, and the third one as the time at which the point defined by the first two coordinates arrives. Suppose that an agent always goes towards the closest non-visited point. In particular, if at some time t a point appears that is closer than all the others, the agent changes direction to aim at the newly arrived point. Let V_t be the number of points visited before time t. Is it true that $\liminf_{t\to\infty} V_t/t > 0$?

ASYMPTOTICS FOR GENERALIZED U-STATISTICS (presented by J. Yukich). U-statistics are a way to bootstrap an r-sample estimator into an n-sample estimator. Given a real-valued function f of r variables, the U-statistic $f_n : \mathbb{R}^n \to \mathbb{R}$ is the average over distinct ordered r-subsamples. Generalized U-statistics of the form $\sum_{i,j} f_n(X_i, X_j, D_n)$ where the X_i are independent and identically distributed in $\mathbb{R}^d$, $D_n = (X_1, ..., X_n)$, and f_n is some translation invariant function are of great interest. Can they be understood using a variation of the stabilization method?

ABOUT INTERFERENCE (presented by P. Morin) Given a set D of n disks, the interference of a point p is defined as the number of disks of D that contain p. The interference of D is the maximum interference over all centers of disks in D.

For a point set V, and a graph G, the set D consists of the disks centered at the points v of V with radius equal to the length of the longest edge adjacent to v. The interference $I(G)$ of the graph G is the interference of that set D.

- Given a point set $V \subset \mathbb{R}$, can we compute at graph G that approximately minimizes $I(G)$?
- Is the following statement true: "for any $V \subset \mathbb{R}^d$, there exists a graph $G = (V, E)$ with $I(G) = O(\sqrt{n})$"
- Is there an algorithm for finding a graph G that approximately minimizes $I(G)$? A $5/4$-approximation is best possible.
- If V consists of n i.i.d. points uniform on $[0, 1]$ the minimum spanning tree has interference $\Theta(\sqrt{n})$. Is there a better graph that gives $o(\sqrt{n})$? What is the interference of the minimum spanning tree of n i.i.d. uniform points in $[0, 1]^d$?
- Let $G^\star$ minimize the interference $I(G)$ for a n i.i.d. points uniform on $[0, 1]^d$. What is $\mathbb{E}[I(G^\star)]$? Previous construction show an upper bound of $O((\log n)^{1/2})$.

DIAMETER OF THE EUCLIDEAN MST (presented by L. Addario-Berry). The minimum spanning tree (MST) is one of the most important sparse graphs. A lot is known about the local properties of minimum spanning trees; much less is known about the typical distances, or about the diameter. In a geometric setting, take n i.i.d. uniform points in the square and use Euclidean distance to weight the edges of a complete graph on n vertices. What can we say about the expected diameter D_n of the corresponding random Euclidean minimum spanning tree? Only the trivial bounds $D_n = O(n)$ and $D_n = \Omega(\sqrt{n})$ are known.

THE EXTRA-COST FOR GUARDING SCULPTURES. (presented by L. Addario-Berry). An art gallery is a simple polygon. One is asked to place a minimal set of guards to that every point of the interior perimeter of the gallery (where the paintings lie) is seen by at least one guard. A related question concern the extra cost needed to also ensure that the interior of the polygon (where the sculptures lie) is also guarded. Addario, Amini, Séréni and Thomassé proved that if n guards are needed for the walls, then the extra cost for sculptures is at most $4n - 6$ [1]. It seems that $n - 2$ extra guards should suffice.

RESILIENT SPANNERS (presented by V. Dujmovic). Highly connected and yet sparse graphs (such as expanders or graphs of high tree-width) are fundamental, widely applicable and extensively studied combinatorial objects. Can we find such graphs that are robust to failures of some of the nodes in the following sense: Given a point set V, $|V| = n$, is it possible to construct a graph $G = (V, E)$ such that for any $V' \subseteq V$, with $|V'| \leq f(n)$ the subgraph G induced on $V \setminus V'$ has a connected component of size $n - o(n)$ that is a sparse (at most $g(n)$ edges) t-spanner? For instance, it is possible with $g(n) = O(n)$ and $f(n) = \sqrt{n}$?

Scientific Progress Made

The schedule was made with only a limited number of presentations in order to leave most of the time for discussions between the participants. In particular, we organized an open problem session on the first day, and working sessions every afternoon. We believe that the workshop was very successful with respect the exchanges it fostered. On top of the informal discussions whose long term impact is difficult to estimate, a number of open problems have been solved, some of which have also already been submitted to a conference or a journal [9, 19]. Most concrete outcomes that have been produced or will be published in the very near future concern groups of people with similar backgrounds, and this is inevitable. However some of the most promising work involve participants with different backgrounds; such collaborations take of course more time to reach a ripe state.

INTERFERENCE GRAPHS. Pat Morin and Luc Devroye have worked on interference in random geometric graphs. They improved the bounds on the interference of graphs on a set V of n uniformly random points in $[0, 1]^d$. In particular, they showed that there exists a connected graph on V that has interference of order $(\log n)^{1/3}$ and that no connected graph on V has interference $o((\log n)^{1/4})$. They also showed that the minimum spanning tree on V has interference $\Theta((\log n)^{1/2})$. Their results have been collected in a manuscript that has been submitted [19].

ROBUST GEOMETRIC SPANNERS. Jit Bose, Pat Morin and Vida Dujmovic have worked on the robustness of highly connected yet sparse graphs. In particular, they initiated the study of such graphs that are in addition geometric spanners. Following the suggestions the open problem proposed by V. Dujmovic, they defined a robustness property and proved that robust spanners must have a superlinear number of edges, even in one dimension. On the positive side, they also give constructions, for any dimension, of robust spanners with a near-linear number edges. These results have already been submitted [9].

IRRIGATION GRAPHS WITH CONSTANT OUT-DEGREE. Full connectivity of a network is of little importance in practice, there will always be some remote and isolated points, and this does not affect the quality of the network from the provider's point of view. What matters is the size of the largest connected component. S. Boucheron, N. Broutin, L. Devroye and G. Lugosi have worked on this question on the model that was the subject of N. Fraiman's lecture, the connectivity of bluetooth graphs. They have proved that even when the range of reach r of a point is only slightly above the connectivity threshold of the random geometric graph, keeping only two random neighbors suffices to ensure that the largest connected component has size $n-o(n)$.

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Bibliography

[1] L. Addario-Berry, O. Amini, J.S. Sereni, and S. Thomassé, , Guarding art galleries: The extra cost for sculptures is linear. In *Proceedings of the 11th Scandinavian workshop on Algorithm Theory*, Lecture Notes in Computer Science vol. 5124, pp. 41–52, Springer, 2008.

[2] D.J. Aldous. Exchangeability and continuum limits of discrete random structures. In *Proceedings of the International Congress of Mathematicians*, 2010.

[3] D. Aldous. Brownian excursions, critical random graphs and the multiplicative coalescent. *The Annals of Probability*, vol. 25, pp. 812–854, 1997.

[4] D. Aldous and V. Limic. The entrance boundary of the multiplicative coalescent. *Electronic Journal of Probability*, vol. 3, pp. 1–59, 1998.

[5] D.J. Aldous and J. Shun. Connected spatial networks over random points and a route-length statistic. arXiv:1003.3700 [math.PR], 2010.

[6] J.L. Bentley and J. Saxe. An analysis of two heuristics for the Euclidean traveling salesman problem. In *Proceedings of the 18th Annual Allerton Conference on Communication, Control, and Computing*, pp. 41–49, 1980.

[7] S. Bhamidi, A. Budhiraja, and X. Wang. Bohman-Frieze processes at criticality and emergence of the giant component. Submitted, 2011. arXiv:1106.1022 [math.PR]

[8] S. Bhamidi, R. van der Hofstad, and J.S.H. van Leeuwaarden. Novel scaling limits for critical inhomogeneous random graphs. *The Annals of Probability*, to appear, 2012. arXiv:0909.1472 [math.PR].

[9] P. Bose, V. Dujmovic, P. Morin and M. Smid Robust geometric spanners. Submitted, arXiv:1204.4679 [CS.CG], 2012.

[10] B. Bollobás and O. Riordan. The diameter of a scale-free random graph. *Combinatorica*, 24:5–34, 2004.

[11] B. Bollobás and O. Riordan. *Percolation*. Cambridge University Press, 2006.

[12] B. Bollobás and O. Riordan. Metrics for sparse graphs. In *Surveys in Combinatorics*, volume 365 of *LMS Lecture Notes Series*, pages 211–287, Cambridge, UK, 2009. Cambridge University Press.

[13] N. Broutin, L. Devroye, N. Fraiman, and G. Lugosi. Connectivity threshold for Bluetooth graphs. Submitted, 2011. arXiv:1103.0351 [math.PR]

[14] M. Bradonjić and I. Saniee. Bootstrap percolation on random geometric graphs. arXiv:1201.2953 [cs.DM], 2012.

[15] F. Chierichetti, S. Lattanzi, and A. Panconesi. Almost tight bounds for rumour spreading with conductance. In *Proceedings of STOC*, 2010.

[16] F. Chung and L. Lu. The diameter of random sparse graphs. *Advances in Applied Mathematics*, 26: 257–279, 2001.

[17] C. Cooper and A. Frieze. The cover time of random geometric graphs. *Random Structures and Algorithms*, 2010.

[18] P. Crescenzi, C. Nocentini, A. Pietracaprina, and G. Pucci. On the Connectivity of Bluetooth-Based Ad Hoc Networks. *Concurrenty and Computation: Practice and Experience*, 21:875–887, 2009.

[19] L. Devroye and P. Morin. A note on interference in random networks. ArXiv:1202.5945 [CS.DS]. Submitted, 2012.

[20] P. Diaconis and S. Janson. Graph limits and exchangeable random graphs. *Rendiconti di Matematica e delle sue Applicazioni*, 28:33–61, 2008.

[21] M. Draief and L. Massoulié. *Epidemics and Rumours in Complex Networks*, volume 369 of *London Mathematical Society Lecture Notes*. Cambridge University Press, Cambridge, 2010.

[22] D. Dubhashi and A. Panconesi. *Concentration of Measure for the Analysis of Randomized Algorithms*. Cambridge University Press, Cambridge, UK, 2009.

[23] D. Dubhashi, O. Häggström, C. Johansson, A. Panconesi, and M. Sozio. Irrigating ad hoc networks in constant time. In *Proceedings of SPAA'05*, pages 106–115, 2005.

[24] R. Durrett. *Random Graph Dynamics*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2006.

[25] P. Erdős aand A. Rényi. On random graphs I *Publ. Math. Debrecen*, vol. 6, pp. 290–297, 1959.

[26] P. Erdős and A. Rényi. On the evolution of random graphs. *Publ. Math. Inst. Hungar. Acad. Sci.*, vol. 5, pp. 17–61, 1960.

[27] M. Franceschetti and R. Meester. *Random Networks for Communication: From Statistical Physics to Information Systems*. Cambridge University Press, 2008.

[28] E.N. Gilbert. Random plane networks. *Journal of the Society for Industrial and Applied Mathematics*, vol 9, pp. 533–543, 1961.

[29] H. Kesten. The critical probability for bond percolation on the square lattice equals $\frac{1}{2}$. *Communications in Mathematical Physics*, 74:41–59, 1980.

[30] L. Lovász and B. Szegedy. Limits of dense graph sequences. *Journal of Combinatorial Theory, Series B*, 96:933–957, 2006.

[31] C. McDiarmid and T. Müller. On the chromatic number of random geometric graphs. *Combinatorica*, vol. 31, pp. 1–66, 2011.

[32] R. Meester and R. Roy. *Continuum Percolation*. Cambridge Tracts in Mathematics. Cambridge University Press, 1996.

[33] M. Penrose. *Random Geometric Graphs*. Oxford Studies in Probability. Oxford University Press, 2003.

[34] O. Riordan and N.C. Wormald. The diameter of sparse random graphs. arXiv:0808.4067 [math.PR], 2008.

[35] J.M. Steele. *Probability Theory and Combinatorial Optimization*. CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1997.

[36] D.J. Watts and S.H. Strogatz. Collective dynamics of 'small-world'networks. *Nature*, 393:440–442, 1998.

Chapter 6

Probabilistic versus Deterministic Techniques for Shared Memory Computation (12w5122)

February 5 - 10, 2012

Organizer(s): Hagit Attiya (Technion), Maurice Herlihy (Brown University), Philipp Woelfel (University of Calgary)

Overview of the Field

Shared memory algorithms exploit the parallelism of multiple computing units. As the number of components and the complexity of their interconnection increases, the difficulty of dealing with the resulting asynchrony and faulty behaviour of this elaborate hardware also increases. In these distributed settings proofs of correctness of proposed solutions and analysis of the complexity of the solutions are challenging. Furthermore, some fundamental algorithmic problems cannot be solved efficiently or at all. Determinism is a significant barrier that limits the power of shared memory computing. For example, a symmetric configuration of several processes may lead to deadlock. Progress will be stalled unless the symmetry can be broken, which is impossible in some deterministic settings. Sometimes, a shared memory computation can be very expensive for a small set of specific initial configurations or under an unfortunate scheduling of the various processes. In deterministic settings, there is no way to insure that these difficult cases do not arise or at least are rare. Probabilistic methods, however, have proved useful to overcome these impasses. The likelihood of problematic scenarios can be made negligible with applications of randomization. Allowing processes to use private random bits helps to break symmetry, to diminish the impact of a malicious schedule or a difficult initial configuration. There is a significant and growing collection of applications of randomization in various multi-process settings. The Ethernet exponential back-off algorithm was an early example that continues to have widespread application. Valiant's randomized routing algorithm [45] is an elegant example of a simple and effective use of randomization. The famous impossibility result [27], stating that it is not always possible to achieve consensus in a system with only one faulty process, has been overcome with increasingly more efficient randomized algorithms. As a consequence of randomized universal constructions, any shared object has a randomized implementation in an asynchronous system with faulty processes (wait free), while (provably) no such deterministic implementations exist.

Many more examples of randomized shared memory algorithms, which help overcome the difficulties associated with asynchrony and faultiness, are known. The results, however, are mostly ad hoc. A comprehensive theory of probabilistic methods for shared memory computation is missing. In general, the power of probabilistic methods and the randomized complexity of many fundamental problems is not well understood. In contrast, over the past 20 years, rich and comprehensive complexity-theoretic results for deterministic shared memory systems have emerged. One example is the infinite wait-free hierarchy that classifies the power of hardware primitives when used by deterministic algorithms. Under randomization, however, the wait-free hierarchy collapses. Similar hierarchical results are not known for systems that can make use of private random bits. Part of the reason is a lack of general techniques to prove lower bounds for randomized algorithms. In the deterministic setting, a wide arsenal of such techniques is known. Examples include algorithmic tools such as covering, valency, scenario, or chain arguments, and mathematical tools such as information theory, Ramsey theory, and algebraic topology. In sequential computing, methods such as Yao's Min-Max principle [46] can be used to derive lower bounds for randomized algorithms from lower bounds for deterministic algorithms. In distributed computing, the situation is more complicated, and generally the usefulness of Yao's principle seems to be limited or at least is not well understood. Another example is a lack of understanding of correctness conditions for randomized algorithm design: The standard correctness condition used in deterministic settings, *linearizability* [36], is not suitable for randomized algorithms.

Recent Developments and Open Problems

In recent years, several new randomized algorithms and lower bounds for fundamental shared memory problems have emerged, and attempts were made to develop a theory for probabilistic methods.

Theory of Probabilistic Shared Memory Algorithms

In the standard shared memory model, each process takes steps (i.e., executes shared memory operations) as determined by its program. The order of steps by concurrent processes can be interleaved arbitrarily. The standard worst-case analysis of deterministic shared memory algorithms assumes that this interleaving occurs in the "worst" possible way. Under such assumptions, many problems don't have any or only very inefficient solutions. Probabilistic algorithms allow processes to make random decisions, and prevent (with high probability) worst-case schedules from occurring. In the analysis of randomized algorithms, adversary models describe how the interleaving of process steps can be influenced by random choices made by processes. The most optimistic assumption, where there is no such influence, is modeled by the *oblivious adversary*, and the most pessimistic reasonable assumption is modeled by the *strong adaptive adversary*. Several adversary models with intermediate strengths were defined (see [9] for an overview of some of them).

A comprehensive study of the relative strength of adversary models is still missing. Most results are ad-hoc, but some problems (most prominently the consensus problem, see Section 6) have been studied extensively in a wide variety of adversary models. For many problems, e.g., consensus, leader election, or mutual exclusion, no very efficient randomized algorithms are known (or are known not to exist) under the most pessimistic system assumptions (strong adaptive adversary). Therefore, more recently the oblivious adversary has become the focus of research interest. For many problems it has been shown that the oblivious (or slightly stronger) adversary model allows significantly more efficient randomized algorithms than the strong adaptive adversary. Examples are consensus [11], leader election [29], or mutual exclusion [23] (see also Section 6).

The standard correctness condition for deterministic shared memory algorithms is *linearizability* [36]. A linearizable implementation can replace an atomic object (which allows every operation to be executed instantaneously) in any deterministic algorithm without affecting the algorithm's worst-case behaviour. A significant amount of research effort has been spent on finding linearizable implementations of fundamental primitives, or to prove the non-existence of such implementations. It was recently observed [32] that replacing atomic objects with linearizable ones in randomized algorithms may not preserve probability distributions of executions; in other words, such a replacement may increase the strength of the underlying adversary model. While a suitable correctness condition, *strong linearizability*, was proposed and investigated for the strong adaptive adversary model [32, 33], not much is known for weaker adversary models.

Yao's Theorem [46] provides a general technique to prove lower bounds for randomized *sequential* algorithms. For example, it follows from that theorem that an expected running time lower bound for a random input (for any given probability distribution) and the fastest possible deterministic algorithm yields the same expected running time lower bound for the fastest possible randomized algorithm and the worst input. In concurrent algorithms, adversarially generated schedules play the same role as inputs in sequential algorithms.

Since in non-oblivious adversary models those schedules are determined dynamically, based on previous random decisions made by processes, it is unclear how Yao's' Theorem can be applied. Recently [30], it was shown how one can apply Yao's principle in the strong adaptive adversary model, but this result is ad-hoc for one particular problem (mutual exclusion). A more comprehensive understanding of the applicability of Yao's principle for randomized concurrent algorithms is still missing.

Randomized Algorithms for Fundamental Problems

The *consensus* problem is among the best studied problems in distributed computing. Processes *propose* different values, and a consensus protocol lets all of them agree on one of the proposed values. The consensus problem is fundamental for distributed computing—consensus objects are very powerful synchronization primitives, and the difficulty of solving consensus helps evaluating the strength of a computational model. For example, the famous Fischer-Lynch-Paterson (FLP) proof [27] applied to the shared memory model [41] states that no deterministic wait-free algorithm can solve consensus using only atomic read/write registers. It has also been known for a long time, that one can solve consensus with randomized wait-free protocols (e.g., [22]). But only in recent years, the complexity of randomized wait-free protocols for different system assumptions (e.g., various adversary models) has been determined more or less accurately [19, 13, 20, 10, 11, 12]. One of the intriguing open question remains how fast one can solve consensus in the oblivious adversary model, i.e., under very optimistic system assumptions.

Consensus can be viewed as a form of *strong leader election*, in which all processes agree on the ID of one of the participating processes, who becomes the leader. *Weak leader election* is a simpler problem, where processes also elect one unique leader, but non-leaders don't need to learn the ID of the leader. This problem can also not be solved deterministically in systems that provide only atomic registers. An efficient randomized solution with logarithmic expected time complexity in the strong adaptive adversary model for this problem has been known since 1992 [1], but recent research efforts have several yielded refined solutions. Examples comprise an efficient *adaptive* algorithm [7], where the time complexity depends only on the number of participating processes as opposed to all processes in the system, and algorithms with almost constant time respectively small space complexity [2, 29, 28] in oblivious or weak adaptive adversary models.

Another classical problem is that of *renaming* [18], where each participating process tries to acquire a distinct name from some given small name space. This problem has deterministic solutions for a large enough name space even if the system provides only atomic registers. However, in such a system, renaming is not possible if the name space is only slightly larger than the number of participating processes (see for example [37, 24]). Moreover, most deterministic renaming algorithms are slow, even if strong synchronization primitives are available and the name space is not tight. (Only very recently a deterministic algorithm with step complexity logarithmic in the number of participating processes was obtained [26]; that algorithm uses registers and compare-and-swap objects.) Recently, several new very efficient randomized algorithms renaming algorithms were devised for different system assumptions [5, 7, 4], the fastest having only double logarithmic expected step complexity even if only atomic registers are available. A recent tight lower bound for randomized renaming algorithms [6] shows that for a polynomial name space only randomized Monte Carlo algorithms can have sub-logarithmic expected step complexity in the strong adaptive adversary model.

The most powerful synchronization primitive is a *mutual exclusion* object [25], which allows processes to serialize accesses to protected resources. Mutual exclusion objects are omnipresent in modern operating systems and concurrent algorithms, and, not surprisingly, finding efficient mutual exclusion algorithms has been of ongoing interest. A recent lower bound [21] showed that using a standard set of synchronization primitives (e.g., registers and compare-and-swap objects), deterministic algorithms cannot achieve a sub-logarithmic complexity with respect to the standard complexity remote-memory-references (RMR) measure. More precisely, the lower bound states that any deterministic algorithm incurs at least $\Omega(\log n)$ RMRs per passage through the critical section, in the worst-case. Since then researchers have tried to answer the question whether randomization can help to circumvent this barrier. While a small improvement is possible under very pessimistic system assumptions (the strong adaptive adversary model) [34, 35, 43], a recent lower bound [30] shows that more than a $\log \log n$ factor cannot be gained. However, it turned out that under optimistic system assumptions (oblivious adversary model), significantly more efficient algorithms are possible [23, 31].

In recent years, research on probabilistic methods for shared memory computing has gained significant momentum and led to several novel solutions and lower bounds for other fundamental problems. Examples are snapshot and max-registers implementations [14, 17] or task allocation protocols [8, 3].

Presentation Highlights and Open Questions

(a) Michael Bender and Seth Gilbert, *Mutual Exclusion in* $O(\log^2 \log n)$ *Time*

Michael and Seth gave a joint talk on the first randomized mutual exclusion algorithm which is exponentially faster than deterministic algorithms [23] in the oblivious adversary model. In particular, their algorithm guarantees that each passage through the critical section costs amortized $O(\log^2 \log n)$ remote memory references (RMRs) with high probability, where n is the number of processes in the system. It guarantees that every process enters the critical section (i.e., deadlock-freedom) with high probability. The algorithm achieves its efficient performance by exploiting a connection between mutual exclusion and approximate counting.

Several open problems were identified, e.g., whether similarly efficient but deterministically deadlock-free algorithms may exist, and whether the true randomized RMR complexity of mutual exclusion is constant in the oblivious adversary model.

(b) Rotem Oshman, *Mutual Exclusion with Faulty Memory*

Modern implementations of computer memory are extremely compact and power-efficient, but this comes at the cost of increased susceptibility to "soft errors", where a bit changes its value spuriously. Rotem presented her work with co-author Moscibroda [42], in which they study the effects of soft memory errors on 2-process mutual exclusion algorithms. Specifically, Rotem asks what degree of fault-tolerance can be achieved using no extra memory compared to classical algorithms such as Peterson and Dekker's algorithms; i.e., three binary registers. Rotem and her co-author suggest a probabilistic model to measure the susceptibility of a (deterministic) mutual exclusion algorithm to memory errors, and show that under this model it is possible to guarantee mutual exclusion with probability 1 when a single register is faulty and may exhibit unboundedly many faults. On the other hand, they proved that *worst-case* (i.e., deterministic) fault tolerance cannot be achieved in this setting. Rotem gave several other related results characterizing the degree of fault tolerance that can be achieved using binary registers.

An open question is what happens if registers are not binary.

(c) Dan Alistarh, *The Complexity of Renaming*

Dan presented results from his recent FOCS paper with Aspnes, Gilbert, and Guerraoui [6], studying the complexity of renaming. The paper provides an individual lower bound of $\Omega(k)$ process steps for deterministic renaming into any namespace of size sub-exponential in k, where k is the number of participants. This bound is tight: it draws an exponential separation between deterministic and randomized solutions, and implies new tight bounds for deterministic fetch-and-increment registers, queues and stacks. The proof of the bound relies on the a reduction from renaming to mutual exclusion. Dan also discussed a global lower bound of $\Omega(k \log(k/c))$ on the total step complexity of renaming into a namespace of size ck, for any $c \geq 1$. This applies to randomized algorithms against a strong adversary, and helps derive new global lower bounds for randomized approximate counter and fetch-and-increment implementations, all tight within logarithmic factors.

The talk and the following discussion raised several open questions: For example, what is the (randomized) complexity of long-lived renaming, where processes can release names that they obtained earlier? Is it possible to exploit initial names assigned to processes? Finally, are there algorithms that circumvent the lower bound by allowing errors or failures with small probability?

(d) George Giakkoupis, *On the Time and Space Complexity of Randomized Test-And-Set*

George described new results on the time and space complexity of randomized Test-And-Set (TAS) implementations from atomic read/write registers in asynchronous shared memory models with n processes [29]. He presented an adaptive TAS algorithm with an expected (individual) step complexity of $O(\log^* k)$, for contention k, against the oblivious adversary, improving a previous (non-adaptive) upper bound of $O(\log \log n)$ [2]. He also presented a modified version of the adaptive RatRace TAS algorithm [7], which improves the space complexity from $O(n^3)$ to $O(n)$, while maintaining logarithmic expected step complexity against the adaptive adversary.

An open problem is whether whether the algorithmic technique that yields the $O(\log^* k)$ upper bound can be applied to other problems, for example to consensus. Another open problem is whether the linear space upper bound is optimal.

(e) Keren Censor-Hillel, *Polylogarithmic Implementations of Restricted-Use Objects*

Implementing objects that only need to support restricted use, such as a bounded number of operations or a bounded number of values, can be strictly faster than general constructions. Keren described polylogarithmic constructions of restricted-use concurrent data structures. This is in contrast to existing linear lower bounds for the unrestricted versions of such objects by Jayanti, Tan, and Toueg [38]. Keren first described her logarithmic max register implementation obtained in collaboration with James Aspnes and Hagit Attiya [14]. Then she gave a restricted-use snapshot object that requires only $O(\log^3 n)$ steps for updates and $O(\log n)$ steps for scans, for systems with n processes. This result is based on work by the same authors together with Faith Ellen [15]. The upper bound seems counterintuitive, since it does not depend on the number of components in the snapshot. The key to this implementation is the construction of a new object consisting of a pair of max registers that supports a scan operation.

An open problem is to close the polylogarithmic gap that usually exists between upper and lower bounds for most objects. Moreover, only very few randomized lower and upper bounds are known (and they are very particular and not for general objects).

(f) Danny Hendler, *Lower Bounds on Restricted-Use Objects*

Danny's joint work with James Aspnes, Hagit Attiya and Keren Censor-Hillel [16] draws a more complete picture by defining a large class of objects for which an operation applied to the object can be "perturbed" L consecutive times, and proving lower bounds on the time and space complexity of deterministic implementations of such objects. This class includes bounded-value max registers, limited-use approximate and exact counters, and limited-use collect and compare-and-swap objects; L depends on the number of times the object can be accessed or the maximum value it can support.

For implementations that use only historyless primitives, Danny and his co-authors proved lower bounds of $\Omega(\min(\log L, n))$ on the worst-case step complexity of an operation, where n is the number of processes; they also prove lower bounds of $\Omega(\min(L, n))$ on the space complexity of these objects. When arbitrary primitives can be used, they proved that either some operation incurs $\Omega(\min(\log L, n))$ memory stalls or some operation performs $\Omega(\min(\log L, n))$ steps.

In addition to these deterministic lower bounds, the work establishes a lower bound on the expected step complexity of restricted-use randomized approximate counting in a weak oblivious adversary model.

An open question remains whether there are expected time lower bounds for randomized implementations of general L-perturpable objects.

(g) Valerie King.

Valerie presented two works in progress. The first was an examination of an often-cited but never published lower bound for resilience for a randomized Byzantine agreement problem. Valerie showed what she thought were the short-comings of method sketched in an unpublished manuscript from the 1980's by Karlin and Yao. Then she proposed a possible fix, in the case of private channels. She also asked the question of whether limiting messages passed by each processor including faulty processors would allow for an algorithm with higher resilience.

The second was a randomized message passing algorithm for asynchronous consensus which uses fewer messages than that given by emulating the shared memory solution.

(h) Eric Ruppert, *The Complexity of Non-Blocking Dictionaries*

The dictionary data structure is fundamental to computer science. It stores a set of elements and provides update operations that add and remove elements, and query operations that can find specific elements, find successors or predecessors, and find minimum and maximum elements. A dictionary implementation that can be accessed concurrently by many processes is called non-blocking if some process must eventually complete its operation. It is known that there is a non-blocking dictionary implementation that uses $O(n + c)$ amortized time per operation, where n is the number of elements in the dictionary and c is the maximum number of processes that ever run concurrently. This is a fairly poor upper bound on the complexity of the problem. Eric discussed possible approaches for improving it.

Several open problems were identified:

- Prove a good upper bound on the amortized step complexity of operations on a non-blocking search tree.
- Prove a good upper bound on the expected depth of a search tree built by concurrent insertions.
- Prove a good upper bound on the expected amortized step complexity of a skip list.
- Extend existing non-blocking data structures to handle predecessor queries, etc.
- Use other data structures as the basis of non-blocking dictionary implementation (e.g. treaps or van Emde Boas trees).
- Create an efficient non-blocking implementation of dynamic perfect hashing.

(i) Jennifer Welch, *Shared Memory Computation Under Probabilistic Churn Models*

Distributed systems with churn, or dynamic distributed systems, allow the processes to join and leave the system at will. Jennifer described a modification of an existing algorithm to implement a shared read-write register in such a system, in which replicas of the state of the implemented object are distributed among the processes [40]. When a process joins the system, it attempts to obtain an up-to-date copy of the data from other processes. Copies of the register are updated by broadcasting information. The consistency condition provided by this algorithm for read operations that are able to return a value is shown to be in the intersection of sequential consistency and a multi-writer variant of regularity.

To model the dynamicity of the system with churn, she and her collaborators used a continuous-time birth-death process which is a special case of continuous-time Markov processes. Given the joining and leaving rates of the processes, she analyzed the likelihood there are no active processes, the mean duration of a time interval containing active processes, and the probability that an era ends while a process is attempting to join.

It was mentioned that some of the results might get improved using Wald's Identity. An open problem is whether the results extend to more realistic assumptions.

Scientific Progress Made and Outcome of the Meeting

We first describe several selected research results that were seeded at the workshop. Many of those resulted in new publications at the top international conferences on theoretical computer science (STOC and FOCS) respectively parallel and distributed computing (PODC, SPAA, and DISC); since we cannot list all of them, we will only discuss some selected ones. After that, we present a number of comments that were made by workshop participants, and which draw a picture of the impact the workshop had on their work and collaborations resulting from the workshop. It should be noted that workshop sparticipant Jared Saia also wrote a blog post [44], summarizing his impressions on the workshop.

Selected Research Results Seeded at the Workshop

- Michael Bender's and Seth Gilbert's talk on mutual exclusion triggered a discussion about whether randomized deadlock-freedom with a small error probability may be desirable. This led to the question whether a similarly efficient algorithm exists that achieves deterministically deadlock-free. Similarly, the algorithm demonstrated that algorithms can be significantly faster in the oblivious adversary model than in the strong adaptive adversary model, and raised the question whether mutual exclusion with constant RMR complexity is possible. The resulting discussions sparked interests of BIRS participants Giakkoupis and Woelfel and ultimately resulted in a new mutual exclusion algorithm that achieves deterministic deadlock-freedom and constant RMR complexity [31], thus answering the open questions from Bender's and Gilbert's talk.

- Discussions on the renaming problem (see also Dan Alistarh's talk) have led to new collaboration among workshop participants Dan Alistarh, James Aspnes, George Giakkoupis, and Philipp Woelfel. Research initiated at the workshop and published at PODC 2013 [5] gives renaming algorithms that circumvent the lower bounds presented by Dan Alistarh in his talk, by allowing a small error probability.
- There are close relations between the asynchronous shared memory model and message passing models. Algorithmic techniques in one model can be applicable in the other one. Ideas from work on randomized shared memory algorithms discussed at the workshop laid the foundations for a randomized, polynomial expected time algorithm to solve asynchronous Byzantine Agreement in a message passing model with a strong adversary that can control up to a constant fraction of the processors. Workshop participants Valerie King and Jared Saia published those results at STOC 2013 [39]. (See also Valerie's comment in Section 6).
- James Aspnes and Keren Censor-Hillel built on the work presented in Keren's talk, and used a randomized techniques to obtain *long-lived* snapshot objects and max registers with poly-logarithmic expected step complexity, whereas previous work allowed such efficiency only by limiting the number of operations on the objects [17].
- Following initial ideas discussed during the workshop, participants George Giakkoupis, Lisa Higham, and Philipp Woelfel, as well as Lisa's and Philipp's PhD student Maryam Helmi worked on understanding the relation between deterministic and randomized progress conditions. They proved that in general any deterministic obstruction-free algorithm, where each process finishes if it executes b uninterrupted steps, can be turned into a randomized wait-free one, with a step complexity that is polynomial in b and the number of processes in the system [28].

Statements from Workshop Participants

Valerie King, Feb 5, 2013: *Dear Nassif and the organizers of BIRS workshop 12w5122,*

The attached paper [39] was just accepted to STOC. It solves a longstanding problem in theoretical computer science. I just wanted to thank you and the organizers of the workshop. It was there, listening to the talks of the workshop on shared memory, in the beautiful surroundings of Banff, that we first had the idea of applying ideas from shared memory to this problem which is not about shared memory. So we have BIRS to thank. [...]

Lisa Higham, Feb 21, 2012 (In an e-mail to Nassif Ghoussoub and the workshop organizers):

[...] *In short, best workshop I have ever attended.*

First off there were talks on the latest by leaders in the research area. Without exception, they were excellent. I learned many new techniques and some surprising results. Also, a few of those beautiful insights that seem so simple and natural —- once someone shows you!!

In my own talk, I presented work from our dept (postdoc Golab, Prof Woelfel and myself). We have shown that the gold standard for correctness of implemented objects (Linearizability), which the community has relied on for 20 years, does not work when the algorithms using these implementations are randomized. Instead we need a stronger property. I followed with our recent work on possibilities and impossibilities for this stronger property. The work caused a stir. I received extremely helpful and encouraging feedback from many participants. The questions and comments have given us new research ideas and directions to last several years, as well as some immediate improvements. For concreteness, here are some directions I intend to pursue:

Rotem Oshram and Maurice Herlihy suggested that our work could be closely connected to (one of the many) notions of refinement mappings used by verification researchers. We need to nail down this connection.

I learned that the Principles of Programming Languages (POPL) community has developed some automated techniques to verify some instances of Linearizability, but have not been able to figure out when these techniques work and when they are inadequate. One of the BIRS participants, Hagit Attiya, pointed this out *and suggested that their impasse may be clarified by our work. Hagit's suggestion is to team up with someone in the POPL community to both communicate our current work and extend it. Furthermore, Hagit is helping to implement this suggested collaboration.*

One conjecture that has eluded us was picked up by at least two other researchers at BIRS, making collaboration on this question a good option.

I would not have learned these connections or made these contacts without the workshop. It was exhilarating (and exhausting) in the best of ways. I am sure I am not alone in feeling an extra charge to my research as a consequence of the meeting. Because of Calgary's strong involvement, I believe the workshop also helped make the UofCalgary more prominent in this community.

Eric Ruppert, Feb 13, 2012: *Here are a few thoughts on the impact the workshop had on me.*

- *I found out a lot about what people in the area are working on currently*
- *After I gave the talk on open problems, a number of people approached me with ideas about how some of the problems could be tackled, and this could lead to future collaborations. Also, I found that one attendee was already working on related problems. Another attendee expressed interest in working on the problems too, and I hope that will lead to a collaboration.*
- *Others shared their open problems, giving me some ideas about what to work on in the future.*

Hagit Attiya, Feb 11, 2012: *Being at BIRS was a wonderful experience: The wonderful surroundings, the convenient facilities, and the warm hospitality of the staff has made the stay very comfortable. (Similar thanks go to the Banff centre staff, in the recreation facilities and the dining room.) I particularly, enjoyed the suite that I received, which was splendid!*

For me it was first and foremost an opportunity to catch up on interesting research in my area. But it was also an opportunity to create new research: With 11 of my collaborators attending the workshop, I could make progress on several papers that I am writing (in one case, a presentation of the work by one of my coauthors has helped to find new applications). I also had a chance to describe some fresh results to relevant researcher and get their feedback. Finally, I got started on one or two new research directions.

Jennifer Welch, Feb 11, 2012: *Thanks again for all your hard work for a great workshop. It was very informative and enjoyable.*

Workshop impact:

- *Opportunity to learn about the latest results in the area of shared memory distributed computing*
- *Sparked ideas for new directions for collaborative research*
- *Provided ideas for new material to cover in my distributed computing course, as well as ideas for challenging new homework and exam problems*

Jared Saia, Feb 10, 2012: *This was one of the strongest workshops that I have been to in the past 10 years. Although I initially had little background in shared memory, I had many excellent outcomes from the workshop, three of which I describe below.*

First, I had very useful feedback on my talk. My talk was on emulating a shared object in a wireless network where an adversary can jam communication between processors. I had several suggestions about ways to extend my result including determining: 1) what happens when there are multiple communication channels available; 2) what happens if one can use signal processing in the event of a jam to determine what the underlying message was in the case where many processors broadcast the sam underlying message; 3) if one can reduce the power consumption necessary to in order to maintain the state of the shared object, perhaps expending more energy only when there is a need to change state; and 4) how to design algorithms that reduce the energy consumption per processor needed to emulate the shared object, as the number of processors grows.

The second major outcome of the workshop was the beginning of collaborations on two separate papers. The first paper is related to extending the research described in my talk along direction 4) described *above. This collaboration began through discussions with Seth Gilbert, a colleague with whom I have not collaborated previously. The second paper is related to reducing communication costs for consensus in the asynchronous communication model. The main idea of this result, borrows a trick frequently used for achieving consensus with shared registers, and adapts it to the message passing model. This type of cross-pollination of ideas between shared memory and message passing models of communication is important but rare. Certainly, it would not be occurring without the impetus of the BIRS workshop. This collaboration is with workshop participants Seth Gilbert, Jim Aspnes and Valerie King.*

The third major outcome was a quick but thorough education about the frontiers of shared memory research. To prepare for the workshop, I read through papers from several of the participants. However, when I arrived, I realized that I would learn about the frontier of research in shared memory, and that this was not something that one could gain from simply reading published papers. Through the workshop, I learned about major open problems, mathematical and algorithmic techniques used to achieve results, and general vision for the field. No collection of papers or even informal group meetings with colleagues at conferences could have achieved anywhere close to the same outcome.

The coordinators for the workshop were excellent, particular Phillip Woefel who went out of his way to make sure everyone had a fun and productive time. In addition to my own positive experiences, I also witnessed many small meetings among researchers discussing problems posed at the workshop after the talks. I am sure that several of these initial collaborations will blossom into interesting results. Finally, the Banff center was extremely well run with excellent hospitality, facilities, logistics, food, and a beautiful natural setting.

Valerie King, Feb 10, 2012: *The workshop gave me great feedback on these problems. A collaboration was started with Jim Aspnes who joined Seth and Jared with me to work on problem 2* [randomized message passing algorithm for asynchronous renaming]). *On problem 1* [randomized Byzantine agreement], *I had the opportunity to see if this audience of experts could see any interpretation of the manuscript which would yield a lower bound. I was encouraged to pursue a publication with a formal proof and given the name of a student who had worked on the problem over 15 years ago, who might have some insight about what was meant (given that the authors cannot tell me themselves–I did try already to contact them about it.)*

Also, Jared and I were inspired to begin work at Banff in a new direction for us working for the first time in the shared me.

Keren Censor-Hillel, Feb 9, 2012: *Workshop impact:*

- *Getting up to date with current research interests in distributed computing in shared memory.*
- *Learning about new techniques for algorithms or for lower bound proofs.*
- *New collaborations.*

Thanks for everything—the workshop was great!

Danny Hendler, Feb 8, 2012: *Apart from being fun, this workshop contributed to me professionally. Following workshop discussions and talks, I have identified with colleagues two promising research directions which we intend to pursue. Moreover, I believe that the new algorithmic ideas and proof techniques I was exposed to in the course of this excellent workshop will benefit my research in the theory of shared-memory systems.*

Participants

Alistarh, Dan (Ecole Polytechnique Federale de Lausanne)
Aspnes, James (Yale University)
Attiya, Hagit (Technion - Israel Institute of Technology)
Bender, Michael (Stony Brook University)
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Bibliography

[1] Yehuda Afek, Eli Gafni, John Tromp, and Paul M. B. Vitányi. Wait-free test-and-set. In *Proceedings of the 6th International Workshop on Distributed Algorithms (WDAG)*, pages 85–94, 1992.

[2] Dan Alistarh and James Aspnes. Sub-logarithmic test-and-set against a weak adversary. In *Proceedings of the 25th International Symposium on Distributed Computing (DISC)*, pages 97–109, 2011.

[3] Dan Alistarh, James Aspnes, Michael A. Bender, Rati Gelashvili, and Seth Gilbert. Dynamic task allocation in asynchronous shared memory. In *Proceedings of the 25th ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 416–435, 2014.

[4] Dan Alistarh, James Aspnes, Keren Censor-Hillel, Seth Gilbert, and Morteza Zadimoghaddam. Optimal-time adaptive strong renaming, with applications to counting. In *Proceedings of the 30th SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, pages 239–248, 2011.

[5] Dan Alistarh, James Aspnes, George Giakkoupis, and Philipp Woelfel. Randomized loose renaming in $O(\log \log n)$ time. In *Proceedings of the 32nd SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, pages 200–209, 2013.

[6] Dan Alistarh, James Aspnes, Seth Gilbert, and Rachid Guerraoui. The complexity of renaming. In *Proceedings of the 52nd Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 718–727, 2011.

[7] Dan Alistarh, Hagit Attiya, Seth Gilbert, Andrei Giurgiu, and Rachid Guerraoui. Fast randomized test-and-set and renaming. In *Proceedings of the 24th International Symposium on Distributed Computing (DISC)*, pages 94–108, 2010.

[8] Dan Alistarh, Michael A. Bender, Seth Gilbert, and Rachid Guerraoui. How to allocate tasks asynchronously. In *Proceedings of the 53rd Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 331–340, 2012.

[9] James Aspnes. Randomized protocols for asynchronous consensus. *Distributed Computing*, 16:165–175, 2003.

[10] James Aspnes. Randomized consensus in expected $O(n^2)$ total work using single-writer registers. In *Proceedings of the 25th International Symposium on Distributed Computing (DISC)*, pages 363–373, 2011.

[11] James Aspnes. Faster randomized consensus with an oblivious adversary. In *Proceedings of the 31st SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, pages 1–8, 2012.

[12] James Aspnes. A modular approach to shared-memory consensus, with applications to the probabilistic-write model. *Distributed Computing*, 25(2):179–188, 2012.

[13] James Aspnes, Hagit Attiya, and Keren Censor. Randomized consensus in expected $o(n \log n)$ individual work. In *Proceedings of the 27th SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, pages 325–334, 2008.

[14] James Aspnes, Hagit Attiya, and Keren Censor-Hillel. Polylogarithmic concurrent data structures from monotone circuits. *Journal of the ACM*, 59(1):2, 2012.

[15] James Aspnes, Hagit Attiya, Keren Censor-Hillel, and Faith Ellen. Faster than optimal snapshots (for a while). In *Proceedings of the 31st SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, pages 375–384, 2012.

[16] James Aspnes, Hagit Attiya, Keren Censor-Hillel, and Danny Hendler. Lower bounds for restricted-use objects: extended abstract. In *24th ACM Symposium on Parallel Algorithms and Architectures (SPAA)*, pages 172–181, 2012.

[17] James Aspnes and Keren Censor-Hillel. Atomic snapshots in $O(\log^3 n)$ steps using randomized helping. In *Proceedings of the 27th International Symposium on Distributed Computing (DISC)*, pages 254–268, 2013.

[18] H. Attiya, A. Bar-Noy, D. Dolev, D. Peleg, and R. Reischuk. Renaming in an asynchronous environment. *Journal of the ACM*, 37(3):524–548, 1990.

[19] Hagit Attiya and Keren Censor. Tight bounds for asynchronous randomized consensus. In *Proceedings of the 39th Annual ACM Symposium on Theory of Computing (STOC)*, pages 155–164, 2007.

[20] Hagit Attiya and Keren Censor. Lower bounds for randomized consensus under a weak adversary. In *Proceedings of the 27th SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, pages 315–324, 2008.

[21] Hagit Attiya, Danny Hendler, and Philipp Woelfel. Tight RMR lower bounds for mutual exclusion and other problems. In *Proceedings of the 40th Annual ACM Symposium on Theory of Computing (STOC)*, pages 217–226, 2008.

[22] Michael Ben-Or. Another advantage of free choice: Completely asynchronous agreement protocols. In *Proceedings of the 2nd SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, pages 27–30, 1983.

[23] Michael Bender and Seth Gilbert. Mutual exclusion with $O(\log^2 \log n)$ amortized work. In *Proceedings of the 52nd Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 728–737, 2011.

[24] Armando Castañeda and Sergio Rajsbaum. New combinatorial topology bounds for renaming: the lower bound. *Distributed Computing*, 22(5-6):287–301, 2010.

[25] E. W. Dijkstra. Solution of a problem in concurrent programming control. *Communications of the ACM*, 8:569, 1965.

[26] Faith Ellen and Philipp Woelfel. An optimal implementation of fetch-and-increment. In *Proceedings of the 27th International Symposium on Distributed Computing (DISC)*, pages 284–298, 2013.

[27] Michael J. Fischer, Nancy A. Lynch, and Mike Paterson. Impossibility of distributed consensus with one faulty process. *Journal of the ACM*, 32(2):374–382, 1985.

[28] George Giakkoupis, Maryam Helmi, Lisa Higham, and Philipp Woelfel. An $O(\sqrt{n})$ space bound for obstruction-free leader election. In *Proceedings of the 27th International Symposium on Distributed Computing (DISC)*, pages 46–60, 2013.

[29] George Giakkoupis and Philipp Woelfel. On the time and space complexity of randomized test-and-set. In *Proceedings of the 31st SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, pages 19–28, 2012.

[30] George Giakkoupis and Philipp Woelfel. A tight RMR lower bound for randomized mutual exclusion. In *Proceedings of the 44th Annual ACM Symposium on Theory of Computing (STOC)*, pages 983–1002, 2012.

[31] George Giakkoupis and Philipp Woelfel. Randomized mutual exclusion with constant amortized RMR complexity on the DSM. Manuscript, under review, 2014.

[32] Wojciech Golab, Lisa Higham, and Philipp Woelfel. Linearizable implementations do not suffice for randomized distributed computation. In *Proceedings of the 43rd Annual ACM Symposium on Theory of Computing (STOC)*, pages 373–382, 2011.

[33] Maryam Helmi, Lisa Higham, and Philipp Woelfel. Strongly linearizable implementations: possibilities and impossibilities. In *Proceedings of the 31st SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, pages 385–394, 2012.

[34] Danny Hendler and Philipp Woelfel. Adaptive randomized mutual exclusion in sub-logarithmic expected time. In *Proceedings of the 29th SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, pages 141–150, 2010.

[35] Danny Hendler and Philipp Woelfel. Randomized mutual exclusion with sub-logarithmic RMR-complexity. *Distributed Computing*, 24(1):3–19, 2011.

[36] M. Herlihy and J. M. Wing. Linearizability: A correctness condition for concurrent objects. *ACM Transactions on Programming Languages and Systems*, 12(3):463–492, 1990.

[37] Maurice Herlihy and Nir Shavit. The asynchronous computability theorem for t-resilient tasks. In *Proceedings of the 25th Annual ACM Symposium on Theory of Computing (STOC)*, pages 111–120, 1993.

[38] Prasad Jayanti, King Tan, and Sam Toueg. Time and space lower bounds for nonblocking implementations. *SIAM Journal on Computing*, 30(2):438–456, 2000.

[39] Valerie King and Jared Saia. Byzantine agreement in polynomial expected time. In *Proceedings of the 45th Annual ACM Symposium on Theory of Computing (STOC)*, pages 401–410, 2013.

[40] Andreas Klappenecker, Hyunyoung Lee, and Jennifer L. Welch. Dynamic regular registers in systems with churn. *Theoretical Computer Science*, 512:84–97, 2013.

[41] M. C. Loui and H. H. Abu-Amara. Memory requirements for agreement among unreliable asynchronous processes. *Advances in Computing Research*, 4:163–183, 1987.

[42] Thomas Moscibroda and Rotem Oshman. Resilience of mutual exclusion algorithms to transient memory faults. In *Proceedings of the 30th SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, pages 69–78, 2011.

[43] Abhijeet Pareek and Philipp Woelfel. RMR-efficient randomized abortable mutual exclusion. In *Proceedings of the 26th International Symposium on Distributed Computing (DISC)*, pages 267–281, 2012.

[44] Jared Saia. BIRS workshop on “probabilisitc versus deterministic techniques for shared memory computation”. `http://jsaia.wordpress.com/2012/02/14/`, 2012.

[45] Leslie G. Valiant. A scheme for fast parallel communication. *SIAM Journal on Computing*, 11(2):350–361, 1982.

[46] Andrew Chi-Chih Yao. Probabilistic computations: Towards a unified measure of complexity. In *Proceedings of the 17th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 222–227, 1977.

Chapter 7

Ordered Groups and Topology (12w5009)

February 12 - 17, 2012

Organizer(s): Steven Boyer (UQAM), Patrick Dehornoy (Caen), Peter Linnell (Virginia Tech.), Akbar Rhemtulla (Alberta), Dale Rolfsen (British Columbia), Adam Sikora (SUNY Buffalo)

Overview of the Field

If the elements of a group can be given a strict total ordering which is invariant under multiplication on the left, the group is said to be left-orderable. Left-orderable groups are also right-orderable, by a possibly different ordering, but if there is an ordering invariant on both sides, one calls the group orderable. For countable groups, left-orderability is equivalent to admitting an effective action on the real line by orientation preserving homeomorphisms.

The study of orderable groups has a long history, dating back to the nineteenth century. Steady progress in understanding orderable groups was made in the twentieth century by the work of many mathematicians, including such notables as O. Hölder, A. A. Vinogradov and W. Rudin. They discovered that many interesting groups are orderable, including free groups, torsion-free abelian groups and fundamental groups of many important topological spaces, such as hyperbolic surfaces and complements of certain hyperplane arrangements. The existence of an ordering on a group implies strong algebraic properties: for example left-orderable groups are torsion-free and obey the zero-divisor conjecture of Kaplansky (still unsolved for torsion-free groups in general); roots of elements of orderable groups are unique, and group rings of orderable groups embed in skew-fields.

In recent years, the theory of orderable groups has attracted much broader interest, in large part due to discoveries of deep connections with topology and the dynamics of group actions on the circle and real line. It was shown by T. Farrell in 1976 that the universal covering of a space X embeds in $X \times I$, respecting the projections, exactly when the fundamental group of X is left-orderable. A major breakthrough in the theory of Artin's braid groups was made in the 1990's when P. Dehornoy showed that the braid groups are left-orderable. This has had strong application in braid and knot theory and inspired discoveries that many other groups which arise in topology also have orderability properties. An important open question is whether the Artin groups of finite type, which generalize the braid groups, are also left-orderable. The answer hinges on the left-orderability of the Artin group of type E_8. It is known that right-angled Artin groups enjoy a 2-sided ordering. These groups have strong connections with topology and geometric group theory.

The fundamental groups of many 3-dimensional manifolds – for example the fundamental group of complements of knots or links in the 3-sphere – are left-orderable. Indeed, in the last few years orderability has proven a very useful tool in 3-manifold theory. For example, 3-manifolds which have particularly nice foliations must have left-orderable fundamental groups. Since many 3-manifold groups are NOT left-orderable, this shows they do not support these nice foliations. Orderability also provides an obstruction to the existence of nonzero degree maps between certain manifolds. Algorithms exist to test the orderability of groups, given generators and relations. A striking application of this, by Calegari and Dunfield, is that the hyperbolic 3-manifold of smallest volume (the so-called Weeks manifold) does not support nice foliations, because its group is not left-orderable. Roberts, Shareshian and Stein constructed an infinite family of 3-manifolds which do not support taut foliations, by showing that their groups cannot act effectively on the real line, or indeed any (possibly non-hausdorff) one-dimensional manifold. In particular, the groups cannot be left-ordered.

Besides the contribution of algebra and orderings to topology, there is a new dynamic in the other direction: topology providing applications to algebra. If G is a group, the set $LO(G)$ of all left-orderings on the group has a natural topology, defined by A. Sikora in 2004. Moreover, there is a natural (conjugation) action of G on $LO(G)$ by homoemorphisms. This is the basis for a recent beautiful argument by D. Witte-Morris, showing that left-orderable amenable groups are locally indicable (meaning any nontrivial finitely generated subgroup has the integers as a quotient group). The argument involves a G-invariant measure on $LO(G)$ and the Poincaré recurrence theorem. A classic result of Burns and Hale shows that all locally indicable groups are left-orderable, but G. Bergman showed that the converse does not hold in general. Another recent result using this topology is P. Linnell's theorem that the number of left-orderings of a group must be either finite or uncountable. By contrast, there exist groups which have a countably infinite number of (two-sided) orderings.

Recent Developments and Open Problems

For many groups G the structure of $LO(G)$, and the subspace $O(G)$ of two-sided orderings, is not known at this time. In general we know that these spaces are compact and totally disconnected, but their exact structure is known for only a few families of groups, for example finitely-generated (non-cyclic) torsion-free abelian groups (for which the spaces are homeomorphic to Cantor sets). It was recently shown that if G is a countable (nonabelian) free group, then $LO(G)$ is also a Cantor set. A stronger result, recently announced by A. Clay, is that there exists a left-ordering of the free group G whose orbit under the G-action is actually dense in $LO(G)$. The structure of $O(G)$ is not known for free groups. Another recent surprise regarding the braid groups B_n is that there exist orderings (constructed by Dubrovin and Dubrovina) which are isolated points in $LO(B_n)$, whereas Dehornoy's ordering is not isolated, and is in fact a limit point of the orbit of the D-D ordering. Another fascinating recent discovery, by A. Navas and C. Rivas, is that for the famous Thompson's group F, the space $O(F)$ consists of a Cantor set, plus four isolated points (the structure of $LO(F)$ is unknown).

Many questions dealing with algebraic and analytic properties of orderable groups have been with us for a long time. A few major ones have been answered recently in the works of Witte-Morris, V. Bludov, A. Glass and others – there exist solvable orderable groups that can not be embedded in divisible orderable groups; necessary and sufficient conditions for a free product of two left-orderable groups with amalgamated subgroup to be left-orderable; existence of finitely presented orderable groups with insolvable word problem. In view of recent result obtained by Alexey Muranov for simple groups, one can now hope to get answers to the following: Are there simple orderable groups in which there is no bound on the number of commutators required to express every element of the group as a product of bounded number of commutators?

Returning to applications of orderable groups to topology, there is very recent evidence of connections between the structure of Heegaard-Floer and similar homology theories and the orderability (or non-orderability) of fundamental groups of 3-manifolds constructed by surgery on knots and links. Ozsváth and Szabó define L-spaces to be 3-manifolds with trivial rational homology and HF homology as simple as possible. It was shown by Boyer, Gordon and Watson that the 3-dimensional Seifert fibred spaces which are L-spaces are exactly those whose fundamental group is NOT left-orderable. It would be remarkable if this is true more generally. The interplay of ordered group theory and the topology of manifolds (and structures on them, such as foliations, fibrations, contact structures) is a new and promising area of research.

Scientific Progress Made

Orderability and 3-manifolds

One of the main themes of the workshop centred on the relationships between L-spaces, left-orderability, and co-oriented taut foliations in 3-dimensional topology. Cameron Gordon's talk surveyed the evidence for his conjecture with Steven Boyer and Liam Watson that an irreducible, rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable. In particular, he described their work which shows that the conjecture holds for non-hyperbolic geometric manifolds as well as for several infinite families of manifolds whose members are generically hyperbolic, such as the 2-fold branched covers of non-split alternating links. Nathan Dunfield reported on his continuing investigation of the conjecture via computer experimentation on small-volume hyperbolic 3-manifolds. Specifically, he has shown that for the 11,031 such manifolds in the Hodgson-Weeks census, at least 27% are L-spaces and at least 2% have left-orderable fundamental groups. So far, these two subsets are disjoint, consistent with the conjecture.

If the fundamental group of an irreducible, rational homology 3-sphere is left-orderable, so is that of any manifold which finitely covers it. This leads to the following question of Boyer-Gordon-Watson: If a 3-manifold is finitely covered by an L-space, is it an L-space? Tye Lidman spoke about his work with Ciprian Manolescu on this question. They study it from the point of view of Seiberg-Witten theory and show that the answer is yes if the manifolds' Heegaard-Floer groups with integer coefficients are torsion-free.

The existence of an L-space surgery on a knot in the 3-sphere places strong constraints on its other surgeries. In the context of the above conjecture, it is natural to ask whether non-left-orderable surgery on a knot places analogous constraints on the non-left-orderability of the fundamental groups of its other surgeries. In his talk, Liam Watson discussed joint work with Adam Clay on this question. They show, for instance, that surgeries on cables of knots admitting a non-left-orderable surgery behave analogously, in this regard, to surgeries on cables of knots admitting an L-space surgery.

The connections between L-spaces and left-orderability with the existence of co-oriented taut foliations in 3-dimensional topology was the subject of three talks. It is easily seen that the fundamental group of a manifold admitting an $\mathbb{R}$-covered, co-oriented foliation is left-orderable. Rachel Roberts explained why the converse is false through the use of examples, constructed with Sergio Fenley, of irreducible, rational homology 3-spheres which have left-orderable fundamental group but which do not contain $\mathbb{R}$-covered foliations. This shows that the relationship between the existence of co-oriented taut foliations and left-orderability is more subtle than one might naively have hoped for.

Steven Boyer described his work with Michel Boileau which shows that graph manifold integer homology 3-spheres, other than the S^3 and the Poincaré homology 3-sphere, admit co-oriented taut foliations. This combines with work of Eftakhary and of Clay-Lidman-Watson to show that the notions of not being an L-space, of having a left-orderable fundamental group, and of admitting a co-oriented taut foliation, coincide for such manifolds.

Adam Clay reported on various aspects of his work with Steven Boyer and with Liam Watson. He introduced the notion of slope-detectability for 3-manifolds with torus boundary components from two points of view: that of left-orders and that of taut foliations. He outlined how recent group-theoretic work of Bludov, Glass and Chiswell allows for the development of left-order gluing conditions for the union of such manifolds. He then described how this can be used to study the connection between taut foliations in rational homology 3-sphere graph manifolds and the left-orderability of their fundamental groups.

Tetsuya Ito, in joint work with K. Kawamuro, introduced a new application of group orderings to *contact structures* on 3-dimensional manifolds. Using Nielsen-Thurston theory, it was proved by Rourke and Wiest that the mapping class group $MCG(\Sigma)$ of an orientable surface Σ with nonempty boundary is left-orderable. On the other hand, contact structures on a 3-manifold M correspond to open book structures (Σ, ϕ), where the monodromy ϕ belongs to $MCG(\Sigma)$. Ito and Kawamura show that if ϕ is sufficiently large with respect to the Nielsen-Thurston ordering, then the corresponding contact structure on M will be a *tight* structure, i.e. not *overtwisted*.

Space of orders

The space of left orders $LO(G)$ was proved by Adam Sikora in 2004 to be a compact Hausdorff space. The concept had been known for some time, including to algebraic geometers. However it was not until 2006 when Dave Morris used $LO(G)$ to prove that amenable left-ordered groups are locally indicable that it attracted much attention. Since then there has been a spate of papers studying $LO(G)$, dealing with questions such as when it is a Cantor set.

In his talk, P. Dehornoy showed how to construct concrete examples of ordered groups G such that the space $LO(G)$ of all left-invariant orderings on G has isolated points, which amounts to constructing finitely generated monoids whose left-divisibility relation turns out to be a linear ordering. To this end, he uses the subword reversing method, a combinatorial approach that, in good cases, enables one to analyze presented monoids, in particular to establish cancellativity results and existence of common multiples. As an application, he explained how to obtain a short proof of earlier results by A. NAVAS and T. ITO about the presented groups $\langle x, y \mid x^m = y^n \rangle$. It is worth noting that this approach gives one more proof of the orderability of Artin's braid group B_3.

THOMAS KOBERDA showed that the automorphism group of a residually torsion-free nilpotent group G acts faithfully on $LO(G)$. In the case G is Gromov-hyperbolic, he explained this theorem in terms of the geometry of G.

CRISTÓBAL RIVAS considered dynamical techniques for studying $LO(G)$, that is by considering G as a subgroup of the orientation preserving homeomorphisms of the real line. He use this to show that if G is a nontrivial free product of groups, then $LO(G)$ has no isolated points.

PETER LINNELL extended the topology on $LO(G)$ to the space of locally invariant orders $LIO(G)$. A locally invariant order is a strict partial order $<$ on the group G with the property either $gx > x$ or $g^{-1}x > x$ for all $g, x \in G$ with $g \neq 1$. DAVE MORRIS suggested an improved topology and this led to some new results on $LIO(G)$. Motivation for studying locally invariant orders is that if G has such an order then its group ring kG has only trivial units, and residually finite word hyperbolic groups always have subgroups of finite index which have locally invariant orders; this was proved by DELZANT.

Following the workshop, as a direct result of mutual discussions, DAVE MORRIS and PETER LINNELL proved a new theorem concerning locally invariant orders: Let G be an amenable group. If G has a LIO, then G is locally indicable.

Braids and knot theory

Artin's braid groups make an important example of orderable groups, and there remain many open questions involving braid orders, and, more generally, orders on mapping class groups, which are natural extensions of Artin's braid groups. In his talk, L. PARIS gave an account of a recent work joint with J. FROMENTIN where the authors describe a simple and efficient algorithm for finding what is known as a σ-definite representative of a braid, namely an expression in the standard Artin generators σ_i with the property that the generator with maximal index occurs only positively or only negatively. The result is directly connected with the question of comparing braids with respect to the standard Dehornoy ordering and it improves on earlier work by S. BURCKEL and P. DEHORNOY and relies on the tricky observation that applying the known methods to a braid and its inverse simultaneously enables one to avoid the difficult case.

W. MENASCO concentrated on another aspect of the standard ordering of braids, namely the connections between the so-called Dehornoy floor of a braid β, which characterizes its position with respect to the successive powers of the central element Δ_n^2, and the knot theoretical properties of the closure of β. Starting from a review of earlier work by A. MALYUTIN and S. NETSVETAEV, he explained the unifying notion of a braid block strand diagram and the general result, due to T. ITO, that, for each such diagram $\mathcal{D}$, there exists a number $n_{\mathcal{D}}$ such that, for every braid β whose floor is at least $m_{\mathcal{D}}$, the closure of β cannot be carried by $\mathcal{D}$. Roughly speaking, this says that, if the absolute value of the floor is large, then the properties of the closure of β can be read simply. Then, W. MENASCO discussed how this approach might lead to a solution of an old conjecture by V. JONES claiming that, for braids with minimal braid index, the algebraic length might be an invariant of the closure.

D. ROLFSEN's talk focused on bi-orderability of knot groups, which are known to be locally indicable, and therefore left-orderable. In joint work with A. CLAY he has shown that if K is a knot in the sphere S^3 and $\pi_1(S^3 \setminus K)$ is bi-orderable then the Alexander polynomial of K has a real positive root. As an application of this result, they showed that a surgery on a knot whose group is bi-orderable cannot produce an L-space in the sense of Ozsváth and Szabó. This is a partial converse to his result with B. PERRON that if all roots of Alexander polynomial are positive real then $\pi_1(S^3 \setminus K)$ is bi-orderable.

Algebraic and dynamic aspects

A. NAVAS, D. MORRIS, J. PRZYTYCKI, D. ROLFSEN and T. KOBERDA discussed orderability of groups using a variety of methods drawn from algebra, analysis and the theory of dynamical systems.

An algebraic criterion for a bi-ordering of a finitely-generated nontrivial group to be preserved by an automorphism of the group was proved by D. ROLFSEN. If $\phi : G \to G$ preserves a bi-ordering, then the induced linear transformation $\phi_* : H_1(G;\mathbb{Q}) \to H_1(G;\mathbb{Q})$ must have a positive real eigenvalue. This is in a sense complementary to the result of T. KOBERDA that for groups G which are residually torsion-free nilpotent (and therefore bi-orderable) every automorphism has some bi-ordering which is *not* preserved.

In his talk, D. MORRIS explained the theory of amenable groups leading to the following result: among amenable groups, a group is left-orderable if and only if it is locally indicable. He discussed also related results, including one stating that a solvable left-orderable group has a recurrent left-order (and, hence, a Conradian one). As an exciting result of interactions at our workshop, Morris proved a new result using dynamical methods: If G is a left-orderable group, and H is a locally nilpotent subgroup of G, then there is a left-invariant order on G, such that the restriction of the order to H is bi-invariant. This is announced in [21]

A. NAVAS discussed his work on left-ordered groups motivated by the question whether one can extend the scope of the above mentioned Morris' result from amenable groups to all groups which do not contain a free non-abelian subgroup. This problem led him to consider random walks on left-ordered groups with probability measures on them. He proved that for any such group there exists an interval in it such that a random walk "crosses" this interval infinitely many times with probability one.

Replacing the condition of associativity of a group product by its left-distributivity leads to the notion of a left distributive system or a “shelf". J. PRZYTYCKI discussed a homology theory for shelfs and speculated about its applications to shelf orderability.

Abstracts of the Lectures

Speaker: **Steve Boyer** (UQAM)
Title: *Graph manifolds which are integral homology 3-spheres and taut foliations*
Abstract: We show that a graph manifold, which is an integral homology 3-sphere and is neither the 3-sphere nor the Poincare homology sphere, admits a taut foliation which is transverse to the fibers in each Seifert piece. This result gives a new proof that such a manifold has a left-orderable fundamental group and is not an L-space. This is joint work with Michel Boileau.

Speaker: **Adam Clay** (UQAM)
Title: *Left-orderability and foliations*
Abstract: Every left-ordering of $\mathbb{Z}\times\mathbb{Z}$ corresponds to a line in the plane. As such, whenever M is a 3-manifold with torus boundary, we can say that every left-ordering of the fundamental group 'detects' a slope on the boundary. The idea of r-decay is to show via calculation that those slopes on the boundary of a 3-manifold which are not detected by a left-ordering correspond to those slopes for which the surgery manifold doesn't have a nice foliation, or is an L-space. In this talk, I will discuss the extent to which we can make precise the association between left-orderings and foliations, and outline how recent group-theoretic work of Bludov, Glass and Chiswell may allow for the development of 'gluing conditions' for foliations of 3-manifolds with torus boundary components. This is joint work with Liam Watson and Steve Boyer.

Speaker: **Patrick Dehornoy** (Caen)
Title: *The ordered structure of the Klein bottle group and subword reversing*
Abstract: The Klein bottle group has a simple but interesting ordered structure, which is connected with the fact that the group is a group of fractions of a Garside monoid in which divisibility is a linear ordering. On the other hand, subword reversing is a combinatorial method relevant for investigating presented groups, in particular those that are groups of fractions. We shall explain how to use this tool in the (easy) case of the Klein bottle group and its ordered structure, with the aim of subsequently applying it to more complex examples.

Speaker: **Nathan Dunfield** (Illinois, Champaign-Urbana)
Title: *L-spaces and left-orderability: an experimental survey*
Abstract: I will discuss the results of some computer experiments on small-volume hyperbolic 3-manifolds. Specifically, for the 11,031 such manifolds in the Hodgson-Weeks census, at least 27% are L-spaces and at least 2% have left-orderable fundamental groups. So far, these two subsets are disjoint, consistent with the

conjecture of Boyer-Gordon-Watson that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable.

Speaker: **Cameron Gordon** (University of Texas)
Title: *L-spaces and left-orderability*
Abstract: We will discuss evidence for the conjecture that a rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable. This is joint work with Steve Boyer and Liam Watson.

Speaker: **Tetsuya Ito** (Tokyo)
Title: *Ordering of mapping class groups and contact 3-manifolds*
Abstract: The mapping class group of a surface with non-empty boundaries have a family of left-orderings called Nielsen-Thurston type orderings. We will show that N-T orderings provide a new criterion for tightness of contact 3-manifolds. This relationship between ordering and contact geometry is based on the open book foliation theory, which was developed by the speaker and Keiko Kawamuro.

Speaker: **Thomas Koberda** (Harvard)
Title: *Faithful actions of automorphisms on the space of orderings of a group*
Abstract: I will sketch the ideas which show that the automorphism group of a residually torsion-free nilpotent group G acts faithfully on the space of left-invariant orderings of G. In the case where G is Gromov-hyperbolic, I will explain this theorem in the context of the geometry of G.

Speaker: **Tye Lidman** (UCLA)
Title: *Left-Orderability and a Seiberg-Witten Smith Inequality*
Abstract: If G is left-orderable, then any subgroup of G is automatically left-orderable as well. In terms of covering spaces, if the fundamental group of Y is left-orderable and Y' covers Y, then the fundamental group of Y' is also left-orderable. Boyer-Gordon-Watson have therefore asked the analogous question for L-spaces. However, the obvious methods fail for technical reasons. We study this question from the point of view of Seiberg-Witten theory and present some results in this direction. This is work in progress with Ciprian Manolescu.

Speaker: **Peter Linnell** (Virginia Tech)
Title: *The spaces of left- and locally invariant orders*
Abstract: I will report on separate work with two of my students Kelli Karcher (doctoral) and Li Hao (undergraduate). In the former we are studying the space of left-orders of polycyclic groups. In the latter we are studying the space of locally invariant orders (LIO) of an arbitrary group G.

Recall that the space of left-orders of a group G is the set of all left-orders of G with the topology given by the subbase $\{< \mid g < h\}$ (for $g, h \in G$). Also an LIO of a group G is a strict partial order $<$ such that for all $r, g \in G$ with $r \neq 1$, either $rg > g$ or $r^{-1}g > g$. Then the space of LIO's on G is defined in the same way as the space of left-orders on G, so is the set of LIO's with the topology given by the subbase $\{< \mid g < h\}$.

Speaker: **William W. Menasco** (SUNY-University at Buffalo)
Title: *The Dehornoy floor and the Markov Theorem without Stabilization*
Abstract: The Markov Theorem without Stabilization (MTWS) [Birman-M] tells us that for a fixed braid index n there are a finite number of "modelled" isotopes (dependent only on n) which take any closed n-braid immediately to a representative of minimal index. Once at minimal index there is a finite number of modelled isotopes (again, dependent only on the value of the braid index) that allows one to jump between conjugacy classes of minimal index. These isotopes which will grow in number as n grows make up the MTWS calculus for closed braids. Connections between the Dehornoy floor and isotopes of the MTWS calculus were first made by T. Ito. In this talk we will expand on these connections by showing a new characterisation of MTWS isotopes for braids. This talk will feature joint work with Doug Lafountain [Aarhus University] and Hiroshi Matsuda [Yamagata University].

Speaker: **Dave Witte Morris** (University of Lethbridge)
Title: *On interactions of amenability with left-orderings*

Abstract: Amenability is a fundamental notion in group theory, as evidenced by the fact that it can be defined in more than a dozen different ways. A few of these different definitions will be discussed, together with some commentary on the theorem that left-orderable amenable groups are locally indicable, and perhaps some speculation on other ways that amenability might be useful in the theory of left-orderings.

Speaker: **Andrés Navas** (USACH, Chile)
Title: *Random walks on left-orderable groups*
Abstract: Given a finitely-generated, left-orderable group endowed with a probability measure supported on a finite system of generators, we are interested on the behavior of typical random products. Among other results, I will sketch the ideas involved in a Polya's like recurrence theorem obtained in collaboration with Deroin, Kleptsyn and Parwani: there exists an interval in the group such that almost every path "crosses" this interval infinitely many times. Potential applications of these ideas will be discussed.

Speaker: **Luis Paris** (Bourgogne)
Title: *A simple and fast method for determining short σ-expressions of braids*
Abstract: Joint work with J. Fromentin. Let $n \in \mathbb{N}$, $n \geq 2$, and $i \in \{1, \dots, n-1\}$. We say that a braid $\beta \in B_n$ is *σ_i-positive* (resp. *σ_i-negative*) if it can be written

$$\beta = \beta_0 \sigma_i \beta_1 \cdots \sigma_i \beta_k \quad (\text{resp. } \beta_0 \sigma_i^{-1} \beta_1 \cdots \sigma_i^{-1} \beta_k)\,,$$

with $k \geq 1$ and $\beta_0, \beta_1, \dots, \beta_k \in B_i$. A celebrated Dehornoy's theorem says that, for any braid $\beta \in B_n \setminus \{1\}$, there exists a unique $i \in \{1, \dots, n-1\}$ such that β is either σ_i-positive, or σ_i-negative, but not both. There are several proofs of this result. Most of them are effective, but the involved algorithms are slow (exponential complexity) and determine σ_i-positive expressions (or σ_i-negative expressions) whose lengths are exponential with respect to the word-length of the original braids. In this talk we present a simple algorithm of quadratic complexity which, given a braid $\beta \in B_n \setminus \{1\}$, determines a σ_i-positive expression (or a σ_i-negative expression) for β, whose length is bounded above by some constant times the word length of β.

Speaker: **Jozef Przytycki** (George Washington University)
Title: *Orderings on Conway algebras, and Tutte algebras; is anything known?*
Abstract: We consider various non-associative binary structures and ask whether they have orderings, and whether orderings lead to some interesting consequences. We concentrate on on entropic property, $(a*b)*(c*d) = (a*c)*(b*d)$, with Conway algebra (including Homflypt polynomial) and Tutte algebra as main examples. Another property of great interest is right self-distributivity, $(a*b)*c = (a*c)*(b*c)$ with quandles, in particular Dehn quandles of surfaces, as premiere examples. We will stress that both structures satisfy "generative property" which we discuss in detail.

Speaker: **Cristóbal Rivas** (ENS-Lyon)
Title: *Left-ordering on free products of groups*
Abstract: Based on the concept of dynamical realization of a left-ordering, we exhibit a dynamical criterion for approximating the giving left-ordering. This criterion is used to show that no left-ordering on a free product of groups is isolated.

Speaker: **Rachel Roberts** (Washington University, St. Louis)
Title: *Manifolds containing no R-covered foliations*
Abstract: We show that there are 3-manifolds which have left-orderable fundamental group but which do not contain R-covered foliations. This is joint work with Sergio Fenley.

Speaker: **Dale Rolfsen** (UBC-Vancouver)
Title: *Ordering Knot Groups*
Abstract: I will discuss orderability of knot groups, that is, fundamental groups of knot complements in 3-space. Howie and Short showed that all knot groups are locally indicable, and therefore left-orderable. In joint work with Bernard Perron and Adam Clay, I'll discuss criteria, involving roots of the Alexander polynomial, determining that certain knot groups are bi-orderable, while many others are not. Sketches of

the proofs will be given. Among the applications: if a knot has bi-orderable group, then surgery on that knot cannot produce an L-space in the sense of Ozsváth and Szabó.

Speaker: **Liam Watson** (UCLA)
Title: *Dehn surgery and left-orderability*
Abstract: In light of the conjectured relationship between L-spaces and manifolds with non-left-orderable fundamental group, it is natural to study the behaviour of left-orderable groups in the context of Dehn surgery. This talk will describe some formal properties of Heegaard-Floer homology in this context, and establish analogous behaviour for left-orderable fundamental groups. This is joint work with Adam Clay.

Questions and Problems

Two problem sessions were organized during the workshop. Several promising ideas appeared, giving rise to a number of conjectures. Here are some of them.

Question 1 (Andrés Navas)

Give a sequence of hyperbolic manifolds M_n such that $\pi_1(M_n)$ is torsion-free, non left-orderable, and the injectivity radius of M_n tends to infinity as $n \to \infty$.

Discussion This probably exists. If this exists, then for large enough n $\pi_1(M_n)$ would be non-LO, torsion-free, and would have the unique product property (UPP). To date no such examples are known.

A group Γ is said to have the unique product property if for all finite sets $\{a_1, \cdots, a_n\} \subset \Gamma$ there exists $a = a_i a_j$ such that if $a = a_k a_l$ then $k = i$ and $j = l$. This is also related to the notion of local orderability. A group Γ is locally orderable if there exists a non-left-invariant partial order $<$ of Γ such that for all $f \in \Gamma$ and all $1 \neq g \in \Gamma$, either $gf > f$ or $g^{-1}f > f$. We have

$$\text{left-orderable} \Rightarrow \text{locally orderable} \Rightarrow \text{UPP} \Rightarrow \text{torsion free.}$$

T. Delzant and Chiswell have also recently proved: Suppose Γ acts by isometries on a δ-Gromov-hyperbolic space X. If for all $x \in X$ and all $1 \neq g \in \Gamma$ we have

$$\mathrm{dist}(gx, x) \geq 6\delta$$

then Γ admits a locally invariant order. As a consequence, if the answer to our question is 'yes,' then we get an example of a locally orderable group that is not left-orderable.

Question 2 (Dave Morris) We have the following theorem: **Theorem**[Howie-Short] If $\Gamma \subset \mathrm{SO}(1,3)$ is a torsion-free lattice (so it's the fundamental group of some finite volume hyperbolic 3-manifold), and there exists a surjective map $\Gamma \to \mathbb{Z}$, then Γ is left-orderable.

Is the same still true for groups $\Gamma \subset \mathrm{SO}(1,n)$? Is there a finite index left-orderable subgroup?

Discussion This is related to the conjecture that for such a Γ in $\mathrm{SO}(1,n)$ there exists a finite index subgroup $G \subset \Gamma$ such that G surjects onto $\mathbb{Z}$. In this case is G left-orderable?

Question 3 (Jozef Przytycki)

Let $M_L^{(n)}$ denote the n-fold cyclic branched cover of the link L in the manifold M. When is $\pi_1(M_L^{(n)})$ left-orderable?

Discussion

In a talk by Dale Rolfsen in the early 2000's, he showed that $\pi_1(M_{4_1}^{(3)}) = \pi_1(M_L^{(2)})$, where L is the Borromean link, is not left-orderable. Also, Przytycki showed:

- If L is a 2-bridge link with $\dfrac{p}{q} = 2m + \dfrac{1}{2k}$ then $\pi_1(M_{p/q}^{(n)})$ is not LO.

- If $\dfrac{p}{q} = n_1 + \dfrac{1}{1+n_2}$ where n_1, n_2 are odd and $n \leq 3$ then $\pi_1(M^{(n)}_{p/q})$ is not LO.

We may also make the following conjectures:

- $\pi_1(M^{(n)}_{5_2})$ is not LO.
- If L is a 2-bridge link and $|H_1(M^{(n)}_L)| < \infty$ then $\pi_1(M^{(n)}_L)$ is not LO.

One may also be tempted to conjecture: If L is a hyperbolic link and $|H_1(M^{(n)}_L)| < \infty$, then $\pi_1(M^{(n)}_L)$ is not LO. This conjecture is not true, however. The even-fold branched covers of the Conway knot were recently shown to have left-orderable fundamental group [Clay, Lidman, Watson].

Question 4 (Dale Rolfsen) Set $\mathrm{LO}(G) =$ the set of all left-orderings of G with the topology defined by Sikora. The subset $\mathrm{O}(G) \subset \mathrm{LO}(G)$ is the space of two-sided orderings (bi-orderings). Let F_n denote the (nonabelian) free group on n generators. Is $\mathrm{O}(F_n)$ a Cantor set or are there isolated bi-orderings? (For the case of left-orderings, $\mathrm{LO}(F_n)$ has no isolated orderings for $n > 2$). If G is the fundamental group of a closed surface of genus > 1 do $\mathrm{LO}(G)$ or $\mathrm{O}(G)$ have isolated points?

Discussion (Thomas Koberda) Given $\{g_1, \cdots g_n\}$ a finite subset of F_n positive in some bi-ordering, can we find a quotient of F_n and pull back orderings from there? For example, if we set $N_i = F/\gamma_i(F)$, then does there exist $i \gg 0$ such that the semigroup generated by $\{g_1, \cdots, g_n\}$ injects into N_i?

Question 5 (Dave Morris) Suppose G_i are finite index subgroups of G. Then the restriction map $\mathrm{LO}(G) \to \mathrm{LO}(G_i)$ is an injection, and define

$$\mathrm{LO}(G)_{f.i.} = \varinjlim \mathrm{LO}(H),$$

where the limit is over all finite index subgroups $H \subset G$.

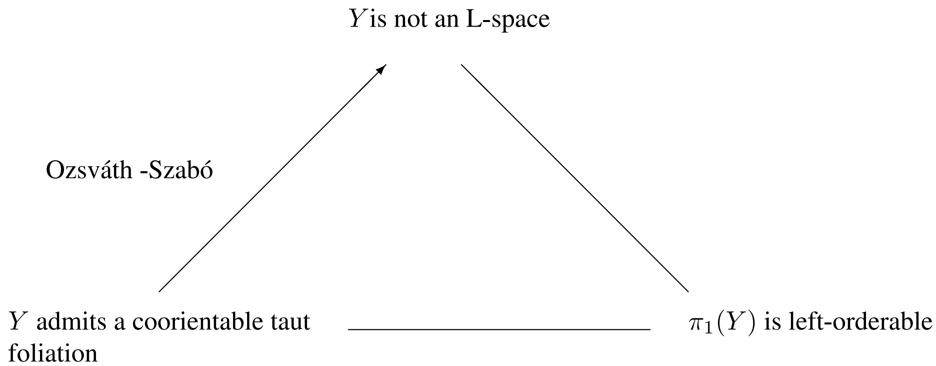
- Is this good for anything?
- Conjecture: If Γ is a lattice in $\mathrm{SO}(1,n) \times \mathrm{SO}(1,n)$ (i.e. $\pi_1(M)$ with $\tilde{M} = \mathbb{H}^2$, not $M_1 \times M_2$) then $\Gamma = \pi_1(M)$ is not virtually left-orderable. Can we prove this using $\mathrm{LO}(G)_{f.i.}$?

Question 6

(Dale Rolfsen) We know that knot groups are locally indicable and therefore left-orderable. Is there an explicit algebraic or geometric way to understand such an ordering? We now know that the torus knot groups $G = \langle a, b | a^p = b^q \rangle$ have isolated orderings in $LO(G)$. What about other knot groups? If $\pi_1(S^3 \setminus K)$ is bi-orderable and the knot is fibred, we know that the Alexander polynomial $\Delta_K(t)$ has at least two positive real roots. Is this true for non-fibred knots?

Question 7 (Liam Watson)

Let Y be a closed, connected, irreducible, orientable 3-manifold. Fill in the edges of the following triangle with logical implications. The left arrow pointing towards the top is a result of Ozsváth and Szabó. The converse is unknown. The bottom edge of the triangle is understood for some special cases. The connection between left-orderability of the fundamental group and L-spaces is not understood in general. However, we now know that everything in the triangle is equivalent for geometric, non-hyperbolic 3-manifolds.



Question 8 (Patrick Dehornoy)

Our understanding of the canonical braid ordering is still quite incomplete. In particular, the deep property that the restriction of that ordering to the submonoid B_∞^+ is a well-ordering has not been fully exploited. One of the most natural approaches consists in introducing, for every positive braid β, the smallest positive braid $\mu(\beta)$ that is conjugated to β. Computing the function μ would lead to a solution of the Conjugacy Problem of braid groups of an entirely new type. As a first step, experimental evidence support the

Conjecture: One has $\mu(\beta\Delta_3^2) = \sigma_1\sigma_2^2\sigma_1\,\mu(\beta)\,\sigma_2^2$ for every β in B_3^+.

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Bibliography

[1] S. Boyer, C. McA. Gordon and L. Watson, On L-spaces and left-orderable fundamental groups, arxiv:1107.5016.

[2] S. Boyer, D. Rolfsen, and B. Wiest, Orderable 3-manifold groups, *Ann. Inst. Fourier* **55** (2005), 243–288.

[3] D. Calegari and N. Dunfield, Laminations and groups of homeomorphisms of the circle, *Inv. Math.* **152** (2003), 149–2004.

[4] A. Clay, T. Lidman and L. Watson, Graph manifolds, left-orderability and amalgamation, arxiv:1106.0486.

[5] A. Clay and L. Watson, Left-orderable fundamental groups and Dehn surgery, arXiv:1009.4176.

[6] ————, On cabled knots, Dehn surgery, and left-orderable fundamental groups, arXiv:1103.2358.

[7] A. Clay and D. Rolfsen, Ordered groups, eigenvalues, knots, surgery and L-spaces, Math. *Proc. Camb. Phil. Soc.* (to appear) arXiv:1004.3615v2, 16 pages.

[8] A. Clay and D. Rolfsen, Densely ordered braid subgroups, *J. Knot Theory and Ramifications*, **16** (2007), 869-877.

[9] M. Dabkovska, M. Dabkowski, V. Harizanov, J. Przytycki, and M. Veve, Compactness of the space of left orders, *J. Knot Theory Ramifications* **16** (2007), 267–256.

[10] P. Dehornoy, with I. Dynnikov, D. Rolfsen, B. Wiest, *Ordering Braids*; Mathematical Surveys and Monographs **148**, American Mathematical Society, 2008.

[11] T. Dubrovina and N. Dubrovin, On braid groups, *Sb. Math.*, **192** (2001), 693-703.

[12] E. Ghys, Groups acting on the circle, *L'Enseignement Math.* **22** (2001), 329–407.

[13] T. Ito, Dehornoy-like left-orderings and isolated left-orderings, arXiv:1103.4669.

[14] T. Ito, Construction of isolated left-orderings via partially central cyclic amalgamation, arXiv:1107.0545.

[15] T. Koberda, Faithful actions of automorphisms on the space of ordering of a group, *New York Journal of Mathematics* **17** (2011), 783–798.

[16] A.I. Kokorin, V.M. Kopyutov, and N.Ya. Medvedev, *Right-Ordered Groups,* Plenum Publishing Corporation (1996).

[17] P. A. Linnell, A. H. Rhemtulla and D. Rolfsen, Invariant group orderings and Galois conjugates, *J. Algebra* **319** (2008), 4891-4898.

[18] D. Witte Morris, Bi-invariant orders on nilpotent subgroups of left-orderable groups, arXiv:1202.4716.

[19] R. Mura and A. Rhemtulla, *Orderable Groups*, Dekker Lecture Notes in Pure and Appl. Math. **27**, 1977.

[20] A. Navas, On the dynamics of (left) orderable groups, *Ann. Inst. Fourier*, **6** (2010) 1685–1740.

[21] A. Navas, A remarkable family of left-ordered groups: central extensions of Hecke groups, *J. Algebra* **328** (2011), 31 - 42.

[22] P. Ozsváth and Z. Szabó, On the Heegaard Floer homology of branched double-covers, *Adv. Math.* **194** (2005), 1–33.

[23] ————, On knot Floer homology and lens space surgeries, *Topology* **44** (2005), 1281–1300.

[24] B. Perron and D. Rolfsen, Invariant ordering of surface groups and 3-manifolds which fibre over S^1 , *Math. Proc. Camb. Phil. Soc.* **141**(2006), 273-280.

[25] C. Rivas, On spaces of Conradian orderings, *J. Group Theory*, **13** (2010), 337–353.

[26] Roberts, R.; Shareshian, J. Non-right-orderable 3-manifold groups. *Canad. Math. Bull.* **53** (2010), 706–718.

[27] Roberts, R.; Shareshian, J.; Stein, M., Infinitely many hyperbolic 3-manifolds which contain no Reebless foliation *J. Amer. Math. Soc.* **16** (2003), 639–679

[28] A.S. Sikora, Topology on the spaces of orderings of groups, *Bull. London Math. Soc.*, **36** (2004), 519–526.

[29] A. A. Vinogradov, On the free product of ordered groups, *Mat. Sbornik N.S.* **25** (1949), 163–168.

Chapter 8

Algebraic K-Theory and Equivariant Homotopy Theory (12w5116)

February 12 - 17, 2012

Organizer(s): Vigleik Angeltveit (Australian National University), Andrew J. Blumberg (University of Texas at Austin), Teena Gerhardt (Michigan State University), Michael Hill (University of Virginia), and Tyler Lawson (University of Minnesota)

Overview of the Field

The study of the algebraic K-theory of rings and schemes has been revolutionized over the past two decades by the development of "trace methods". Following ideas of Goodwillie, Bökstedt and Bökstedt-Hsiang-Madsen developed topological analogues of Hochschild homology and cyclic homology and a "trace map" out of K-theory that lands in these theories [15, 8, 9]. The fiber of this map can often be understood (by work of McCarthy and Dundas) [27, 13]. Topological Hochschild homology (THH) has a natural circle action, and topological cyclic homology (TC) is relatively computable using the methods of equivariant stable homotopy theory. Starting from Quillen's computation of the K-theory of finite fields [28], Hesselholt and Madsen used TC to make extensive computations in K-theory [16, 17], in particular verifying certain cases of the Quillen-Lichtenbaum conjecture.

As a consequence of these developments, the modern study of algebraic K-theory is deeply intertwined with development of computational tools and foundations in equivariant stable homotopy theory. At the same time, there has been a flurry of renewed interest and activity in equivariant homotopy theory motivated by the nature of the Hill-Hopkins-Ravenel solution to the Kervaire invariant problem [19]. The construction of the norm functor from H-spectra to G-spectra involves exploiting a little-known aspect of the equivariant stable category from a novel perspective, and this has begun to lead to a variety of analyses. One of the exciting aspects of this conference was an effort to grapple with various perspectives on equivariant stable homotopy theory in the context of real applications.

Recent Developments in Algebraic K-Theory

Algebraic K-theory is a field of wide mathematical interest, lying in the intersection of algebraic topology, algebraic geometry, and number theory. A number of speakers at the workshop reported on exciting recent developments in the study of algebraic K-theory and related invariants which were informed by or involved equivariant homotopy theory.

Real algebraic K-theory

In the study of topological K-theory, Atiyah's Real K-theory gives rise to a G-equivariant spectrum KR, where $G = Gal(\mathbb{C}/\mathbb{R})$ [2]. The underlying non-equivariant spectrum of KR is equivalent to KU, representing periodic complex K-theory. The spectrum of G-fixed points of KR is equivalent to KO, representing periodic real K-theory. Lars Hesselholt and Ib Madsen have developed an analogous theory, called Real algebraic K-theory. Lars Hesselholt reported on these recent developments at the workshop. They associate to a pointed exact category with strict duality $(\mathcal{C}, T, 0)$ a G-equivariant spectrum $KR(\mathcal{C}, T, 0)$ that they call the Real algebraic K-theory of $(\mathcal{C}, T, 0)$. The underlying non-equivariant spectrum is equivalent to Quillen's algebraic K-theory spectrum $K(\mathcal{C}, 0)$ [28]. The spectrum of G fixed points is equivalent to the Hermitian K-theory of $(\mathcal{C}, T, 0)$. To construct the spectrum $KR(\mathcal{C}, T, 0)$, Hesselholt and Madsen have developed a new variant of Waldhausen's $S_\bullet$-construction which they call the Real Waldhausen construction [36]. They also introduce a G-equivariant spectrum $KR^{\oplus}(\mathcal{C}, T, 0)$, called the Real direct sum K-theory of $(\mathcal{C}, T, 0)$. This spectrum is essential for understanding the G-equivariant homotopy type of $KR(\mathcal{C}, T, 0)$. It uses a variant of Segal's Γ-category construction that Hesselholt and Madsen call the Real Γ-category construction [31]. Hesselholt and Madsen have proven the following theorem:

Theorem 1. *If $\mathcal{C}$ is split-exact, then there is a canonical weak equivalence of G-spectra*

$$KR^{\oplus}(\mathcal{C}, T, 0) \simeq KR(\mathcal{C}, T, 0).$$

They define the Real algebraic K-theory groups of $(\mathcal{C}, T, 0)$ to be the bi-graded family of equivariant homotopy groups:

$$KR_{p,q}(\mathcal{C}, T, 0) = [S^{p,q}, KR(\mathcal{C}, T, 0)]_G.$$

Here $S^{p,q}$ is the virtual G-equivariant sphere $S^{\mathbb{R}^{p-q}} \wedge S^{i\mathbb{R}^q}$, where $S^{i\mathbb{R}}$ denotes the sign representation. If (A, L, α) is a ring with antistructure and $(\mathcal{C}, T, 0)$ is the category of finitely generated projective right A-modules, with the induced duality structure, then the main theorem identifies the groups $KR_{p,0}(\mathcal{C}, T, 0)$ with the Hermitian K-groups of (A, L, α), defined by Karoubi.

Progress towards $TC(MU)$

Thom spectra in general, and MU in particular, are vitally important spectra carrying rich structure. Andrew Blumberg described joint work with Angeltveit, Gerhardt, Hill, and Lawson which generalizes previous work of Blumberg, Cohen, Schlichtkrull [5] on the topological Hochschild homology of Thom spectra. In particular, Blumberg described several different symmetric monoidal products on G-spaces. The "commutative monoids" for these various symmetric monoidal products are the infinite loop space analogue of the flavors of E_∞ ring spectra for which the E_∞-operad is modeled by linear isometries on a possibly incomplete universe. Blumberg described how to make the earlier constructions of $THH(M_f)$ into an equivariant construction, producing a genuine S^1-equivariant commutative ring spectrum.

This new construction of THH arises by introducing an equivariant version of Hopkins' construction of the Thom spectrum of a map $f\colon X \to BGL_1S^0$ [1]. Blumberg described how one can apply the Hill-Hopkins-Ravenel norm technology to mirror this equivariantly, landing not in maps to BGL_1S^0 but rather in maps to $BGL_1S^0_G$. Coupled with a new description of THH as the left adjoint to the forgetful functor from S^1-commutative ring spectra to ordinary commutative ring spectra, this produces a model of $THH(M_f)$ as an equivariant Thom spectrum that has the right equivariant homotopy type for all finite subgroups. Blumberg also indicated that this new model of THH could be extended to construct TC relative to ground rings other than the sphere spectrum; this resolves an old question in the area, and opens the door to new computational approaches.

Representation rings and K-theory

Gunnar Carlsson reported on the completion of a program relating algebraic K-theory and the representation theory of Galois groups. Several results in algebraic K-theory, such as Thomason's descent theorem and the Quillen-Lichtenbaum conjectures, assert that the algebraic K-groups of a field F should be assembled from the algebraic K-groups of its algebraic closure $\overline{F}$ and the action of the Galois group G_F. Carlsson's program aimed to recover the entire homotopy type of the spectrum $K(F)$ from $K(\overline{F})$, which is understood by work of Quillen and Suslin, and the representation theory of the Galois group [28, 35].

If F contains an algebraically closed field k, Carlsson constructed a K-theory spectrum $K(Rep_k(G_F))$ from the category of representations of G over k. The homotopy groups are closely related to the representation ring of G_F, and descent theory provides it with a natural map $K(Rep_k(G_F)) \to K(F)$. Carlsson's conjecture has been that, upon applying an appropriately "derived" notion of completion, this map becomes an equivalence.

The main theorem described in this talk is a proof of this result. The proof first uses a calculation for the case of a Laurent polynomial ring $k[t^{\pm 1}]$ and extensions obtained by adjoining roots of t. It then applies the Bloch-Kato theorem, which shows that algebraic K-groups of fields (appropriately completed) are generated in degrees 1 and 2, with specific descriptions of the generating elements. Finally, there is an "algebraic to geometric" spectral sequence relating the homotopy groups of a completed spectrum to the appropriate derived completion in algebra.

Localization sequences in THH

Mike Mandell reported on joint work with Andrew Blumberg establishing certain localization sequences in THH. In earlier work [6], they established certain localization sequences in algebraic K-theory, the most important example being the cofibration sequence

$$K(\mathbb{Z}) \to K(ku) \to K(KU).$$

The main result described in this talk is a corresponding localization sequence for THH [7]. If R is a discrete valuation ring with residue field k and fraction field F, there is a cofibration sequence

$$THH(k) \to THH(R) \to THH(F)$$

compatible with the corresponding cofibration sequence for algebraic K-theory.

Ausoni-Rognes and Hesselholt have conjectured that there should be a similar localization sequence involving $THH(ku)$ [4]. The most obvious approach does not work because $THH(KU)$ is not connective, so the homotopy fiber of the map $THH(ku) \to THH(KU)$ is something strange and definitely not $THH(\mathbb{Z})$.

Instead, Blumberg and Mandell work with THH of spectral categories. Let $\mathcal{C}$ be the category of finite cell ku-modules. For $X, Y \in \mathcal{C}$ one can define a connective spectral category $\mathcal{C}^\Gamma$ by

$$\mathcal{C}(X,Y)^\Gamma(n) = |\mathcal{C}(X, \bigvee_{S^n_\bullet} Y)|.$$

Then one can recover $THH(ku)$ by applying the Bökstedt version of the cyclic bar construction to the category $S_\bullet N^w_\bullet \mathcal{C}^\Gamma$. Here $N^w_\bullet$ is the nerve of the subcategory where the maps are all the weak equivalences, and $S_\bullet$ is the Waldhausen construction.

If we instead use the category $S_\bullet N^q_\bullet \mathcal{C}^\Gamma$, where $N^q_\bullet$ means taking the nerve of the category where all the maps become weak equivalences after inverting the Bott element, we get a spectrum $W^\Gamma THH(ku|KU)$.

The homotopy fiber of this map is the cyclic bar construction on the category $S_\bullet N^w_\bullet (\mathcal{C}^\Gamma)^q$ consisting of torsion ku-modules. By a devissage theorem, they identify this with $THH(\pi_0 ku) = THH(\mathbb{Z})$. Hence there is a cofiber sequence

$$THH(\mathbb{Z}) \to THH(ku) \to W^\Gamma THH(ku|KU).$$

Moreover, this cofiber sequence is compatible with the Dennis trace.

K-theoretic assembly maps, Rips complexes, and equivariant phantom maps

Dan Ramras described recent progress on Loday's assembly map and the integral Novikov conjecture: If G is a discrete, torsion free group, then the map

$$\alpha\colon BG_+ \wedge K(R) \to K(R[G])$$

is injective in homotopy groups. He began by recasting the problem into a geometric one, analogous to Segal's description of the K-theory of a space [32]. This allowed more geometric tools and approaches to be brought to bear. In particular, Ramras considered several families of groups that have buildings with particularly nice geometric structure. The additional geometry allowed, for this family of groups, descent-style arguments showing the Novikov conjecture. Ramras finished with several conjectures, based on of Rips complexes, which would establish the Novikov conjecture in a wide variety of cases.

Recent Developments in Equivariant Stable Homotopy Theory

Complementing the talks on algebraic K-theory proper were a series of talks on new work in the foundations of equivariant stable homotopy theory, as well as applications of recent foundational work to other areas.

G-spectra as presheaves of spectra

Bert Guillou and Peter May have developed a model of the category of G-spectra as a category of enriched presheaves of spectra. Both researchers presented on these results at the workshop. By a result of Schwede and Shipley, any stable model category is equivalent to a category of presheaves enriched in a chosen category of spectra [30]. However, the domain category can be rather mysterious. Guillou and May give an explicit construction of the domain category in their case by applying an infinite loop space machine, $\mathbb{K}$, to an elementary category of finite G-sets enriched in permutative categories, $G\mathcal{E}$. They prove the following.

Theorem 2. *Let G be finite. The the category $\mathbb{K}(G\mathcal{E})$ is equivalent to $G\mathcal{B}$.*

Here $G\mathcal{B}$ denotes an enriched version of the Burnside category of G. This new model extends a description of the homotopy category given in [20], recasting equivariant stable homotopy theory in terms of elementary point-set level categories of G-spans and nonequivariant spectra.

This work requires a number of ingredients of independent interest, such as the theory of classifying G-spaces for equivariant bundles. Guillou and May also define and give examples of genuine permutative G-categories, and more generally E_∞ G-categories. Further contributions of the work include:

1. equivariant infinite loop space theory and infinite loop space machines,
2. the equivariant Barratt-Priddy-Quillen theorem,
3. the tom Dieck splitting theorem for suspension G-spectra,
4. equivariant algebraic K-theory, and
5. pairings of permutative G-categories.

Equivariant commutative ring spectra

Bjørn Dundas reported on work (in part by his student Stolz) aimed at providing foundational underpinnings to study the redshift conjecture and the answering the question: "What are the slices of the equivariant THH spectrum?" Rognes' redshift conjecture asserts that K-theory increases chromatic (telescopic) complexity; this is supported by calculation in the cases for $n = 0$ and $n = 1$ [3]. The conjecture suggests studying iterated K-theory and consequently iterated THH and TC. In previous work, Dundas (with Brun, Carlsson, and Douglas) has studied iterated THH for commutative ring spectra in terms of tensoring with higher tori, and associated "TC-like" limit constructions [10, 11]. In this talk, Dundas described a model structure on commutative ring spectra (constructed by Stolz) which provides a formal home for interpreting the equivariant nature of these tensor constructions and, more generally, the equivariant nature of smash powers of ring spectra. This work leads to interesting equivariant filtrations on smash powers.

Global equivariant homotopy theory

Both Anna-Marie Bohmann and Stefan Schwede reported on work aimed at constructing a "global" equivariant stable homotopy theory.

Anna-Marie Bohmann's report was devoted to conceptual foundations for these categories. She motivated global equivariant homotopy theory as describing a family of compatible G-equivariant homotopy types as G varies, with the goal of understanding "change of groups" phenomena.

In Bohmann's version, a global spectrum is a compatible family of equivariant spectra. To make sense of this, one needs compatibility of the universes for maps of groups, and Bohmann described a categorical framework for this work, based on universe-indexed spectra such as those employed by Lewis-May-Steinberger and Elmendorff-Kriz-Mandell-May [20, 14]. Given an appropriately compatible family of G-equivariant universes for all groups G, a global equivariant spectrum consists of a section of a certain functor of categories. She also described the roles of several canonical examples in the global equivariant world, including the sphere spectrum, the equivariant K-theory spectrum, and equivariant bordism theories.

By contrast, Stefan Schwede described a notion of G-equivariant spectra based a new model structure on the orthogonal spectra introduced in Mandell-May-Schwede-Shipley [25]. Orthogonal spectra are equivalent to certain "enriched" functors on equivariant vector spaces; this is based on previous observations published by Shimakawa, and played an important role in the solution of the Kervaire invariant problem [33, 19].

Stefan also reported on some calculational work in global equivariant homotopy theory. Global equivariant homotopy groups take values in the category of global functors. These have the feature that, unlike Mackey functors or abelian groups, they are not rationally semisimple, and so rational equivariant homotopy types do not naturally decompose as products of Eilenberg-Mac Lane objects.

The bulk of Schwede's talk focused on a particular example: the homotopy groups of the symmetric powers of the sphere spectrum. Schwede completely computed π_0 as a global functor (showing the fantastically simple solution in the global context), and he used this to produce explicit examples of nontrivial extensions naturally occurring in rational global homotopy theory.

An algebraic model for rational G-spectra

Brooke Shipley and John Greenlees both reported on their joint work on developing models for rational G-spectra. The category of rational spectra, with no group action, is Quillen equivalent to the category of $\mathbb{Q}$-DG modules [30]. By previous work of Schwede and Shipley we also know that, given certain technical conditions, any rational stable homotopy theory with a single generator is Quillen equivalent to the category of DG modules over some $\mathbb{Q}$-DGA (or over a DG category in the case of a set of generators). This result applies to free rational G-spectra, but it is only an existence result and we would like a small, explicit algebraic model.

The first talk, by Brooke Shipley, focused on the category of free $\mathbb{Q}$-G-spectra, which is Quillen equivalent to $H\mathbb{Q}[G]$-module spectra, where $H\mathbb{Q}[G] = H\mathbb{Q} \wedge G_+$. If G is finite then $\pi_* H\mathbb{Q}[G]$ is concentrated in degree 0 and we simply get $\mathbb{Q}$-DG modules with a G-action.

If G is an arbitrary connected compact Lie group, we can use Koszul duality in spectra:

$$H\mathbb{Q}[G] \rightsquigarrow F(BG_+, \mathbb{Q})$$

The latter is commutative and the homotopy is a polynomial algebra concentrated in even degrees, hence formal. From this we get the following result:

Theorem 3 (Greenlees-Shipley)**.** *For any connected compact Lie group G we have a Quillen equivalence*

$$\text{free-}\mathbb{Q}\text{-}G\text{-}Sp \simeq_Q \text{torsion DG } H\mathbb{Q}^*(BG)\text{-modules.}$$

In the nonconnected case, let N be the identity component of G and $W = G/N$ the component group. Then we can combine the above result with the simpler behavior for finite groups to get the following:

Theorem 4 (Greenlees-Shipley)**.** *Define $\tilde{B}N = EG/N$, which has a W-action. Then we have a Quillen equivalence*

$$\text{free-}\mathbb{Q}\text{-}G\text{-}Sp \simeq_Q \text{torsion DG } H^*(\tilde{B}N)\langle W\rangle\text{-modules,}$$

where $H^*(\tilde{B}N)\langle W\rangle$ denotes the twisted group ring.

The second talk, by John Greenlees, discussed the case where we no longer assume that the G-action is free. John started with a conjecture:

Conjecture 1. *For any compact Lie group G we have*

$$\mathbb{Q}\text{-}G\text{-}Sp \simeq_Q DG\mathcal{A}(G)$$

for some "nice" abelian category $\mathcal{A}(G)$ of injective dimension equal to the rank of G.

There are several applications of this.

1. It enables us to do calculations by using an Adams short exact sequence or spectral sequence with finitely many rows.
2. It lets us construct of G-spectra algebraically.
3. It has applications to other theories such as G-equivariant elliptic cohomology.

The idea is as follows: $\mathcal{A}(G)$ should be some category of sheaves over the category of subgroups $Sub(G)$ with fiber over H capturing H-geometric isotropy information.

When G is a torus, then they have verified their conjecture.

Theorem 5 (Greenlees-Shipley)**.** *There is a Quillen equivalence*

$$\mathbb{Q}\text{-}G\text{-}Sp \simeq_Q DG\mathcal{A}(G)$$

for G a torus.

The Gap theorem at 3

Ravenel described ongoing, and largely conjectural, work with Hill and Hopkins concerning the 3-primary Arf-Kervaire problem, and specifically the survival of the family $\beta_{3^i/3^i}$ in the Adams-Novikov spectral sequence. The Hill-Hopkins-Ravenel solution to the Kervaire invariant one problem (at the prime two) used several equivariant techniques which port over directly to the odd primary case. In particular, there is a natural slice filtration described by Hill-Hopkins-Ravenel for any finite group, and the norm machinery allows the construction of commutative ring spectra for larger groups.

Ravenel described a large snag: we do not have a 3-primary analogue of the spectrum $MU_{\mathbb{R}}$ of Real bordism. This was the start of the Kervaire solution, as from this C_2-equivariant spectrum one can build a C_8-equivariant spectrum that detects the Kervaire classes and for which their non-existence follows from straightforward computations. The desired properties of a C_3-analogue, called MU_A, are fairly simple:

1. The underlying spectrum should be $MU \wedge MU$ with a kind of "reduced regular" action.
2. The geometric fixed points should carry the "universal formal group law in which the 3-series is zero".

Assuming the existence of such a spectrum, and basic properties connecting it to MU, Ravenel sketched out a proof of the 3-primary analogue of the "Gap Theorem": the homotopy group π_{-2} of any regular representation suspension of MU_A (or its norm to C_9) is torsion free. In particular, coupled with a periodicity theorem (provable via homotopy fixed point arguments), we see that only finitely many of the classes $\beta_{3^i/3^i}$ survive the Adams-Novikov spectral sequence.

Equivariant A_∞ bundle theory

John Lind described work using "rigid" models of infinite loop space theory to study bundle theory [21]. Based on work of Blumberg, Cohen, and Schlichtkrull [5], there now exist various categories of "spaces" with a symmetric monoidal product such that monoids and commutative monoids model A_∞ and E_∞ spaces. This is akin to the situation with modern categories of spectra (and there is a strong mathematical analogy in the technology used). Lind is applying this technology to study principal fibrations where the structure group acting is an A_∞ space; this is already interesting non-equivariantly, as it allows us to talk about bundles of spaces that are "groups up to coherent homotopy" without having to fixed an equivalent group. This talk focused also on the extension to the equivariant setting (for both finite and compact Lie groups). Lind described versions of the standard classification results in this context and sketched applications to equivariant twisted cohomology theories and modeling iterated algebraic K-theory classes.

Modeling stable n-types

Angelica Osorno described a joint project with Niles Johnson studying the relationship between homotopy n-types and higher category theory. A homotopy n-type is a space X whose homotopy groups vanish in degrees above n for all choices of basepoint. It is a classical result that groupoids model homotopy 1-types, in the sense that the classifying space and fundamental groupoid functors establish an equivalence between their homotopy categories. In higher category theory this is known as the "homotopy hypothesis" and has long been a motivating principle.

Johnson and Osorno have extended this result to an equivalence between stable homotopy 1-types and Picard groupoids. A Picard groupoid is a symmetric monoidal groupoid in which every object has a weak inverse under the monoidal structure. Using an algebraic description of Picard groupoids, they have identified the Postnikov data associated to a stable 1-type:

1. the group π_0 is the set of isomorphism classes,
2. π_1 is the automorphism group of the unit object, and
3. the unique k-invariant is determined by the twist automorphism.

Their ongoing work has also explored the case for $n = 2$, where they expect stable homotopy 2-types to be modeled by Picard bigroupoids. In this direction, they have already identified a Picard bigroupoid which acts as the homotopy cofiber of a map between Picard groupoids.

Fusion categories and field theories

Chris Douglas, Chris Schommer-Pries, and Noah Snyder have explored the relationship between fusion categories and 3-dimensional topological field theories. Chris Douglas reported on this work at the workshop. Fusion categories are monoidal categories that have the nice properties of the category of representation of a finite group:

1. each object has a dual,
2. there are finitely many simple objects, and
3. any object decomposes into a finite sum of simple objects.

In particular, fusion categories are a type of tensor category. Any fusion category gives rise to a 3-dimensional topological field theory.

A key question about the algebraic structure of a fusion category is whether the double dual operation is trivial, as it is in the representation category of a finite group. The following is known:

Theorem 6 (Etingof-Nikshych-Ostrik)**.** *The quadruple dual is trivial.*

Etingof, Nikshych, and Ostrik also conjecture that the double dual is trivial. While this interesting question remains open, Douglas reported on new perspectives on this question provided by the work of Douglas, Schommer-Pries and Snyder. This question corresponds to the question of whether the 3-manifold invariants of the associated field theory depend on a spin structure. Douglas then connected the problem to various other structures on 3-manifolds, linking the problem to classically known homotopy computations.

Scientific Progress Made

By bringing together the experts in equivariant homotopy and algebraic K-theory, the workshop established several large projects which helped define the scope of the field for the next several years. In particular, several dominant themes arose:

1. Understand G-equivariant infinite loop space machines and more generally what is meant by a G-symmetric monoidal categories.
2. Use new constructions and approaches in equivariant homotopy to compute algebraic K-groups.

The first point was spearheaded by the talks of Guillou and May. Their talks outlined a construction of a G-equivariant infinite loop space machine on G-permutative categories (along the way, discussing what is meant by a G-permutative category). This meshed with a philosophy expounded by Hill for G-symmetric monoidal categories as symmetric monoidal categories for which we have "products indexed by G-sets". The talk and philosophy underscore several big, outstanding questions in equivariant homotopy: how to reconcile G-equivariant as diagrams indexed by the category G and other notions of G-objects with symmetric monoidal structures. This should give rise to new interpretations of previously confusing topics (such as the difference between Green and Tambara functors), and allow a very natural explanation of the Hill-Hopkins result about equivariant localizations [18].

The second point comes from the specifics of constructions of THH as an S^1-equivariant spectrum. Blumberg, Dundas, Hesselholt, and Mandell all spoke about such constructions and the computational ramifications. Together they provide a picture of equivariant homotopy which is computationally approachable. Tethered to the models described by the first point, we see new way to interpret the homotopy groups provided by trace methods. The modern constructions spell out the connection quite clearly and cleanly.

Based on the work presented at this conference and some of the collaborations initiated, we are optimistic that the new foundations of equivariant stable homotopy theory wil facilitate and support continued progress in the use of trace methods to understand algebraic K-theory.

Outcome of the Meeting

This meeting gathered together experts from around the world in the areas of equivariant stable homotopy theory and algebraic K-theory. Recent advancements in these areas were presented at the workshop, and the talks all sparked lively discussion. Time was also set aside for participant discussion, and extensive collaboration took place during the week. A number of participants commented how unique and valuable it was to have this meeting of experts. We felt that the meeting was a tremendous success, far exceeding our hopes and goals for the week.

Participants

Angeltveit, Vigleik (Australian National University)
Blumberg, Andrew (University of Texas at Austin)
Bohmann, Anna Marie (Vanderbilt University)
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Douglas, Christopher (Oxford University)
Dundas, Bjorn (University of Bergen)
Gerhardt, Teena (Michigan State University)
Greenlees, John (University of Sheffield)
Guillou, Bertrand (University of Illinois at Urbana-Champaign)
Hesselholt, Lars (Nagoya University)
Hill, Michael (UCLA)
Lawson, Tyler (University of Minnesota)
Lind, John (Johns Hopkins University)
Mandell, Michael A. (Indiana University)
May, Peter (University of Chicago)
Osorno, Angelica (University of Chicago)
Ramras, Daniel (New Mexico State University)
Ravenel, Douglas (University of Rochester)
Schwede, Stefan (Universitaet Bonn)
Shipley, Brooke (University of Illinois, Chicago)

Bibliography

[1] M. Ando, A. Blumberg, D. Gepner, M. Hopkins, and C. Rezk. Units of ring spectra and thom spectra. Preprint, arXiv:0810.4535

[2] M. F. Atiyah K-theory and reality *Quart. J. Math. Oxford Ser (2)*, 17:367–386, 1966

[3] C. Ausoni and J. Rognes. Algebraic K-theory of topological K-theory. *Acta Math.*, **188**(1): 1–39, 2002.

[4] C. Ausoni and J. Rognes. Algebraic K-theory of the fraction field of topological K-theory. October, 2008, unpublished.

[5] A. Blumberg, R. Cohen, and C. Schlichtkrull Topological Hochschild homology of Thim spectra and the free loop space *Geom. Topol.* **14**(2), 1165-1242, 2010.

[6] A. Blumberg and M. Mandell. The localization sequence for the algebraic K-theory of topological K-theory. *Acta Math.*, **200**(2):155–179, 2008.

[7] A. Blumberg and M. Mandell Localization theorems in topological Hochschild homology and topological cyclic homology *Geom. Topol.* **16**, 1053–1120 (2012).

[8] M. Bökstedt. Topological Hochschild homology. Preprint.

[9] M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. *Invent. Math.*, **111**(3), 465–539, 1993.

[10] M. Brun, G. Carlsson, and B. Dundas Covering homology *Adv. Math.* **225**(6): 3166–3213, 2010.

[11] G. Carlsson, C. Douglas, and B. Dundas Higher topological cyclic homology and the Segal conjecture for tori *Adv. Math.* **226**(2), 1823–1874, 2011.

[12] W. Chacholski and J. Scherer. *Homotopy theory of diagrams*. Mem. of the Amer. Math. Soc. **155** (736), 2002.

[13] B. I. Dundas. Relative K-theory and topological cyclic homology. *Acta Math.* **179** (2) (1997), 223–242.

[14] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Rings, modules, and algebras in stable homotopy theory. *Mathematical Surveys and Monographs* **47**, Amer. Math. Soc., Providence, RI, 1997. With an appendix by M. Cole.

[15] T.G. Goodwillie. Relative algebraic K-theory and cyclic homology. *Ann. of Math. (2)*, **124**(2), 347–402, 1986.

[16] L. Hesselholt and I. Madsen. On the K-theory of finite algebras over Witt vectors of perfect fields *Topology* **36** (1997), 29–101.

[17] L. Hesselholt and I. Madsen. On the K-theory of local fields *Ann. of Math. (2)* **158** (2003), 1–113.

[18] M. A. Hill and M. J. Hopkins. On the localization of equivariant commutative ring spectra. In preparation.

[19] M. A. Hill and M. J. Hopkins and D. C. Ravenel. On the non-existence of elements of Kervaire invariant one. arXiv:0908.3724.

[20] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[21] J. Lind. Diagram spaces, diagram spectra, and spectra of units. arXiv:0908.1092.

[22] S. Lunøe-Nielsen and J. Rognes. The topological Singer construction. arXiv:1010.5633.

[23] I. Madsen. Algebraic K-Theory and traces. *Current Development in Mathematics*, International Press Inc., Boston (1995), 191–321.

[24] M. A. Mandell and J. P. May. *Equivariant orthogonal spectra and S-modules*. Mem. of the Amer. Math. Soc. **159** (755), 2002.

[25] M. A. Mandell and J. P. May and S. Schwede and B. Shipley Model categories of diagram spectra. *Proc. London Math. Soc.* **82** (2001), 441–512.

[26] M. A. Mandell and B. Shipley. A telescope comparison lemma for THH *Top. Appl.,* **117** (2002), 161–174.

[27] R. McCarthy. Relative algebraic K-theory and topological cyclic homology. *Acta Math.*, **179**(2),197–222, 1997.

[28] D. Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[29] C. Schlichtkrull. Units of ring spectra and their traces in algebraic K-theory. *Geom. & Top.* **8** (2004), 645-âĂŞ673.

[30] S. Schwede and B. Shipley. Stable model categories are categories of modules. *Topology*, **42**, 103–153, 2003b.

[31] G. Segal. Categories and cohomology theories. *Topology*, **13**, 293–312, 1974.

[32] G. Segal. K-homology theory and algebraic K-theory. In *K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975)*, pages 113–127. Lecture Notes in Math., Vol. 575. Springer, Berlin, 1977.

[33] K. Shimakawa. A note on Γ_G-spaces. *Osaka J. Math.*, **28**,(2):223–228, 1991.

[34] B. Shipley. Symmetric spectra and topological Hochschild homology. *K-theory* **19** (2) (2000), 155–183.

[35] A. Suslin. On the K-theory of algebraically closed fields. *Invent. Math.*, **73**,(2):241–245, 1983. ISSN 0020-9910.

[36] F. Waldhausen Algebraic K-theory of topological spaces. I. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1* pages 35–60. Proc. Sympos. Pure Math., XXXII, 1978.

Chapter 9

Outstanding Challenges in Combinatorics on Words (12w5068)

February 19 - 24, 2012

Organizer(s): James Currie (University of Winnipeg), Jeffery Shallit (University of Waterloo)

Overview of the Field

Combinatorics on words is a relatively new area of research in discrete mathematics. It studies the properties of words (sequences), either finite or infinite, over a finite alphabet. The perspective on words can be variously algebraic, combinatorial, or algorithmic. The field is characterized by its manifold connections to topics, not only in mathematics, but also in other scientific disciplines. Inside mathematics, connections exist to certain parts of algebra (e.g., combinatorial group theory and semigroups), probability theory, number theory, and discrete symbolic dynamics. In other sciences, connections exist to crystallography, and to DNA sequencing. Combinatorics on words has also been notably connected to and motivated by theoretical computer science, e.g., automata theory and pattern matching algorithms.

The Norwegian number theorist A. Thue [22, 23] was the first to study combinatorial problems on sequences of symbols for their own sake. Among other things, he established the existence of an infinite word on two symbols (three symbols) having no cubes (squares, respectively) as factors. Here, a cube refers to three consecutive repetitions of the same word. Thue's research was the origin of research on pattern avoidability in words. Such results have had profound applications, e.g., in Burnside-type questions in algebra.

Later, over several decades, several papers on combinatorics on words appeared, mainly as tools of solving problems on different areas. A typical example is a paper by M. Morse [17] in the 1930's, where the motivation arose from symbolic dynamics. This work initiated research on the subword complexity of words. Another example is provided by the Russian mathematician A. I. Shirshov who while studying polynomial rings, discovered a fundamental unavoidability result of words, subsequently known as Shirshov's theorem [21].

Combinatorics on words is now listed in the AMS Mathematics Subject Classification as a distinct topic, under the code 68R15. The emergence of fundamental results on sequences in different domains of mathematics is one of the motivations for this identification of combinatorics on words as an area of research. Van der Waerden's theorem [24] and the periodicity lemma of Fine and Wilf [9] are further examples of this phenomenon. The former was stated in terms of colorings of the natural numbers, and the latter as a periodicity criterion for real functions. However, their most natural formulation is as properties of words. Systematic research on words started in the 1950's in two different research communities: the Moscow and the French schools. The Moscow school was initiated by P. S. Novikov and S. I. Adjan in connection to their research on the Burnside problem and the establishment of the undecidability of the word problem for groups [1]. At around the same time M. P. Schützenberger created the French school as part of his research on the theory of codes [20]. Both schools can be viewed as pioneers of the field. The decidability of the satisfiability problem for word equations by G. S. Makanin [16] is an important sample outcome. Over the last 20 years or so combinatorics on words has developed into a quickly growing topic of its own; a few textbooks [13, 14, 15, 2] have also emerged as very influential.

Recent Developments

Several striking results have been achieved in this area recently. Among these are the resolution of long-standing problems and conjectures:

- A 1972 conjecture by F. Dejean [8] stated a precise bound on the size of unavoidable repetitions in infinite words. This conjecture was finally confirmed through the work of M. Rao, J. Currie, N. Rampersad, and A. Carpi [19, 7, 4].
- The centralizer of a language is the maximal language commuting with it. The question, raised in 1970 by J. H. Conway [6], whether the centralizer of a rational language is always rational has been negatively answered with a celebrated result [12]. In fact, even complete co-recursively enumerable centralizers exist for finite languages.
- The satisfiability of word equations with constants is in PSPACE [18]. It follows from the proof of that result that the satisfiability of word equations with constants is in NP if one shows that the minimal solutions of a word equations are single exponential in the size of the equation if they exist.
- The solution of J.-P. Duval's 1982 conjecture and the Ehrenfeucht-Silberger problem about the relation between the period of a word and the maximum length of its unbordered factors by T. Harju, S. Holub, and D. Nowotka in 2004 settled long-standing questions [10, 11].
- Does there exist an infinite word over a finite subset of N such that no three consecutive blocks of the same size and the same sum exist? G. Pirillo and S. Varricchio raised that question 1994 in the context of semigroup theory. L. Halbeisen and N. Hungerbauhler formulated that problem in different terminology in 2000 independently of G. Pirillo and S. Varricchio. Just recently that question was affirmatively answered by J. Cassaigne, J. D. Currie, L. Schaeffer, and J. Shallit [5].

In addition, tools in several subareas are clearly coming to maturity. At one point, the connections between combinatorics on words and transcendence results seemed to be one-way only, and somewhat ad hoc; today, this connection is better understood, and several tools have emerged in this intersection of discrete mathematics with algebra. Major progress has also been made on variations of the run-length problem, which ties together combinatorics on words with ideas from data compression. As a final example, properties of automatic sequences, expressed in a certain logic, have extremely recently been shown to be decidable.

Presentation Highlights

Run-length and maximal exponent problems

Videotaped lecture

For the first of our videotaped lectures, *Maximal Exponent Repeats*, **Maxime Crochemore** presented an overview of the run-length problem and related questions. This talk was a summary of basic issues related to repetitions in strings, concentrating on algorithmic and combinatorial aspects. This area is important both from theoretical and practical point of view. Repetitions are highly periodic factors (substrings) in strings and are related to periodicities, regularities, and compression. The repetitive structure of strings leads to higher compression rates, and conversely, some compression techniques are at the core of fast algorithms for detecting repetitions. There are several types of repetitions in strings: squares, cubes, and maximal repetitions also called runs. For these repetitions, we distinguish between the factors (sometimes qualified as distinct) and their occurrences (also called positioned factors). The combinatorics of repetitions is a very intricate area, full of open problems. For example we know that the number of (distinct) primitively-rooted squares in a string of length n is no more than $2n\theta(\log n)$, and is conjectured to be n, and that their number of occurrences can be $\theta(n \log n)$. Similarly we know that there are at most $1.029n$ and at least $0.944n$ maximal repetitions and the conjecture is again that the exact bound is n.

Some other notable talks related this area

Golnaz Badkobeh gave a talk *Fewest repetitions vs maximal-exponent powers in infinite binary words*
Abstract: The exponent of a word is the ratio of its length over its smallest period. The repetitive threshold $r(a)$ of an a-letter alphabet is the smallest rational number for which there exists an infinite word whose finite factors have exponent at most $r(a)$. This notion was introduced in 1972 by Dejean who gave the exact values of $r(a)$ for every alphabet size a as it has been eventually proved in 2009. The finite-repetition threshold for an a-letter alphabet refines the above notion. It is the smallest rational number FR$t(a)$ for which there exists an infinite word whose finite factors have exponent at most FR$t(a)$ and that contain a finite number of factors with exponent $r(a)$.

It is known from Shallit (2008) that FR$t(2) = 7/3$. With each finite-repetition threshold is associated the smallest number of $r(a)$-exponent factors that can be found in the corresponding infinite word.

It has been proved by Badkobeh and Crochemore (2010) that this number is 12 for infinite binary words whose maximal exponent is $7/3$.

In this article we give some new results on the trade-off between the number of squares and the number of maximal-exponent powers in infinite binary words, in the three cases where the maximal exponent is $7/3$, $5/2$, and 3. These are the only threshold values related to the question.

Jamie Simpson spoke on *Matching fractions*
Abstract: If x is an infinite word and k is a non-negative integer then the kth matching fraction of x, written $\mu(k)$, is the following limit, if it exists:

$$\mu(k) = lim_{n\to\infty} \frac{|\{i \in [1, n] : x[i] = x[i + k]\}|}{n}. \tag{9.1}$$

The idea of a matching fraction was suggested by Peter Pleasants and has been investigated by Keith Tognetti and colleagues. In particular they showed that if x is a Sturmian word on alphabet $\{a, b\}$ whose density of as is α then

$$\mu(k) = \max(|1 - 2\{\alpha\}|, |1 - 2\{k\alpha\}|). \tag{9.2}$$

Here $\{\alpha\}$ is the fractional part of α.

I will review what's known about matching fractions and suggest some areas for investigation.

Diophantine approximation

Monday morning was devoted to an exposition of topics relating combinatorics on words and automatic sequences to diophantine approximation.

Boris Adamczewski gave a lecture *Combinatorics on words and Diophantine approximation*
Abstract: A very fruitful interplay between combinatorics on words and Diophantine approximation comes up with the use of numeration systems.

Finite and infinite words occur naturally in Number Theory when one considers the expansion of a real number in an integer base or its continued fraction expansion. Conversely, with an infinite word w on the finite alphabet $\{0, 1, \ldots, b-1\}$ one can associate the real number ξ_w whose base-b expansion is given by w. Many problems are then concerned with the following question: how the combinatorial properties of the word w and the Diophantine properties of the number ξ_w may be related?

In this talk, I will survey some of the recent advances on this topic. I will also try to point out new challenges.

A closely related talk was given by **Jason Bell** speaking *On automatic sequences*.

Logic and decision problems

Several workshop members work in the area of decision problems and logic, and the existence or non-existence of algorithms for solving problems in combinatorics on words. Of course, word problems go back to the beginning of the theory of algorithms, since the Post Correspondence Problem (PCP) is phrased in the laguage of strings, or alternatively, as a question related to morphisms. Some talks addressed issues of this flavour:

Jeffrey Shallit spoke on *On k-automatic sets of rational numbers*
Abstract: In this talk I will describe how automata can accept sets of rational numbers. Applications include deciding various questions about automatic sequences and the enumeration of various properties. There are many open questions.

Yuri Matiyasevich, famous for his solution of Hilbert's tenth problem, gave a brief lecture on *Arithmetization of words and related problems*

Complexity measures

As noted in our overview of the field, the work of Morse and Hedlund led to the study of factor complexities of various classes of words. This remains a major theme in combinatorics on words, with several open problems of long standing.

Videotaped lecture

A presentation by **Amy Glen** entitled *On a generalisation of trapezoidal words* was recorded:
Abstract: The *factor complexity function* $C_w(n)$ of a finite or infinite word w counts the number of distinct *factors* of w of length n for each $n \geq 0$. A finite word w of length $|w|$ is said to be *trapezoidal* if the graph of its factor complexity $C_w(n)$ as a function of n (for $0 \leq n \leq |w|$) is that of a regular trapezoid (or possibly an isosceles triangle); that is, $C_w(n)$ increases by 1 with each n on some interval of length r, then $C_w(n)$ is constant on some interval of length s, and finally $C_w(n)$ decreases by 1 with each n on an interval of the same length r. Necessarily $C_w(1) = 2$ (since there is one factor of length 0, namely the *empty word*), so any trapezoidal word is on a binary alphabet. Trapezoidal words were first introduced by A. de Luca (1999) when studying the behaviour of the factor complexity of *finite Sturmian words*, i.e., factors of infinite "cutting sequences", obtained by coding the sequence of cuts in an integer lattice over the positive quadrant of $\mathbf{R}^2$ made by a line of irrational slope. Every finite Sturmian word is trapezoidal, but not conversely. However, both families of words (trapezoidal and Sturmian) are special classes of so-called *rich words* – a wider class of finite and infinite words characterised by containing the maximal number of palindromes – recently introduced by the speaker, together with J. Justin, S. Widmer, and L.Q. Zamboni (2009).

In this talk, I will introduce a natural generalisation of trapezoidal words over an arbitrary finite alphabet A consisting of at least two distinct letters, called *generalised trapezoidal words* (or *GT-words* for short). In particular, I will discuss some combinatorial properties of this new class of words when $|A| \geq 3$ and I will show that, unlike in the binary case ($|A| = 2$), not all GT-words are rich in palindromes, but we do have a neat characterisation of those that are.

This is joint work with Florence Levé (Université de Picardie – Jules Verne, France).

Some other notable talks related this area

Words form basic objects in a variety of areas. Free groups, for example, are naturally viewed as sets of words, related by combinatorial rules. Within combinatorics on words, several now classical constructions comprise standard examples, notably the Thue-Morse word and the Sturmian words. However, open problems concerning these classes remain open. The following trio of talks by young researchers addressed various classical objects, with a unifying theme of complexity (of factors, or otherwise):

Thomas Stoll (Université d'Aix-Marseille) addressed *Thue-Morse at polynomial subsequences*
Abstract: Consider the Thue-Morse sequence (t_n) on the symbols $\{0, 1\}$. A celebrated result of Gelfond (1967/68) gives an asymptotic formula for the number of 0's and 1's in any fixed linear subsequence of TM, i.e., (t_{an+b}). In quite recent work, Mauduit and Rivat (2009) found precise formulas for quadratic polynomials. The cases of cubes and higher-degree polynomials remain elusive so far. From work of Dartyge and Tenenbaum (2006) it follows that there are $\gg N^{2/h!}$ symbols "0" (or "1") in any subsequence indexed by a polynomial of degree h. The aim of the present talk is to give an overview about the various results known in this area, pose some (old and new) conjectures, and – to improve by elementary/combinatorial means the general lower bound to $\gg N^{4/(3h+1)}$.

Eric Rowland spoke on *Counting equivalence classes of words in* F_2
Abstract: In the last decade several papers have appeared concerning the size of an equivalence class of words in a free group under its automorphism group. A central theme of the area is that information about the equivalence class of a word can be obtained from statistics of its (contiguous) subwords. Here we are interested in the free group on 2 generators. We give a new characterization of words of minimal length, and we introduce a natural operation that "grows" words from smaller words. The growth operation gives rise to a notion of maximally minimal words (in a way that can be made precise), which we call root words. Equivalence classes containing root words have special structure, and the hope is that understanding this structure will lead to an exact enumeration of equivalence classes in F_2 containing a minimal word of length n.

Julien Leroy gave a lecture entitled *The S-adic conjecture)*
Abstract: An infinite word is S-adic if it can be obtained by successive iterations of morphisms belonging to the set S. Sturmian words are well-known examples of S-adic words with card(S) $= 4$. The S-adic conjecture tries to determine the link that should exist between S-adicity and sub-linear factor complexity. More precisely, it says that there is a stronger notion of S-adicity which is equivalent to sub-linear factor complexity, i.e., an infinite word would have a sub-linear complexity if and only if it is "strongly S-adic".

In this talk, I will present some recent results about that conjecture. First I will present some examples that allow to reject some natural ideas that one could have. Then I will briefly explain a general method to compute an S-adic expansion of any uniformly recurrent infinite word with sub-linear complexity. That method allows to solve the conjecture in the particular case of uniformly recurrent infinite words with first difference of complexity bounded by 2.

Pattern avoidance

A classical topic in combinatorics on words is the avoidance of patterns such as repetitions (squares, cubes, overlaps) and images of words. Work of this sort contributed to the solution of the Burnside problem for groups. However, analogously to number theory, the study of pattern avoidance is a source of simply stated, yet apparently intractable problems. While repetitions involve the concatenation of identical factors, several researchers are now interested in generalizations where subsequent factors are equivalent up to order, or under various transformations. Several stimulating talsk on this theme were presented:

Juhani Karhumäki presented a lecture *Combinatorics on words and k-abelian equivalence*
Abstract: We define a new equivalence relation on words which is properly in between the equality and commutative (abelian) equality. We call it k-abelian equality. We say that two words are k-abelian equal if they contain each factor of length k equally many times, and in addition start with a common prefix of length k-1. This allows to have better and better approximations of problems based on equality of words. We report basic properties of these equivalence relations, and analyze k-abelian variants of some well known problems on words. In particular we consider problems dealing local and global regularities in infinite words, as well as the existence of certain type of repetition-free words. It turns out that there exist a lot of challenging open problems in this area.

Antonio Restivo gave a talk showing the state of the art in an area related to the critical factorization theorem of Fine and Wilf in his lecture *Sturmian Words and Critical Factorization Theorem*
Abstract: We prove that characteristic Sturmian words are extremal for the Critical Factorization Theorem (CFT) in the following sense. If $px(n)$ denotes the local period of an infinite word x at point n, we prove that

x is a characteristic Sturmian word if and only if $px(n)$ is smaller than or equal to $n+1$ for all $n \geq 1$ and it is equal to $n+1$ for infinitely many integers n.

This result is extremal with respect to the CFT since a consequence of the CFT is that, for any infinite recurrent word x, either the function $px(n)$ is bounded, and in such a case x is periodic, or $px(n) \geq n+1$ for infinitely many integers n. As a byproduct of the techniques used in the paper we extend a result of of Harju and Nowotka stating that any finite Fibonacci word f_n for $n \geq 5$ has only one critical point. Indeed we determine the exact number of critical points in any finite standard Sturmian word.

Young researcher **Robert Mercas** presented a different perspective on results of Fine and Wilf: *On Pseudo-repetitions in words*

Abstract: The notion of repetition of factors in words was studied already from the beginnings of the combinatorics on words area. One of the recent generalizations regarding this concept was introduced by L. Kari et al., and considers a word to be an f-repetition if it is the iterated concatenation of one of its prefixes and the image of this prefix through an anti-/morphic involution f. In this paper, we extend the notion of f-repetitions to arbitrary anti-/morphisms, and investigate a series of algorithmic problems arising in this context. Further, we present a series of results in the fashion of the Fine and Wilf theorem for f-repetitions, when f is an iso(anti)morphism.

Several other speakers contributed to this topic:

Dirk Nowotka spoke on *Avoidability under Permutations*; **Arturo Carpi** (Universita Degli Studi Di Perugia) gave a lecture on *Unrepetitive walks in digraphs (and the repetition threshold)*; **Sébastien Ferenczi** presented *Word combinatorics of interval-exchange transformations for every permutation***Anna Frid** gave a brief problem involving *Morphisms on Permutations*

Word equations

Several speakers addressed the state of decision procedures for word equations.

Štěpán Holub spoke on *Length types of word equations*

Abstract: One of the 'rules of thumb' for solving a word equation is to guess the length vector of the solution and then construct the corresponding equivalence on the positions. The talk will comment on this method and indicate some less trivial implications.

Luca Zamboni presented work tying combinatorics on words to ultrafilter constructions in algebra: *Partition regularity and words*

Abstract: Van der Waerden's theorem states that given a finite partition of the natural numbers N, at least one element of the partition contains arbitrarily long arithmetic progressions (i.e., is a.p.-rich). In fact, in the hypothesis N can be replaced by any subset X of N which is itself a.p.-rich. In other words, the property of being a.p.-rich is partition regular, i.e., cannot be destroyed under finite partition. However given a partition regularity property P (e.g., being a.p.-rich) and a subset X of N (e.g., X =set of all prime numbers), the question of determining whether X has property P is often quite difficult (even if either X or its complement must have property P). In the context of combinatorics on words it is natural to consider X to be the set of all occurrences of a given factor in some nice infinite word (i.e., the set of all occurrences of 0 in the Thue-Morse word, or the set of all occurrences of 0100 in the Fibonacci word). In this talk we will consider different partition regularity properties in the context of words. It turns out that many well known words (including Sturmian words and words generated by substitution rules) provide a rich setting for studying different partition regularity properties. And conversely, some deep dynamical and or arithmetical questions concerning these words may be reformulated in terms of partition regularity. One such example is the strong coincidence condition (SCC) conjectured for irreducible primitive Pisot substitutions. We will describe an equivalent reformulation of SCC whose difficulty lies in the arithmetic properties of the associated Thomas-Dumont numeration systems. This is based on joint work with M. Bucci and S. Puzynina.

Volker Diekert , in his talk *Local Divisors*, highlighted the relevance of combinatorics on words to group-theoretic questions.

Abstract: In my talk I will speak about several new results and simpler proofs of known results in formal language theory using the concept of a local divisor. The results encompass recent joint work with Manfred Kufleitner and Benjamin Steinberg on Krohn-Rhodes Theorem.

Other talks

Researchers at the workshop were extremely generous in volunteering talks. We also heard presentations by

- **Nicolas Bédaride** *Piecewise isometries and words*
- **Srecko Brlek** *The last talk on the Kolakoski sequence*
- **Svetlana Puzynina** *Locally catenative sequences and Turtle graphics*

Open problems

A session on open problems identified several questions:

1. If u, v, w are primitive words, define $p(u, v, w)$ to be the integer k such that
$$(u^* \text{ shuffle } v^*) \cap w^* = (w^k)^*.$$
Then define $P(u, v) = \{p(u, v, w) : w \text{ primitive }\}$. Characterize the u, v such that $P(u, v) = \{0, 1\}$.
2. If x is a prefix of infinitely many square-free words over $\{1, 2, 3\}$ and y is a suffix of infinitely many square-free words over $\{1, 2, 3\}$, must there exist some squarefree word xuy over $\{1, 2, 3\}$?
3. Does the paperfolding word contain arbitrarily large Abelian powers?
4. Can the iterated hairpin completion of a singleton w (a) be regular (b) be context-free but not regular?
5. Improve the bounds on run lengths and sums of exponents of runs:
$$\begin{array}{rcl} 0.9445757|x| \leq & \text{runs}(x) & \leq 1.029|x| \\ 2.035|x| \leq & \text{sum of exponents of runs(x)} & \leq 4.1|x| \\ 0.406|x| \leq & \text{3-runs}(x) & \leq 0.5|x|. \end{array}$$
6. Are 2-Abelian cubes 2-avoidable?

Scientific Progress Made and Outcome of the Meeting

Several researchers have reported papers and research progress related to this one week workshop. In particular:

- The paper S. Ferenczi *The self-dual induction for every interval exchange transformation* (in preparation) was improved through several discussions in Banff.
- The paper S.Ferenczi, L. Zamboni *Clustering words* (http://arxiv.org/abs/1204.1541) was started during the (homeward) train trip from Banff and Montreal.
- Steffen Kopecki and Volker Diekert worked at BIRS on the combinatorics of hairpin completions. These combinatorial problems about words are inspired by biochemical processes in DNA computing where hairpin formations arise naturally. During the workshop they sharpened some of their results which became part of the revised version of their paper

  - Volker Diekert, Steffen Kopecki, Victor Mitrana *Deciding Regularity of Hairpin Completions of Regular Languages in Polynomial Time*.

  The paper was just accepted for publication in *Information and Computation* on April 17, 2012 and will contain an acknowledgement to the inspiring workshop at BIRS.

- The recent BIRS workshop was a great opportunity for Boris Adamczewski to advance his collaboration with Jason Bell. In particular, it allowed them to
  - (Almost) Finish the writing of a joint paper on diagonals of multivariate algebraic functions.
  - Discuss a current project on Mahler's functions
  - Discuss various new questions. They also restarted a joint project with Berthé and Zamboni.
- In addition Bell was able to
  - Work on characterization of subsets of $\mathbb{R}$ accepted by Buchi automata with respect to two multiplicatively independent bases (with Julien Leroy and Emilie Charlier.
  - Begin work on various questions of Jeff Shallit on k-automatic sequences with specific properties.
- Štěpán Holub solved the workshop open problem on the paperfolding word, shortly after his return from BIRS.
- Thomas Stoll also participated in discussion regarding the paperfolding word, and also
  - Answered a question of Shallit asking whether it is possible to give a bound for $\min(n : t_{k_1 n} = e_1, t_{k_2 n} = e_2)$ where t_n is the Thue-Morse sequence and k_1, k_2 are arbitrary distinct integers and e_1, e_2 all of the four possibilities. He got the result during the BIRS workshop and they are now in progress toward finding the natural generalized result (with two students, one in Marseille and one from Stanford).
  - Also with Shallit, Stoll found during BIRS the proof of a conjecture of Eric Rowland that the 2-kernel oft_{n+l} is of size $f(l)$ where $f(l)$ satisfies some nice explicit recursive relations and is k-regular.
  - Researchers from France (Stoll, Cassaigne, Ochem) were also inspired to think about the (difficult) problem of avoiding sumsquares in words over finite alphabets.

Researchers were extremely enthusiastic about the workshop, and the wonderful atmosphere at BIRS. Thanks to BIRS for putting this resource at the disposal of the international mathematical community.

Participants

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Bibliography

[1] S. Adjan & P.S. Novikov, Infinite periodic groups I–III. (Russian) *I*zv. Akad. Nauk SSSR Ser. Mat. **3**2 (1968).

[2] Jean-Paul Allouche and Jeffrey Shallit, *Automatic Sequences: Theory, Applications, Generalizations.* Cambridge University Press, 2003.

[3] Jean Berstel, *A*xel Thue's papers on repetitions in words: a translation, Publications du LCIM **2**0, Université du Quebec à Montréal (1994)

[4] A. Carpi, On Dejean's conjecture over large alphabets, Theoret. Comput. Sci. 385 (2007) 137–151.

[5] J. Cassaigne, J. Currie, L. Schaeffer, J. Shallit, Avoiding Three Consecutive Blocks of the Same Size and Same Sum, arXiv:1106.5204v3

[6] *R*egular Algebra and Finite Machines, Chapman Hall, 1971.

[7] J. D. Currie, N. Rampersad, *A* proof of Dejean's conjecture, Math. Comp. **8**0 (2011), 1063–1070.

[8] F. Dejean, Sur un théorème de Thue, J. Combin. Theory Ser. A 13 (1972) 90–99.

[9] N. J. Fine and H. S. Wilf, Uniqueness theorem for periodic functions. *P*roc. Amer. Math. Soc., /textbf 16 (1965) 109–114.

[10] T. Harju and D. Nowotka, Periodicity and unbordered words: A proof of the extended Duval conjecture, *J*. ACM **5**4(4) (2007).

[11] S. Holub, A proof of the extended Duval conjecture, *T*heor. Comput. Sci., **3**39 (2005) 61–67.

[12] M. Kunc, Regular solutions of language inequalities and well quasi-orders, *P*roc. of ICALP 2004, 870–888, 2004.

[13] M. Lothaire, *C*ombinatorics on Words, Encyclopedia of Mathematics and its Applications **1**7, Addison-Wesley, Reading (1983).

[14] M. Lothaire, *A*lgebraic Combinatorics on Words, Cambridge University Press, Cambridge (2002).

[15] M. Lothaire, *A*pplied Combinatorics on Words, Cambridge University Press, Cambridge (2005).

[16] G. S. Makanin, The problem of solvability of equations in a free semigroup, *M*at. Sb. (N.S.) **1**03(145) (1977) 147–236

[17] M. Morse, Recurrent geodesics on a surface of negative curvature, *T*rans. AMS **2**2 (1921), 84–100.

[18] W. Plandowski, Satisfiability of word equations with constants is in PSPACE, *J*. ACM **5**1(3) 483–496 (2004).

[19] M. Rao, Last Cases of Dejean's Conjecture, *T*heor. Comput. Sci. **4**12 (2011), 3010–3018.

[20] M. P. Schutzenberger, Une theorie algebrique du codage, *S*éminaire Dubreil-Pisot, 1955–1956, Exposé No. 15.

[21] A. I. Shirshov, *B*ases of free Lie algebras, Algebra i Logika **1**, 1962, 14–19.

[22] Axel Thue, Über unendliche Zeichenreihen, *N*orske Vid. Selsk. Skr. I. Mat. Nat. Kl. Christiana No. 7 (1906).

[23] A. Thue, Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen, Norske Vid. Selsk. Skr. I. Mat. Nat. Kl. Christiana 1 (1912) 1–67.

[24] B. L. van der Waerden, Beweis einer Baudetschen Vermutung, *N*ieuw. Arch. Wisk. **1**5 (1927) 212–216.

Chapter 10

Operator structures in quantum information theory (12w5084)

February 26 - March 2, 2012

Organizer(s): Mary Beth Ruskai (Tufts University); Marius Junge (Univeristy of Illinois); David Kribs (University of Guelph); Patrick Hayden (McGill University); Andreas Winter (University of Bristol)

Overview of the Field

Physicists have long recognized that the appropriate framework for quantum theory is that of a Hilbert space $\mathcal{H}$ and in the simplest case the algebra of observables is contained in $\mathcal{B}(\mathcal{H})$. This motivated von Neumann to develop the more general framework of operator algebras and, in particular C^*- algebras and W^*-algebras (with the latter also known as von Neumann algebras). In the 1970's these were used extensively in the study of quantum statistical mechanics and quantum field theory.

Although at the most basic level, quantum information theory (QIT) is expressed using matrix algebras, interactions with the environment play a critical role. This requires the study of open quantum systems in which the effects of noise can be studied. Operator spaces are used implicitly starting with the identification of completely positive (CP) trace-preserving maps (CPT maps) with quantum channels. During the past five years, the role of operator spaces has played an increasingly important and more explicit role.

In many situations, even when the underlying Hilbert space is finite dimensional, many interesting questions, particularly those involving channel capacity require asymptotic limits for infinitely many uses of the channel. In many situations, generic properties are important. Random matrices, free probability, and high dimensional convex analysis have all played an important role.

The first workshop on operator structures held at BIRS in February 2007, brought together experts in these areas and people working in quantum information theory. This was followed by a second workshop at the Fields Institute in July, 2010. The use of operator structures is now an integral component of work in QIT, as demonstrated in the presentations at this workshop.

Some of the major open questions in 2007, particularly the so-called "additivity conjecture" have now been solved, in part due to activity begun during that workshop. Nevertheless, many open questions remain and discussions of these questions were as important a component of this workshop as the presentation of recent developments. In view of the diverse interdisciplinary nature of the group, some expository talks were included and all speakers endeavored to make their talks accessible to this diverse group.

Major Topics of Discussion

Operator systems

In operator spaces, the role of norms plays a fundamental role. Recently, it has been recognized that structures in which order plays the central role are also important. These are known as operator systems. The workshop began with an expository talk by Vern Paulsen showing how this viewpoint provides a new perspective on such familiar topics as separable and entangled states using tensor products of operator spaces.

Graphs have long been used in QIT in a variety of ways, beginning with the concept of stabilizer states, used in quantum error correction and in one-way quantum computing. Winter's talk described recent work on "non-commutative graphs" which a quantum generalization of the Lovasz theta functions has important implications for the zero-error capacity problem. This can be framed in terms of operator systems and Hilbert modules.

Bell inequalities and quantum XOR games

Bell inequalities have long played a fundamental tool in distinguishing quantum correlations from classical ones. The ground-breaking paper [24] used techniques from operator spaces and tensor norms to resolve a long-standing open question in physics by showing that tripartite systems could have unbounded violations of Bell inequalities. Their work raised a number of open questions, including one about the form of the states which achieved these violations and showed that it could be framed as equivalent to a 30 year-old open question about Banach algebras. This was subsequently resolved by Briet et al [5]. This work also recognized that Bell inequalities could be viewed as a quantum analogue of XOR games in computer science. A series of talks on Thursday described the developments in this area, demonstrating the interplay between different viewpoints building on each other to provide striking progress on a series of complex questions.

The session about "Bell inequalities and functional analysis techniques" was divided into four talks. The first one, given by David Perez-García, was an introduction to the topic. He spent most of the time motivating the interest of studying Bell inequalities, or equivalently multi prover one round games. He did it from a historical point of view, starting from the independent discovery of these objects in the foundations of quantum mechanics (Bell, 60's) and in the study of inapproximability results in computational complexity (Arora, Raz, 90's). He then moved into recent applications of this concept in a Quantum Information context: quantum key distribution, certifiable random number generation and position based quantum cryptography. In the last part of the talk, he sketched why tensor norms and operator spaces are the natural mathematical framework to deal with Bell inequalities.

The second talk, given by Carlos Palazuelos, focused on the recent use of operator spaces to analyze quantum multi prover one round games. He showed how, using a non-commutative version of Grothendieck's theorem, one can obtain interesting results for a particular type of these games (called rank-one games): first an efficient algorithm to approximate the value of this games to a constant precision; second, the lack of a perfect parallel repetition result in this context.

The third talk was given by Jop Briet. He presented a new proof (in collaboration with T. Vidick) of a result by Perez-Garcia, Wolf, Palazuelos, Villanueva and Junge (2008) showing that there can exist unbounded violations to tripartite correlation Bell inequalities. The techniques were based on tensor norms and random estimates, as opposed to the operator space flavor of the original proof. With their new tools, they were able to improve the old result in all possible directions (smallest number of all parameters appearing in the problem), and even show the (almost) optimality of their result.

The fourth talk was given by Tobias Fritz. He presented a connection between some Bell inequality problems and some objects in operator algebra theory. This allowed him to reprove some known results (i.e about the semidefinite hierarchy of Navascues, Pironio and Acin) and to show a connection between Tsirelson problem and Connes embedding problem.

The participants liked a lot the way this session was developed and the smooth transition into talks, which helped non-experts in the area participate actively also in the lively discussions that took place during this session.

Reformulation of major conjectures in operator algebras

A major open question in operator algebras is known as the "Connes embedding problem" raised by Alain Connes [11] in the mid 70's. This problem has been extensively studied by operator algebraists, leading to a number of reformulations of the problem such as that of Kirchberg [26]. Recent work in quantum information theory [19] has connected it to other conjectures, including one known as Tsirelson's problem. A resolution of this problem would have deep implications in several areas of mathematics as discussed in the reviews [8, 22]

Because the original problem is somewhat technical, we focus on Tsirelson's question. Roughly speaking, this asks whether two commuting algebras of observables can always be represented as the tensor product of two algebras of observables. Because this question is closely connected to issues about non-locality which play a major role in QIT there has been a burst of interest in the Connes embedding problem from an entirely new direction, and new conjectures which have implications for this collection of problems. Some of these were discussed in the talk by Tobias Fritz.

A new and different reformulation of the problem was presented by Haagerup and Musat in their work on factorizable maps. Although this work originated in operator algebras, it has important implications for QIT. In particular, they showed that a question about unital quantum channels known as the asymptotic quantum Birkhoff conjecture (raised as an open question in the 2007 BIRS workshop) is false, by showing that it is false for any non-factorizable channel. They also described several new classes of unital quantum channels. And they resolved another long-standing open question by exhibiting a class of channels that are extreme points of the convex set of unital quantum channels which are not extreme points of either the set of unital CP maps or the set of trace-preserving CP maps.

Paulsen's talk concluded with a reformulation of the Connes embedding problem in the language of operator systems. He also gave a presentation in the open problems session with implications for this question.

Developments in LOCC

The set of protocols refereed to as LOCC (local operations and classical communication) play a major role in QIT and are usually described in terms of a physical process. A "local operation" is easily described as the tensor product of trace-decreasing CP maps on a tensor product space, for which it suffices to consider operations in which one term is the identity within a larger protocol. However, a proper description of "classical communication" is rarely given; it involves a pair of operations which can be written in the simplest case as

- a measurement is performed using a formal object called an "instrument" which records the outcome in a classical algebra

$$(\mathfrak{A}\otimes\mathcal{I})(\rho^{AB}) = \sum_k |\phi_k^A\rangle\langle\phi_k^A| \otimes |k\rangle\langle k| \otimes \rho_k^B = \bigoplus_k |\phi_k^A\rangle\langle\phi_k^A| \otimes \rho_k^B$$

- an operation $\mathfrak{B} = \bigoplus_k \mathfrak{B}_k$ conditioned on the classical algebra so that

$$(\mathcal{I}\otimes\mathfrak{B})\circ(\mathfrak{A}\otimes\mathcal{I})(\rho^{AB}) = \bigoplus_k |\phi_k^A\rangle\langle\phi_k^A| \otimes \mathfrak{B}_k(\rho_k^B)$$

By contrast a separable operation is merely a sum of local operations

$$\sum_k \mathfrak{A}_k \otimes \mathfrak{B}_k$$

Although it has long been known that there are separable operations which are not LOCC, it is not at all trivial to provide examples of separable operations and show that they can not be rewritten as LOCC in some way.

Since orthogonal product states can always be distinguished by separable operations, the existence of an orthonormal basis of product states which can not be distinguished by LOCC leads to a special class of separable maps which do not come from LOCC operations. The first such example was given in [3]. Mancinska and Oozols present recent work [9] which gave a new and simpler proof of this result and resolved an open question raised in [3]. Their approach also leads to a new measure of non-locality.

Subsequently, they joined Chitambar and Winter [10] in a very nice clarification of the LOCC issues in precise mathematical terms, as well as a precise formulation of the use of a sequence of LOCC operations on multiple copies of a state in a process known as distillation. We anticipate that this will make the field more accessible to people working in operator algebras by enabling major questions to be stated in mathematical as well as physical language.

One session was devoted to an overview and discussion of the long-standing open question of whether NPT bound entanglement exists. If a bipartite state is separable, it is known the the partial transpose must be positive semi-definite, known as the PPT condition. However, it is also known that for $d > 2$ this is only a necessary, but not a sufficient condition for separability. And it is not possible to distill entanglement from entangled states which have this so-called PPT property. Such states are said to have bound entanglement. It is a long-standing open question as to whether or not states which are not PPT (called NPT) can always be distilled to a maximally entangled state. This question can be reformulated as one about positive maps on operator algebras. Thus, there was intense interest in this topic. The session began with an overview of the problem and what is known about it by Michael Horodecki, including some reformulations and related questions. John Watrous then explained [1] why there are states from which entanglement can be distilled from n copies of a state, although none can be distilled using fewer copies. This result demonstrates the difficulty of the analysis which would to resolve this question. The participants then engaged in a lively discussion about the problem until dinner.

Channels and capacity

Quantum Shannon theory is much richer than its classical counterpart because of the many different ways in which quantum systems can be used to transmit both classical and quantum information. Graeme Smith gave an overview of this subject. He explained how there were even situations in which the tensor product of a pair of channels with zero capacity could have non-zero capacity.

Because the capacity for transmission of quantum information can rarely be calculated exactly, obtaining good bounds is extremely important. Marius Junge explained how the completely bounded entropy can be used to obtain both upper and lower bounds on the capacity of certain types of channels.

Collins review recent results related to the study of the behavior of typical quantum channels, and techniques to study them, including free probability and Weingarten calculus. He described quantities related to additivity problems and discussed two different models of randomness: channels defined via Haar isometries and random unitary channels with i.i.d. Haar unitary operators.

In his thesis Renner introduced the concept of max- and min-entropy which proved to be very useful tools in quantum crytopgraphy. These quantities satisfy the usual properties of entropy and one can recover the von Neumann entropy as a limit; however, they are not smooth. Subsequently smoothed versions were introduced [26] which proved extremely useful, not only in cryptography, but in the study of channel capacity. However, all of these notions were defined only on matrix algebras.

Scholz gave an overview of these concepts and the issues involved in extending them to the more general setting of von Neumann algebras. Jencova in her talk discussed quantum channels defined between general finite dimensional C^*-algebras, and a related notion called "quantum combs" that has been utilized lately in the physics literature to describe quantum networks. Hayden outlined a conjecture of his on the approximation of operators on tensor product spaces, and the intersection of operator subspaces in particular, motivated by problems in multiuser quantum information theory.

Computational complexity

QIT arose from the realization that a quantum computer could solve certain types of problems more efficiently than classical computers. There is a large body of work discussing the various types of problems on which a quantum speed-up can be attained. This has led to new complexity classes for the different types of problems a quantum computer can solve.

The classical PCP theorem in computer science says, roughly speaking, that it is hard (in fact, NP-hard) to find approximate solutions to certain NP-complete problems such as SAT, even given a fairly weak sense of approximation. Matt Hastings explained this result and discussed the quantum PCP conjecture, which conjectures that approximation of certain quantum problems is even harder (so-called QMA-hard). He described a possible program for attacking the quantum PCP conjecture. This program involves showing results that forbid topological order in systems with certain interaction graphs, and it leads to some interesting conjectures in topology and C*-algebras.

Many optimization problems over polynomials are NP-hard to solve exactly, but can be approximated using methods such as the semi-definite programming (SDP) hierarchies of Lasserre, Parrilo, and others. Aram Harrow explained how explain how quantum techniques tools from QIT yield hardness results, as well as ways to prove the effectiveness of SDP hierarchies based on work in [16, 4].

Other

Other talks included Szkola's presentation on the construction of positive operator valued measures for detecting a true quantum state among a finite number of hypothetic ones. Her algorithm can be viewed as a quantum generalization of the maximum likelihood method to this setting, in that it recovers the classical rule in the special case of commuting states. Eisert discussed analysis of the timing of dissipative quantum processes that can be sometimes used to protect quantum information from noise. And Gross in his talk presented an analysis of the quantum time evolutions that can be realized in one-dimensional quantum lattice systems, showing there is a single invariant that takes a value in an abelian group which solves three basic classification problems for such systems.

Open Problems

The open problems session was chaired by Ruskai who began with a brief recap of problems from the 2007 meeting noting which had been solved and which remained open.

The following problems were presented. Some of the presenters wrote up the details, which are included in the Appendix below.

1. L. Macinskia and M. Ozols defined a measure of non-locality for an orthogonal set of vectors on a bipartite space, and explained its significance for state discrimination. Since the parameter is difficult to compute explicitly in most situations, a method for finding good bonds is an important open problem.

2. Vern Paulsen described a generalization of numerical range called the "j-th matrix range" of a matrix. It is an open question whether or not an associated limit as $j \to \infty$ is 0.

   The question is important because it is equivalent to a long-standing open question about operator systems known as the Smith-Ward problem. It is rather surprising that there are equivalent formulations which reduce to questions about systems of dimension 2 or dimensions 3. Furthermore, these reformulations are conjectured to have implications for both the Connes embedding problem and Tsirelson's problem.

3. Andreas Winter raised questions about the Petz condition for sufficiency and equality in strong subadditivity. In particular, he asked for conditions under which one could obtain precise bounds for the decrease in monotonicity of the relative entropy under CPT maps. This has a number of important applications, including improving the bounds on the squashed entanglement described in Christandl's presentation on the work in [7]. Subsequently, this problem was solved by Li and Winter.

4. S. Szarek raise a question with arises in recent work [2] about the size of ancilla dimensions which lead to separable states. Related results about the ancilla dimensions for PPT states and class of states defined by another necessary condition for separability have recently appeared [**?**].

5. Aram Harrow described a question about the trace distance of certain recently defined families of states from the separable states.

6. David Perez-Garcia pointed out that recently proved existence theorems about unbounded violations of Bell inequalities do not give explicit bounds on the dimension required to obtain violations of a given size. Finding such bounds would have many important implications as indicated in Section **??**sect:Bell above.

7. Matthias Christandl raised the following question. A channel can be called PPT if its Choi-Jamiolkowski state is PPT. It is well-known that this is a necessary, but not sufficient condition for a channel to be entanglement breaking. Is the composition of two PPT channels entanglement breaking?

8. Arleta Szokla raised the following question during her talk. The quantum extension of the Chernoff distance as well as its operational meaning within quantum binary hypothesis testing were provided a few years ago. These results were presented at the BIRS workshop in 2007 (07w5119). In the meantime, a quantum analogue of the multiple Chernoff distance has been considered as well. It is conjectured, that similarly to the binary special case, it represents an achievable bound on the error exponent in quantum multiple hypothesis testing. The conjecture has been proven for large classes of finite sets of quantum states (quantum hypotheses) giving a strong evidence in favor of the conjecture. However, a general proof is still missing. In contrast to the classical multiple hypothesis testing as well as the quantum binary special case, in multiple quantum hypothesis testing asymptotically optimal tests are not known explicitely. From mathematical point of view, quantum tests are represented by positive operator valued measures (POVM) corresponding to decompositions of identity into positive elements of a given *-algebra. The recently introduced quantum maximum likelihood type tests turn out to be asymptotically optimal under some technical restrictions on the set of hypothetic density operators. They were presented at this workshop and the potential of the corresponding construction algorithm to provide a general solution was been discussed as one of the open problems.

9. The Connes embedding conjecture, Tsirelson's problem and several reformulations and related questions were discussed in Section 10 above.

10. The NPT problem was discussed in Section 10 above.

11. The quantum PCP conjecture was the subject of Matt Hastings talk

In addition to these specific problems, many speakers included open questions in their talks.

L. Macinskia and M. Ozols: A Problem on State Discrimination with LOCC

Consider a set of orthonormal vectors $S = \{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ in a bipartite vector space $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$.

Definition For any $a \in \mathrm{Pos}(\mathbb{C}^{d_A})$ and $b \in \mathrm{Pos}(\mathbb{C}^{d_B})$, let $G_{ij} := \langle\psi_i|(a \otimes b)|\psi_j\rangle$ where $i, j \in \{1, \ldots, n\}$. We say that $\eta > 0$ satisfies *the nonlocality constraint for S* if

$$\eta \cdot \left(\frac{\max_k G_{kk}}{\sum_{j=1}^{n} G_{jj}} - \frac{1}{n} \right) \leq \max_{i \neq j} \frac{|G_{ij}|}{\sqrt{G_{ii} G_{jj}}} \tag{10.1}$$

holds for all $a \in \mathrm{Pos}(\mathbb{C}^{d_A})$ and $b \in \mathrm{Pos}(\mathbb{C}^{d_B})$ such that $G_{ii} > 0$ for all $i \in \{1, \ldots, n\}$.

The above definition is significant due to the following theorem [9]

Theorem 10.0.1. *Let $\eta > 0$ be any constant that satisfies the nonlocality constraint for S. Then any quantum protocol aiming to perfectly discriminate the states from S using only local quantum operations and classical communication (LOCC) errs with probability*

$$p_{\mathrm{error}} \geq \frac{2}{27} \frac{\eta^2}{n^5}. \tag{10.2}$$

Problem 10.0.2. *For which sets of orthogonal vectors $S \subset \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ can one explicitly find an $\eta > 0$ satisfying the nonlocality constraint (particularly interesting is the case when S consists of orthogonal* product *vectors)? Only very few examples are known where an explicit value of $\eta > 0$ satisfying the nonlocality constraint has been found. Hence, besides a complete characterization even any new examples would be very useful. Especially useful are examples with large values of $\eta > 0$, since those translate into stronger lower bounds on the error probability (see the above theorem). Also it would be helpful to come up with a* general *approach for finding $\eta > 0$ satisfying the nonlocality constraint.*

Examples

1. *Standard basis.* When S is standard basis, no $\eta > 0$ satisfies the nonlocality constraint, since we can always choose matrices a, b to be trace one and diagonal with distinct diagonal elements. For such a and b, we have $|G_{ij}| = 0$ for all i, j but $\max_k G_{kk} > \frac{1}{n}$ and hence Equation (10.1) is not satisfied for any $\eta > 0$.

2. *Domino states.* Consider the following orthonormal product basis $S \subset \mathbb{C}^3 \otimes \mathbb{C}^3$, first introduced in [**?**]:

$$|\psi_1\rangle = |1\rangle|1\rangle, \tag{10.3}$$
$$|\psi_{2,3}\rangle = |0\rangle|0 \pm 1\rangle, \tag{10.4}$$
$$|\psi_{4,5}\rangle = |2\rangle|1 \pm 2\rangle, \tag{10.5}$$
$$|\psi_{6,7}\rangle = |1 \pm 2\rangle|0\rangle, \tag{10.6}$$
$$|\psi_{8,9}\rangle = |0 \pm 1\rangle|2\rangle, \tag{10.7}$$

where $|i \pm j\rangle := (|i\rangle \pm |j\rangle)/\sqrt{2}$. In [9] it has been shown that $\eta = \frac{1}{8}$ satisfies the nonlocality constraint for S. In fact when $S \subset \mathbb{C}^3 \otimes \mathbb{C}^3$ is a set of orthogonal product states, a complete characterization of when there exists $\eta > 0$ satisfying the nonlocality constraint is known [13]. For these cases an explicit value of η that satisfies the nonlocality constraint has been given in citeCLMO .

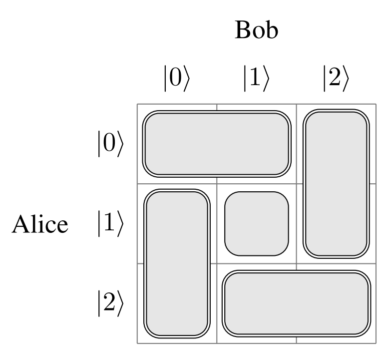

Figure 10.1: Graphical representation of the domino states.

V. Paulsen: j-th numerical range

Statement of the problem

I will first state the problem and then discuss its ramifications.

Given an element T of a C*-algebra $\mathcal{A}$ the **j-th matrix range of T** is defined as the following set of $j \times j$ matrices:

$$W^j(T) = \{\Phi(T) : \Phi : \mathcal{A} \to M_j \text{ unital, completely positive } \}.$$

When $T \in M_n$ then by Stinespring's theorem, one sees that $W^j(T)$ consists of all $j \times j$ matrices that one can get by compressing the matrix $T \otimes I_E$ to a j-dimensional subspace.

The problem that we are interested in is:

How much does knowledge of $W^j(T)$ determine T or $W^n(T)$ for $n >> j$?

Precisely, let

$$W^{n,j}(T) = \{\sum_{k=1}^{K} A_k^* X_k A_k : X_k \in W^j(T), A_k \in M_{j,n}, \sum_{k=1}^{K} A_k^* A_k = I_n, \exists K\},$$

which is in some sense the $n \times n$ matrices that can be constructed as images under unital completely positive maps of elements of $W^j(T)$.

In fact $W^{n,j}(T)$ is the intersection of the sets $W^n(R)$ over all operators R, with $W^j(R) = W^j(T)$ and there is an operator R with $W^j(R) = W^j(T)$ such that $W^n(R) = W^{n,j}(T)$.

Now let

$$\alpha_{n,j} = \sup\{d(T, W^{n,j}(T)) : T \in M_n, \|T\| = 1\},$$

where $d(T, W^{n,j}(T)) = \inf\{\|T - Y\| : Y \in W^{n,j}(T)\}$ is the distance from T to the set $W^{n,j}(T)$. Thus, $\alpha_{n,j}$ is a measure of the maximum distance that an $n \times n$ matrix can be from $W^{n,j}(T)$ and gives us a measure of how little information about T is contained in $W^j(T)$.

Problem 10.0.3. *Is* $\lim_{j\to+\infty}[\sup\{\alpha_{n,j} : n \geq j\}] = 0$*?*

Consequences

It seems very unlikely that this limit could be zero, but this is the only obstruction to settling a 30+ year old problem. On the other hand, requiring this limit to be zero is, essentially, a statement about, asymptotically, how well knowing all finite dimensional compressions of an operator gives knowledge of the operator. In that sense it fits quite well with some information theoretic questions.

The problem that determining the above limit would settle was motivated by the desire to generalize a result of Weyl.

Given a separable, infinite dimensional Hilbert space $\mathcal{H}$, let $B(\mathcal{H})$ denote the bounded operators on $\mathcal{H}$ and let $K(\mathcal{H})$ denote the compact operators on $\mathcal{H}$. Since the compact operators are an ideal there is also a quotient algebra, $B(\mathcal{H})/K(\mathcal{H}) = Q(\mathcal{H})$ often called the Calkin algebra. We let $\pi : B(\mathcal{H}) \to Q(\mathcal{H})$ denote the quotient map. When one talks about the **essential spectrum of an operator** T**,** they really mean the spectrum of $\pi(T)$.

Similarly, by the **essential j-th matrix range of T** we mean $W^j(\pi(T))$ the j-th matrix range of $\pi(T)$.

Weyl proved that given any $T \in B(\mathcal{H})$ there exists a compact operator $K \in K(\mathcal{H})$ such that $W^1(T + K) = W^1(\pi(T))$.

Smith-Ward extended this result by proving that given any j there is $K_j \in K(\mathcal{H})$ such that

$$W^k(T + K_j) = W^k(\pi(T)), 1 \leq k \leq j.$$

The Smith-Ward problem asks: Does there always exists $K \in K(\mathcal{H})$ such that $W^J(T+K) = W^j(\pi(T))$ holds for all j?

The answer is yes for many operators T but is unknown in general.

Here is the connection between the Smith-Ward problem, the above problem and other questions.

First, given $T \in B(\mathcal{H})$, let $\mathcal{S}^{n,j}(T) = \{A \in M_n : W^j(A) \subseteq W^j(T)\}$. It turns out that $\mathcal{S}^{n,j}(T) = \cup\{W^n(R) : W^j(R) = W^j(T)\}$ where the union is over all such operators R and in fact, there is always an operator R such that $W^j(R) = W^j(T)$ and $\mathcal{S}^{n,j}(T) = W^n(R)$.

Theorem 10.0.4. *The following are equivalent:*

1. *The Smith-Ward problem has an affirmative answer,*
2. $\lim_{j\to+\infty} \sup\{\alpha_{n,j} : n \geq j\} = 0$,
3. *for every* $T \in B(\mathcal{H})$, $\lim_{j\to+\infty}[\sup\{d(\mathcal{S}^{n,j}(T), W^n(T)) : n \geq j\}] = 0$,
4. *for every 3 dimensional operator system* $\mathcal{S}$ *and every UCP map* $\phi : \mathcal{S} \to Q(\mathcal{H})$ *there exists a UCP map* $\psi : \mathcal{S} \to B(\mathcal{H})$ *such that* $\pi \circ \psi = \phi$,
5. *for every 3 dimensional operator system* $\mathcal{S}$*, for every C*-algebra* $\mathcal{A}$*, for every two-sided ideal* $J \subseteq \mathcal{A}$*, for every UCP map* $\phi : \mathcal{S} \to \mathcal{A}/J$ *there exists a UCP map* $\psi : \mathcal{S} \to \mathcal{A}$ *such that* $\pi \circ \psi = \phi$.

Statements (4) and (5) are known to be true for every *two* dimensional operator system and there are *four* dimensional operator systems for which (4) and (5) are known to be false. So the Smith-Ward problem is the only remaining dimension.

Similarly, in the theory of operator spaces, there are lifting properties known to hold for all one dimensional operator spaces, to fail for three dimensional operator spaces, but the case of two dimensional operator spaces is trickier. This shift in dimension comes about from the need to make operator spaces into operator systems. In particular, it is known that Pisier's OH(n) is not exact for every $n \geq 3$, but it is open for $n = 2$, which is the case needed for the Smith-Ward problem.

Why should I care?

Requiring the limit in Problem 1 to be zero is, essentially, a statement about, asymptotically, how well knowing all finite dimensional compressions of an operator gives knowledge of that operator. In that sense it fits quite well with some information theoretic questions and likely has some equivalent interpretations in that language.

Maybe not so surprisingly, the Smith-Ward problem has subtle connections with Connes embedding problem, a.k.a., the Tsirelson problem.

Here are two not too outlandish conjectures.

Conjecture 1: If Connes' embedding is true, then Smith-Ward is true.

After all, both reduce to asymptotic questions about finite dimensional approximations. Thus, if one could show that this conjecture is true and that the limit in the first problem is not zero, then that would give a route to showing that Connes' embedding is false.

Conjecture 2: Assuming that Connes' embedding is true and that Smith-Ward is true leads to a contradiction.

Our recent work almost proves this second conjecture. If, indeed, the Smith-Ward problem has an information theoretic interpretation, then this would give an example of Tsirelson being true implying the violation of something.

Andreas Winter See [21].

S. Szarek: Almost sure entanglement of induced states.

Let $\mathcal{H}$ be an n-dimensional Hilbert space, $\mathcal{H}_a$ an ancilla space of dimension s, and let ψ be a (random) unit vector uniformly distributed on the sphere of $\mathcal{H} \otimes \mathcal{H}_a$. Then $\rho = \mathrm{tr}_{\mathcal{H}_a}|\psi\rangle\langle\psi|$ is a (random) induced state on $\mathcal{H}$. If, further, $n = d^2 > 1$ and $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d$, one may ask whether ρ is typically entangled, PPT etc., with respect to this particular splitting.

It has been known since 2006 [17, 1] that, for large d, ρ is typically (i.e., with probability close to 1) separable if s/d^4 is sufficiently large and typically entangled if $s = d^2$. More recently, it has been established in [3] that a fairly sharp transition from "generic entanglement" to "generic separability" occurs when s is, up to a logarithmic factor, of order d^3. Here we ask a related question:

For which ancilla dimensions is the probability of separability exactly zero?

Partial results in this direction are listed in section 7.1 of [2] (where the reader is referred for further details on what follows). As pointed out there, it follows from known facts that the above happens when $s \leq (d-1)^2$, while for $s \geq d^2$ the probability in question is strictly between 0 and 1. However, determining the *precise* threshold for almost sure entanglement seems to require a new idea, and perhaps more than one idea: in the case $d = 2$ the transition occurs when s increases from 2 to 3, with the reasons for almost sure entanglement when $s = 2$ and for non-zero probability of separability when $s = 3$ being quite different. Actually, it is not right away clear that there *is* a threshold: while it is elementary to show that, for a fixed s, the probability of separability is a nonincreasing function of d (Lemma 4.3 in [3]), its monotonicity with respect to s (for fixed d) is not immediately apparent (at least to the authors of [3]) and would also be interesting to establish. However, the weaker property needed for the existence of a threshold (i.e., almost sure entanglement for given d, s_0 implies almost sure entanglement for the same d and all $s < s_0$) is relatively easy to show.

Harrow

In [12] Doherty, Parrilo and Spedalieri gave a family of relaxations of the set of separable states (denoted Sep). The k'th level of their hierarchy yields the set of density matrices ρ^{AB} that can be extended to $\sigma^{AB_1 \cdots B_k}$ where $\rho^{AB} = \sigma^{AB_1}$, $\sigma^{B_1 \cdots B_k}$ is supported on the symmetric subspace and σ is PPT across all partitions. Call this set DPS(k).

The open question is to determine the maximum trace distance of an element of DPS(k) from Sep. It is known from [6] that if each subsystem has dimension d, then this distance is at most $O((d/k)^2)$. But is this tight? Do there exists states in DPS(k) that are far from Sep whenever $k < d$?

It is known that the antisymmetric Werner state on d by d dimensions is (d-1)-extendable and is far from Sep, but this state is not PPT. It is also known (from the original DPS paper) that there exist, for any d and k, states which are is DPS(k), but not Sep; however, no nonzero lower bound is known on the distance. Finally, the construction of [7] can give a state on d dimensions that is in DPS(log d) and is far from separable. So the optimal value of k can be anywhere between O(log d) and O(d).

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Bibliography

[1] G. Aubrun and S. J. Szarek, Tensor products of convex sets and the volume of separable states on N qudits. *Phys. Rev. A* **73** (2006) 022109.

[2] G. Aubrun, S. Szarek and D. Ye, Entanglement thresholds for random induced states. arXiv:1106.2264.

[3] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters. Quantum nonlocality without entanglement. *Phys. Rev. A* **59** (1999), 1070–1091

[4] Boaz Barak, Fernando G.S.L. Brandao, Aram W. Harrow, Jonathan A. Kelner, David Steurer, Yuan Zhou , Hypercontractivity, Sum-of-Squares Proofs, and their Applications arXiv:1205.4484

[5] Jop Briet, Harry Buhrman, Ben Toner A generalized Grothendieck inequality and entanglement in XOR games arXiv:0901.2009

[6] Jop Briet, Thomas Vidick, Explicit lower and upper bounds on the entangled value of multiplayer XOR games arXiv:1108.5647

[7] squashed entanglement

[8] Valerio Capraro, A Survey on Connes' Embedding Conjecture, arXiv:1003.2076

[9] Andrew M. Childs, Debbie Leung, Laura Mancinska, Maris Ozols, A framework for bounding nonlocality of state discrimination arXiv:1206.5822

[10] Eric Chitambar, Debbie Leung, Laura Mancinska, Maris Ozols, Andreas Winter Everything You Always Wanted to Know About LOCC (But Were Afraid to Ask) arXiv:1210.4583

[11] A. Connes, Classification of injective factors, *Ann. of Math.* **104** (1976), 73-115.

[12] , Doherty, Parrilo and Spedalieri (DPS) quant-ph/0308032

[13] Y. Feng and Y. Shi. Characterizing locally indistinguishable orthogonal product states. *IEEE Trans. Inf. Theor.* **55** (2009) 2799–2806.

[14] Tobias Fritz arXiv:1008.1168 [pdf, ps, other] Tsirelson's problem and Kirchberg's conjecture *Rev. Math. Phys.* **24**, (2012) 1250012.

[15] D. Gross, V. Nesme, H. Vogts, R.F. Werner, Index theory of one dimensional quantum walks and cellular automata *Commun. Math. Phys.* **310**, 419-454 (2012) arXiv:0910.3675

[16] Aram W. Harrow, Ashley Montanaro Testing product states, quantum Merlin-Arthur games and tensor optimisation arXiv:1001.0017

[17] P. Hayden, D. Leung and A. Winter, Aspects of generic entanglement. *Comm. Math. Phys.* **265** (2006) 95-117.

[18] Unbounded violations of bipartite Bell Inequalities via Operator Space theory arXiv:0910.4228

[19] M. Junge, M. Navascues, C. Palazuelos, D. Perez-Garcia, V. B. Scholz, R. F. Werner Connes' embedding problem and Tsirelson's problem *J. Math. Phys.* **52** 012102 (2011) arXiv:1008.1142

[20] E. Kirchberg, On semi-split extensions, tensor products and exactness of group C^*- algebras, *Inv. Math.* **112** (1993), 449-489.

[21] Ke Li, Andreas Winter, Relative entropy and squashed entanglement arXiv:1210.3181

[22] Ozawa, About the QWEP conjecture, *Intern. J. Math.* **15** (2004), 501-530.

[23] (Navascues, Owari and Plenio) arxiv: 0906.2731

[24] D. Perez-Garcia, M.M Wolf, C. Palazuelos, I. Villanueva, M. Junge Unbounded violation of tripartite Bell inequalities *Comm. Math. Phys.* **279**, 455 (2008) arXiv:quant-ph/0702189

[25] Renato Renner Security of Quantum Key Distribution arXiv:quant-ph/0512258

[26] Marco Tomamichel, Roger Colbeck, Renato Renner Duality Between Smooth Min- and Max-Entropies *IEEE Trans. on Inf. Theory* **56** (2010), 4674-4681 arXiv:0907.5238

[27] John Watrous, Many copies may be required for entanglement distillation arXiv:quant-ph/0312123

Chapter 11

Banach space theory (12w5019)

March 4 - 9, 2012

Organizer(s): Razvan Anisca (Lakehead), Steve Dilworth (South Carolina), Edward Odell (UT Austin), Bünyamin Sarı (North Texas)

The workshop was largely motivated by the recent extraordinary work of Argyros and Haydon [AH] (discussed below) which, following on the fundamental work of Gowers and Maurey in the 1990's, continued the discovery of the incredible variety of possible Banach space structure. [AH] is connected strongly with operator theory and Banach algebras. The last couple of years were very fruitful for researchers and many stunning results were presented at the workshop and new consequences were observed as well. The testimonials of the participants, which included numeruous young researchers and graduate students, attest to the great success of the workshop. The following presents a brief overview in which it is possible to cite only a selection of highlights. Much more can be found in the files and the videos and we expect more developments as these are studied.

Scalar plus compact and Invariant subspace problems

A key theme of the workshop and one underlying a number of presentations, dates back to the remarkable construction in 1980 of J. Bourgain and F. Delbaen [BD]. They constructed a Banach space X with X^* isomorphic to ℓ_1, yet c_0 does not embed into X. The example struck researchers as quite special and too limited to be useful in solving other open problems. After almost 20 years some researchers [A], [7] began to think otherwise. Two years ago, S.A. Argyros and R. Haydon used the BD-construction to solve a famous problem in Banach spaces.

Given a specific classical example of a Banach space, it is usually quite easy to construct many nontrivial bounded linear operators $T \in \mathcal{L}(X)$. But just given that X is separable and infinite dimensional, this is not at all clear. Over 35 years ago Lindenstrauss [10] asked if such an X existed so that $\mathcal{L}(X) = \{\lambda I + K : \lambda$ is a scalar and K is a compact operator$\}$. Here I is the identity operator on X. In their remarkable example of a space X not containing an unconditional basic sequence W.T. Gowers and B. Maurey [GM] proved that for their space all operators had the form $\lambda I + S$, where S is strictly singular (i.e., not an isomorphism restricted to any infinite dimensional subspace). But the "scalar plus compact" problem remained open. Then Argyros and Haydon [AH] constructed a space X with the "scalar plus compact" property. X is formed using the BD technique and thus X^* is isomorphic to ℓ_1. Shortly after that D. Freeman, E. Odell and Th. Schlumprecht [FOS] proved that if X^* is separable then X embeds into an isomorphic predual of ℓ_1. The proof, again, adopted the BD construction. It is amazing that the class of preduals of ℓ_1 is so large and yet small and that confining oneself to this class, the "scalar plus compact" problem could be solved.

Spaces X satisfying the "scalar plus compact" property are also of interest to operator theorists in that every operator $T \in \mathcal{L}(X)$ must admit a nontrivial invariant subspace. Furthermore $\mathcal{L}(X)$ is separable, and from the construction, is amenable as a Banach algebra.

Richard Haydon gave the first talk, outlining the BD-construction. In particular he presented a new way of focusing on families of compact subsets of $[\mathbb{N}]^{<\omega}$, during the construction, to consolidate numerous previous technical arguments such as appear in [AH] and [FOS]. He also mentioned a new example of a space X, with the "scalar plus compact" property, that contains an isomorph of ℓ_1. **Dan Freeman** followed with a discussion of [FOS] and then a newer result [AFHORSZ1]. The latter is that any separable superreflexive space can be embedded into an isomorphic predual X of ℓ_1 with the "scalar plus compact" property. X shares the properties of [AH]. Namely all $T \in \mathcal{L}(X)$ admit nontrivial invariant subspaces, $\mathcal{L}(X)$ is separable and amenable. Furthermore X is somewhat reflexive (every infinite dimensional subspace of X contains an infinite dimensional reflexive subspace). He also mentioned a more general but much more technically difficult result, by the same seven authors ([AFHORSZ2], in preparation). If X^* is separable and does not contain ℓ_1 as a complemented subspace, then X embeds into an ℓ_1 predual with the "scalar plus compact" property.

In [AH] the authors also show that their space X can be constructed to be HI (hereditarily indecomposable). This means that for all $Y \in X$ if $Y = Z \oplus W$ then Z or W must be finite dimensional. **Matthew Tarbard**, a student at Oxford, discussed his construction of some new remarkable spaces. He was motivated by the question as to whether any HI isomorphic predual of ℓ_1, with the "scalar plus strictly singular" property must have the "scalar plus compact" property. He showed this to be false. Indeed one can obtain such spaces with the Calkin algebra, $\mathcal{L}(X)/K(X)$ having any finite dimension. Here $K(X)$ denotes the ideal of compact operators on X. It was pointed out by W.B. Johnson during Tabbard's talk that his examples solved a longstanding open problem concerning which $T \in \mathcal{L}(X)$ are commutators, i.e., $T = UV - VU$ for some $U, V \in \mathcal{L}(X)$. The problem solved is that there is a space X and $T \in \mathcal{L}(X)$ with T lying in a proper ideal (here the ideal of strictly singular operators) that is not a commutator.

Another major breakthrough was presented by **Spiros Argyros** [3]. He gave an example of an HI reflexive space X so that every $T \in \mathcal{L}(X)$ admits a nontrivial invariant subspace. Moreover this holds for all $T \in \mathcal{L}(Y)$, $Y \subseteq X$. This construction is not of the BD type. Nor does X have the "scalar plus compact" property. The strictly singular operators on every subspace of X form a nonseparable ideal. Furthermore the only spreading models of X are c_0 and ℓ_1, and such exist in all subspaces. This then solves another open problem in that the spreading models are stabilized within $\{\ell_p : 1 \leq p < \infty\} \cup \{c_0\}$ and yet do not form an "interval." Also every $T \in \mathcal{L}(X)$ either commutes with a non-zero compact operator or else $T^3 = 0$. All in all, quite a remarkable space. The construction uses Tsirelson ideas under constraints, motivated by earlier constructions in [OS1], [OS2].

Despoina Zisimopoulou showed how to adapt the BD construction to obtain certain $\mathcal{L}_\infty$ sums of Banach spaces. In particular, she constructs Z_p, a $\mathcal{L}_\infty$ sum of ℓ_p, so that $Z_p \sim Z_p \oplus \ell_p$ and every $T \in \mathcal{L}(Z_p)$ satisfies, for some λ, $T - \lambda I$ is horizontally compact, i.e., the restriction to every horizontally block subspace is compact.

Ioannis Gasparis showed that the BD constructions can be performed within $C(K)$ spaces as well. Using (a dual version of) the BD construction, he showed an interesting example of an ℓ_1-predual which is isomorphic to a subspace of $C(\omega^\omega)$ but it is neither isomorphic to c_0 nor contains a copy of $C(\omega^\omega)$. It follows from the literature that this space is not isomorphic to a complemented subspace of a $C(K)$ space.

As mentioned it can be difficult to construct nontrivial operators on Banach spaces. **Antonis Monoussakis** [M,P-B] and **Anna Pelczar-Barwacz** [KM,P-B] discussed two papers in which strictly singular non-compact operators are constructed in two settings. One is certain mixed Tsirelson (and HI) spaces and the other is in certain asymptotic ℓ_p spaces.

Thomas Schlumprecht discussed a recent paper with Daws, Haydon, and White [DHSW] which explored the Banach algebra structure of $\ell_1(\mathbb{Z})$. The latter is a Banach algebra under convolution:

$$(f * g)(n) = \sum_{k \in \mathbb{Z}} f(k) g(n - k) \ .$$

$\ell_1(\mathbb{Z})$ is, as a Banach space, the dual of uncountably many nonisomorphic Banach spaces. It is said to be a dual Banach algebra if it is realized as the dual of X so that the product is separately ω^*-continuous. They consider preduals where the bilateral shift is ω^*-continuous (equivalently the above natural convolution is separately ω^*-continuous) and produce an uncountable number of such preduals. They use Banach space theory (Szlenk indices) to show that, as Banach spaces, the preduals are all isomorphic to c_0 and go on to construct many other nonisomorphic preduals.

Operator ideals and Commutators

The workshop also featured the dissemination of new and significant results in two related topics that have become very active in Banach spaces theory during the last few years: the study of the lattice of closed ideals in the Banach algebra of bounded linear operators on a Banach space, and the study of commutators on Banach spaces.

Until a few years ago, the only Banach spaces X for which the lattice of closed ideals was completely described were the spaces ℓ_p, with $1 \leq p < \infty$, in which case the ideal of compact operators $\mathcal{K}(X)$ was the only non-trivial ideal in $\mathcal{L}(X)$. Laustsen, Loy and Read [LLR] and Laustsen, Schlumprecht and Zsak [LSZ] added two new members to the family of Banach spaces for which the lattice of closed ideals in $\mathcal{L}(X)$ is completely understood: $X = (\oplus_n l_2^n)_{c_0}$ and $X = (\oplus_n l_2^n)_{\ell_1}$, respectively. In both cases it was shown that there exist only one other non-trivial closed ideal besides the ideal of compact operators: the ideal of operators on X that factor through c_0 (respectively ℓ_1). These results prompted a renewed interest in this topic and the workshop presented a good forum for communicating interesting new results in this direction:

- we have already described above the talks of **R. Haydon** and **M. Tabard** who presented some remarkable constructions of Banach spaces whose Calkin algebra can have any given finite dimension; therefore the lattice of closed ideals in the algebra of bounded linear operators on these spaces can have any given finite cardinality.

- there were two talks, by **Andras Zsak** and **Denny Leung**, that discussed the (unique) maximal closed ideals in the algebra of bounded linear operators on some Banach spaces which share a similar structure of the complemented subspaces with the spaces from [LLR], [LSZ]: $X = (\oplus_n \ell_1^n)_{c_0}$ and $X = (\oplus_n \ell_\infty^n)_{\ell_1}$, respectively. Namely, these spaces are known to have only two different isomorphic types of complemented subspaces, the whole space X or c_0 (respectively ℓ_1), and these makes them good candidates for investigation in this direction of research. In his talk, **A. Zsak** presented a more general approach, starting with spaces of the form $X = (\oplus_n E_n)_{c_0}$ where E_n's are finite dimensional spaces. The results obtained are in the form of interesting dichotomies about sequences of operators into L_1, including a dichotomy theorem for random matrices. Similarly, **D. Leung** discussed the general case of spaces of the form $X = (\oplus_n E_n)_{\ell_1}$.

- **Bentuo Zheng** [LSZh] discussed results regarding the structure of the closed ideals in the algebra of bounded linear operators on a class of p-regular Orlicz sequence spaces ℓ_M which are close (in a certain sense) to ℓ_p. After obtaining some structural results about these spaces, it was shown that the immediate successor of the ideal of compact operator $\mathcal{K}(\ell_M)$ in $\mathcal{L}(\ell_M)$ is the closed ideal generated by the formal identity from ℓ_M into ℓ_p, and that the maximal closed ideal in $\mathcal{L}(\ell_M)$ is of the form $M = \{T : \ell_M \to \ell_M \mid Id_{\ell_M} \text{ does not factor through } T\}$. It should be noted that there are several other known examples of Banach spaces X for which $M_X = \{T : X \to X \mid Id_X \text{ does not factor through } T\}$ is the unique maximal closed ideal in $\mathcal{L}(X)$.

- It is also the case for $C[0, \omega_1]$, where ω_1 is the first uncountable ordinal. This new result was part of **Niels Laustsen**'s talk, which also contained some other interesting results regarding the closed ideals in $\mathcal{L}(C[0, \omega_1])$. A maximal closed deal in $\mathcal{L}(C[0, \omega_1])$ was identified before in the literature, using a representation of operators on $C[0, \omega_1]$ as scalar-valued $[0, \omega_1] \times [0, \omega_1]$-matrices. The new result discussed in the workshop provides a matrix-free characterization of this ideal and, in the same time, implies that there is no other maximal ideal in $\mathcal{L}(C[0, \omega_1])$. The talk provided also a list of equivalent conditions describing the strictly smaller ideal of operators with separable range.

The workshop also featured remarkable new results on the important and difficult problem of classifying the commutators in the algebra of bounded linear operators on a Banach space X. When studying derivations on a general Banach algebra $\mathcal{A}$, a natural problem that arises is to classify the commutators in the algebra, that is, elements of the form $AB - BA$. This problem is hard to tackle on general Banach algebras. Nevertheless one has a better chance in the special case of the algebra of bounded linear operators on a Banach space X since there is hope that the underlying structure of the space X could provide useful information regarding the operators acting on X. For a Banach space X for which there is a unique maximal closed ideal in $\mathcal{L}(X)$, one can hope to obtain a complete classification of the commutators on the space. The natural conjecture is that the only operators on X that are not commutators are the ones of the form $\lambda I + S$, where S belongs to the unique maximal ideal in $\mathcal{L}(X)$ and $\lambda \neq 0$ ([DJ]). This was known to be true for the spaces ℓ_p, when $1 < p < \infty$ (by an old result of Apostol), and there has been some remarkable progress in this direction during the last three years, with the conjecture being verified for the spaces ℓ_1, ℓ_∞, L_p $(1 \leq p < \infty)$ ([18], [DJ], [DJS]).

- As mentioned earlier, it was pointed out by W. B. Johnson during the workshop that, as a byproduct of his results, **M. Tabard** has produced an example of a Banach space X that admits an operator T in the maximal closed ideal of $\mathcal{L}(X)$ which is not a commutator, thus providing a negative answer to the conjecture mentioned above.
- **Detelin Dosev** presented some recent progress on the problem of verifying the above conjecture for the spaces $C(K)$. The results obtained rely on a theorem of Kalton about decomposition of Borel measures on an infinite compact metric space.
- In his talk, **Gideon Schechtman** presented some interesting results regarding the converse of the following well-known fact: if an $n \times n$ matrix A is a commutator ($A = BC - CB$, where B, C are also $n \times n$ matrices) then the trace of A is zero, in which case it is clear that $\|A\| \leq 2\|B\|\|C\|$. The question discussed in the talk is the following: assuming that A is an $n \times n$ matrix with trace zero, is it possible to find $n \times n$ matrices B, C such that $A = BC - CB$ and $\|B\|\|C\| \leq K\|A\|$ for some absolute constant K? The talk provided a weaker estimate. Namely, the above holds for $K = K_\epsilon n^\epsilon$ for every $\epsilon > 0$, where K_ϵ depends only on ϵ.

Gowers' classification program

After proving his famous dichotomy theorem that every (separable) Banach space contains either a hereditarily indecomposable subspace or a subspace with an unconditional basis, Gowers proposed a classification program for separable Banach spaces. The aim of this program is to produce a list of classes of infinite dimensional Banach spaces such that

(a) the classes are hereditary, i.e., stable under taking subspaces (or block subspaces),

(b) the classes are inevitable, i.e., every infinite dimensional Banach space contains a subspace in one of the classes,

(c) the classes are mutually disjoint,

(d) belonging to one class gives some information about the operators that may be defined on the space or on its subspaces.

Gowers also proved a second dichotomy theorem which asserts that every Banach space contains either a quasi-minimal subspace or a subspace with an unconditional basis such that disjointly supported block subspaces are totally incomparable (i.e., neither embeds into the other). Recall that a space X is minimal if it embeds into all of its subspaces, and quasi-minimal if it has no pair of totally incomparable subspaces. Gowers deduced from these dichotomies a list of four inevitable classes of Banach spaces; HI spaces, no disjointly supported subspaces are isomorphic, strictly quasi-minimal (i.e., quasi-minimal with no minimal subspace) with an unconditional basis, and minimal spaces. In a series of papers, Ferenczi and Rosendal further refined this to a list of six inevitable classes;

1. HI and tight by range,
2. HI, tight, and sequentially minimal,
3. Tight by support,
4. Has unconditional basis, tight by range, and quasi-minimal.
5. Has unconditional basis, tight, and sequentially minimal
6. Has unconditional basis and minimal.

We will refer to **Valentin Ferenczi**'s talk for the notion of tightness, and for a nice survey of the Gowers' classification program. The above list is obtained via some dichotomy theorems, and it wasn't clear if there are examples of spaces in class 2 and in class 4. Ferenczi presented a recent result (joint with Th. Schlumprecht) showing that a subspace of a variant of the original Gowers-Maurey space is indeed a class 2 space. No example of class 4 is yet known.

Spreading models

The notion of a spreading model has proven to be a very useful tool in the geometry of Banach spaces. Roughly speaking, a spreading model of a Banach space X is another Banach space with a spreading basis whose norm is obtained by stabilizing at infinity the norm on a non-degenerate sequence of vectors in X. Such a stabilization is possible thanks to Ramsey theory. In general, a spreading model is not a subspace but it is finitely represented in the space in a special way. The initial segments of the basis are $(1+\varepsilon)$-equivalent to a finite subsequence of the generating sequence in X. What type of spreading models must exist on a given Banach space X was a central problem which have been widely investigated since they were first introduced in the 70's. For instance, every space has a spreading model with an unconditional basis. However, Odell and Schlumprecht [OS1] constructed a reflexive space admitting no c_0 or ℓ_p spreading models. One can iterate the process and ask the same question for spreading models of a spreading model of X and so on. Argyros, Kanellopoulos and Tyros [AKT1, AKT2] introduced a generalized notion of spreading models based on sequences indexed by some special families of finite subsets of natural numbers (called $\mathcal{F}$-sequences) which yields, among other things, a natural framework to study the iterated spreading models. They have also constructed a space, generalizing Odell-Schlumprecht's example, so that none of the n-fold iterated spreading models are isomoprhic to c_0 or ℓ_p. In his talk **Kostantinos Tyros** explained the fundamentals of $\mathcal{F}$-spreading models. This notion depends on plegma families, sequences of interlacing subsets, in a regular thin family $\mathcal{F}$ which satisfy strong Ramsey properties. The combinatorics of such families is promising in that essentially the order of $\mathcal{F}$ determines the $\mathcal{F}$-spreading models, and one gets an increasing transfinite hierarchy of spreading models (one for each countable ordinal).

Nonlinear theory

One of the exciting research programs in Banach space theory is the nonlinear theory of Banach spaces. Besides the intrinsic geometric interest, some of the problems are motivated by the applications to theoretical computer science. Parallel to the linear theory, the main focus is the nonlinear classification of Banach spaces. The linear operators are replaced by Lipschitz or uniformly continuous maps. The problems of interest are Lipschitz, uniform or coarse embedding of metric spaces in normed spaces, or such an embedding of a Banach space into another.

Florent Baudier spoke on joint with Fernando Albiac on Lipschitz embeddings between L_p spaces and snowflaked versions. In particular, he presented the joint work with F. Albiac that for $1 \leq p < q$, ℓ_p with the snowflake metric $d_p^{p/q}$ Lipschitz embeds into ℓ_q, and for $0 < p < q \leq 1$, (ℓ_p, d_p) Lipschitz embeds into (ℓ_q, d_q). These are analogs of the function space versions due to Mendel and Naor in which case one gets isometric embeddings.

Alejandro Chavez-Dominguez took on the question of how one could possibly define tensor products of a metric space with a Banach space. In the Banach space setting, there is a close duality relationship between tensor norms and operator ideals. Chavez-Dominguez presented his endeavor of developing a general theory of tensor norms on spaces of Banach-space-valued molecules on metric spaces. Some of the classical results carry over to this new setting. There is a natural notion of a reasonable norm, and among the reasonable norms there is a smallest one and a largest one. A new projective tensor product of a metric space X and a Banach space E was introduced which in the case $E = \mathbb{R}$ coincides with the classical Arens-Eells space. The dual of this tensor product is isometric to the space of Lipschitz mappings from X to E^*. Lipschitz quotient maps between metric spaces induce Lipschitz quotient maps between corresponding projective tensor products. A norm on spaces of molecules can be defined which is a predual for the space of Lipschitz p-summing operators from X to E^*. Moreover, a Hilbertian norm on spaces of molecules can be defined, and it is in duality with the ideal of maps that factor through a subset of Hilbert space.

Denka Kutzarova spoke on joint work with Stephen Dilworth, Gilles Lancien, and N. Lovasoa Randrianarivony on (nonlinear) uniform quotients of Banach spaces [DKLR]. Recently, Lima and Randrianarivony [3] proved that the uniform quotients of ℓ_p $(1 < p < 2)$ are the same up to isomorphism as the linear quotients of ℓ_p, answering a problem that had been open for over a decade. Their proof made essential use of Property (β) of Rolewicz, which is an asymptotic property of Banach spaces whose definition involves the metric but not the linear structure of the space, and which therefore lends itself nicely to the nonlinear theory. In this talk it was shown that if $T\colon X \to Y$ is a uniform quotient then the modulus of asymptotic smoothness of

Y essentially dominates the β-modulus of X. It follows that certain reflexive spaces, e.g $(\sum \ell_{p_n})_{\ell_2}$, where $p_n \to \infty$, cannot be expressed as uniform quotients of any space with Property (β), and also that ℓ_q is not a subspace of any uniform quotient of an ℓ_p sum of finite-dimensional spaces if $q > p$. The separable spaces that are isomorphic to spaces with Property (β) are precisely the reflexive spaces X such that both X and X^* have Szlenk index equal to ω.

Operators on Banach spaces

Alexey Popov reported on joint work with Laurent Marcoux and Heydar Radjavi on almost invariant subspaces of operators $T \in \mathcal{L}(X)$. A closed subspace $Y \subset X$ is a half-space if Y and X/Y are infinite-dimensional and is *almost invariant* if $T(Y) \subset Y + F$, where F is finite-dimensional. It is an open question as to whether every operator has an almost invariant half-space. It was shown that all polynomially compact operators on reflexive spaces and bi-triangular operators on Hilbert space have almost invariant half-spaces and that if A is a norm-closed subalgebra of $\mathcal{L}(X)$ which admits a complemented almost invariant half-space then A admits an invariant half-space.

Haskell Rosenthal discussed an approach to the Hyperinvariant Subspace Problem. He defined a subspace of $L_2[0, 2\pi]$ consisting of functions whose Fourier coefficients satisfy a certain growth condition. Some properties of the right shift operator T on this space were presented and it was conjectured that T and T^{-1} have no common invariant subspace.

The aim of **Kevin Beanland**'s talk was to use descriptive set theoretic results in order to provide a uniform version of the classical result of Davis-Figiel-Johnson-Pelczynski regarding the factorization of weakly compact operators on Banach spaces. The main result presented is of the following form: if X is a separable Banach space and Y is either a Banach space with a shrinking basis or the space $C(2^{\mathbb{N}})$, then all operators from a Borel (in the strong operator topology) collection $\mathcal{B}$ of weakly compact operators from X to Y factor through the same separable reflexive space $Z_{\mathcal{B}}$.

Isometric theory of Banach spaces

Christian Rosendal spoke on joint work with Valentin Ferenczi on bounded subgroups of the general linear group of X, where X is a separable Banach space [30]. This work is motivated by the famous *Banach Rotation Problem*: if the isometry group of X is transitive, does it follow that X is linearly isometric (or even isomorphic) to a Hilbert space? It was shown that if X is a complex HI space without a Schauder basis then the isometry group of X acts *nearly trivially* on X, i.e., it restricts to the identity on a closed subspace of finite codimension. This result yields the first known example of a Banach space with no maximal bounded subgroup of its general linear group (equivalently, a space with no equivalent *maximal norm* in the sense of Pełczyński and Rolewicz) and answers a question of Wood raised in the eighties. It also provides the first known example of a complex super-reflexive Banach space which does not admit an equivalent almost transitive norm, answering another longstanding open problem.

Jesus Castillo spoke about spaces U of *universal disposition* for a class $\mathfrak{M}$ of Banach spaces [ACCGM]: i.e., if $A, B \in \mathfrak{M}$ and $u\colon A \to U$ and $i\colon A \to B$ are isometric embeddings then there exists an isometric embedding $u'\colon B \to U$ such that $u = iu'$. It was shown that a space of Kubis obtained from the push-out construction is a space of density character $\aleph_1$ which is of universal disposition for the class of separable Banach spaces (and under CH is unique up to isometry). Several properties of spaces of universal disposition for the classes of separable and of finite-dimensional Banach spaces were presented. A Banach space X is *automorphic* if every isomorphism between subspaces of X can be extended to an automorphism of X. The only known automorphic spaces are $c_0(I)$ and $\ell_2(I)$. It was shown that HI spaces with certain additional properties, if they exist, would be automorphic.

Random Matrices and High-Dimensional Convexity

Nicole Tomczak-Jaegermann spoke about random matrices associated to high-dimensional convex bodies, with applications to the areas of computational geometry, compressed sensing, especially approximate reconstruction problems and the Restricted Isometry Property (RIP), and random matrix theory [ALPT-J1, ALPT-J2]. Consider an $n \times N$ random matrix M with columns $X_1, \ldots, X_N$ that are independent copies of an isotropic log-concave vector X. A deviation inequality for the norm of the restriction of M to the collection of k-sparse vectors was presented. This result was used to give an answer to a question of Kannon, Lovász and Simonovits, namely a concentration inequality for the convergence of the empirical covariance matrix $(1/N)\sum_{i=1}^{N} X_i \otimes X_i$ to I_n as $N \to \infty$. It was shown that the spectral properties of M are similar to those of random matrices with independent Gaussian entries and that M has the RIP of order m provided $m \leq cn/\log^2(CN/n)$. RIP properties for matrices with independent rows were also presented.

Participants

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Bibliography

[ALPT-J1] Radosław Adamczak, Alexander E. Litvak, Alain Pajor and Nicole Tomczak-Jaegermann, *Quantitative estimates of the convergence of the empirical covariance matrix in log-concave ensembles*, J. Amer. Math. Soc. **23** (2000), no. 2, 535–561.

[ALPT-J2] Radosław Adamczak, Alexander E. Litvak, Alain Pajor and Nicole Tomczak-Jaegermann, *Restricted Isometry Properties of matrices with independent columns and neighborly polytopes by random sampling*, Constr. Approx. **34** (2011), no. 1, 61–88.

[A] S.A. Argyros, *The dual of the Bourgain-Delbaen space*. Israel J. Math. 117 (2000), 239–259.

[ACCGM] A. Avilés, F. Cabello, J. M. F. Castillo, M. González, and Y. Moreno, *On Banach spaces of universal disposition*, J. Funct. Anal. **261** (2011), 2347–2361.

[AFHORSZ1] S.A. Argyros, D. Freeman, R.G. Haydon, E. Odell, Th. Raikoftsalis, Th. Schlumprecht, and D.Z. Zisimopoulou, *Embedding uniformly convex spaces into spaces with very few operators*, J. Funct. Anal. **262** (2012), 825-849.

[AFHORSZ2] S.A. Argyros, D. Freeman, R.G. Haydon, E. Odell, Th. Raikoftsalis, Th. Schlumprecht, and D.Z. Zisimopoulou, *HI extensions of $\mathcal{L}_\infty$ spaces and spaces with very few operators*.

[AH] S.A. Argyros and R. Haydon, *A hereditarily indecomposable $\mathcal{L}_\infty$-space that solves the scalar-plus-compact problem*. Acta Math. 206 (2011), no. 1, 1–54.

[AKT1] S.A. Argyros, V. Kanellopoulos, and K. Tyros, *Higher order spreading models*, arXiv:1202.6390.

[AKT2] S.A. Argyros, V. Kanellopoulos, and K. Tyros, *Finite order spreading models*, arXiv:1105.2732.

[AM] S.A. Argyros and P. Motakis, *The invariant subspace property of the strictly singular operators in a reflexive space*, preprint.

[BD] J. Bourgain and F. Delbaen, *A class of special $\mathcal{L}_\infty$ spaces*. Acta Math. 145 (1980), no. 3-4, 155–176.

[DHSW] M. Daws, R. Haydon, Th. Schlumprecht, and S. White, *Shift invariant preduals of $\ell_1(\mathbb{Z})$*, Israel J. Math., to appear.

[DKLR] S. J. Dilworth, Denka Kutzarova, G. Lancien, and N. Lovasoa Randrianarivony, *Asymptotic Geometry of Banach Spaces and Uniform Quotient Maps*, preprint, 2012.

[D] D. Dosev, *Commutators on ℓ_1*, J. Funct. Analysis **256**(2009), 3490-3509.

[DJ] D. Dosev and W. B. Johnson, *Commutators on ℓ_∞*, Bull. London Math. Soc. **42**(2010), 155-169.

[DJS] D. Dosev, W. B. Johnson and G. Schechtmann, *Commutators on L_p, $1 \le p < \infty$*, 2011, preprint.

[FR] Valentin Ferenczi and Christian Rosendal, *On isometry groups and maximal symmetry*, arXiv:1106.5020

[FOS] D. Freeman, E. Odell, and Th. Schlumprecht, *The universality of ℓ_1 as a dual space*. Math. Ann. 351 (2011), no. 1, 149–186.

[GM] W.T. Gowers and B. Maurey, *The unconditional basic sequence problem*. J. Amer. Math. Soc. 6 (1993), no. 4, 851–874.

[H] R. Haydon, *Subspaces of the Bourgain-Delbaen space*. Studia Math. 139 (2000), no. 3, 275–293.

[KM,P-B] D. Kutzarova, Antonis Manoussakis, A. Pelczar-Barwacz, *Isomorphisms and strictly singular operators in mixed Tsirelson spaces*, JMAA, to appear.

[LLR] N. J. Laustsen, R. Loy and C. Read, *The lattice of ideals in the Banach algebra of operators on certain Banach spaces*, J. Funct. Analysis **214**(2004), 106-131.

[LSZh] P-K. Lin, B. Sarı, and B. Zheng, *Norm closed ideals in the algebra of bounded linear operators on Orlicz sequence spaces*, preprint.

[LSZ] N. J. Laustsen, T. Schlumprecht and A. Zsak, *The lattice of closed ideals in the Banach algebra of operators on a certain dual Banach space*, J. Operator Th. **56:2**(2006), 391-402.

[LR] Vegard Lima and N. Lovosoa Randrianarivony, *Property (β) and uniform quotient maps*, Israel J. Math., in press.

[L] J. Lindenstrauss, *Some open problems in Banach space theory*, Seminaire Choquet. Initiation ‘a l’analyse, 15 (1975-1976), No 18, 9 pp.

[M,P-B] Antonis Manoussakis and A. Pelczar-Barwacz, *Strictly singular non-compact operators on a class of HI spaces*, preprint.

[OS1] E. Odell and Th. Schlumprecht, *On the richness of the set of p’s in Krivine’s theorem*. Geometric aspects of functional analysis (Israel, 1992–1994), 177—198, Oper. Theory Adv. Appl., 77, Birkhäuser, Basel, 1995.

[OS2] E. Odell and Th. Schlumprecht, *A Banach space block finitely universal for monotone bases*. Trans. Amer. Math. Soc. 352 (2000), no. 4, 1859–1888.

[T] Matthew Tarbard, *Hereditarily indecomposable, separable $\mathcal{L}_\infty$ spaces with ℓ_1 dual having few operators, but not very few operators*, preprint.

Chapter 12

Advances in hyperkähler and holomorphic symplectic geometry (12w5126)

March 11 - 16, 2012

Organizer(s): Marco Gualtieri (University of Toronto), Jacques Hurtubise (McGill University), Daniel Huybrechts (University of Bonn), Eyal Markman (University of Massachusetts), Ruxandra Moraru (University of Waterloo), Justin Sawon (University of North Carolina)

Overview of the Field

Hyperkähler geometry, defined by Calabi in 1978, is endowed with such a high degree of structure that it has occupied a special status within many fields of geometry simultaneously. As a result, breakthroughs in hyperkähler geometry have come from Riemannian geometry, algebraic geometry, symplectic geometry, integrable systems, and quantum field theory in physics.

From a Riemannian point of view, hyperkähler metrics are Ricci-flat Kähler metrics with a surplus of constant spinors. They are also endowed with a natural family of real symplectic forms, making hyperkähler geometry amenable to methods of symplectic geometry, such as the symplectic quotient construction. Within algebraic geometry, they may be viewed as holomorphic symplectic manifolds, a fact which has led to the construction of many examples of hyperkähler manifolds, and which accounts for the appearance of hyperkähler geometry in the study of algebraically completely integrable systems. They have been of interest to physicists because they are required in supersymmetric quantum field theories, but also for the reason that hyperkähler geometry describes the behaviour of a topological field theory related to 4-dimensional Yang-Mills theory. Perhaps the most interesting hyperkähler manifold is Hitchin's moduli space of Higgs bundles, a central object of study in the geometric Langlands programme.

In the last five years, there have been several explosive developments in the study of hyperkähler manifolds, coming from different lines of inquiry by researchers working in very different fields of mathematics.

- **The classification or Torelli problem.** Holomorphic symplectic geometry is the algebro-geometric manifestation of hyperkähler geometry. By Yau's Theorem, every compact Kähler holomorphic symplectic manifold admits a hyperkähler metric. In two dimensions, K3 surfaces and complex tori are the only examples. One of the main focuses of algebraic geometers working in this area has been to extend to higher dimensions the well-understood classification theory of K3 surfaces. For instance, moduli of deformations of irreducible symplectic manifolds are described by a period map, which is known to be a local isomorphism (Beauville 1983). Huybrechts' (1999) proof of the surjectivity of the period map uses twistor spaces, and is a fine example of the interplay of the differential geometric and algebraic approaches. A complete understanding of the global behaviour of the period map has so far been out of reach, but there has been recent significant progress on this "Global Torelli Theorem" by Verbitsky (2009).

- **Mirror symmetry for hyperkähler manifolds.** An elliptic K3 surface is a K3 surface fibred over $\mathbb{P}^1$ with generic fibre an elliptic curve. The higher-dimensional analogue of this is a holomorphic Lagrangian fibration, which is a holomorphic symplectic manifold fibred by n-dimensional complex tori over $\mathbb{P}^n$. Hyperkähler rotation turns this into a special Lagrangian fibration, a structure of central importance in the Strominger-Yau-Zaslow formulation of Mirror Symmetry; moreover, this is currently the only way known of producing special Lagrangian fibrations on compact manifolds. Matsushita (1999) proved that Lagrangian fibrations are the only possible fibrations on holomorphic symplectic manifolds. A major goal is to prove the "hyperkähler SYZ conjecture", which states that these fibrations always exist, at least after a small deformation of the manifold (Matsushita, Verbitsky 2008). Singular fibres of Lagrangian fibrations were recently classified by Hwang and Oguiso (2009), who are now working on understanding the local structure of the fibration near the singular fibres.

- **The study of compact hyperkähler manifolds.** Compact hyperkähler manifolds are exceedingly rare, and their complete classification is one of the main outstanding problems in the field. Examples are found using algebraic geometry, by constructing holomorphic symplectic varieties. Almost all examples of higher-dimensional holomorphic symplectic manifolds can be described in terms of moduli spaces of stable sheaves on K3 or abelian surfaces. Such moduli spaces can sometimes be singular, and it came as a surprise in 1998 when O'Grady constructed explicit symplectic desingularizations of some of these spaces. Subsequent investigations by Lehn and Kaledin (2004) revealed that this behaviour is rare: the moduli spaces considered by O'Grady are essentially the only ones that admit symplectic desingularizations. The topology and birational geometry of O'Grady's spaces remains a topic of interest (Rapagnetta, Nagai 2010).

- **Rozansky-Witten theory.** Hyperkähler and holomorphic symplectic geometry figures prominently in many physical theories. For example, Rozansky-Witten theory (1997) used a hyperkähler manifold X as the target space for a 3-dimensional sigma model. The theory was originally used to study knot and 3-manifold invariants, but recently, the theory has been revisited by Kapustin, Rozansky, and Saulina (2009), who discovered that by studying the boundary conditions in the original Rozansky-Witten theory, one uncovers the rich structure of an extended topological field theory, which lends an unexpected (and poorly understood) structure on the category of holomorphic Lagrangian submanifolds of X, connecting them with the theory of matrix factorizations in the study of noncommutative algebra.

- **Wall-crossing and explicit hyperkähler metrics.** One of the long-standing mysteries of modern geometric analysis is that while the theorem of Yau implies that Calabi-Yau and hyperkähler metrics exist, the metric is not explicitly known even in the simplest interesting cases on compact manifolds. Even for non-compact manifolds, few explicit hyperkähler metrics are known, and those that are are celebrated examples such as the Gibbons-Hawking metric, frequently used in various branches of geometry and physics. Recently, using results of Kontsevich-Soibelman (2008) on wall-crossing formulas describing enumerative invariants on Calabi-Yau manifolds, physicists Gaiotto, Moore, and Neitzke (2009) were able to explicitly compute hyperkähler metrics on certain noncompact varieties in terms of these enumerative invariants. This work provides a completely new viewpoint on the relationship between the Riemannian aspect of hyperkähler geometry and the algebro-geometric aspect.

- **The geometry/topology of hyperkähler quotients.** The hyperkähler quotient construction of Hitchin-Karlhede-Lindström-Roček (1987) generalizes symplectic reduction, and has been the main method of constructing non-compact hyperkähler manifolds. Surprisingly, many equations arising in gauge theory (certain anti-self-dual equations, the Bogomolny equations, Nahm's equations) can be naturally interpreted as hyperkähler moment maps. The corresponding moduli spaces (of instantons, magnetic monopoles, Higgs bundles, etc.) thereby inherit hyperkähler metrics. Understanding the geometry and topology of hyperkähler quotients will therefore shed some light on the geometry and topology of many interesting physical moduli spaces; contributors in this area include Hitchin, Roček, Swann, and Nakajima-Yoshioka. In addition to the many results of Hausel and his collaborators, a recent development has been a criteria for desingularization of a quotient by Jeffrey-Kiem-Kirwan (2009), who also investigate the surjectivity of an analogue of the Kirwan map to the cohomology of a hyperkähler quotient.

A great deal is known about hyperkähler quotients constructed from torus actions on flat quaternionic vector spaces, which are known as "toric hyperkahler manifolds" or "hypertoric varieties". Bielawski-Dancer (2000) classified such hyperkähler quotients in terms of hyperplane arrangements; they also determined a Kähler potential and gave an explicit local form of the metric. Konno (1999, 2000) studied the cohomology rings. Harada-Proudfoot (2004) computed the equivariant cohomology. Hausel-Proudfoot (2005) used hypertoric varieties to help understand the cohomology rings of general hyperkähler quotients. The topology of hypertoric varieties continues to be a topic of great interest, with recent results of Braden-Proudfoot (2009) on the intersection cohomology ring, and of Stapledon (2009) on orbifold cohomology and Ehrhart polynomials of Lawrence polytopes.

Objectives of the workshop

The meeting was intended as a 5-day workshop involving the main researchers in the fields above, both faculty and postdoctoral, together with graduate students who have taken up the subject in their doctoral work. For students and postdocs, the meeting was to provide two parallel opportunities: first, to learn well-established aspects of the theory of holomorphic symplectic and hyperkähler manifolds, and second, to be exposed to the plethora of open questions ripe for investigation, deriving from the recent advances listed above.

There were two main scientific objectives. The first was to summarize and understand the recent developments and main questions within the different groups of researchers focusing on hyperkähler geometry, including the recent work on the Torelli theorem, the algebro-geometric study of holomorphic Lagrangian fibrations, and the study of hypertoric varieties. The second aim was to introduce new ideas coming from physics in the recent papers of Kapustin-Rozansky-Saulina and Gaiotto-Moore-Neitzke and to work on some of the many open questions resulting from this seminal work. Specifically, the three main questions which must be resolved are whether the categorical structure they obtain on holomorphic Lagrangians can be made mathematically precise, whether the structure can be computed in known special cases, and what impact this has on the study of holomorphic symplectic manifolds, and possibly on holomorphic Poisson manifolds.

Overview of the meeting

Here are the abstracts of the talks, in alphabetical order by speaker surname:

Speaker: **Philip Boalch** (Centre National de la Recherche Scientifique)
Title: “Irregular connections, Dynkin diagrams, and fission"

Abstract: I’ll survey some results (both old and new) related to the geometry of hyperkahler moduli spaces of irregular connections on curves. If time permits this will include: 1) new nonlinear group actions generalising the well known actions of the mapping class/braid groups on character varieties, 2) new ways to glue Riemann surfaces together to obtain (symplectic) generalisations of the complex character varieties of surfaces, and 3) a precise conjecture that the Hilbert scheme of points on any 2d meromorphic Hitchin system is again a Hitchin system.

Speaker: **Sabin Cautis** (Columbia University)
Title: “Flops and about"

Abstract: Stratified flops of type A, D and E show up in the birational geometry of holomorphic symplectic varieties. For example, by a result of Namikawa all Springer resolutions of the closure of nilpotent orbits are related by a sequence of such flops.

Two varieties related by such a flop are expected to have equivalent derived categories. Concentrating on the A and D types, I will discuss the geometry of such flops, explain how they induce derived equivalences and speculate on various open questions.

Speaker: **François Charles** (IRMAR – Université de Rennes 1)
Title: "Some arithmetic aspects of specialization of Néron-Severi groups for holomorphic symplectic varieties"

Abstract: For a given family of smooth projective complex varieties, the theory of variations of Hodge structures gives a precise description of the variation of the Picard number of the members of the family. In the case of families of holomorphic symplectic varieties, the two following properties are well-known. On the one hand, the Picard number of a very general member of the family is equal to the Picard number of the generic fiber of the family. This is an easy consequence of Baire's theorem. On the other hand, if the family is not isotrivial, the locus in the parameter space of varieties with Picard number strictly bigger than that of the generic fiber is topologically dense. This is a consequence of the local Torelli theorem and of Lefschetz' theorem on $(1,1)$-classes which was pointed out by M. Green.

The goal of this talk is to investigate the extent to which the behaviour described above still appears in the arithmetic situation where the parameter space is replaced with the ring of integers of a number fields. This amounts to investigating specialization of Néron-Severi groups for holomorphic symplectic varieties defined over number fields after reduction to a finite field.

In this situation, the first result above does not hold. However, we will describe precisely the extent to which the Picard number can be forced to jump after specialization to a finite field. If time allows, we will describe a proof of the arithmetic analog of the theorem of Green in the special case of products of elliptic curves and discuss its arithmetic significance.

These problems have implications outside of arithmetic geometry, as was pointed out by recent results on the existence of rational curves on K3 surfaces by Bogomolov-Hassett-Tschinkel and Li-Liedtke. We will describe how they can be used to get an algorithm that allows one to compute the Picard number of any holomorphic symplectic variety.

Speaker: **Sergey Cherkis** (University of Arizona)
Title: "Doubly-periodic monopoles and their moduli spaces"

Abstract: A monopole wall is a solution of the Bogomolny equation on $\mathbb{R}\times T^2$; in other words it is a doubly periodic monopole. Moduli spaces of monopole walls are hyperkähler and, when the dimension is minimal, deliver examples of gravitational instantons. We formulate spectral description of a monopole wall of any given charges and use it to compute the dimension of its moduli space.

The Nahm transform maps a monopole wall to a monopole wall establishing the isometry between their respective moduli spaces. We find $SL(2,\mathbb{Z})$ group action on monopole walls, such that the Nahm transform is its S element. We conclude by identifying all monopole walls with four real moduli, up to this $SL(2,\mathbb{Z})$ equivalence.

These results are obtained in collaboration with Richard Ward.

Speaker: **Andrew Dancer** (Oxford University)
Title: "Implosion for hyperkähler manifolds"

Abstract: Implosion is an abelianisation construction in symplectic geometry, due to Guillemin, Jeffrey and Sjamaar. In this talk we describe joint work with Frances Kirwan and Andrew Swann on developing an analogous construction for hyperkähler spaces.

Speaker: **Tamás Hausel** (Oxford University)
Title: "Symmetries of SL(n) Hitchin fibres"

Abstract: In this talk we show how the computation of the group of components of Prym varieties of spectral covers leads to cohomological results motivated by mirror symmetry on the cohomology of moduli space of Higgs bundles and in turn to cohomological results on the moduli space of stable bundles on curves originally due to Harder-Narasimhan. This is joint work with Christian Pauly.

Speaker: **Daniel Huybrechts** (University of Bonn)
Title: "Chow groups and stable maps"

Speaker: **Jun-Muk Hwang** (Korea Institute for Advanced Study)
Title: "Webs of Lagrangian tori in projective symplectic manifolds"

Abstract: For a Lagrangian torus A in a simply-connected projective symplectic manifold M, we prove that M has a hypersurface disjoint from a deformation of A. This implies that a Lagrangian torus in a compact hyperkaehler manifold is a fiber of an almost holomorphic Lagrangian fibration, giving an affirmative answer to a question of Beauville's. Our proof employs two different tools: the theory of action-angle variables for algebraically completely integrable Hamiltonian systems and Wielandt's theory of subnormal subgroups. This is a joint-work with Richard Weiss.

Speaker: **Emanuele Macrì** (The Ohio State University)
Title: "Projectivity and birational geometry of Bridgeland moduli spaces"

Abstract: In this talk we will present a construction of a family of nef divisor classes on every moduli space of stable complexes in the sense of Bridgeland. For a generic stability condition on a K3 surface, we will prove that these classes are ample, thereby generalizing a recent result of Minamide, Yanagida, and Yoshioka.

We will apply this construction to describe a region in the ample cone of a moduli space of Gieseker-stable sheaves on a K3 surface and to study its birational geometry.

This is joint work in progress with Arend Bayer.

Speaker: **Dimitri Markushevich** (Universite Lille 1)
Title: "Some examples of Prym Lagrangian fibrations"

Abstract: The objective of the talk is to describe several constructions of holomorphically symplectic varieties equipped with Lagrangian fibrations. The constructions are related to the variations of mixed Hodge structures, and the fibers of the obtained Lagrangian fibrations are intermediate Jacobians of algebraic varieties. Several examples are produced when the latter algebraic varieties are conic bundles and the intermediate Jacobians are Prym varieties of double covers. A work is in progress on their (partial) compactification.

Speaker: **Yoshinori Namikawa** (Kyoto University)
Title: "On the structure of homogeneous symplectic varieties of complete intersection"

Abstract: If X is a symplectic variety embedded in an affine space as a complete intersection of homogeneous polynomials, then X coincides with a nilpotent orbit closure of a semisimple Lie algebra. Moreover, if X is a homogeneous symplectic hypersurface, then $\dim X = 2$ and X is an A_1-surface singularity.

Speaker: **Andrew Neitzke** (University of Texas at Austin)
Title: "Spectral networks"

Abstract: I will describe some objects called "spectral networks." A spectral network is a set of paths drawn on a punctured Riemann surface, obeying some local conditions. Spectral networks arise naturally in a new construction of the hyperkähler structure on moduli spaces of Higgs bundles with gauge group SU(N). This is joint work with Davide Gaiotto and Greg Moore.

Speaker: **Kieran G. O'Grady** (Rome (Sapienza))
Title: "Vector-bundles and zero-cycles on $K3$ surfaces"

Abstract: Let X be a projective complex $K3$ surface. Let $A^q(X)$ be the Chow group of codimension-q cycles on X modulo rational equivalence. Beauville and Voisin singled out a class $c_X \in A^2(X)$ of degree 1: it is represented by any point lying on an arbitrary rational curve (an irreducible curve whose normalization is rational). The class c_X has the following remarkable property:

$$\text{Let } D_1, D_2 \in A^1(X)\text{: then } D_1 \cdot D_2 \in \mathbb{Z}c_X.$$

Moreover $c_2(X) = 24c_X$. (Conjecturally the Chow ring of Hyperkähler varieties has similar properties.) In particular one has the *Beauville-Voisin ring* $A^0(X) \oplus A^1(X) \oplus \mathbb{Z}c_X$. Huybrechts proved that if E is a spherical object in the bounded derived category of X then the Chern character of E belongs to the Beauville-Voisin ring provided $\mathrm{Pic}(X)$ has rank greater than 2 or $c_1(E) \equiv \pm 1$ (mod rk(E)) if $\mathrm{Pic}(X) = \mathbb{Z}$. A rigid simple vector-bundle on X is a particular case of spherical object - in fact the key case in the proof of Huybrecths' result. One may summarize Huybrechts' result as follows. Let F_1, F_2 be rigid vector-bundles on X (the additional hypotheses mentioned above are in force): then $c_2(F_1) = c_2(F_2) + ac_X$ where $a := (\deg c_2(F_1) - \deg c_2(F_2))$. We believe that the following more general statement (with no additional hypotheses) holds. Let $\mathfrak{M}_1^{st}$ and $\mathfrak{M}_2^{st}$ be moduli spaces of stable pure sheaves on X with Mukai vectors v_1 and v_2 respectively. Suppose that $\dim \mathfrak{M}_1^{st} = \dim \mathfrak{M}_2^{st}$ i.e. v_1 and v_2 have equal norm with respect to Mukai's pairing: then the subset of $A^2(X)$ whose elements are $c_2(F_1)$ where $[F_1] \in \overline{\mathcal{M}_1^{st}}$ (the closure of $\mathfrak{M}_1^{st}$ in the moduli space of semistable sheaves) is equal to the subset of $A^2(X)$ whose elements are $c_2(F_2) + ac_X$ where $[F_2] \in \overline{\mathcal{M}_2^{st}}$ and $a := (\deg c_2(F_1) - \deg c_2(F_2))$ (notice that $(\deg c_2(F_1) - \deg c_2(F_2))$ is independent of F_1 and F_2). We will prove that the above statement holds under some additional assumptions.

Speaker: **Keiji Oguiso** (Keio University)
Title: "K3 surface automorphisms and hyperkähler automorphisms inspired by complex dynamics"

Abstract: I would like to discuss some nature of automorphisms of K3 surfaces and compact hyperkähler manifolds from the following basic and natural aspects in complex dynamics with concrete examples:

(1) topological entropy;
(2) Tits' alternatives;
(3) fixed point set;
(4) relations with ambient spaces.

Speaker: **Nicholas Proudfoot** (University of Oregon)
Title: "Quantizations of conical symplectic resolutions"

Abstract: The most studied example of a conical symplectic resolution is the cotangent bundle M of the flag manifold G/B, which resolves the nilpotent cone in Lie(G). Much of what goes under the name "geometric representation theory" is the study of this resolution, called the Springer resolution. Here are two cool features of this subject:

- If you construct a deformation quantization of M and take global sections, you get the ring of global (twisted) differential operators on the flag variety, which is isomorphic to a central quotient of the universal enveloping algebra of Lie(G). This allows you to study representations of Lie(G) in terms of sheaves on M.

- There is a natural action of "convolution operators" on the cohomology of M which provides a geometric construction of the regular representation of the Weyl group of G. This action can be promoted to a braid group action on a category by replacing cohomology classes with sheaves.

I will make the case that these two phenomena fit neatly into a theory that applies to arbitrary conical symplectic resolutions, including (for example) quiver varieties, hypertoric varieties, and Hilbert schemes of points on ALE spaces.

This is joint work with Braden, Licata, and Webster.

Speaker: **Brent Pym** (University of Toronto)
Title: “Residues of Poisson structures and applications"

Abstract: A holomorphic Poisson manifold is foliated by symplectic leaves, and the locus consisting of all leaves of dimension $2k$ or less is called the $2k^{th}$ degeneracy locus. In recent work with Marco Gualtieri, we explain that a Poisson structure has natural residues along its degeneracy loci, which are direct analogues of the Poincaré residue of a meromorphic volume form. As applications, we prove that the anti-canonical divisor along which a generically symplectic Poisson structure degenerates is singular in codimension two, and provide new evidence in favour of Bondal's conjecture that the $2k^{th}$ degeneracy locus of a Poisson Fano variety has dimension $\geq 2k+1$.

Speaker: **Giulia Saccà** (Princeton University)
Title: “Fibrations in abelian varieties and Enriques Surfaces"
Abstract: I will discuss two classes of fibrations in abelian varieties that can be associated to a linear system on an Enriques surface. The first class corresponds to Lagrangian subvarieties of certain HK manifolds, whereas the second class corresponds to singular symplectic subvarieties of certain singular moduli spaces of sheaves. The second class is a joint work in progress with E. Arbarello and A. Ferretti.

Speaker: **Misha Verbitsky** (SU-HSE, Faculty of Maths)
Title: “Trisymplectic manifolds"

Abstract: A trisymplectic structure on a complex 2n-manifold is a triple of holomorphic symplectic forms such that any linear combination of these forms has rank 2n, n or 0. We show that a trisymplectic manifold is equipped with a holomorphic 3-web and the Chern connection of this 3-web is holomorphic, torsion-free, and preserves the three symplectic forms. We construct a trisymplectic structure on the moduli of regular rational curves in the twistor space of a hyperkähler manifold.

Presentation Highlights/Scientific Progress Made

Examples of recent breakthroughs:

1. In the last five years, a discovery in quantum field theory about the counting of stable configurations of particles and black holes has shown itself to be an extremely powerful mathematical tool, with ramifications far beyond its initial applications in physics.

   The discovery concerns a “wall-crossing formula” which explains how certain numerical invariants, such as the number of stable configurations in a system, change when the parameters describing the system pass through a “wall”, which is a real hypersurface in the parameter space.

   In recent work of Gaiotto, Neitzke, and Moore, the physical meaning of the formula was explained in a remarkable way using the theory of supersymmetric sigma models: they interpret the jumping of invariants as a change in the behaviour of instanton contributions to a hyperKähler metric; the wall-crossing formula is then viewed as a sum of multi-instanton contributions to this metric. In this way, enumerative invariants in algebraic geometry become involved in the definition of a smooth, non-algebraic object, namely a Riemannian metric.

   At BIRS, Andrew Neitzke presented this new research to an audience of experts and it brought tremendous excitement to many of them. Perhaps the greatest promise that it holds is the possibility that, using the wall-crossing ideas, it might actually be possible to explicitly determine the famous hyperKähler metrics on the Hitchin moduli space of Higgs bundles — an object of central study in differential geometry, algebraic geometry, and number theory through the geometric Langlands programme.

2. Jun-Muk Hwang's talk was a spectacular unveiling of his recent solution to a conjecture of Beauville concerning the existence of Lagrangian foliations in holomorphic symplectic manifold. In a tour-de-force, Hwang combined a beautiful argument using the theory of integrable systems with intricate estimates coming from the theory of discrete groups. The audience immediately appreciated that this was a landmark achievement and there were many discussions afterward about possible extensions of the argument.

3. Kieran O'Grady is well-known in the field of holomorphic symplectic geometry as one of the primary innovators that sets the direction for the field. One of the main sources of examples of holomorphic symplectic manifolds are moduli spaces of sheaves – in his talk, O'Grady explained his recent detailed study of certain homological properties of moduli spaces of sheaves on K3 surfaces. His discovery, which extends the work of several other researchers in attendance such as Huybrechts, is a precise formula which relates the possible cohomological supports for different moduli spaces over the same K3.

4. Tamàs Hausel presented his recent work with Pauly concerning the detailed behaviour of Prym varieties of spectral covers in the $SL(n)$ Hitchin system. The work he explained at BIRS was instrumental in his recent work with de Cataldo and Migliorini [3], which appeared in the Annals of Mathematics.

5. The presentation by François Charles was a very interesting one for the conference, as it dealt with holomorphic symplectic geometry over fields other than the complex numbers. His work was shocking to many in the audience, because of how differently the varieties of interest behave in this situation as compared to the usual complex case. Naturally, this opened the eyes of many attendees and suggested many other questions about whether known results for complex K3 surfaces extended to number fields.

Outcome of the Meeting

We had *thirty-eight* participants. The participants consisted of main researchers in the area, both faculty and postdoctoral, and of graduate students who have taken up the subject in their doctoral work. It was important to us that the workshop provide an opportunity for graduate students to interact with experts in the field. We were very successful in that respect, with the participation of *seven* graduate students from the Universities of Toronto, Princeton, Roma Tre, and Pierre-et-Marie-Curie (Paris VI), and from the Institut Fourier, *two* of whom gave talks on their research. There were also *four* postdoctoral researchers, from the Universities of Toronto, Columbia, and Bonn, and from ETH Zürich.

Hyperkähler and holomorphic symplectic geometry figures prominently in many physical theories, and the work of physicists has, on numerous occasions, uncovered new structures and relationships that have advanced our understanding of some of the objects appearing in the field. We therefore also had the participation of physicists. In fact, two talks were given by physicists, Prof. Neitzke and Prof. Cherkis. The talk of Prof. Neitzke, in particular, sparked the interest of many mathematicians in attendance.

The workshop stimulated a number of fruitful discussions and collaborations. Those we are aware of are the following: Dr. Charles and Prof. Huybrechts intensively discussed Maulik's proof of the Tate conjecture for K3 surfaces [4] during the workshop (they tried to simplify his proof and get rid of additional assumptions on the degree and the characteristic). This eventually led to a wonderful paper [1] by Dr. Charles, which was accepted for publication in *Inventiones Mathematicae* in November 2012. Dr. Charles mentions BIRS in his acknowledgements. The ongoing collaboration between Profs. Moraru and Verbitsky also benefited from the conference, and some of the work they discussed at BIRS will appear shorly.

The BIRS meeting also provided an ideal environment for the collaboration between M. Gualtieri and B. Pym, who completed their paper concerning holomorphic Poisson structures [2] during the conference; also B. Pym spoke on this work at the conference.

Another interesting and unexpected benefit of the BIRS research environment was an impromptu after-hours lecture series given by A. Odesskii concerning the famous Feigin–Odesskii holomorphic Poisson structures. This was a subject of intense interest for a group of seven participants including M. Gualtieri, B. Pym and J. Fisher.

The participants were very enthusiastic about the scientific content of the workshop, as well as the facilities and breathtaking natural setting of BIRS. Moreover, the warm hospitality and professionalism of the staff were very much appreciated.

Participants

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Bibliography

[1] F. Charles, The Tate conjecture for K3 surfaces over finite fields, to appear in *Invent. Math.*. Preprint: arXiv:1206.4002.

[2] M. Gualtieri and B. Pym, Poisson modules and degeneracy loci, to appear in *Proceedings of the London Mathematical Society*. Preprint: arXiv:1203.4293

[3] M. A. de Cataldo, T. Hausel and L. Migliorini, Topology of Hitchin systems and Hodge theory of character varieties: the case A_1, *Ann. of Math.* **(2) 175** (2012), no. 3, 1329-1407.

[4] D. Maulik, Supersingular K3 surfaces for large primes. Preprint: arXiv:1203.2889.

Chapter 13

Challenges and Advances in High Dimensional and High Complexity Monte Carlo Computation and Theory (12w5105)

March 18 - 23, 2012

Organizer(s): David Ceperley (University of Illinois Urbana-Champaign), Yuguo Chen (University of Illinois Urbana-Champaign), Radu V. Craiu (University of Toronto), Xiao-Li Meng (Harvard University), Antonietta Mira (University of Lugano), Jeffrey Rosenthal (University of Toronto)

Overview of the Field

It is commonly recognized in the literature that the only possible way to estimate many realistic highly structured and high dimensional statistical models that properly describe the real world and the complex interactions among the variables that come into play, is by using computational tools such as Monte Carlo methods. The development and application of Monte Carlo methods has been an active research area for the last two decades. Many useful Monte Carlo techniques have been proposed in the literature, including Markov chain Monte Carlo (MCMC), sequential Monte Carlo, adaptive MCMC, perfect sampling, and quantum Monte Carlo.

While these Monte Carlo algorithms have turned into standard tools over the past decade (with dedicated software developed to implement them, such as Winbugs, www.mrc-bsu.cam.ac.uk/bugs, and numerous packages within the R project, www.r-project.org), they still face difficulties in handling less regular problems such as those involved in deriving inference for high-dimensional models, data sets that contain severe collinearity, massively multimodal target distributions, hierarchical latent variable models, or rapidly evolving phenomena. Two of the main problems encountered when using MCMC in these challenging settings are: 1) it is difficult to design a Markov chain that efficiently samples the state space of interest; and 2) the resulting MCMC estimators have high variance thus producing less reliable inference and poor forecasts. As a result of these challenges, while the central keyword of the nineties was “convergence diagnostic” for MCMC algorithms, the research in this area now focuses on adaptive algorithms and variance reduction techniques.

On the contrary to the lack of new methods for complex problems, users of standard Monte Carlo methods sometimes suffer from an embarrassment of riches, in that there are many different related algorithms available which are all asymptotically valid. For example, the Metropolis requires a choice of proposal distribution, with some choices working much better than others. An important start on better understanding these choices was the pioneering paper of Roberts, Gelman, and Gilks [1], which theoretically determined the proposal distribution from among a fixed parametric family for random-walk Metropolis algorithms. This led to numerous follow-up papers. However, much work – both theoretical and practical – remains to be done regarding choice of proposal distribution and other MCMC tuning issues.

Another class of challenges in the MCMC world aims at reconciling the methodological advances in the computational tools with the applications where evolving and complex phenomena are overly present. Evolving systems clearly embrace a large variety of applied problems and it is thus useful to develop software and packages that allow the practitioners to implement in an almost automatic way the recent advances in the literature without having to write their own code. Currently there are many R routines to implement MCMC algorithms (such as MCMCpack, mcmc, MCMCglmm and mcclust), or to perform convergence diagnostic (such as BOA and CODA). There is a single routine to implement adaptive MCMC (AMCMC) and there is nothing available to reduce the variance of the resulting estimators mostly because the tools that have been so far proposed in the literature are not sufficiently general to motivate a dedicated software.

The area of exact Monte Carlo sampling with Markov chains, also known as perfect sampling, has had a meteoric rise due to the seminal concept of coupling from the past introduced by Propp and Wilson [2]. While the method is spectacularly successful in a number of applications involving finite state spaces, especially in statistical physics and point processes, its use remains problematic in statistical models where the parameter has a continuous (possibly unbounded) state space. It would be valuable to find a good trade-off between the classical MCMC methods whose implementability is very general but suffer from lack of reliable convergence diagnostics, and perfect sampling algorithms that eliminate the need for convergence diagnostics but require ingenious solutions for wider applicability.

Recent Developments and Open Problems

MCMC has had periods of rapid methodological development since its second birth in the early nineties as the main computational tool for performing Bayesian inference. Recently, one can notice that the incentive for further improvements and innovation comes mainly from three broad directions: the high dimensional challenge, the quest for adaptive procedures that can eliminate the cumbersome and often frustrating process of tuning “by hand” the simulation parameters for a complex MCMC design and the need for flexible theoretical support, arguably required by all recent developments as well as many of the traditional MCMC algorithms that are widely used in practice. Each of these directions is discussed below.

High Dimensional Challenge for MCMC. We have reached a time when collecting large volume of data for increasingly complex models is becoming standard in a number of high-impact scientific explorations in genetics, medicine, astronomy, to name just a few. When the state space is very high dimensional, not only it is difficult to develop MCMC samplers that are able to move across the space, but once a reasonable candidate has been build, it is difficult to assess its performance. Possible approaches involve a “divide and conquer” strategy in which analyses based on smaller data subsamples are performed and must be combined in the last stage of the analysis. Alternatively, one may wish to replace costly and lengthy numerical calculations that are asymptotically unbiased with bias-controlled approximations that can be much faster to obtain.

Approximate Bayesian Computation. The challenges of dealing with high-volume data and complex models reflect on our ability to express the latter mathematically. For instance, in astronomy and genetics it is not unusual to encounter a “black-box” model (usually the result of an extremely complex computer model) for which one can obtain likelihood values corresponding to the parameters fed into the model, but no general mathematical formulation is available. This feature makes the traditional calculation of posterior probabilities impossible and thus Bayesian inference and, consequently, MCMC algorithms must be implemented in a very different form known as Approximate Bayesian Computation (ABC). A most challenging aspect of ABC-based inference is determining which model, among a number of competing ones, is most suitable for the data at hand.

Ergodicity of Adaptive MCMC. For many users of MCMC methods, it becomes quickly apparent that one must carefully tune the transition kernel parameters in order to produce an efficient sampling algorithm. The class of Adaptive MCMC (AMCMC) allows the modification of the Markov chain's transition kernel automatically and "on the fly", i.e. it allows the simulation process to self-adjust at each iteration n based on the information provided by all the samples drawn by that time. Such an approach violates the Markovian property as the subsequent realizations of the chain depend not only on the current state but also on all past realizations. This implies that one can validate theoretically this approach only if one is able to prove from first principles that the adaptive algorithm is indeed sampling from the stationary distribution π.

Exploration and Exploitation for Adaptive MCMC. The designer of an AMCMC algorithm has to keep in mind two crucial tasks the adaptive scheme must achieve. First, the algorithm must exploit efficiently the information obtained each time it modifies its parameters, and second, it must continue to explore and search for new regions of the sample space that have significant importance with respect to the density (or probability mass function) of π. Striking the right balance between these two tasks is the main methodological challenge for any AMCMC algorithm.

The practical implementation of AMCMC samplers requires careful consideration in those cases when the target distribution is multimodal. For instance, a posterior distribution may be multimodal if the sampling distribution is represented as a mixture. The latter occurs when we assume that the population of interest is heterogeneous or when such a formulation is a convenient representation of a non-standard density. It is well known that MCMC sampling from multimodal distributions can be extremely difficult as the chain can get trapped in one region of the sample space due to areas of low probability (bottlenecks) between the modes. Therefore, a large amount of effort has been devoted to designing efficient MCMC sampling methods for multimodal target distributions.

Theoretical Developments. The theoretical developments follow closely the methodological advances described above. For practical implementation and interpretation, a statistical estimator must be provided with an estimator of its variance. The MCMC estimators are no exception, although the dependence between samples can pose challenges to proving a central limit-type theorem. The CLT is known to hold when the MCMC chain is geometrically ergodic. However, it is rather difficult to prove for a general and moderately complex sampling algorithm that its underlying Markov chain is geometrically ergodic. Recent developments strive to simplify the conditions under which geometric ergodicity holds.

Presentation Highlights

High Dimensional Data. Small sample data with large number of variables often occur in genetics, and **Chiara Sabatti** from Stanford University described Bayesian models that can be particularly fruitful in analyzing these data, as well as the computational challenges that they pose. **David van Dyk** from Imperial College London, has focused his talk on the analysis of massive data sets that quickly become standard in astronomy. In this context MCMC requires careful use of existent strategies and the design of new ones. For instance, in the case of high-energy spectral analysis the classical Gibbs sampling is inefficient so a partially collapsed Gibbs sampler is proposed instead. This strategy reduced conditioning in the Gibbs scheme and improves convergence. In the case of generalized linear models with a high number of predictors, Bayesian methods for variable selection pose a great challenge. **Faming Liang** from Texas A&M University introduced a novel Bayesian subset regression model which incorporates idea from regularized likelihood methods within the Bayesian paradigm.

Adaptive MCMC. Jeffrey Rosenthal from the University of Toronto presented the fundamental theoretical results one can use to prove ergodicity for adaptive MCMC and he introduced an interesting counterexample where an intuitively appealing adaptive Gibbs sampler fails to converge, thus clearly proving the need for rigorous theoretical backing of any adaptive MCMC algorithm. **Éric Moulines** and **Gersende Fort** from Telecom Paris-Tech shared a talk on two papers, one on adaptive tempering and the other one on equi-energy sampling. **Nando de Freitas** from University of British Columbia has shown some interesting connections between the field of Bayesian optimization and AMCMC.

Approximate Bayesian Computation. The challenges of performing model selection within the ABC framework have been clearly stated by **Christian Robert** from Universite Paris-Dauphine. His discussion has focused on theoretical results related to the importance of the summary statistics in computing the ABC Bayes factor for two competing models. The difficulties of ABC computation in high dimensions have been discussed by **Scott Sisson** from University of New South Wales. Possible remedies based on regression adjustment and a novel marginal-adjustment strategy were proposed.

Sequential Monte Carlo. **Jun Liu** from Harvard University presented two ideas to improve the efficiency of sequential Monte Carlo methods. One is to allow lookahead in the sequential sampling so that future information can be used in generating samples and computing the weights. The other is to design a sequential rejection control sampler on lower resolution spaces in order to effectively sample the target distribution on a high resolution space. **Gareth Roberts** from University of Warwick presented recent work on a sequential importance sampler which provides online unbiased estimation for irreducible diffusions (that is ones for which the reduction to the unit diffusion coefficient case by the Lamperti transform is not possible).

Theoretical Developments. Novel convergence results for commonly-used Gibbs samplers were reported by **James Hobert** from the University of Florida. He has shown conditions that guarantee the geometrical ergodicity for a Gibbs sampler that is used for performing Bayesian quantile regression. Similarly, geometric ergodicity is shown for the Gibbs sampler used in Bayesian linear mixed models. A novel method for proving geometric ergodicity results for hierarchical models has been introduced by **Krzysztof Latuszynski** from University of Warwick.

New MCMC algorithms. While perfect sampling may be difficult to implement in Bayesian analyses, **Duncan Murdoch** from University of Western Ontario proposed to use near perfect samplers to gain insights on the performance of MCMC samplers. The talk by **Zhiqiang Tan** from Rutgers University compared the efficiency of resampling and subsampling strategies in MCMC.

Topics in Bayesian Inference. **Helene Massam** from York University considered the calculation of Bayes factors for hierarchical loglinear models for discrete data given under the form of a contingency table with multinomial sampling. Motivated by the invariance of copulas to monotone transformations, **Francois Perron** from University of Montreal proposed Bayesian inference for the copula based on ranks. The ABC method is applied to estimate an Archimedean copula using a nonparametric Bayesian approach.

Interactions Between Scientists and Statisticians

Over the past half century, physicists and chemists have made significant contributions to the Monte Carlo literature. Several key Monte Carlo algorithms, such as the Metropolis Algorithm, Gibbs sampler, and the Wang-Landau algorithm, are first invented by physicists and chemists, but are made more general by statisticians years later. The flow of knowledge is mainly unidirectional from physics to statistics, and usually it may take approximately 10 to 15 years before the cross-fertilization process bears fruits. One of the precise aims of this workshop is to shorten this time by having statisticians and physicists together exchanging ideas. Moreover, we believe that statisticians have something to offer to physicists so that the flow of knowledge can be bidirectional.

Lines of research originated in the physics literature that have been recently explored by some of the participants are the zero-variance principle which is a control variates variance reduction technique and the well-tempered ensemble which is the biased ensemble sampled by well-tempered metadynamics. The seminal papers by Assaraf and Caffarel [3] and Per, Snook and Russo [4] were published in the physics literature and propose a very interesting and effective variance reduction method for Monte Carlo simulation. Their original idea, named “Zero Variance", could be useful also to statisticians who make wide use of Monte Carlo and MCMC simulation. Similarly, the papers by Parrinello and his collaborators on the so called “Well Tempered Metadynamics" [5] [6] could offer to statisticians a new way of designing Metropolis-Hastings type algorithms and move between very separated modes in a quite effective way.

Unfortunately the physics literature is in general not easy to read for a statistician since it contains notation and definitions that might not be standard (at least to the statistician) and with which he/she is typically not familiar. This is a serious problem that researchers encounter when trying to transfer knowledge from one scientific area to a different one. Indeed, the first issue in cross fertilization attempts, is to be aware of some interesting principle or idea that can be successfully transferred from one area to another. The second difficulty is the lack of a common background and “vocabulary" that allows the transfer flow of knowledge to be fast and effective.

Two of the speakers invited were Prof. Michele Parrinello (Italian physicist, 2009 Dirac medal, well know for his works in molecular dynamic, affiliated both with the ETH, Zurich, and the USI, Lugano) and Prof. Roland Assaraf (chemist by background, Université Pierre et Marie Curie, Paris VI, expert in quantum and fermion Monte Carlo and efficient MCMC). Besides their brilliant presentations, many conference participants reported that interacting with them was very enlightening and interesting to have a different prospective and to reach a valuable cross-fertilization among disciplines and fields. We are confident that, having these “foreign" experts available for research discussions during the conference, will foster fast scientific progress not only among the statistics and probability community, but also among the chemists and physicists since cross-fertilization is a bi-directional pathway.

Outcome of the Meeting

By bringing together an international group composed of statisticians, physicists, chemists, mathematicians, and computer scientists working in the Monte Carlo field, this meeting provided a platform for exchanging ideas and developing collaborations that will have a positive impact for years to come. Such exchanges have been at the origin of several directions in Monte Carlo and their benefit is still reverberating within scientific communities. This workshop gave participants an update on the current state of the field and identified new challenges in Monte Carlo computation and theory.

The feedback we received from workshop participants is overwhelmingly positive. Some participants feel that this is the “best structured workshop" that they have ever attended, “large enough to have significant content, but intimate such that it was possible to chat with everyone." The participants like the congenial atmosphere at the workshop which is “very conducive to discussions and planting the seeds for new research directions and projects." The balance of structured and unstructured time “allowed for intense exchanges".

The participants are happy to see a mix of experts with different interests (from theory to applications) in Monte Carlo. They enjoyed the presentations and found that the topics are interesting and stimulating. Some participants feel that “it was especially good to see how some serious applications were making substantial use of the theory for guidance."

Many participants said that they learned interesting ideas from the talks, and from informal discussions with the participants. Several participants brought back some ideas from the workshop that they plan to investigate in the near future. Some learned ideas of specific directions for the project they are currently working on. Some participants collaborated with each other at the workshop to write research papers or develop ideas for future publications.

All participants are thankful for superb facilities that the Banff Centre provides, from the accommodations, cafeterias, to the conference space with its video-recording system. BIRS has also been very helpful through the organization of the workshop. We would like to conclude the report by thanking BIRS for making the workshop such a wonderful experience for every participant.

Participants

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Bibliography

[1] G. O. Roberts, A. Gelman and W. R. Gilks, Weak convergence and optimal scaling of random walk Metropolis algorithms, *The Annals of Applied Probability* **7** (1997), 110–120.

[2] J. G. Propp and D. B. Wilson, Exact sampling with coupled Markov chains and applications to statistical mechanics, *Random Structures and Algorithms* **9** (1996), 232–252.

[3] R. Assaraf and M. Caffarel, Zero-variance principle for Monte Carlo algorithms, *Physical Review Letters* **83** (1999), 4682–4685.

[4] M. C. Per, I. K. Snook and S. P. Russo, Zero-variance zero-bias quantum Monte Carlo estimators for the electron density at a nucleus, *Journal of Chemical Physics* **135** (2011), 134112.

[5] A. Barducci, M. Bonomi and M. Parrinello, Metadynamics, *Computational Molecular Science* **1** (2011), 826–843.

[6] G. A. Tribelloa, J. Cuny, H. Eshet and M. Parrinello, Exploring the free energy surfaces of clusters using reconnaissance metadynamics, *Journal of Chemical Physics* **135** (2011), 114109.

Chapter 14

Algebraic Stacks: Progress and Prospects (12w5027)

March 25 - 30, 2012

Organizer(s): Patrick Brosnan (University of Maryland), Roy Joshua (Ohio State University), Hsian-Hua Tseng (Ohio State University)

Overview of the field

The theory of stacks originated with the work of Grothendieck and his students in the 60s (see [Gir65]) as an extension of the notion of an algebraic variety and is closely related to the notion of Grothendieck topologies, introduced by Grothendieck. An affine algebraic variety may be viewed as the zero locus of a collection of polynomial functions. For this to make sense over any commutative ring rather than a field, one needs to allow nilpotents and zero divisors in the ring. These are called affine schemes, and schemes are obtained by gluing together affine schemes making use of the Zariski topology. It is possible to consider objects that look locally like schemes, but globally are not schemes, by performing the above gluing in a different Grothendieck topology. One may view such objects as contravariant functors from the category of schemes to sets satisfying certain gluing conditions. The algebraic spaces introduced by Michael Artin are examples of such functors. Good examples of such functors are the moduli-functors appearing in moduli problems. Such a functor will associate to each scheme U, a family of certain algebraic objects over U satisfying certain conditions. Unfortunately, the fact that these are functors to sets means one has to ignore all automorphisms of these algebraic objects, or in other words one is considering only isomorphism classes of such objects rather than the objects themselves. Since the category of schemes is not closed under colimits, it means that many such functors are usually not representable by schemes or even realized as algebraic spaces.

The idea of algebraic stacks is to consider similar functors, but without modding out by the automorphisms. This makes it necessary to consider what are functors *upto natural isomorphism to the category of groupoids* or *lax-functors to groupoids*. Such lax functors that satisfy certain gluing conditions are stacks and those that look *locally like affine schemes, in a suitable Grothendieck topology* are algebraic stacks. The gluing conditions correspond to *descent data* as in Grothendieck's theory of faithfully flat descent. When the topology used is the étale topology, one obtains what are called Deligne-Mumford stacks: the objects called orbifolds are special cases of Deligne-Mumford stacks that are *smooth* (in a certain sense) and which are generically schemes. For example, if G is a finite group acting on a smooth scheme X, with stabilizers that are trivial generically, the resulting quotient stack $[X/G]$ is an orbifold. Observe that the quotient stack $[X/G]$ is not simply the quotient space X/G, which ignores the stabilizers: instead the quotient stack $[X/G]$ should be viewed as an object sitting over X/G which also keeps track of all the stabilizers. The theory of orbifolds and their cohomology has been of significant interest in recent years, especially to the algebraic topologists.

The class of Artin stacks is much bigger than Deligne-Mumford stacks. As a simple example, if G is a linear algebraic group acting on a smooth scheme X, one obtains the quotient stack $[X/G]$ which classifies principal G-bundles together with a G-equivariant map to X. If the stabilizers for the G-action are not finite, the resulting stack is an Artin stack that is not Deligne-Mumford. As the above example of quotient stacks shows, Artin stacks occur far more commonly than Deligne-Mumford stacks: yet it is only in the last 10 years or so that such stacks have begun to be studied in detail.

Recent developments

After a rather slow beginning, with a few notable papers in the 60s by Mumford and Deligne and Mumford (see [Mum65], [DM69]), and by Michael Artin (see [Ar74]) in the 70s, the theory of algebraic stacks gained acceptance as a main-stream mathematical tool in the last 15 years or so, because of the wide-spread applications that were discovered in diverse fields as mathematical physics, geometric representation theory (especially centered around the geometric Langlands' correspondence), higher topoi and categories, differential graded algebraic geometry and also algebraic topology. In retrospect this is hardly surprising, since algebraic varieties and schemes are often too restrictive to contain solutions of many important problems. The notion of algebraic stacks being a vast generalization of schemes, it is in fact possible to solve many problems in this more general framework which cannot be solved in a more classical setting. Several objects of fundamental importance in mathematics have been constructed only in the framework of stacks.

- For example, many moduli-spaces that cannot be constructed in the framework of schemes have been constructed in the setting of algebraic stacks as in the work of Deligne and Mumford, Mumford and Knudsen (see [MFK94], [MK76]) as well as Kontsevich [Kont95].

- Connections with mathematical physics and Gromov-Witten theory. This is an area where the use of stacks has been highly successful and useful. For example, construction of a virtual fundamental class associated to certain moduli-spaces of stable maps, has been carried out elegantly in stack-theoretic contexts as in the work of Behrend and Fantechi (and also Li and Tian): see [BF97] and [LT98]. Mathematical physics continues to be a source of interesting and challenging problems and recent work of Witten and others have established new connections of this area with mathematical physics: for example between the geometric Langlands' program which requires algebraic stacks just to get off the ground with mathematical physics.

- Some of the applications, notably to Gromov-Witten theory and mathematical physics, necessitated the development of cohomology-homology theories for algebraic stacks, extending them from schemes. The last 15 years or so, saw the development of various cohomology theories for Deligne-Mumford stacks as in the work of Toen, Chen and Ruan: see [T99] and [CR04]. In addition various technical tools to study Deligne-Mumford stacks have been sharpened: for example, the Quot functor, which is extremely useful in algebraic geometry, has only been recently constructed for Deligne-Mumford stacks by Olsson and Starr: see [OS03]. Considerable work on intersection theory on algebraic stacks was done also in view of these applications: currently there is a workable theory of Chow-groups and higher-Chow groups for all Artin stacks. (See [18], [J02], [J07] and [Kr99].

  The study of vector bundles on any geometric object, whether it is a topological space, an algebraic variety or an algebraic stack is of fundamental importance. K-theory is the cohomology theory that studies vector bundles. Several results on the algebraic K-theory of schemes have been extended in recent years to algebraic stacks. For example, in a series of papers, Angelo Vistoli and his collaborators studied this for quotient stacks: see [VV1], [VV2]. Joshua then was able to extend Thomason's basic theorem relating étale cohomology and algebraic K-theory with finite coefficients to fairly general Artin tacks: see [J03]. Moreover [J10] extends several of the basic results in K-theory and G-theory to stacks and differential graded stacks.

- l-adic Derived categories, t-structures and perverse sheaves for algebraic stacks. Though there exist several works from the 90s on this, for example see [J93] and [Be03], the whole machinery of Grothendieck-Verdier duality, t-structures perverse sheaves in the setting of l-adic derived categories were extended in full generality to Artin stacks satisfying certain (mild) finiteness conditions in recent work of Laszlo and Olsson: see [LO08]. Applications to the Langlands' program made this development essential and quite useful.

- Derived and higher stacks and higher categories This is an area of active research by several groups. The manuscript, *Pursuing stacks* by Grothendieck has been a guiding force in the development of this area. Another source of motivation comes from study of *virtual phenomena* in Gromov-Witten theory while another distinct source of motivation was the work of Hopkins and Miller on topological modular forms followed upon by Lurie's work The work of Hirschowitz and Simpson on higher categories (especially higher Segal categories) is a somewhat different approach.
- Connections and techniques of an arithmetical nature. Here the main technique seems to be using the Brauer groups and important recent work in this area is due to Johann Dejong and Max Lieblich. Several impressive results have been obtained using this methods in recent years.
- Toric stacks Just like toric varieties form a nice class of algebraic varieties that can be studied rather easily using combinatorial data, toric stacks have emerged in recent years as a nice class of Artin stacks which can be studied combinatorially. Work in this area seems closely connected with logarithmic geometry: in Martin Olsson's thesis (see [Ol]) provides an explicit connection between log-schemes and algebraic stack, where he constructed a moduli stack of log-structures.

Participation

The workshop planned to and succeeded in bringing together several of the leading experts in related fields along with several post-docs and advanced graduate students working in these areas. In fact most of the recent Ph. Ds working related fields were invited and participated in the workshop. In addition, several graduate students working on related areas also participated.

One of our goals was to combine various camps of mathematicians working in aspects of geometric representation theory, differential graded algebraic geometry, algebraic topology and mathematical physics and who make use of algebraic stacks and stack theoretic techniques from possibly diverse points of view so as to promote exchange of ideas between these various camps. We believe we succeeded in this based on the comments we have received from the participants.

Presentation Highlights

Toric and quotient stacks: The talks by Satriano, Geraschenko and Krishna fell into this area. Geraschenko considered the (old) question of deciding whether a variety with quotient singularities is the quotient of a smooth variety by a finite group. He spoke on results from his joint work with Satriano which answers the question affirmatively for torus quotients.

Satriano considered the notion of "stacky resolution" of a scheme X. Roughly speaking this is a morphism $\mathcal{X} \to X$ from an Artin stack $\mathcal{X}$ to X which is an isomorphism over the smooth locus of X and exhibits X as a good moduli space of $\mathcal{X}$ (in the sense of J. Alper). He gave results proving that stacky resolutions exists for certain X. He also gave some applications of stacky resolutions, the general nature of which is proving a statement for X by proving it for the stacky resolution $\mathcal{X}$ and "push the result to X".

A. Krishna's talk was on equivariant algebraic cobordism which could be viewed as a variant of algebraic cobordism for quotient stacks.

Higher categories and stacks. Both Simpson and Kapranov (who replaced Rydh who could not attend) spoke on closely related results on higher Segal categories.

K-theory and G-theory of stacks. Joshua's talk was on the K-theory and G-theory of algebraic and dg-stacks. After giving an overview, he discussed several more recent results. This involved recent and ongoing computations on the K-theory of toric stacks joint with A. Krishna and also several results of a basic nature on K-theory and G-theory of dg-stacks as well. He concluded with results on a site called the iso-variant étale site that is a replacement for the étale site of a coarse moduli space which may not always exist. This talk had close connections with the talk of Jarod Alper on his construction of a good moduli space for Artin stacks.

Edidin's talk was on producing a new λ-ring structure on the rational K-theory of the inertia stack of toric Deligne-Mumford stacks. Kai Behrend discussed an operator on the Grothendieck groups of algebraic stacks defined by taking the inertia stacks of a given stack. He presented results towards understanding the eigenvalue spectrum of this operator.

Moduli problems. Alper talked about a weak generalization of the Keel-Mori theorem giving conditions on a non-separated algebraic stack which guarantee the existence of a good moduli space. Given a finite group G, if $M_g(G)$, denotes the locus in M_g, consisting of curves which admit an effective action by G, Perroni discussed numerical invariants of the G-action to distinguish irreducible components of $M_g(G)$.

Gerbes, Deformation theory. Vistoli discussed an extension of Nori's fundamental group schemes to schemes/stacks over fields which need not have a base rational point. Tseng discussed a duality between étale gerbes and a pair of disconnected space and a $U(1)$-gerbe on it.

Wise considered the question of classifying deformations and obstructions by cohomology groups. He showed that the ideal result valid for deforming smooth schemes still holds true in the singular case if one uses a suitable Grothendieck topology.

Results of an arithmetic nature. Lieblich discussed a work in progress which shows that the moduli space of supersingular $K3$ surfaces is uniruled. His method involves producing rational curves via moduli spaces of twisted sheaves, which is quite surprising.

Gromov-Witten theory, connections with mathematical physics and quantum geometry. Bryan discussed some results of his Ph.D. student Simon Rose which gives a formula for the number of hyperelliptic curves in an Abelian surface A in terms of quasi-modular form. The formula was derived by first relating the counts of hyperelliptic curves in A with genus 0 Gromov-Witten invariants of the *orbifold* $[A/\pm 1]$, which is then related to Gromov-Witten invariants of the Kummer $K3$ surface via the so-called crepant resolution conjecture. Since the Gromov-Witten theory of $K3$ is explicitly solved by the Yau-Zaslow formula, the aforementioned relations combined to give the result.

Mann talked about a problem involving quantum D-modules for toric nef hypersurfaces. Jarvis discussed recent work with Drew Johnson, Amanda Francis, and Rachel Suggs on the Landau-Ginzburg Mirror Symmetry Conjecture for orbifolded Frobenius algebras for a large class of invertible singularities, including arbitrary sums of loops and Fermats with arbitrary symmetry groups.

Other talks. Noohi discussed classification of forms of an algebraic stack in terms of certain cohomology of a 2-group. He also explained the classification of forms of weighted projective stacks using his result. Vakil talked on stabilization of discriminants in the Grothendieck ring.

Participant Feedback

(Dhillon) There were many valuable interactions and great talks but probably the most valuable for me was some conversations with Zinovy and Angelo regarding essential dimension. In particular their latest paper (arXiv:1103.1611v2). I expect to be able adapt the arguments in the paper to prove the genericity theorem for the moduli stack of vector bundles. The cited arxiv paper provides a powerful tool for controlling the essential dimension of polystable loci.

(Edidin) This is the second Banff conference I've attended. At the first conference in 2008 I met Jarvis and Kimura and we began a collaboration that continues to date. Once again, it looks like I'll be able to gain new collaborators by attending a Banff conference. I think that having participants share a dormitory and common meals definitely helps foster mathematical discussion.

(Kimura) The workshop gave me an opportunity to get together with my collaborators, Dan Edidin and Tyler Jarvis, to continue our work on orbifold and virtual K-theory, power operations, and hyperKahler resolutions. This is generally quite difficult for us to do since we are geographically separated.

(Reichstein) I am a recent convert to algebraic stacks. I attended the workshop to learn more about them from the experts. My own background is in algebraic groups and invariant theory. I connected best to the talks related to these areas, but enjoyed a number of the others as well.

There were two surprising things I learned during the workshop. One was from Dan Edidin, who has reworked my 1988 Ph.D. thesis in a very satisfying way (some of this work Dan did jointly with his postdoc Yogesh More). My thesis was inspired by earlier work of Francis Kirwan. Given the action of a reductive linear group G on an algebraic variety X, the idea is to construct a sequence of blow ups $X_n - \to \ldots \to X_2 \to X_1 = X$ with smooth G-equivariant centers, which "improves" the properties of the GIT quotient $X^{ss}//G$. (Here X^{ss} denotes the open subset of semistable points in X). For example, the initial quotient $X_1^{ss}//G$ may only be categorical, while we may want the final quotient X_n^{ss}/G to be geometric. The final theorem I got was quite nice, but it required some (common in GIT but nevertheless) awkward choices/assumptions along the way, having to do with a linearization of the original action, and carrying this linearization up the chain of blow ups. Dan adopted my resolution procedure to a more general (non-GIT) framework, where the conclusion has to do with the existence of a “good quotient" for a certain G-invariant open subset of X, and no linearizations are involved at any stage.

Another surprise for me was an application of the work I have been involved in on classifying (stably) Cayley simple and semisimple linear algebraic groups. A group G is defined to be Cayley if admits a (partial) algebraic analogue of the exponential map; G is stably Cayley, if $G \times T$ is Cayley for some split algebraic torus T. Stable Cayliness is equivalent to the character lattice of G being quasi-permutation. Surprisingly, this same condition came up in Kai Behrend’s work on the inertia operator on the K-group of stacks. Kai is my colleague at UBC but I was not previously aware of this connection; I intend to think about it some more.

(Simpson) The stacks conference was a great occasion to meet a lot of people whose work I knew of, but whom I hadn’t met in person before; and also to see some colleagues whom I hadn’t seen for many years, as well as some who I meet more regularly of course. I learned a lot about several subjects, perhaps foremost among them, the relationship between stacks and different aspects of mirror symmetry. This includes a very interesting discussion with Jim Bryan about hypergeometric functions and Givental’s theorem. Other new things include different aspects of Brauer groups, and also Kapranov’s talk on higher Segal conditions which might well bear some analogies with a seemingly unrelated subject of 2-metric spaces that I have been thinking about. I also hope to have been able to provide some helpful replies to people’s queries about higher categories, parabolic bundles, Higgs bundles and fundamental groups. While none of my students came, I did make contact with someone at the place where my student will be going for a post-doc next year, so that should be really helpful.

The setting was really wonderful (this was my first visit to Banff), the food was awesome, and everything was extremely well organized.

(Vakil) A number of conversations helped me both in my ongoing work, and in understanding that will lead to future work. Most substantively, (i) discussions with Angelo Vistoli on Chow groups of infinite symmetric products will substantively help a paper I’m writing; (ii) discussions with Donu Arapura answered some important questions I had in this paper as well, and will lead to him visiting me in the fall; (iii) a series of discussions with Matt Satriano led to an ongoing project (we’ll see how it goes). But I had a number of other mathematical significant discussions, and from experience I know that some of them will have more impact on my work than the ones I’ve already listed; but it’s hard to know in advance which!

Bibliography

[Ar74] M. Artin, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165–189.

[BF97] K. Behrend and B. Fantechi, *The Intrinsic Normal Cone*, Invent. Math. **128** (1997), 45–88.

[Be03] K. Behrend, *Derived l-adic categories for algebraic stacks*, Mem. Amer. Math. Soc., **163** (2003) no. 774, viii+93.

[CR04] W. Chen and Y. Ruan, *A new cohomology theory for orbifolds*, Comm. Math. Phys. **248** (2004), 1–31.

[DM69] D. Mumford and P. Deligne, *The irreducibility of the space of curves of a given genus*, IHES Publ. Math. 36 (1969), 75–109.

[EG] D. Edidin and W. Graham, *Equivariant intersection theory*, Invent. Math, **131**, (1998), 595-634.

[Gir65] J. Giraud, *Cohomologie non-abeliene*, Die Grundlehren der mathematischen Wissenschaften, Band 179. Springer-Verlag, Berlin-New York, 1971. ix+467 pp.

[Hop02] M. Hopkins, *Algebraic Topology and Modular Forms*, in "Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)", 291–317, Higher Ed. Press, Beijing, 2002.

[J93] R. Joshua, *The derived category and intersection cohomology of algebraic stacks*, in "Algebraic K-theory and Algebraic Topology (Lake Louise, AB, 1991)", 91–145, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 407, Kluwer Acad. Publ., Dordrecht, 1993.

[J03] R. Joshua, *Riemann-Roch for algebraic stacks: I*, Compositio Math. **136** (2003), no. 2, 117–169.

[J02] R. Joshua, *Higher Intersection Theory on Algebraic Stacks:I and :II*, K-Theory, **27** (2002), no. 2, 134–195. and **27** (2002), no. 3, 197–244.

[J07] R. Joshua, *Bredon-style homology, cohomology and Riemann-Roch for algebraic stacks*, Adv. Math. **209** (2007), no. 1, 1–68.

[J10] *K-theory and G-theory of dg-stacks*: to appear in The Proceedings of the regulator Conference, Barcelona, (2010).

[Kont95] M. Kontsevich, *Enumeration of rational curves via torus actions*, in "The moduli space of curves (Texel Island, 1994)", 335–368, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.

[Kr99] A. Kresch, *Cycle groups for Artin stacks*, Invent. Math. **138** (1999), no. 3, 495–536.

[LO08] Y. Laszlo and M. Olsson, *The six operations for sheaves on Artin stacks I*, Publ. Math. IHES, **107**, (2008), 109-168.

[LT98] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc. **11** (1998), no. 1, 119–174.

[Lur04] J. Lurie, *Derived Algebraic Geometry*, PhD thesis, MIT, 2004.

[Lur09] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, 170. Princeton University Press, Princeton, NJ, 2009. xviii+925 pp.

[Lur09a] J. Lurie, *A survey of elliptic cohomology*, in “Algebraic topology”, 219–277, Abel Symp., 4, Springer, Berlin, 2009.

[Mum65] D. Mumford, *Picard groups of moduli problems*, in “Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)”, 33–81, Harper & Row, New York, 1965.

[MK76] F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves. I: Preliminaries on "det" and "Div"*, Math. Scand. **39** (1976), no. 1, 19–55.

[MFK94] D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*. Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) (Results in Mathematics and Related Areas (2)), 34. Springer-Verlag, Berlin, 1994.

[Ol] M. Olsson, *Logarithmic geometry and algebraic stacks*, Ann. Sci. d’ENS 36 (2003), 747-791

[OS03] M. Olsson and J. Starr, *Quot functors for Deligne-Mumford stacks*, Comm. Algebra **31** (2003), no. 8, 4069–4096.

[S96] C. Simpson, *Algebraic (geometric) n-stacks*, arXiv:alg-geom/9609014.

[T99] B. Toen, *Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford*, K-Theory **18** (1999), no. 1, 33–76.

[TVa07] B. Toen and M. Vaquié, *Moduli of objects in dg-categories*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 3, 387–444.

[TV05] B. Toen and G. Vezzosi, *Homotopical algebraic geometry. I. Topos theory*, Adv. Math. **193** (2005), no. 2, 257–372.

[TV08] B. Toen and G. Vezzosi, *Homotopical algebraic geometry. II. Geometric stacks and applications*, Mem. Amer. Math. Soc. **193** (2008), no. 902, x+224 pp.

[VV1] G. Vezzossi and A. Vistoli, *Higher Algebraic K-theory of group actions with finite stabilizers*, (2000)

[VV2] G. Vezzossi and A. Vistoli, *Higher Algebraic K-theory for actions of diagonalizable groups*, Invent. Math, (2002)

Final List of participants

Total number of participants including organizers: 33

	Name	Affiliation
1	Alper, Jarod	Columbia University
2	Arapura, Donu	Purdue University
3	Behrend, Kai	University of British Columbia
4	Beke, Tibor	University of Massachusetts Lowell
5	Bergh, Daniel	Stockholm University
6	Brosnan, Patrick	University of Maryland
7	Bryan, Jim	University of British Columbia
8	Chen, Jiun-Cheng	National Tsing-Hua University
9	Dhillon, Ajneet	University of Western Ontario
10	Edidin, Dan	University of Missouri
11	Geraschenko, Anton	California Institute of Technology
12	Jarvis, Tyler	Brigham Young University
13	Joshua, Roy	Ohio State University
14	Kapranov, Mikhail	Yale University
15	Kimura, Takashi	Boston University
16	Krishna, Amalendu	Tata Institute Of Fundamental Research
17	Lieblich, Max	University of Washington
18	Lowrey, Parker	University of Western Ontario
19	Mann, Etienne	Universite de Montpellier 2
20	Moret-Bailly, Laurent	UniversitÃl’ de Rennes 1
21	Morrison, Andrew	University of British Columbia
22	Nitsure, Nitin	Tata Institute of Fundamental Research
23	Noohi, Behrang	University of London
24	Perroni, Fabio	Universitaet Bayreuth
25	Reichstein, Zinovy	University of British Columbia
26	Satriano, Matthew	University of Michigan
27	Sheshmani, Artan	University of British Columbia
28	Simpson, Carlos	University of Nice
29	Sorkin, Adam	University of California Davis
30	Tseng, Hsian-Hua	Ohio State University
31	Vakil, Ravi	Stanford University
32	Vistoli, Angelo	Scuola Normale Superiore di Pisa
33	Wise, Jonathan	Stanford University

Chapter 15

Stochastic Analysis & Stochastic Partial Differential Equations' (12w5023)

April 1 - 6, 2012

Organizer(s): Robert C. Dalang, Davar Khoshnevisan, Yimin Xiao

The Banff 5-day workshop 12w5023 was held from Monday April 1 to Friday April 6, 2012 at the conference site in Banff International Research Station in Alberta. The conference was mainly concerned with the fine analysis of stochastic partial differential equations, hereforth referred to as SPDEs, for the sake of brevity. This topic has experienced tremendous growth, particularly in the past decade for at least two significant reasons: First, a number of central open problems of this area have been solved, and/or are on the verge of being solved; and second, the topic has continued input from other scientific disciplines, including but not limited to mathematics. Therefore, the organizers felt, and still feel, that the conference was timely.

The main objectives of this meeting were to bring together some of the leading figures in the analysis of SPDEs, together with select highly-promising young researchers in order to present their recent findings, as well as identify key research problems/areas within the general topic of SPDEs and related fields. The organizers feel strongly that these objectives were met, and that the conference was a success.

The conference was organized roughly as follows: On Monday April 1 through Thursday April 5, the morning talks began at 9:00 a.m. with several half-hour research talks by various leading experts. There was also one 1-hour plenary talk every day in order to introduce to the younger audience aspects of the "big picture" in the modern going-ons of research in SPDEs. These talks were delivered, in this order, by Professors Carl Mueller [University of Rochester], Michael Röckner [Bielefeld University], David Nualart [University of Kansas], and Robert Dalang [Ecole Federale de Lausanne]. The evenings of Monday, Tuesday, and Thursday, and all of Friday were dedicated to informal breakout research sessions wherein the audience would spend 5–20 minutes describing one, or a series, of open problems and/or directions of modern research interest. These breakout sessions were informal but highly well-attended, and have led to current potential research collaborations among various combinations of the participants. The following is a brief, but still more detailed, synopsis of the lectures at the conference.

Carl Mueller opened the conference with a plenary 1-hour talk on a central open problem for a family of SPDEs that arise in population genetics. The problem is basically the following: Consider the SPDE

$$\frac{\partial}{\partial t}u(t\,,x) = \kappa\frac{\partial^2}{\partial x^2}u(t\,,x) + \rho(u(t\,,x))\,\dot{W}(t\,,x), \tag{15.0.0.1}$$

subject to zero initial data, where $\kappa > 0$ denotes a "viscosity term," $\dot{W}$ denotes space-time white noise, and $\rho(x) = |x|^{\alpha}$. Clearly, $u(t\,,x) = 0$ is a solution; the question is whether or not $u(t\,,x) = 0$ is the only solution. The particular case that $\alpha = 1/2$ arises most prominently in population genetics, where any solution has the law of the socalled "Brownian density process." Professor Mueller presented his recent non-uniqueness result [joint with L. Mytnik and E. Perkins]: If $\alpha < 3/4$ then the solution is not unique. A major open problem that was introduced is to see whether or not uniqueness holds among all "physical solutions," in this case "all nonnegative solutions." More recently, another participant of this workshop, Mr. Y.-T. Chen, has completed a thesis under the supervision of Professor Edwin Perkins [The University of British Columbia]. Mr. Chen's thesis proves that if we add nonvoid immigration term to the SPDE (15.0.0.1), then the solution is not unique even among non-negative solutions.

Wenbo Li's lecture gave a bird's-eye view of a number of recent developments in the general theory of "small-value probabilities." A number of novel connections of this topic to SPDEs, branching processes, tauberian theory, Gaussian processes, statistical mechanics [Edwards model] etc. were pointed out. Roughly speaking, the area of small-value probabilities is concerned with the asymptotic behavior of probabilities of the type $\mathrm{P}\{W \le \epsilon\}$ as $\epsilon \downarrow 0$, where W is an interesting non-negative random variable. Among other things, Wenbo Li introduced his striking work [joint with Q.-M. Shao] on d-parameter Brownian sheet W: There exist universal constants c_1 and c_2 such that for all $\epsilon > 0$ sufficiently small,

$$\exp\left(-c_1 \left|\log(1/\epsilon)\right|^d\right) \le \mathrm{P}\left\{\sup_{(s,t)\in[0,1]^d} W(s\,,t) < \epsilon\right\} \le \exp\left(-c_2 \left|\log(1/\epsilon)\right|^d\right).$$

A vast array of open problems were presented.

Daniel Conus presented his research on intermittency and chaos for various stochastic systems. In particular, he presented his work [joint with M. Joseph & D. Khoshnevisan] which shows that the solution to the stochastic heat equation (15.0.0.1) with $\alpha = 1$ and initial data [say] $u(0\,,x) \equiv 1$—this is the socalled parabolic Anderson model of mathematical physics—has the following "KPZ scaling property": There exist positive and finite universal constants c_1 and c_2—depending only on the time variable $t > 0$—such that almost surely for all R large,

$$\exp\left(c_1 \frac{(\log R)^{2/3}}{\kappa^{1/3}}\right) \le \sup_{|x|<R} u(t\,,x) \le \exp\left(c_2 \frac{(\log R)^{2/3}}{\kappa^{1/3}}\right).$$

Sandra Cerrai lectured on her work [joint in part with M. Friedlin] on 2-D stochastic Hamiltonian systems of the type

$$\mu \frac{\partial^2}{\partial t^2} q^\mu(t) = b\left(q^\mu(t)\right) + A_0 \frac{\partial}{\partial t} q^\mu(t) + \sigma\left(q^\mu(t)\right) \dot{W}(t),$$

subject to $q^\mu(0) := q \in \mathbf{R}^2$ and $\partial q^\mu(t)/\partial t := p \in \mathbf{R}^2$. Here, $\mu > 0$ is a positive parameter and $\dot{W}$ denotes white noise on $[0\,,\infty)$. When the the real parts of the eigenvalues of A_0 are strictly negative, then the preceding is a generalized 2-D Langevin equation [$\partial q^\mu/\partial t :=$ friction], and Friedlin [2005; also, Chen] have shown that the following Kramers–Smoluchowski approximation is valid for every $T, k > 0$:

$$\lim_{\mu\downarrow 0} \mathrm{E}\left(\sup_{t\le T} \left|q^\mu(t) - q(t)\right|^k\right) = 0,$$

where q solves the stochastic differential equation $\mathrm{d}q = (b \circ q)\,\mathrm{d}t + (\sigma \circ q)\,\dot{W}$. Cerrai's talk addressed the remaining case which corresponds to when the Hamiltonian system is describing *charged* particles in a magnetic field. She showed how the Hamiltonian system needs to be regularized, in that case, and presented a homogenization theorem for the regularized equation. As a consequence of this development, Cerrai showed a type of "propagation of chaos" result.

Raluca Balan's lecture revolved around her work on novel linear SPDEs that are driven by fractional noises. An example of her work is the following: Consider the SPDE

$$\frac{\partial}{\partial t} u(t\,,x) = (\mathcal{L}u)(t\,,x) + \dot{W}(t\,,x), \tag{15.0.0.2}$$

with zero initial data, where $\mathcal{L} := -(-\Delta)^{\beta/2}$ denotes the fractional Laplacian of order $\beta/2$, and $\dot{W}$ is a Gaussian noise that is white in space, and whose temporal covariance kernel is of a fractional Brownian motion type,

$$R_H(t\,,s) := H(2H-1)\int_0^t \mathrm{d}u \int_0^s \mathrm{d}v\ |u-v|^{2H-2}. \tag{15.0.0.3}$$

Balan presented a necessary and sufficient condition for the existence of a random field solution to (15.0.0.2) [joint with C. Tudor]. These works were shown to be connected to a new sort of probabilistic potential theory, in particular, to weighted local times functionals

$$L_t(\varphi) := \beta \int_0^t \mathrm{d}r \int_0^t \mathrm{d}s\ |r-s|^{2H-2}\varphi(X_r - Y_s),$$

where X and Y are two independent β-stable Lévy processes. An important ingredient of the existence proof rested on developing a new maximum principle for a corresponding weighted potential kernel. Related results on hyperbolic equations were also presented.

John Walsh's lecture closed the Monday lecture sessions, and contained a new method for the numerical analysis of linear SPDEs of the type,

$$\frac{\partial}{\partial t}u(t\,,x) = \frac{\partial^2}{\partial x^2}u(t\,,x) + \dot{W}(t\,,x), \tag{15.0.0.4}$$

where $\dot{W}$ denotes space-time white noise on $[0\,,1]\times\mathbf{R}$, and the SPDE has a "nice" initial value. It is well known that solution is Hölder continuous of any order $< 1/2$ in the x variable and any order $< 1/4$ in the t variable.

If one discretizes (15.0.0.4)—with respective mesh sizes Δt and Δx for the time and space variables—then the universal error-rates of Davies and Gaines show that the error rate in the resulting numerical scheme is of rough order $\max\{(\Delta t)^{1/4}\,,(\Delta x)^{1/2}\}$. It follows that we need to adopt $\Delta t \approx (\Delta x)^2$ for best results. Walsh's lecture was concerned about the practical problem of making more precise the meaning of "$\approx$" in the preceding discussion. In other words, Walsh proved that there typically is a canonical choice of $c > 0$ such that $\Delta t = c(\Delta x)^2$ is optimal; the value of c depends, among other things, on the particulars of the numerical method that is being used. For instance, when one applies the Crank–Nicholson numerical scheme for solving (15.0.0.4), then Walsh's striking optimal choice is $\Delta t = (\Delta x)^2/(\pi-2)$. A series of open problems were also introduced.

Michael Röckner's was the second 1-hour plenary talk of the conference. His lecture was on the regularization of ODEs and PDEs by noise [joint with V. Bogachev, G. Da Prato, N.V. Krylov, E. Priola, and S. Shaposhnikov]. As an example of this general theory, Röckner presented the following ∞-dimensional SDE/SPDE: Let H be a separable Hilbert space, $B : H \to H$ a nice vector field, and $\sigma : H \to L(H)$ measurable. Then, consider the stochastic differential equation,

$$\mathrm{d}X_t^x = B(X_t^x)\,\mathrm{d}t + \sigma(X_t^x)\,\mathrm{d}W_t^x,$$

subject to $W_0^x := x \in H$. Then, it was shown that $p_t f(x) := \mathrm{E}f(X_t^x)$ solves the Fokker–Planck equation, $(\mathrm{d}/\mathrm{d}t)\,p_t f(x) = p_t(Lf)$, and the operator L has a second-order part [because of the noise] which has a regularizing effect on the Fokker–Planck equation. Underlying this theory lies a new method of characteristics for PDEs, which now involves also ideas from the Itô calculus. This method is shown to lead to a uniqueness theorem for the Fokker–Planck equation. Röckner went on to show how to extend this theory in order to establish the existence of pathwise solutions to various infinite-dimensional SDEs that are driven by a "large" white-noise forcing term.

Yaozhong Hu presented a novel Feynman–Kac representation for parabolic Anderson model (15.0.0.1) [joint with D. Nualart and J. Song], where $\dot{W}$ now denotes a Gaussian noise with covariance form

$$\mathrm{E}\dot{W}(t\,,x)\dot{W}(s\,,y) = R_{H_0}(s\,,t)\cdot\prod_{j=1}^{d} R_{H_i}(x_i\,,y_i),$$

where R_H was defined in (15.0.0.3). Hu's talk established exactly when the preceding SPDE has a [Stratonovich] solution, and that when there is a solution, it has a Feynman–Kac formulation. The sufficient condition for the existence and uniqueness of a weak solution was shown to be: $H_0, \ldots, H_d > \frac{1}{2}$; and $2H_0 + \sum_{i=1}^{d} H_i > d+1$. In that case, the solution exists, $u(t\,,x)$ has a finite moment generating function near the origin, $(t\,,x) \mapsto u(t\,,x)$ is Hölder continuous a.s., and the law of $u(t\,,x)$ is absolutely continuous with respect to the Lebesgue measure. In the case that $\mathrm{E}\dot{W}(t\,,x)\dot{W}(s\,,y) = R_H(s\,,t)Q(x\,,y)$ for a bounded, jointly Lipschitz continuous Q, and $1/4 < H < 1/2$, Hu showed the existence of a Feynman–Kac formula.

Le Chen's lecture was on stochastic heat equations of the type (15.0.0.1), where ρ is Lipschitz and satisfies the lower cone condition $\inf_x |\rho(x)/x| > 0$. Motivated by statistical mechanics [where $\rho(x) \propto x$ and $u(0\,,x) = \delta_0(x)$], Chen showed his analysis of the preceding SPDE [joint with R. Dalang], wherein they prove that the preceding SPDE has a unique random-field solution provided that $u(0\,,\cdot)$ is a termpered signed measure.

Chen also presented a solution to an open problem of Conus and Khoshnevisan (2012) about the existence of *L^2-intermittency fronts* of the solution. An example of his general theorem is the following: Consider the parabolic Anderson model wherein $\rho(x) = \lambda x$ for some $\lambda > 0$. Then, the lower and upper L^2-intermittency fronts agree and are equal to $\lambda^2/2$; more precisely, for all $\alpha > \lambda^2/2$ and $\beta < \lambda^2/2$,

$$\liminf_{t\to\infty} \frac{1}{t} \sup_{|x|>\alpha t} \log \mathrm{E}\left(|u(t\,,x)|^2\right) < 0, \quad \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|>\beta t} \log \mathrm{E}\left(|u(t\,,x)|^2\right) > 0.$$

This material was borrowed from Mr. Chen's PhD thesis under the supervision of Professor R. Dalang [Ecole Federal de Lausanne], and has now been completed.

Leonid Mytnik's lecture was about his work on the multifractal analysis of 1-dimensional super processes with β-stable branching. Let $\{X_t\}_{t\geq 0}$ denote a $(2\,,d\,,\beta)$ super process with β-stable branching, in the sense of Dawson, Perkins, ... Mytnik considers the case that $d = 1$ and $\beta < 1$; this is precisely when $t \mapsto X_t$ is a discontinuous measure-valued process with pure jumps of the form $r\delta_x$, and yet $X_t(\mathrm{d}x) \ll \mathrm{d}x$ a.s. In that case, we can $X_t(x)$ for $X_t(\mathrm{d}x)/\mathrm{d}x$, and $X_t(x)$ solves the SPDE

$$\frac{\partial}{\partial t} X_t(x) = \frac{\partial^2}{\partial x^2} X_t(x) + [X_{t-}(x)]^{1/(1+\beta)}\, \dot{L}(t\,,x),$$

where $\dot{L}$ denotes a $(1+\beta)$-stable Lévy noise with no negative jumps [Mytnik, 2002]. Perkins and Mytnik (2003) proved that X_t is a.s. continuous for all $t > 0$ fixed, and Fleischmann and Wachtel (2010) established that whenever $0 < \eta < \eta_c := 2/(\beta+1) - 1$, X_t is a.s. Hölder continuous of order η, and that η_c is an optimal choice. Finally, Mytnik introduced two novel results. The first [joint with Fleischmann and Wachtel] shows that if x and $t > 0$ are fixed then the optimal Hölder exponent at x is

$$\bar{\eta}_c := \min\left[\eta_c + \frac{1}{1+\beta}\;,\; 1\right].$$

For his second main theorem, Mytnik defined the space $C^\eta(x)$ as the space of all functions $f : \mathbf{R} \to \mathbf{R}$ for which we can find a finite constant c and a polynomial P_x of degree $\leq \lfloor \eta \rfloor$ such that $|f(y) - P_x(y)| \leq c|x-y|^\eta$, globally. Define $H(x) := \sup\{\eta : X_t \in C^\eta(x)\}$ as a new measure of optimal Hölder regularity. Then Mytnik showed that if $\eta < 1$ and X_0 is a finite measure, then the measure X_t has the following multifractal behavior: For every open set $U \subset \mathbf{R}$ and $\eta \in [\eta_c\,, \bar{\eta}_c]$,

$$\dim_{\mathrm{H}} \{x \in U :\; H(x) = \eta\} = (\beta+1)(\eta - \eta_c),$$

a.s. on $\{X_t(U) > 0\}$.

Martina Zähle's presented SPDEs driven by gradient noises of the type

$$\partial_t u = -\mathcal{A}^\theta u + F(u) + G(u) \cdot \dot{Z},$$

where $\mathcal{A}$ is the generator of an ultra-contractive semigroup $\{P_t\}_{t\geq 0}$ that has positive and finite spectral dimension, $\theta \leq 1$, and $\dot{G}$ is an arbitrary [possibly random] element of the function space $C^{1-\alpha}([0\,,T]\,,H_2^{\theta\beta}(\mu)^*)$. The main result of this lecture is that the preceding SPDE has a pathwise unique mild solution in the Sobolev space $W^\gamma([0\,,T]\,,H_2^{\theta\delta}(\mu))$ for a suitable choice of $\delta > 0$ [joint with M. Hinz and E. Issoglio].

Franceso Russo discusses his recent on stochastic and non-stochastic porous media equations [joint with V. Barbu, M. Röckner, N. Belaribi, and F. Cuvelier]. These are models of self-organized criticality that are supposed to describe, for example, the evolution of snow flakes. The general form of the model is

$$\frac{\partial}{\partial t}u(t\,,x) = \frac{1}{2}\Delta(\beta(u)) + u(t\,,x)\dot{W}(t\,,x), \tag{15.0.0.5}$$

where $\dot{W}$ denotes a Gaussian noise that is white in time and possibly nice in x. In the non-random setting, one obtains the PDE $\partial_t u(t\,,x) = \frac{1}{2}\Delta(\beta(u))$, which is supposed to be solved in $L^1(\mathbf{R}^d)$ for example because it is known the the solution, if any, can have discontinuities when the function β is large. An important example is when $\beta(u) = uH(u - e_c)$ for a nice function H and a "critical parameter" e_c. Russo presented a theorem that states that the preceding PDE has a weak solution when β is continuous. Moreover, that solution can be characterized as degenerate [versus non-degenerate] if and only if a certain explicitly-defined function $\Phi = \Phi_\beta$ vanishes at zero. Furthermore, there is a corresponding Fokker–Planck equation. Relations to the stochastic PDE with multiplicative noise were also introduced. Most remarkably, it was pointed out that the stochastic problem hinges on a stochastic Fokker–Planck equation that is of independent interest.

Martina Hoffmanová introduced the main findings of her PhD thesis [under the supervision of Professor A. Debussche] which has quite recently been approved and completed. This talk's main results are on wellposedness problems for kinetic solutions to degenerate parabolic SPDEs such as

$$\mathrm{d}u + \mathrm{div}(B(u))\,\mathrm{d}t = \mathrm{div}(A(x)\nabla u)\,\mathrm{d}t + \Phi(u)\,\mathrm{d}W,$$

where W is a cylindrical Brownian motion in a separable Hilbert space H, $A : \mathbf{T}^n \to \mathbf{R}^{N\times N}$ is a smooth and symmetric positive semidefinite matrix, $B : \mathbf{R} \to \mathbf{R}^N$ is a C^1 flux function of at-most polynomial growth, and $\Phi(z) : H \to L^2(\mathbf{T}^N)$ has linear growth and is coordinatewise Lipschitz for every $z \in L^2(\mathbf{T}^N)$. Using related non-random conservation laws, Hoffmanová introduced a notion of a *kinetic solution*, and went on to prove that if the initial function u_0 is in $L^p(\Omega\,, L^p(\mathbf{T}^N))$, then the degenerate SPDE (15.0.0.5) has a unique kinetic solution that is continuous in its initial data.

Xia Chen studied the stationary parabolic Anderson model,

$$\frac{\partial}{\partial t}u(t\,,x) = \frac{1}{2}\Delta u(t\,,x) + u(t\,,x)V(x),$$

for $x \in \mathbf{R}^d$ and $t \geq 0$, subject to $u(0\,,x) = 1$, where the random potential V has any one of the following four types: (i) V is a stationary Gaussian process with mean zero and covariance function; (ii) V is fractional white noise; that is, $V(x) = \partial^d W^H(x)/\partial x_1 \cdots \partial x_d$, for a fractional Brownian motion W^H with Hurst vector $H := (H_1\,,\ldots\,,H_d)$; (iii) $\{V(x)\}_{x\in\mathbf{Z}^d}$ is a spatial white noise; and (iv) V has the following representation in terms of a white noise W on $\mathbf{R}^d$: $V(x) = \int_{\mathbf{R}^d} \|y - x\|^{-p}\,W(\mathrm{d}y)$.

For case (i) it had been conjectured by Carmona and Molchanov that if the spectral density of V at x behaves as $C/\|x\|^\alpha$ as $\|x\| \to 0$, for some $0 < \alpha < 2 \wedge d$, then $\log u(t\,,0) \sim Ct(\log t)^{(4-\alpha)/(2-\alpha)}$ as $t \to \infty$. Chen shows that this conjecture is incorrect; the correct form is the following:

$$\log u(t\,,0) \sim \kappa(\alpha)[c(d\,,\alpha)]^{2/(4-\alpha)} \cdot t(\log t)^{2/(2-\alpha)} \qquad \text{as } t \to \infty, \tag{15.0.0.6}$$

where $\kappa(\alpha)$ is an explicitly-defined numerical quantity that depends solely on α, and $c(d\,,\alpha)$ is the optimal constant in the following Sobolev inequality:

$$\iint\limits_{\mathbf{R}^d\times\mathbf{R}^d} \frac{[f(x)f(y)]^2}{\|x-y\|^\alpha}\,\mathrm{d}x\,\mathrm{d}y \leq c(d\,,\alpha)\cdot\|f\|_2^{4-\alpha}\|\nabla f\|_2^\alpha,$$

valid for all $f \in W^{1,2}(\mathbf{R}^d)$. Similar results are shown for cases (ii) and (iv). In both of these cases, (15.0.0.6) holds with the respective choices $\alpha := 2p - 2\sum_{j=1}^{d} H_j$ [for (ii)] and $\alpha = 2p - d$ [for (iv)]. Finally, Chen showed that in case (iii) [white noise], (15.0.0.6) has to be adjusted as follows:

$$\log u(t\,,0) \sim \frac{1}{2}\left(\frac{3}{2}\right)^{2/3} t(\log t)^{2/3} \qquad \text{as } t \to \infty.$$

In all cases, the proofs involved a delicate large-deviations analysis of the Feynman–Kac formula, $u(t\,,0) = \mathrm{E}(\exp\{\int_0^t V(B_s)\,\mathrm{d}s\} \,|\, V)$, where B denotes a Brownian motion.

David Nualart delivered the third 1-hour plenary talk of the workshop. Nualart's lecture began with a brief overview of the Malliavin calculus, and in particular, the Nourdin–Viens formula for the density of elements of Malliavin's probabilistic Sobolev space $\mathbf{D}^{1,2}$. Nualart's lecture then proceeded by showing how one can apply these ideas from Malliavin's calculus in concrete problems of SPDEs. As a first example, Nualart presented [a more general form of] the following theorem: Suppose u solves (15.0.0.1) with nice initial data, where $\dot{W}$ is white in time and spatially homogeneously correlated with a spectral density f that satisfies $\int_{\mathbf{R}^d}(1+z^2)^{-\eta} f(z)\,\mathrm{d}z < \infty$ for some $\eta \in (0\,,3/4)$, then $u(t\,,x)$ has an absolutely-continuous distribution with a density function that satisfies the heat-kernel bounds

$$\frac{c_0}{t^{1-\eta}} \exp\left(-\frac{-(z-m)^2}{c_1 t}\right) \le p(z) \le \frac{c_2}{t} \exp\left(-\frac{-(z-m)^2}{c_3 t}\right).$$

Next, Nualart discusses degenerate SPDEs of the form

$$\mathcal{H}X_t(x) = -\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(h(y-x)X_t(x)\right) \frac{\partial^2}{\partial t \partial x} W(t\,,y)\,\mathrm{d}y + \sqrt{X_t(x)}\frac{\partial^2}{\partial t \partial x} V(t\,,x),$$

where $\mathcal{H} := \partial_t - \partial^2_{xx}$ denotes the heat operator W and V are independent Brownian sheets, and $h \neq 0$ is a nice function. Such SPDEs arise as continuum limits of interacting particle systems [Dawson, Vaillancourt, Wang, ...]. An open question in this area is to establish the Hölder continuity of the solution $X_t(x)$. Recently (2011) Li, Wang, Xiong, and Zhou have proved that the solution is Hölder continuous for any index $< 1/10$ in t and $< 1/2$ in x. Nualart showed how one can use the Malliavin calculus in a clever way in order to establish the desired Hölder continuity in t of an arbitrary index $< 1/4$ [joint work with Lu and Hu].

Frederi Viens began his lecture with an accessible introduction to the Malliavin calculus and Stein's equation. He then showed how to obtain the following remarkable inequality: If $X \in \mathbf{D}^{1,2}$ is an otherwise arbitrary centered random variable and Z is a standard normal random variable, then

$$\sup_{z\in\mathbf{R}} \left|\mathrm{P}\{X > z\} - \mathrm{P}\{Z > z\}\right| \le \mathrm{E}\left(|1 - G_X|\right), \tag{15.0.0.7}$$

where G_X is the random variable that is defined uniquely via the Nourdin–Peccati integration by parts formula $\mathrm{E}(X\varphi(X)) = \mathrm{E}(\varphi'(X)G_X)$, valid for all smooth and bounded functions φ. Viens then showed how one can apply the inequality (15.0.0.7) to the study of a family of continuous 1-dimensional polymer measures. Namely, he considered the parabolic Anderson model

$$\frac{\partial}{\partial t}u(t\,,x) = \frac{\partial^2}{\partial x^2}u(t\,,x) + u(t\,,x)\frac{\partial}{\partial t}W(t\,,x), \tag{15.0.0.8}$$

with $t > 0$ and $x \in \mathbf{R}^d$ and $u(0\,,x) \equiv 1$, where W is a centered Gaussian process with covariance form

$$\mathrm{E}\dot{W}(t\,,x)\dot{W}(s\,,y) = \min(s\,,t)Q(x-y),$$

for a bona fide correlation function Q on $\mathbf{R}^d \times \mathbf{R}^d$ that is bounded [i.e., $Q(0) < \infty$]. The solution to (15.0.0.8) exists and is related to the partition function Z_t^W of a 1-D polymer in random environment W, where $Z_t^W := \mathrm{E}(\exp\{\int_0^t W(\mathrm{d}s\,\mathrm{d}b_s)\} \,|\, W)$, for an independent Brownian motion b. The polymer measure is then $\tilde{P}$, whose Radon–Nikodým derivative is $\mathrm{d}\tilde{P}/\mathrm{dP} = (Z_t^W)^{-1}\exp\{\int_0^t W(\mathrm{d}s\,\mathrm{d}b_s)\}$, where P denotes the

Wiener measure. [The "curvilinear stochastic integral" in question is known to exist.] An important question in this area of statistical mechanics is to understand the behavior of the variance of $\log Z_t^W$ as $t \to \infty$.

When $d = 1$ and W is space-time white noise, it is generally believed that $\mathrm{Var}(\log Z_t^W) = t^{(2/3)+o(1)}$ for large t. This conjecture has been verified in recent work by Balazs, Quastel, and Seppäläinen in the case that $u(0\,,x)$ is the exponential of an independent two-sided Brownian motion [this is the invariant measure]. When $Q(0) = 0$—say when $Q(x) = 1 - x^{2\alpha} + o(x^{2\alpha})$ for $x \approx 0$—it is believed that $\mathrm{Var}(\log Z_t^W) = t^{2\chi(\alpha)+o(1)}$ as $t \to \infty$, where $1/3 \le \chi(\alpha) \le 1/2$. Moreoever, it is believed that all values of $\chi(\alpha)$ in this interval are achievable for different models.

Viens proves that if $\inf_{x\in\mathbf{R}^d} Q(x) > 0$, then the conjecture always holds with $\chi(\alpha) = 1/2$. The idea is to write $X := \log Z_t^W$, and estimate the relavant random quantity G_X that arises in (15.0.0.7), for this particular random variable X, using Mehler's formula.

Samy Tindel's lecture concluded the invited research talks of Wednesday, and was concerning rough SDEs of the type

$$\mathrm{d}Y_t = V_0(Y_t)\,\mathrm{d}t + \sum_{j=1}^d V_j(Y_t)\,\mathrm{d}B_t^j, \tag{15.0.0.9}$$

where the V_i's are bounded and smooth vector fields, and B is a d-dimensional Gaussian process such as fractional Brownian motion [fBm] with Hurst index $\frac{1}{4} < H < 1/2$. Rough stochastic differential equations of type (15.0.0.9) have been solved recently [Cass, Fritz, Victoir], for example where B is fBm with Hurst index $\frac{1}{4} < H < \frac{1}{2}$. However, smoothness has eluded prior attempts. Tindel introduced his recent work [joint with T. Cass, M. Hairer, and C. Litterer] in which they show how the solution to (15.0.0.9) exists and is fairly generally continuous, for instance if B is fBm with $\frac{1}{4} < H < \frac{1}{2}$ or if the covariance function R of B satisfies some regularity conditions [enough to ensure strong local non-determinism, for instance]. Some of the key ingredients of the proof were also presented; one particularly noteworthy ingredient was shown to be the following: The Jacobian $\boldsymbol{J}$ that corresponds to (15.0.0.9) itself solves the following rough-path differential equation:

$$\boldsymbol{J}_t = \boldsymbol{I} + \int_0^t DV_0(Y_u)\boldsymbol{J}_u\,\mathrm{d}u + \sum_{j=1}^d \int_0^t DV_j(Y_u)\boldsymbol{J}_u\,\mathrm{d}B_u^j.$$

Robert Dalang began the Thursday lectures with his 1-hour plenary talk on hitting probability estimates for solutions to SPDE systems of the form

$$\mathcal{L}u^l = b^l(u) + \sum_{j=1}^d \sigma_{l,j}(u)\dot{W}_j \qquad (1 \le l \le d) \tag{15.0.0.10}$$

were $\mathcal{L}$ acts on the spatial variable $x \in \mathbf{R}^k$, the time variable is $t \in [0\,,T]$, and $\mathcal{L}$ can denote either the heat operator $\partial_t - \partial^2_{xx}$ or the wave operator $\partial^2_{tt} - \partial^2_{xx}$. The functions b^l and $\sigma_{j,l}$ are Lipschitz continuous, and the initial function u_0 [also u_0' for the wave case] are assumed to be deterministic and given. Finally, $\dot{W}_j$'s are i.i.d. Gaussian noises; when $k = 1$, they are assumed to be space-time white noise and when $k \ge 2$, they are assumed to be white in time and colored in space according to a Riesz kernel, viz.,

$$\mathrm{E}\dot{W}_i(t\,,x)\dot{W}_j(s\,,y) = \frac{\delta_0(t-s)}{\|x-y\|^\beta}\cdot\delta_{i,j},$$

where $\beta \in (0\,,2\wedge k)$ in order to ensure the existence and uniqueness of a solution.

The lecture addresses the following question: Given a d-dimensional set A and a hypercube $I \times J \subset (0\,,\infty) \times \mathbf{R}^k$, when is there positive probability that there exists some random $(t\,,x) \in I \times J$ such that $u(t\,,x) \in A$? If such a point exists then we say that A is *nonpolar*; else it is *polar*.

The presented answer depends on whether or not $\mathcal{L}$ is the wave operator or the heat operator. In the case of the wave operator, Dalang showed [joint work with M. Sanz–Solé] that

$$A \text{ is polar if } \dim_{\mathrm{H}}(A) < d - \left(\frac{2+2k}{2-\beta}\right) \text{ and non polar if } \dim_{\mathrm{H}}(A) > d - \left(\frac{2+2k}{2-\beta}\right).$$

By contrast, when $\mathcal{L}$ is the heat operator [joint work with D. Khoshnevisan and E. Nualart],

$$A \text{ is polar if } \dim_{\mathrm{H}}(A) < d - \left(\frac{4+2k}{2-\beta}\right) \text{ and nonpolar if } \dim_{\mathrm{H}}(A) > d - \left(\frac{4+2k}{2-\beta}\right).$$

The proofs hinge on developing detailed "heat-kernel estimates" for random variables of the form $u(t\,,x)$ and $(u(t\,,x)\,,u(s\,,y))$, together with a great deal of Malliavin's calculus, probabilistic potential theory, much of which were carefully introduced.

Martin Grothaus presented a lecture on an algebraic SPDE that was derived for a concrete problem in industrial mathematics for textile such as diapers, disposable clothes in hospitals, etc. [in collaboration with the Fraunhofer Institute and others]. Grothaus began his lecture with a detailed description of the underlying problem, and derived a nonlinear SPDE of the form $m\partial^2_{tt}X = \left[\partial_s(\lambda\partial_s X) - b\partial^4_{ssss}X + f(X\,,\partial_t X)\right]\mathrm{d}t + g(X, \partial_t X)\,\mathrm{d}W$, subject to a certain algebraic norm-one condition on $\partial_s X(s\,,t)$. Grothaus showed how one can rewrite a simplified version of the preceding, more succinctly, as an infinite-dimensional SDE of the form

$$\mathrm{d}X(t) = (L(t)X(t) + F(t))\,\mathrm{d}t + G\,\mathrm{d}W(t), \tag{15.0.0.11}$$

subject to $X(t_0) = \xi$, where the operator L leads us to a 2-parameter evolution system $\partial_t U(t\,,\tau)\varphi = L(t)U(t\,,\tau)\varphi$, and the noise has a covariance operator that satisfies $\int_{t_0}^t \mathrm{Tr}\,[U(t\,,r)GQG^*U(t\,,r)^*]\,\mathrm{d}r < \infty$ for all $t \in [t_0\,,T]$. Grothaus then showed [joint work with B. Baur and T. T. Mai] that, under some regularity conditions, (15.0.0.11) has a unique mild solution which is, more significantly, an analytic solution. The preceding does not address the algebraic constraint on the original SPDE. In order to address that matter, Grothaus showed then how J. P. Aubin's work on viability theory can be utilized in the present setting. In order to address that matter, Grothaus showed that a solution to any equation of the form $X'(t) = f(X(t))$ is viable in K—that is, $X(t) \in K$ for all $t_0 \le t \le T$—if and only if

$$\liminf_{h\downarrow 0} \frac{\text{dist}(x + hf(x)\,,K)}{h} = 0 \qquad \text{for every } x \in K.$$

Finally, a stochastic version of this result was also briefly mentioned [De Prato and Frankowska]; that result is what is needed in order to build in the algebraic constraints into the original problem.

Leif Döring's lecture revolved around his solution to an old problem in the structure theory of self-similar Markov processes, and its use in the analysis of symbiotic branching processes. Specifically, he presented a complete characterization theorem [joint with M. Barczy] of self-similar Markov processes in terms of a weak solution to a [quite complicated] SDE, thereby also characterizing the solution to the following system that was introduced earlier by Etheridge and Fleischmann:

$$\begin{aligned}
\mathrm{d}u(t\,,k) &= \Delta u(t\,,k)\,\mathrm{d}t + \sqrt{\beta u(t\,,k)v(t\,,k)}\,\mathrm{d}B^1_t(k),\\
\mathrm{d}v(t\,,k) &= \Delta v(t\,,k)\,\mathrm{d}t + \sqrt{\beta u(t\,,k)v(t\,,k)}\,\mathrm{d}B^2_t(k),
\end{aligned}$$

where B^1 and B^2 denote correlated Brownian motions with $\rho := \text{Corr}(B^1_t\,,B^2_t)$, and the initial states u_0 and v_0 are assumed to be nonnegative.

This model reduces to mutually-catalytic branching process when $\rho = 0$, to the parabolic Anderson model when $\rho = 1$, and to a $2\times$stepping-stone model when $\rho = -1$. Döring described his work on the behavior of the solution as $\beta \to \infty$: When $\rho = -1$ the solution converges to the voter model; when $\rho = 0$ it converges to a "monster process" [Mytnik and Klenke]; and when $\rho \in (-1\,,1)$, it converges to a "generalized monster process" [joint work with Mytnik]. It was shown how these questions reduce to problems about duality relations. A series of [very hard] open questions were also posed.

Peter Imkeller lectured on his joint work with N. Perkowski, wherein they devise a Fourier-analytic approach of pathwise integration as a possible alternative to other integration theories against rough functions. This idea can be summarized roughly as follows: If $f \in L^2[0\,,1]$, then we can develop f in terms of the

Schauder basis $\varphi_n(t) := \int_0^t \chi_n(s)\,\mathrm{d}s$, where the χ_n's are Haar functions. Imkeller showed that whenever f is Hölder continuous of order α, we have a pointwise bound,

$$\left|\sum_{k>K}\sum_{\ell=0}^{2^k-1}\left(\int \chi_{2^k+\ell}\,\mathrm{d}f\right)\varphi_{2^k+\ell}\right| \le C2^{-\alpha K}\|f\|_\alpha,$$

for a universal finite constant C, and all $K \geq 1$. Thus, the preceding holds true for all $f \in \mathcal{H} :=$ the closure of C^α with respect to the norm $\|\cdots\|_\alpha$, as well. From this, Imkeller deduced that the map

$$f \mapsto \left(c_n^{-\alpha}\int_0^1 \chi_n\,\mathrm{d}f\right)$$

defines an isomorphism between $\mathcal{H}$ and a sequence space, where the c_n's have a concrete, though somewhat complicated, numerical form. Extensions to other base spaces than C^α were also mentioned, in particular, to Besov spaces $\mathbf{B}^\alpha_{p,q}$.

Finally, an argument was sketched that described how one can plan to construct integrals of the form $\int_0^1 g\,\mathrm{d}f$ for rough functions g and $f \in \mathcal{H}$, using the sequence-space ideas together with methods of rough-path theory.

Jan van Neerven introduced stochastic reaction-diffusion equations of the form

$$\frac{\partial}{\partial t}u(t\,,\xi) = \mathcal{A}u(t\,,\xi) + f(t\,,\xi\,,u(t\,,\xi)) + g(t\,,\xi\,,u(t\,,\xi))\mathcal{R}\dot{W}(t\,,\xi), \qquad (15.0.0.12)$$

where the space variable ξ takes values in a bounded open subset O of $\mathbf{R}^d$, $\dot{W}$ dentoes space-time white noise, and $\mathcal{R}$ is a Radonifying separator from $L^2(O)$ to $L^q(O)$ when $d \geq 2$, and $\mathcal{R} :=$ the identity map on $L^2(O)$ when $d = 1$.

Van Neerven addressed the question of global existence of solutions to (15.0.0.12) by rewriting the problem as one about stability of an SDE on a UMD Banach space E:

$$\mathrm{d}X(t) = [\mathcal{A}X(t) + F(t\,,X(t))]\,\mathrm{d}t + G(t\,,W(t))\,\mathrm{d}W(t),$$

Stability theorems were presented that show that if $\mathcal{A}^n \to \mathcal{A}$ in a suitable sense, then the resulting solutions $X^{(n)}$, killed at suitable stopping times, converge to X, killed at a suitable stopping time. And convergence holds in the space $L^0(\Omega\,; B_b([0\,,T]\,;E))$. Moreoever, one can control the behavior of the stopping times well enough to ensure the following result.

Under natural regularity assumptions on $\mathcal{A}$, f, if $X_0 \in L^p$ for some p sufficiently large [explicit bounds were shown], then (15.0.0.12) has a global solution that is in $L^p(\Omega\,; C([0\,,T] \times \bar{O}))$ [joint work with M. Kunze].

Lluís Quer–Sardanyons presented his work [joint with A. Deya and M. Jolis] on the stochastic heat equation. Let W be an $L^2 := L^2[0\,,1]$-valued Brownian motion with [a finite-trace] nuclear covariance Q, and consider the random Stratanovich-type integral equation

$$Y_t = S_t\psi + \int_0^t S_{t-s}\left(f(Y_s)\circ \mathrm{d}W_s\right), \qquad (15.0.0.13)$$

where $\{S_t\}_{t\geq 0}$ denotes the L^2-semigroup corresponding to $-\Delta$ and $\psi \in L^2$.

Let X denote the mild solution to

$$\mathrm{d}X_t = \Delta X_t\,\mathrm{d}t + V_t^1\,\mathrm{d}t + V_t^2\,\mathrm{d}W_t,$$

where $X_0 = \psi$ and V^i's are continuous-in-L^2 vector fields. Quer–Sardanyons proved that one can always construct the curvilinear stochastic integral $\int_0^t S_{t-s}(f(X_s)\circ \mathrm{d}W_s)$ by first mollifying the white noise and solving the preceding heat equation to obtain X^ϵ—where ϵ is the mollifier parameter–and then letting $\epsilon \downarrow 0$

in order to deduce that $\int_0^t S_{t-s}(f(X_{t-s}) \circ \mathrm{d}W_s) := \lim_{\epsilon \downarrow 0} \int_0^t S_{t-s}(f(X_{t-s}^{\epsilon}) \circ \mathrm{d}W_s)$ exists in probability. Moreover, that limits was shown to be equal to

$$\int_0^t S_{t-s}\left(f(X_s) \cdot \mathrm{d}W_s\right) + \int_0^t S_{t-s}\left(V_s^2 \cdot f'(X_s) \cdot P\right) \mathrm{d}s,$$

where P is a certain polynomial for the covariance Q, and the first stochastic integral is an Itô integral. Using this stability result, Quer–Sardanyons showed that if $f, f' \in L^\infty$, then (15.0.0.13) has a unique L^2-valued solution. Moreover, Quer–Sardanyons showed that one can use the correlational rough-path analysis of Tindel and Gubinelli (2010) in order to establish that if $f \in C_b^3$ then the solution is Hölder continuous.

Tusheng Zhang's lecture presented a uniqueness theorem for the invariant measure of SPDEs with two reflecting walls [joint work with J. Yang]. Specifically, Zhang considered an SPDE of the form.

Let h^1 and h^2 denote two reflecting walls on the state space S^1 with the properties that: (H1) $h^1(x) < h^2(x)$ for all $x \in S^1$; and (H2) $\partial_{xx}^2 h^j \in L^2(S^1)$ for $j = 1, 2$. The SPDEs that was studied seeks to find a random function $u(t\,, x)$ such that $h^1(x) \le u(t\,, x) \le h^2(x)$ for $t > 0$ and $x \in S^1$, and u satisfies

$$\frac{\partial}{\partial t} u(t\,,x) = \frac{\partial^2}{\partial x^2} u(t\,,x) + f(u(t\,,x)) + \sigma(u(t\,,x))\, \dot{W}(t\,,x) + \eta - \xi, \tag{15.0.0.14}$$

where $f, g : S^1 \to \mathbf{R}$ are Lipschitz continuous, $u(0\,,\cdot) \in C(S^1)$, $\dot{W}$ denotes space-time white noise, and η and ξ are random measures that are a part of the solution and satisfy $\int_{\mathbf{R}_+ \times S^1} (u - h^1)\, \mathrm{d}\eta = \int_{\mathbf{R}_+ \times S^1} (h^2 - u)\, \mathrm{d}\xi = 0$. In the one-sided case (say, $h_1 \equiv 0$ and $h_2 \equiv \infty$) the preceding becomes an SPDE with reflection. In that case: When the noise is additive, Nualart and Pardoux have shown that there exists a unique solution; and in the multiplicative case when σ is non-constant Donati Martin and Pardoux established existence, and Xu and Zhang proved uniqueness.

Zhang's lecture presented an argument based on the respective theories of Krylov-Bogolyubov [for existence] and Mueller [for uniqueness] in order to prove that, under the stated assumptions, (15.0.0.14) always has a unique invariant measure.

Annie Millet's lecture on the stochastic Cahn–Hilliard and Allan–Cahn equations concluded the research talks of the conference. As a sampler of the theory presented in this talk, let us consider a nice convex domain $O \subset \mathbf{R}^d$ with piecewise smooth boundary, and denote by ν its outward normal vector. Millet introduced the SPDE

$$\partial_t u = -\rho\Delta\left(\Delta u - f(u)\right) + \left(\Delta u - f(u)\right) + \sigma(u)\dot{W}, \tag{15.0.0.15}$$

for $(t\,, x) \in [0\,, T] \times O$, subject to

$$\frac{\partial}{\partial \nu} u = \frac{\partial}{\partial \nu} \Delta u = 0 \qquad \text{on } [0\,,T] \times \partial O.$$

The function f is assumed to be a third-degree polynomial with positive leading coefficient; for instance, $f = F'$, where $F(u) = (1 - u^2)^2$ denotes the free energy for a double-well potential. Then Millet showed that if $u_0 \in L^q(O)$ for some $q \ge 6$ and σ is Lipschitz continuous with $|\sigma(u)| = O(|u|^\alpha)$ as $|u| \to \infty$ for some $\alpha \in (0\,, 1/9)$, then for all $T > 0$, (15.0.0.15) admits a unique pathwise solution $u \in L^\infty([0\,,T]; L^q(O))$ [joint work with A. Antonopoulos and G. D. Karali]. Similar results were presented for the stochastic Cahn–Hilliard equation.

In the case that $O = (0\,,\pi)^d$ is the open torus, then more information about the solution is available. For example: (i) If u_0 is continuous, then so is u; (ii) If $u_0 \in C^\gamma$ for some $\gamma \in (0\,,1)$, then u is Hölder continuous in its space variable; (iii) If $d = 1, 2, 3$, u_0 is continuous, and $|\sigma(x)| > 0$ for all $x \in O$, then the law of $u(t\,,x)$ is absolutely continuous for all $t > 0$ and $x \in O$.

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Chapter 16

Open Dynamical Systems: Ergodic Theory, Probabilistic Methods and Applications (12w5050)

April 8 - 13, 2012

Organizer(s): Wael Bahsoun (Loughborough University, UK), Chris Bose (University of Victoria, Canada), Gary Froyland (University of New South Wales, Australia)

Overview of the Field

Ergodic theory as a mathematical discipline refers to the analysis of asymptotic or long-range behaviour of a dynamical system (a map or flow on a state space, or more generally a group or semigroup action on a state space) using probabilistic methods. The word 'ergodic' used in this context dates back to the origins of the field in statistical mechanics by reference to the *ergodic hypothesis* (that time averages of observables along orbits be deemed equal space averages) and to the celebrated ergodic theorems of Birkhoff and von Neumann describing conditions under which this property holds. It is a close cousin to topological dynamics, the study of continuous actions on a topological space, and to smooth dynamics, the study of differentiable actions on smooth manifolds. Because of this, it has become quite common in recent years to use the term *measurable dynamics* in place of 'ergodic theory.' There is a rich and productive synergy between the three fields since many interesting examples can be approached through any one, or more typically, a combination of these paradigms. In the case of ergodic theory a key object is an invariant or stationary measure against which observables are integrated to obtain the system statistics and asymptotic behaviour. Determining the properties of such measures is key step for many problems in the field.

Open dynamical systems is currently an active subbranch of dynamical systems research. The difference between an open system and a more traditional 'closed' dynamical system is very simple. In a closed dynamical system, the map (or a flow) maintains its image in the state space for all time, whereas in an open system, orbits may eventually escape from the state space, typically by falling into a 'hole' in the space. An everyday example is the dynamics of a billiard ball on a table with a pocket (the hole). Other mechanisms for terminating orbits, for example, diffusion through a boundary or similar stochastic perturbations from a closed system are studied within the broad classification of open dynamics.

The notion of an open dynamical system, and for the moment we focus specifically a map with a hole, was introduced by Pianigiani and Yorke in 1979: [12]. In a series of papers from the 1990's, Collet, Martinez and collaborators developed the framework for a general theory applicable to the case of uniformly hyperbolic Markov maps with a hole. They proved existence, and studied ergodic properties of, the associated conditionally invariant measure, the so-called Yorke-Pianigiani measure (the natural analogue of an invariant measure for a closed system). A representative paper from this corpus would be: Collet, Martinez and Schmitt (1994):

[4]. Only slightly later, Chernov, Markarian and collaborators extended the framework to cover a wide range of natural hyperbolic settings in, for example: Chernov, Markarian, and Troubetzkoy (1998): [3]. During this decade, a further extension to include maps that are non-uniformly or weakly hyperbolic has been undertaken (eg. Demers (2005): [18]). A survey of results from this period is contained in Demers and Young (2006): [6]. Thus we observe, in a broad sense, the development of the theory for open systems retracing the evolution of the theory for closed dynamical systems, starting with a few interesting examples, followed by the general case for expanding systems, evolving to hyperbolic systems and finally, more recently to the significant challenge presented by non-uniformly hyperbolic systems.

Recent Developments

While open systems are extremely natural models for certain applications, there has recently been an emphasis on introducing open 'components' in a closed system to study subtle dynamical properties of the closed system. By isolating parts of the phase space, mechanisms for mass transfer between regions or components inside the system, and rates of mixing between the components can sometimes be determined. The key idea is that infrequent transitions leading to slow mixing between quasi-stable states are similar to infrequent escapes from one open component system and subsequent re-injection into another component. A particularly transparent description of this phenomenon is discussed in the recent work of Gonzalez-Tokman, Hunt and Wright (2011): [8] . A selection of papers treating various aspects of current research in open dynamics would also include:

Demers, Wright and Young (2010): [7]

Keller and Liverani (2009): [10]

Bunimovich and Yurchenko (2011): [2]

Bruin, Demers and Melbourne (2010): [1]

The mathematical tools for a rigorous analysis of open and closed dynamical systems cross all three dynamical paradigms: measurable, topological and smooth. These theoretical tools have found applications in non-equilibrium statistical mechanics, molecular dynamics and 'fluid flow,' including ocean and atmospheric dynamics. At the same time, the development of numerical schemes for applications, motivated by rigorous theoretical results is an important and active area of research with implications for both theoretical and applied studies.

Finally, another exciting area of research is the development of both theoretical and computational tools for non-autonomous (time varying) systems . This is particularly important for some key applications such as atmospheric or ocean flow modelling, where the time scale of change in the parameters defining the dynamics (temperature, pressure, moisture etc) is on the same order or smaller than the observational time scale to be addressed by the model. Again, there are challenging applications and compelling theoretical questions in this area, for both closed and open dynamical systems.

Presentation Highlights

A workshop participants gave 35 talks throughout the week, representing all the major threads of pure and applied work in the subject area. The talks can be broadly classified into the following areas. The notation [V] indicates that an online video of the talk is archived on the BIRS website at `http://www.birs.ca/videos/` Search for workshop 12w5050.

Theoretical Developments in Open Dynamical Systems

Mark Demers, Fairfield University. Title: Dispersing billiards with holes, [V]

Carl Dettmann, University of Bristol. Title: Escape and diffusion through small holes

Andrew Ferguson, University of Bristol. Title: The dimension of some sets generated by systems with holes, [V]

Rainer Klages, Queen Mary University of London. Title: Where to place a hole to achieve a maximal diffusion coefficient, [V]

Some of the earliest questions in the field of open dynamical systems involved computing the probability of survival (and related to this the so-called *escape rate* of mass from the system as orbits enter the hole and are terminated). Moving to the finer structure of the system, we may wish to compute the distribution of mass remaining in the system after a fixed number of iterations (suitably renormalized to account for the loss of mass up to that time) and if possible, the limiting distribution as the number of iterations goes to infinity. This limiting distribution is known as the conditionally invariant measure for the open system. Just as in the case of closed systems, the transfer operator (also known as the Ruelle-Perron-Frobenius operator) plays a central role in determining dynamical properties of the open system. For example, the function $f^* \geq 0$ is the density of an absolutely continuous conditionally invariant measure, and $-\log \rho$ is its escape rate iff

$$P_T f^* = \rho f^*,$$

where $0 < \rho < 1$ is the dominant eigenvalue and f^* is the associated eigenfunction for the transfer operator. One well-known approach for finding f^* is through spectral perturbation. Start with the transfer operator for the closed system, then introduce a (small) hole in the space and prove continuity of the spectral projections associated to *peripheral eigenvalues* of the associated open transfer operators; in effect, the invariant density for the closed system (with eigenvalue 1) perturbs to an eigenfunction for $\rho \approx 1$ if the hole is sufficiently small. The key step for making this work is to find an appropriate Banach space of functions, invariant under the action of the transfer operators and for which the transfer operators are, in some sense, uniformly quasi-compact. An important ingredient is the result of Keller-Liverani [11], which provides the theoretical framework over which to build the method.

Ironically, one of the most intuitively attractive open systems, a billiard table with a hole, has proved to be one of the most difficult to treat rigorously with these methods, owing to the prevalence of singular points and unbounded derivative arising from grazing collisions. Demers' presentation described recent joint work with H.K. Zhang, using a new class of anisotropic Banach spaces on which the closed billiard flow, for a wide range of dispersive configurations, allows for a spectral gap in the associate transfer operator. From this it follows that certain open perturbations (=small holes satisfying some reasonable technical assumptions) will admit absolutely continuous conditionally invariant measures. The holes may be on the interior of the table, or in the boundary of the table.

This spectral picture leads another set of natural questions: suppose we study a closed system with a family of holes that 'shrink to a point'. What kind of continuity properties are present for the escape rate and the absolutely continuous conditionally invariant measure? Does the latter converge to an invariant measure, and in what sense? What is the relation between the escape rate and the size of the hole? A rough heuristic goes like this: orbits will follow the closed dynamics until they fall into the hole. By the ergodic theorem, the probability that a typical orbit will fall into the hole should be about the value of the invariant density at the hole, multiplied by the area of the hole. Therefore, up to first order, the 'linear response' of the escape rate to the size of the hole should be the value of the density at the hole. Although this heuristic misses some important points (the obits may fall into the hole long before they conform with the statistics specified by the ergodic theorem, for example) the heuristic is approximately correct. The presentations by Dettmann and Klages address some of these issues, including aspects of the *location* of the hole and higher order estimates of the response function with respect to the hole size.

The *survivor set* is the set of points whose orbit never falls into the hole. This set supports invariant measures for the open system. Motivated by an early number-theoretic investigation by Hensley [9], Ferguson estimates the dimension of the survivor set, compared to the original state space. Both box dimension and Hausdorff dimension were considered and the dependence of each of these on escape rate was discussed. Ferguson's collaborators on this work are T. Jordan and M. Rams.

Applications

Dynamical systems are frequently constructed as models of physical systems; examples of recent interest include ocean and atmospheric flows, trajectories of spacecraft, planetary motion, or models of biological or medical processes. Of course the foundational example in the field is the study of billiards already discussed above. In all of these examples the role of time evolution is clear. Much more subtle use of dynamics has led to important links with probability and stochastic processes, coding theory, aperiodic order (or quasicrystals) and analytic number theory. In these areas, the notion of time evolution is not nearly so transparent.

The following presentations indicate a diverse range of modern applications of ergodic theory to other areas such as physics and engineering.

Kevin Lin, University of Arizona. Title: Reliability of Driven Oscillator Networks, [V]

James Meiss, University of Colorado Boulder. Title: Transport in Transitory Systems: Mixing in Droplets, [V]

Yuzuru Sato, Hokkaido University, Japan. Title: Noise-induced phenomena and their applications, [V]

Paul Tupper, Simon Fraser University. Title: Using the Lorentz gas to resolve a paradox of state-dependent diffusion, [V]

Meiss described an interesting micro-mixing process where small droplets of raw material are placed in a homogenous fluid flow, the goal being to induce laminar mixing of the material inside the droplet. Techniques from the area of non-autonomous flows and coherent structures can be used to study the internal mixing process inside the drop as the drop is transported by the ambient flow through a sinuous channel. This reports on joint work with B. Mosovsky.

Lin address the question of *reliability* in pulse coupled phase oscillator networks by introducing random dynamical systems models. Both stability and instability results were presented.

Tupper discussed an intriguing paradox about a particle moving in a box where there are two disjoint regions with different diffusion constants. The question is, how much time does the particle spend in each part of the box? 'Physicists' have argued that they spend equal amounts of time, whereas mathematical results would suggest more time is spent in the low-diffusion component. By modelling the trajectory by a random Lorenz gas the speaker shows that both conclusions can be attained, depending on the ratio of free volume fractions in the two regions.

T. Sato discussed various noise-induced phenomenon in dynamical systems such as stochastic resonance, noise-induced synchronization and noise-induced chaos. The results of various numerical experiments were presented. This reports on joint work with D. Albers and Y. Tasaka.

Computational Methods in Ergodic Theory

In order to apply the powerful machinery available from ergodic theory to a 'real dynamical system' one first needs to develop numerical schemes to compute or at least estimate one or more of the key quantities arising in the theory. These can include: invariant measures, Lyapunov exponents, entropy, invariant sets and so-on. The desire to develop tools based on ergodic theory are almost as old as the subject itself. For example, already in the 1950's S. Ulam [77] proposed a simple discretization scheme for the transfer operator associated to a map that can be used to compute invariant measures that is still used today in many applications and whose convergence in various settings is still the subject of active research. One should be clear here what the goal is: to develop tractable numerics that can be *proved* to approximate the object or quantity in question, and if possible with *a priori* error bounds. A well-known example is the following. Birkhoff's theorem guarantees that the orbit of almost every point in the phase space will be distributed proportionally to some invariant measure. The problem is that an orbit is an infinite object, difficult or impossible to compute with sufficient precision even when truncated and the qualification of 'almost every' is with respect to an unknown measure (often the one you are trying to compute). So this is an intractable numerical scheme, but none the less one that is frequently used to get a non-rigorous feel for the character of an invariant measure when nothing rigorous is available. Generally, we aim to do better than this.

For closed, discrete time systems, the transfer operator (also called the Perron-Frobenius operator) allows us to study the dynamics acting, not on individual points in the state space, but on functions defined over the state space. For continuous time systems, the infinitesimal generator of the associated group or semigroup action plays a similar role. Estimating dynamical properties is therefore frequently done by studying the related linear operator (transfer or infinitesimal) acting on appropriately chosen Banach spaces of functions over the state space. A similar approach can be applied to open dynamical systems.

The following presentations treated a broad range of approaches and an up-to-date discussion of some of the issues at the forefront of research in numerical ergodic theory.

James Yorke, University of Maryland. Title: Partial Control, [V]

Oscar Bandtlow, Queen Mary University of London Title: Lagrange-Chebyshev approximation of transfer operators, [V]

Rua Murray, University of Canterbury. Title: Numerical approximation of conditionally invariant measures, [V]

Oliver Junge, TU Muňchen. Title: Estimating long term behavior of flows without trajectory integration: An infinitesimal generator approach, [V]

Michael Dellnitz, University of Paderborn. Title: The Computation of Invariant Sets via Newton's Method, [V]

Leonid Bunimovich, Georgia Institute of Technology. Title: Finite-time dynamics, [V]

Erik Bollt, Clarkson University. Title: Basis Markov Systems, Estimated Transfer Operators of Open Systems, and Absorbing Markov Chains, [V]

Robyn Stuart, University of New South Wales. Title: Almost-invariance in Open Dynamical Systems, [V]

Many computational procedures in ergodic theory are based on a Galerkin-type method of projection of the transfer operator to a finite-dimensional subspace. Ulam's method [77] is such an example, where the subspace is piecewise constant functions with respect to some finite partition (usually boxes). Oscar Bandtlow described a method based on projection to polynomials via Lagrange-Chebyshev interpolation. For sufficiently smooth maps, it is possible to prove convergence and establish convergence rates of the Lagrange-Chebyshev approximation for the invariant density. In a similar vein, Erik Bollt introduced the concept of *basis Markov* for a dynamical system, wherein the transfer operator exactly preserves a finite dimensional functional subspace of L^1, in much the same way that a Markov partition allows for a SFT representation of the dynamics. Examples related to the Hénon map were given. Finally, Oliver Junge explained how these techniques can be used to study *flows* by using a Galerkin approximation of the infinitesimal generator. This allows one to avoid the computationally costly step of computing long flow trajectories over continuous time. Projections to spaces spanned by piecewise constant, Chebyshev and Fourier bases were each discussed. Oliver's talk presented joint work with G. Froyland and P. Koltai.

Robyn Stuart introduced the notion of *invariance ratio* for a subset of an open dynamical system. Subsets with large invariance ratio escape the system slowly and an upper bound on the invariance ratio is the escape rate for the system. A Markov chain interpretation was given, and using this, a computational method for estimating the maximum invariance ratio was developed. Application to computation of almost invariant sets in closed systems was briefly discussed. This talk was based on joint work with G. Froyland and P. Pollett.

Jim Yorke's presentation described a method to control a dynamical system despite the presence of noise or disturbances. One can imagine this as a kind of dynamical game theory, since the disturbances can be thought of as either random noise or as purposeful, hostile efforts of an opponent. The mathematical problem is to keep trajectories inside some specified region of the phase space despite the disturbances. Yorke showed this is possible in some cases, even when the applied control is constrained to be smaller than the applied disturbance. The method involves computation of a *safe set* in phase space; the idea is to always return to the safe set after each iteration. An algorithm for computing safe sets was described and an example application for the Duffing oscillator was given. The work presented is joint with Juan Sambrano, Samuel Zambrano, and Miguel A. F. Sanjuan.

Rua Murray reported on recent work that applies methods from convex optimization to the problem of computing conditionally invariant measures and escape rates in open systems. A bi-level approach is used, wherein one first specifies the escape rate, then computes an associated conditionally invariant measure for that rate through a moment-constrained maximum entropy problem. This work is joint with C. Bose.

Michael Dellnitz focussed on *invariant sets* and their computation. Various box-subdivision algorithms were described that are frequently used to computed invariant submanifolds such as global attractors. Applications to multi-objective optimization were described. A new set-based method analogous to Newton's method for root finding was described. The technique shows particular promise for cases where the invariant set is *unstable* for the global dynamics. This is joint work with Baier, Hessel-von Molo, Kevrekidis and Sertl.

Leonid Bunimovich discussed a notion of *finite-time* escape dynamics. It is known that the position of the hole typically affects the escape rate, even if the holes have the same size. It turns out that for two different hole positions, it is possible to determine in finite time which has the faster escape rate. This is interesting since the escape rate itself is only determined in the limit as time tends to infinity; still the *order* of escape rates can be compared in finite time. Application to dynamical networks was discussed.

Non-autonomous Dynamical Systems - theory and applications

In some realistic applications, time-varying parameters governing the flow or transformation on the state space necessitate modelling by a non-autonomous system. While the ergodic theory of non-autonomous systems parallels that of autonomous dynamics in many ways, there are important differences. Stable and unstable fibrations, a foundation of geometric analysis for an autonomous map or a flow, are now equivariant, time-dependent structures. Other dynamical objects such as Lyapunov exponents and Oseledets subspaces can be used in alternative ways to describe non-autonomous or random dynamics. Invariant or almost invariant objects arising in autonomous flows have non-autonomous analogues called *coherent structures*, features that move around in the state space under time evolution but that may still represent barriers to mixing and relaxation to 'equilibrium,' a concept that also has to be reinterpreted compared to the autonomous setting.

Sanjeeva Balasuriya, Connecticut College. Title: Flow barriers and flux in non-autonomous flows, [V]

Cecilia González Tokman, University of Victoria. Title: A semi-invertible operator Oseledets theorem, [V]

Simon Lloyd, University of Sao Paulo. Title: Slowly decaying modes for skew-product systems, [V]

William Ott, University of Houston. Title: Memory loss for time-dependent dynamical systems, [V]

Kathrin Padberg-Gehle, TU Dresden. Title: Finite-time entropy: a probabilistic approach for measuring nonlinear stretching, [V]

Shane Ross, Virginia Tech. Title: Geometric and probabilistic descriptions of chaotic phase space transport

Naratip Santitissadeekorn, Clarkson University. Transfer operator approach for finite-time coherent sets identification, [V]

Tom Watson, UNSW Title: Computing Oseledets subspaces: A short overview, [V]

A key theoretical tool for non-autonomous systems is the Oseledets Multiplicative Ergodic Theorem for matrix cocyles. There is both an invertible and a non-invertible version of this classical theorem. At this meeting, an alternative setting wherein the fixed time maps need not be invertible, but the timing sequence or *base* is invertible was discussed. This is the so-called semi-invertible case and an Oseledets-type splitting can still be obtained. Extensions from the matrix cocycle setting to operators on an infinite-dimensional Banach space are important for many applications.

Cecilia González-Tokman presented a semi-invertible multiplicative ergodic theorem that for the first time can be applied to the study of transfer operators associated to the composition of piecewise expanding maps randomly chosen from a set of cardinality of the continuum. This is one possible infinite-dimensional setting alluded to above. This work (joint with A. Quas) extends the range of semi-invertible systems that can be handled theoretically. Naratip Santitissadeekorn discussed the use of variational techniques for the computation of finite-time coherent states with applications to geophysical and atmospheric models. This reports on joint work with G. Froyland and A. Monahan. Simon Lloyd looked at cocycles built from a countable collection of expanding interval maps with a SFT base. This is joint work with G. Froyland and A. Quas.

Tom Watson gave an overview of numerical challenges encountered in computing or estimating the equivariant spaces (the so-called Oseledets subspaces) derived in a semi-invertible setting. This is recent joint work with G. Froyland, T. Huels and G. Morriss.

Sanjeeva Balasuriya discussed time-varying stable and unstable manifolds as *flow barriers* that are related to the boundaries of coherent states. Methods for locating these objects at a given time slice and estimating the flux across them (unlike the autonomous case, these are not true barriers to flow) were presented. An application to the non-autonomous double gyre example was presented.

Shane Ross gave a brief overview of geometric and probabilistic methods that have been successfully applied in the case of autonomous systems, and showed how these can be recast for aperiodic time-dependent or what he called *data-based* models. A brief overview of connections to concepts such as symbolic dynamics, chaos, coherent sets, and optimal control was given, with highlights to some recent applications in areas such as celestial mechanics, musculoskeletal biomechanics, ship capsize prediction, and atmospheric microbe transport.

William Ott introduced the notion of memory loss in dynamical systems as an example of exponential decay of transient behaviour. When the system is non-autonomous, one cannot expect a fixed limiting measure to attract all measures under the action of the flow, but rather there should be a path of time-evolving measures that attract all others under evolution of the flow. Various settings in which this sort of result holds were discussed and the situation for open dynamics was briefly touched on. The main results presented in this talk are joint work with M. Stenlund, L.S. Young, C. Gupta and A. Török.

Kathrin Padberg-Gehle introduced us to a new method for estimating nonlinear *streching* processes in a flow, using what she called finite-time entropy. This provides another tool (along with finite-time Lyapunov exponents, and spectral methods based on approximate transfer operators) for studying mixing in non-autonomous flows. The method is based on estimating the entropy growth of a small, localized density under evolution of the transfer operator. This is joint work with G. Froyland.

Theoretical Developments in Ergodic Theory; Closed Systems

Theoretical research in ergodic theory with applications to important physical and scientific problems continues at a very high rate. The importance of new theoretical results for closed systems to modelling and theoretical progress for open systems should now be clear from the foregoing discussion. In this section we describe a group research talks on theoretical ergodic theory.

Viviane Baladi, Ecole Normale Superieure. Title: On the Whitney-Holder differentiability of the SRB measure in the quadratic family, [V]

Henk Bruin, University of Surrey. Title: Renormalization and Thermodynamic Formalism in Subshifts, [V]

Peyman Eslami, Concordia University. Title: On the acim-stability of piecewise expanding dynamical systems, [V]

Pawel Gora, Concordia University. Title: Random map model of metastable system, [V]

Nicolai Haydn, Southern California. Title: The almost-sure invariant principle for uniformly strong mixing measures, [V]

Carlangelo Liverani, University of Rome Tor Vergata. Title: Partially hyperbolic systems close to trivial extensions, [V]

Ian Melbourne, University of Surrey. Title: Convergence and asymptotics of moments for billiards and Lorentz gases

Matt Nicol, University of Houston. Title: Erdos Renyi limit laws for dynamical systems, [V]

Mark Pollicott, University of Warwick. Title: Asymptotics of geodesic flows, [V]

Dalia Terhesiu, University of Rome, Tor Vergata. Title: A new technique for sharp mixing rates associated with infinite measure preserving systems, [V]

Roland Zweimuëller, University of Vienna. Title: Systems with holes as a tool for recurrent infinite measure systems, [V]

A common theme in ergodic theory is 'stability' of dynamical objects, like invariant measures, with respect to perturbation of the system. Viviane Baladi derived weak differentiability (Whitney-Holder regularity) for the path of Sinai-Ruelle-Bowen (SRB) measures arising from a smooth curve of maps within the quadratic family. This work extends similar results due to Ruelle, for example, (with a stronger notion of regularity) for paths of uniformly hyperbolic maps.

Carlangelo Liverani presented a dynamical model of the heat equation built as a small perturbation of a smooth expanding map of the circle times the identity. Exponential decay of correlation results are obtained with respect to the SRB measure. This is joint work with Dolgopyat and De Simoi.

Henk Bruin reported on the connection between renormalization processes and phase transitions in one-dimensional dynamics. A motivating example of the Thue-Morse substitution and Feigenbaum maps was investigated.

Peyman Eslami presented a new Lastota-Yorke inequality that can be applied to maps with periodic turning points. As an application, he showed how this leads to spectral stability type results for associated open systems. Some of the work presented is joint with M. Misiurewicz and P. Gora.

Pawel Gora presented a random map model for almost invariant states and showed the residence time in the two almost invariant domains can be computed from the 'gate sizes' and the relative probabilities that one passes through a gate.

Invariant measures on shift spaces over finite or infinite alphabets are uniformly strong mixing if they satisfy a weak kind of mixing condition. Nicolai Haydn showed that in this situation the information function satisfies the almost-sure invariance principle. This extends previously known results such as the CLT and the weak invariance principle already known in this context.

Mark Pollicott spoke about geodesic flow on surfaces of negative curvature. Various limit laws on the rate at which orbits flow into a cusp were presented, starting with an early (1982) result of Sullivan and, more recently, an estimate on the maximum displacement up the cusp in time T that turns out to be equivalent to a classical result in continued fractions. In the second half of the talk, a dynamical proof of the convergence of the Schottley-Klein function was sketched.

It has been known for some time that a central limit theorem holds for Axiom A diffeomorphisms and also for uniformly hyperbolic systems with exponentially decaying return tails. Ian Melbourne presented recent work (joint with A. Torok) extending these results to non-uniformly hyperbolic maps with polynomial decay of correlation (i.e., polynomial decay of return tails). In this case the correct normalization is $\sqrt{n \log n}$ for the central limit theorem to hold.

Matt Nicol presented large deviation results of Erdös Renyi type, where averages are computed over blocks of consecutive random variables in a stationary sequence. For i.i.d. random variables a classical result shows that there is a critical block size necessary to obtain a value strictly between zero and the essential supremum of of the random variable. Dynamical versions of the result for maps having large deviation principles with rate functions were presented and illustrated with intermittent maps of Liverani, Saussol, Vaienti type. This talk reports on joint work with M. Denker.

Two speakers addressed systems having *infinite invariant measures* (and no finite invariant measure). Typically, these maps are studied by first *inducing* to a suitable subset where the first return map is well behaved, and has a finite invariant measure, and then 'pulling back' this finite measure through the dynamics. Dalia Terhesiu presented improved estimates on correlation decay rates for these systems using a functional operator renewal theorem (related to the induced map), while Roland Zweimuëlle showed how maps with holes can be used to study asymptotics like distributional limits for infinite measure preserving systems.

Preparation and publication of Ergodic Theory, Open Systems and Coherent Structures

Prior to the meeting date, the organizers contracted with Springer Verlag for production of a book entitled *Ergodic Theory, Open Systems and Coherent Structures* in order to bring together in one place some of the various threads of these new research covered at this workshop. At the time of writing, we have accepted 10 peer-refereed book chapter submissions for this project, representing some of the best cutting-edge research reported on during the conference. We anticipate publication in late 2013 under the series heading *Proceedings in Mathematics and Statistics*. See `http://www.springer.com/series/8806` for further information.

Outcome of the Meeting

A key goal of the meeting was to bring together both theoretical and applied researchers at one time, in one place for a free exchange on recent scientific results and current challenges. In this respect, the workshop was a great success, drawing widely from both the applied and theoretical research community. Of the forty-three participants arriving in Banff during the week of April 3, sixteen are internationally recognized for their applied work while twenty-five work in more theoretical/foundational areas of the field. Two participants are famous for outstanding work on both theoretical and applied problems! Twenty six participants arrived from outside of North America, travelling from Europe, Asia, South America, Australia and New Zealand. Seven participants were trainees at the time of the workshop, including graduate students or postdocs.

In terms of tangible progress, we know from the submissions to the springer book publications that a number of projects were initiated and/or substantially influenced through direct interaction between researchers while at the workshop. The organizers also know of at least five researchers who began new projects as a result of their experience during the workshop (in addition to the material submitted to the book publication).

The organizers feel our goal to bring together in a meaningful and scientifically productive way both theoretical and applied researchers in the field, to mix young, innovative researchers with established research leaders (end even the inclusion of some pioneers in the field!) along with significant representation by students and postdocs, was a solid success. This BIRS workshop was widely and enthusiastically supported by an strong international contingent of researchers in the field, and we thank the BIRS organization for their large role in bringing it to fruition.

Final Thoughts

Near the end of the meeting the organizers asked for feedback from participants at the workshop. We collected some comments that future organizers might want to think about and we take this opportunity to share them.

We scheduled a large number of talks, including some one-hour presentations, between 8:30am and 5:30 pm. This made for some very long, intense days. It was suggested that one long talk in the evening, after dinner, might have allowed for a more relaxing schedule during the day. The organizers took some care to make thematic blocks in the schedule, pairing up talks of a similar type. This was appreciated and commented on by participants.

On Wednesday afternoon the workshop organized four options for field trips. The aim was to have something interesting and appropriate for everyone. However, some participants observed that a single large field trip would have given them a better opportunity to talk with colleagues in an informal setting and without the time pressure of the regular workshop schedule.

Participants

Bahsoun, Wael (Loughborough University)
Baladi, Viviane (Ecole Normale Superieure, CNRS)
Balasuriya, Sanjeeva (University of Adelaide)
Bandtlow, Oscar (Queen Mary, University of London)
Berger, Arno (University of Alberta)
Bollt, Erik (Clarkson University)
Bose, Chris (University of Victoria)

Bruin, Henk (University of Vienna)
Bunimovich, Leonid A. (Georgia Institute of Technology)
Dellnitz, Michael (University of Paderborn)
Demers, Mark (Fairfield University)
Dettmann, Carl (The University of Bristol)
Eslami, Peyman (Concordia University)
Ferguson, Andrew (University of Bristol)
Froyland, Gary (University of New South Wales)
Gonzalez Tokman, Cecilia (University of Victoria)
Gora, Pawel (Concordia University)
Haydn, Nicolai (University of Southern California)
Islam, Shafiqul (University of Prince Edward Island)
Junge, Oliver (Technische Universitat Munchen)
Klages, Rainer (Queen Mary University of London)
Lin, Kevin (University of Arizona)
Liverani, Carlangelo (University of Rome Tor Vergata)
Lloyd, Simon (University of Sao Paulo)
Meiss, James (University of Colorado, Boulder)
Melbourne, Ian (University of Warwick)
Murray, Rua (University of Canterbury - New Zealand)
Nicol, Matthew (University of Houston)
Ott, William (University of Houston)
Padberg-Gehle, Kathrin (TU Dresden)
Persson, Tomas (Lund University - Sweden)
Pollicott, Mark (University of Warwick)
Quas, Anthony (University of Victoria)
Ross, Shane (Virginia Tech.)
Santitissadeekorn, Naratip (University of North Carolina - Chapel Hill)
Sato, Yuzuru (Hokkaido University)
Stuart, Robyn (UNSW)
Terhesiu, Dalia (University of Vienna)
Troubetzkoy, Serge (Institut de Mathématiques de Luminy (UMR 6206))
Tupper, Paul (Simon Fraser University)
Watson, Tom (University of New South Wales)
Yorke, James Alan (University of Maryland)
Zweimuller, Roland (University of Vienna)

Bibliography

[1] H. Bruin, M. Demers and I. Melbourne, Existence and convergence properties of physical measures for certain dynamical systems with holes. *Ergod. Th. and Dynam. Sys.*, **30 (3)**, (2010), 687-728.

[2] L. Bunimovich and A. Yurchenko, Where to place a hole to achieve a maximal escape rate. *Israel J. Math*, **152**, (2011), 229-252.

[3] Chernov, Markarian, and Troubetzkoy, Conditionally invariant measures for Anosov maps with small holes. *Ergod. Thry Dynam. Syst.* **18** (1998), 1049-73

[4] Collet, Martinez and Schmitt, The Yorke-Pianigiani measure and the asymptotic law on the limit Cantor set of expanding systems. *Nonlinearity*, **7**, (1994),1437-43.

[5] M. Demers, Markov extensions and conditionally invariant measures for certain logistic maps with small holes. *Ergod. Thry Dynam. Syst.* **25** (2005), 1139-71.

[6] M. Demers and L.S. Young, Escape rates and conditionally invariant measures, *Nonlinearity*, **19** (2006), 377-397.

[7] M. Demers, P. Wright and L.S. Young, Escape rates and physically relevant measures for billiards with small holes. *Comm. Math. Phys.*, **294, (2)** (2010), 253-288.

[8] C. Gonzalez-Tokman, B. Hunt and P. Wright (2009), Approximating invariant densities for metastable systems. *Ergod. Th. and Dynam. Syst.*, **31**, (2011), 1345-1361.

[9] Hensley, D., Continued fraction Cantor sets, Hausdorff dimension, and functional analysis. J. Number Theory 40 (1992), no. 3, 336-358.

[10] G. Keller and C. Liverani, Rare events, escape rates, quasistationarity: some exact formulae. *J. Stat. Phys.*, **135, (3)** (2009)519-534.

[11] Keller, G. and Liverani, C., *Stability of the spectrum for transfer operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 28 (1999), no. 1, 141-152.

[12] Pianigianni and Yorke, Expanding maps on sets which are almost invariant: decay and chaos, *Trans. Amer. Math. Soc.* **252**, (1979), 351-66

[13] Ulam, S.M., *A Collection of Mathematical Problems* , Interscience, New York, Interscience Tracts in Pure and Applied Math., **8**, 1960.

Chapter 17

Geometric Structures on Manifolds (12w5121)

April 15 - 20, 2012

Organizer(s): Ian Hambleton (McMaster University), Alexei Kovalev (Cambridge University), Ronald J. Stern (UC Irvine)

The geometry and topology of manifolds is a large research area, making connections with many flourishing specialties such as algebraic topology, symplectic geometry, gauge theory, knot and links, and differential geometry. The purpose of this meeting was to bring together a broad selection of researchers from many flourishing areas of current work in the more geometric aspects of topology, in order to promote awareness of new developments across the whole field.

This meeting was a sequel to our highly successful meetings "Topology" 05w5067 and "Topology" 07w5070 held at BIRS (August 27 - Sept. 1, 2005 and February 25 - March 2, 2007, respectively) both of which had similar objectives and scope. The strongly positive comments we received from the participants at that time encouraged us to think that a meeting with this broader scope was a valuable service to the mathematics research community. We limited the talks to 5 per day and 45 minutes each, allowing ample time for informal interactions. The speakers were asked to address a broad audience and most of them did this very successfully. A good number of the talks were given by younger mathematicians. In general the atmosphere was very creative, and also those who did not give a lecture had the chance to explain their ideas in numerous discussions in smaller groups. As organizers, we were very pleased with the high scientific level of the talks, and with the energy and enthusiasm of all the participants.

Overview of the Field

The overall problem is to classify manifolds of a fixed dimension in a variety of inter-related categories, e.g. topological, smooth, symplectic, complex, as well as those that support geometric structures reflected in its fundamental group. At our last meeting we focused on the recent work of Perleman on geometric structures on 3-manifolds. This year we focused on higher dimensions, with emphasis on the notoriously difficult classification of simply-connected smooth 4-manifolds, the topological classification of 4-manifolds with specific fundamental groups, geometric structures on 4-manifolds, and the study of mirror symmetry and the re-emerging study of G_2 and $Spin(7)$ manifolds.

Presentation Highlights

The format of the meeting was designed to promote interaction and discussion, as well as exposure of all the participants to the above-mentioned themes. We encouraged several of the speakers to provide a broad overview of their respective areas of geometry/topology. These included a broad overview by Robert Bryant of current research in the active area of G_2 and $Spin(7)$ manifolds. This area begins with the motivational observation (due to M. Wang and based on the famous classification of Riemannian holonomy due to Berger) that a connected, oriented Riemannian spin manifold (M^n, g) can have the dimension of its parallel spinor fields equal to 1 only when n is one of 1, 7, or 8.

The case $n = 1$ is, of course, trivial; the case $n = 7$ only occurs for Riemannian 7-manifolds whose holonomy is $G_2 \subset SO(7)$, the so-called 'proper' G_2-manifolds; and the case $n = 8$ only occurs for Riemannian manifolds whose holonomy is $Spin(7) \subset SO(8)$, the so-called 'proper' $Spin(7)$-manifolds. (If the holonomy of a Riemannian 7-manifold is a proper subgroup of G_2, then it is locally a product, and we say that it is an 'improper' G_2-manifold, with similar terminology for the 8-dimensional case.) Thus G_2-manifolds are characterized (and specified) by the choice of a 3-form $\sigma \subset \Omega^3_+(M^7)$ (where $\Omega^3_+(M^7) \subset \Omega^3(M^7)$ consists of the 'definite' 3-forms, which satisfy a point-wise non-degeneracy condition that is described in the talk). Such a form defines a canonical metric g_σ and orientation $\star_\sigma$ (i.e., Hodge star operation), and σ defines a (possibly improper) G_2-manifold structure if and only if σ and $\star_\sigma \sigma$ are closed forms on M. This system of differential equations is characterized in terms of its solvability and the known existence results are described, both in the cases of local solutions and closed 7-manifolds (particularly the spectacular results of Joyce and Kovalev proving existence in these cases). This leads to several obstructions.

There were several talks on the (possible and hoped-for) extensions of gauge theoretic techniques to the study of 7- and 8-manifolds. There is an enormous amount of work to be done there to make it into a useful theory in analogy with the application of geometric techniques that have been so successful in 3- and 4-dimensions. But it is clear that there are some talented young people working on this.

Mirror symmetry has taken a central role and there were two important contributions represented at this meeting. Tim Perutz provided insights into his work on the arithmetic aspects of homological mirror symmetry. Homological mirror symmetry (HMS) relates the Fukaya category of a degenerating Calabi-Yau manifold to the derived category of coherent sheaves on its mirror partner. Ordinarily, these categories are taken with coefficients in an algebraically closed field containing the complex numbers. It turns out that HMS for Calabi-Yau manifolds should have a refinement in which the mirror is a variety defined over a far smaller ring. In this way, the symplectic topology of one manifold is tied to the arithmetic geometry of its mirror. Joint work with Yankı Lekili proves a test case, arithmetic mirror symmetry for the 2-torus, which says in part that the Fukaya category of exact Lagrangian curves on the once-punctured 2-torus is equivalent, over the ring of integers, to the differential-graded category of perfect complexes on the nodal Weierstrass curve $y^2 + xy = x^3$.

Einstein 4-manifolds have taken a central role in 4-manifolds as a possible structure to place on many smooth 4-manifolds. This was highlighted by the beautiful talk of Claude LeBrun on the classification of Einstein 4-manifolds with complex structure. An Einstein metric is, by definition, a Riemannian metric of constant Ricci curvature. Not all smooth compact 4-manifolds admit such metrics, and the difference between existence and non-existence primarily depends on the smooth structure rather than the homeotype of the manifold in question. During this conference recent existence, uniqueness, and classification results concerning Einstein metrics smooth compact 4-manifolds were presented, contrasting these results with the wildly different results that have emerged regarding Einstein metrics in other dimensions.

To put the study of geometric structures on 4-manifolds in context: a *geometry* consists of a complete 1-connected Riemannian manifold X such that $G = Isom(X)$ acts transitively on X and has discrete subgroups Γ (lattices) which act freely on X, with finite covolume ($vol(X/\Gamma < \infty$).

Jonathan Hillman described some earlier work on the characterization of geometric 4-manifolds up to homotopy (or better). In dimension 4 there are 18 geometries with cocompact lattices. In 12 cases the model X is contractible, and a closed 4-manifold M is TOP s-cobordant to X/Γ if and only if $\pi_1(M) \cong \Gamma$ and $\chi(M) = \chi(\Gamma)$. (However the possible groups and values of χ are not known for the geometries H^4 and $H^2(C)$, and for the irreducible $H^2 \times H^2$-lattices.)

Three have compact models, and such manifolds are well understood. However complete algebraic invariants are not yet known when $X = S^2 \times S^2$ and $|\Gamma| = 4$ (i.e., $\chi = 1$). The key problem for the geometry $S^3 \times E^1$ is to show that if M is a closed 4-manifold with $\chi(M) = 0$ and $\pi_1(M) \cong Z/qZ \times_s Z$ then M is homotopy equivalent to a mapping torus. (This has been verified in the non-orientable cases, by Davis and Weinberger.)

Jonathan Hillman then went on to describe recent progress on the remaining two geometries $S^2 \times E^2$ and $S^2 \times H^2$. Every closed 4-manifold M with $\pi_2(M) \cong Z$ is either (i) homeomorphic to CP^2; (ii) TOP s-cobordant to the total space of an S^2-bundle; (iii) simple homotopy equivalent to the total space of an

RP^2-bundle; or (iv) simple homotopy equivalent to the total space of an S^2-orbifold bundle over a flat or hyperbolic 2-orbifold with non-empty singular locus. (All the S^2- and RP^2-bundle spaces are geometric, as are most of the orbifold bundle spaces.) The recent work establishes case (iv). The key step involves a careful choice of model for the second Postnikov stage $f_M : M \to P_2(M)$, and comparison of the images of the fundamental classes of manifolds related by "Gluck reconstruction".

In a completely different direction, unusual and intriguing results were presented by Liviu Nicolaescu concerning the complexity of random smooth functions on compact manifolds. It turns out there is an interesting relationship between the distribution of critical values of a random smooth function on a compact m-dimensional and the distribution of eigenvalues of a random, real, symmetric $(m+1) \times (m+1)$-matrix.

Recent Developments and Open Problems

Given that this meeting covered a rather large portion of the overall research in the topology and geometry of manifolds, it is most concise to give the abstracts of the remaining talks and discussions presented at the meeting.

D. V. Alekseevsky: Compact cohomogeneity one Kähler and Kähler–Einstein manifolds

The talk is based on a joint work with A.Loi and F.Zuddas and also on a joint work with V.Cortes, K. Hasegawa and Y. Kamishima.

Let G be a compact semisimple Lie group. A cohomogeneity one compact G-manifold M with the orbit space $M/G = [0,1]$ is determined by a triple (H_0, L, H_1) of subgroups, where H_i, $i = 0, 1$ are the stabilizers of the singular orbits , $L \subset H_0 \cap H_1$ is the stabilizer of a regular orbit and the coset spaces $S_i := H_i/L$ are spheres.

If $(g, J, \omega = g \circ J)$ is an invariant Kähler structure on M, then regular orbits $S = G/L$ carry an invarinat CR structure. We study Kähler cohomogeneity one G-manifolds under assumption that this CR structure is Levi non degenerate and unique, that it the distribution $(JT)^{\perp} \subset TS$ associated with the normal vector field T of S is the unique invariant contact structure on $S = G/L$. Such Kähler structures are called ordinary.

In this case, the image $F = \mu(S) = G/K$ of a regular orbit S under the moment map is a flag manifold with an invariant complex structure and $\pi : S = G/L \to F = G/K$ is the Sasaki principal S^1-bundle over F associated with the Reeb vector field Z of the contact form $\theta := g \circ JT$.

We describe cohomogeneity one manifolds $M = M(H_0, L, H_1)$ which admit an ordinary invariant Kähler structure in terms of painted Dynkin diagrams associated with the corresponding flag manifold $F = G/K$. Using results by F. Podesta and A. Spiro, we give an explicit description of invariant Kähler structures on M in terms of appropriate functions of one variable.

We classify cohomogeneity one G-manifolds which admit an invariant Kähler-Einstein metric under assumption that the associated flag manifold $F = G/K$ is defined by Dynkin diagram with ≤ 2 black nodes, or, equivalently, the center $Z(K)$ has dimension ≤ 2.

Scott Baldridge: Coisotropic Luttinger Surgery on Symplectic 6-Manifolds

We introduce a surgery operation on symplectic manifolds called coisotropic Luttinger surgery, which generalizes Luttinger surgery on Lagrangian tori in symplectic 4-manifolds. In this talk we will show how to use it to produce infinitely many distinct symplectic 6-manifolds X with $c_1(X) = 0$ which are not of the form $M^4 \times T^2$ and survey some of the recent results in this area. This work is joint with Paul Kirk.

Robert Bryant: G_2 and $Spin(7)$-manifolds

Interspersed among the talks, I had a number of interesting conversations with the other attendees. They inspired me to work out some interesting examples of special holonomy in the case of metric connections with skew-symmetric, closed torsion, a class of problems that has turned up in the physics literature, generalizing the usual background of a Riemannian metric to include a so-called 'H-field' as an extra feature of the theory. In particular, I was able to show, by constructing examples, that every connected subgroup of $SO(4)$ occurs as the holonomy of such a connection in dimension 4, (even the 1-dimensional 'skew lines' in a maximal torus, though it turns out that there is only a finite-parameter family of such examples). Also, I was able to classify the pairs (g, H), metric and 3-form, in dimension 3 for which the holonomy of both of the connections $\nabla_g + H^\sharp$ and $\nabla_g - H^\sharp$ reduces to $SO(2)$. This was asked of me by Gil Cavalcanti in connection with a problem he has been working on. It turns out that these exist and depend on one function of one variable, up to diffeomorphism. I am interested in pursuing this to see whether there are any useful or interesting restrictions on holonomy for such pairs (g, H) in higher dimension.

Olivier Biquard: Desingularization of Einstein orbifolds

Can one desingularize an Einstein orbifold ? the question is well studied in the Kähler case, but very little is known in general. For example, can one desingularize the quotient of the 4-sphere by an involution with 2 fixed points ? We will give a beginning of answer to this question.

Gil Cavalcanti: SKT geometry

An SKT structure on a manifold is a Hermitian structure for which the metric and complex structure are parallel for a connection whose torsion is skew and closed. If this torsion vanishes, this is simply a Kahler structure. At first, it appears that the introduction of torsion spoils several properties shared by Kahler manifolds. I will show how one can use the general framework from generalized complex geometry to develop the theory of SKT manifolds and show that these manifolds actually share several of the properties of (generalized) Kahler manifolds.

Weimin Chen: On Seifert fibered 4-manifolds

In the study of 4-dimensional S^1-manifolds, an important technique has been the so-called "Pao's replacement trick". Originally, Pao used this technique to construct nonlinear S^1-actions on the 4-sphere. Fintushel adapted Pao's trick to give a smooth classification of simply connected 4-dimensional S^1-manifolds. Later, Baldridge gave further applications of Pao's trick to the Seiberg–Witten invariants of S^1-manifolds.

Pao's trick requires that the circle group has a fixed point. In this talk, we will consider 4-dimensional S^1-manifolds without fixed points (called Seifert fibered 4-manifolds). Several results showing the rigidity of such smooth structures will be discussed. Main Theorems: (1) If the Seifert fibered 4-manifold has nonzero Seiberg–Witten invariant, then the fundamental group must have infinite center. (2) Given any finitely presented group with infinite center, there exist at most finitely many smoothly distinct orientable Seifert fibered 4-manifolds realizing the group.

Jim Davis: Topological Rigidity

A group G has a cocompact manifold model for $E_{fin}G$ if there is a G-manifold M with M/G compact, with M^F contractible for all finite subgroups F of G, and with M having the G-homotopy type of a G-CW-complex. Examples of such arise from discrete subgroups of isometries of manifolds with non-negative curvature, or CAT(0) examples of Mike Davis.

Two such cocompact manifold models are G-homotopy equivalent. G satisfies equivariant rigidity if any such G-homotopy equivalence is G-homotopic to a G-homeomorphism. For G torsion-free, a group satisfies equivariant rigidity if and only if M/G satisfies the Borel Conjecture.

Frank Connolly, Qayum Khan, and I are systematically attacking the problem of equivariant rigidity, using, among other ingredients, the Farrell-Jones Conjectures in K- and L-theory. A sample result is:
Theorem: *All H_1-negative involutions on a torus T^n are conjugate to a smooth action. If $n \equiv 0, 1 \pmod 4$ or if $n = 2, 3$ then all H_1-negative involutions on T^n are topologically conjugate. Otherwise there are an infinite number of conjugacy classes of such actions.*

A involution on a space is H_1-negative if it induces multiplication by -1 on the first homology.

Matt Hedden: Recent progress on topologically slice knots

I'll discuss the world of topologically slice knots: those knots which bound topologically flat embedded disks in the 4-ball. These knots generate a particularly interesting subgroup of the smooth concordance group of knots which offers insight into the distinction between the smooth and topological categories in dimension 4. I'll discuss the first examples of (two-) torsion elements in this topologically slice subgroup. This is joint work with Se-Goo Kim and Charles Livingston.

Ludmil Katzarkov: From Higgs bundles to stability conditions

In this talk we will make an analogy between gauge theoretic and categorical invariants. We will outline possibility for constructing new algebro geometric and symplectic invariants.

Matthias Kreck: Codes from 3- and 4 - manifolds

This is a continuation of my previous work with Puppe where we studied the relation between self-dual codes and 3-manifolds with involution. Recently Manin showed interest in out work and asked for realization of non self-dual codes, the typical case, and also for an extension to codes over other fields. In both directions we proved several new results which might make up for a reasonable talk.

Yankı Lekili: Floer theoretically essential tori in rational blowdown

We compute the Floer cohomology of monotone tori in the Stein surfaces obtained by a linear plumbing of cotangent bundles of spheres, also known as the Milnor fibre associated with the complex surface singularity of type A_n. We next study some finite quotients of the A_n Milnor fibre which coincide with the Stein surfaces that appear in Fintushel and Stern's rational blowdown construction. We show that these Stein surfaces have no exact Lagrangian submanifolds by using the already available and deep understanding of the Fukaya category of the A_n Milnor fibre coming from homological mirror symmetry. On the contrary, we find Floer theoretically essential monotone Lagrangian tori, finitely covered by the monotone tori that we studied in the A_n Milnor fibre. We conclude that these Stein surfaces have non-vanishing symplectic cohomology. Joint work with Maksim Maydanskiy.

Tim Nguyen: Quantum Chern-Simons Theory and Perturbative Renormalization

Our goal is to give an informal discussion of the ideas involved in Kevin Costello's formalism for perturbatively quantizing gauge theories using BV geometry. For the sake of illustration, we will use Chern-Simons theory as a model example in this discussion. In this particular case, the path integral yields topological invariants generalizing those obtained by Axelrod-Singer and Kontsevich through Feynman diagramatic analysis.

Timothy Perutz: Arithmetic aspects of homological mirror symmetry

Homological mirror symmetry (HMS) relates the Fukaya category of a degenerating Calabi–Yau manifold to the derived category of coherent sheaves on its mirror partner. Ordinarily, these categories are taken with coefficients in an algebraically closed field containing the complex numbers. I'll explain why HMS for Calabi-Yau should have a refinement in which the mirror is a variety defined over a far smaller ring. In this way, the symplectic topology of one manifold is tied to the arithmetic geometry of its mirror. Joint work with Yankı Lekili proves a test case, arithmetic mirror symmetry for the 2-torus, which says in part that the Fukaya category of exact Lagrangian curves on the once-punctured 2-torus is equivalent, over the ring of integers, to the differential-graded category of perfect complexes on the nodal Weierstrass curve $y^2 + xy = x^3$.

Mihaela Pilca: Lowest eigenvalue of the Dirac operator on Kähler manifolds and special holonomy

On compact Kähler manifolds there exist lower bounds for the first eigenvalue of the Dirac operator (acting on the eigenbundles of the spinor bundle under the action of the Kähler form), which are given by the so-called refined Kirchberg inequalities. Our main result gives a complete description of the limiting manifolds, i.e. those on which this bound is attained. More precisely, we obtain Riemannian products of a Calabi–Yau manifold and a twistor space over a positive quaternion-Kähler manifold. One of the main tools used is the study of the corresponding eigenspinors, which turn out to be exactly the sections in the kernel of a natural first order operator adapted to the Kähler structure, namely the so-called Kählerian twistor (or Penrose) operator.

Nikolai Saveliev: Seiberg-Witten theory on a homology $S^1 \times S^3$

The usual count of solutions to the Seiberg-Witten equations on a 4-manifold X only gives a smooth invariant when $b^2_+(X) > 0$ (with wall crossing when $b^2_+(X) = 1$.) If X is a homology $S^1 \times S^3$, we provide an index-theoretic term which, when added to the count of Seiberg-Witten solutions, gives a smooth invariant of X. The definition of this term is based on our extension of the Atiyah–Patofi–Singer index theory for first order differential operators, from manifolds with cylindrical ends to manifolds with more general periodic ends. We will discuss some examples in which we have been able to make calculations of our invariant. This is joint work with Danny Ruberman and Tom Mrowka.

Andras Stipsicz: Knots in Lattice homology

We introduce a filtration on the lattice homology of a negative definite plumbing tree associated to a further vertex and show how to determine lattice homologies of surgeries on this last vertex. We discuss the relation with Heegaard Floer homology.

Nathan Sunukjian: The relationship between surgery and surface concordance in 4-manifolds.

Understanding embedded surfaces in 4-manifolds is a starting point to understanding smooth structures on 4-manifolds. One reason for this is because by performing some sort of surgery on a surface, it is sometimes possible to change the smooth structure of the 4-manifold without altering its homeomorphism class. A considerable hurdle to this approach is that it can be very difficult to find appropriate surfaces to perform the surgery on. Surface concordance is one way of trying to organize the surfaces in a 4-manifold. In this talk we will, (1) compute the concordance group for a class of surfaces, and (2) explain when surgery on concordant surfaces gives the same result.

Andrew Swann: Multi-moment maps and applications to special holonomy

Suppose a group acts on a manifold preserving a closed differential form. In work with Thomas Bruun Madsen we introduced a notion of multi-moment map that extends the established concept of moment map in symplectic geometry, and, in contrast to previous definitions, shares the basic existence properties. The concept applies particularly well to torus actions on manifolds of holonomy G_2 or $Spin(7)$. One application is an interesting correspondence between such structures and four-manifolds equipped with triples of symplectic forms.

Andrei Teleman: Gauge theoretical approach in the classification of class VII surfaces.

The classification of complex surfaces is not finished yet. The most important gap in the Kodaira–Enriques classification table concerns the Kodaira class VII, e.g. the class of surfaces X having $\mathrm{kod}(X) = -\infty$, $b_1(X) = 1$. These surfaces are interesting from a differential topological point of view, because they are non-simply connected 4-manifolds with definite intersection form. The main conjecture which (if true) would complete the classification of class VII surfaces, states that any minimal class VII surface with $b_2 > 0$ contains b_2 holomorphic curves. We explain a new approach, based on ideas from Donaldson theory, to prove existence of curves on class VII surfaces.

My method starts with the observation that the lack of curves (or more precisely the lack of a "cycle" of curves) implies the appearance of a smooth compact connected component in a certain moduli space of stable bundles ($\mathrm{PU}(2)$-instantons) over the given surface. For $b_2 \leq 2$ I showed that the presence of such a component leads to a contradiction, proving the existence of cycle for this class of surfaces. I have recentely showed that if a refined form of the Grothedieck–Riemman–Roch theorem for proper maps of (non-Kählerian) complex manifolds was true, the method would generalize to arbitrary b_2. Such refined forms of the Grothedieck–Riemman–Roch have been considered by Jean-Michel Bismut and Julien Grivaux. Extending my results to arbitrary b_2 (which is "work in progress") would completely solve the classification problem up to deformation equivalence.

Bibliography

[1] A. Teleman: *Instantons and holomorphic curves on class VII surfaces*, Annals of Mathematics, 172 (2010), 1749–1804.

[2] A. Teleman: Donaldson Theory on non-Kählerian surfaces and class VII surfaces with $b_2 = 1$, Invent. math. 162, 493–521, 2005

[3] A. Teleman: Gauge theoretical methods in the classification of non-Kählerian surfaces, "Algebraic Topology - Old and New" (the Postnikov Memorial Volume), Banach Center Publications Vol. 85, 2009.

Thomas Walpuski: A conjectural G2 Casson invariant

Given a bundle over a G_2-manifold one can consider a special class of connections called G_2-instantons. Formally, G_2-instantons are rather similar to flat connection on 3-manifolds. In particular they can be understood as critical points of a G_2 Chern-Simons functional. It is conjectured that counting G_2-instantons will lead to a G_2 analogue of the Casson invariant. I will discuss some problems that arise in this project and how they could possibly be attacked. Finally, I will explain how one could hope to compute the conjectural Casson invariant in some rather special cases.

Katrin Wehrheim: How to construct 4-manifold invariants via the symplectic category

Combining Gay-Kirby's theory of Morse 2-functions and the theory of pseudoholomorphic quilts, we develop a procedure for constructing 4-manifold invariants depending on the choice of

- a homotopy class $X \to S^2$ (or in fact to a general Riemann surface);
- a "symplectization functor" from the 2+1 bordism category to the symplectic category (such as the ones used in constructions of 3-manifold invariants - arising from representation spaces or symmetric products).

The construction proceeds by

- interpreting the Cerf diagram of a Morse 2-function (with connected fibers of sufficiently high genus) as string diagram of X in the 2+1+1 bordism bicategory;
- applying the "symplectization functor" to the surface fibers and elementary 3-cobordisms, and associating canonical Floer classes to the 4-dimensional 2-morphisms labeling cusps and crossings of the Morse 2-function;
- evaluating the quilt invariant on the resulting string diagram in the symplectic 2-category.

Invariance of the resulting integer can be proven by translating the moves between Morse 2-functions into moves between quilt diagrams, where a fundamental strip shrinking isomorphism reduces them to verifying

- two axioms on the composition of Lagrangian correspondences;
- one nontrivial axiom on a local holomorphic curve count

In particular, this provides an approach for verifying the invariance of Perutz' Lagrangian matching invariants.

Outcome of the Meeting

Several new collaborations ensued during and after this meeting. In particular Tim Perutz talked through an idea with Tim Nguyen which might become the basis of a collaboration. Jim Davis made considerable progress with some of his collaborators present at the meeting. Andrew Swann met with Andrew Dancer and commented that the ability to meet away from their distractions of their departments allowed them to finish off a number of remaining points in a joint project they have and were pleasantly surprised by how much progress they made on a paper that is now very near completion. A collaboration between Matt Hedden and Nathan Sunujkian began as a direct result of the workshop, the outcome of which could be quite exciting. Later in the summer, Matthias Kreck and Ian Hambleton answered a question from the talk of Baldridge and Kirk, by showing how to "recognize" by explicit invariants which 6-manifolds are diffeomorphic to the products of surfaces and simply-connected 4-manifolds.

As pointed out earlier, another not well understood area in complex surface theory is the classification of Class VII surfaces. Andrei Teleman, using gauge theoretic input provided the best to date classification of these surfaces. Rather than using Seiberg-Witten theory the key new input is to use ideas from Gromov-Witten theory to count only rational curves in a given homology class. However, this talk engendered discussions with Nikolai Saveliev concerning his work with Ruberman concerning their use of Seiberg-Witten theory to study integral rational homology $S^1 \times S^3$ that may well finish off the classification.

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Chapter 18

Composite Likelihood Methods (12w5046)

April 22 - 27, 2012

Organizer(s): Harry Joe (University of British Columbia), Nancy Reid (University of Toronto), Peter Xuekun Song (University of Michigan), David Firth (University of Warwick), Cristiano Varin (University Ca' Foscari)

Overview of the Field

Composite likelihood methods are extensions of the Fisherian likelihood theory, one of the most influential approaches in statistics. Such extensions are generally motivated by the issue of computational feasibility arising in the application of the likelihood method in high-dimensional data analysis. Complex dependence presents substantial challenges in statistical modelling and methods and in substantive applications. The idea of projecting high-dimensional complicated likelihood functions to low-dimensional computationally feasible likelihood objects is methodologically appealing. Composite likelihood inherits many of the good properties of inference based on the full likelihood function, but is more easily implemented with high-dimensional data sets. This methodology is, to some extent, an alternative to the Markov Chain Monte Carlo method, and its impact is unbounded.

The literature on both theoretical and practical issues for inference based on composite likelihood continues to expand quickly; the field of extremal processes for spatial data, of particular importance for climate modelling, is one of the most recent examples of an area where composite likelihood inference is both practical and efficient.

The first international workshop on composite likelihood methods was held at the University of Warwick in April 2008. It attracted participants from all over the world and was widely viewed as very successful. Following the workshop, a special issue of the journal *Statistica Sinica* devoted to composite likelihood was announced; it was published in January 2011. This issue includes two long overview papers, one of which is devoted to applications in statistical genetics; several papers developing new theory for inference based on composite likelihood; new results in the application of composite likelihood to time series, spatial processes, longitudinal data and missing data. The methodology has drawn considerable attention in a broad range of applied disciplines in which complex data structures arise. Some notable application areas include, statistical genetics, genetic epidemiology, finance, panel surveys, computer experiments, geostatistics and biostatistics.

Presentation Highlights

In the opening presentation, Varin described complex likelihoods where the ordinary likelihood function is difficult to evaluate or to specify. However, in many of these situations it is however possible to compute marginal or conditional densities for subsets of the data.

Terminology that he set for the workshop, and which are used for this report, are the following.

- pseudo-likelihood: any function of parameter and data that behaves in “some respect" as a likelihood;
- composite likelihood: one of many examples of pseudo-likelihoods based on terms that are logarithms of marginal and conditional densities;
- quasi-likelihood: different meanings, the two most common are (a) Wedderburn’s quasi-likelihood and variants (statisticians), (b) quasi-likelihoods as misspecified likelihoods (econometricians);
- limited information methods: used in psychometrics as inference procedures based on low-dimensional margins.

Let $\boldsymbol{\theta}$ be a parameter vector of a parametric model for an observation $\mathbf{y}$, a realization of random m-vector $\mathbf{Y}$. For independent and identically distributed replications $\mathbf{y}_1, \mathbf{y}_2, \ldots$, let n be the sample size. Then

- the composite log-density based on K different margins or conditional distributions has the form

$$cl(\boldsymbol{\theta}, \mathbf{y}) = \sum_{k=1}^{K} w_k l_{A_k}(\boldsymbol{\theta}, \mathbf{y}), \quad l_{A_k}(\boldsymbol{\theta}, \mathbf{y}) = \log f_{A_k}(y_j, j \in A_k; \boldsymbol{\theta}) \text{ for margin } A_k$$

- for a sample of size n, the maximum composite log-likelihood estimator $\hat{\boldsymbol{\theta}}$ maximizes $\sum_{i=1}^{n} cl(\boldsymbol{\theta}, \mathbf{y}_i)$.
- the composite score function is the partial derivative of the composite log-density with respect to the parameter vector:

$$\mathbf{u}(\boldsymbol{\theta}, \mathbf{y}) = \sum_{k=1}^{K} w_k \nabla_{\boldsymbol{\theta}} l_{A_k}(\boldsymbol{\theta}, \mathbf{y})$$

- sensitivity or Hessian matrix: $\mathbf{H}(\boldsymbol{\theta}) = \mathrm{E}_{\boldsymbol{\theta}}\{-\nabla_{\boldsymbol{\theta}} \mathbf{u}(\boldsymbol{\theta}, \mathbf{Y})\}$
- variability matrix: $\mathbf{J}(\boldsymbol{\theta}) = \mathrm{Var}_{\boldsymbol{\theta}}\{\mathbf{u}(\boldsymbol{\theta}, \mathbf{Y})\}$
- Godambe information: $\mathbf{G}(\boldsymbol{\theta}) = \mathbf{H}(\boldsymbol{\theta})\mathbf{J}^{-1}(\boldsymbol{\theta})\mathbf{H}(\boldsymbol{\theta})$. As $n \to \infty$, $\mathbf{G}^{-1}(\boldsymbol{\theta})$ is the asymptotic covariance matrix of $n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ under some regularity conditions. In the case of observation of a single time series or random field, the asymptotics depend on ergodicity conditions and the above n is replaced by the observation length m.
- The composite likelihood version of AIC, as given in Varin and Vidoni (2005), is

$$\mathrm{CL}-\mathrm{AIC} = -2\sum_i cl(\hat{\boldsymbol{\theta}}, \mathbf{y}_i) + 2\mathrm{tr}(\mathbf{J}(\hat{\boldsymbol{\theta}})\mathbf{H}^{-1}(\hat{\boldsymbol{\theta}}))$$

- The composite likelihood version of BIC, as given in Gao and Song (2010), is

$$\mathrm{CL}-\mathrm{BIC} = -2\sum_i cl(\hat{\boldsymbol{\theta}}, \mathbf{y}_i) + (\log n)\mathrm{tr}(\mathbf{J}(\hat{\boldsymbol{\theta}})\mathbf{H}^{-1}(\hat{\boldsymbol{\theta}}))$$

Some challenges covered by the workshop and summarized in Varin’s talk included the following.

- Design issues: how do we select the set of marginal or conditional sets A_k, and how should they be combined through choice of weights w_k; this was discussed in presentations of Lindsay and others.
- Uncertainty estimation: $\hat{\boldsymbol{\theta}}$ is straightforward to obtain by inputting the negative of $\sum_i cl(\boldsymbol{\theta}, \mathbf{y}_i)$ into a numerical minimizer, and this can yield an estimate $\mathbf{H}$ at $\hat{\boldsymbol{\theta}}$, but estimation of the variability matrix $\mathbf{J}$ needed for $\mathbf{G}^{-1}$ and standard errors for components are $\hat{\boldsymbol{\theta}}$ can be computationally challenging.
- Calibration: this comes up in several contexts, including calibration of test statistics with nuisance parameters, as discussed in Salvan’s talk.

- Robustness to model misspecifications: examples of different robustness ideas were included in talks of Jordan and Xu.
- Prediction: how does composite likelihood for prediction compare with composite likelihood for estimation.
- Software development.

The most common form of composite marginal or conditional likelihood that has been used in applications is pairwise likelihood. Several presentations considered other sets of margins or conditional distributions; the talks on spatial extremes suggested that there can be gains in efficiencies with triple-wise or trivariate composite likelihood combined with increased (but feasible) computational time.

Lindsay compared various designs (choices of marginal or conditional distributions) such as pairwise, conditional pairs, and hybrids (combination of one-wise and pairwise likelihoods).

There are connections with the use of two-stage (and multi-stage) estimating equations in the copula modelling literature; the method is called inference functions for margins (IFM) in Joe (1997) — first univariate parameters are estimated from univariate likelihoods and then dependence parameters are estimated from higher-dimensional likelihoods.

In Vidoni's presentation, for predictive densities, another consideration is how to weight different components of cl; the best choices for estimation and prediction might not be the same. In Molenberghs' presentation, missing data were handled with inverse probability weighting. In family-based studies of genetic markers, Briollais and Choi had adjustments for ascertainment, for censoring, and for missing data.

Several notations of calibration came up in the workshop. In Salvan's presentation, this meant finding a constant c for the weights w_k in the composite likelihood so that for inference

$$2 \sum_i [cl(\hat{\boldsymbol{\theta}}, \mathbf{y}_i) - cl(\boldsymbol{\theta}_0, \mathbf{y}_i)]$$

can be adjusted suitable to recover an approximate χ^2 distribution when the true parameter is $\boldsymbol{\theta}_0$. In Ribatet's presentation: a similar adjustment was mentioned for quasi-Bayes, with composite likelihood replacing likelihood, and the quasi-posterior or composite posterior distribution. Ng's presentation mentioned that if the weights are all multiplied by a common constant with $w_k \to cw_k$ for all k, then in the CL-AIC, $\mathrm{tr}(\mathbf{JH}^{-1}) \to c\,\mathrm{tr}(\mathbf{JH}^{-1})$ which implies that the penalty term $\mathrm{tr}(\mathbf{JH}^{-1}) = \mathrm{tr}(\mathbf{HG}^{-1})$ should not be interpreted as the effective number of parameters.

For estimation of $\mathbf{J}(\boldsymbol{\theta}) = \mathrm{E}[\mathbf{u}(\boldsymbol{\theta}, \mathbf{Y})\, \mathbf{u}^\top(\boldsymbol{\theta}, \mathbf{Y})]$, is some form of direct estimation better or is it better to estimate the inverse Godambe matrix $\mathbf{G}^{-1}$ of the maximum composite likelihood estimator via an appropriate jackknife or (parametric) bootstrap? Lindsay mentioned that approximately orthogonal pieces can make calculation of $\mathbf{J}$ or $\mathbf{G}$ simpler.

Over the workshop, in addition to those topics mentioned above, there was discussion of the theory of composite likelihood for incomplete data (Molenberghs), survey data and analysis of composite score equations from a design viewpoint (Yi) and model comparisons (Ng). A wide range of applications were covered within the special themed sessions on spatial statistics, multivariate extremes, psychometrics, genetics/genomics as well as other sessions. Applications included spatial-temporal data in air pollution and health (Bai); spatial-temporal data in fMRI (Kang); spatial extremes (Genton, Ribatet, Padoan); extreme rainfall events (Huser, Davison); ecology (Lele); Gaussian graphical models (Gao); random graph models for networks (Bellio); linkage disequilibrium, recombination rates, penetrance in genetics (Larribe, Choi, Briollais); genetic networks (Song); psychometrics and latent variable models (Moustaki, Vasdekis, Maydeu-Olivares); panel multinomial probit models for transportation choices (Bhat); multivariate times series of traffic accidents (Karlis).

There was also a session with demonstrations of software, followed by discussion. Moustaki gave a demonstration of some software for latent variable modeling used in psychometrics; these include implementations of limited information methods. Padoan gave a demonstration of an R package `CompRndFld`, Composite Likelihood for Random Fields, co-authored with Moreno Bevilacqua. Ribatet gave a demonstration of his R package `SpatialExtremes`, Modelling Spatial Extremes; in this package, max-stable processes are fit to spatial data using composite likelihood.

Recent Developments and Open Problems

For the study of efficiencies of various designs, it is helpful to have some models where composite likelihood with higher-dimensional margins are computationally feasible. There are models, such as random effects and structural equation models, for which some theoretical comparisons might be feasible.

There is still an open question about the sense in which composite likelihood estimation is more robust? For robustness against misspecification of the full distribution, this seems difficult to make precise. For studying the robustness of estimating equations based on low-dimensional log-likelihoods, it might be easier to make further study of IFM for copula models because one can more readily come up with different models where some set of margins are fixed and others can vary.

As mentioned in one talk, the case of $\mathbf{H}(\boldsymbol{\theta}) = \mathbf{J}(\boldsymbol{\theta})$ is called *information-unbiased* (a term from Lindsay 1982, Biometrika). Care must be taken in understanding this definition. More clearly, this is written as $\mathbf{H}(\boldsymbol{\theta}) \equiv \mathbf{J}(\boldsymbol{\theta})$ for all $\boldsymbol{\theta}$ in a parameter space.

In the discussion on the Friday morning of the workshop, it was mentioned that for pairwise likelihood, $\mathbf{J}(\boldsymbol{\theta}) - \mathbf{H}(\boldsymbol{\theta})$ tends to be positive definite when the parameter $\boldsymbol{\theta}$ represents positive dependence; see the Appendix of Ribatet et al (2012) for the context of a Markov process. The intuitive explanation is that $\mathbf{J}(\boldsymbol{\theta}) - \mathbf{H}(\boldsymbol{\theta})$ involves covariances among different terms in the pairwise log-likelihood and $\mathbf{H}(\boldsymbol{\theta})$ involves the sum of the variance terms of the pairwise log-likelihood. Subsequent to the conference, it was checked that for the case of the parameter space with $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ representing independence in the components of $\mathbf{Y}$, then $\mathbf{H}(\boldsymbol{\theta}_0) = \mathbf{J}(\boldsymbol{\theta}_0)$ under some assumptions, and furthermore, sometimes $\mathbf{G}^{-1}(\boldsymbol{\theta}_0) = \mathbf{H}^{-1}(\boldsymbol{\theta}_0)\mathbf{J}(\boldsymbol{\theta}_0)\mathbf{H}^{-1}(\boldsymbol{\theta}_0)$ matches inverse Fisher information and sometimes it doesn't.

An example was given by Xu (see also in Cox and Reid 2004; Joe and Lee 2009) for the 3-parameter exchangeable multivariate normal distribution where $\mathbf{H}(\boldsymbol{\theta}) \neq \mathbf{J}(\boldsymbol{\theta})$ but the maximum pairwise likelihood estimator is the same as the maximum likelihood estimator. More generally for structured models based on the multivariate normal distribution, the maximum pairwise likelihood estimator could be different from the maximum likelihood estimator depending on (a) the structural forms of the mean vector and covariance matrix, and (b) whether some parameters are assumed fixed or known. The relation of the maximum pairwise likelihood estimator and maximum likelihood estimator for these types of structured models was mentioned as a research problem in Maydeu's presentation.

The above example means that one cannot say that the composite likelihood estimate is not fully asymptotic efficient if $\mathbf{H}(\boldsymbol{\theta}) \neq \mathbf{J}(\boldsymbol{\theta})$.

Some other theoretical issues are the following.

- Does it matter if there is not a multivariate distribution compatible with, for example, bivariate margins? This depends on the inferences (e.g., joint tail probabilities) to be obtained from the model.
- How do we ensure identifiability of parameters?
- Can connections to weighted likelihoods provide additional insight?
- Is the composite likelihood ratio test preferable to Wald-type test?
- When is composite marginal likelihood preferred to composite conditional likelihood?
- For large p, small or moderate n asymptotics: is there consistency?

Outcome of the Meeting

With more opportunities for discussion in this second composite likelihood conference, there is a clearer picture of some of the challenges for composite likelihood. Some of these have been mentioned above.

Outside of the presentations, there were many opportunities for researchers to discuss their work in composite likelihood and give feedback to each other. Subgroups of the participants had other overlapping interests so there was also much discussion of other topics during meals etc. This workshop will lead to many future collaborative research efforts among the participants.

From the discussion of computing software for general use for composite likelihood, the conclusion seemed to be that creating software to cover the many existing applications of composite likelihood is premature until we have a clearer theoretical understanding of the construction of composite likelihood. However, a web page could be created to (a) collect the software packages related to composite likelihood, (b) suggest a common format for development of further R packages with composite likelihood estimation, providing a standardized user interface to enable easier application of composite likelihood methods.

Finally, invited sessions in other mainstream conferences of international statistical societies will continue the dissemination of research results on composite likelihood methods. An example is a session on composite likelihood at the World Congress in Probability and Statistics in July 2012.

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Bibliography

[1] Cox, D.R. and Reid, N. (2004). A note on pseudolikelihood constructed from marginal densities. *Biometrika*, 91, 729–737.

[2] Ribatet, M., Cooley, D. and Davison, A.C. (2012). Bayesian inference from composite likelihoods, with an application to spatial extremes. *Statistica Sinica*, 22, 813–845.

[3] Gao, X. and Song, P. X.-K. (2010). Composite likelihood Bayesian information criteria for model selection in high-dimensional data. *J. Amer. Statist. Assoc*, 105, 1531–1540.

[4] Joe, H., (1997). *Multivariate Models and Dependence Concepts*, Chapman & Hall, London.

[5] Joe, H. and Lee, Y. (2009). On weighting of bivariate margins in pairwise likelihood. *J. Multivariate Analysis* 100, 670–685.

[6] Lindsay, B.G. (1982). Conditional score functions: some optimality results. *Biometrika*, 69, 503–512.

[7] Varin, C. and Vidoni, P. (2005). A note on composite likelihood inference and model selection. *Biometrika* 92, 519–528.

Chapter 19

Manifolds with special holonomy and their calibrated submanifolds and connections (12w5024)

April 29 - May 4, 2012

Organizer(s): Bobby Acharya (King's College London / ICTP), Robert Bryant (UC Berkeley / MSRI), Spiro Karigiannis (University of Waterloo), Naichung Conan Leung (Chinese University of Hong Kong / IMS)

Overview of the field

The *holonomy* H of an oriented Riemannian manifold (M, g) of dimension n is a compact Lie subgroup of $\mathrm{SO}(n)$, which is a global invariant that is intimately related to the Riemann curvature tensor of g, via the Ambrose-Singer theorem. More precisely, its Lie algebra $\mathfrak{h}$ is generated by the Riemann curvature tensor R of the metric. Because of this, metrics with reduced holonomy (a proper subgroup of $\mathrm{SO}(n)$) have restrictions on their curvature, which makes them interesting solutions to certain prescribed curvature equations. Note that the holonomy condition is actually a *first order* condition on the metric, which automatically implies a second order condition. In 1955, Marcel Berger classified the possible Riemannian holonomy groups that can occur. In the case that M is not locally reducible and not locally symmetric, he found that only seven possible holonomy groups could occur. These groups are summarized in Table 19.1.

Holonomy group	n	Name	Remarks
$\mathrm{SO}(n)$	n	oriented Riemannian	generic
$\mathrm{U}(m)$	$2m$	Kähler	complex and symplectic
$\mathrm{SU}(m)$	$2m$	Calabi-Yau	Ricci flat and Kähler
$\mathrm{Sp}(m)$	$4m$	hyperKähler	Calabi-Yau (Ricci flat) in an S^2 family of ways
$\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$	$4m$	quaternionic-Kähler	positive Einstein (but *not* Kähler)
G_2	7	G_2 manifolds	Ricci-flat, related to octonion algebra
$\mathrm{Spin}(7)$	8	$\mathrm{Spin}(7)$ manifolds	Ricci-flat, related to octonion algebra

Table 19.1: The possible Riemannian holonomy groups

In particular, the last two examples are called the *exceptional* holonomy groups, as they occur in particular dimensions and are related to exceptional structures in algebra (the octonions.) It was initially thought that, although Berger could not exclude these possibilities, they would not actually occur. This was proved to not be the case, as Bryant found the first local examples in the 1980's, followed by complete non-compact examples by Bryant-Salamon and independently by groups of physicists, and later compact examples by Joyce in 1994. The last five holonomies in Table 19.1 (all but the generic and Kähler holonomies) are often called *special* holonomies. They are also characterized by the fact that they admit parallel or Killing spinors, which are important ingredients in theories of physics that incorporate supersymmetry. As a result, such metrics have long been of intense interest in physics. All metrics with reduced holonomy come equipped with one or more differential forms which are parallel with respect to the Levi-Civita connection. These forms are intimately related to the concept of a *calibration*, which we discuss next.

A *calibration* α on a Riemannian manifold (M, g) is a closed k-form satisfying a certain inequality related to the metric g, called the *comass one condition*. Given a calibration, we say that an oriented k-dimensional submanifold L of M is a *calibrated submanifold* if the form α restricts on L to the induced volume form. A theorem of Harvey–Lawson says that such submanifolds are always minimal (that is, they have vanishing mean curvature). Again we see an example where a first order condition automatically implies a second order condition. In addition, the known examples of calibrations for which there exist many non-trivial calibrated submanifolds are defined on manifolds with special holonomy. The most studied calibrations are summarized in Table 19.2.

Ambient manifold	k	Calibration form	Name of calibrated submanifolds
Kähler	$2p$	$\frac{1}{p!}\omega^p$	Kähler submanifolds
Calabi-Yau	$m = \frac{n}{2}$	$\mathrm{Re}(e^{i\theta}\Omega)$	special Lagrangian submanifolds of phase $e^{i\theta}$
G_2	3	φ	associative submanifolds
G_2	4	ψ	coassociative submanifolds
$\mathrm{Spin}(7)$	4	Φ	Cayley submanifolds
quaternionic-Kähler	4p	Υ^p	quaternionic-Kähler submanifolds

Table 19.2: The known calibrations which admits many calibrated submanifolds

Many explicit examples of calibrated submanifolds are known, both in Euclidean spaces and in complete non-compact manifolds. These usually involve a high degree of symmetry, often reducing the non-linear system of partial differential equations to an integrable ordinary differential equation. There are also examples of Kähler and special Lagrangian submanifolds in compact ambient manifolds. A very small subset of the mathematicians who have worked in this area include Bryant, Joyce, Harvey, and Lawson. It is interesting to note that calibrated submanifolds seem to belong to two different families, exhibiting strikingly different behaviour. For example, work of McLean has shown that the moduli spaces of special Lagrangian and of coassociative submanifolds are smooth and unobstructed, and intrinsically defined. This is definitely not the case for Kähler, associative, and Cayley submanifolds. Properties of the moduli spaces of special Lagrangian submanifolds were also extensively studied by Hitchin. The two families of calibrated submanifolds are called *instantons* and *branes* by Leung and his collaborators, who derive many common properties of such submanifolds. Calibrated submanifolds also play a crucial role in the phenomenon known as *mirror symmetry*, which we discuss further below.

Finally, on manifolds with special holonomy, one can consider (real or complex) vector bundles over them, and in some cases, the parallel (calibrating) differential forms allows one to define a special class of connections. Such *calibrated connections* are essentially defined by the fact that the curvature 2-form F_A of the connection A lies in the Lie algebra $\mathfrak{h}$ of the holonomy group H, which is a natural subspace of the space of 2-forms. Examples include Hermitian-Yang-Mills or Hermitian-Einstein connections on bundles over Kähler manifolds, and the Donaldson-Thomas connections on G_2 and $\mathrm{Spin}(7)$ manifolds. These connections can all be viewed as higher-dimensional analogues of the anti-self-dual connections (ASD) over Riemannian 4-manifolds, which were the key ingredients in the spectacular success of the work of Donaldson, Taubes, Uhlenbeck, and others in the early 1980's. Calibrated connections in the G_2 and $\mathrm{Spin}(7)$ settings are still not as well understood as they are in the Kähler setting.

Manifolds with special holonomy are believed to exhibit the phenomenon of mirror symmetry, which is currently best understood in the hyperKähler and Calabi-Yau cases, but which for the exceptional cases is at present more mysterious. Due to work of Strominger–Yau–Zaslow, Gukov–Yau–Zaslow, and others, it is expected that understanding the geometric aspects of mirror symmetry will involve studying the moduli spaces of calibrated submanifolds that these manifolds possess, as well as the moduli spaces of ASD connections or their exceptional holonomy analogues. A bit more precisely, it is expected that under certain conditions, a manifold M with special holonomy will fibre over a base space B, with the generic fibre being a calibrated torus in M. We know that such a fibration *must* have singular fibres, which are very difficult to deal with analytically and geometrically. The idea is that the "mirror" $\widehat{M}$ should be obtainable by "dualizing" the smooth part of this fibration, and then somehow compactifying to deal with the singular fibres. Although a great deal of progress has been made in the case of mirror symmetry for Calabi-Yau manifolds, by Gross, Ruan, Seidel, and others, we are still far from a complete proof of the "Strominger-Yau-Zaslow conjecture" (which is not even precisely formulated.) There is much less progress on mirror symmetry for the exceptional holonomy groups, other than some important early papers by Acharya and other physicists.

Another important aspect is the role of spinors and Dirac operators in these settings. Spin geometry seems to be natural for describing many of these structures. For example, we have already mentioned that the Ricci-flat manifolds that have special holonomy admit parallel spinors. Work of Harvey and others shows that calibrations can be obtained as the "square" of a spinor. It is likely that spinors may also prove to be fruitful in the study of calibrated connections.

Objectives of the workshop

It is important to note that mirror symmetry was (successfully) predicted by physicists, working in string theory and M-theory. The intuition of the physics community has been absolutely essential for so much of the spectacular progress that has been made in differential geometry and algebraic geometry since the 1970's, led by Atiyah, Bott, Witten, Yau, and others, and this continues to be the case. We name just a few of the research areas in geometry that are heavily influenced and inspired by physics: Lorentzian geometry (Positive Mass Theorem, Penrose Inequality), connections on vector bundles (Yang-Mills theory, Donaldson theory, Seiberg-Witten theory), enumerative algebraic geometry (Gromov-Witten invariants, quantum cohomology), Einstein metrics (the AdS / CFT correspondence, stability of vector bundles), and many others.

It is therefore vitally important for the geometry community to continue to maintain and to expand their dialogue with the physics community, for the mutual benefit of both parties. This particular BIRS workshop served just such a purpose, gathering together geometers and physicists, and allowing them to work together in an environment that was ideally suited for making important research advances. We expect that there will be several new collaborations arising from this workshop. It was also very important to us that we invited a significant number of young researchers, both graduate students and postdoctoral fellows, to expose them to some of the current research in this area, and to enable them to meet other mathematicians and begin new research projects. No less than 8 out of the 35 participants were either graduate students or postdoctoral fellows.

Summary of the talks

There were a number of talks given at the workshop. Here are the abstracts, in alphabetical order by speaker surname:

Speaker: **Bielawski, Roger (University of Leeds)**
Title: **Pluricomplex geometry and quaternionic manifolds**
Abstract: I will describe a new type of geometric structure on complex manifolds. It can be viewed as a deformation of a hypercomplex structure, but it also leads to special types of hypercomplex and hyper-Kähler geometry. These structures have both algebro-geometric and differential-geometric descriptions, and there are interesting examples arising from physics. Moreover, a class of pluricomplex manifolds leads to quaternionic-Kähler metrics, generalising the $\mathrm{SO}(3)$-invariant self-dual Einstein examples of Hitchin.

Speaker: **Chen, Yunxia (Chinese University of Hong Kong)**
Title: **Minuscule representation bundles on surfaces with ADE singularities**

Abstract: The minimal resolution of a surface with a simple singularity has a bunch of (-2)-curves as its exceptional locus, whose dual graph is a Dynkin diagram of type ADE. In this talk, we construct minuscule representations of the corresponding Lie algebra using configurations of (-1)-curves. Then we build extension bundles over the resolution using their associated line bundles satisfying: (i) they can be descended to the singular surface and (ii) they carry natural tensorial structures. Using (ii) we construct ADE Lie algebra bundles so that the original vector bundles become minuscule ADE representation bundles over our surface with ADE singularities.

Speaker: **Cherkis, Sergey (University of Arizona)**
Title: **Octonions, monopoles, and knots**
Abstract: Witten's approach to Khovanov homology of knots is based on the five-dimensional system of equations, which we call the Haydys–Witten equations. We formulate a dual seven-dimensional system of equations. It can be formulated on any G_2 holonomy manifold and is a close cousin of the monopole equation of Bogomolny. The octonions play a central role in our view of the Haydys–Witten equations and in the transform relating the five- and seven-dinemsional systems.

Speaker: **Conlon, Ronan (McMaster University)**
Title: **A theorem of existence for asymptotically conical Calabi-Yau manifolds**
Abstract: Asymptotically conical (AC) Calabi-Yau manifolds are Ricci-flat Kähler manifolds that resemble a Ricci-flat Kähler cone at infinity. I will describe an existence theorem for AC Calabi-Yau manifolds which, in particular, yields a refinement of an existence theorem of Tian and Yau for such manifolds. I will also discuss some examples. This is ongoing work with Hans-Joachim Hein.

Speaker: **Dunajski, Maciej (University of Cambridge)**
Title: G_2 **geometry, twistor theory, and cuspidal cubics**
Abstract: We establish a twistor correspondence between seven-parameter families of rational curves in a surface, and certain G_2 structures on moduli spaces of such curves. There are several explicit examples — e.g. the space of all cuspidal cubic curves in $\mathbb{P}^2$ gives rise to a homogenous co-calibrated G_2 structure on $\mathrm{SU}(2,1)/\mathrm{U}(1)$.

Speaker: **Gayet, Damien (Université Lyon I)**
Title: **Smoothing moduli spaces of associative submanifolds**
Abstract: It is known that deforming a closed associative Y (respectively an associative Y with boundary in a fixed coassociative X) as an associative is an elliptic problem of vanishing index (respectively of index given by the topology of the normal bundle in TX over the boundary of Y). I will explain two ways to ensure smoothness of the moduli space of local associative deformations of Y. The first way is to assume metric conditions on Y and the second is to perturb the G_2 structure in the realm of closed G_2 structures (respectively the boundary X).

Speaker: **Grigorian, Sergey (Stony Brook University)**
Title: **Deformations of** G_2**-structures with torsion**
Abstract: We consider non-infinitesimal deformations of G_2-structures on 7-dimensional manifolds and derive a closed expression for the torsion of the deformed G_2-structure. We then specialize to the case where the deformation lies in the 7-dimensional representation of G_2 and is hence defined by a vector v. In this case, we explicitly derive the expressions for the different torsion components of the new G_2-structure in terms of the old torsion components and derivatives of v. In particular this gives a set of differential equations for the vector v which have to be satisfied for a transition between G_2-structures with particular torsions. For some specific torsion classes we then explore the solutions of these equations.

Speaker: **Harland, Derek (Loughborough University)**
Title: **Instantons and Killing spinors**
Abstract: I will present some new examples of instantons on manifolds with real Killing spinors and their

cones. Examples of manifolds admitting real Killing spinors include nearly Kähler 6-manifolds, nearly parallel G_2-manifolds in dimension 7, Sasaki-Einstein manifolds, and 3-Sasakian manifolds. For each of these classes of manifolds, I will exhibit a connection on the tangent bundle which has reduced holonomy and which solves the appropriate instanton equation. I will also discuss new 1-parameter families of instantons on the cones over real Killing spinor manifolds: these generalise various examples that appeared in the physics literature, and can be lifted to solutions of heterotic supergravity.

Speaker: **Ivey, Thomas (College of Charleston)**
Title: **Austere submanifolds in complex projective space**
Abstract: A submanifold M in Euclidean space $\mathbb{R}^n$ is austere if all odd-degree symmetric polynomials in the eigenvalues of the second fundamental form (in any normal direction) vanish. Harvey and Lawson showed that this condition is necessary and sufficient for the normal bundle of M to be special Lagrangian in $T\mathbb{R}^n \cong \mathbb{C}^n$. A similar result was proved by Karigiannis and Min-Oo for submanifolds in S^n, with TS^n carrying a Calabi-Yau metric due to Stenzel. In this joint work with Marianty Ionel, we determine conditions under which the normal bundle of a *CR*-submanifold in $\mathbb{CP}^n$ is special Lagrangian with respect to the Stenzel metric on $T\mathbb{CP}^n$. We give examples in the case of hypersurfaces in $\mathbb{CP}^2$, and some nonexistence results in the totally real case.

Speaker: **Kovalev, Alexei (University of Cambridge)**
Title: **Asymptotically cylindrical $\mathrm{Spin}(7)$ manifolds**
Abstract: Riemannian manifolds with asymptotically cylindrical ends are essential ingredients in gluing theorems and also have a natural interpretation as having a ‘boundary at infinity’. I will report on recent progress in constructing examples of asymptotically cylindrical 8-manifolds with special holonomy $\mathrm{Spin}(7)$. The method uses parts of Joyce’s construction of compact $\mathrm{Spin}(7)$ manifolds modified in some important ways and can also be compared at some points with the known constructions of asymptotically cylindrical manifolds with holonomy G_2 and $\mathrm{SU}(n)$.

Speaker: **Lotay, Jason (University College London)**
Title: **Deforming G_2 conifolds**
Abstract: Two natural classes of G_2 manifolds are those which either have non-compact ends asymptotic to cones or have isolated conical singularities. Examples of the former are given by the first complete examples of G_2 manifolds due to Bryant and Salamon, and the latter play an important role in M-Theory. By the fundamental work of Joyce, a compact G_2 manifold M has a smooth moduli space of deformations of dimension $b^3(M)$. I will describe a natural extension of this result to the two aforementioned types of G_2 conifolds. In particular, I will show the stark contrast between the deformation theories in each case and give some applications. This is joint work with S. Karigiannis.

Speaker: **McKay, Benjamin (University College Cork)**
Title: **Some soliton solutions of a flow for G_2-structures**
Abstract: The Laplacian coflow of a G_2-structure is the flow in which the 4-form evolves by its Laplacian. Spiro Karigiannis, Mao-Pei Tsui, and I found a few examples of solitons for this flow.

Speaker: **Mettler, Thomas (Mathematical Sciences Research Institute)**
Title: **Holonomy reduction of 2-Segre structures**
Abstract: The Weyl metrisability problem on a projective surface M corresponds to finding holomorphic curves in a certain quasiholomorphic fibre bundle over M. In this talk I will show that there is a similar correspondence for reducing the holonomy group of a torsion-free 2-Segre structure on an even dimensional manifold.

Speaker: **Pacini, Tommaso (Scuola Normale Superiore)**
Title: **Gluing constructions for special Lagrangian conifolds in $\mathbb{C}^m$**
Abstract: I will present some recent gluing results as in my preprint “Special Lagrangian conifolds, II”, available on the arXiv.

Speaker: **Parton, Maurizio (Universita di Chieti-Pescara)**
Title: $\mathrm{Spin}(9)$**, complex structures, and vector fields on spheres**
Abstract: Joint work with Paolo Piccinni and Victor Vuletescu. Although holonomy $\mathrm{Spin}(9)$ appears to be a very restrictive condition, weakened holonomy $\mathrm{Spin}(9)$ conditions have been proposed and studied in the last years. In this setting, a basic problem is to have a simple algebraic formula for the canonical 8-form Φ whose stabilizer is $\mathrm{Spin}(9)$, as happens for instance in the $\mathrm{Sp(n)} \cdot \mathrm{Sp(1)}$, G_2, and $\mathrm{Spin}(7)$ cases. I will show a nice relation between Φ and a family of almost complex structures J associated to the $\mathrm{Spin}(9)$ structure, leading to an algebraic formula for Φ. I will then show how the existence of more than 7 independent tangent vector fields on spheres is all the fault of $\mathrm{Spin}(9)$, more precisely, all the fault of the J's. Finally, if time permits, the case of metrics which are locally conformal to a parallel $\mathrm{Spin}(9)$ metric will be discussed.

Speaker: **Sá Earp, Henrique (Universidade Estadual de Campinas)**
Title: **Perspectives on** G_2**-instantons**
Abstract: Solutions to the Hermitian Yang–Mills problem over A. Kovalev's asymptotically cylindrical Calabi-Yau $3-$folds induce instantons over compact $7-$manifolds with holonomy group G_2, obtained by a twisted gluing procedure. Moreover, algebro-geometric monad constructions developed by M. Jardim can be used to generate numerous concrete examples of such G_2-instantons. I will present a survey of that study, punctuated by some open questions ranging from naïve to quite ambitious.

Speaker: **Smith, Aaron (University of Waterloo)**
Title: **A theory of multiholomorphic maps**
Abstract: In recent decades the phenomena associated to pseudoholomorphic curves in Kähler manifolds have led to the discovery of a number of interesting invariants of symplectic manifolds. I will introduce the generalizing framework of multiholomorphic mappings of which the theory of pseudoholomorphic curves forms one of a few families of examples. This is a theory pertaining to mappings (between Riemannian manifolds) which satisfy a particular PDE describing the intertwining of geometric data on domain and target. The higher-dimensional scenario is characterized by a significant amount of rigidity. This will be seen in particular on a family of examples of multiholomorphic maps which involve maps from a 3-manifold into a G_2 manifold. There are close relations to calibrated geometry and mathematical physics.

Speaker: **Wang, Mu-Tao (Columbia University)**
Title: **A separation of variables Ansatz for special Lagrangian submanifolds**
Abstract: I shall discuss new constructions of special Lagrangian submanifolds and self-similar solutions of Lagrangian mean curvature flows based on a separation of variables ansatz. A similar construction for Ricci solitons will also be discussed.

Speaker: **Warren, Micah (Princeton University)**
Title: **Calibrated geometry in the optimal transportation problem**
Abstract: It has been observed that Monge-Ampère equations are related to notions of special Lagrangian submanifolds in manifolds with signature (n, n). We discuss optimal transportation problems and find that there is a natural metric on the product manifold associated to a given problem. With respect to this metric, the graph of the solution to the optimal transportation problem is a calibrated current. We will say precisely what this means and discuss the many analogies between this setting and the Calabi-Yau setting.

Speaker: **Witt, Frederik (Universität Münster)**
Title: **A variational problem for spinors**
Abstract: We introduce a natural functional on the universal spinor bundle and discuss the Euler-Lagrange equation of the associated variational problem.

Speaker: **Ye, Rugang (University of California at Santa Barbara)**
Title: **The Laplacian flow**

Abstract: The Laplacian flow, introduced by R. Bryant, is a natural evolution equation and serves to deform closed G_2 structures to torsion-free G_2 structures which produce G_2 holonomy. We'll present short-time existence of the Laplacian flow, its stability around torsion-free G_2 structures, long time convergence under a small torsion condition, and the smoothness of the limit map of the Laplacian flow. Additional geometric properties of the Laplacian flow will also be discussed.

Informal discussions arising from the meeting

As we hoped, there were a large number of informal discussions, encouraged both by our scheduling large blocks of open time and by the excellent facilities at BIRS. Of course, it is in the nature of informal discussions that we don't have extensive records of them, but what follows is a list of some of the open problems suggested during the informal discussions:

- Regarding the talk of Yunxia Chen, it would be desirable to generalize this work to del Pezzo surfaces. Namely, it would be of interest to construct ADE bundles over a family of del Pezzo surfaces.
- Regarding the talk by Ronan Conlon, it would be very interesting to study the relations of his work to mirror symmetry.
- From the talk by Damien Gayet, could we generalize his work of smoothing the moduli spaces of associative submanifolds in G_2 manifolds to other cases, for instance to complex subvarieties in Kähler manifolds, or to Cayley submanifolds in $\mathrm{Spin}(7)$ manifolds?
- From the talk by Thomas Ivey, it would be interesting to generalize their work to translations of normal bundles.

Conclusion

As mentioned above, there were many informal discussions, in particular a great deal of exchanges between experts in different fields. Although it is too early to say exactly what collaborations will arise from this workshop, we are confident that the meeting allowed a significant cross-fertilization between fields, and in particular allowed several of the younger participants to advance their research programme thanks to the advice of more senior scholars.

All of the participants were very enthusiastic about BIRS: the natural setting, the infrastructure and the warmth, hospitality and professionalism of the staff were all very much appreciated.

Participants

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Bibliography

[1] M. Atiyah and E. Witten, “M-theory dynamics on a manifold of G_2 holonomy”, *Adv. Theor. Math. Phys.* **6** (2002), 1–106.

[2] C. Bär, “Real Killing spinors and holonomy”, *Comm. Math. Phys.* **154** (1993), 509–521.

[3] A.L. Besse, *Einstein Manifolds*, Springer-Verlag, New York, 1987.

[4] R.L. Bryant, “Some remarks on G_2 structures”, *Proceedings of Gökova Geometry-Topology Conference 2005*, 75–109.

[5] R. Harvey and H.B. Lawson, “Calibrated geometries”, *Acta Math.* **148** (1982), 47–157.

[6] D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford University Press, 2000.

[7] R. C. McLean, “Deformations of calibrated submanifolds”, *Comm. Anal. Geom.* **6** (1998), 705–747.

Chapter 20

Connections Between Regularized and Large-Eddy Simulation Methods for Turbulence (12w5063)

May 13 - 18, 2012

Organizer(s): Eliot Fried (McGill University), Bernard Geurts (University of Twente), Bill Layton (University of Pittsburgh), Robert Moser (University of Texas, Austin), Ugo Piomelli (Queen's University)

Overview of the Field

Fluid flows are typically classified as laminar or turbulent. While the glassy, regular flow of water from a slightly opened tap is laminar, the sinuous, irregular flow of water from a fully opened tap is turbulent. In a laminar flow, the velocity and other relevant fields are deterministic functions of position and time. Photos taken at different times, no matter how far, of steady laminar flow from a tap will be identical. In a turbulent flow, the velocity and other relevant fields manifest complex spatial and temporal fluctuations. A video of steady turbulent flow from a tap will exhibit a constantly changing pattern and many length and time scales. In nature and technology, laminar flows are more the exception than the rule. Fluvial, oceanic, pyroclastic, atmospheric, and interstellar flows are generally turbulent, as are the flows of blood through the left ventricle and air in the lungs. Flows around land, sea, and air vehicles and through pipelines, heating, cooling, and ventilation systems are generally turbulent, as are most flows involved in industrial processing, combustion, chemical reactions, and crystal growth.

Turbulence enhances transport and mixing. Turbulent ocean currents can increase the encounter rate between fish larvae and their prey. Without turbulence, the fuel and air injected into the cylinder of an internal combustion engine would mix too slowly to be effective. Birds extract energy from turbulent winds to soar for great distances without flapping their wings. Turbulence can be detrimental as well. The efficiencies of vehicles, pipelines, and industrial equipment are all hindered by turbulence. Turbulence can also cause structural fatigue, generate unwanted noise, and distort the propagation of electromagnetic signals.

These observations highlight the importance of turbulence research. Our ability to predict and control turbulence and, thus, to intensify or suppress its effects as circumstances warrant is contingent on our understanding of the underlying mechanisms. Turbulence is also immensely interesting from a purely scientific perspective and is a great source of fundamentally important, challenging problems for physicists, engineers, and mathematicians. Moreover, various methods and tools developed in the field have found applications in other fields, including nonlinear optics, nonlinear acoustics, pattern formation, image processing, data compression, and econophysics.

The difficulty of achieving a physically meaningful analytical description of turbulence is well-recognized. The earliest efforts to do so, dating to works published by Boussinesq and Reynolds in the late 19th century, employed a statistical approach based upon additively splitting the flow variables entering the Navier–Stokes (NS) equations into mean and fluctuating components. The resulting Reynolds-averaged Naver–Stokes (RANS) equations include an additional stress, the Reynolds stress, that represents all interactions between the mean flow and the turbulence. A model for these interactions closes the equations. The statistical approach provides the foundation for the RANS-based simulation methods now used extensively in commercial computational fluid dynamics codes. See Alfonsi [1] for a recent review of RANS-based methods. Although the computational cost of these methods is essentially independent of the Reynolds number, they are not without drawbacks. Most importantly, they cannot produce solutions to the NS equations. Further, since all fluctuations are averaged out, RANS-based methods cannot provide information about turbulent flow structures. They are also often found to be overly dissipative and, thus, to produce unnaturally sluggish flows. Finally, because they are based on questionable closure conditions and rely on ad hoc choices of parameters, their predictive capacity is limited.

In contrast to RANS-based simulation, direct numerical simulation (DNS) of the NS equations resolves all the eddy scales in a turbulent flow. However, for DNS of isotropic turbulence in an incompressible fluid at Reynolds number Re, memory needs and CPU usage scale with $\mathrm{Re}^{9/4}$ and Re^3, respectively. DNS simulations of realistic applications—which typically involve Reynolds numbers well in excess of 10^7—are therefore far beyond the capacity of currently available supercomputers. Even so, DNS is highly valuable from the research perspective. In particular, it provides benchmark results against which results from other methods can be compared, at moderate Reynolds numbers.

A less computationally intensive approach is provided by large eddy simulation (LES), in which a low-pass filter is used to eliminate eddy scales smaller than some width h. For LES, with filter width chosen equal to the Taylor microscale, of isotropic turbulence in an incompressible fluid, memory needs and CPU usage scale with $\mathrm{Re}^{3/2}$ and Re^2, respectively. Like averaging, filtering generates a closure problem. Interactions between the resolved and unresolved scales are accounted for by the subfilter stress arising from filtering. Closure requires an expression for the subfilter stress in terms of the resolved scales. To ensure consistency with the NS equations, this stress must become negligible as the filter width h tends to the Kolmogorov length η—the smallest dynamically relevant scale present in a turbulent flow. Although LES is now being applied to practical problems, the achievable complexity is limited by computer power, particularly in case of high-Reynolds wall-bounded flows. Even assuming that computer power continues to grow according to Moore's law, further advances in LES will hinge on developing improved subgrid models and numerical methods.

LES couples the resolution error tied to neglecting the small scales and the truncation error tied to discretization. This complicates the development of subfilter models. In an ideal LES, the net error associated with filtering and discretization will vanish when compared to filtered DNS results. Explicit LES is based first on developing subfilter models and then on constructing numerical methods that minimize the combined error. Implicit LES is based on discretization schemes in which truncation errors serve implicitly as subfilter models. While subfilter models used in explicit LES tend to be loosely motivated by statistical considerations, discretization schemes used in implicit LES are chosen for reasons of numerical expediency. Stability concerns arise for both explicit and implicit LES. Sagaut [2], Geurts [3], John [4], Berselli et al. [5], and Grinstein et al. [6] provide up-to-date descriptions and assessments of the rich spectrum of contemporary LES approaches.

It is possible, at least away from walls, to derive LES models of very high-order accuracy. However, these models also lead to correspondingly intense computational complexity for their numerical solution. Some also introduce yet more questions on the already difficult problem of specifying needed and sometimes artificial local boundary conditions for the inherently nonlocal flow averages. For these two (and other) reasons, there has been a resurgence of interest in basing simulations of turbulent flows on much simpler regularizations of the Navier–Stokes equations rather than on full models of local averages—i.e., LES models. Initially these were developed as pure theoretical tools. The influx of ideas from LES and the added constraint that the regularization be amenable to numerical simulation have breathed new ideas and thus new life into this thread of turbulence modeling.

Considerable attention is now being given to an alternative class of turbulence models that stem from direct, inviscid regularizations of the NS equations. The earliest example of such a model is the Leray [7] regularization. More recent examples include the Clark [8] model, the Bardina [9] model, the NS-α model of Chen et al. [10, 11, 12], the NS-$\alpha\beta$ model of Fried and Gurtin [13], and the NS-ω model of Layton et al. [14]. In these models, the filtered convective flux in the NS equations is modified directly. This produces regularizations that retain the many salient properties of the NS equations. For instance, Foias et al. [15] have shown that solutions to the NS-α model preserve, in an appropriately generalized sense, the circulation behavior of solutions to the NS equations and have also demonstrated that the NS-α equations possess a global attractor with finite dimension. Fried and Gurtin [13] have established Lyapunov relations for the NS-$\alpha\beta$ equations along with specializations for the NS-α equations. Global regularity results for solutions to the NS-α equations has been obtained by Foias et al. [15] and Marsden and Shkoller [16].

In a recent study of the quality of Leray, Bardina, NS-α models (along with recent modifications of the Leray and Bardina models), Geurts et al. [17, 18] showed that these regularized models can be interpreted as LES models. This exciting development indicates the potential for a useful dialog between researchers in the LES community and researchers working on mathematically-based regularizations of the NS equations.

Recent Developments and Open Problems

DNS is currently a highly useful research tool. Nevertheless, there is a consensus that decades might pass before computational resources become sufficient to use DNS for practical applications. Most fluid mechanicians subscribe to the belief that, in the meantime, methods that resolve the most energetically significant modes of turbulent flow at computational costs lower than DNS provide ... promising alternatives to DNS. Regularized turbulence models share common features with ...NS. Their mathematical structure provides a foundation for careful analysis. Such analysis should provide a foundation for the construction of improved LES models.

There are many significant questions that need to be addressed, including:

1. Where is the common ground between DNS and LES?
2. What are the distinguishing features of current regularized turbulence models and how do they agree with and differ from features of current LES models?
3. What general properties should regularized turbulence models possess?
4. To what extent should regularized models satisfy physical principles such as frame-indiffference and thermodynamic compatibility?
5. Is it possible to identify the subclass of regularized turbulence models that implicitly embody effective subfilter stresses?
6. For regularized turbulence models that do implicitly embody effective subfilter stresses, does the LES perspective provide insight regarding the physical nature of the underlying regularization?
7. How should boundary conditions for regularized equations such as the NS-α equation, which involves fourth order spatial derivatives, be formulated?
8. For regularized models that give rise to higher-order evolution equations, what is the meaning of the additional boundary conditions at the subfilter level in the LES context?
9. Can advanced LES approaches such as the variational multiscale model be adapted to and implemented for regularized turbulence models ?
10. To what extent should compatibility of models/regularizations and solution algorithms play a role in selection or development of LES?

Presentation Highlights

The proceedings began with an overview talk given by Edriss Titi, the primary message of which was that turbulence is a phenomenon that is most suitably described by long-time averages. Building toward this point, Titi discussed the distinguishing features of the Smagorinsky, Euler-α, NS-α, Euler–Voigt, NS–Voigt, and Bardina models. He also made connections with shell models and the nonlinear Schröding...er equation. Regarding flows in bounded domains, Titi described a strategy for Navier–Stokes-α equations which involves setting $\alpha = 0$ in the viscous sublayer (so that it is the NS equations that are solved there, subject to the no-slip boundary condition) and solving the NS-α equations elsewhere. Reynolds stresses calculated using this strategy are found to be in excellent agreement with those obtained from NS based simulations. When applied to pipe flow, choosing α to be 1–2% of the pipe width is found to be quite accurate. Despite its numerical feasibility, the challenges associated with establishing analytical results for this strategy are significant. This presents an important opportunity for mathematicians.

Bill Layton addressed challenges associated with the central role of legacy codes in applications-oriented simulations of turbulent flows. Most often, the inner workings of such codes are not completely understood by their users. Moreover, the developers of these codes are from generations passed and, hence, are not available for consultation. Nevertheless, many industries would face significant disruptions without these legacy codes. The objective of Layton's work on this topic is to facilitate the use of modern methods involving filtering and dealiasing in conjunction with legacy codes. To be practical, this must be achieved at low cost in both computer time and programmer effort. Importantly, the algorithms involved is so doing generate new models of turbulence and there is a strong need to understand the properties of these models, which presents an opportunity for analysis.

Bernard Geurts provided a comprehensive, unified assessment of a broad spectrum of approaches to regularizing the nonlinear convective terms entering the NS equations, with emphasis on quantifying commutation errors that accompany such regularizations. Generally, these models appear to be a bit too energetic in the small scales. Standard practice has been to compare results from simulations performed using models with results from DNS of the NS equations. However, Geurts demonstrated convincingly that it is more meaningful to compute errors relative to suitably filtered DNS data. Geurts also promoted the utility of performing tests designed to push models to their limits. In particular, Geurts showed that the Leray model is more robust than the modified Leray model.

Roel Verstappen focused on eddy-viscosity modeling within the context of LES. Based on the intuitively appealing assumption that that energy transfer should not exceed eddy dissipation, Verstappen presented an analytically-based strategy for determining the eddy viscosity with the objective of ensuring that—consistent with the insight provided in the talk of Bernard Geurts—the solution of the LES equations yields the best possible approximation of the filtered velocity field. Here, the primary objective is to reduce artifacts generated by over-damping (associated with excessively large values of the eddy viscosity). Verstappen also presented an analysis of the scale truncation properties of the regularization obtained by his approach.

Motivated by challenges associated with the modeling of volcanic eruptions, Luigi Berselli presented results based on a reduced model for suspensions of solid particles in compressible fluids. In such models, the governing equations are coupled via drag terms and a dimensionless parameter of key importance is the Stokes number, which incorporates the difference between the characteristic velocities of the air and the suspended particles. Aside from addressing the challenges associated with the effects of compressibility, Berselli's work highlights the importance of problems involving the coupling between turbulence and additional physical processes, in particular the diffusion of suspended particles.

Lars Röhe described recent advances based on the variational multiscale (VMS) method with particular emphasis on local projection stabilization (LPS). LPS is based on introducing an operator that subtracts the large scales from the symmetric part of the velocity gradient tensor. Röhe showed that, for coarsely resolved finite-element simulations of channel flow, LPS drastically reduces spurious oscillations in discrete approximations. A primary advantage of LPS stems from the symmetry of the stabilization terms. Moreover, LPS is also found to preserve the favourable stability and approximation properties of classical residual-based stabilization techniques but avoids difficulties associated with strong coupling of velocity and pressure in the stabilization terms. Röhe also discussed effective multiscale eddy viscosities and focusing boundary conditions.

Volker John brought the audience up to date on the current status of the analysis of the existence and uniqueness of solutions to the flow equations arising from regularized turbulence models and the numerical analysis of convergence properties for discretizations of those equations. A key difficulty with analyses arises due to the presence of a large constant in the Gronwall inequality. John placed special emphasis on the two-scale VMS method. He also emphasized the importance of analyzing statistics and attractors for regularized models, noting in the process that the gap between practice and theory is large and, thus, the need for increased attention to analysis. This led to the discussion of the need to design methods based on invariant measures. It was noted that although invariant measures for the NS equations exist, the physical relevance of these measures is unknown. Mathematical analysis aimed at clarifying said relevance would represent a very important advance.

Traian Illescu presented a new LES method for solving the quasigeostophic model for barotropic ocean circulation. The strategy employs a closure strategy based on approximate deconvolution. To determine the eddy viscosity, Illescu advocates matching the dissipation scale to the grid scale. Results from simulations for symmetric double-gyre wind-...driven circulation in a shallow ocean basin were presented and discussed. It was shown that approximate deconvolution captures the correct flow features on coarse meshes. A challenge, common to all quasigeostophic models, is associated with providing boundary conditions for what amounts to the (scalar) vorticity field. In the simulations described by Illescu, this field was set to zero on the boundary of the flow domain.

Jonathan Graham began by recalling a key hypothesis underlying the derivation of the NS-α model via Lagrangian averaging, namely Taylor's frozen-in turbulence hypothesis—which amounts to stipulating that the small scales advect with the large scales. As a consequence of this hypothesis, phase-locking can occur and large eddies can tumble like rigid bodies. At sufficiently low Reynolds numbers, viscous dissipation effectively "breaks-up" artificial rigid bodies. However, at sufficiently high Reynolds numbers, this is no longer the case and, due to the presence of artificial rigid bodies, the NS-α model fails to function like an LES (inasmuch as undesirably high resolutions are needed to achieve fidelity with DNS). To counteract this difficulty, Graham suggested a strategy based on filtering rigid-body motions. A method for applying this strategy to the magnetohydrodynamic NS-α equations was then described and numerical results based on that method were presented.

Hans Kuerten presented results from detailed direct numerical simulations of turbulent channel flow and compared these to LES using the Leray model. Kuerten concentrated on near-wall turbulence with focus on providing an understanding of the extent to which the model captures flow characteristics predicted by conventional DNS. Particular attention was paid to the influence of the inverse filtering operation. He showed that direct use of the Leray model for wall-bounded turbulence is not effective — more research is needed in this direction.

Leo Rebholz described studies of the impact of an adaptive nonlinear filtering on the Leray model. The method is motivated largely by the desire to avoid spurious filtering of coherent striations. This is achieved by allowing for local adjustment of the filtering operation. Rebholz described an unconditionally stable time-stepping scheme for his method. The primary advantage of this scheme is that the filtering operation is effectively linear at each time step and, thus, is decoupled from other aspects of the numerical strategy. It was, however, observed that the filtering method is not invariant with respect to Galilean changes of observer. A simple solution to this problem involves modifying the filter so that it depends not on the velocity but instead on the difference between the velocity and its filtered counterpart. Importantly, Rebholz noted that the analysis needed to establish the unconditional stability of his time-stepping scheme would go through even with this modification. A somewhat more difficult and challenging question arose in connection with the breakdown of mass balance that accompanies the proposed filtering method, leading to a very lively discussion.

Xavi Trias explored the connections between a general class of spectrally consistent regularized models for turbulent flow and DNS. He also established connections with small-large and small-small VMS models. As with the NS-α and NS-$\alpha\beta$ models, the models considered by Trias preserve the symmetry and conservation properties of the NS equations. Of particular interest is the C_4 regularization, in which the convective term in the NS equations is replaced by a fourth-order approximation involving the residual of a self-adjoint linear filter. Trias observed that the C_4 regularization yields what can be viewed as a "parameter-free" turbulence model. However, one potentially undesirable feature of the model is that it does not preserve the Galilean invariance of the NS equations. Strategies for the partial recovery of Galilean invariance were described. These involve the addition of a hyperviscous term which has the benefit of reducing spurious backscatter.

Helene Dalmann focused on establishing a sound mathematical approach to determine the suitability of any proposed turbulence model. The approach hinges on developing an appropriate error functional and, thus, an understanding of which statistics are most appropriate. In particular, Dalmann studied the properties of the eddy-viscosity based LES model of Roel Verstappen. While it was previously known that Verstappen's eddy viscosity vanishes for the 13 laminar flows previously detailed by Vr...eman, Dalmann's analysis showed that there exist other laminar flows for which Verstappen's eddy viscosity vanishes. An error functional that takes into account this additional information was presented, as were numerical results from homogeneous decaying turbulence and channel flow.

James Riley described results associated with dispersion-induced area stretching of scalar isosurfaces in turbulent flows. The surface evolution equations that emerge from these considerations relate the scalar normal velocity and curvature of the evolving surfaces and, therefore, resemble closely interfacial evolution equations from models for phase transformations. This connection suggests interesting connections with curvature-driven surface evolution equations.

Jonathan Gustafsson spoke on integral invariants that arise for homogeneous, isotropic, decaying turbulent flows of incompressible fluids. Although somewhat disjoint from the central theme of the workshop, Gustafsson's results dovetail with issues raised by Titi and others regarding invariant measures. Moreover, Gustafsson's presentation raised awareness of the importance of understanding the spatial decay properties of various correlations and of mathematical challenges associated with extending results to flows on bounded domains.

Assad Oberoi described the residual-based variational multiscale (RBVM) formulation of LES. In this formulation, a projection operator is used to separate the solution of the NS equations into coarse and fine scales. While the coarse scale equations are solved numerically, the fine scale equations are solved analytically. Specifically, an algebraic approximation for the fine scale velocities is derived wherein they are expressed in terms of the residual of the NS operator applied to the coarse scale solution. Oberai's investigations were prompted by two complementary observations. First, while the Smagorinsky model does a good job of capturing Reynolds stress transfer terms, it does not capture the cross stress transfer terms. Second, while VMS captures cross-stress transfer terms, it underpredicts Reynolds stress transfer terms. To remedy this, Obera...i adds an additional, dynamic Smagorinksy, eddy-viscosity term, leading to a mixed model capable of accurately modeling all components of the subgrid stress.

Gantumur Tsogtgerel established the well-posedness of the NS-$\alpha\beta$ equations for bounded flows. Like the NS-α equations, the NS-$\alpha\beta$ equations involve fourth-order spatial derivatives of the filtered velocity and, thus, require additional boundary conditions. For flow past a slip-free, impermeable surface, the requirement that the filtered velocity vanish at the boundary is augmented by what is called the "wall-eddy condition." This condition involves both the filtered vorticity and its gradient, along with the length scale β and the wall-eddy length ℓ. Physically, this condition accounts for the generation of vorticity in the viscous sublayer. Since the NS-α model arises on setting β to α in the NS-$\alpha\beta$ model, Tsogtgerel's results apply directly to the NS-α equations for bounded flows. A related mathematical challenge concerns flows in domains involving free surfaces.

Tae-Yeon Kim presented results from numerical simulations for the NS-$\alpha\beta$ equations, based on comparisons with DNS for the NS system. Both homogeneous, isotropic, decaying isotropic turbulence and turbulent channel flow were considered. Results concerning energy spectra and the alignment of vorticity structures were presented.

Scientific Progress Made

1. The balance between dispersive and dissipative models was discussed at length.

2. Dispersive models: Understanding of dispersive models as represented in the talks has advanced rapidly over the last years. A consensus on their strengths and weaknesses seems to be developing in the community.

3. Dissipative models: These had previously been considered a blunt instrument. However, the workshop presented several steps forward in selectivity that seem promising for development. No consensus will develop until these new possibilities are explored.

4. In informal discussions, many supported the idea of mixed models that are syntheses of both types.

5. In numerical tests,the behavior of the models on the standard test problems, such as channel flow, seems to be now well understood. Most surviving models produce acceptable answers on all but the coarsest

grids. This could be due to the evolution of the models, to experience with the essential difficulties of benchmark problems or to both.

6. Predictability and computability of statistics is still an important open problem.

7. Other open problems noted include: wall-flow interactions, flow+ model sensitivities and how they cascade through the system, coupled NS systems and simulation of turbulence in complex flows, and the gap between problems for which precise analysis and computational testing can be done and the needs of simulation of industrial flows.

8. Several approaches of relating the averaging radius to the mesh width and the model micro scale were discussed.

9. Technical questions in the numerical analysis of LES were discussed, especially the Reynolds number dependence of the rate constant in Gronwall's inequality.

10. Many stressed that knowing when/how a model fails can be valuable information.

Outcome of the Meeting

There is a unanimous consensus among all participants that the workshop was successful on all grounds and, moreover, that it would be most useful to hold a follow-up meeting within 2–3 years.

Importantly, the workshop facilitated contacts between the regularized turbulence and LES communities, contacts which have already led to new collaborations. As a noteworthy example, Tae-Yeon Kim and Eliot Fried are currently working with Traian Illescu to develop a relatively inexpensive finite-element method with optimal convergence rates, based on the idea of the continuous-discontinuous Galerkin method, for the single-layer stationary quasigeostropic equations for the large scale wind-driven ocean circulation. As another example, Denis Hinz, Giulio Giusteri, and Eliot Fried are working with James Riley to analyze structure functions for homogeneous, isotropic decaying turbulence obtained using regularized turbulence models. Additionally, Tae-Yeon Kim, and Eliot Fried are working on methods for simulating turbulent flows involving suspended particles. Edriss Titi and Bernard Geurts are joining forces to extend regularization to rotation and stratification with applications in technology and in climate modeling. Particle-laden flow near walls is being pursued further by Hans Kuerten, Cees van der Geld and Bernard Geurts in a new collaboration. The issue of commutator errors on non-uniform meshes and the question how to deal with violation of conservation principles is considered by Roel Verstappen and Bernard Geurts.

Additionally, discussions during the workshop exposed a variety of fundamental questions and problems for future study, including:

1. Can we develop reliable strategies for predicting flow development and ensemble-averaged flow properties?

2. Is it possible to develop analytical techniques to improve upon error estimates that currently lead to Gronwall-type inequalities involving impractically large constants?

3. What connections can be established between solution-dependent filters and NS?

4. Can we quantify the influences of artificial dissipation on subgrid scale behavior?

5. How might we incorporate methods from uncertainty quantification to determine upper bounds on the errors that accompany any model?

6. Insofar as the inclusion of flow physics, what constitute suitable models and how can we test model suitability?

7. What are suitable numerical methods?

8. Can we develop a mathematical theory for reliability?

9. How might we begin to develop methods and algorithms to compute invariant measures in the form of probability distribution functions?

10. With increases understanding of regularization and LES models, how can we blend such models in sensible, desirable, and correct ways?

11. What considerations should be emphasized in developing cost functions to determine optimal choices of parameters such as eddy viscosities?

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Bibliography

[1] G. Alfonsi, Reynolds-averaged Navier - Stokes equations for turbulence modeling, *Applied Mechanics Reviews* **62** (2009), 040802.

[2] P. Sagaut. *Large Eddy Simulation for Incompressible Flows*: *An Introduction*, Springer, Berlin, 2001.

[3] B.J. Geurts: 2001. Elements of Direct and Large-Eddy Simulation. Edwards Publishing. ISBN: 1-930217-07-2, 2003.

[4] V. John, *Large Eddy Simulation of Turbulent Incompressible Flows*: *Analytical and Numerical Results for a Class of LES Models*, Springer, Berlin, 2004.

[5] L.C. Berselli, T. Iliescu and W.J. Layton, *Mathematics of Large Eddy Simulation of Turbulent Flows*, Springer, Berlin, 2006.

[6] F.F. Grinstein, L.G. Margolin and W.J. Rider, *Implicit Large Eddy Simulation*, Cambridge, New York, 2007.

[7] J. Leray, Sur les movements d'un fluide visqueux remplaissant l'espace, *Acta Mathematica* **63** (1934), 193–248.

[8] A. Clark, J.H. Ferziger and W.C Reynolds, Evaluation of subgrid-scale models using an accurately simulated turbulent flow, *Journal of Fluid Mechanics* **91** (1979), 1–16.

[9] J. Bardina, J.H. Ferziger and W.C. Reynolds. Improved subgrid scale models for large eddy simulation, Paper number 80, in: *13th Fluid and Plasma Dynamics Conference*, Snowmass, 14–16 July 1980. AIAA, New York, 1980.

[10] S. Chen, C. Foias, D.D. Holm, E. Olson, E.S. Titi and S. Wynne, The Camassa–Holm equations as a closure model for turbulent channel and pipe flow. *Physical Review Letters* **81** (1998), 5338–5341.

[11] S. Chen, C. Foias, D.D. Holm, E. Olson, E.S. Titi and S. Wynne, The Camassa–Holm equations and turbulence, *Physica D* **133** (1999), 49–65.

[12] S. Chen, C. Foias, D.D. Holm, E. Olson, E.S. Titi and S. Wynne, A connection between the Camassa–Holm equations and turbulent flows in channels and pipes, *Physics of Fluids* **11** (1999), 2343–2353.

[13] E. Fried and M.E. Gurtin, A continuum mechanical framework for turbulence giving a generalized Navier–Stokes-α equation with complete boundary conditions, *Theoretical and Computational Fluid Dynamics* **22** (2008), 433–470.

[14] W.J. Layton, C.C. Manica, M. Neda, L.G. Rebholz, Numerical analysis and computational comparisons of the NS-alpha and NS-omega regularizations, *Computer Methods in Applied Mechanics and Engineering* **199** (2010), 916–931.

[15] C. Foias, D.D. Holm and E.S. Titi. The three dimensional viscous Camassa-Holm equations, and their relation to the Navier-Stokes equations and turbulence theory, *Journal of Dynamics and Differential Equations*, **14** (2002), 1–35.

[16] J.E. Marsden and S. Shkoller, Global well-posedness for the Lagrangian averaged Navier–Stokes (LANS-α) equations on bounded domains, *Philosophical Transactions of the Royal Society A*: *Mathematical*, *Physical and Engineering Sciences* **359** (2001), 1449–1468.

[17] B.J. Geurts, D.D. Holm, Regularization modeling for large-eddy simulation. *Phys. of Fluids*, **15** (2003) L13.

[18] B.J. Geurts, A.K. Kuczaj and E.S. Titi, Regularization modeling for large-eddy simulation of homogeneous isotropic decaying turbulence, *Journal of Physics A*: *Mathematical and Theoretical* **41** (2008), 344008.

Chapter 21

Optimal Transportation and Differential Geometry (12w5118)

May 20 - 25, 2012

Organizer(s): Alessio Figalli (The University of Texas at Austin), Young-Heon Kim (University of British Columbia)

Overview of the Field

Optimal mass transportation can be traced back to Gaspard Monge's famous paper of 1781: 'Mémoire sur la théorie des déblais et des remblais'. The problem there is to minimize the cost of transporting a given distribution of mass from one location to another. Since then, it has become a classical subject in probability theory, economics and optimization.

At the end of the 80's, the seminal work of Brenier [7, 8] paved a way to connect optimal mass transportation to partial differential equations and related areas. On the one hand, his theory was followed by McCann's displacement convexity and Otto's differential geometry of the space of probability measures, making the theory of mass transportation applicable to wide range of problems in differential geometry, geometric and functional inequalities, and nonlinear diffusions. On the other hand, it stimulated Caffarelli, Urbas, and many others, to develop regularity theory of Monge-Ampère equations.

These two related directions have seen unexpected recent progresses: One of the highlights is the breakthrough of Lott and Villani [58] and Sturm [72, 73] who have characterized singular spaces with lower bounded Ricci curvature, solving one of the well-known open problems in Riemannian geometry. Also, Ma, Trudinger and Wang [65] extended the regularity theory for Monge-Ampère to a more general class of Monge-Ampère type equations, which surprisingly led the discovery of a new type of curvature, called Ma-Trudinger-Wang curvature, which is nowadays an object of many investigations in different directions.

Recent Developments and Open Problems

Ricci curvature on metric measure spaces, and geometry of the space of probability measures

Links from optimal transport to geometric analysis, including to the theory of Ricci curvature and Ricci flow, take their origin in the work of Otto and Villani [68] and Cordero-Erausquin, McCann, and Schmuckenschlager [17], and have received even more attention after the recent works of Lott and Villani [58], Sturm [72, 73], McCann and Topping [64], Lott [57] and Topping [74]. The possibility to define useful analogs of such concepts in a metric measure space setting has been a tantalizing goal, only partly realized so far. Still this progress, together with the original contribution due to Otto [67] on the formal Riemannian structure of the Wasserstein space and its applications to PDE's, is having a strong impact on the research community.

For other but closely related aspects of optimal transportation, there are studies of Sturm, Gigli, Ohta, Von Renesse, Ambrosio, Savarè, and others, on the heat flow on metric spaces spaces. Recently, Gigli, Kuwada and Ohta [39] have obtained a fundamental result showing the equivalence between gradient flow of energy functional and that of entropy functional on Alexandrov spaces as the underlying space. Apart from allowing to extend the Otto's work on gradients flow in Wasserstein spaces [67, 2] to Alexandrov spaces for constructing solutions to the heat equations (see also [38]), a simple corollary of this equivalence is that solutions to the heat equations become instantaneously Lipschitz continuous in space (before this result, only Hölder continuity was known).

After this work, this theory has been pushed even further by Ambrosio, Gigli and Savarè in a series of papers [3, 4] where they can study fine properties of Sobolev functions on metric measure spaces, they introduce a notion of Riemannian Ricci curvature for metric spaces, they prove equivalence between horizontal and vertical derivatives, etc.

Regularity of optimal transportation, fully nonlinear partial differential equations, and Riemannian geometry

The smoothness of optimal transport maps is an important issue in transportation theory since it gives information about qualitative behavior of the map, as well as simplifying computations and algorithms in numerical and theoretical implementations (see [20, 21] for discussions on this issue). Thanks to the results of Brenier [7, 8] and McCann [63], it is well known that the potential function of the map satisfies a Monge-Ampère type equation, an important fully nonlinear second order elliptic PDE arising in differential geometry. In the case of the quadratic cost function in Euclidean space, pioneering papers in this field are due to Delanoë [18], Caffarelli [9, 10, 11, 12], and Urbas [77]. More recently, Ma, Trudinger and Wang [65, 24] discovered a mysterious analytical condition, now called the Ma-Trudinger-Wang condition (or simply MTW condition) to prove regularity estimates for general cost functions [19, 21, 27, 25, 76, 26, 20, 60, 61, 47]. Costs functions which satisfy such a condition are called regular. At this point, Loeper [10, 56] gave a geometric description of this regularity condition, and he proved that the distance squared on the sphere is a uniformly regular cost, giving the first non-trivial example on curved manifolds. Moreover, he also showed that nonnegativity of MTW curvature is necessary for smoothness of optimal transportation maps (or simply optimal maps): more precisely, there are discontinuous optimal maps even between smooth distributions whenever the manifold has negative curvature at one point [10], and moreover the sole positivity of the sectional curvature is not enough for regularity [26, 31].

The Ma-Trudinger-Wang tensor is reinterpreted by Kim-McCann [45, 46] in an intrinsic way, and they show that it can be identified as the sectional curvature tensor on the product manifold equipped with a pseudo-Riemannian metric with signature (n, n). Recently, Kim, McCann and Warren [48] have found a pseudo-metric with respect to which the graphs of optimal maps give volume maximizing space-like Lagrangian submanifolds, thus giving some hope for relating optimal transportation theory to submanifold theory and symplectic geometry.

In addition, recent results of Loeper and Villani [58, 80] and Figalli, Rifford and Villani [30, 31, 32, 33, 34] show that the regularity condition on the square distance of a Riemannian manifold implies geometric results, like the convexity of the cut-loci. These developments show fruitful interactions of analysis and geometry around optimal transportation.

A wide open problem of regularity theory is to understand the nature of discontinuity/singularity set of optimal maps when the MTW condition is not satisfied, e.g., the distance squared cost on negatively curved Riemannian manifolds. As Villani asked in his book [21], does such set have nice geometry or does it show fractal nature? Partial results in this direction have been recently proved in [23, 24], but a complete answer to this problem is still missing.

Geometric and functional inequalities

A further line of research, which takes its origin in McCann's proof of Brunn-Minkowski inequality via optimal transport [62], is to apply optimal transport to prove geometric and functional inequalities (such as isoperimetric or Sobolev inequalities). For instance, Figalli, Maggi and Pratelli [29] were able to exploit the optimal transport proof of the Wulff inequality (an anisotropic version of the isoperimetric inequality) to prove a sharp stability estimate, solving a long-standing open problem in crystals shape formation (see also [28]).

More recently, Castillon [14] and Chang and Wang [15] have used optimal transport maps to prove some versions of Micheal-Simon and Alexandrov-Fenchel inequalities.

One of the main open problems in this area is to understand whether one can find optimal transport proofs of relevant inequalities also on manifolds (for instance, for proving the isoperimetric inequality on the sphere or the hyperbolic space), which may then lead to new results with wider applications.

Presentation Highlights

Theoretical aspects of optimal transportation

The classical optimal transport problem and distances between measures

A basic question in optimal transport is whether an optimal transport map exists. This depends of course on the cost function, and the Monge problem with "cost=distance" is one of the most difficult and challenging cases.

Stefano Bianchini has described a recent work where he shows that the Monge-Kantorovich problem with any convex cost function has always a solution which is induced by a transport map. This includes in particular the case "cost=distance", as well as much more degenerate cases. The proof is based on a decomposition of Sudakov's type.

Another important topic in optimal transport is how to use transportation distance to construct gradient flows in the space of measures.

In this direction, Matthias Erbar has presented a joint work with Karl Theodor Sturm [ES], where they introduce a new transportation distance between probability measures that is built from a Lévy jump kernel. It is defined via a non-local variant of the Benamou-Brenier formula. They study geometric and topological properties of this distance, in particular they prove existence of geodesics. For translation invariant jump kernels they can identify the semigroup generated by the associated non-local operator as the gradient flow of the relative entropy w.r.t. the new distance and show that the entropy is convex along geodesics.

Nassif Ghoussoub has presented his recent work on the Monge-Kantorovich problem with symmetry. This kind of problem has strong relations with self-dual Lagrangians, convex analysis, and monotone operators. In particular he can obtain important variants of Brenier's polar decomposition theorem [8].

Jonathan Korman has presented a joint work with Robert McCann [52] where they consider a variant of the classical Monge-Kantorovich problem by imposing a constraint on the joint measures: find an optimal one among all joint measures with fixed marginals, which are dominated by a given measure. They show uniqueness of the solution, and explicitly compute some examples.

Brendan Pass has formulated and studied the problem of aligning a continuum of marginals as efficiently as possible. In his formulation, he looks for the stochastic process with prescribed single time marginals which minimizes the expectation of a certain functional. This problem is a natural extension of a multi-marginal optimal transportation problem previously studied by Gangbo and Swiech [37], and in his talk he has shown how to obtain existence, uniqueness, and characterization results for the quadratic cost function.

Regularity of optimal transportation and nonlinear PDEs.

A number of speakers discussed topics around Ma, Trudinger and Wang conditions on regularity of optimal transportations and fully nonlinear elliptic Hessian equations. In addition, important extensions of this theory have been presented.

Jun Kitawaga presented a joint work [49] with Micah Warren, where they consider regularity for Monge solutions to the optimal transport problem when the initial and target measures are supported on the embedded sphere, and the cost function is the Euclidean distance squared. Gangbo and McCann [36] have shown that when the initial and target measures are supported on boundaries of strictly convex domains in R^n, there is a unique Kantorovich solution, but it can fail to be a Monge solution. In the case when one deals with the sphere with measures absolutely continuous with respect to surface measure, they could obtain two different types of conditions on the densities of the measures to ensure that the solution given by Gangbo and McCann is indeed a Monge solution, and obtain higher regularity as well.

Alexander Kolesnikov has presented his recent study on the optimal transportation mapping $\nabla\Phi : R^d \mapsto R^d$ pushing forward a probability measure $\mu = e^{-V}\ dx$ onto another probability measure $\nu = e^{-W}\ dx$, which follows a line of research previously investigated in [13, 16]. Following a classical approach of E. Calabi he introduces the Riemannian metric $g = D^2\Phi$ on R^d and studies spectral properties of the metric-measure space $M = (R^d, g, \mu)$. He proves, in particular, that M admits a non-negative Bakry-Emery tensor provided both V and W are convex. If the target measure ν is the Lebesgue measure on a convex set Ω and μ is log-concave, he can show that M is a $CD(K, N)$ space. Applications of these results include some global dimension-free a priori estimates of $\|D^2\Phi\|$. With the help of comparison techniques on Riemannian manifolds he estimates the diameter of M in terms of the dimension and the diameter of Ω.

Jiakun Liu presented his recent work on the study of the general case of the light reflection problem, showing how it is related to a nonlinear optimization problem. This problem involves a fully nonlinear PDE of Monge-Ampere type, subject to a nonlinear boundary condition, and generalizes previous works of Xu-Jia Wang [81, 82] in the special far field case, which are related to the reflector antenna design problem.

Neil Trudinger has shown how to develop the fundamentals of a local regularity theory for prescribed Jacobian equations which extend the corresponding results for optimal transportation equations. In this theory the cost function is extended to a generating function through dependence on an additional scalar variable. In particular he can recover in this generality the local regularity theory for potentials of Ma, Trudinger and Wang, along with the subsequent development of the underlying convexity theory.

Geometry of Wasserstein spaces

Luigi Ambrosio has presented a joint work with Gigli and Savaré [4] where they compare several notion of weak (modulus of) gradient in metric measure spaces and prove their equivalence. This equivalence is part of the "calculus program" they developed, largely based on tools from optimal transportation theory. In particular, they prove density in energy of Lipschitz maps in Sobolev spaces independently of doubling and Poincaré inequality assumptions on the metric measure space.

Nicola Gigli has shown that every metric measure space (complete, separable endowed with a Borel locally finite measure) has a natural first order differentiable structure: it is possible to speak about differential and gradients of Sobolev functions. As an application, he has introduced a general definition of distributional Laplacian and shown that on spaces with Ricci curvature bounded from below, for the Laplacian of the distance function, the standard comparison estimates hold.

Benoit Kloeckner has illustrated that classical ideas in metric geometry can be used to generalize Hausdorff dimension in a way that distinguishes many Wasserstein spaces [50, 51]. One nice consequence is that the Wasserstein space of a compact manifold can never be Lipschitz embedded in the Wasserstein space of a compact manifold of lower dimension.

Kazumasa Kuwada has presented a joint with Karl Theodor Sturm [53] where new monotonicity in time of a time-dependent transportation cost between distribution of diffusion processes is shown under Bakry-Emery's curvature-dimension condition on a Riemannian manifold. This result is an analog of the L^p-Wasserstein contraction of heat distributions under lower Ricci curvature bound. The cost function comes from the total variation distance between heat distributions on the space forms. As a corollary, they obtain a comparison theorem for the total variation distance between heat distributions. They also obtain an explicit expression of the cost function. It leads to a time-independent transportation cost which is non-increasing in time for heat distributions even on a negatively curved space. In addition, they show that their monotonicity is stable under the Gromov-Hausdorff convergence of the underlying space under a uniform curvature-dimension and diameter bound.

Paul Lee has presented some recent work [54] on interpolation of measures from a Hamiltonian point of view: he has discussed displacement interpolations from the point of view of Hamiltonian systems and give a unifying approach to various known results.

Shin-Ichi Ohta has discussed the notion of Ricci curvature in Finsler geometry and, as a recent application, shown generalizations of the Cheeger-Gromoll type splitting theorem [66].

Tommaso Pacini has presented a joint work with Wilfrid Gangbo and Hwa-Kil Kim [35], discussing differential forms and symplectic geometry on Wasserstein spaces. More precisely, he has talked about differential calculus on Wasserstein spaces, de Rham cohomology, symplectic structures, and Hamiltonian systems.

Tapio Rajala has talked about some recent work on interpolation measures with bounded density in $CD(K,N)$-spaces [69, 70, 71]. He has explained how using only the convexity-inequality for the critical entropy-functional in a $CD(K,N)$-space one can construct geodesics in the Wasserstein space along which all the measures have bounded density. He also discussed some applications of these "good" geodesics.

Giuseppe Savarè has presented a joint work with Ambrosio and Gigli [3] on the links between the displacement convexity of entropy functionals and the characterizations of their gradient flows in Wasserstein spaces in terms of a family of evolution variational inequalities. In the particular case of the logarithmic entropy the above properties are strictly related to the linearity and the contractivity of the flow. Various properties and applications to metric measure spaces with Ricci curvature bounded from below have also been discussed.

Riemannian geometry

Jerome Bertrand has illustrated a new proof of Alexandrov's theorem on the Gauss curvature prescription of Euclidean convex body, based on mass transport and the classical theory of convex bodies duality [5]. In particular, this proof does not rely on PDEs method nor convex polyhedra theory. With this approach, it is also possible to treat the case of equivariant convex bodies in the (Lorentzian) Minkowski space.

Simon Brendle has discussed minimal tori in S^3, and has shown how to prove Lawson conjecture on the fact that any embedded minimal torus in S^3 is congruent to the Clifford torus [6].

Yi Wang has reported some recent joint work with Alice Chang in which they generalize Michael-Simon inequality and partially generalize the Aleksandrov-Fenchel inequalities for quermassintegrals from convex domains in the Euclidean space to a class of non-convex domains [15]. In the talk she has also discussed about optimal constants of the inequality in some special cases.

Guofang Wei has presented a joint with Wylie [83] where they extended several comparison results (in particular, the Bishop-Gromov volume comparison) for manifolds with lower Ricci curvature bound to smooth metric measure spaces with Bakry-Emery Ricci tensor bounded from below. She also discussed several applications of this. In particular, Peng Wu uses it to show that the infimum of the potential function of a gradient steady Ricci soliton grows linearly. Jointly with P. Wu, for a large class of gradient steady Ricci solitons, she has also obtained optimal growth estimate for the potential function, and show the volume grows at most like polynomial of degree n.

William Wylie has presented joint works with Chenxu He and Peter Petersen [40, 41, 42, 43] where they show that any algebraic Ricci soliton on a left invariant Lie group can be extended to a homogeneous warped product Einstein space. This extends a result of Lauret for solvable groups. It also provides the existence of many new homogeneous smooth metric measure spaces which are m-Quasi Einstein manifolds. They also give a strong characterization of the geometric structure of such spaces.

Results in other related areas

Beatrice Acciaio has shown how optimal transport is related to robust pricing and trajectories inequalities [1]. Robust pricing basically corresponds to considering extremal pricing rules coming from possible pricing measures which satisfy marginal constraints. This problem is naturally connected to optimal transportation. Mathematically the crucial difference is that in her setting transport plans are required to be martingales. She has discussed the advantages of relating the robust pricing problem to the theory of mass transportation. In particular, she has shown that the duality theorem from optimal transport can be used to establish new robust super-replication results. This dual point of view also provides new insights on classical martingale inequalities, such as a (new) sharp version of the classical Doob maximal inequality.

Outcome of the Meeting

The present meeting between experts in optimal transportation and Riemannian geometry has come at a time that saw an explosion of interests in the link between the two fields.

The workshop brought together mathematicians working on optimal transport, Riemannian geometry, metric geometry, geometric flows, and geometric inequalities. The audience included specialists in all these fields.

By bringing together researchers from a range of different fields with common interests, this meeting has showcased some of the recent progresses and set the stage for future developments, while stimulating new collaborations, new questions, and new lines of research. By making these connections, we believe that the meeting has accelerated the rate of progress within these two important areas of mathematics , having a lasting impact through influencing new directions for future research.

In addition to all the directions of research already discussed previously, this workshop has allowed us to find two new very promising and interesting directions to investigate.

The first one has been suggested by Nassif Ghoussoub: in his talk he has shown strong links between convex analysis, self-dual lagrangians, cyclical monotonicity, monotone operators, and idempotent measure preserving maps. It looks natural now to try to understand what are the sharp conditions on the cost function in order to obtain optimal maps, and what are their regularity properties. In addition, as pointed out by Ambrosio, when specialized to the Coulomb interaction cost this new theory of Ghoussoub seems to have very important potential applications in theoretical physics.

The second one has been suggested by Neil Trudinger: in his talk he has shown how, by considering generating functions on R^{2n+1}, one can include in a unifying setting not only Monge-Ampère type equations arising from optimal transport, but also for instance equations coming from reflector antenna problems. A natural question is how to extend the results known up to now for optimal transport maps, to this more general setting, and it seems to give an interesting and challenging line of research.

Participants

Acciaio, Beatrice (London School of Economics)
Ambrosio, Luigi (Scuola Normale Superiore Pisa)
Bertrand, Jerome (IMT)
Bianchini, Stefano (SISSA)
Brendle, Simon (Stanford University)
Castillon, Philippe (Universite de Montpellier)
Erbar, Matthias (MSRI, Berkeley and Postdoc at MSRI, Berkeley and from 01-2014 Scuola Normale Superiore, Pisa, Italy))
Figalli, Alessio (The University of Texas at Austin)
Ge, Yuxin (Universite Paris Est Creteil)
Gharib, Saman (University of British Columbia)
Ghoussoub, Nassif (University of British Columbia)
Gigli, Nicola (University of Nice)
Guan, Pengfei (McGill University)
Kim, Young-Heon (University of British Columbia)
Kitagawa, Jun (University of Toronto)
Kloeckner, Benoit (Universite Grenoble 1)
Kolesnikov, Alexander V (Higher School of Economics)
Korman, Jonathan (University of Toronto)
Kuwada, Kazumasa (Ochanomizu University)
Lee, Paul (University of California at Berkeley)
Li, Junfang (University of Alabama at Birmingham)
Liu, Jiakun (Princeton University)
Maximo, Davi (University of Texas)
Merigot, Quentin (Universite de Grenoble / CNRS)
Munn, Michael (University of Missouri)
Nguyen, Van-Hoang (Universite Pierre et Marie Curie - Paris 6)
Ohta, Shin-ichi (Kyoto Univesity)
Pacini, Tommaso (SNS)

Pass, Brendan (University of Alberta)
Rajala, Tapio (Scuola Normale Superiore Pisa)
Rifford, Ludovic (University of Nice)
Savare, Giuseppe (University of Pavia (Italy))
Trudinger, Neil (Australian National University)
Wang, Yi (Princeton University)
Wei, Guofang (University of California, Santa Barbara)
Wylie, William (Syracuse University)

Bibliography

[1] B. Acciaio, M. Beiglbock, F. Penkner, W. Schachermayer and J. Temme. A trajectorial interpretation of Doob's martingale inequalities. *Annals of Applied Probability*, to appear.

[2] L. Ambrosio, N. Gigli and G. Savaré, *Gradient flows in metric spaces and in the Wasserstein space of probability measures*. Birkhäuser, 2004.

[3] L. Ambrosio, N. Gigli and G. Savaré, Calculus and heat flow on metric measure spaces and applications to spaces with Ricci curvature bounds from below. Preprint, 2011.

[4] L. Ambrosio, N. Gigli and G. Savaré, Metric measure spaces with Riemannian Ricci curvature bounded from below. Preprint, 2011.

[5] J. Bertrand, Prescription of Gauss curvature using optimal mass transport. Preprint, 2011.

[6] S. Brendle, Embedded minimal tori in S^3 and the Lawson conjecture. Preprint, 2012.

[7] Y. Brenier, Décomposition polaire et réarrangement monotone des champs de vecteurs. *C. R. Acad. Sci. Paris Sér. I Math.* 305 (1987), no. 19, 805-808.

[8] Y. Brenier, Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44 (1991), 375-417.

[9] L. A. Caffarelli, A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. *Ann. of Math.*, 131 (1990), 129-134.

[10] L. A. Caffarelli, Some regularity properties of solutions of Monge Ampère equation. *Comm. Pure Appl. Math.*, 44 (1991), no. 8-9, 965-969.

[11] L. A. Caffarelli, The regularity of mapping with a convex potential. *J. Amer. Math. Soc.*, 5 (1992), 99-104.

[12] L. A. Caffarelli, Boundary regularity of maps with convex potentials II. *Ann. of Math.*, 144 (1996), 453-496.

[13] L. A. Caffarelli, Monotonicity properties of optimal transportation and the FKG and related inequalities. *Comm. Math. Phys.*, 214 (2000), no. 3, 547-563.

[14] P. Castillon,Submanifolds, isoperimetric inequalities and optimal transportation. *J. Funct. Anal.* 259 (2010), no. 1, 79-103.

[15] S.-Y. A. Chang and Y. Wang, On Aleksandrov-Fenchel inequalities for k-convex domains. *Milan J. Math.*, 79 (2011), no. 1, 13-38

[16] D. Cordero-Erausquin, Some applications of mass transport to Gaussian-type inequalities. *Arch. Rat. Mech. Anal.* 101 (2002), 257-269.

[17] D. Cordero-Erausquin, R. J. McCann, and M. Schmuckenschlager, A Riemannian interpolation inequality à la Borell, Brascamp and Lieb. *Invent. Math.* 146 (2001), 219-257.

[18] P. Delanoë, Classical solvability in dimension two of the second boundary value problem associated with the Monge-Ampère operator. *Ann. Inst. Henri Poincaré-Anal. Non Lin.*, 8 (1991), 443-457.

[19] P. Delanoë, Gradient rearrangement for diffeomorphisms of a compact manifold. *Diff. Geom. Appl.*, 20 (2004), 145-165.

[20] P. Delanoë and Y. Ge, Regularity of optimal transportation maps on compact, locally nearly spherical, manifolds. *J. Reine Angew. Math.* 646 (2010), 65-115.

[21] P. Delanoë and G. Loeper, Gradient estimates for potentials of invertible gradient mappings on the sphere. *Calc. Var. Partial Differential Equations*, 26 (2006), no. 3, 297-311.

[22] M. Erbas and K.-T. Sturm, Gradient flows of the entropy for jump processes. Preprint, 2012.

[23] A. Figalli, Regularity properties of optimal maps between nonconvex domains in the plane, *Comm. Partial Differential Equations*, 35 (2010), no. 3, 465-479.

[24] A. Figalli and Y.-H. Kim, Partial regularity of Brenier solutions of the Monge-Ampère equation, *Discrete Contin. Dyn. Syst. Series A*, 28 (2010), no. 2, 559-565.

[25] A. Figalli, Y.-H. Kim and R. J. McCann, Hölder continuity and injectivity of optimal maps. Preprint, 2011.

[26] A. Figalli, Y.-H. Kim and R.J. McCann, Regularity of optimal transport maps on multiple products of spheres. Preprint, 2010. To appear in *J. Eur. Math. Soc. (JEMS)*

[27] A. Figalli and G. Loeper, C^1-regularity of solutions of the Monge-Ampère equation for optimal transport in dimension two. *Calc. Var. Partial Differential Equations*, 35 (2009), no. 4, 537-550.

[28] A. Figalli, F. Maggi and A. Pratelli, A refined Brunn-Minkowski inequality for convex sets. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26 (2009), no. 6, 2511-2519.

[29] A. Figalli, F. Maggi and A. Pratelli, A mass transportation approach to quantitative isoperimetric inequalities. To appear in *Invent. Math.*

[30] A. Figalli and L. Rifford, Continuity of optimal transport maps on small deformations of S^2. *Comm. Pure Appl. Math.*, 62 (2009), no. 12, 1670-1706.

[31] A. Figalli, L. Rifford and C. Villani, On the Ma-Trudinger-Wang curvature on surfaces. *Calc. Var. Partial Differential Equations*, 39 (2010), no. 3-4, 307-332

[32] A. Figalli, L. Rifford and C. Villani, Nearly round spheres look convex. *Amer. J. Math.*, 134 (2012), no. 1, 109-139.

[33] A. Figalli, L. Rifford and C. Villani, Tangent cut loci on surfaces. *Differential Geom. Appl.*, 29 (2011), no. 2, 154-159.

[34] A. Figalli, L. Rifford and C. Villani, Necessary and sufficient conditions for continuity of optimal transport maps on Riemannian manifolds. *Tohoku Math. J.*, (2), 63 (2011), no. 4, 855-876.

[35] W. Gangbo, H.K. Kim and T. Pacini, Differential forms on Wasserstein space and infinite-dimensional Hamiltonian systems. *Mem. Amer. Math. Soc.* 211 (2011), no. 993, vi+77 pp.

[36] W. Gangbo and R.J. McCann, Shape recognition via Wasserstein distance. *Quart. Appl. Math.*, 58 (2000), no. 4, 705-737.

[37] W. Gangbo and A. Swiech, Optimal maps for the multidimensional Monge-Kantorovich problem. *Comm. Pure Appl. Math.*, 51 (1998), no. 1, 23-45.

[38] N. Gigli, On the Heat flow on metric measure spaces: existence, uniqueness and stability. To appear in *Calc. Var. Part. Diff. Eq.*.

[39] N. Gigli, K. Kuwada and S.I. Ohta, Heat Flow on Alexandrov spaces. *Comm Pure Appl. Math.*, to appear.

[40] C. He, P. Petersen and W. Wylie, On the classification of warped product Einstein metrics. Preprint, 2010.

[41] C. He, P. Petersen and W. Wylie, Warped product Einstein metrics over spaces with constant scalar curvature. Preprint, 2010.

[42] C. He, P. Petersen and W. Wylie, Warped Product Rigidity. Preprint, 2011.

[43] C. He, P. Petersen and W. Wylie, The space of virtual solutions to the warped product Einstein equation. Preprint, 2011.

[44] Y.-H. Kim, *Counterexamples to continuity of optimal transportation on positively curved Riemannian manifolds.* Int. Math. Res. Not. IMRN 2008, Art. ID rnn120, 15 pp.

[45] Y.-H. Kim and R. J. McCann, Continuity, curvature, and the general covariance of optimal transportation. *J. Eur. Math. Soc. (JEMS)* 12 (2010) 1009-1040

[46] Y.-H. Kim and R.J. McCann. Appendices to original version of *Continuity, curvature, and the general covariance of optimal transportation*. Preprint at arXiv:math/0712.3077v1.

[47] Y.-H. Kim and R. J. McCann, Towards the smoothness of optimal maps on Riemannian submersions and Riemannian products (of round spheres in particular). To appear in *J. Reine Angew. Math.*

[48] Y.-H. Kim, R.J. McCann and M. Warren, Pseudo-Riemannian geometry calibrates optimal transportation. *Math. Res. Lett.* 17 (2010) 1183-1197.

[49] J. Kitawaga and M. Warren, Regularity for the optimal transportation problem with Euclidean distance squared cost on the embedded sphere. Preprint, 2011.

[50] B. Kloeckner, A geometric study of Wasserstein spaces: Euclidean spaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 9 (2010), no. 2, 297-323.

[51] B. Kloeckner, A generalization of Hausdorff dimension applied to Hilbert cubes and Wasserstein spaces. Preprint, 2011.

[52] J. Korman and R.J. McCann, Optimal transportation with capacity constraints. Preprint, 2012.

[53] K. Kuwada and K.-T. Sturm, Monotonicity of time-dependent transportation costs and coupling by reflection. Preprint, 2012.

[54] P. Lee, Displacement interpolations from a Hamiltonian point of view. Preprint, 2012.

[55] G. Loeper, On the regularity of solutions of optimal transportation problems. *Acta Math.* 202 (2009), no. 2, 241-283.

[56] G. Loeper, Regularity of optimal maps on the sphere: The quadratic cost and the reflector antenna. *Arch. Ration. Mech. Anal.* 199, no 1 (2011), 269-289,

[57] J. Lott, Optimal transport and Perelman's reduced volume. *Calc. Var. Partial Differential Equations* 3 (2009) 49-84.

[58] G. Loeper and C. Villani, Regularity of optimal transport in curved geometry: the nonfocal case. *Duke Math. J.* 151, no 3 (2010), 431-485.

[59] J. Lott and C. Villani, Ricci curvature for metric-measure spaces, *Ann. of Math.* 169, no 3 (2009), 903-991.

[60] J. Liu. Hölder regularity of optimal mappings in optimal transportation. *Calc Var. Partial Differential Equations*, 34 (2009), 435-451.

[61] J. Liu, N. Trudinger and X.-J. Wang, Interior $C^{2,\alpha}$-regularity for potential functions in optimal transportation, To appear in *Comm. Partial Differential Equations.*

[62] R. J. McCann, *A Convexity theory for interacting gases and equilibrium crystals.* Ph. D. thesis, Princeton University (1994).

[63] R. J. McCann, Polar factorization of maps on Riemannian manifolds. *Geom. Funct. Anal.*, 11 (2001), 589-608.

[64] R.J. McCann and P. Topping, Ricci flow, entropy, and optimal transportation. *Amer. J. Math.* 132 (2010) 711-730.

[65] X.-N. Ma, N. Trudinger and X.-J. Wang, Regularity of potential functions of the optimal transport problem. *Arch. Ration. Mech. Anal.*, 177(2): 151-183, 2005.

[66] S.-I. Ohta, Splitting theorems for Finsler manifolds of nonnegative Ricci curvature. Preprint, 2012.

[67] F. Otto, The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26 (2001), no. 1-2, 101-174.

[68] F. Otto and C. Villani, Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.* 173 (2000), 361-400.

[69] T. Rajala, Local Poincaré inequalities from stable curvature conditions on metric spaces *Calc. Var. Partial Differential Equations*, 44 (2012), no. 3-4, 477-494.

[70] T. Rajala, Interpolated measures with bounded density in metric spaces satisfying the curvature-dimension conditions of Sturm. *J. Funct. Anal.*, 263 (2012), no. 4, 896-924.

[71] T. Rajala, Improved geodesics for the reduced curvature-dimension condition in branching metric spaces *Discrete Contin. Dyn. Syst.*, to appear.

[72] K.-T. Sturm, On the geometry of metric measure spaces I, *Acta Math*. 196, no.1, 65-131 (2006)

[73] K.-T. Sturm, On the geometry of metric measure spaces II, *Acta Math*. 196, no. 1, 133-177 (2006).

[74] P. Topping, L-optimal transportation for Ricci flow. *J. Reine Angew. Math.* 636 (2009) 93-122.

[75] N. Trudinger and X.-J. Wang, On the second boundary value problem for Monge-Ampère type equations and optimal transportation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 8 (2009), no. 1, 143-174.

[76] N. Trudinger and X.-J. Wang, On strict convexity and continuous differentiability of potential functions in optimal transportation. *Arch. Ration. Mech. Anal.* 192 (2009), no. 3, 403-418.

[77] J. Urbas, On the second boundary value problem for equations of Monge-Ampère type. *J. Reine Angew. Math.*, 487 (1997), 115-124.

[78] C. Villani, *Topics in Optimal Transportation*, Graduate Studies in Mathematics, vol. 58. American Mathematical Society, Providence, RI, 2003.

[79] C. Villani, *Optimal Transport, Old and New*, Grundlehren des mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 338, Springer-Verlag, Berlin-New York, 2009.

[80] C. Villani, Stability of a 4th-order curvature condition arising in optimal transport theory. *J. Funct. Anal.*, 255 (2008), no. 9, 2683-2708.

[81] X.-J. Wang, On the design of a reflector antenna, *Inverse Problems* 12 (1996), 351-375.

[82] X.-J. Wang, On the design of a reflector antenna II. *Calc. Var. PDE* 20 (2004), 329-341.

[83] G. We and W. Wylie, Comparison geometry for the Bakry-Emery Ricci tensor. *J. Differential Geom.* 83 (2009), no. 2, 377-405.

Chapter 22

Arithmetic geometry of orthogonal and unitary Shimura varieties (12w5011)

June 3 - 8, 2012

Organizer(s): Fabrizio Andreatta (University of Milano), Jan Bruinier (Technical University, Darmstadt), Eyal Goren (McGill University)

In June 2012, a workshop dedicated to the arithmetic geometry and applications of orthogonal and unitary Shimura varieties was held at the Banff International Research Station, located in Banff in the Canadian Rockies. The workshop, attended by about 40 participants, from three continents and many countries, represented a surge of interest in this topic. At the time we had applied for the workshop, that is a few years earlier, one could say that the study of these particular Shimura varieties was undergoing revival and so we had thought it wise to take toll of recent and undergoing research concerning these varieties, with special emphasis on subjects that had received attention recently. For example, special cycles, generating series and special values of L functions (the Kudla Programme); integral models and arithmetic intersection theory; compactifications and Arakelov theory with "singularities at infinity".

In the time that had passed since, the impression that these particular areas are blooming has proven itself. Many important developments have been made exactly along the lines mentioned above and many of those by the participants of the workshop. Certain collaborations had their origins in this workshop, for instance the work by Andreatta, Goren, Howard and Madapusi-Pera. As many of the people attending the conference were already involved in collaborative research with other participants, the workshop certainly encouraged the continuation of these collaborations, for example: Goren-de Shalit, Bruinier-Howard-Yang, Kudla-Rapoport, Terstiege-Rapoport, Burgos-Gil-Kramer and more.

The following is a selection of titles and abstract for some of the talks. This selection gives an excellent outline of the contents of the workshop and its focus.

- Steve Kudla: "Special cycles for unitary groups: the unramified case".

  In the first part of this lecture I will review the definition of arithmetic special cycles on Shimura varieties for unitary groups of signature $(n-1,1)$ and explain how the arithmetic 0-cycles arise in the computation of their height pairings. In the second part of talk, I will review the structure of the Rapoport-Zink space used in the p-adic uniformization of the supersingular locus of such a varieties in the case of an inert prime. Finally, I will give the definition of the analogous special cycles in this RZ space and explain their properties.

- Ben Howard: A Gross-Zagier theorem for higher weight modular forms.

  I'll talk about an extension of the Gross-Zagier theorem to higher weight modular forms, expressing the height pairings of special cycles on unitary Shimura varieties to the central derivatives of Rankin-Selberg L-functions. This is joint work with Jan Bruinier and Tonghai Yang.

- Keerthi Madapusi: Regular integral models for orthogonal Shimura varieties and the Tate conjecture for K3 surfaces in finite characteristic.

  We construct regular integral canonical models for Shimura varieties of orthogonal type with maximal parahoric level, and we show that certain moduli spaces of polarized K3 surfaces can be viewed as open sub-schemes of such integral models. Using a result of Kisin, this then implies the Tate conjecture for K3 surfaces in odd characteristic p, as long as they admit a polarization of degree indivisible by p^2. The same methods also work to prove the Tate conjecture for cubic fourfolds in odd characteristic.

- Michael Rapoport: On the geometry of unitary Shimura varieties in the ramified case.

  I will explain structure theorems for the formal moduli space of p-divisible groups of Picard type of signature $(1, n-1)$ for a ramified quadratic extension of $\mathbb{Q}_p$. The underlying reduced scheme possesses a stratification by Deligne-Lusztig varieties for symplectic groups over $\mathbb{F}_p$; the strata are parametrized by simplices in the Bruhat-Tits building of a p-adic unitary group. This is joint work with U. Terstiege and S. Wilson; the results are analogous to the results of I. Vollaard and T. Wedhorn in the case of an unramified quadratic extension of $\mathbb{Q}_p$.

- Brian Smithling: Moduli descriptions of some local models for Shimura varieties.

  Local models are certain schemes introduced to model the étale-local structure of p-adic integral models of Shimura varieties. A general definition of them in the setting of PEL Shimura varieties was given by Rapoport and Zink; recently Pappas and Zhu have formulated a general group-theoretic definition of them for tamely ramified groups and established many good properties for them. Unfortunately the local models do not in general admit ready moduli-theoretic interpretations. To facilitate applications, it is therefore of interest to describe to describe them in moduli-theoretic terms when possible. In the case of orthogonal and ramified unitary groups, such a description has been proposed by Pappas and Rapoport. I will report on some progress towards proving their conjecture.

- Jurg Kramer: Arithmetic intersections on modular curves.

  In our talk we will report on asymptotic formulas for the arithmetic self-intersection of the relative dualizing sheaf equipped with the Arakelov metric on modular curves attached to congruence subgroups as the level tends to infinity. In case of the modular curve $X_0(N)$ (N squarefree and not divisible by $2, 3$) such results are due to A. Abbes and E. Ullmo. We will present analogous results for the modular curve $X_1(N)$ (for suitable squarefree N), which then enable us to compute the Faltings height of the associated Jacobian $J_1(N)$ asymptotically (as N tends to infinity).

- Ulf Kuhn: Modularity of generating series for arithmetic Hecke correspondences.

  The aim of the talk is to explain our approach to Kudla's conjectures for the case of the product of two modular curves. The major difficulties in this situation are of analytical nature. We present a mild modification of this Green function that satisfies the requirements of being a Green function in the sense of Arakelov theory on the natural compactification in addition. Only this allows us to define arithmetic special cycles and to show that the generating series of those modified arithmetic special cycles is as predicted by Kudla's conjectures a modular form with values in the first arithmetic Chow group. Moreover its intersection with the arithmetic canonical class yields essentialy the derivative of an Eisenstein series. This is joint work with Rolf Berndt: http://xxx.uni-augsburg.de/abs/1205.6417 http://www.math.uni-hamburg.de/home/kuehn/berndt-kuehn-part-II.pdf

- Ellen Eischen: p-adic families of Eisenstein series for unitary groups

  Special values of certain L-functions can be expressed in terms of values of Eisenstein series at points on the Shimura variety for U(n,n). One approach to p-adically interpolating values of these L-functions relies on construction of a p-adic family of Eisenstein series. In this talk, I will explain how to construct such a family of Eisenstein series, and I will explain how to p-adically interpolate certain values of both holomorphic and non-holomorphic Eisenstein series on U(n,n).

- Matthew Greenberg: Triple product p-adic L-functions for balanced weights.

  The conjecture of Gross and Prasad and its refinements constitute a framework for generalizing the famous formula of Gross and Zagier, relating the height of a Heegner point on an elliptic curve to the central derivative of an associated L-function, to the context of orthogonal and unitary Shimura varieties. In this talk, I would like to discuss work-in-progress with Marco Seveso dealing with p-adic analogues of these conjecture in a low-dimensional (though very rich!) test scenario. Specifically, I'll describe a construction of a triple product p-adic L-function in the case of "balanced weights." The key tools are existing classical special value formulae due to Gross-Kudla, Boecherer-Schulze-Pillot, and Ichino, and the Ash-Stevens theory of p-adic deformation of arithmetic cohomology.

Examination of these abstracts shows recurring themes that intersect and branch over and over again: integral models, cycles, L-functions, Hecke correspondences, automorphic forms. As such, one could say that this pattern, together with the all important sibling Galois representations, is a fair image of much of the arithmetic geometry of Shimura varieties. However, a closer examination, shows that the particular results discussed in these talks, and in other talks of the workshops, form a net of closely related and inter-dependent results. This explains well the relevance of this workshop to the area of orthogonal and unitary Shimura varieties in general, but more than that, it explains the service for the particular group that attended - a group that is responsible for many of the exciting developments in these area (unfortunately, the limit on the number of participants, as well as unavailability of some of the people we have invited, resulted in a partial representation of the key players).

Participants

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Chapter 23

Descriptive Set Theory and Operator Algebras (12w5099)

June 17 - 22, 2012

Organizer(s): Edward Effros (UCLA), George Elliott (Toronto), Ilijas Farah (York), Andrew Toms (Purdue)

Overview of the Field

Descriptive set theory is the theory of definable sets and functions in Polish spaces. It's roots lie in the Polish and Russian schools of mathematics from the early 20^{th} century. A central theme is the study of regularity properties of well-behaved (say Borel, or at least analytic/co-analytic) sets in Polish spaces, such as measurability and selection. In the recent past, equivalence relations and their classification have become a central focus of the field, and this was one of the central topics of this workshop.

One of the original co-organizers of the workshop, Greg Hjorth, died suddenly in January 2011. His work, and in particular the notion of a turbulent group action, played a key role in many of the discussions during the workshop.

Functional analysis is of course a tremendously broad field. In large part, it is the study of Banach spaces and the linear operators which act upon them. A one paragraph summary of this field is simply impossible. Let us say instead that our proposal will be concerned mainly with the interaction between set theory, operator algebras, and Banach spaces, with perhaps an emphasis on the first two items.

An operator algebra can be interpreted quite broadly ([4]), but is frequently taken to be an algebra of bounded linear operators on a Hilbert space closed under the formation of adjoints, and furthermore closed under either the norm topology (C*-algebras) or the strong operator topology (W*-algebras). The theory of these algebras began with the work of Murray and von Neumann in the 1930s, and has since expanded to touch much of modern mathematics, including number theory, geometry, ergodic theory, and topology.

The connection between descriptive set theory and functional analysis can be traced back at least as far as Mackey's work on group representations in the 1950s. There he recognized a fundamental obstruction to obtaining a satisfactory decomposition of a unitary representation of a second countable locally compact group as a direct integral of unitary equivalence classes of irreducible representations: the space of infinite-dimensional representations did not naturally carry the structure of a standard Borel space. This led to his "Type I iff smooth dual" conjecture for such groups, ultimately proved with C*-algebra theory by James Glimm in 1961. These early connections have given way to a range of powerful new results connecting descriptive set theory to ergodic theory and to the theory of C*- and W*-algebras, results which have tantalizing prospects for the future.

Recent Developments and Open Problems

Recently, there has been great progress at the interface of operator algebras, descriptive set theory, and ergodic theory. Here we single out examples of four types, and explain how our workshop will advance research in each area. (We emphasize the first two types, as they exhibit particularly strong connections between these fields. And of course, the topics covered at the workshop will not be restricted to these four.)

I. Borel reducibility and the complexity of classification problems.

Mackey's opinion that Borel equivalence relations yielding non-standard Borel spaces are simply "unclassifiable" has more recently been countered with a rich theory of cardinality for such relations. Given Polish spaces X and Y carrying equivalence relations E and F, respectively, one says that (X, E) is *Borel reducible* to (Y, F) if there is a Borel map $\Theta : X \to Y$ with the property that

$$xEy \Leftrightarrow \Theta(x)F\Theta(y).$$

In words, assigning invariants to F-classes is at least as difficult as assigning them to E-classes; one writes $E \leq_B F$. There are infinitely many degrees of complexity in this picture, and the relationships between them remain murky in places. There are, however, some standout types: E is *classifiable by countable structures*, if, roughly, it is no more complex than the isomorphism relation on countable graphs; E is *turbulent* if there is a Borel reduction from the orbit equivalence (OE) relation of a turbulent group action into E; E is *below a group action* if there is a Borel reduction from E into the OE relation of a Polish group action (see [15], [2], [14]).

Several results regarding the Borel complexity of C*- and W*-algebras and group actions upon them have recently emerged. Sasyk-Törnquist ([29, 30, 28]) have established turbulence for isomorphism of von Neumann factors of all types, while Kerr-Li-Pichot ([20]) have done the same for various types of group actions on standard probablity spaces and the hyperfinite II_1 factor. On the C*-algebra side, Farah-Toms-Törnquist ([13], [12]) have proved that the isomorphism relation for unital nuclear simple separable C*-algebras (the primary object's in G. A. Elliott's K-theoretic classification program, [27], [8]) is turbulent yet below a group action, and have established a similar result for metrizable Choquet simplices.

Our workshop will bring together these and other researchers to work on new questions in Borel reducibility, such as assessing the complexity of exact and non-exact C*-algebras and non-commutative L_p spaces (and perhaps finding therein an instance of the Kechris-Louveau conjecture ([19]) concerning E_1 and group actions), and determining whether various classification functors in functional analysis (K-theory for AF algebras, say) have Borel computable inverses.

II. Measure preserving group actions.

Three steadily weaker notions of equivalence for free ergodic actions of countable groups on a standard probability space are conjugacy, orbit equivalence, and von Neumann equivalence (isomorphism of the von Neumann algebra crossed products associated to $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$). The last of these is very weak, as any two actions of ICC amenable groups are equivalent in this sense. Nevertheless Popa has since 2002 developed a remarkable deformation/rigidity theory ([24], [25]) which has allowed him and subsequently many others to establish orbit equivalence from von Neumann equivalence for a wide array of non-amenable groups. This has led to striking results in the descriptive theory of orbit equivalence relations, including the proof (due to Ioana and Epstein, [9], [16]) that every countable non-amenable group admits continuum many orbit inequivalent actions, giving a strong converse to the Connes-Feldman-Weiss Theorem ([6]). More recently, Hjorth has used techniques derived from Popa-Ioana to prove that there are continuum many mutually $\leq_B$-incomparable Borel equivalence relations (see I. above).

We will address several problems in this theory at our workshop, including the extension of von Neumann rigidity to new and larger classes of groups, and the application of these results to gain finer descriptive understanding of orbit equivalence relations.

III. Banach spaces.

The descriptive theory of Banach spaces is another active area under this proposal's umbrella. Rosendal, Ferenczi and Louveau are the prime actors. Recent results include the proof that the isomorphism problem for separable Banach spaces is equivalent to the maximally complicated analytic equivalence relation in the Borel hierarchy, and a partial classification of Banach spaces in terms of minimal subspaces. The second item is part of Gowers' program to classify Banach spaces by finding characteristic spaces present in every space. That program will be pursued further at our workshop.

IV. The structure of C^*-algebras.

C^*-algebra theory has seen many old problems solved lately using set theory as a fundamental tool. (It must be said that the set theory involved is not really descriptive, but we nevertheless have another important interaction between set theory and functional analysis.) These results include the proof by Farah ([10]) and Phillips-Weaver ([23]) that the question of whether all automorphisms of the Calkin algebra are inner is independent of ZFC, and the Akemann-Weaver proof of the consistency of a counterexample to Naimark's problem ("Must a C^*-algebra with only one irreducible representation up to unitary equivalence be isomorphic to the compact operators on some Hilbert space?"), see [1]. Further questions to be addressed at our workshop include the possibility that a solution to Naimark's problem is consistent, and the question of whether the Calkin algebra admits a K_1-reversing automorphism.

Timeliness and Relevance.

The progress described above has led to a tremendous amount of new collaboration and dialogue between functional analysts and descriptive set theorists, albeit through a multitude of largely independent projects. That is why a 5-day workshop at BIRS on Descriptive Set Theory and Functional Analysis will be especially effective: we will not only disseminate research and lay the groundwork for progress on major problems in the field–any BIRS workshop should do as much–but also give new coherence to this interdisciplinary field. Success in this last goal will prove particularly helpful to young researchers wanting to enter the field, as they will get a panoramic view of its research and be able to discuss their own research with a cast of faculty never before assembled at a single meeting.

The profile of interdisciplinary research in set theory and functional analysis has been rising steadily. For instance, Texas A&M University hosted a 5-day conference on the topic in August, 2010, and there have been three Appalachian Set Theory Workshops (a NSF funded series) by Kechris, Törnquist and Farah discussing several of the recent results described in this proposal. Our BIRS workshop, however, was an order of magnitude more significant than these events, not least because of the quality of the participants. They included 7 ICM speakers and the present or erstwhile editors of *Journal of the American Mathematical Society*, *Fundamenta Mathematicae*, *Journal of Symbolic Logic*, *Bulletin of Symbolic Logic*, *Canadian Journal of Mathematics*, *Journal of Functional Analysis*, *Pacific Journal of Mathematics*, and *Journal of Operator Theory*.

As for timeliness, we would point out, in addition to the conference activity mentioned above, that most of the significant results motivating this workshop have appeared in the last five years.

Schedule of the Workshop.

The following lectures were delivered over the course of the week.

Monday, June18

1. Dima Shlyakhtenko, *Free monotone transport*
2. Bradd Hart, *Model theory of tracial von Neumann algebras*
3. Vern Paulsen, *The Kadison–Singer problem*
4. Asger Törnquist, *A Fraisse-theoretic approach to the Poulsen Simplex*
5. Stuart White, *Perturbations of crossed products*

Tuesday, June19

1. Simon Thomas, *A descriptive view of unitary group representations*
2. Stefaan Vaes, II_1 *factors with a unique Cartan decomposition*
3. Juris Steprans, *Topological centres of group actions*
4. David Kerr, *Independence and entropy in topological dynamics*
5. Ed Effros, *Some personal reflections on QFA (Quantized Functional Analysis)*

Wednesday, June20

1. Tristan Bice, *Calculus of projections in C*-algebras*
2. Justin Moore, *Spatial models of Boolean actions*
3. Cyril Houdayer, *A class of* II_1 *factors with an explicit abelian amenable subalgebra*

Thursday, June 21

1. Jesse Peterson, *Stabilizers of ergodic actions of lattices and commensurators*
2. N. Christopher Phillips, *Outer automorphisms of the Calkin algebra*
3. Problem session
4. Aleksandra Kwiatkowska, *Boolean actions on groups of isometries*
5. Caleb Eckhardt, *Amenable group C*-algebras*
6. Martino Lupini, *Non-classification of automorphisms of C*-algebras up to unitary equivalence*

Friday, June 22

1. Hiroshi Ando, *Finite-type Polish groups and Popa's problem*
2. Todor Tsankov, *Generic representations of abelian groups*
3. Simon Wasserman, *Factorial representations of nonseparable C*-algebras*

Questions and Problems.

The following problems were collected during the Thursday problem session.

(1) (Caleb Eckhardt) For a discrete, amenable group Γ; Is $I(\Gamma) := \ker(C^*(\Gamma) \longrightarrow \mathcal{C})$ ever the unique maximal ideal?

(2) (George Elliott) Let Γ be a discrete group. Is Γ type 1 if and only if $C^*(\Gamma)$ has continuous trace?

(3) (Ilijas Farah) What is the complexity of isomorphism of separable C^*-algebras?

Remark 1.

An upper bound for this classification problem was found shortly after the workshop (see 23).

(4) (N. Christopher Phillips) How does the complexity of isomorphism of separable C^*-algebras compare with the complexity of complete isometric isomorphism of:

- non-self-adjoint subalgebras.
- von Neumann algebras with separable preduals.

Remark 2 (Vern Paulsen)**.** *The complexity is very large; At least as bad as complete isometric isomorphism of all Banach spaces.*

(4') (N. Christopher Phillips) Complexity of complete isometric isomorphism of AF algebras vs. triangular AF algebras.

Remark 3.

While AF algebras are classifiable by countable structures (more precisely, by K_0), no reasonable classification of non-selfadjoint algebras is known (see [26]). A proof that the isomorphism relation of triangular algebras is not classifiable by countable structures would show that no reasonable classification is possible.

(5) (Justin Moore) Does $\{C^\infty$ functions $M \longrightarrow U(1)\}$, where M is a compact smooth manifold, have the point realisation property?

Remark 4. *It is known that for all k, $\{C^k$ functions $[0,1] \longrightarrow U(1)\}$ does not have the point realisation property.*

(6) (Vern Paulsen) Let G be a countable discrete group with a pure state on $\ell^\infty(G)$ that uniquely determines a pure state on $L(G)$. Does this extend uniquely to $B(\ell^2(G))$?

Remarks 1.
- *This is an open question for all countable, discrete groups G.*
- *It is hard for $\mathbb{Z}$, but might be easier for more complicated groups.*
- *q points (always extend uniquely to $B(\ell^2(G))$) and special filters may be relevant.*
- *Related to $L(G)$ being pavable in the Anderson sense.*
- *Could ask the global question: Do all pure states extend uniquely?*
- *The motivation for this question comes from Kadison–Singer problem ([5]),*

(7) (N. Christopher Phillips) Does $L(\ell^p)/K(\ell^p)$ have outer automorphisms?

Remark 5. *In some cases, the answer is certainly no (e.g. when the quotient is $\mathcal{C}$). $p = \infty$ might be interesting.*

(8) (Ilijas Farah) Is there an analogue of turbulence/some criteria for non-reducibility to unitary conjugacy of normal operators?

(9) (Wilhelm Winter) Does every trace on the countable ultraproduct of unital, separable, tracial C^*-algebras come from an ultraproduct of traces?

Remarks 2.
- *(Ilijas Farah) The answer is no if the continuum hypothesis is assumed. The counterexample is commutative and has many traces (uses cardinalities).*
- *The following may be relevant: Unique trace; UHF-algebras; Property SI.*
- *Matui-Sato have positive answers.*

(10) (Bradd Hart) What are the values of σ_n in II_1 factors, where

$$\sigma_n := \sup_{\substack{x_1,\dots,x_n \\ \|x_j\|\le 1}} \inf_{\substack{y \\ \|y\|\le 1}} \left(\|y^*y - 1\|_2 + |\operatorname{tr}(y)| + \sum_{j=1}^{n} \|[x_j, y]\|_2\right)?$$

Remarks 3. – *This is known for free groups and hyperfinite II_1 factors.*

– *If they are all true (i.e. $\sigma_n = 0$ for all n) then this is property Γ (implies existence of a nontrivial central sequence).*

(11) (Justin Moore) Conjecture: Let $\mathbb{T}$ be a free binary system on one generator, then there is an idempotent finitely additive probability measure on $\mathbb{T}$ with respect to Arens products.

Remarks 4. – *A positive answer implies that the Thompson group is amenable, and potentially the above is a stronger statement.*

– *Are there algebraic conditions on a binary system that imply the existence of an idempotent measure?*

(12) (Simon Thomas) Conjecture: Unitary equivalence on representations of $\mathbb{F}_2$ is strictly more complicated than unitary equivalence of representations of $\mathbb{Z}$.

(13) (Martino Lupini) Is the relation of conjugacy on the automorphism group of the CAR-algebra Borel?

(14) (Hiroshi Ando) Is it true that $\mathcal{U}(\ell^2)_p = \{u \in \mathcal{U}(\ell^2) : u - 1 \in S^p(\ell^2)\}$ is $\mathcal{U}_{fin}$?

Remark 6. *Shlossberg-Megrelishvili showed that ℓ^p is UR if and only if $1 \leq p \leq 2$.*

Scientific Progress Resulting from the Workshop.

This workshop was particularly successful in starting new collaborations and finding solutions to prominent open problems.

Isomorphism relation of separable C*-algebras

One of the central problems about the complexity of the isomorphism relation of separable C*-algebras is whether it is Borel-reducible to an orbit equivalence relation of a Polish group action. A partial positive answer was given by Farah–Toms–Törniquist ([13]) using a Borel version of Kirchberg's $\mathcal{O}_2$-embedding theorem. A novel approach to this problem was suggested by Vern Paulsen during the workshop. In a rapid email interchange in the week following the workshop a positive answer to this problem was given by Elliott, Farah, Paulsen, Rosendal (who could not attend the workshop but joined the email correspondence), Toms and Törnquist. The construction, while technically simpler than the earlier one by Farah–Toms–Törnuist, is general enough to show that the isometry of non-self-adjoint operator algebras as well as the complete isometry of operator spaces are Borel-reducible to orbit equivalence relations.

Traces on ultrapowers

Recently Matui-Sato made a conceptual breakthrough in the fine structure of nuclear C*-algebras ([21], [22]). They introduced excision techniques to the stably finite classification programme, which simplifies an important technical argument of Winter ([32], [31]). For separable simple unital nuclear C*-algebras with finitely many extremal traces they found a method for extracting the critical large central sequences required to run these arguments from the central sequences found in tracial von Neumann closures. In this way Matui-Sato where able to very an implication of the important Toms-Winter regularity conjecture for simple unital nuclear C*-algebras with finitely many extremal traces.

Our work has focused on weakening the assumption on the trace space: prior to the BIRS workshop Toms and White had produced an outline for extending Matui-Sato to the case of where the tracial state space has a zero dimensional compact extreme boundary. Whilst at BIRS, Toms-White-Winter were able to find a marriage of these new techniques with earlier techniques of Winter ([32], [31]) and produced a strategy for extending to a finite dimensional compact extreme boundary. Subsequent to the workshop this strategy has been completed and a paper is in preparation.

Tristan Bice has made progress on problem (9) from the list, by constructing a ZFC example of a separable C*-algebra and a trace on its ultrapower that does not arise as an ultraproduct of traces.

Classification of automorphisms of C*-algebras

At the time of the BIRS meeting, Lupini could prove nonclassification for C*-algebras that contain a central sequence which is not uniformly central. He then asked if all non continuous trace C*-algebras have this property, and George Elliott observed that at least non type I C*-algebras do.

Model theory of metric structures

A topic that has prominently emerged in the last few years is applications of model theory to operator algebras. Model theory of metric structures was developed by Ben–Ya'acov, Berenstein, Henson and Usvyatsov ([3]). It was adapted to operator algebras in [11]. Bradd Hart gave a well-received talk on applications of this logic to C*-algebras and tracial von Neumann algebras. At the moment it is not clear how these methods can be applied to other operator algebras, most importantly to type III von Neumann algebras. In conversations with Hiroshi Ando and Dima Shlyakhtenko, Hart started developing an approach to this problem.

Outcome of the Meeting

From the point of view of the organizers (who have some experience with BIRS workshops!), this was and extremely productive meeting. The number of new collaborations begun and problems solved was very high, and this is largely attributable to having met one of the workshop's original goals: bring together a diverse collection of researchers in functional analysis and set theory in hopes that their collective knowledge will allow them to solve problems hitherto out of reach for either field on its own.

The progress made at the workshop is expected to be pushed further at the Fields Institute Program on Forcing this fall, and in particular at the 5-day workshop on applications of set theory to C^*-algebras from September 10-14. Another event that should solidify gains promised by our BIRS workshop is the Oberwolfach meeting on C^*-algebras, dynamics, and classification in early November. Overall, we consider the workshop to have been very successful.

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Bibliography

[1] C. Akemann and N. Weaver. Consistency of a counterexample to Naimark's problem. *Proc. Natl. Acad. Sci. USA*, 101(20):7522–7525, 2004.

[2] H. Becker and A.S. Kechris. *The descriptive set theory of Polish group actions*. Cambridge University Press, 1996.

[3] I. Ben Yaacov, A. Berenstein, C.W. Henson, and A. Usvyatsov. Model theory for metric structures. In Z. Chatzidakis et al., editors, *Model Theory with Applications to Algebra and Analysis, Vol. II*, number 350 in London Math. Soc. Lecture Notes Series, pages 315–427. Cambridge University Press, 2008.

[4] B. Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[5] P.G. Casazza and J.C. Tremain. The Kadison-Singer problem in mathematics and engineering. *Proc. Natl. Acad. Sci. USA*, 103(7):2032–2039 (electronic), 2006.

[6] A. Connes, J. Feldman, and B. Weiss. An amenable equivalence relation is generated by a single transformation. *Ergodic Theory Dynamical Systems*, 1(4):431–450 (1982), 1981.

[7] Edward G. Effros. Classifying the unclassifiables. In *Group representations, ergodic theory, and mathematical physics: a tribute to George W. Mackey*, volume 449 of *Contemp. Math.*, pages 137–147. Amer. Math. Soc., Providence, RI, 2008.

[8] G.A. Elliott and A.S. Toms. Regularity properties in the classification program for separable amenable C^*-algebras. *Bull. Amer. Math. Soc. (N.S.)*, 45(2):229–245, 2008.

[9] I. Epstein. Orbit inequivalent actions of non-amenable groups. preprint, arXiv:0707.4215v2, 2007.

[10] I. Farah. All automorphisms of the Calkin algebra are inner. *Annals of Mathematics*, 173:619–661, 2011.

[11] I. Farah, B. Hart, and D. Sherman. Model theory of operator algebras II: Model theory. preprint, arXiv:1004.0741, 2010.

[12] I. Farah, A.S. Toms, and A. Törnquist. The descriptive set theory of C*-algebra invariants. *IMRN*, to appear. Appendix with Caleb Eckhardt.

[13] I. Farah, A.S. Toms, and A. Törnquist. Turbulence, orbit equivalence, and the classification of nuclear C*-algebras. *J. Reine Angew. Math.*, to appear.

[14] G. Hjorth. When is an equivalence relation classifiable? In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, Documenta Mathematica, pages 23–32, 1998.

[15] G. Hjorth. *Classification and orbit equivalence relations*, volume 75 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2000.

[16] Adrian Ioana, Alexander S. Kechris, and Todor Tsankov. Subequivalence relations and positive-definite functions. *Groups Geom. Dyn.*, 3(4):579–625, 2009.

[17] A.S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate texts in mathematics*. Springer, 1995.

[18] A.S. Kechris. The descriptive classification of some classes of C^*-algebras. In *Proceedings of the Sixth Asian Logic Conference (Beijing, 1996)*, pages 121–149. World Sci. Publ., River Edge, NJ, 1998.

[19] A.S. Kechris and A. Louveau. The structure of hypersmooth Borel equivalence relations. *Journal of the American Mathematical Society*, 10:215–242, 1997.

[20] David Kerr, Hanfeng Li, and Mikaël Pichot. Turbulence, representations, and trace-preserving actions. *Proc. Lond. Math. Soc. (3)*, 100(2):459–484, 2010.

[21] H. Matui and Y. Sato. Strict comparison and Z-absorption of nuclear C*-algebras. Preprint, arXiv:1111.1637, 2011.

[22] H. Matui and Y. Sato. Z-stability of crossed products by strongly outer actions II. Preprint, arXiv:1205.1590, 2012.

[23] N.C. Phillips and N. Weaver. The Calkin algebra has outer automorphisms. *Duke Math. Journal*, 139:185–202, 2007.

[24] Sorin Popa. On a class of type II_1 factors with Betti numbers invariants. *Ann. of Math. (2)*, 163(3):809–899, 2006.

[25] Sorin Popa. Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups. I. *Invent. Math.*, 165(2):369–408, 2006.

[26] S.C. Power. *Limit algebras: an introduction to subalgebras of C^*-algebras*, volume 278 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1992.

[27] M. Rørdam. *Classification of nuclear C^*-algebras*, volume 126 of *Encyclopaedia of Math. Sciences*. Springer-Verlag, Berlin, 2002.

[28] R. Sasyk and A. Törnquist. An anti-classification theorem for von Neumann factors. preprint, 2008.

[29] R. Sasyk and A. Törnquist. Borel reducibility and classification of von Neumann algebras. *Bulletin of Symbolic Logic*, 15(2):169–183, 2009.

[30] R. Sasyk and A. Törnquist. Turbulence and Araki-Woods factors. *J. Funct. Anal.*, 259(9):2238–2252, 2010.

[31] W. Winter. Decomposition rank and $\mathcal{Z}$-stability. *Invent. Math.*, 179(2):229–301, 2010.

[32] W. Winter. Nuclear dimension and $\mathcal{Z}$-stability of pure C*-algebras. *Invent. Math.*, 187(2):259–342, 2012.

Chapter 24

Eigenvalues/singular values and fast PDE algorithms: acceleration, conditioning, and stability (12w5021)

June 24 - 29, 2012

Organizer(s): Oscar P. Bruno (Computing and Mathematical Sciences, Caltech), Michael Haslam (Department of Mathematics and Statistics, York University), Mark Lyon (Department of Mathematics & Statistics, University of New Hampshire), Catalin Turc (Department of Mathematical Sciences, New Jersey Institute of Technology)

Overview of the Field

Partial Differential Equation (PDE) theory constitutes a cogent set of theoretical and computational methods that enable qualitative and quantitative understanding in vast areas of science and engineering, including the fields of physics, chemistry, biology, economics and ecology, amongst many others. With increasing computational power, the ambitions to produce physically faithful numerical solutions have been raised to exceedingly high levels; in recent years it has become clear, however, that the advances in computer technology alone will not enable accurate solution of complex scientific PDE problems. It is the sound techniques of numerical analysis, grounded in solid theoretical foundations that will unleash the computational power of modern computer systems in the area of computational science.

While many high-quality tools are currently available for the numerical solution of Partial Differential Equations, a large number of important problems have remained intractable, or nearly so, due to the shear scale of the computer power their solutions require. Interestingly, most of the difficulties that hinder applicability and/or performance of numerical methods concern the structure of spectra (eigenvalues and singular values) of various discrete operators associated with numerical solvers.

Thus, unfavorable spectral distributions give rise to

- Lack of stability or restricted stability of explicit time-domain solvers, including Finite-Difference solvers, Finite-Element solvers, Discontinuous Galerkin solvers, Finite Volume solvers, methods based on Radial Basis Functions, Spectral solvers, etc.
- Slow convergence of iterative linear-algebra methods for implicit time-stepping.
- Large iteration counts in iterative algorithms for elliptic problems, such as those based on domain decomposition, overlapping meshes, etc.
- Accuracy limitations owing to ill-conditioning in high-frequency solvers based on use of Oscillatory Basis Functions.

- Oversampling requirements to meet accuracy tolerances in acceleration methods for frequency-domain and time-domain integral-equation solvers.
- Slow convergence of iterative linear-algebra solvers for integral-equation methods.
- Conditioning difficulties arising from presence of geometric singularities and associated accuracy limitations for both volumetric and integral-equation PDE solvers.

While it is clear that these difficulties have common sources, which relate to spectral distributions of various operators, such connections have not been systematically explored. We believe such studies, which lie at the heart of the contributions presented in this workshop, are greatly beneficial to a wide range of areas in computational PDE. For example, the existence of geometric singularities gives rise to spectral distributions that affect the convergence of iterative steady state solvers as well as the time steps of the time-domain solvers, and thus, an understanding of one impacts on the other. Recent advances in QR decomposition techniques to resolve conditioning issues that have been successfully implemented for Radial Basis Function solvers, further, could yield significant improvement for other methodologies including time-domain, frequency-domain and steady state methods based on Oscillatory Basis Functions, Discontinuous Galerkin and, indeed, most of the approaches listed above. Similarly, recent progress on elimination of time-step constraints on the basis of Alternating-Direction approaches might be applicable to remedy or eliminate stability restrictions arising in the area of Discontinuous Galerkin or other methods.

Recent Developments and Open Problems

A number of important new ideas have recently emerged in the field of Numerical Solution of Partial Differential Equations, many of which relate to eigenvalues/singular-values of the associated discrete systems. These include topics concerning the following areas.

Stability - Regardless of the accuracy of a particular method for time-dependent problems, if it is not in some measure stable it will not be useful: failing stability, the numerical approximation of a bounded solution becomes infinite (and thus totally inaccurate) in finite time. The stability of the system is indeed governed by certain eigenvalues and singular values. Recent ideas, including temporal sub-cycling in high-order Discontinuous Galerkin methods and boundary projections in Fourier based methods, can nearly (even completely, in some cases) overcome stability constraints without sacrificing computational efficiency.

High-Order/Spectral Accuracy - Relying on methods that ensure agreement of solutions to their Taylor/Fourier expansions to some (adequately high) order n, higher-order/spectral methods generally use fewer unknowns to reach a prescribed solution accuracy. At the same time the eigenvalues of the resulting systems are often correlated to the order of the method and the problems associated with poorly behaved eigenvalues are exacerbated as the order is increased. Further, the eigenvalues are often very sensitive to the specific geometry and/or mesh of the problem. The advent of certain embedded-boundary methods, high-order integral methods, and overlapping meshes has greatly broadened the applicability of high-order and spectral methods.

Fast Iterative Methods - Relying on ideas related to the Fast Fourier Transform, use of approximate inverses (preconditioners) to accelerate convergence of iterative methods and low-rank matrix approximations, the advent of fast integral-equation solvers and multi-grid methods, as well as wavelet-based and fast direct solvers has inaugurated a new research direction in the field of numerical solution of PDEs. The overall computational cost of these methods is often tied directly to behavior of eigenvalues. Poorly behaved eigenvalues often lead to the requirement of larger numbers of linear-algebra iterations which can then only be reduced by finely tuned preconditioners. Emerging hybrid methods, which employ, say, a combination of a fast algorithm and a classical method (e.g., a hybrid of integral equation solvers and high-order finite-element approximations), are promising and currently a subject of much activity.

Presentation Highlights

Highlights of the presentations are included below, organized according to the thematic areas of the workshop. This organization is to some extent arbitrary, since the topics are inherently interconnected—as the presentations themselves made it abundantly clear, and as necessitated by the nature of the workshop.

Presentation highlights in the area of *Stability and Efficiency* The presentation by **B. Henshaw** on high-order accurate algorithms for overlapping grids, touched on several of the themes of the workshop. The focus was the use of overlapping grid meshes that can be applied to both static and dynamic geometries to avoid several complications. The overlapping grids allow greater structure to the meshes and dramatically reducing mesh anomalies. Generation of the mesh becomes less cumbersome for complicated geometries, and the eigenvalues of the resulting system of equations are generally more predictable and better behaved. Thus the stability of the solvers can be significantly better than alternate methods if the geometry is nominally complex. These overlapping mesh methods are adaptable and can be utilized by essentially any mesh dependent PDE solver. The presentation by **S. Jin**, in turn, concerned Semiclassical Computation of High Frequency Waves in Heterogeneous Media. In this presentation semiclassical Eulerian methods were presented, which can be used to evaluation of the evolution of high frequency waves through heterogeneous media without explicit numerical resolution of the small wavelengths. The method is based on the classical Liouville equation in phase space, with discontinous Hamiltonians; the presentation provided relevant interface conditions consistent with the correct transmissions and reflections.

The adaptability of overlapping meshes was once again considered in the presentation by **N. Albin** entitled *Fourier Continuation Methods Long-Range Propagation and Transport*. Fourier Continuation methods, essentially a method to overcome the Gibbs' phenomenon for non-periodic functions, was applied to the solution of PDEs in complex geometries. A key component of the geometric flexibility of this effort was the overlapping mesh approach. On the other hand, the idea of using generalized Fourier methods in of itself is a promising idea which addresses accuracy, stability, and the computational speed of algorithms. A different perspective on Fourier Continuation methods was presented by **M. Lyon**, who presented new methods for evaluation of FC continuation functions, giving rise to algorithms that are both fast and are capable of spectral accuracy throughout the computational domain. Fourier methods for non-periodic problems were also presented by **R. Platte** and **A. Gelb** in their presentation entitled *A hybrid Fourier-polynomial method for partial differential equations*. Similarly, **R. Braverman** presented a discussion of methods based on local Fourier bases obtained from use of windowing functions. All of these presentations showed that Fourier methods could be employed in a manner which allows the solver to avoid time step inherent in Chebyshev methods (due to the tight boundary discretizations in those methods) were to be used. In a related context, the presentation by **T. Driscoll** considered various powerful functions provided by Chebyshev-based methods that can be accessed from the Matlab application *Chebfun*, and the presentation by **S. Lau** demonstrated a multidomain spectral-tau method for the solution of the three dimensional helically reduced wave equation. Finally, the use of adaptive meshes applied to the pricing of financial derivatives was discussed in a the presentation by **C. Christara** entitled *Adaptive and high-order PDE pricing of financial derivatives*. A space-time adaptive and high-order method was developed for valuing options using a PDE approach. Both finite difference and finite element methods were considered for the spatial discretization of the PDE, while classical finite differences were used for the time discretization. To control the space error, an adaptive gridpoint distribution based on an error equidistribution principle was used.

A major concern for stability is the way in which the boundary conditions are enforced, especially for high-order methods. Many presenters, including those listed above addressed the manner in which the boundary conditions can be enforced within their specific methodologies throughout the presentation. Additionally, **T. Hagstrom** (Southern Methodist University) focused on the specifics of applying boundary conditions in the context high-order differentiation schemes and the resulting difficulties along with varies partial resolutions that have been developed. Significant progress has been made to stabilize high-order differentiation schemes by controlling the behavior at the boundary and this presentation overview of strategies in this area will be important for all workshop participants working on high-order methods to consider. A discussion of stability in the context of integral equation methods was presented by **X. Antoine** for the iterative solution of the Helmholtz equation in by means of finite element methods. In particular, the presentation introduced certain *Shifted Laplace Preconditioners*, which can stabilize the conditioning of the associated weak formulation, with controlled dependence on the wave number.

Presentation highlights in the area of *High-Order/Spectral Accuracy* Several issues related to the stability of high-order numerical methods have been discuss in the workshop. Amongst these, **Victor Dominguez** presented a convergence analysis of a Nyström method for 3D BIEs in acoustic scattering problems, in which he outlined a much needed rigorous analysis of a high-order Nyström method for the combined Boundary

Integral Equation (BIE) that arises in 3D sound-soft acoustic scattering by a smooth obstacle S.

The workshop hosted several presentations on the design and analysis of efficient numerical methods for solution of PDEs in periodic media via periodic and quasi-periodic Green's functions. These are time-honored challenging problems, whose solution has escaped the scientific computing community in several important cases. Most notably, the issues of Wood anomalies (that is frequencies for which classical quasi-periodic Green's functions put forth in the literature do not exist) in 3D is not entirely resolved. In this connection, **M. Siegel** presented a small-scale decomposition for 3D boundary integral computations with surface tension, in which he showcased an efficient, non-stiff boundary integral method for 3D interfacial flow with surface tension, with an application to porous media flow. For these problems, the velocity of the interface is typically given in terms of the Birkhoff-Rott integral, and a new method to compute this efficiently by Ewald summation was presented. The stiffness was removed by developing a small-scale decomposition. In order to develop this small scale decomposition, the problem was formulated using a generalized isothermal parametrization of the free surface.

Several important contributions related to rapidly convergent quasi-periodic Green's functions throughout the frequency spectrum were highlighted in the presentation by **S. Shipman** presented an efficient solver for acoustic and electromagnetic scattering problems in three-dimensional periodic media. The speaker presented an accurate and efficient numerical method, based on integral Nyström discretizations, for the solution of three dimensional wave propagation problems in piece-wise homogeneous media that have two-dimensional (in-plane) periodicity (e.g. photonic crystal slabs). The approach uses (**1**) A fast, high-order algorithm for evaluation of singular integral operators on surfaces in three-dimensional space, and (**2**) A new, representation of the three-dimensional quasi-periodic Green's functions, which, based on use of infinitely-smooth windowing functions and equivalent-source representations, converges super-algebraically fast throughout the frequency spectrum—even for high-contrast problems and at and around the resonant frequencies known as Wood anomalies. A related periodic problem was discussed by **J. Tausch**: in this case, an eigenvalue problem on periodic waveguides was considered, where periodicity was enforced through appropriate uses of *Dirichlet-to-Neumann maps*. **M. Melenk**, in turn, considered the convergence properties of numerical methods for boundary value problems for the Helmholtz equation at large wave numbers k. At the heart of the analysis lies a decomposition of solutions into two components: the first component is an analytic, but highly oscillatory function and the second one has finite regularity but features wavenumber-independent bounds. An important contribution results: for a conforming high order finite element method quasi-optimality is guaranteed provided (a) the approximation order p is selected as $p = O(\log k)$ and (b) the mesh size h is such that kh/p is small. **J. Ovall** considered issues concerning error estimation and adaptivity for finite element approximation of eigenvalues, with a focus on the use of efficient hp finite elements that robust with respect to singularities and near-singularities in the eigenfunctions, degeneracies and near-degeneracies in the spectrum, and discontinuities or other undesirable behavior in the differential operator.

A focus also developed in the workshop on the general area of efficient numerical methods for fluid-dynamics. **J-C. Nave** discussed a variety of numerical methodologies for fluid-dynamics and other PDE problems and interface problems, including methodologies based on level set methods and finite-difference schemes. In particular, Nave presented an approach that augments the level set function values by gradient information, and evolves both quantities in a fully coupled fashion. This procedure allows the algorithm to maintain the coherence between function values and derivatives, while exploiting the extra information carried by the derivatives. The method is of comparable quality to WENO schemes, but with optimally local stencils. In addition, structures smaller than the grid size can be located and tracked, and the extra derivative information can be employed to obtain simple and accurate approximations to the curvature. **G. Wright**, in turn, presented an efficient methodology based on Radial Basis Functions in conjunction with Partitions of Unity for solution of atmospheric transport (advective) fluid motions.

Several contributors discussed eigenvalues of important classical operators. **N. Nigam** discussed recent progress on eigenvalue problems related to the Laplace operator. In particular, discussion dealt with sharp bounds on the eigenvalue of the Laplace-Beltrami operator of closed Riemannian surfaces of genus higher than one, and eigenvalue problems for the Laplacian, with mixed Dirichlet-Neumann data. **P. Monk** (University of Delaware) discussed the "Interior Transmission Problem" (ITP), which gives rise to a non-standard eigenvalue problem. Properties of the ITP were discussed in several applications and numerical schemes for computing the related eigenvalues were explored. Remarkably, transmission eigenvalues can be observed from far field data, and the resulting eigenvalues can be used to estimate properties of the scatterer. **M. Costa-**

bel and M. Dauge presentation, in turn, concerned the Cosserat eigenvalue problem, that is, the Dirichlet problem for the Lame equations of linear elasticity, where the bulk modulus is considered as the eigenvalue parameter. This problem is notoriously difficult from the computational point of view; this presentation included recent progress on theoretical and on numerical aspects of this problem.

Another very important area of research that was featured in the workshop is efficient solutions of PDEs with a large number of possibly random parameters. The presentation given by **M. Ganesh** concerned a model reduction algorithm for parametrized multiple particle electromagnetic configurations. In this approach, a parameterized multiple scattering wave propagation model in three dimensions is considered as a function of the parameters in the model—describing the location, orientation, size, shape, and number of scattering particles as well as properties of the input source field such as the frequency, polarization, and incident direction. For such dynamic parameterized multiple scattering models, the standard discretization procedures are prohibitively expensive due to the computational cost associated with solving the full model for each online parameter choice. In the work presented, an iterative offline/online reduced basis approach for a boundary element method was proposed in order to simulate a parameterized system of surface integral equations reformulation of the multiple particle wave propagation model. In a related connection, **Y. Chen** presented certified fast algorithms for electromagnetic problems based on the use of reduced basis methods. In particular, like Ganesh's method, this approach was concerned with offline/online solution strategies for electromagnetic scattering problems to effect dramatic reductions in the computational costs for parameterized PDEs. The discussion concerned a posteriori error estimation to control the accuracy of the reduced basis methods as well as generalization of the applicability of the approach through use of domain decomposition.

Presentation highlights in the area of *Fast Iterative Methods* The area of numerical solutions for wave propagation problems in the high-frequency regime has been extremely active in the past decade. Our workshop hosted several speakers that have made key contributions in this area. **S. Chandler-Wilde, S. Langdon and E. Spence** considered numerical-asymptotic boundary integral methods for problems of high-frequency acoustic scattering, and discussed the authors' recent progress on the design and analysis of hybrid numerical-asymptotic boundary integral methods for boundary value problems for the Helmholtz equation that model time harmonic acoustic wave scattering in domains exterior to impenetrable obstacles. Their presentation included new results on the analysis of highly oscillatory boundary integral operators and on the high-frequency asymptotics of scattering problems as well as the fundamental question of whether the Helmholtz equation is sign-indefinite.

One important issue that several speakers in our workshop addressed is the design of numerical solutions of PDEs that are based on formulations of the PDEs whose spectral properties lead to rapidly convergent algorithms. For instance, boundary integral equations are a viable approach for the solution of linear PDEs as they offer an important dimensional reduction, yet the classical integral equations of many PDEs may not possess good spectral properties. Several of our participants have been made significant progress in the design and implementation of hybrid integral formulations that exhibit nearly optimal spectral properties. The presentation by **D. Levadoux** presented certain inherently well-conditioned formulations for time-harmonic scattering/transmission problems of electromagnetism. These equations form a family of source integral equations for the solution of time-harmonic Maxwell scattering and/or transmission problems. Regardless of the composition of the obstacle—metallic, full dielectric or coated with an impedance layer—the general methodology presented by the speaker is able to guide the construction of well-conditioned integral equations.

The very important issue of spectral properties of boundary integral operators for scattering problems has been recently investigated in relation to convergence properties of linear algebra solvers. In connection with spectral properties of various possible formulations of PDEs, nonnormality is a well studied subject in the context of partial differential operators. Yet, only little is known for boundary integral operators.The presentation by **T. Betcke**, on nonnormality of boundary integral operators in acoustic scattering, addressed issues related to the lack of normality of boundary integral operators and the impact of this property on the convergence of associated iterative solvers. In particular, recent results presented by the speaker for the analysis of spectral decompositions and nonnormality of boundary integral operators on general domains were discussed. One particular application is the analysis of stability constants for boundary element discretizations.

Boundary integral techniques have been extensively used for simulation of wave propagation phenomena, including scattering by penetrable and non-penetrable obstacles. The collection of potentials and integral operators associated to a particular operator (the acoustic wave propagation at fixed frequency, for instance, leads to Helmholtz's operator) can be used to build a Calderón Calculus—which amounts to a set of rules to handle potentials and operators, leading to well posed integral operators of the first and second kind, resonance-free combined field formulations, preconditioners, etc. In the presentation by **F. J. Sayas**, a recently completed fully discretized Calderón Calculus for the two dimensional Helmholtz equation was discussed. This full discretization can be understood as a highly non-conforming Petrov-Galerkin discretization of the continuous calculus, based on two staggered grids, Dirac delta distributions substituting acoustic charge densities and piecewise constant functions substituting dipole densities. It was pointed out in the presentation that the frequency-domain approximations and the convolution quadrature black-box introduced by C. Lubich can be combined, yielding a simple approach for simulation of scattering of transient waves in the plane.

Other workshop highlights The workshop fostered a wide variety of research connections and the development of collaborations in ways traditional conferences do not—ample time afforded opportunities for meaningful exchanges leading, in particular, to consolidation of ideas, methods, and even research efforts amongst the participants. The organizers believe that these informal discussions played a significant role towards making the workshop a great success.

Participants

Akhmetgaliyev, Eldar (California Institute of Technology)
Albin, Nathan (Kansas State University)
Antoine, Xavier (Universite de Lorraine)
Betcke, Timo (University College London)
Braverman, Elena (University of Calgary)
Bruno, Oscar (California Institute of Technology)
Chandler-Wilde, Simon (University of Reading)
Chen, Yanlai (University of Massachusetts Dartmouth)
Christara, Christina (University of Toronto)
Costabel, Martin (University of Rennes)
Dauge, Monique (University of Rennes)
Dominguez, Victor (Public University of Navarre)
Driscoll, Tobin (University of Delaware)
Ganesh, Mahadevan (Colorado School of Mines)
Gelb, Anne (Arizona State University)
Hagstrom, Tom (Southern Methodist University)
Haslam, Michael (York University)
Henshaw, William (Lawrence Livermore National Laboratory)
Jin, Shi (Shanghai Jiao Tong University, China and University of Wisconsin, Madison)
Kim, Tatiana (University of Bath)
Langdon, Stephen (University of Reading)
Lau, Stephen (University of New Mexico)
Levadoux, David (ONERA)
Leykekhman, Dmitriy (University of Connecticut)
Lyon, Mark (University of New Hampshire)
Melenk, Markus (Vienna University of Technology)
Monk, Peter (University of Delaware)
Nave, Jean-Christophe (McGill University)
Nigam, Nilima (Simon Fraser University)
Ovall, Jeffrey (University of Kentucky)
Platte, Rodrigo (Arizona State University)
Sayas, Francisco-Javier (University of Delaware)
Shipman, Stephen (Louisiana State University)
Shirokoff, David (McGill University)

Siegel, Michael (New Jersey Institute of Technology)
Spence, Euan (University of Bath)
Tausch, Johannes (Southern Methodist University)
Turc, Catalin (Case Western Reserve University)
Wright, Grady (Boise State University)

Chapter 25

Torsion in the Homology of Arithmetic Groups: Geometry, Arithmetic, and Computation (12w5075)

July 1 - 6, 2012

Organizer(s): Frank Calegari, Paul Gunnells, Akshay Venkatesh

This report briefly recalls the main objectives of the workshop, gives a detailed list of the talks presented there, and finally some responses from participants.

Summary of the objectives of the workshop

The goal of the workshop was to bring together topologists, geometers, and number theorists with the intent of exploring connections between topology, analysis, and number theory, with the special focus on topics described in the section below. We had hoped that the balance between the lectures and free time as well as the intimate setting of the Banff Research Station will stimulate many informal discussions and collaborations. We were delighted to see that these objectives were fulfilled.

We would like to thank BIRS staff for their hospitality and efficiency.

To briefly recall, the main point of the workshop was the remarkable fact — whose significance is only beginning to be appreciated and analyzed — that the cohomology of arithmetic groups contain a lot of torsion, and the conjectured relationships of the Langlands program extend to describe this torsion.

The workshop played a significant role in popularizing this circle of ideas and questions, and the topic has attracted even more attention since the completion of the workshop – on the arithmetic side, the work of Peter Scholze; on the analytic side, much more work on extending the use of analytic torsion to the noncompact case by Jonathan Pfaff and others. Indeed, we believe that a sequel workshop would be appropriate in view of these developments.

Summary of talks delivered at the meeting

The talks at the workshop reported on many new results related to the above list of topics. There were also some expository talks from experts, on both the automorphic and representation theory sides, designed to help build bridges between the different topics. Finally, there was an evening problem session that both summarized open questions mentioned in the talks and also generated new ideas. In the following we summarize the contents of each talk. Junior speakers are indicated by a bullet (•).

Frank Calegari (Northwestern) spoke on *Torsion in the cohomology of arithmetic groups*. He gave an overview of how torsion in the cohomology of arithmetic groups interacts with questions in the Langlands program, with a particular emphasis on the (mostly conjectural) relationship between cohomology and Galois representations.

Simon Marshall (IAS) (•) presented *Multiplicities of cohomological automorphic forms on* GL_2 [**?**]. He discussed some ideas related to the theory of p-adically completed cohomology developed by Frank Calegari and Matthew Emerton. If F is a number field which is not totally real, he showed how to use these ideas to prove a strong upper bound for the dimension of the space of cohomological automorphic forms on GL_2 over F which have fixed level and growing weight.

Nicolas Bergeron (Jussieu) (•) gave the talk *Growth of torsion homology for arithmetic groups*. His talk addressed the question, "When does the amount of torsion in the homology of an arithmetic group grow exponentially with the covolume?" Presenting joint work with Akshay Venkatesh [**?**], he gave many examples where this is so, and conjectured precise conditions under which this is expected to be true.

John Voight (Dartmouth) spoke on *Minimal isospectral, nonisometric orbifolds*. By revisiting a construction due to Vignéras, he exhibited the minimal pairs of orbifolds and manifolds of dimension 2 and 3 arising from arithmetic Fuchsian and Kleinian groups that are isospectral but nonisometric. This was joint work with Peter Doyle and Benjamin Linowitz.

Werner Müller (Bonn) gave a talk entitled *Analytic torsion and the growth of torsion in the cohomology of arithmetic groups* (cf. [**?**]). Analytic torsion is an invariant of a compact Riemannian manifold and a flat vector bundle over this manifold. It is defined in terms of the spectrum of the Laplace operators on forms twisted by the flat bundle. Bergeron and Venkatesh used the analytic torsion to study the growth of the torsion in the cohomology of towers of congruence subgroups with coefficients in fixed strongly acyclic flat bundle. His talk addressed the complementary case, where the lattice is fixed and the flat bundle varies. He also discussed the finite volume case.

Akshay Venkatesh (Stanford) spoke on *A torsion Jacquet–Langlands correspondence.* This talk presented joint work with Frank Calegari [**?**] that investigates relationships between torsion classes in the cohomology of split arithmetic groups and their inner forms.

Toby Gee (Imperial College) (•) gave the talk *p-adic Hodge-theoretic properties of etale cohomology with mod p coefficients, and the cohomology of Shimura varieties* [**?**]. He presented new results about the etale cohomology of varieties over a number field or a p-adic field with coefficients in a field of characteristic p, and gave some applications to the cohomology of unitary Shimura varieties. This was a report on joint work with Matthew Emerton.

Jean Raimbault (Jussieu) (•) spoke on *Asymptotics of analytic and homological torsion for congruence subgroups*. The goal of the talk was to describe work extending the cocompact computations of Bergeron–Venkatesh to Bianchi groups, which have $\mathbb{Q}$-rank 1 [**?**].

Nathan Dunfield (UIC) presented the talk *Integer homology* 3*-spheres with large injectivity radius*. Conjecturally, the amount of torsion in the first homology group of a hyperbolic 3-manifold must grow rapidly in any exhaustive tower of covers (see Bergeron-Venkatesh and F. Calegari-Venkatesh). In contrast, the first betti number can stay constant (and zero) in such covers. Here "exhaustive" means that the injectivity radius of the covers goes to infinity. In this talk, Dunfield explained how to construct hyperbolic 3-manifolds with trivial first homology where the injectivity radius is big almost everywhere by using ideas from Kleinian groups. He then related this to the recent work of Abert, Bergeron, Biringer, et. al. This was a report on joint work with Jeff Brock [**?**].

Thomas Church (Stanford) (•) spoke on *Stability and instability in the homology of arithmetic groups*. He surveyed both classical and recent results on stability in the homology of arithmetic groups, including the rational stability proved by Borel, and the more complicated central stability in the mod-p homology of congruence subgroups proved by Putman. He described a conjecture, joint with Benson Farb and Andrew Putman [**?**], that predicts a certain stability in the unstable homology of $\mathrm{SL}_N(\mathbf{Z})$ and its congruence subgroups.

Philippe Elbaz-Vincent (Grenoble) gave a talk entitled *Computations of the cohomology of modular groups and applications*. The focus of the talk was explicit techniques to compute the cohomology groups of $\mathrm{GL}_N(\mathbf{Z})$ and $\mathrm{SL}_N(\mathbf{Z})$ for $N = 5, 6, 7$, and their homology groups with coefficient in their Steinberg modules. He gave applications to algebraic K-theory and number theory. Part of the talk was based on joint work with H. Gangl (Durahm) and C. Soule (CNRS & IHES) [**?**].

Aurel Page (Bordeaux) (•) spoke on *Presentations of Arithmetic Kleinian Groups*. He presented an algorithm to compute a fundamental domain and presentation for arithmetic Kleinian groups. This was a report on his Ph.D. thesis [**?**].

Mehmet Sengun (Warwick) (•) presented the talk *On the integral cohomology of Bianchi groups*. In the first part of his talk, he presented numerical data investigating the asymptotic growth of torsion in the homology of cofinite Kleinian groups. In the second part, He presented some data which suggests a speculative connection between even Galois representation of Q and torsion in the homology of Bianchi groups [**?**].

Alexander Rahm (Galway) (•) gave the talk *Accessing the Farrell-Tate cohomology of discrete groups*. He introduced a method to explicitly determine the Farrell-Tate cohomology of discrete groups. His method allows one to show that the Farrell-Tate cohomology of the Bianchi groups is completely determined by the numbers of conjugacy classes of finite subgroups, and allowed him to give a conceptual description of the integral cohomology ring structure of the Bianchi groups.

Dan Yasaki (UNC Greensboro) (•) spoke on *Some explicit* $\delta = 1, 2$ *computations*. This was a preliminary report on some recent computations (joint with Paul Gunnells) of torsion in the integral cohomology for $\Gamma_0(N) \subset \mathrm{GL}_2(\mathcal{O}_F)$, where F is the complex cubic field of discriminant -23 ($\delta = 1$) and where F is the cyclotomic field of fifth roots of unity ($\delta = 2$) (cf. [**?**, **?**]). The results provided computational evidence in support of conjectures of Bergeron–Venkatesh about the growth of torsion with covolume.

In addition to these talks, a discussion session **Computational challenges** was held.

Feedback and response from the conference

After the completion of the workshop, the organizers received many positive comments from both seniore and junior participants. The junior participants in were happy about being able to present their work to more senior colleagues. All participants expressed excitement about being together in an environment that gave many opportunities to interact with experts from different fields, all with a common interest in the geometry, topology, arithmetic, and analysis surrounding torsion phenomena in the cohomology of arithmetic groups. and gives some flavor of the energy of the conference. Based on these comments, we expect new colloborations to arise from the workshop.

To give some flavor of the energy of the conference we include a link to a blog writeup by one of the participants:

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https://quomodocumque.wordpress.com/2012/07/19/
torsion-in-the-homology-of-arithmetic-groups-and-an-iwasawa-algebra-puzzle/
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At the completion of the workshop, it was agreed that a similar workshop should be held in the near future. This is especially pressing in light of Scholze's recent breakthrough in connecting Galois representations to torsion classes. The organizers plan to submit an application for such a follow-up workshop in the near future.

Participants

Bergeron, Nicolas (Institut de Mathematiques de Jussieu)
Calegari, Francesco (Northwestern University)
Chinta, Gautam (City University of New York)
Church, Thomas (Stanford University)
Doud, Darrin (Brigham Young University)
Dunfield, Nathan (University of Illinois (Urbana-Champaign))
Elbaz-Vincent, Phillipe (Grenoble University)
Ellenberg, Jordan (University of Wisconsin)
Gee, Toby (Imperial College London)
Geraghty, David (Princeton University)
Gunnells, Paul (University of Massachusetts Amherst)
Koch, Sophie (Vienna)
Lipnowski, Michael (Stanford University)

Marshall, Simon (Northwestern University)
Mueller, Werner (University of Bonn)
Page, Aurel (Universite Bordeaux 1)
Pollack, David (Wesleyan University)
Rahm, Alexander D. (National University of Ireland at Galway)
Raimbault, Jean (Universite Pierre et Marie Curie (Paris 6))
Savitt, David (University of Arizona)
Sengun, Mehmet Haluk (University of Warwick)
Speh, Birgit (Cornell University)
Venkatesh, Akshay (Stanford University)
Voight, John (Dartmouth College)
Wong, Siman (University of Massachusetts Amherst)
Yasaki, Dan (The University of North Carolina at Greensboro)

Chapter 26

Rigidity Theory: Progress, Applications and Key Open Problems (12w5069)

July 15 - 20, 2012

Organizer(s): Walter Whiteley, York University, Canada, Robert Connelly, Cornell University, USA, Tibor Jordán, Eötvös University, Budapest, Hungary, Stephen Power, University of Lancaster, UK, Ileana Streinu, Smith College, USA

Overview of the Field

The rigidity and flexibility of a structure, either man-made in buildings, linkages, and lightweight deployable forms, or found in nature ranging from crystals to proteins, is critical to the form, function, and stability of the structure. The mathematical theory of 'rigidity and flexibility' is developing methods for the analysis and design of man-made structures, as well as predictions of the behavior of natural structures such as proteins. We live in 3-dimensions, and a fundamental problem is to develop results for 3-dimensions which are as good and as efficient as the recently developed theory for structures in 2-dimensions.

Rigidity theory probes some of the fundamental geometry of space, distance and metrics. With the multiple applications of this theory, results which yield new descriptions, new constructions, new analyses and new algorithms spread out to impact on our understanding, our designs, our predictions and our connections within and among these areas of application.

Rigidity theory has both geometric and combinatorial aspects: what happens at particular configurations, and what 'almost always' happen as we search over all configurations. The methods used now draw on a wide range of geometric and combinatorial techniques, ranging from matroid theory, through positive semi-definite programming, representation theory of symmetry groups, and functional analysis to algebraic and projective geometry.

The well-developed theory for plane frameworks highlights the central 100 year old problem of of characterizing the graphs of generic infinitesimally rigid bar and joint frameworks in 3-D. The characterization of graphs which are generically globally rigid in 3-space has recently risen to be another of these key unsolved problems. These were central topics to the talks and the discussions during the workshop.

Recent Developments and Open Problems

Over the last four years, since the previous BIRS workshop, recent work from a new generation of contributors to rigidity theory have developed new methods, and probed new problems connected to the rigidity and flexibility of new classes of frameworks, under symmetry and periodicity conditions, for specialized families of examples, as well as for broader problems in CAD constraints. Along with the new focus on such topics, several key problems have been resolved - notably the Molecular Theorem, confirming that methods developed for body-bar frameworks apply directly to the study of generic molecular structures arising in material science and biochemistry. These results have potential extensions and applications to the study of built structures, protein structures with symmetry (such as protein dimers) or repetitive structures such as beta sheets or crystals.

The week-end after our workshop was devoted to a specific focus on inductive constructions in the framework of a 2-day workshop. This combination of a 5-day workshop and a 2-day workshop has proven very fruitful for our community.

Some key factors in the recent accelerated progress have been: (i) the arrival of a new generation of researchers, at multiple centers around the world; (ii) insights and new questions coming from fields of application; (iii) the evolving community and interactions of people from multiple communities which has been supported by a series of small and medium sized gatherings. Two of the gatherings contributing to these developments were previous meetings at BIRS.

The objectives for the workshop included:

1. With the rapid progress outlined above, including: global rigidity; body-bar and molecular rigidity; periodic rigidity; rigidity under symmetry; transfer of results among metrics and to other surfaces, it was a central activity to share what is known and identify key open problems that should be addressed.

2. With the range of applications, one objective was to make connections of current results, the key questions, and the answers that might matter in these areas, such as algorithmic and inductive characterizations; decompositions (Assur Graphs, rigid components); stability (global rigidity); short time scale motions (fixed lattice) versus slower deformation of materials (flexible lattice, fixed or variable volume);

3. The workshop highlighted key conjectures: sufficient connectivity in 3-space for generic rigidity and generic global rigidity (12 connectivity may be sufficient); proposed inductive constructions for all isostatic frameworks in 3-space; proposed characterization of all globally rigid frameworks in 3-space; and some new conjectures in all the areas were born;

4. The workshop helped the community develop new larger scale mathematical programs for further work on key questions such as: 3-space rigidity; rigidity of infinite frameworks, with and without periodicity; flexibility of surfaces in higher dimensions; the impact of symmetry on global rigidity and applications.

Presentation Highlights

For this meeting, the proposed focus was on developments within rigidity theory itself over the range of concepts and structures outlined above. Some connections were drawn to a variety of applications, and insights into rigidity theory brought from ongoing work in these areas of application with a few key people from these areas invited to the workshop. The list of invitees included key people from a number of these interdisciplinary collaborations, as well as all the core researchers in the many facets of rigidity theory, across all the generations.

The workshop started by a special session, effectively moderated by one of the organizers, Ileana Streinu. In this session each participant gave a short (4-5 minutes) talk, in which he or she could offer a glimpse of an upcoming longer presentation, or pose an open problem, or introduce a topic of interest. This unusual start supported immediate interactions, from the first lunch, as people immediately identified some areas of shared interest and possible questions they might be able to answer.

After this session the following talks were scheduled for Monday, to present a range of connections to unusual topics from old-hands, and to introduce some new researchers into our larger community.

Helmuth Stachel: Self-motions of the Kokotsakis tesselation,

Ileana Streinu: Rigidity and Origami

Hans-Peter Schroecker: New Results in Algebraic Kinematics: Linkages, Factorization, Discrete Mathematics

Andrea Micheletti: Engineering Tensegrities: Some Recent Studies

Oleg Karpenkov: Strata of Tensegrity Frameworks.

The program for the rest of the week was organized around some focus areas which featured new results, as well as opportunities to hear some overview talks.

Tuesday was devoted to Global Rigidity, Semidefinite Programming and related topics with an open discussion session on problems and connections and several lectures. The lectures:

Steven Gortler: A survey of Global and Universal Rigidity, with new results

Anthony Man-Cho So: Graph Realization via Schatten p-Quasi Norm Minimization

Igor Pak: Global Rigidty and Graph Colouring

Abdo Alfakih: Universal Rigidity of bar frameworks in General Position: Trilateral and Chordal graphs

Michael Thorpe: Why is the Maxwell count so good for the Rigidity Threshold in Homogeneous Networks?

Overall, this area of global rigidity generated a number of lively discussions which continued on through the workshop and afterwards. The paper [1] was refined through the continuing conversations over the week (and beyond).
Wednesday included some shorter updates on recent work - along with afternoon walks through the wonderful countryside around BIRS.

Alexander Gaifullin: Sabitov polynomials for polyhedra in four dimensions

Tiong-Seng Tay and Shin-ichi Tanigawa: Body-bar-hinge-rod frameworks

Stephen Power: Frameworks on Surfaces

Wendy Finbow-Singh: Isostatic almost spherical frameworks via disc decomposition

John Owen: Specialisation of Generic Frameworks

Shisen Luo: Lower Bound For The Rank of Rigidity Matrix of 4-Valent Graphs (participation via Skype)

Thursday was devoted to Periodic and Symmetric Frameworks :

Ciprian Borcea: Periodic Rigidity

Shin-ichi Tanigawa: Matroids of Gain Graphs in Applied Discrete Geometry

Viktória Kaszanitzky: Gain-sparsity and Symmetric Rigidity in the Plane

There was an extended discussion following the first talk, during which the group explored number of possible definitions of rigidity and flexibility for infinite frameworks with and without initial periodic structure. As a companion, a set of counterexamples were also explored, refining the choices and resulting in several documents posted to the workshop wiki over the next few weeks by Stephen Power and Walter Whiteley (see below).

We had a long discussion and open problem session on Friday morning. Before the workshop and throughout the week, a wiki site was used to post a number of conjectures and open problems. There were regular updates, including posting of presentations and revised conjectures. This was quite fruitful and continued to be available for some months following the workshop.

Scientific Progress Made and Outcome of the Meeting

Many collaborative groups were formed during the workshop. It is notable that many of these involved the younger researchers who were at BIRS for the first time. Some groups have reported progress on projects whose origin, direction or momentum can be traced back to the workshop.

- Anthony Man-Cho So reports: "For the topic I spoke at the workshop (Graph Realization via Schatten p-Quasi Norm Minimization), the paper has just been accepted to INFOCOM 2013 (acceptance rate is 17 percent)."

- Audrey Lee-St.John reports: "Louis Theran and I continued working on matroids and pebble games for body-and-cad rigidity (discussion started in Toronto at Fields in 2011). Also, Shin-ichi Tanigawa and I started discussing an algorithm for optimizing rigid systems (motivated by applications to control theory)."

- Brigitte Servatius reports: “I’d like to state that it was good to be able to interact with the young researchers, in particular Oleg Karpenkov, Viktoria Kaszanitzky and Shi-Ichi Tanigawa. I talked to them about several open problems for which I do have some partial results such as the Jackson - Jordán - Szabadka conjecure [[2], Conjecture 5.9] or the generalization of the molecular theorem which would allow more than two bodies to share a hinge. The discussion surrounding Conjecture 5.9 finally enabled Herman Servatius and me to answer the following question of Jackson and Jordán [3] on mechanisms: is it true that for a generic realization of a mechanism the operation of 1-extension may be performed without restricting the motion? We worked out a counterexample to this question. This example is mentioned in the latest version of [3] and we are writing up a short paper giving not just a counterexample but a more general answer to the question."

- Tony Nixon reports: “Bernd Schulze and I have been looking at symmetric frameworks on surfaces and so far we have focused on orbit graph constructions for half-turn symmetric frameworks on the cylinder.

  There will be a workshop on geometric and topological graph theory in Bristol from 15th-19th April, 2013 where a number of the participants at the BIRS workshop have been invited to speak. In this workshop we anticipate that many of the ongoing rigidity problems arising from the BIRS meeting will be presented and developed.”

- Meera Sitharam reports:

  “Having understood affine rigidity (recent results of Steve Gortler et al.) better from the workshop, I am now using it towards the open problem I posed in the 5-minute talk in the workshop that arose from a problem in machine learning called dictionary learning. Collaboration with Steve Gortler is likely both on this problem and the more general problem of characterizing rigidity and global rigidity when the equivalence and congruence used in defining these properties is based on general groups rather than the Euclidean group.

  Jialong Cheng and I have almost worked out a purely combinatorial, explicit algorithmic characterization of a matroid (for independence and closure) and shown that it is an abstract rigidity matroid in 3D. We know that the rigidity matroid in 3D is at least as restrictive than this matroid. If it turns out to be strictly less restrictive, that disproves a conjecture about abstract rigidity matroids. If not, it gives a combinatorial 3D rigidity characterization.”

- Ciprian Borcea reports: “My visit to BIRS, for the workshop on Rigidity Theory: Progress, Applications and Key Open Problems was useful, stimulating and pleasant at the same time. I believe that the workshop was successful in addressing and disseminating recent results in rigidity theory. I welcomed the chance to survey some of the new developments on periodic frameworks. This area has reached a certain "critical mass" and has interesting outstanding open problems. I mentioned in my talk the wider relation between rigidity, periodicity and sparsity and the topic of ultrarigidity. It is a matter of satisfaction to report that, under the direct stimulus of the workshop, I obtained - in collaboration with Ileana Streinu - new results on periodic volume and symplectic frameworks.”

- Stephen Power reports:

  “The workshop facilitated my collaboration with John Owen and Tony Nixon. It lead ultimately to an improvement of work that was in progress in July 2012 and which is now submitted to a research journal and is on the ArXiv [6].

  The workshop was also important in enabling conversations for rapid and informed progress concerning new rigidity theory projects with my postdoctoral research associate, Dr Derek Kitson. This is a new collaboration.

  The workshop was also invaluable in providing myself, and also Derek Kitson, with an up-to-date view from specialists on a range of topics, such as periodic rigidity, global rigidity and universal rigidity. I now expect these themes to feature in my future work.”

- Robert Connelly reports:

"I think the talk by Igor Pak about applying global rigidity ideas to unique colorability problems was especially interesting and I think the discussions at BIRS were a help to him."

- Walter Whiteley reports: "To me, the several key outcomes were:

  (i) the morning discussion of infinite frameworks / the talk of Ciprian Borcea and the discussion of ways to the rigidity approach infinite frameworks. Stephen Power and myself uploaded a sequence of documents to the wiki of the workshop to show the ongoing development, including a summary of definitions and connections for infinite frameworks with incidental (and forced) periodic structure. This focuses on infinitesimal rigidity and infinitesimal flexibility.

  From these Stephen Power created an example showing that there is a countably infinite bar-joint framework which is continuously flexible (has a finite flex) yet has no infinitesimal flex. We also explored simple 3-d examples which illustrate the distinction between strongly sequentially rigid infinite frameworks and weakly sequentially rigid frameworks. These tubes continued to be explored in the lounge, with physical models made of polyhedron, over the rest of the week. See some related simulations at: $http://wiki.iri.upc.edu/index.php/Symmetric_linkages$.

  (ii) the discussions around incidental symmetry - frameworks which are initially symmetric but may have motions which break the symmetry, or go to a reduced symmetry. There are some new collaborations involving Bernd Schulze, Louis Theran and others on this topic. There has also been a growing collaboration between Bernd, Adnan Sljoka (another of my students who just graduated) and the group represented by Lluis Ros from Barcelona. They have been using their path continuation techniques to explore the finite motions / configuration space of some of the structures (most recently the tubes I was presenting at the workshop as examples). "

- Tibor Jordán reports:

  "With my co-authors, who were also present at the workshop, I have made substantial progress on some joint projects and papers during the meeting. These include new results on globally linked pairs in generic two-dimensional frameworks (joint work with Bill Jackson [3]), frameworks with dihedral symmetry (joint work with Viktória Kaszanitzky and Shin-ichi Tanigawa [4]) as well as universally rigid graphs and frameworks in one-dimensional space (joint work with Viet-Hang Nguyen [5])."

- Louis Theran reports: "Tony Nixon and I have been working on the question of generic rigidity of frameworks supported by surfaces with no isometries. An inductive approach seems promising.

  Also, Audrey Lee-St John and I have been working on some things relating to body-CAD and matroids. Bernd Schulze and I have also been looking a bit at generic incidental symmetry in the plane."

- Bill Jackson reports: " Viet Hang Nguyen and myself began discussing the rigidity of d-dimensional body-direction-length frameworks in Banff. We shall continue this research. We have used a recursive construction to characterize rigidity in the cases when the bodies are either rigid or direction rigid and are now working on the case when the bodies are length rigid."

It is clear that, overall, the workshop supported new collaborations, and supported the development of all the graduate students and post doctoral fellows who participated. The series of workshops, about four years apart, has played a major role in expanding the community. In the Spring of 2013 a workshop in Bristol, England will provide follow up for a number of the mathematical topics which arose during this five day workshop and the follow-on two day workshop.

Participants

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Cheng, Jialong (University of Florida)

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Finbow-Singh, Wendy (St. Mary's University)
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Kitson, Derek (Lancaster University)
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Theran, Louis (Aalto University)
Thorpe, Michael (Arizona State University)
Whiteley, Walter (York University)

Bibliography

[1] Steve Gortler and Dylan Thurston, *Generic Global Rigidity in Complex and Psuedo-Euclidean Spaces*, arXiv:1212.6685.

[2] Bill Jackson and Tibor Jordán, and Zoltán Szabadka, *Globally linked pairs of vertices in equivalent realizations of graphs*, Discrete and Computational Geometry, Volume 35, pg. 493-512, 2006.

[3] Bill Jackson and Tibor Jordán, and Zoltán Szabadka, *Globally linked pairs of vertices in rigid frameworks*, Egerváry Research Group, Budapest, TR-2012-19, 2012.

[4] Tibor Jordán and Viktória Kaszanitzky, and Shin-ichi Tanigawa, *Gain-sparsity and symmetry-forced rigidity in the plane*, Tech. report: Egerváry Research Group, Budapest, TR-2012-17, 2012.

[5] Tibor Jordán and Viet-Hang Nguyen, *On universally rigid frameworks on the line*, Tech. report: Egerváry Research Group, Budapest, TR-2012-10, 2012.

[6] Anthony Nixon and John Owen, and Stephen Power, *A Laman theorem for frameworks on surfaces of revolution*, arXiv:1210.7073, 2012.

Chapter 27

Tissue Growth and Morphogenesis: from Genetics to Mechanics and Back (12w5048)

July 22 - 27, 2012

Organizer(s): Christian Dahmann (Dresden University of Technology), James J. Feng (University of British Columbia), Len M. Pismen (Technion, Israel Institute of Technology)

Overview of the Field

Tissue growth and morphogenesis are fundamental processes in developmental biology. On the cellular level, cell size growth, cell division, and cell shape changes are controlled by signaling molecules that, in turn, are expressed according to a genetic blueprint. On the tissue level, different cell types arrange themselves in spatial patterns that eventually form the tissue or organ. Evidently, morphogenesis involves biochemical and mechanical mechanisms on several length and time scales, and thus has attracted the attention of scientists from several fields, including genetics, cell biology, soft matter physics and mechanics of solids and fluids.

Past work on growth and morphogenesis has followed three distinct lines, corresponding to molecular, cellular and tissue scales. Most of our current understanding of tissue growth and morphogenesis has come from molecular-genetic studies, which elucidate what genes carry the blueprint of how the organism will develop, and how the blueprint is implemented through pathways and networks of causal relationships among signaling proteins. Besides its genetic aspect, it has long been recognized that morphogenesis is also a physical process that involves the adhesion, deformation and movement of cells in a temporal-spatial framework. On the cellular scale, physical scientists have undertaken studies on the intracellular mechanisms that cause prototypical deformation of cells such as apical constriction and convergent-extension, on how like cells migrate and aggregate and dissimilar ones segregate into domains, and on cell adhesion to an extracellular matrix or substrate. On the coarser tissue level, relevant to growth, development, and wound healing, continuum mechanics models have been developed. But they tend to be simplistic and lack a connection to molecular and genetic mechanisms. Feedback of mechanics on chemical processes in development and pattern formation remains largely unexplored, although its importance is widely recognized.

A good example to illustrate the unique complexity of developmental processes is cell sorting [1, 2, 3]. During embryogenesis, cells destined for different organs must organize themselves into domains and structures. First and foremost, these domains are determined by the so-called selector genes, which are expressed in different regions so as to establish an initial boundary between the cells. With cell proliferation and tissue deformation, these boundaries are liable to distortion, and cells of different lineage may intermingle (Fig. 27.1). Yet in successful development, these boundaries typically retain their integrity; cells that transgress into the other side are promptly brought back. The maintenance of boundaries in development

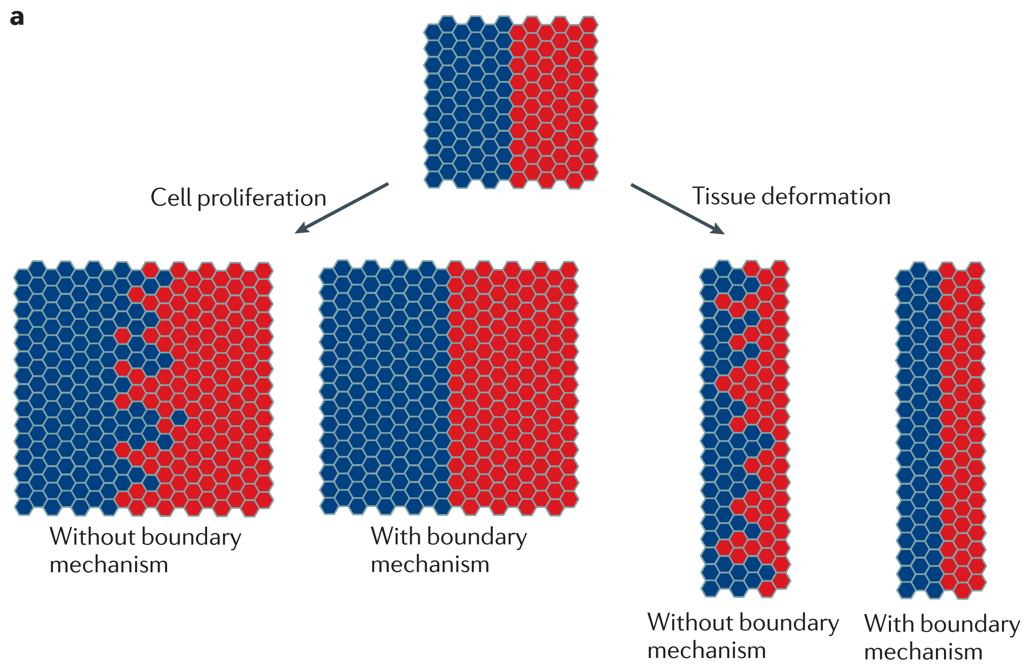

Figure 27.1: Schematic showing the maintenance of straight boundaries between cells of different lineage against challenges posed by proliferation and tissue deformation. Adapted from Ref. [3], first published by Nature Publishing Group, a division of Macmillan Publishers Limited.

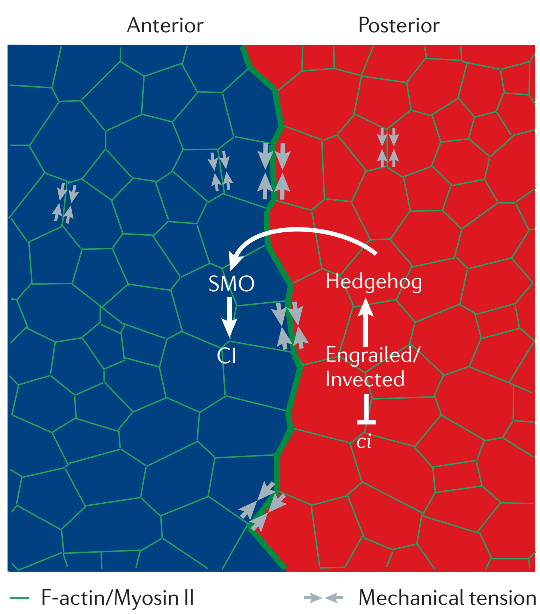

Figure 27.2: The Hodgehog signaling pathway that moderates the maintenance of the anterior-posterior (A-P) boundary on the wing disc of *Drosophila melanogaster*. Adapted from Ref. [3], first published by Nature Publishing Group, a division of Macmillan Publishers Limited.

turns out to require intricate cooperation among gene expression, molecular signaling and the physics of force generation and cell and tissue deformation. For the *Drosophila* wing primordium, geneticists have more or less clarified the signaling network via knock-out studies (Fig. 27.2), and the downstream effect appears to be the enrichment of actin and myosin along the boundary between the anterior and the posterior domains. Yet, we have only a rudimentary understanding of how this translates into mechanical forces and how the forces in turn maintain the boundary. Using an energy-minimization procedure, it has been shown that increasing the tension on the cell bonds does tend to reduce the roughness of the boundary [2, 4]. Quite unrelated to the modern genetic studies, there have been ideas of drawing an analogy between the spontaneous segregation of cell colonies and that of immiscible fluids. This so-called differential adhesion hypothesis has been under continuing debate [1, 5]. Despite the progress on separate fronts, an integrated understanding, encompassing the temporal-spatial distribution of the morphogens and the mechanical deformation of the tissue as a whole, has yet to be established. A consensus is forming that such an integrative approach calls for the close interaction between geneticists, biophysicists and mathematicians.

So far research dealing with the genetic, cellular and tissue aspects of morphogenesis has largely progressed independently, with insufficient conversation among the different communities of scientists. Yet, it is clear from the brief overview above that the problem at hand is multidisciplinary and calls for a coordinated approach that integrates insights and knowledge from the different approaches. One can envision a close coupling of the different length scales via propagation of information in both directions. For example, genetic-molecular pathways may predict or rationalize the abundance of certain signaling molecules such as the Hedgehog. This information can then be utilized on the cellular level to explain apical constriction. In the meantime, intracellular mechanical and geometric cues will affect the reaction-diffusion of active chemical messengers that modifies the genetic-molecular expression in return.

The latest advances on the three scales (genetic, cellular and tissue) and the emerging efforts at integrating them suggest that the time is ripe for a concerted assault. The general objective of this workshop is to bring together the leading researchers in tissue growth and morphogenesis across several disciplines, including biologists, physicists and mathematicians, to foster awareness and cross-disciplinary transfer of ideas on this burgeoning topic.

Recent Developments and Open Problems

New challenges are arising in measuring the mechanical properties of living organisms at different scales and in modeling them mathematically. Rapid development of experimental tools, such as high-resolution luminescence and fluorescence microscopy for molecule tracking, laser ablation, probing using automated tuning and matching (ATM) devices, and local traction measurements for mechanical response, provides abundant data of increasing resolution. These results are often not explained by cell- and tissue-level mechanical models. Physicists and mathematicians may not understand these data sufficiently well to choose a suitable modeling framework. The disconnect between the genetic, cellular and tissue scales is manifested by the following outstanding problems:

- Understanding the role of mechanical forces in morphogenesis through balances and feed-backs between genetic and mechanical inputs;

- Interaction between processes on different scales (subcellular, multicellular) and multiscale modeling (from molecular effectors to multicellular shape);

- Building a full morphogenetic pathway from molecules to shape via sequences of emerging properties, e.g. translating the activity of a microtubule regulator into morphogenetic out-put;

- Solving biologically important inverse problems, e.g. finding ways to construct a desired morphogenetic pattern;

- Developing tools to test hypothesis in different organisms and on different scales, and identifying similarities and differences between animals, plants, yeasts, etc;

- Comparing non-matched data sets, e.g. from different animals of the same or different phenotypes or between experimental and modeling data.

Clearly, there is a need for integrating biomolecular and mechanical studies, and for developing multiscale models that bridge the genetic, cellular and tissue levels. Integrating two or all three scales may be achieved by applying novel multiscale techniques being developed in fluid and solid mechanics, such as particle-based models incorporating intracellular transport of signaling molecules into mechanical representation of deformation of multiple cells or tissues.

Presentation Highlights

We had some 40 presentations covering the biological and mathematical aspects of cell and tissue dynamics. Despite the great diversity and scope of these talks, several key topics emerged that were notable for the fact that there were multiple groups working on them from different angles, and the interaction at the workshop provided a particularly fruitful forum for taking stock of recent achievements and looking ahead at the next steps. These are highlighted in the following.

(1) Collective motion of cell colonies. The coordinated motion of a group of cells is important for a large number of physiological and biological processes, including embryogenesis and wound healing. The key element is the signaling and coordination among cells, through both biochemical and mechanical pathways. Several speakers presented their latest experimental findings and theoretical models on this.

Pascal Silberzan spoke about the collective motility of epithelial cells that maintain strong adhesions between them during their migration. They grow epithelial (MDCK) cells within the apertures of micro-stencil previously put on the substrate. The removal of the stencil triggers the migration without damaging the border cells. This collective motility is characterized by Particle Image Velocimetry, and features long-range coordinated displacements of large groups of cells well within the monolayer that are well described by a simple model of self-propelled interacting particles. In a second stage, the edges of these epithelia roughen drastically and exhibit a strong directional fingering led by a cell of different phenotype (a 'leader cell') initially not discernible from the others. Interestingly, similarly looking leader cells are found in a large number of different situations in morphogenesis or local invasion from tumors. Silberzan focused on the properties of the migration fingers, which have been characterized by using a variety of physical techniques (image analysis, force measurements, laser photoablation) together with the mapping of the biochemical activity of migration-involved small GTPases.

Xavier Trepat discussed mechanical waves during tissue growth in his talk. Essential features of morphogenesis, wound healing and certain epithelial-derived diseases involve expansion of an epithelial monolayer sheet. Epithelial expansion is driven by mechanical events that remain largely unknown. Using the micropatterned epithelial monolayer as a model system, the Trepat group discovered an unexpected mechanical wave that propagates slowly to span the monolayer, traverses intercellular junctions in a cooperative manner, and builds up differentials of mechanical stress. Essential features of wave generation and propagation are captured by a minimal model based on sequential fronts of cytoskeletal reinforcement and fluidization. These findings establish a novel mechanism of long range cell guidance, symmetry breaking, and pattern formation during monolayer expansion.

Cristian Dahmann and Daiki Umetsu presented two talks on the fascinating process of cell sorting. Dahmann discussed signals and mechanics guiding cell sorting in animal development. The sorting out of cells with different identities and fates during animal development is an important process to organize functional tissues and organs. Previous hypotheses have explained the sorting-out of cells by differences in cell adhesion or surface tension. However, the mechanisms that guide cell sorting in animal development remain poorly understood. Dahmann and coworkers studied the mechanisms underlying cell sorting at compartment boundaries in *Drosophila*. Results show that the establishment of compartment boundaries in the developing *Drosophila* wing requires signaling by the Hedgehog, BMP, and Notch pathways. Recent data indicate that these pathways control cell sorting by locally increasing mechanical tension at cell junctions along the compartment boundaries. Cell sorting at compartment boundaries therefore provides an excellent model system to study the interplay between signaling and cell mechanics in animal development. Umetsu further reported live imaging that elucidates dynamics of cell sorting at lineage restriction boundaries in *Drosophila*. He combined long-term live imaging with quantitative image analysis to demonstrate clear differences in the dynamic behavior of cells at the compartment boundaries compared to cells further away. These differences in dynamic behavior result from distinct mechanical properties of cell bonds along the compartment boundaries.

These novel experimental findings were matched by several theoretical modeling efforts of collective cell motion. The models of Michael Koepf and Nir Gov are both inspired by the cell motion and pattern formation in the experiments. In collaboration with Len Pismen, Koepf formulated a generic continuum model of a polarizable active layer with neo-Hookean elasticity and chemo-mechanical interactions. Homogeneous solutions of the model equations exhibit a stationary long-wave instability when the medium is activated by expansion, and an oscillatory short-wave instability in the case of compressive activation. Both regimes are investigated analytically and numerically. The long-wave instability initiates a coarsening process, which provides a possible mechanism for the establishment of permanent polarization in spherical geometry. Nir Gov presented a model for the evolution of the outer contour of cellular aggregates. Such circumstances occur during wound-healing, cancer growth and morphogenesis. He demonstrated that there is a feedback between the cell shape at the culture contour and the motile activity of the cell. This feedback can give rise to a Turing-type instability, resulting in the spontaneous formation of cellular "fingers". When these fingers undergo tip-splitting the morphology can become branched. This simple mechanism may explain many observed patterns in embryogenesis, where there are multiple chemical cues that regulate this instability and refine the resulting shapes. Finally, Joshua Shaevitz discussed collective pattern formation in groups of moving bacteria. He reviewed his group's recent work on studying force production, motion control and coordination on the molecular, cellular and population scales using the model social bacterium Myxococcus xanthus, a fascinating organism that takes advantage of multi-cellular, coordinated motility to feed on colonies of bacteria in large, pack-like groups and to form giant, spore-filled fruiting bodies during starvation.

(2) Cytoskeletal dynamics: structure and mechanics. Going down to the intracellular length scale, several talks were devoted to the mechano-biochemical dynamics of the filamental actin and molecular motors that drive the machinery of cell motion and deformation. Again both experimental and theoretical approaches are well represented at the workshop.

In his talk entitled "Actin polymerization and depolymerization is critical for force generation and epithelial invagination", Adam Martin explored the role of actin networks coupled to adhesive complexes in morphogenetic processes such as epithelial invagination. While the role of actin assembly and disassembly/turnover is well established for individual cell migration, the importance of actin turnover for the coordinated movement of an epithelial sheet is not known. To examine the importance of actin turnover, Martin and coworkers performed live imaging and quantitative analysis of F-actin during *Drosophila* gastrulation. Unexpectedly, they found that F-actin levels decrease during apical constriction, suggesting net F-actin depolymerization during constriction. Using various drugs to block actin depolymerization and barbed end growth, they saw a transition from a general disruption of contractility in all cells with high doses of cytochalasin D to a mesoderm specific disruption in cell-cell connections at low doses. At low cytochalasin D concentrations, neighboring actomyosin networks continually lost and reformed connections, resulting in an unbalanced "tug-of-war" between cells of the mesoderm. They proposed that F-actin turnover is critical for apical constriction and is required for cellular contraction and to maintain attachments between cells.

Fred Mackintosh spoke about active stresses and self-organization in intra/extracellular networks. He described recent advances in theoretical modeling of such networks and in experiments on reconstituted in vitro actomyosin networks and living cells. His group was able to elucidate how internal force generation by motors can lead to dramatic mechanical effects, including strong mechanical stiffening. Furthermore, stochastic motor activity can give rise to diffusive-like motion in elastic networks. This can account for both probe particle motion and microtubule fluctuations observed in living cells. He also showed how the collective activity of myosin motors generically organizes actin filaments into contractile structures, in a multistage non-equilibrium process. This can be understood in terms of the highly asymmetric load response of actin filaments: they can support large tensions, but they buckle easily under pico-Newton compressive loads.

Len Pismen's presentation focused on strain dependence of cytoskeleton elasticity. He described the mechanosensing mechanism in actin-myosin networks based on the tension dependence of the motor detachment rate and a possibility of rupture of actin filaments under strain. Both effects, operating at different orientations to the applied strain, induce orientational anisotropy of the network. The theory is applied to explain quantitatively the drastic reduction of the elastic modulus under oscillatory strain observed in the experiment.

Dimitrios Vavylonis's study is centered on the organization of actin filaments, myosin motors, and actin filament cross-linkers and on the kinetics of this organization in single cells. His group developed numerical simulations modeling actin filaments as semiflexible polymers polymerized by formins, pulled by myosin, and represented cross-linking as an attractive interaction. The simulations show that actin cross-linkers regulate actin-filament orientations inside actin bundles and organize the actin network. These simulations reproduce experimental observations of D. Laporte and J.-Q. Wu that mutations and changes of the concentrations of cross-linker alpha-actinin in live cells causes the nodes to condense into different morphologies, forming clumps at low cross-linking and extended meshworks at high concentrations. To study the process of stress fiber formation in animal cells, they carried out 2D Monte Carlo simulations on the kinetics of stress-fiber network formation observed in experiments. This work supports a hierarchical process of self-organization involving components drawn together from distant parts of the cell followed by progressive stabilization and alignment by cross-linker and other proteins.

Yuan Lin presented another computational study of actomyosin bundles and stress fibers. Using a combined finite element-Langevin dynamics (FEM-LD) approach, he described the shape fluctuations of extended objects like bio-filament and cell membrane, and developed methods to implement a stochastic partial differential equation formulation in finite element (FEM) simulations of the mechanical behavior of bio-polymer networks where, besides entropy, the finite deformation of filament has also been taken into account. The validity of the proposed finite element-Langevin dynamics (FEM-LD) approach is verified by comparing simulation results with those from renowned FEM software as well as various theoretical predictions. As an application, they applied their method to investigate the mechanical response of an actin filament network. The results clearly and quantitatively demonstrate that entropy dictates how such actin network responds at small strains while elasticity gradually takes over as the dominant factor as deformation progresses.

(3) Cell interaction and migration through extracellular matrix. Cells migrate as part of their normal physiological function or, in the prominent case of cancer cells, to metastasize to other organs. For both the fundamental and medical significance of this process, several groups have concentrated on the process of cell-matrix interactions during cell migration.

Daphne Weihs delivered a talk entitled "Initial stages of metastatic penetration require cell flexibility and force application". She explained that the process of invasion is of special importance in cancer metastasis, the main cause of death in cancer patients. Cells typically penetrate a matrix by degrading it or by squeezing through pores. While various mechanisms of invasion have been studied, the mechanics and forces applied by cells especially during the initial stages of metastatic penetration are unknown. Her group evaluated cell-substrate mechanics when an impenetrable substrate is indented by cells in a controlled way. During indentation, cells change shape and apply mechanical forces depending on the substrate stiffness as well as on the metastatic potential of the cells. While increased metastatic potential (MP) results in cells being more pliable internally and externally, allowing rapid changes in morphology, at the same time those cells are also able to apply stronger forces. They proposed a model for the cell-indentation mechanism and highlight a special role for the nucleus.

Josef Käs raised the following questions: Do cancer cells care about physics? Are fundamental changes in a cell's material properties necessary for tumor progression? In a review of the state of the art, he noted that in tumour biology an overwhelming complexity arises from the diversity of tumours and their heterogeneous molecular composition. Nevertheless in all solid tumours malignant neoplasia, i.e. uncontrolled growth, invasion of adjacent tissues and metastasis, occurs. Recent results indicate that all three pathomechanisms require changes in the active and passive cellular biomechanics. Therefore, passive and active biomechanical behaviour of tumour cells, cell jamming, cell demixing and surface tension-like cell boundary effects are key factors to stabilize or overcome compartment boundaries. He discussed how insights into changes of these properties during tumour progression may lead to selective treatments.

Ben Fabry spoke about the physical principles of cell migration in three dimensions (3D). He described how many cancer cells, stem cells, fibroblasts, and cells of the immune system are able to migrate through dense connective tissue. Cell migration through a connective tissue matrix depends strongly on the mesh size, the mechanical properties, and the functionalization of the matrix with adhesive ligands. Moreover, different cell types employ fundamentally divergent (e.g. mesenchymal or amoeboid) migration strategies in 3D. Despite these differences, there are also striking similarities. He proposed a treatment of 3D migration as a super-diffusive random process with directional and speed persistence. Moreover, unlike migration in a 2D environment, the ability to generate traction forces and to direct these forces non-isotropically is key to understand how cells can overcome the steric hindrance of the matrix for efficient 3D migration.

(4) Dorsal closure: new data and numerical modeling. Dorsal closure (DC) is a morphogenetic process during the development of the embryo of the fruit fly *Drosophila*. It is a complex, multi-faceted process that involves close coordination among molecular signaling, intracellular and intercellular force generation and whole-tissue deformation. As such it has received much attention from developmental biologists, and more recently mathematicians and physicists have started to rationalize the observations into models.

Nicole Gorfinkiel discussed contractile activity of amnioserosa (AS) cells during dorsal closure. She pointed out that long-standing questions in developmental biology such as morphogenesis, tissue homeostasis and organ growth cannot be understood using the reductionist approach of classical genetics but will require more systems biology approaches. Thus her lab focuses on how coordinated movements of cells in the context of embryonic development emerge from the integration of events occurring at the molecular, cellular and tissue scales. Using dorsal closure as a model system and employing quantitative and biophysical tools, she presented data showing that the AS generates one of the main driving forces of DC through the apical contraction of its individual cells. Apical contraction in AS cells is pulsed with oscillations in apical cell area correlating with the activity of dynamic and intermittent actomyosin networks that flow across the apical cortex of these cells. But how do AS cells switch from a predominantly fluctuating behaviour to a predominantly contractile behavior? Quantitative analysis suggested that the decision to contract is taken at the single cell level, and that both changes in the architecture of the actin cytoskeleton and in the activity of Myosin II motors underlie the transition from a fluctuating to a contractile behavior.

Jerome Solon continued on the theme of tissue contraction during DC. With high-resolution selective plane illumination microscopy (SPIM), Solon's group has obtained a 3D view of the cellular remodeling occurring during DC. This revealed three different phases of contractions. During the first phase, the cells remain flat and elongated and generate pulses of contraction to power the closure. The arrest of these pulses correlates with the start of a second phase where the tissue is now remodeling itself by shrinking the volume of individual cells. At the same time, they observed the occurrence of cell delamination. The volume shrinkage and cell delamination determine the dynamics of DC at that stage and allow the proceeding of the closure. In the last phase, the cells undergo a squamous-columnar transition by elongating towards the basal surface and will eventually invaginate. A key finding is that the cells exert a tight mechanical control of DC by means of shrinking and cell apoptosis in a complementary fashion and that the amnioserosa tissue adopts different strategies to contract upon experiencing tension-pressure during the process.

Shane Hutson's work tested the hypothesis that oscillatory changes in AS apical area are driven by stretch-induced contractions—with the contraction phase of one cell driven by the contraction of neighboring cells. Using holographic laser-microsurgery to nearly instantaneously isolate a single amnioserosa cell, he found that not all isolated cells immediately collapse their apical area. Cells that were expanding just prior to isolation pause or even continue expansion for 30-60 seconds before collapsing. These results contradict a previous quantitative model for the cell shape oscillations that coupled stretch-induced contractions with a highly strained epithelium. The new results instead suggest that oscillatory shape changes are more cell autonomous and that the tensed epithelium is nevertheless under low strain. Hutson then presented a revised model that is able to reconcile prior experiments with the new results.

The talk of Dan Kiehart focused on new observations that show a unique frequency behavior of amnioserosa cells that ingress during the course of closure. Using molecular and classical genetics, pharmacology, laser surgery and biophysical modeling, his group has investigated cell and tissue kinematics (cell shape changes and movements) and dynamics (the forces that drive such kinematics) during dorsal closure. A variety of signaling pathways are required for closure, and he discussed how these pathways regulate the assembly and function of these distinct actomyosin arrays in order to generate the emergent behaviors that drive closure.

Feng and coworkers presented a mathematical model on dorsal closure that strives to incorporate the key experimental observations. They represent the amnioserosa by 81 hexagonal cells that are coupled mechanically through the position of the nodes and the elastic forces on the edges. Besides, each cell has radial spokes on which myosin motors can attach and exert contractile forces on the nodes, the myosin dynamics itself being controlled by a signaling molecule. In the early phase, amnioserosa cells oscillate as a result of coupling among the chemical signaling, myosin attachment/detachment and mechanical deformation of neighboring cells. In the slow phase, we test two "ratcheting mechanisms" suggested by experiments: an internal ratchet by the myosin condensates in the apical medial surface, and an external one by the supracellular actin cables encircling the amnioserosa. The model predictions suggest the former as the main contributor to cell and tissue area reduction in this stage. In the fast phase of dorsal closure, cell pulsation is arrested, and the cell and tissue areas contract consistently. This is realized in the model by gradually shrinking the resting length of the spokes.

(5) Mathematical modeling across length and time scales. While the biologists presented latest observations and increasingly quantitative measurements, the mathematicians and physicists contributed outstanding talks on modeling the mounting and sometimes bewildering experimental data. A few notable highlights are given below.

Alain Goriely gave a systematic review of the state of the art in his talk entitled “Mathematical methods and challenges in the theory of biological growth". The review covered ten types of models, from the conceptually simple to the technically formidable. These include: fluid flows, plastic flows, a nonlinear solid with evolving reference configuration, a multiplicative decomposition of the deformation gradient, a discrete assembly of interacting springs with evolving reference lengths, a set of interacting and dividing spheres, or a manifold with evolving connection and metric. Befitting the context of the Banff workshop, he identified and compared various attempts to describe growth in biological structures. Are they equivalent? Is one better suited for a certain type of analysis (experimental, mechanical, mathematical)? He pointed out the importance of matching the modeling approach to the special features of the target process and concluded with a list of mathematical/scientific challenges and open problems in the theory of growth.

Hisao Honda spoke about three-dimensional cell model for tissue morphogenesis. His starting point is the development of an equation of motion for cell behavior, which would be a powerful tool for understanding cell morphogenesis. He used a geometrical model (Dirichlet or Voronoi geometry) to describe polygonal cell patterns, and made a cell boundary-shortening model to describe an epithelial tissue. The model was useful to understand epithelial cell properties. A two-dimensional cell vertex dynamics model has been extended to 3D and applied to investigate morphogenesis of mammalian blastocyst from morula, and cell intercalation observed in sea urchin gut elongation, amphibian gastrulation, *Drosophila* germ band extension, among other processes. The 3D cell model has been useful in explaining tissue morphogenesis in terms of cell behavior, liquid secretion of cells, contraction of specific edges of cells and apical extension/contraction of cells.

Wayne Brodland focused on multi-scale modeling in his talk “From genes to morphogenetic movements". A multi-scale model was developed with the goal of tracing the sequence of causal events from gene expression to neurulation, a crucial tissue reshaping process that occurs during early embryo development. During this process, a sheet of tissue reshapes and rolls up to form a tube, the precursor of the spinal cord and brain; errors in this process are a common source of birth defects. The model assumes that, at the sub-cellular level, genes regulate the construction and operation of specific structural components. They, in turn, affect the cell level forces generated. Cell-level computational models were then developed to explore how planar collections of cells (i.e., tissues) with various properties would generate and respond to forces, and to derive a system of cell-based constitutive equations. These equations were then calibrated for specific tissue types, locations and developmental stages against tensile tests of real embryonic tissues. A whole-embryo finite element simulation of neurulation was then carried out. In this model, tissues were modeled using "super elements" that represented tens of cells and that were governed by the calibrated constitutive equations. Ultimately, it was possible to trace how genes affect structural components and how these changes subsequently affect cell properties, then tissue properties, tissue mechanical interactions, tissue motions and finally whether the resulting embryonic phenotype is normal or abnormal.

On a larger spatial scale, Kasia Rejniak presented an integrative IBCell model for normal and malignant remodeling of epithelial tissues. The computational model IBCell (Immersed Boundary model of a eukaryotic Cell) is used to first reconstruct in a quantitative way the development of a normal tissue (such as epithelial acini grown in 3D culture), and then to investigate how perturbations in model parameters lead to formation of abnormal structures (such as tumor-like acinar mutants). Epithelial cell self-organization into a hollow polarized acinus is achieved in the model by a combination of cell proliferation, cell-cell adhesion, and self-secretion of ECM proteins. In contrast, by changing the ratio between cell-cell and cell-ECM adhesion receptors expressed on cell membrane, the model produces aberrant, mutant-like morphologies. Thus, by using this computational framework to test cell intrinsic sensitivity to extrinsic cues, she identified core trait alternations in cells expressing a mutant HER2 receptor, such as a loss of negative feedback from autocrine secreted ECM. Finally, she discussed how the IBCell model can be used to investigate other cell process involved in epithelial morphogenesis and carcinogenesis.

Outcome of the Meeting

The study of tissue morphogenesis has an increasingly diverse constituency, made up of cell and developmental biologists, biophysicists, mathematicians and engineers. The workshop has included mathematical experts on modeling and computations, leading biophysicists specializing in cell-level measurements and characterization, and geneticists and developmental biologists actively exploring the connection between gene expression and tissue mechanics. Geographically, the workshop gathered scientists from Canada, US, Israel, Spain, Netherlands, Germany, France, China, Hong Kong and Japan, at different stages of their careers, and provided a rare opportunity for exchange among people who would otherwise not have interacted regularly across the disciplinary boundaries. In particular, we have included a significant number of younger researchers, with the aim of connecting them with more established research groups.

The exchange of ideas have benefited modellers as well as experimenters. The former gathered the latest observations and insights from the top labs in this field, and heard about novel processes and phenomena that may lead to new directions of research. The latter learned about new physical conceptualization of biological processes and powerful computational tools that can help make sense of experimental data. The relatively small size of the workshop facilitated many informal but in-depth discussions among the attendees. Many new connections were made that promise fruitful collaborations in the future. Although it is difficult to quantify the value and outcome of such connections, all agree that this had been a most stimulating and rewarding gathering.

On the scientific side, we have set out a number of objectives that have been successfully fulfilled:

(1) To survey the state of the art on tissue growth and morphogenesis, and highlight it as an important emerging area in applied mathematics. Experimentalists have described new observations and measurements on processes such as morphogen spreading, collective cell migration, gastrulation movements, dorsal closure, cell sorting, and force sensing. Theoreticians and numerical analysts have summarized the predictive capabilities of their models, including continuum and discrete numerical methods. The two sides have compared notes and come up with a common understanding of our current knowledge in this area.

(2) To identify the most pressing scientific problems. Experimentalists have underscored several phenomena that cannot yet be readily rationalized or explained, e.g. collective cell motion and dorsal closure. Key missing pieces in our understanding include actuation of mechanical forces by chemical signaling, and mechanical feedback on gene expression. Theoreticians have identified multi-scale modeling and computation as a critical problem, and recommended explorations of multi-scale approaches to prototypical events such as the apical constriction in epithelial invagination.

(3) To develop promising research strategies for the future. In view of the list of high-priority research problems, the workshop has conducted an inventory of the cutting-edge observational, modeling and computational techniques, and suggested strategies for tackling the problems. For example, it has become clear that modeling cell sorting and dorsal closure needs a more integrative approach that adequately reflects the intricacies of the signaling pathways. On the experimental side, quantitative measurement of intracellular forces is considered a promising route to unraveling the complexities of tissue development.

(4) To facilitate the pursuit of these strategies by initiating interdisciplinary collaborations. The workshop saw broad and lively discussions of all the theoretical developments, computational methodologies and experimental discoveries. Out of the formal and informal interactions, new opportunities have emerged for collaborations between researchers with complementary skills and expertise. In particular, younger scientists have benefited from direct interactions with more established experts, and from the opportunity to integrate or distinguish themselves in this exciting multi-disciplinary community.

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Bibliography

[1] G. Wayne Brodland, The differential interfacial tension hypothesis (DITH): a comprehensive theory for the self-rearrangement of embryonic cells and tissues, *J. Biomech. Eng.* **124** (2002), 188–197.

[2] Maryam Aliee, Jens-Christian Röper, Katharina P. Landsberg, Constanze Pentzold, Thomas J. Widmann, Frank Jülicher and Christian Dahmann, Physical mechanisms shaping the *Drosophila* dorsoventral compartment boundary, *Curr. Biol.* **22** (2013), 967–976.

[3] Christian Dahmann, Andrew C. Oates and Michael Brand, Boundary formation and maintenance in tissue development, *Nature Rev. Genetics* **12** (2011), 43–55.

[4] Katharina P. Landsberg, Reza Farhadifar, Jonas Ranft, Daiki Umetsu, Thomas J. Widmann, Thomas Bittig, Amani Said, Frank Jülicher and Christian Dahmann, Increased cell bond tension governs cell sorting at the *Drosophila* anteroposterior compartment boundary, *Curr. Biol.* **19** (2009), 1950–1955.

[5] Albert K. Harris, Is ceil sorting caused by differences in the work of intercellular adhesion? A critique of the Steinberg hypothesis, *J. Theor. Biol.* **61** (1976), 267–285.

Chapter 28

Conformal and CR Geometry (12w5072)

July 29 - August 3, 2012

Organizer(s): Spyros Alexakis (University of Toronto), Robin Graham (University of Washington), Kengo Hirachi (University of Tokyo), Paul Yang (Princeton University)

Overview of the Field

Conformal and CR geometry has been a rich topic of study going back to the work of Cartan in the early 20th century. It continues to be the source of interesting new problems as well as to spawn new developments in related areas. It is characterized as a meeting ground of researchers from a wide variety of backgrounds in geometry, analysis, and algebra.

Cartan's algebraic point of view on the topic has inspired the development of the theory of parabolic geometries, curved geometries modeled on homogeneous spaces G/P, where G is a semisimple Lie group and P a parabolic subgroup. This side of the subject draws heavily on representation theory. A general theory of parabolic geometries has been developed, and much recent effort has been devoted to the study of specific examples of the theory beyond the basic examples of the conformal and CR cases. Topics here include BGG sequences and symmetric examples of special geometries. Even for conformal and CR geometry, only recently has important progress been made towards a good understanding of holonomy reduction.

Another side of the subject also with an algebraic flavor concerns the study of asymptotics and its connection with underlying geometric objects. Poincaré-Einstein metrics were introduced formally as a tool to study conformal geometry of the boundary at infinity; an analogous correspondence is manifested in other parabolic geometries. These metrics were subsequently studied as models for the celebrated AdS-CFT correspondence in theoretical physics, and have since evolved to become an arena for a wide class of geometric and analytic questions. Challenges of an analytic flavor are to construct families of such metrics and study their dependence on the conformal structure on the boundary. Many tools that have been developed in this connection (for example the renormalized volume) have found far-reaching applications, some of which were presented in the conference.

Conformal geometry has long been a fertile area for the study of variational problems. The Yamabe problem of finding a constant scalar curvature metric in a conformal class reduces to solving an nonlinear equation involving the conformal Laplacian and remains a model problem inspiring much work in geometric analysis. More recently higher-order and fully nonlinear analogues of this variational problem have been studied, for example in connection with BransonâĂŹs Q-curvature and the σ_n-curvatures.

The conformal/CR invariant Laplacian and BGG sequences provide new set up for the index theorems. The heat kernel asymptotics of the conformal Laplacian gives a conformal index, and the conformal variation of the functional determinant of conformal Laplacian gives the Q-curvature. These studies can be extend to a lager class of invariant operators. In CR geometry, the de Rham complex can be refined to Rumin complex, which is an example of BGG sequences, and one may define the analytic torsion of the complex. This gives a variant of Ray-Singer metric in CR geometry.

The Workshop

The workshop brought together researchers from a variety of backgrounds who find common ground in conformal and CR geometry. There were 42 participants, representing a wide spread of experience levels from senior researchers to postdocs and graduate students. The number of lectures was limited to 22 to leave plenty of time for discussion and collaboration among and between the various research groups present.

A one hour expository lecture by a leading researcher known to be an excellent speaker opened the morning session of each of the three full days. Each of these lectures presented an overview of one aspect of the subject to the diverse audience and provided background for other lectures in that area. Seven lectures of 30 minutes each were delivered by young researchers on their work. The remaining twelve lectures were each of 50 minutes duration and reported on recent progress.

The discussion and interaction during the open periods was lively. It included significant mentoring of junior participants by senior researchers, in which questions were answered in detail and issues raised in the talks were elaborated. Other discussion led to new projects and collaboration. Among such new projects initiated as a result of interations at the workshop were the following.

Bent Orsted and Yoshihiko Matsumoto started a collaboration on the total Q-curvature for CR manifolds. The talk of Orsted presented a general framework on the variation of global CR invariants at the flat model, but no examples of CR invariants were discussed. It turns out that an example of such a global invariant is given by the total Q-curvature for partially integrable CR structures, as constructed via Matsumoto's ACHE metric.

Based on the talk of Jeffrey Case on the Q'-curvature on 3-dimensional CR manifolds, Hirachi found a way to define Q'-curvature in terms of the ambient metric that can be formulated in all dimensions. The integral of Q'-curvature gives a global invariant of strictly pseudo convex domains in $\mathbf{C}^n$. This provides another example of an invariant for which the framework of Orsted can be applied.

Talks

The topics covered by the workshop are well-represented by the talks given. We have divided the talks into five sometimes overlapping subject areas.

1. **Algebraic Aspects and Parabolic Geometries**

Michael Eastwood delivered a one hour lecture presenting an overview of conformal and CR geometry from the perspective of representation theory. He explained Cartan's description of geometry modeled on homogenous spaces G/P, where G is semisimple and P is a parabolic subgroup — such geometries are now called *parabolic geometries*. The examples include conformal and CR geometries. Manifolds with such structures admit invariant operators acting on irreducible bundles induced by the representations of the parabolic P; these operators play essential roles in parabolic geometry. Some of these operators form natural complexes, e.g., the de Rham complex, the deformation complex, the Rumin complex; they can be constructed in a unified manner and are called *Bernstein-Gelfand-Gelfand complexes*. There are other invariant operators that are not operators in the complexes, e.g., Yamabe operators, Paneitz operators and more generally GJMS operators. These are constructed via the Fefferman-Graham ambient metric and/or tractor calculus. These invariant operators are important objects of study and some of the subsequent talks considered various aspects related to them.

The lecture of *Bent Orsted* served to highlight how the representation theory of conformal and CR geometry can be used to study analytical problems that arise in these fields. He explained (in part based on joint work with N. M. Møller) the application of representation theory in the study of natural functionals in conformal and CR geometry. The functionals may be thought of as certain functions on the space of metrics or complex structures, and he gave examples of stationary points and local extrema. In particular for conformal and CR spheres, he treated determinants of the Yamabe operator and total Q-curvature (the integral of Q-curvature). The problem can be reduced to the study of eigenvalues of intertwinors between the bundles of metrics or infinitesimal deformations thereof and representation theory gives a complete answer for the sphere. Among other things, this provides an alternate proof of a result of Okikiolu on the determinant of Yamabe operator.

Andreas Cǎp presented results of joint work with Rod Gover concerning projective compactness of affine connections and pseudo-Riemannian metrics. They introduce a parameter measuring an order of projective compactness, which turns out to describe asymptotic behavior of volume growth near the boundary. In the case that the order is 1 or 2, they relate holonomy reductions to the condition that a defining density for the boundary be a solution of a first BGG operator.

Pawel Nurowski reported on joint work with Ian Anderson concerning explicit formulae for Fefferman-Graham ambient metrics of the conformal structures defined by Nurowski from the data of a generic 2-plane field in 5 dimensions. The explicit ambient metrics have remarkable properties in addition to the fact that they are explicit: among other things their expansions can be made to terminate at any desired order.

Katharina Neusser presented a joint work with Robert Bryant, Michael Eastwood and Rod Gover on a general method for constructing complexes of invariant differential operators on manifolds endowed with geometric structures given by bracket generating distributions. For certain cases, using the distribution, the constructed complexes provide fine resolutions of the sheaf of locally constant functions which can serve as alternatives to the de Rham complex. In the case of parabolic geometries, this construction recovers the Bernstein-Gelfand-Gelfand complexes associated to the trivial representation; however, the new constructions are relatively simple and avoid most of the machinery of parabolic geometry.

Matthias Hammerl presented a result of joint work with Cǎp, Gover and Graham. The result identifies the infinitesimal holonomy of the Fefferman-Graham ambient metric of a conformal structure with the holonomy of the normal connection of its standard tractor bundle. The result generalizes recent work of Graham/Willse concerning ambient extension of parallel tractor fields. The proof uses previous work of Cǎp/Gover expressing ambient objects in terms of tractors.

Travis Willse gave a talk concerning highly symmetric generic 2-plane fields on 5-manifolds. Cartan solved the equivalence problem for generic 2-plane fields in 5 dimensions, and in the process showed that the maximal possible symmetry group was the split real form of the exceptional group G_2. Willse's work concerned examples with large but sub-maximal symmetry group. He identified explicitly the Fefferman-Graham ambient metric of the associated Nurowski conformal structure and showed that the ambient holonomy as well as the conformal holonomy are both equal to the Heisenberg 5-group.

2. **Geometric Analysis and PDE**

Andrea Malchiodi gave a one hour talk which was an overview of both older and more recent work on the conformal and CR Yamabe problems. While the classical Yamabe problem was the most well-known (and very well-studied) PDE naturally arising in conformal geometry, its resolution left open the question of compactness of the space of solutions in the positive case. The lecture reviewed the recent positive and negative results on this question. It also introduced the analogous question in CR geometry and reviewed the results obtained there.

Pierre Albin gave a talk on the conformal surgery and compactness of isospectral surfaces. He first reviewed the work of Melrose and Osgood-Phillips-Sarnak on the compactness of isospectral closed surfaces. He then explained the work of Hassel-Zelditch on the notion of relative isospectrality for the Dirichlet Laplacian for exterior domains, making use of the identity of the trace of heat kernels. Next, the work of Borthwick and Perry on isoresonant surfaces (for which the resolvents have the same scattering poles) was recalled. He then introduced the class of bordered surfaces with funnel ends and cusp ends, and formulated the main result (joint work with Aldana and Rochon) stating that for such complete surfaces which coincide outside a compact set, the relatively isospectral ones form a compact set in the C^∞ topology. The main tool is Cheeger-Gromov criteria for compactness of metrics, which requires control of the injectivity radius. To control the cusp ends, he introduced the notion of conformal surgery, in which a delicate analysis is used to verify that the relative determinant behaves correctly under a suitable limit of the conformal surgery. This analysis reduces the compactness result to that of Borthwick-Perry discussed earlier.

Antonio Ache presented recent work on removability of singularities for obstruction-flat metrics with constant scalar curvature. The obstruction tensor, which originates in the work of Fefferman-Graham on the ambient metric, is a higher-order conformally invariant symmetric 2-tensor which exists only in even dimensions. Ache presented analogues of results that are known to hold for minimal submanifolds and for Ricci-flat metrics, using the well-known results of Leon Simon as a point of departure.

Yi Wang presented work showing how the finiteness and suitable smallness of Q-curvature for a complete conformaly flat 4-manifold controls the geometry of the manifold via an isoperimetric inequality. The lecture brought in the theory of quasiconformal mappings to construct appropriate bi-Lipschitz parametrizations.

3. **Poincaré-Einstein Manifolds**

Colin Guillarmou gave a one hour lecture which was an outline of recent work on renormalized volume for Poincaré-Einstein metrics. Starting from a discussion of PE metrics (M, g) as generalizations of hyperbolic space with an induced conformal structure on the sphere at infinity, he discussed the asymptotics of the metric g at the boundary ∂M. He then introduced the renormalized volume functional, which is well-defined when the boundary dimension n is odd. The rest of his lecture was devoted to joint work with S. Moroianu and Schlenker on the case of even-dimensional boundaries. In that case the renormalized volume is a function of the metric in the conformal class at the boundary. Its critical points and extrema relate to the well-studied v_n-functional. He concluded by showing that for even-dimensional boundaries, the space of Cauchy data at the boundary (Dirichlet and Neumann, in the asymptotic expansion at ∂M) of suitably normalized PE metrics is a Lagrangian submanifold of the cotangent space of the space of conformal structures on ∂M.

Maria del Mar González gave a talk on fractional order conformally covariant operators in the setting of Poincaré-Einstein spaces. This is a topic which has generated a lot of activity in the PDE community due to the analytic work of Caffarelli-Sylvestre. She first explained the point of view of Caffarelli-Sylvestre in which they viewed the fractional order equation as a weighted second order equation in the upper half space. She discussed her joint work with Alice Chang in which they extended the energy identity for the fractional order operators whose leading term is of the form Δ^γ, $\gamma \leq 1$, to the setting of Poincaré-Einstein spaces. She then explained her joint work with J. Qing on the solution of the analogue of the Yamabe equation for the fractional order operator. The main result is the existence of energy minimizing solutions when the Sobolev quotient is strictly smaller than that of the hyperbolic space; a particular instance arises when the boundary is non-umbilic somewhere. A key analysis is the extension of the Hopf Lemma to the setting of this equation which provides the basic boundary regularity for the minimizing solution. As a consequence, they obtain a sharp weighted Sobolev trace inequality for such Poincaré-Einstein spaces. A key open question is whether there is a positive mass theorem for this operator. In the last part of the lecture, she briefly outlined on-going joint work with Frank-Monticelli-Tan on an extension of this operator to the CR setting when the CR structure is the boundary at infinity of a complete Kähler-Einstein metric.

The lecture of *Jie Qing* introduced a new aspect of the correspondence between horospherically convex hypersurfaces whose boundary at infinity bounds a domain (at infinity), and complete conformal metrics over these subdomains.

Andreas Juhl gave a lecture concerning his "building blocks" for the GJMS operators. In previous work he showed that the GJMS operators can be expressed by universal formulae in terms of second order building block operators. In this lecture he discussed a relation between the building block operators for a metric g and the corresponding operators for the metric in one higher dimension which arises as the compactification of the associated Poincaré-Einstein metric in normal form.

Rod Gover lectured on collaborative work with Andrew Waldron and Emanuele Latini concerning a calculus for boundary problems on Poincaré-Einstein manifolds. The work studies extension problems for differential forms formulated in terms of solving suitable Proca-type problems in the bulk. Among the many consequences are holographic formulae for the conformally invariant operators and generalized Q-curvatures of Branson-Gover on the boundary at infinity.

4. **CR Geometry**

Sagun Chanillo presented a joint work with Hung-Lin Chiu and Paul Yang on the CR Paneitz operator P_4 on 3-dimensional CR manifolds. P_4 is a unique 4th order CR invariant differential operator with leading part Δ_b^2. This talk gave an intimate link between the spectrum of P_4 and the global embeddability of 3-dimensional compact CR manifolds into complex manifolds, which is a main problem in CR geometry. (For higher dimensions the embeddability is know to be equivalent to the integrability of the CR structure.) Chanillo discussed the proof that if P_4 is non-negative and the CR-Yamabe constant is positive, then the CR manifold is embeddable. He also discussed the converse.

Jan Slovák presented a joint work with Gerd Schmalz on free CR-distributions. These are a geometric structure generalizing generic real n^2-codimensional submanifolds in $\mathbf{C}^{n+n^2}$. This is another instance of parabolic geometry; in the lowest dimensional case $n = 1$, it agrees with usual 3-dimensional CR manifolds. Free CR-distributions admit properties very similar to conformal and CR geometries: for example, the Fefferman construction gives a natural circle bundle with a conformal structure, which is modeled on skew-Hermitian matrices in the same way as the conformal 4-dimensional case. The construction of basic invariants and invariant operators of the geometry is carried out by using the classical exterior calculus and the cohomological data known for the parabolic geometry defined by these distributions.

Jeffrey Case lectured on a collaboration with Paul Yang, in which they considered a variant of the Paneitz operator P_4 on 3-dimensional CR manifolds. Based on the fact that $\ker P_4$ contains all CR pluriharmonic functions, they defined a 4th order differential operator P_4' acting on pluriharmonic functions — it generalizes the operator discovered by Branson, Fontana and Morpurgo on the sphere S^3. As an analog of the derivation of Q-curvature from P_4, they defined a Q'-curvature from P_4' and studied the question of finding a contact form with constant Q'-curvature.

Yoshihiko Matsumoto presented recent results on the asymptotics of asymptotically complex hyperbolic Einstein (ACHE) metrics, the CR counterpart of well-studied asymptotically hyperbolic Einstein metrics. An important aspect regarding ACHE metrics is that the boundary CR structures are just "partially integrable" in general, and it is surprising that a local obstruction to the existence of smooth solutions to the Einstein equation can occur only for non-integrable structures. As an application, he constructed CR invariant powers of the sub-Laplacian and Q-curvature for partially integrable CR manifolds via the Graham-Zworski approach.

5. **Index Theory**

Xianzhe Dai discussed his joint work with X. Huang on intersection R-torsion for manifolds with conical singularities. He first reviewed the half analytic torsion of Cheeger as a conformal invariant for even dimensional manifolds. The Ray-Singer conjecture/theorem says that the analytic torsion equals the R-torsion for closed manifolds. For manifolds with conical singularities, the analytic torsion can be defined using Cheeger's L^2 theory while the R-torsion is defined in terms of the intersection cohomology theory of Goresky-MacPherson. Dai and Huang showed that there is a new geometric contribution from the conical singularity in the Ray-Singer conjecture. They then prove a formula for the intersection R-torsion of a finite cone and use it to introduce a family of spectral invariants which is closely related to Cheeger's half torsion, a concept which has found application in conformal geometry.

Zhiqin Lu spoke about his joint work with Chiung-ju Liu on the Tian-Yau-Zelditch (TYZ) expansion of the Bergman kernel on singular Riemann surfaces. It is well-known that the Bergman kernel defined from the sections of the m-th powers of an ample bundle over a smooth manifold has an expansion in powers of m, called the TYZ expansion. As a first step to generalize the TYZ expansion to a degenerating family of compact Riemann surfaces, they tried to give upper and lower C^0-estimates of the Bergman kernel, which amounts to studying the leading term of the expansion. A proof was given for the case when a single Riemann surface in the family has one ordinary double point.

Participants

Antonio Ache, University of Wisconsin
Pierre Albin, University of Illinois Urbana-Champaign
Spyros Alexakis, University of Toronto
Eric Bahuaud, Stanford University
Andreas Čap, University of Vienna
Jeffrey Case, Princeton University
Alice Chang, Princeton University
Sagun Chanillo, Rutgers University
Jih-Hsin Cheng, Academica Sinica Taipei
Xianzhe Dai, University of Calfornia Santa Barbara
Luca di Cerbo, Duke University
Michael Eastwood, Australian National University
Maria del Mar González, Universitat Politècnica de Catalunya
A. Rod Gover, University of Auckland
Robin Graham, University of Washington
Colin Guillarmou, École Normale Supérieure Paris
Matthew Gursky, University of Notre-Dame
Matthias Hammerl, University of Vienna
Kengo Hirachi, University of Tokyo
Dmitry Jakobson, McGill University
Thalia Jeffres, Wichita State University
Andreas Juhl, Uppsala University
Zhiqin Lu, University of Calfornia Irvine
Farid Madani, Universität Regensburg
Andrea Malchiodi, SISSA
Yoshihiko Matsumoto, University of Tokyo
Stephen McKeown, University of Washington
Brendan McLellan, Center for Quantum Geometry of Moduli Spaces
Katharina Neusser, Australian National University
Pawel Nuroski, University of Warsaw
Bent Ørsted, Aarhus University
Jie Qing, University of California Santa Cruz
Nicholas Reichert, Princeton University
Katja Sagerschnig, Australian National University
Jan Slovák, Masaryk University
Petr Somberg, Charles University
Vladimír Souček, Charles University
George Sparling, University of Pittsburgh
Dennis The, Australian National University
Yi Wang, Princeton University
Travis Willse, Australian National University
Paul Yang, Princeton University

Talks

The following is the list of talks, in the order they were delivered.

Colin Guillarmou: On the renormalized volume
Jie Qing: Hypersurfaces in hyperbolic space and conformal metrics on domains in sphere
Pierre Albin: Compactness of relatively isospectral sets of surfaces
Xianzhe Dai: Cheeger's half torsion and cone
Zhiqin Lu: On the Tian-Yau-Zelditch expansion on Riemann surfaces
Michael Eastwood: Conformal and CR geometry from the parabolic viewpoint
Andreas Juhl: On the building blocks of GJMS-operators
Bent Orsted: Extremal properties of natural functionals in conformal and CR geometry
Pawel Nurowski: More explicit Fefferman-Graham metrics with G_2 holonomy
Jan Slovák: Free CR-distributions
Andreas Căp: Projective compactness
Rod Gover: Conformal geometry, holography, and boundary calculus
Matthias Hammerl: Ambient and conformal holonomy
Kathatina Neusser: Some complexes of differential operators
Andrea Malchiodi: Recent progress on the Yamabe problem
Sagun Chanillo: Embedding CR 3-manifolds
Maria del Mar González: Fractional order operators in conformal geometry
Yi Wang: Quasiconformal mappings, isoperimetric inequality and finite total Q-curvature
Antonio Ache: Asymptotics in the study of obstruction-flat metrics
Yoshihiko Matsumoto: Asymptotics of ACH-Einstein metrics
Jeffrey Case: A Paneitz-type operator for CR pluriharmonic functions
Travis Willse: Highly symmetric generic 2-plane fields on 5-manifolds and Heisenberg 5-group holonomy

Chapter 29

Workshop on Syzygies in Algebraic Geometry, with an exploration of a connection with String Theory (12w5117)

August 12 - 17, 2012

Organizer(s): Lawrence Ein (University of Illinois at Chigago), David Eisenbud (University of California, Berkeley), Gavril Farkas (Humboldt Universität zu Berlin), Irena Peeva (Cornell University)

Overview of the Field

Free resolutions are often naturally attached to geometric objects. A question of prime interest has been to understand what constraints the geometry of a variety imposes on the corresponding Betti numbers and structure of the resolution. In recent years, free resolutions have been also impressively used to study the birational geometry of moduli spaces of curves.

One of the most exciting and challenging long-standing open conjectures on free resolutions is the Regularity Eisenbud-Goto Conjecture that the regularity of a prime ideal is bounded above by its multiplicity. The conjecture has roots in Castelnuovo's work, and is known to hold in only a few cases: the Cohen-Macaulay case, for curves, and for smooth surfaces.

In the 80's, Mark Green [11], [12] conjectured that the the minimal resolution of the canonical ring a non-hyperelliptic curve X of genus g satisfies the N_p property if and only if the Clifford index of X is greater than P. Recall that N_1 says that the homogenous ideal of X in $\mathbf{P}^{g-1}$ is generated by quadrics. For $p \geq 2$, N_p means that the first $p-1$ steps of the minimal resolution of the homogenous ideal of X are given by matrices of linear forms. Using geometry of the Hilbert schemes of K3 surfaces, Voisin [20], [21] showed that the Green conjecture holds for generic curves. In the last few years, by the work of Schreyer, Texidor, Aprodu and Farkas [17]. [19], [1], [2], we know that Green's conjecture holds also in many other cases. For higher dimension variety, the spectacular recent work of Birkar, Cascini, Hacon and Mckernan [4] showed that the canonical ring of a smooth projective variety is always finitely generated, but we also know that the canonical ring may require generators of very high degrees. An alternative approach to study the syzygies of higher dimensional varieties would be to study the asymptotic syzygies of embeddings given by very positive line bundles. For curves, a conjecture of Green and Lazarsfeld [13] would predict essentially that the shape of the minimal resolution of a very high degree embedding is controlled by the gonality of the curve. The conjecture is again known for many cases by the work of Aprodu and Voisin [3], [1]. In higher dimensions, our knowledge is quite limited. Pareschi [16] has done very nice work on the N_p-property of abelian varieties. Snowden [18] has obtained interesting results for Segre varieties. Ottoviani and Paoletti [14] have proved sharp results on the N_p-property of the Veronese embedding of $\mathbf{P}^2$. See also the work of Bruns, Conca and Römer [5], [6]. In [7], Ein and Lazarsfeld showed that if X is smooth complex projective variety of dimension n and L_d is a line bundle of of the form $L_p = \mathcal{O}_X(K_X + (n+1+p)A + B)$ where A is very ample, B is nef and p is a nonnegative, then L_p satisfies the property N_p . More generally, we can consider a line bundle of the form $L_d = \mathcal{O}_X(dA + B)$, where A is an ample divisor and B is a given fixed divisor. Let $r(L_d) = h^0(L_d) - 1$. Observe that with $d >> 0$ one has

$$r(L_d) \ = \ O(d^n).$$

In particular, the length of the minimal resolution $E_\bullet(X, L_d)$ for the coordinate ring of X grows like a polynomial of degree n in d. So in dimensions two and higher, statements such as the one above that are linear in d ignore most of the syzygy modules that occur. It therefore seemed interesting to ask whether one can say anything about the overall shape of $E_\bullet(X, L_d)$ for $d \gg 0$. Denote by $K_{p,q}(X, L_d)$ be the space of minimal generators of $E_P(X, L_d)$ of degree $p + q$. Let S be the polynomial ring of $\mathbf{P}^{r_d}$, then

$$E_p(X; L) \ = \bigoplus_q \ K_{p,q}(X; L_d) \otimes_k \ S(-p-q).$$

By elementary consideration, one sees that

$$K_{p,q}(X, L_d) = 0 \ \text{ for } \ \ q > n+1.$$

In [8], Ein and Lazarsfeld obtain the following non-vanishing theorem. They show that there are constants $C_1, C_2 > 0$ with the property that if d is sufficiently large then

$$K_{p,q}(X, L_d) \neq 0$$

for every value of p satisfying

$$C_1 d^{q-1} \leq p \leq r_d - C_2 d^{n-1}.$$

This result gives the rough shape of the minimal resolution. Ein and Lazarsfeld further conjecture that these results are asymptotically sharp. This means that we expect a vanishing result which says that if we fix an integer q such that $2 \leq q \leq n$, then we can find a positive constant C with the property that if d is sufficiently large then

$$K_{p,q}(X, L_d) = 0 \ \text{ for } \ p \leq C d^{q-1}.$$

In the case for $X = \mathbf{P}^n$ and $L_d = \mathcal{O}_{\mathbf{P}^n}(d)$, one has very precise conjecture. This case is particularly interesting, because one can study the question using techniques from the cohomology of homogenous vector bundles and representation theory. In particular, in the very interesting work [15] Ottaviani and Rubei study these bundles using quiver representations. Another natural problem is determine the size of the Betti numbers of the minimal resolution. In a recent preprint, Ein, Erman and Lazarsfeld [9] show that a random Betti table with a fixed number of rows, sampled according to a uniform choice of Boiz-Söderberg coefficients [ES], for a fixed q, the distribution of $k_{p,q}$ would converge to a normal distribution after some normalization. They conjecture that as $d \to \infty$ the distribution of $k_{p,a}(X, L_d)$ would also converges to a normal distribution. This is know to hold when X is a curve. Assuming that K_X is ample, it would also be natural to ask whether one can understand the intrinsic geometry of X by studying the syzygies of modules of the form $\bigoplus_s H^0((mK_X + sL_d)$, where m is a positive integer. This would parallel the Green-Lazarsfeld gonality conjecture for curves.

The second focus in the workshop was matrix factorizations and their applications in physics. We organized a 10-lecture short course on that topic. A "matrix factorization" (of size n) of a function f (say on some variety) is a pair of $n \times n$ matrices A, B such that $AB = f \cdot \mathrm{Identity}$. A classic example is the expression of the determinant of a matrix as the product of the matrix with its cofactor matrix. Eisenbud introduced matrix factorizations in the 80's to describe the asymptotic behavior of minimal free resolutions over hypersurfaces in affine or projective space (or, equivalently, Cohen-Macaulay modules over hypersurfaces). For example, if S is polynomial ring and $f \in S$, any free resolution over $S/(f)$ becomes eventually periodic of period 2, with maps given by matrices $\overline{A}, \overline{B}$ such that any lifting to matrices A, B over S is a matrix factorization of f. In recent years it was discovered by physicists that matrix factorizations were useful in several new contexts:

- To provide supersymmetric boundary conditions for Landau-Ginzburg theories;
- To describe objects of the category of D-branes related to superpotentials in Landau-Ginzburg theories;
- As a tool in the theory of noncommutative crepant resolutions of singularities.

Recent developments on Free Resolutions and related topics

In this section we provide short summaries of recent research conducted by participants in the workshop. They give an excellent overview of some of the current research in the area. The summaries are ordered alphabetically by the last name of the author.

Christine Berkesch's research focuses on homological invariants in combinatorial algebraic geometry, specifically within the areas of hypergeometric systems in D-module theory and free resolutions in commutative algebra.

Hypergeometric series include many familiar functions, such as trigonometric and Bessel functions. Their importance stems not only from their appearances throughout mathematics, but physics and engineering as well. Their differential annihilators can be studied via algebraic analysis, and due to work of Gelfand, Kapranov, and Zelevinsky, they fit nicely into the context of toric geometry. Berkesch is working with J. Forsgård and L. Matusevich in order to obtain a complete and computationally explicit understanding of the solution spaces of these equivariant hypergeometric systems, or GKZ-systems. She is also establishing with L. Matusevich and U. Walther a framework to translate essential properties between GKZ-systems and the classical hypergeometric systems studied by Euler, Gauss, Appell, and Horn, among others. The methods for these projects involve integrating homological and combinatorial tools with complex analysis and tropical geometry. In addition, Berkesch is considering with E. Miller and S. Griffith a generalization of GKZ-systems due to Kapranov, which replaces the torus with an arbitrary reductive group. The goal of this project is to construct a theory of holonomic families and Euler–Koszul homology in this setting, generalizing the work of Matusevich, Miller, and Walther.

While computer algebra systems can compute individual examples of these homological objects, classifying their collective properties is a powerful approach to understanding many structural and enumerative invariants in algebraic geometry. In this area, Berkesch has focused on describing the numerics of free resolutions for several rings and algebraic varieties, including local rings, projective hypersurfaces, and, most recently, toric varieties. She is currently working with D. Erman and G.G. Smith to understand complexes over a Cox ring that have irrelevant homology. While various methods are employed, the answers sought parallel recent ground-breaking results of Eisenbud and Schreyer for projective space, which not only confirmed the long-standing Multiplicity Conjectures of Herzog, Huneke, and Srinivasan, but also exhibited a beautiful duality between the numerics of free resolutions and the cohomology of coherent sheaves.

Mats Boij together with Jonas Söderberg initiated the study of the cone of Betti tables of graded Cohen-Macaulay modules over the polynomial ring. The original purpose was to prove the Multiplicity Conjecture by Huneke and Srinivasan, and they proposed a conjectural structure of the cone which would imply this conjecture. These conjectures were proven by David Eisenbud and Frank-Olaf Schreyer and later Boij and Söderberg could extend these results to the case of modules that are not necessarily Cohen-Macaulay. In order to get a better understanding of the situation when the polynomial ring comes with a different grading Boij has has been working together with Gunnar Fløystad on the bigraded case and together with Gregory G. Smith on the cone of Hilbert functions in other gradings.

Together with Juan Migliore, Rosa Maria Miró-Roig, Uwe Nagel and Fabrizio Zanello, Boij has studied graded level algebras. In particular they wrote "*On the shape of a pure O-sequence*" which addresses several aspects of monomial level algebras.

Together with Juan Migliore, Rosa Maria Miró-Roig and Uwe Nagel, Boij has studied the Weak Lefschetz property of Artinian complete intersections.

Together with Fabrizio Zanello, Boij worked on consequences of Green's Hyperplane Restriction Theorem on h-vectors of Gorenstein algebras.

Jesse Burke's research interests are in complete intersection rings. One particular focus has been using matrix factorizations to study the singularity category of a complete intersection. Using a theorem of Orlov, the singularity category of a complete intersection is equivalent to the singularity category of a suitably defined non-affine hypersurface Y. In detail, Y is the hypersurface in a smooth scheme X defined by the vanishing of a regular global section W of a line bundle $\mathcal{L}$ on X. The singularity category of Y is, in turn, equivalent to the homotopy category of "twisted" matrix factorizations associated to the triple $(X, \mathcal{L}, W)$. An object in the latter category consists of a pair of algebraic vector bundles E_1, E_0 on X and morphisms $E_1 \to E_0$ and $E_0 \to E_1 \otimes \mathcal{L}$ such that each of the two evident compositions is multiplication by W.

Using this machinery, Mark Walker and Burke have redeveloped the theory of stable supports for modules over complete intersections in a more geometric manner. Additionally, from such a matrix factorization a free resolution may be constructed. Exploring these resolutions is another topic of interest for Burke.

Finally, Burke is interested in Boij-Söderberg theory. In recent work with Christine Berkesch, Daniel Erman and Courtney Gibbons, Boij-Söderberg theory was extended from the case of a polynomial ring to a simple class of hypersurface rings.

Andrei Căldăraru's research is concentrated in two main areas. Both directions of research are influenced by the study of open problems in physics, especially mirror symmetry and string theory. The first direction of research is focused on the interactions of string theory with birational geometry (a fairly classical subject in algebraic geometry). The second direction involves applying ideas and techniques of a very new field, derived algebraic geometry, to the study of problems in algebraic geometry, some new (inspired by string theory), and some old (the Hodge-de Rham degeneration).

One of the main ideas that string theory has brought to modern algebraic geometry is that we should not always study individual spaces separately, but rather study them up to the equivalence relation on smooth projective varieties given by the notion of so-called derived equivalence. There is another important equivalence relation in algebraic geometry, that of birationality, and this relationship has been extensively studied for over two centuries. A major question is to understand the relationship between these equivalence relations. In joint work with Lev Borisov, Căldăraru constructed the first known example of two non-birational Calabi-Yau varieties of dimension three which are derived equivalent. He is currently interested in understanding the gauged non-linear sigma models from physics, which appear to explain when such examples can appear.

In a second direction, Căldăraru has been working on understanding fundamental properties of the Hochschild homology and cohomology of algebraic varieties. These are fundamental invariants which, in the case of the spaces that appear in mirror symmetry (Calabi-Yau's) are mirror to the usual, singular homology and cohomology. Căldăraru introduced the so-called Mukai pairing on Hochschild homology, which is the analogue of the usual Poincaré pairing on cohomology, and showed that it satisfies many remarkable properties. At the moment Căldăraru is studying interpretations of Hochschild (co)homology from the perspective of derived algebraic geometry, in particular their relationship with derived intersection theory. In a related area, Căldăraru is expecting that ideas from derived intersection theory can be applied to give geometrical interpretations of results of Deligne-Illusie on the degeneration of the Hodge-de Rham spectral sequence and similar results by and Barannikov-Kontsevich-Sabbah on the twisted de Rham complex.

Izzet Coskun discovered positive, geometric rules for computing restriction coefficients for all classical flag varieties. The symplectic and orthogonal flag varieties embed in the ordinary flag variety by inclusion. The induced map in cohomology carries information important in representation theory, combinatorics and algebraic geometry. The image of a Schubert class under this map is a non-negtaive sum of Schubert classes, whose coefficients are called restriction coefficients. Using degenerations, a positive, geometric rule for computing these coefficients was discovered.

The Hilbert scheme of points $\mathrm{Hilb}_n(P^2)$ parameterizes length n subschemes of P^2. $\mathrm{Hilb}_n(P^2)$ is a smooth, irreducible variety of dimension $2n$ that contains the locus of n unordered points as a dense open subset. $\mathrm{Hilb}_n(P^2)$ is a very important parameter space that plays a crucial role in algebraic geometry, combinatorics and mathematical physics. In joint work with, Daniele Arcara, Aaron Bertram and Jack Huizenga, Coskun studied the birational geometry of $\mathrm{Hilb}_n(P^2)$, describing the ample and effective cones, the stable base locus decomposition and the explicit sequence of flips and divisorial contractions between the models. Most interestingly, modular interpretations of the models that occur in MMP were discovered. For small n, all the models are moduli spaces of Bridgeland stable objects in the derived category. In fact, there is a very precise correspondence between Mori walls and Bridgeland walls in the stability manifold. In joint work with

Aaron Bertram, these results have been extended to other surfaces such as $P^1 \times P^1$ and Hirzebruch surfaces.

Steven Dale Cutkosky has made a study of asymptotic properties of ideals. Although various properties of ideals such as regularity can be very complicated, they do tend to behave well for large powers. An example is the regularity of homogeneous ideals, which becomes a linear polynomial for large n, as shown by Cutkosky, Herzog and Trung, and independently by Kodiyalum. Finally, this is a reflection of the fact that the Rees algebra of powers of ideals is a finitely generated algebra. There are other interesting numerical functions of ideals for which one can also ask if they have a good expression for high powers of an ideal. An example is the regularity of saturated powers. Geometrically, saturation corresponds to the notion of sections over a punctured neighborhood of a point of an ideal sheaf. The algebra of saturated powers of an ideal is in general not finitely generated, so we can not expect as good an answer in this case. However, the limit of the regularity of the saturation of I^n divided by n always exists, as was shown by Cutkosky, Ein and Lazarsfeld. However, Cutkosky gave an example that shows that even for a homogeneous prime ideal of a smooth curve in P^3, the limit can be irrational.

Cutkosky has shown with Ha, Srinivasan and Theodorescu that the lengths of the quotients of the saturated powers by the ordinary powers of a homogeneous ideal has a limit, and that this limit can be irrational. More recently, Cutkosky has extended this theorem to local rings, and generalized saturation. If I and J are ideals in a ring R, then the generalized saturated power $I_n(J)$ is the saturation $I^n : J^\infty$, which corresponds to the geometric notion of taking section of I^n on the complement of the vanishing locus of the ideal J. Specifically, if R is a local ring with some very mild conditions (for instance a local ring of a point on an algebraic variety over a field of characteristic zero, or a local ring of a complex analytic variety) and I, J are ideals in R, then the limit of multiplicities of quotients of the generalized saturated powers of I by the ordinary powers has a limit.

Cutkosky's recent research also includes simplification of algebraic mappings and resolution of singularities.

Hailong Dao works on a number of questions in commutative algebra. One of his most recent works involves new invariants and classifications of subcategories of modules over a commutative noetherian ring. With Takahashi, he classified all resolving categories of modules over a complete intersection R using the "grade consistent functions" from $\mathrm{Spec} R$ to non-negative integers.

Another recent topic is about non-commutative desingularizations. He proved that an isolated hypersurface singularity in dimension three which is a UFD admits no non-commutative crepant resolutions in the sense of Van den Bergh. He and Huneke gave a simple proof of a characterization for all such resolutions over certain cA_n singularities. Most recently, he and Iyama, Takahashi, Vial proved that over certain algebras R, the existence of a faithful module whose endomorphism ring has finite global dimension forces R to have only rational singularities, extending work by Stafford-Van den Bergh.

Another topic is about solving simple equations in the semi-ring of vector bundles over the punctured spectrum of a regular local ring. He showed that some weak form of multiplicative cancellation holds, for example the equation $XY = nY$ for $Y \neq 0$ and an integer n yields $X = n$.

Daniel Erman has been working on three recent projects on the structure of free resolutions and their applications to algebraic geometry.

First, David Eisenbud and Daniel Erman provide a robust categorical foundation for the duality theory introduced by Eisenbud and Schreyer to prove the Boij-Söderberg conjectures describing numerical invariants of syzygies. The new foundation extends the reach of the theory substantially. More explicitly, Eisenbud and Erman construct a pairing between derived categories that simultaneously categorifies all the functionals used by Eisenbud and Schreyer.

Second, Lawrence Ein, Daniel Erman and Rob Lazarsfeld prove a Law of Large Numbers type result for high degree syzygies. They present a conjecture to the effect that the ranks of the syzygy modules of a smooth projective variety become normally distributed as the positivity of the embedding line bundle grows. Then to render the conjecture plausible, they prove a result suggesting that this is in any event the typical behavior from a probabilistic point of view.

Third, Daniel Erman and Melanie Matchett Wood prove a generalized Bertini theorem over a finite field, thus extending a result of Poonen from the ample case to the semiample case. Free resolutions, Castelnuovo–

Mumford regularity, and other homological techniques play an essential role in the proof.

Gunnar Fløystad has been recently working on the Boij-Söderberg theory. Pure free resolutions are graded free resolutions over the polynomial ring S of the form

$$S(-d_0)^{\beta_0} \leftarrow S(-d_1)^{\beta_1} \leftarrow \cdots \leftarrow S(-d_r)^{\beta_r}.$$

Their Betti diagrams have proven to be of fundamental importance in the study of Betti diagrams of graded modules over the polynomial ring. Their significance were put to light by the Boij-Söderberg conjectures which describes such Betti diagrams, up to multiplication by a rational number. Theses conjectures were later proven in full generality by D.Eisenbud and F.-O. Schreyer.

More generally, a complex $F^\bullet$ of free modules over the polynomial ring S, for instance a free resolution, comes with a triplet of numerical homological invariants: i) Its graded Betti numbers, ii) the Hilbert functions of its homology modules, and iii) the Hilbert functions of the cohomology modules, where these modules are defined as the modules of the dual complex $Hom_S(F^\bullet, \omega_S)$. The current research of Fløystad is inspired by the following general problem: Up to rational multiple, what sets of triplets of numerical homological invariants can occur?

Fløystad is currently working on the following approach: On the category of graded S-modules, there is the standard duality functor $\mathbf{D} = Hom_S(-, \omega_S)$. Restricting to a subcategory of graded S-modules, the squarefree S-modules, there is also Alexander duality $\mathbf{A}$. The composite functor $\mathbf{A} \circ \mathbf{D}$ *rotates the three homological invariants*, and has order three on the derived category of squarefree modules, up to translation. He then studies complexes of free squarefree modules $F^\bullet$ such that (when considered as singly graded modules), both $F^\bullet, \mathbf{A} \circ \mathbf{D}(F^\bullet)$ and $(\mathbf{A} \circ \mathbf{D})^2(F^\bullet)$ are pure. This is called a *triplet of pure complexes*. (That $F^\bullet$ is a pure resolution of a Cohen-Macaulay squarefree module, the classical case, corresponds to the second and third complex being linear.) Such complexes have unique sets of Betti numbers (just as in the classical case) up to scalar multiple. He has partly been able to construct such triplets of complexes using recent constructions of Berkesch, Erman, Kumini, and Sam of tensor complexes. Inspired by their approach he is furthermore working on transferring the conjectured existence of triplets of pure complexes, to a conjecture on the existence of certain classes of Tate resolutions over the exterior algebra.

Hans-Chrsitian Graf v. Bothmer recently had a joint project with Sosna. The bounded derived category of coherent sheaves $D^b(X)$ on a smooth projective variety X (always over $\mathbf{C}$ in the following) may be viewed as a categorification of the Grothendieck group of X or the Chow ring of X, both of which tend to be very intricate objects in their own right already. Moreover, according for example to Kontsevich, Orlov and Tabuada there the intuition that $D^b(X)$ should be a version of the non-commutative motive of X, with decompositions of the (classical) Chow motive $h(X)$ of X being reflected in a suitable sense by semi-orthogonal decompositions of $D^b(X)$. Recently (see, for example Rouquier, Kawamata and Kuznetsov) a number of results as well as conjectures try to link semi-orthogonal decompositions in derived categories to the birational geometry of X, including very subtle features such as the rationality or irrationality of X which do not seem to be detected by sheaf-cohomological (non-categorical) data. However, the best understood examples considered so far mainly consist of varieties close to the toric and rational-homogeneous ones as well as some Fano hypersurfaces. It seems however, that the optimism radiated by existing conjectures is not reflected in the data one can sample from these varieties.

It follows from Serre duality that the derived category of a Calabi-Yau manifold is indecomposable, that is, it does not admit any non-trivial semi-orthogonal decomposition. Furthermore, according to Okawa, varieties of general type with globally generated canonical bundle do not have exceptional objects. However, on surfaces of general type X with $p_g = q = 0$ every line bundle is exceptional and one may hope that interesting semi-orthogonal decompositions exist which may yield a nontrivial testing ground for existing conjectures. The classical Godeaux surface is such an example. Guletskii and Pedrini have shown that the Chow motive of the classical Godeaux surface splits as a direct sum of Lefschetz motives $h(X) \simeq 1 \oplus 9\mathbf{L} \oplus \mathbf{L}^2$. The Grothendieck group of X is $\mathbf{Z}^{11} \oplus \mathbf{Z}/5$, hence X does not admit a full exceptional sequence, since the existence of the latter would imply that the Grothendieck group is free. One may therefore conjecture that $D^b(X)$ has an exceptional sequence of length 11 corresponding to the "trivial commutative part of the motive" and some nontrivial genuinely non-commutative semi-orthogonal complement to this sequence. This expectation turns out to be correct and is the main result of Böhning, v. Bothmer and Sosna; they prove that if X is the classical Godeaux surface, then there exists a semi-orthogonal decomposition $D^b(X) = \langle \mathcal{A}, \mathcal{L}_1, \ldots, \mathcal{L}_{11} \rangle$ where $(\mathcal{L}_1, \ldots, \mathcal{L}_{11})$ is an exceptional sequence of maximal length consisting of line bundles on X and $\mathcal{A} \neq 0$ is the right orthogonal to this sequence. The existence of the above decomposition answers Kuznetsov's Non-vanishing Conjecture about the Hochschild homology of an admissible subcategory, in the negative. In fact, the Hochschild homology of $\mathcal{A}$ is zero, but $\mathcal{A}$ itself is not. Böhning, v. Bothmer and Sosna also produce explicit nonzero objects in $\mathcal{A}$.

Remke Kloosterman is mostly interested in the syzygies of the ideal of the singular locus of projective varieties. In the case of (reducible) cuspidal plane curve C of degree d he showed that the maximal degree for a syzygy of $\sqrt{I(C_{sing})}$ is $5/6d$ and that the number of syzygies of maximal degree equals half the degree of the Alexander polynomial of C. He used this to find a new upper bound for the degree of the Alexander polynomial of a cupsidal plane curve.

Moreover, he showed that the equisingular and equianalytic deformation space of a singular plane curve of degree at least 13 with nonconstant Alexander polynomial is not T-smooth, i.e., the tangent space to these deformation spaces have bigger dimension than expected.

In the case of a degree d nodal hypersurface $X \subset PP^{2k}$ he determined the minimal number of nodes $n(k,d)$ such that $h^{2k}(X) > 1$ holds: A general hyperplane section X_H of X is a smooth hypersurface in PP^{2k-1} such that the Hodge structure on $H^{2k-2}(X_X, ZZ)_{prim}$ contains a onedimensional sub-Hodge structure of type $((k-1)/2, (k-1)/2)$. In particular, X_H corresponds to a point of the Noether-Lefschetz locus $NL_{d,2k-1}$. By the work of Green, Voisin and Otwinowska one can associate an ideal with the tangent space of the Noether-Lefschetz locus of X_H. By comparing the syzygies of this ideal and the ideal of of the singular locus of X one can show that if X has $n(k,d)$ nodes then X_H lies on the largest component of the Noether-Lefschetz locus. By the work of Otwinowska it follows that X_H contains a linear space of dimension $k-1$ and that X containes a linear space of dimension k. From this it follows that $n(k,d) = (d-1)^k$. This extends a result of Cheltsov for the case $k = 2$.

Yusuf Mustopa is an algebraic geometer with interests in commutative and noncommutative algebra. His recent projects focus on the following topics and their interrelations: the geometry of curves, syzygies of algebraic subsets of projective space, vector bundles, and the representation theory of associative algebras.

One major part of his current program is the study of Ulrich bundles, which are the "best-behaved" arithmetically Cohen-Macaulay bundles. These been the subject of intense investigation in recent years, and they occur naturally in a wide variety of algebraic and algebro-geometric topics, including determinantal descriptions of hypersurfaces, the computation of resultants, Boij-Söderberg theory, and generalized Clifford algebras. In previous joint work with Emre Coskun and Rajesh Kulkarni, Yusuf has applied recent work on the geometry of curves to study Ulrich bundles on quartic surfaces and del Pezzo surfaces. This body of results includes the existence of a linear Pfaffian representation of any smooth quartic surface, the existence of low-dimensional irreducible representations of the Clifford algebra of a ternary quartic form, and a characterization of Chern classes of Ulrich bundles on del Pezzos in terms of vector bundles on curves. In current joint work with Kulkarni, he is using techniques from the theory of generalized Clifford algebras to exhibit smooth threefolds of any given degree d in projective 4-space admitting stable Ulrich bundles of ranks both arbitrarily high and relatively low.

The other major part centers around the syzygy bundle associated to a subvariety of projective space, which governs the fine structure of the equations cutting out the subvariety. A classical result of Ein-Lazarsfeld says that the syzygy bundle associated to a smooth curve of genus at least 2 embedded in projective space by a complete linear series of sufficiently high degree is slope-stable. Lazarsfeld and Yusuf have recently generalized this to smooth projective surfaces, and they are presently working with Ein to extend it to varieties of dimension 3 or higher.

Wenbo Niu's recent research focuses on singularities of a generic link of an algebraic variety. Given an algebraic variety X, one can construct a variety Y linked to X so that X and Y are irreducible components of a complete intersection. The idea of linkage can be traced back to the nineteenth century. The modern form of linkage was first introduced by Peskine and Szpiro. The theory of generic linkage, in which, roughly speaking, Y is assumed to be taken as general as possible, was built and developed by Huneke and Ulrich in the past twenty years. Since X and Y are linked via a complete intersection, the properties of X can be compared to the ones of Y, which in turn provides a way to better understand X and Y. By the efforts of many mathematicians, including besides those mentioned above, Hartshorne, Migliore, Rao and others, many important features, such as divisor classes, Cohen-Macaulayness, normality, Serre's conditions S_k and R_k, have been studied for linkage. Meanwhile in the past twenty years the theory of singularities of algebraic varieties has been developed vastly. In particular, singularities arising from the minimal model program have played a central role and many modern techniques have been introduced. In his recent project, Wenbo is interested in understanding how the singularities of X can be compared to the singularities of Y. By using resolution of singularities and multiplier ideal sheaves he is able to produce a machinery to study the singularities of Y. As a consequence, he gives a criterion when Y has rational singularities. He also shows that log canonical threshold increases and log canonical pairs are preserved in generic linkage.

Giorgio Ottaviani worked in the field of algebraic geometry, especially about moduli of vector bundles, homogeneous vector bundles and group actions, small codimension subvarieties. The techniques of homogeneous vector bundles can be used to study the syzygies of homogeneous varieties, starting from the basic case of the Veronese variety.

Ottaviani is recently interested in tensor decomposition and tensor rank . The rank of a tensor t is the minimum number of decomposable tensors t_i needed to express $t = \sum_i t_i$. This is called a decomposition of t. When t is a symmetric tensor, this amounts to the classical Waring decomposition of a polynomial as a sum of powers of linear forms. On the geometric side, a tensor of rank k lies in the k-secant variety of the variety of decomposable tensors. Decomposable tensors fill the Veronese variety, (respectively the Grassmann variety, the Segre variety) in the symmetric (resp. skew-symmetric, general) case. The equations of the k-secant varieties of these classical varieties are still unknown in general. Such equations can be found, for small values of k, by minors of certain contraction maps, called flattenings or, more generally, Young flattenings. A nice example is the 3-secant variety of the cubic Veronese embedding of the plane, parametrizing polynomials which are cubes, like x_0^3. Its 3-secant variety is a hypersurface in P^9, which parametrizes polynomials which are sum of three cubes, like the Fermat cubic $x_0^3+x_1^3+x_2^3$. Its equation is a polynomial of degree 4, called the Aronhold invariant, which can be expressed as the generator of the pfaffian ideals of a 9×9 skew-symmetric contraction matrix. Equations of this kind are useful to effectively perform the tensor decomposition.

A very recent application, joint with J.M. Landsberg, is a lower bound for the complexity of the algorithm of matrix multiplication.

Stepan Paul's current research involves trying to understand the structure of multi-graded betti tables using recently developed tools in Boij-Söderberg theory, and seeking to apply new results in different contexts. Multi-graded polynomial rings arise naturally as the total coordinate rings for smooth toric varieties, carrying a grading by the Picard group. A recent pre-print from David Eisenbud and Daniel Erman establishes a foundation for starting to extend Eisenbud-Schreyer duality (in the standard $\mathbf{Z}$-graded case) to the toric case. Stepan Paul has been thinking about how to build on this foundation, keeping in mind the slightly pathological case where the toric variety in question has a non-simplicial cone of effective divisor classes.

Multi-graded betti tables also become useful when investigating the coordinate ring for a Veronese embedding of a projective variety. For example, the degree-d Veronese embedding of $\mathbf{P}^n$ into $\mathbf{P}^N$, where $N = \binom{d+n}{n} - 1$, induces a $\mathbf{Z}^n$-grading on $R = \mathbf{k}[x_1, \ldots, x_N]$. Stepan Paul has been considering how a suitable multi-graded version of Boij-Söderberg theory might answer open questions about the free resolution of the coordinate ring as an R-module.

Mihnea Popa currently works on problems related to generic vanishing theory and derived categories of coherent sheaves. He has been thinking about the invariance of Hodge numbers under derived equivalence, a problem inspired by mirror symmetry and string theory, and was able to prove it together with C. Schnell in the case of smooth projective threefolds. He is currently trying to prove the appropriate generalization to the case of stringy Hodge numbers for singular threefolds, using the framework of derived categories of smooth stacks introduced by Kawamata. On a related note, he is also thinking about the non-negativity of stringy Hodge numbers, a problem posed by Batyrev. In a different direction, he has proved together with C. Schnell results generalizing the classical framework of generic vanishing theory to the setting of $\mathcal{D}$-modules and local systems, and is applying them to study the singular cohomology algebra of irregular varieties.

Maria Evelina Rossi has been working on the classification of Artinian Gorenstein local rings. A celebrated paper by H. Bass in 60's outlines the ubiquity of Gorenstein rings: they play a fundamental role in many theories and constructions in Commutative Algebra and Algebraic Geometry. The common aim of some recent papers is to prove structure's theorems and to classify, up to analytic isomorphisms, Artinian local rings which are Gorenstein of given multiplicity. B. Poonen and A. Iarrobino proved that there exists a finite number of isomorphism classes of Artinian algebras of multiplicity at most 6. J. Elias and G. Valla proved the existence of codimension 2 complete intersection algebras of multiplicity 10 with infinitely many isomorphism classes. In this direction other results come from D. A. Cartwright, D. Erman, M. Velasco, and B. Viray. Recently, jointly with Elias, we proved that the classification of Artinian Gorenstein local rings with socle degree 3 can be reduced to the classification of K-standard graded algebras with socle degree 3 or, equivalently, to the projective classification of cubic forms in $\mathbf{P}^n$. As a consequence, a complete classification arises in the case of embedding dimension three. Part of these unexpected results have been extended to compressed level K-algebras.

In the one-dimensional case the problem becomes much more difficult, even if one considers local complete intersections of codimension two.

The motivating goal behind this research come from classical problems:

- the study of the irreducibility and the smoothness of the punctual Hilbert scheme parameterizing 0-dimensional subschemes of fixed degree in $\mathbf{P}^n$
- the study of the rationality of the Poincare' series of K-algebras
- the study of free resolutions over Gorenstein algebras
- the study of the Hilbert function of complete intersection curve singularities.

Mike Roth's recent research uses ideas from asymptotic algebraic geometry to prove results in diophantine approximation. As an example, motivated by the Bombieri-Lang conjecture one should study how local positivity of a line bundle influences the local accumulation of rational points. Together with David McKinnon, Roth introduces an approximation constant measuring how well an algebraic point on a variety can be approximated by rational points. and shows that this constant has great formal similarity with the Sesadri constant. Most importantly they show that the classic approximation theorems on the line — the theorems of Liouville and K.F. Roth — generalize to inequalities between these invariants valid for all projective varieties.

The research of **Steven Sam** can be roughly classified into three different topics, listed below.

In Boij–Söderberg theory the goal is to construct modules with linear resolutions on special classes of varieties and study the cone of Betti tables over quadric hypersurface rings. Boij–Söderberg theory is a significant new approach to free resolutions over polynomial rings and cohomology tables of sheaves over projective space. In joint work with Berkesch, Erman, and Kummini, he produced the following three results: (1) an interpretation of a certain partial ordering arising in this theory, (2) a functorial construction for pure resolutions, which are at the heart of the theory, and (3) a description of the situation for regular local rings and local hypersurface rings.

To extend this theory to other varieties, the former is related to the existence of pure resolutions and the latter is related to the existence of Ulrich modules, and hence to elimination theory via work of Eisenbud–Schreyer. This theory gives a numerical decomposition of modules and sheaves into basic ones, and a "duality" between free resolutions and cohomology tables. These are mysterious and demand a conceptual explanation. A basic step towards this is to understand the situation for other varieties. The current work on quadric hypersurfaces has led to connections with classical invariant theory and twisted commutative algebras, which is current joint work with Snowden and Weyman.

Sam has been also working on Discriminants. The goal is to understand the relationship between "discriminantal degeneracy loci" and Abelian varieties connected to Vinberg's theory of θ-representations. This is part of a larger project that originates with groundbreaking work of Bhargava on counting number fields and elliptic curves of bounded discriminant, rank, etc. While many orbit spaces of θ-representations admit natural moduli interpretations, the "sporadic" examples are opaque. In work with Gruson and Weyman, he established a general framework for discovering moduli interpretations of Vinberg's θ-representations using ideas from degeneracy loci and free resolutions. This connects to many topics in classical algebraic geometry and sheds new light on constructions and theorems.

Sheaf cohomology has been used successfully many times to study equations for "nice" varieties. Nonnormality is a major technical hurdle, and appears in important examples like nilpotent orbits in semisimple

Lie algebras. This project aims to produce tools and examples to understand the "degree of non-normality" which is a first step toward overcoming such obstacles. The main focus is on varieties arising in linear algebraic or representation-theoretic situations and to use tools from representation theory. He found a pattern for the nature of the non-normality of the "Kalman varieties" which will hopefully provide a framework for understanding other examples of interest.

Hal Schenck uses algebraic and computational methods to study problems at the interface of algebra, geometry, and combinatorics. His research has three main themes: Toric varieties, Geometric modelling: implicitization and approximation theory, Hyperplane arrangements.

Toric varieties are objects defined by the discrete geometric data of a fan, and here Schenck studies the interplay between the fan and various algebraic objects associated to the fan (such as the Chow ring), as well as properties of the homogeneous coordinate ring associated to a divisor. For example, he recently gave a description of certain classes in equivariant Chow cohomology for a nonsimplicial fan. The result differs substantially from the simplicial case. Interestingly, this ties into work in applied mathematics, as the equivariant Chow ring is an algebra over a polynomial ring on the ambient space of the fan, and corresponds to continuous splines on the fan. In a second paper, the Cartan-Eilenberg spectral sequence is used to obtain sufficient conditions for freeness of splines on the fan, a topic of much interest in approximation theory.

A second applied topic involves rational surface modelling and implicitization. Work of Simis-Vasconcelos on approximation complexes and Rees algebra techniques show that obtaining the syzygies of some incomplete linear system is a crucial step in determining the implicit equations. In recent work with Seceleanu and Validashti, Schenck studies this question for a four dimensional basepoint free subspace of sections of bidegree $(2,1)$ on $\mathbf{P^1} \times \mathbf{P^1}$. The result is a complete classification of all possible minimal free resolutions (there are six types), as well as a beautiful dictionary between syzygies on the sections, and singularities of the resulting surface in $\mathbf{P^3}$.

On the final topic of hyperplane arrangements, Schenck (in joint work with Terao and Yoshinaga) has recently proven an inductive criterion for splitting of the module of vector fields on $\mathbf{P^3}$ which are tangent to an arrangement of smooth plane curves with quasihomogeneous singularities. There should be many generalizations possible here.

Frank-Olaf Schreyer has been recently working on the three projects listed below.

Joint work with Alessandro Chiodo, David Eisenbud, and Gavril Farkas is on syzygies of torsion bundles and the geometry of the level ℓ modular variety over M_g. They formulate, and in some cases prove, three statements concerning the purity or, more generally, the naturality of the resolution of various modules one can attach to a generic curve of genus g and a torsion point of ℓ in its Jacobian. These statements can be viewed an analogues of Green's Conjecture and we verify them computationally for bounded genus. They then compute the cohomology class of the corresponding non- vanishing locus in the moduli space $R_{g,\ell}$ of twisted level ℓ curves of genus g and use this to derive results about the birational geometry of $R_{g,\ell}$. For instance, they prove that $R_{g,3}$ is a variety of general type when $g > 11$ and the Kodaira dimension of $R_{11,3}$ is greater than or equal to 19. In the last section we explain probabilistically the unexpected failure of the Prym-Green conjecture in genus 8 and level 2.

Schreyer has a joint project in progress with Madhusudan Manjunath and John Wilmes on the Toppling Complex. Let G be a undirected connected graph possibly with multiple edges on vertices $x_1, \ldots, x_n$. Let Δ_G be its laplacian and let J_G be the corresponding binomial ideal in the polynomial ring $K[x_1, ..., x_n]$. The authors describe the minimal free resolution of the toppling ideal J_G in terms of oriented connected partitions of the graph with a unique sink.

Frank-Olaf Schreyer and Laura Costa study Ulrich bundles on ACM varieties of codimension 2. They establish the existence of Ulrich bundles on determinantal varieties.

Eric Sharpe's research is in physics. A generalization of mirror symmetry, known as "(0,2) mirror symmetry," is being studied. Existence of this generalization has long been conjectured, and would be a powerful computational tool in studies of compactifications of heterotic string theory, specified by a compact Kähler manifold X together with a holomorphic vector bundle $\mathcal{E} \to TX$ satisfying the constraints

$$\det \mathcal{E}^* \;\cong\; K_X, \;\; \mathrm{ch}_2(TX) \;=\; \mathrm{ch}_2(\mathcal{E}).$$

(0,2) mirror symmetry exchanges pairs $(X, \mathcal{E})$, $(X', \mathcal{E}')$, and reduces to ordinary mirror symmetry between X and X' in the special case that $\mathcal{E} = TX$, $\mathcal{E}' = TX'$.

One part of this effort involves understanding an analogue of quantum cohomology, involving the sheaf cohomology groups $H^*(X, \wedge^* \mathcal{E}^*)$. These have a natural product structure

$$H^p(X, \wedge^q \mathcal{E}^*) \times H^{p'}(X, \wedge^{q'} \mathcal{E}^*) \longrightarrow H^{p+p'}(X, \wedge^{q+q'} \mathcal{E}^*),$$

and a trace $H^{\mathrm{top}}(X, \wedge^{\mathrm{top}} \mathcal{E}^*) \longrightarrow \mathbf{C}$, which are used to define a 'quantum' sheaf cohomology ring, analogous to quantum cohomology rings.

Recent work computes quantum sheaf cohomology rings in the special case that X is a toric variety and $\mathcal{E}$ is a deformation of the tangent bundle TX, checking and extending results in the physics literature. The results agree with the ordinary quantum cohomology relations described by Batyrev in the special case $\mathcal{E} = TX$.

Ian Shipman works in algebraic geometry and representation theory. He is currently exploring two circles of ideas. First, suppose that a reductive group G acts on a smooth quasi-projective algbraic variety X. Given a G-equivariant line bundle L (a linearization) on X, GIT provides an open set X^s of semistable points together with a stratification of the complement $X \setminus X^s$. The geometry of this stratification, and the action of G on it has been used in two recent preprints by Halpern-Leistner and Ballard-Favero-Katzarkov to relate the derived categories of G equivariant coherent sheaves on X^{ss} and X. In an preprint in preparation, Shipman and Dan Halpern-Leistner will use, under certain circumstances, this relationship to construct derived autoequivalences and to explain relations in the group of derived autoequivalences. Furthermore, this picture can be sometimes be used to construct derived equivalences among the various GIT quotients of X. In the setting of the abelian McKay correspondence, there is a variation of GIT connecting an abelian quotient singularity to its crepant resolutions and this gives rise to derived equivalences. Shipman and Halpern-Leistner are working to construct the crepant resolutions (when they exist) by first constructing different t-structures on the noncommutative resolution of the singularity, then obtaining the resolution as a moduli space of simple objects, following Bridgeland.

Shipman is also interested in the connection between the representation theory of finite dimensional algebras and the geometry of various varieties parameterizing their modules. Of particular interest is the degeneration order. A general linear group G acts on the standard affine variety parameterizing framed modules of a fixed dimension over a finite dimensional algebra. The orbits for this action are in bijection with the isomorphism classes of modules. A module is said to degenerate to another if the closure of the orbit corresponding to the first module contains the orbit corresponding to the second module. There is a representation theoretic criterion for when this happens, involving an auxilliary module. However bounds on the dimension of this module, or constraints on its structure are unknown. With Ryan Kinser, Shipman is developing a technique involving the loop group $G((t))$ of G to address such questions.

Gregory Smith has been working recently with Victor Lozovanu. In higher-dimensional algebraic geometry, vanishing theorems are indispensable for uncovering the deeper relations between the geometry of a subvariety and its defining equations. Given scheme-theoretic equations for a nonsingular subvariety, Victor Lozovanu and Gregory G. Smith prove that the higher cohomology groups for suitable twists of the corresponding ideal sheaf vanish. From that result, they obtain linear bounds on the multigraded Castelnuovo-Mumford regularity of a nonsingular subvariety, and new criteria for the embeddings by adjoint line bundles to be projectively normal. Their techniques also yield a new Griffiths-type vanishing theorem for vector bundles. To be more precise, let X be a nonsingular complex projective variety with canonical line bundle K_X. A subvariety $Y \subseteq X$ with ideal sheaf $\mathcal{I}_Y$ is *defined scheme-theoretically* by the divisors $D_1, \ldots, D_r$ on X if $Y = D_1 \cap \cdots \cap D_r$ and the map $\bigoplus_{j=1}^r \mathcal{O}_X(-D_j) \to \mathcal{I}_Y$ determined by the D_j is surjective. For a nonsingular subvariety $Y \subseteq X$ of codimension e, the following is their main theorem: Let L be a line bundle on X and let m be a nonnegative integer. If Y is defined scheme-theoretically by the nef divisors $D_1, \ldots, D_r$ and $L \otimes \mathcal{O}_X(-(m+1)D_{s_1} - D_{s_2} - \cdots - D_{s_e})$ is a big, nef line bundle for all subsets $\{s_1, s_2, \ldots, s_e\} \subseteq \{1, \ldots, r\}$, then we have $H^i\big(X, \mathcal{I}_Y^{m+1} \otimes K_X \otimes L\big) = 0$ for all $i > 0$.

Under the assumption that each D_j belongs to the linear system for some power of a single globally generated line bundle, Lozovanu and Smith recover the vanishing theorem of Bertram, Ein, and Lazarsfeld. In particular, this additional hypothesis induces an ordering on the subsets $\{s_1, \ldots, s_e\}$ and it is enough to consider the unique maximal subsets.

Part of **Mark Walker's** current research interests concern the singularity category (also known as the stable category) of complete intersections and the theory of affine and non-affine matrix factorizations.

Using a theorem of Orlov, the singularity category of a complete intersection is equivalent to the singularity category of a suitably defined non-affine hypersurface Y. In detail, Y is the hypersurface in a smooth scheme X defined by the vanishing of a regular global section W of a line bundle $\mathcal{L}$ on X. The singularity category of Y is, in turn, equivalent to the homotopy category of "twisted" matrix factorizations associated to the triple $(X, \mathcal{L}, W)$. An object in the latter category consists of a pair of algebraic vector bundles E_1, E_0 on X and morphisms $E_1 \to E_0$ and $E_0 \to E_1 \otimes \mathcal{L}$ such that each of the two evident compositions is multiplication by W.

Using this machinery, Jesse Burke and Walker have redeveloped the theory of stable supports for modules over complete intersections in a more geometric manner. Additional applications include devising a notion of Chern classes for modules over a complete intersection, taking values in a suitably defined version of Hochschild homology for twisted matrix factorizations. Among other things, Walker hopes to use these Chern classes to establish the vanishing of Dao's so-called "η invariant" for pair of modules over a complete intersection with an isolated singularity.

Xin Zhou's research interest lies in the study of asymptotic behaviors of syzygies. The first line of research he pursued is the asymptotic nonvanishing behavior of syzygies. Ein and Lazarsfeld, in their paper "Asymptotic syzygies of algebraic varieties", describes the nonvanishing of almost all syzygy groups asymptotically, when the embedding is defined by higher and higher multiples of an ample line bundle. They proved a noneffective result for general varieties and an effective result for projective spaces. In "Effective non-vanishing of asymptotic adjoint syzygies", using a variant of their method, Xin Zhou proves an effective result for adjoint syzygies of an arbitrary variety which specializes to the effective result for projective spaces of Ein and Lazarsfeld.

The other focus of Xin Zhou's research is the asymptotic behavior of syzygy functors of Veronese embeddings. For Veronese embeddings, the syzygy groups are functorial in the underlying vector space. Hence, they have functorial decompositions as Schur functors. In "Asymptotics of syzygy functors" (in prepration), Mihai Fulger and Xin Zhou start by counting the Schur functors in $\otimes S^d$, $\mathrm{Sym}^p\mathrm{Sym}^d$ and $\wedge^p\mathrm{Sym}^d$. Then they apply these results to the study of the functors $\mathbf{K}_{p,1}(d)$, $\mathbf{K}_{p,0}(b;d)$ and show that the functors have maximal scales of growth possible with respect to the parameter d, fixing the other parameters. They also study other interesting asymptotic behaviors present when varying the first parameter following a restriction method for Koszul cohomology suggested by Rob Lazarsfeld.

Presentation Highlights

The following six one-hour presentations on topics in the area of Syzygies in Algebraic Geometry took place at the workshop. They cover recent important developments in that area.

Marian Aprodu gave a one-hour talk on *Vector bundles and syzygies*, with emphasis on Green's Conjecture for curves on arbitrary $K3$ surfaces. He discussed recent developments in the theory of syzygies using vector bundle techniques. The talk was partly based on joint works with **G**avril Farkas.

Aldo Conca gave a one-hour talk on *Koszul algebras and their syzygies*. In a joint paper with Avramov and Iyengar it was shown that the syzygies of Koszul algebras behave very much as the syzygies of algebras with quadratic monomial relations; for example, the highest degree of an i'th syzygy of a Koszul algebra is at most $2i$. Conca presented new results concerning this analogy. He also discussed some open problems concerning the syzygies of modules over a Koszul algebra.

Steven Cutkosky gave a one-hour talk on *Multiplicities Associated to Graded Families of Ideals*. He had proved that limits of multiplicities associated to graded families of ideals exist under very general conditions. Most of the results hold for reduced excellent equicharacteristic local rings, with perfect residue fields. He discussed a number of applications, including a "volume = multiplicity" formula, generalizing formulas of Lazarsfeld and Mustata and of Ein, Lazarsfeld and Smith, and a proof that the epsilon multiplicity of Ulrich and Validashti exists as a limit for ideals in rather general rings, including analytic local domains. He presented a generalization of this to generalized symbolic powers of ideals, proposed by Herzog, Puthenpurakal and Verma. He also gave an asymptotic "additivity formula" for limits of multiplicities, and a formula on limiting growth of valuations, which answers a question posed by the author, Kia Dalili and Olga Kashcheyeva. The proofs are inspired by a philosophy of Okounkov, for computing limits of multiplicities as the volume of a slice of an appropriate cone generated by a semigroup determined by an appropriate filtration on a family of algebraic objects.

Robert Lazarsfeld gave a one-hour talk on *Asymptotic syzygies of algebraic varieties*. He discussed joint work with **L**awrence Ein and others concerning the asymptotic behavior of the syzygies of algebraic varieties as the positivity of the embedding line bundle increases.

Giorgio Ottaviani gave a one-hour talk on *The syzygies of Veronese embeddings*. The resolution of the Veronese embedding of P^n is a basic algebraic object, not yet completely understood. It is a prototype for the behaviour of Betti numers of more general varieties, as shown by Ein and Lazarsfeld. It is well known that the resolution of the Veronese embedding can be in principle computed by the cohomology of certain homogeneous bundles. The category of homogeneous bundles on P^n is equivalent to the category of representations of a certain quiver with commutativity relations. This gives a combinatorial point of view for the computation of the cohomology of the relevant homogeneous bundles. Ottaviani discussed these ideas having in mind the example of the Veronese embedding of P^2.

Frank-Olaf Schreyer gave a one-hour talk on *Syzygies of torsion bundles and the geometry of the level ℓ modular variety over M_g*. The talk was on joint work with Alessandro Chiodo, David Eisenbud, and Gavril Farkas. The authors formulate, and in some cases prove, three statements concerning the purity or, more generally, the naturality of the resolution of various modules one can attach to a generic curve of genus g and a torsion point of ℓ in its Jacobian. These statements can be viewed an analogues of Green's Conjecture and we verify them computationally for bounded genus. Schreyer focused on the unexpected failure of the Prym-Green conjecture in genus 8 and level 2, which they can establish probabilistically. It is expected that the Prym-Green conjecture fails for all genera which are powers of 2 and a theoretical explanation of this fact which has been discovered computationally is still to be found.

The other focus in the workshop was a Short Course on Matrix Factorizations and String Theory. The Short Course consisted of ten one-hour lectures.

The first two lectures in the Short Course were algebraic. **Ian Shipman** gave a talk on Orlov's theorem and related results. Orlov's theorem describes a precise relationship between two triangulated categories related to a projective hypersurface: the derived category of coherent sheaves, and the equivariant stable derived category of its affine cone. Shipman discussed semiorthogonal decompositions, the the equivalence between the equivariant stable derived category and a category of matrix factorizations, and the proof of the theorem. **Andrei Caldararu** gave a talk on curved algebras, curved dg-algebras, and curved A_∞ algebras. He also discussed the twisted complexes construction, and how matrix factorizations can be viewed as analogues of complexes of projective modules for a certain curved algebra. The remaining eight lectures in the Short Course were:

- **David Berenstein**: "From quivers and superpotentials to algebras and representation theory"
- **Paul Aspinwall**: "The topological B-model and superpotentials"
- **Eric Sharpe**: "Boundary terms in 2d theories and matrix factorization"
- **Dave Morrison**: "D-brane algebras"
- **David Berenstein**: "Conjectures about superpotential algebras"
- **Sheldon Katz**: "Computation of superpotentials for D-branes"
- **Paul Aspinwall**: "Matrix factorization on the quintic"
- **Dave Morrison**: "Matrix factorizations in physics (Summary talk)".

The Workshop

The physicists and mathematicians working on the edge of physics discovered new generalizations and results about the matrix factorizations; and the language of the mathematicians and the physicists slowly diverged. The goal of our workshop was to bring the mathematics culture and the physics culture closer together, and to promote the exchange of new results between the two groups. We organized a 10-lecture short course on matrix factorizations. The following eight physicists and mathematical physicists participated in the workshop: Paul S. Aspinwall, David Berenstein, Andrei Caldararu, Sheldon Katz, David R. Morrison, Stepan Paul, Eric Sharpe.

The Workshop provided an opportunity to exchange open problems, share ideas, and explore in new directions on free resolutions in Algebraic Geometry and Commutative Algebra (and related topics).

We had a lively exchange of ideas and methods which will foster further research.

Participants

Aprodu, Marian (Inst. Math. Romanian Academy)
Aspinwall, Paul (Duke University)
Avramov, Luchezar (University of Nebraska)
Berenstein, David (University of California at Santa Barbara)
Berkesch, Christine (Duke University)
Boij, Mats (KTH Royal Institute of Technology)
Burke, Jesse (Bielefeld University)
Caldararu, Andrei (University of Wisconsin, Madison)
Conca, Aldo (University of Genova)
Coskun, Izzet (UIC)
Cutkosky, Steven Dale (University of Missouri - Columbia)
Dao, Hailong (University of Kansas)
Ein, Lawrence (University of Illinois at Chigago)
Eisenbud, David (Mathematical Sciences Research Institute)
Erman, Daniel (University of Michigan)
Farkas, Gavril (Humboldt Universität zu Berlin)
Floystad, Gunnar (University of Bergen, Norway)
Hartshorne, Robin (UC Berkeley)
Katz, Sheldon (University of Illinois)
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Lazarsfeld, Robert (Stony Brook)
Morrison, David (University of California, Santa Barbara – Math)
Murfet, Daniel (University of Melbourne)
Mustopa, Yusuf (Boston College)
Niu, Wenbo (Purdue University)
Ottaviani, Giorgio (University of Florence)
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Peeva, Irena (Cornell University)
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Roth, Michael (Queen's University)
Sam, Steven (University of Wisconsin, Madison)
Schenck, Hal (University of Illinois Urbana Champaign)
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Sharpe, Eric (Virginia Tech)
Shipman, Ian (Massachusetts Institute of Technology)
Smith, Gregory G. (Queen's University)
Stillman, Mike (Cornell University)
v. Bothmer, Hans-Christian (Universität Hamburg)
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Bibliography

[1] Marian Aprodu, Green–Lazarsfeld gonality conjecture for a generic curve of odd genus, *Int. Math. Res. Not.* **63** (2004),3409–3416.

[2] , Marian Aprodu and Gavril Farkas, Green's conjecture for curves on arbitrary K3 surfaces *Compos. Math.* **147** (2011), no. 3, 839-851.

[3] Marian Aprodu and Claire Voisin, Green–Lazarsfeld's conjecture for generic curves of large gonality, *C. R. Math. Acad. Sci. Paris* **336** (2003), 335-339.

[4] Caucher Birkir, Paolo Cascin, Christopher Hacon and James Mckernan, Existence of minimal modles for varieties of log general type, *J. Amer. Math. Soc.* **(**23) (2010), 405-468.

[5] Winfried Bruns, Aldo Conca, Tim Römer, Koszul homology and syzygies of Veronese subalgebras, to appear in *Math. Ann.*

[6] Winfried Bruns, Aldo Conca, Tim Römer, Koszul cycles, preprint, *arXiv:1009.1230*

[7] Lawrence Ein and Robert Lazarsfeld, Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension, *Invent. Math.* **111** (1993), 51–67.

[8] Lawrence Ein and Robert Lazarsfeld, Asymptotic syzygies of algebraic varieties, to appear in *Invent. Math.*

[9] Lawrence Ein, Daniel Erman and Robert Lazarsfeld, Asymptotics of random Betti Tables, preprint, *arXiv:1207.5467*.

[10] David Eisenbud and Frank Schreyer, Betti numbers of graded modules and cohomology of vector bundles, *J. Amer. Math. Soc.* **22** (2009), 859–888.

[11] Mark Green, Koszul cohomology and the geometry of projective varieties, *J. Diff. Geom.* **19** (1984), 125–171.

[12] Mark Green, Koszul cohomology and the geometry of projective varieties, II, *J. Diff. Geom.* **20** (1984), 279–289.

[13] Mark Green and Robert Lazarsfeld, On the projective normality of complete linear series on an algebraic curve, *Invent. Math.* **83** (1985), 73 – 90.

[14] Giorgio Ottaviani and Rafaella Paoletti, Syzygies of Veronese embeddings, *Compos. Math.* **125** (2001), 31–37.

[15] Giorgio Ottaviani and Elena Rubei, Quievers and cohomology of homogenous vector bundles, *Duke Math. J.* **132** (2006), 459-508.

[16] Giuseppe Pareschi, Syzygies of abelian varieties, *Journal of the AMS* **13** (2000), 651– 664.

[17] Frank Schreyer, Syzygies of canonical curves and special linear series, *Math. Ann.* **275** (1986), 105–137.

[18] Andrew Snowden, Syzygies of Segre varieties and Δ functors, to appear.

[19] M. Texidor i Bigas, Green's conjecture for generic r-gonal curves of genus $g \geq r+3$, *Duke Math. J.* **111** (2002) 195-222.

[20] Claire Voisin, Green's generic syzygy conjecture for curves of even genus lying ona K3 surface, *J. Eur. Math. Soc.* **4** (2002), 363-404.

[21] Claire Voisin, Green's canonical syzygy conjecture for generic curves of odd genus, *Compos. Math.* **141** (2005) 1163-1190.

[22] Xin Zhou, thesis in preparation.

Chapter 30

New Trends and Directions in Combinatorics (12w5001)

August 19 - 24, 2012

Organizer(s): Penny Haxell (University of Waterloo), Michael Krivelevich (Tel Aviv University), Benny Sudakov (University of California, Los Angeles)

Overview

Combinatorics, or Discrete Mathematics, is a fundamental mathematical discipline, focusing on the study of discrete objects and their properties. Although Combinatorics is probably as old as the human ability to count, the field has experienced tremendous growth in the last fifty years, partially spurred by its tight connections with other growing branches of science, most notably Theoretical Computer Science. While in the past many of the basic combinatorial results were obtained mainly through ingenuity and detailed reasoning, the modern theory has grown out of this early stage, and often relies on deep, well-developed and sophisticated tools. Faithfully reflecting the mature status of the field are deep and prolific interconnections between its various branches – one frequently uses probabilistic considerations to attack a problem in Extremal Graph Theory, or relies on extremal results to make advances in a discrete geometric problem; examples of such fruitful cooperation abound.

In recent years Combinatorics appears to have made a qualitative leap forward, with sizeable or even astonishing progress having been achieved in a variety of directions. Some of the key areas of Combinatorics to have developed greatly in these years are Extremal Graph Theory, Extremal Set Theory, Ramsey Theory, and Probabilistic Combinatorics. In what follows we provide a brief outline of these fields, which were the focus of attention over the course of this workshop.

Extremal Graph Theory explores the question of how large a graph can be if a certain substructure is forbidden. This is one of the most important branches of modern Graph Theory, with a variety of methods and arguments applied, including linear algebraic arguments, analytic tools, probabilistic considerations, and the regularity method. A concrete example of an extremal problem in which there has been striking progress in recent years is the Turán problem for which the forbidden subgraph is bipartite. This is a notoriously difficult problem that has eluded researchers for many years, in sharp contrast to the case in which the chromatic number is at least three (which by now is very well understood). However, recent new advances involving sophisticated algebraic and geometric constructions have given fresh insight on the topic, and promise that much more can be achieved.

Extremal Set Theory problems are usually formulated and studied for families of sets satisfying given restrictions. Problems of this type are especially appealing, in part due to the fact that they frequently arise in a variety of applications in diverse fields of Mathematics, Computer Science, and Coding and Information Theory. Recent advances have resulted from several new sophisticated methods such as hypergraph regularity, the stability approach, and Fourier analytic techniques. These approaches have been developed over the past decade, and are now reaching the point of maturity that can make them standard general (and very powerful) tools for researchers. Again to give just one example, significant new progress on Turán problems for hypergraphs has been achieved using some of these methods.

Ramsey Theory is undoubtedly one of the most central branches of modern mathematics, studying quantitatively the phenomenon that every sufficiently large object, chaotic as it may be, must contain a well-structured sub-object. Quite a few questions from Ramsey theory, including estimates on the so called Ramsey numbers, can be cast and viewed as problems in Extremal Graph or Set Theory. Probabilistic arguments are essential here, and their importance and applicability cannot be overestimated. The recent development of analytic tools, used in conjunction with the probabilistic methods, has inspired a great deal of progress in this area in the last few years, leading to the resolution of a number of long-standing conjectures.

Probabilistic Combinatorics studies probability spaces of discrete structures. In quite a few cases these probability spaces are deep and complex and are studied entirely for their own sake, yet probabilistic considerations are indispensable in many other areas of Combinatorics, most notably for extremal problems and Ramsey Theory. They also form a mathematical foundation for the design and analysis of algorithms involving randomness, and more generally for addressing the fundamental role of randomness in Theoretical Computer Science, especially in Algorithmics and Complexity. Recently, there has been an impressive stream of novel and exciting results and concepts in the field. Some of them include development of sophisticated techniques, deepening our understanding of the phase transition in various models of random graphs; important advances in extremal properties (Turán and Ramsey-type problems) of random graphs; analysis of the so-called controlled random processes; and applications of sophisticated tools from Extremal Graph Theory (such as the Sparse Regularity Lemma) to random graphs, to mention just a few.

It should be stressed that the above mentioned main fields of Combinatorics, as well as several of its other branches, are tightly intertwined, and in quite a few cases progress in one of the directions soon results in some equally exciting progress in another. One aim of the workshop was to encourage interaction between researchers in different fields, to facilitate the exchange of ideas and enable the introduction of novel techniques to different areas of Combinatorics. Moreover, certain new directions of combinatorial research were highlighted, most notably the study of combinatorial problems in an algebraic setting.

In the remainder of this report we present in detail some of the advances presented at the workshop.

Extremal Graph Theory RAINBOW TURÁN PROBLEMS

Shagnik Das *joint with C. Lee and B. Sudakov*

The rainbow Turán problem, first introduced by Keevash, Mubayi, Sudakov and Verstraëte in 2007, beautifully combines two central and oft-studied branches of Extremal Graph Theory, namely Turán Theory and Graph Colouring. We say an edge-coloured graph is coloured *properly* if no two adjacent edges share a colour, and we say it is *rainbow* if every edge receives a unique colour. The rainbow Turán problem asks, for a given graph H, how many edges a properly edge-coloured graph on n vertices can have if it does not contain a rainbow copy of H. In their paper, Keevash, Mubayi, Sudakov and Verstraëte determine asymptotically the rainbow Turán number for any non-bipartite graph. However, as is the case in classical Turán Theory, the problem appears to be much more difficult for bipartite graphs. The original authors remark that the problem is most interesting in the case of even cycles, due to a connection to a problem in Additive Number Theory. As observed by Mubayi and Verstraëte during this workshop, this case also has implications for the Turán number of linear cycles in 3-uniform hypergraphs.

In this talk, we discuss some recent results on the rainbow Turán number for even cycles. We show that any properly edge-coloured graph on n vertices with $O\left(n^{1+(1+\varepsilon_k)\ln k/k}\right)$ edges contains a rainbow cycle of length $2k$, where $\varepsilon_k \to 0$ as $k \to \infty$. This improves the previous best-known bound of $O\left(n^{3/2}\right)$, and is significantly closer to the lower bound of $\Omega\left(n^{1+1/k}\right)$. We also narrow the gap between the upper and lower bounds on the size of properly edge-coloured graphs without rainbow cycles of any length.

CHROMATIC NUMBER, CLIQUE SUBDIVISIONS, AND THE CONJECTURES OF HAJOS AND ERDOS-FAJTLOWICZ
Jacob Fox *joint with C. Lee and B. Sudakov*

A *subdivision* of a graph H is any graph formed by replacing edges of H by internally vertex disjoint paths. This is an important notion in graph theory, e.g., the celebrated theorem of Kuratowski uses it to characterize planar graphs. For a graph G, we let $\sigma(G)$ denote the largest integer p such that G contains a subdivision of a complete graph of order p. Clique subdivisions in graphs have been extensively studied and there are many results which give sufficient conditions for a graph G to have large $\sigma(G)$. For a given graph G, let $\chi(G)$ denote its chromatic number. A famous conjecture made by Hajós in 1961 states that $\sigma(G) \geq \chi(G)$. Dirac proved that this conjecture is true for all $\chi(G) \leq 4$, but in 1979, Catlin disproved the conjecture for all $\chi(G) \geq 7$. By considering random graphs, Erdos and Fajtlowicz in 1981 showed that the conjecture actually fails for almost all graphs.

We revisit Hajós' conjecture and study to what extent the chromatic number of a graph can exceed the order of its largest clique subdivision. Let $H(n)$ denote the maximum of $\chi(G)/\sigma(G)$ over all n-vertex graphs G. Erdos and Fajtlowicz showed that almost all graphs on n vertices satisfy $\sigma(G) = O(n^{1/2})$ and $\chi(G) = \Theta(n/\log n)$. Thus it implies that $H(n) = \Omega(n^{1/2}/\log n)$. Erdos and Fajtlowicz conjectured that this bound is tight up to a constant factor so that $H(n) = \Theta(n^{1/2}/\log n)$. This conjecture says that the random graph is essentially the strongest counterexample to the Hajós' conjecture. We verify the Erdos-Fajtlowicz conjecture.

Extremal problems in Eulerian digraphs

Hao Huang *joint with J. Ma, A. Shapira, B. Sudakov and R. Yuster*

One of the central themes in graph theory is to study the extremal graphs which satisfy certain properties. There are many classical results in this area. For example, any undirected graph G with n vertices and m edges has a subgraph with minimum degree at least m/n, and thus G also contains a cycle of length at least $m/n + 1$. It is natural to ask whether such results can be extended to digraphs. However, it turns out that graphs and digraphs behave quite differently, and many classical results for graphs are often trivially false when extended to general digraphs. Therefore it is usually necessary to restrict to a smaller family of digraphs to obtain meaningful results. One such very natural family is Eulerian digraphs, in which the in-degree equals out-degree at every vertex. In this talk, we discuss several natural parameters for Eulerian digraphs and study their connections. In particular, we prove the following result.

Theorem 7. *For any Eulerian digraph G with n vertices and m edges, the minimum feedback arc set (the smallest set of edges whose removal makes G acyclic) has size at least $m^2/2n^2 + m/2n$, and this bound is tight.*

Using this result, we show how to find subgraphs of high minimum degrees, and also the existence of long cycles in Eulerian digraphs. In particular, we prove that every Eulerian digraphs with n vertices and m edges contains a cycle of length at least $\Omega(m^2/n^3)$. This verifies a conjecture of Bollobás and Scott for dense digraphs.

On k-color-critical n-vertex graphs with fewest edges

Alexander Kostochka *joint with M. Yancey*

A graph G is k*-critical* if it has chromatic number k, but every proper subgraph of G is $(k-1)$–colorable. Let $f_k(n)$ denote the minimum number of edges in an n-vertex k-critical graph. We give a lower bound, $f_k(n) \geq F(k, n)$, that is sharp for every $n = 1 \pmod{k-1}$. It is also sharp for $k = 4$ and every $n \geq 6$. The result improves the classical bounds by Gallai and Dirac and subsequent bounds by Krivelevich and Kostochka and Stiebitz. It establishes the asymptotics of $f_k(n)$ for every fixed k. It also proves that the conjecture by Ore from 1967 that for every $k \geq 4$ and $n \geq k+2$, $f_k(n+k-1) = f(n) + \frac{k-1}{2}(k - \frac{2}{k-1})$ holds for each $k \geq 4$ for all but at most $k^3/12$ values of n. We give a polynomial-time algorithm for $(k-1)$-coloring a graph G that satisfies $|E(G[W])| < F_k(|W|)$ for all $W \subseteq V(G)$, $|W| \geq k$. We also present some applications of the result.

One of the corollaries of our theorem is a half-page proof of the theorem due to Grötzsch that every triangle-free planar graph is 3-colorable.

Asymptotic structure of graphs with the minimum number of triangles

Oleg Pikhurko *joint with A. Razborov*

Let $g(m,n)$ be the smallest number of triangles in a graph with n vertices and m edges. Let us consider the asymptotic question, that is, what is the limit

$$g(a) = \lim_{n\to\infty} \frac{g\big(\lfloor a\binom{n}{2}\rfloor, n\big)}{\binom{n}{r}}?$$

While it is not difficult to show that the limit exists, determining $g(a)$ is a much harder task that was accomplished only recently by Razborov (with previous partial results obtained by Bollobás, Erdos, Lovász, Mantel, Simonovits, and others).

The following construction gives the value of $g(a)$. Choose t and $c \in \left[\frac{1}{t+1}, \frac{1}{t}\right]$ such that the complete $(t+1)$-partite graph of order $n \to \infty$ with t largest parts each of size $(c+o(1))n$ has edge density $a + o(1)$. Partition the vertex set $[n] = \{1,\ldots,n\}$ into $t+1$ parts $V_1,\ldots,V_{t+1}$ with $|V_1| = \cdots = |V_t| = \lfloor cn \rfloor$ for $i \in [t]$. Let G be obtained from the complete t-partite graph $K(V_1,\ldots,V_{t-1},U)$, where $U = V_t \cup V_{t+1}$, by adding an arbitrary triangle-free graph on U with $|V_t|\,|V_{t+1}|$ edges. Let $\mathcal{C}H$ consist of all graphs obtained this way.

We prove the following result.

Theorem 8. *For every $\varepsilon > 0$ there are $\delta > 0$ and n_0 such that every graph G with $n \geq n_0$ vertices and at most $(g(a)+\delta)\binom{n}{3}$ triangles, where $a = e(G)/\binom{n}{2}$, can be made isomorphic to some graph in $\mathcal{C}H$ by changing at most $\varepsilon\binom{n}{2}$ adjacencies.*

A PROOF OF OHBA'S CONJECTURE

Bruce Reed *joint with J. Noel and H. Wu*

List colouring is a variation on classical graph colouring. An instance of list colouring is obtained by assigning to each vertex v of a graph G a list $L(v)$ of *available colours*. An *acceptable colouring* for L is a proper colouring f of G such that $f(v) \in L(v)$ for all $v \in V(G)$. When an acceptable colouring for L exists, we say that G is *L-colourable*. The *list chromatic number* χ_ℓ is defined in analogy to the chromatic number:

$$\chi_\ell(G) = \min\{k : G \text{ is } L\text{-colourable whenever } |L(v)| \geq k \text{ for all } v \in V(G)\}.$$

List colouring was introduced by Vizing, and independently by Erdos, Rubin, and Taylor, and researchers have devoted a considerable amount of energy towards its study ever since.

A graph G has an ordinary k-colouring precisely if it has an acceptable colouring for L where $L(v) = \{1,2,\ldots,k\}$ for all $v \in V(G)$. Therefore, the following bound is immediate:

$$\chi \leq \chi_\ell.$$

At first glance, one might expect the reverse inequality to hold as well. It would seem that having smaller intersection between colour lists could only make it easier to find an acceptable colouring. However, this intuition is misleading, there are bipartite graphs with arbitrary high list-chromatic number.

The problem of determining which graphs satisfy $\chi_\ell = \chi$ is well studied; such graphs are said to be *chromatic-choosable*.

For example, the famous List Colouring Conjecture claims that every line graph is chromatic-choosable. It first appeared in print in a paper of Bollobás and Harris, but had also been formulated independently by Albertson and Collins, Gupta, and Vizing. Galvin showed that the List Colouring Conjecture is true for line graphs of bipartite graphs. Kahn proved that the List Colouring Conjecture is asymptotically correct.

We will present a proof of the following conjecture of Ohba:

Conjecture 2 (Ohba). *If $|V(G)| \leq 2\chi(G)+1$, then G is chromatic-choosable.*

ON A CONJECTURE OF ERDOS AND SIMONOVITS ON BIPARTITE TURÁN NUMBERS

Jacques Verstraëte *joint with P. Allen, P. Keevash and B. Sudakov*

Let $\mathcal{C}_k = \{C_3, C_5, \ldots, C_k\}$ for an odd integer k denote the family of all odd cycles of length at most k and let $\mathcal{C}$ denote the family of all odd cycles. Erdos and Simonovits conjectured that for every family $\mathcal{F}$ of bipartite graphs, there exists k such that $\text{ex}(n, \mathcal{F} \cup \mathcal{C}_k) \sim \text{ex}(n, \mathcal{F} \cup \mathcal{C})$ as $n \to \infty$. This conjecture was proved by Erdos and Simonovits when $\mathcal{F} = \{C_4\}$, a result that has since been extended to certain families of even cycles. In this paper, we give a general approach to the conjecture using Scott's sparse regularity lemma. Our approach proves the conjecture for complete bipartite graphs $\mathcal{F} = \{K_{2,t}\}$ and $\mathcal{F} = \{K_{3,3}\}$: we obtain more strongly that for any odd $k \geq 5$,

$$\text{ex}(n, \mathcal{F} \cup \{C_k\}) \sim \text{ex}(n, \mathcal{F} \cup \mathcal{C})$$

and we show further that the extremal graphs can be made bipartite by deleting very few edges. It is natural to ask whether for large enough n, extremal $\mathcal{F} \cup \{C_k\}$-free graphs are exactly bipartite. We prove that this is true for those of large enough minimum degree, for instance, if $k \geq 5$ is odd and G is a $\{C_4, C_k\}$-free n-vertex non-bipartite graph, then G has a vertex v such that $d(v) \leq \sqrt{2n/5} + o(\sqrt{n})$. In contrast, these results do not extend to triangles – the case $k = 3$ – and we give an algebraic construction for odd $t \geq 3$ of $K_{2,t}$-free C_3-free graphs with substantially more edges than an extremal $K_{2,t}$-free bipartite graph on n vertices. Our general approach to the Erdos-Simonovits conjecture is effective based on some reasonable assumptions on the maximum number of edges in an m by n bipartite $\mathcal{F}$-free graph.

THE TURÁN NUMBER OF SPARSE SPANNING GRAPHS
Raphael Yuster *joint with N. Alon*

For a graph H, the *extremal number* $ex(n, H)$ is the maximum number of edges in a graph of order n not containing a subgraph isomorphic to H. Let $\delta(H) > 0$ and $\Delta(H)$ denote the minimum degree and maximum degree of H, respectively. We prove that for all n sufficiently large, if H is any graph of order n with $\Delta(H) \leq \sqrt{n}/40$, then $ex(n, H) = \binom{n-1}{2} + \delta(H) - 1$. The condition on the maximum degree is tight up to a constant factor. This generalizes a classical result of Ore for the case $H = C_n$, and resolves, in a strong form, a conjecture of Glebov, Person, and Weps for the case of graphs. A counter-example to their more general conjecture concerning the extremal number of bounded degree spanning hypergraphs is also given.

The proof of the main result is constructive. We consider the equivalent packing version of the problem where G is a graph with $n - \delta - 1$ edges, where $\delta = \delta(H)$. It suffices to prove that G and H pack. The proof constructs a bijection $f : V(G) \to V(H)$ such that for all $(u, v) \in E(G)$, $(f(u), f(v)) \notin E(H)$ (the packing property). The initial step consists of packing the large degree vertices of G (in decreasing order), and some random subsets of independent non-neighbors of them, with appropriate vertices of H. This, however, can only be done iteratively until some point, where one has to switch to a more "global-approach" of packing the remaining vertices all at once. We make sure, by maintaining certain invariants (whp), that when arriving at this final stage, the global packing can indeed succeed as it satisfies (whp) sufficient conditions for a perfect matching between the yet-unpacked vertices of G and the yet-unpacked vertices of H that does not violate the packing property.

Extremal Set Theory

2-CANCELLATIVE CODES, AN ALGEBRAIC CONSTRUCTION FOR A HYPERGRAPH TURÁN PROBLEM
Zoltan Furedi

There are many instances in Coding Theory when codewords must be restored from partial information, like defected data (error correcting codes), or some superposition of the strings. These lead to superimposed codes, a close relative of group testing problems.

There are lots of versions and related problems, like Sidon sets, sum-free sets, union-free families, locally thin families, cover-free codes and families, etc. We discuss here *cancellative* codes, esp. 2-cancellative uniform hypergraphs.

A family $\mathcal{F}$ is called 2-*cancellative* if for all distinct four members A, B, C, D we have

$$A \cup B \cup C \neq A \cup B \cup D.$$

Our main result is to determine the order of magnitude of the largest $2r$-uniform 2-cancellative hypergraph on n vertices. The proof of the lower bound is an almost explicit construction obtained by a combination of the probabilistic and the algebraic methods.

ON THE CONSTRUCTION OF 3-CHROMATIC HYPERGRAPHS WITH FEW EDGES
Heidi Gebauer

A *hypergraph* is a pair (V, E), where V is a finite set whose elements are called *vertices* and E is a family of subsets of V, called *hyperedges*. A hypergraph is n*-uniform* if every hyperedge contains exactly n vertices. An r-coloring of a hypergraph (V, E) is a mapping $c : V \to \{1, \dots, r\}$. An r-coloring c is proper if no edge in E is monochromatic under c. The *chromatic number* $\chi(H)$ of a hypergraph H is the minimum r such that H admits a proper r-coloring. A hypergraph H is r*-chromatic* if $\chi(H) = r$.

The minimum number $m(n)$ of hyperedges in a 3-chromatic n-uniform hypergraph has been widely studied in the literature. Erdos found that $2^{n-1} \leq m(n) \leq O(n^2 2^n)$. The lower bound was improved in a sequence of publications: Schmidt showed that $m(n) \geq (1 - \frac{2}{n})2^n$, then Beck increased the factor $(1 - \frac{2}{n})$ to $n^{\frac{1}{3}-o(1)}2^n$. A simpler proof of Beck's result was given by Spencer. The best known lower bound is due to Radhakrishnan and Srinivasan, who proved that $m(n) \geq 0.7 \left(\frac{n}{\ln n}\right)^{\frac{1}{2}} 2^n$. In 2009 Pluhár gave a very short and elegant proof that $m(n)$ is at least $2^n f(n)$ where $f(n)$ goes to infinity. (In his paper, $f(n)$ is roughly $n^{\frac{1}{4}}$.)

The upper bound of Erdos is still the best we know. The proof uses the probabilistic method and does not indicate how the corresponding hypergraph can be constructed. We investigate a constructive upper bound on $m(n)$. The only previous result we are aware of is the trivial bound $m(n) \leq \binom{2n-1}{n}$, which is achieved by the hypergraph on the vertex set $V = \{v_1, \dots, v_{2n-1}\}$ where the edge set consists of all subsets of V of cardinality n. In this talk we give an explicit construction of a 3-chromatic n-uniform hypergraph with at most $2^{(1+o(1))n}$ hyperedges. Our technique can also be used to describe n-uniform hypergraphs with chromatic number at least $r + 1$ and at most $r^{(1+o(1))n}$ hyperedges, for every $r \geq 3$.

LAGRANGIANS OF INTERSECTING FAMILIES AND TURÁN NUMBERS OF HYPERGRAPHS
Dan Hefetz *joint with P. Keevash*

It is a central problem in Extremal Combinatorics to determine the Turán number of an r-graph F, that is, the maximum possible number of edges in an F-free r-graph on n vertices (at least for sufficiently large n). The special case of graphs (that is, $r = 2$) is mostly solved. However, to date very little is known if $r \geq 3$, even asymptotically. Recent years have witnessed increasing interest and the development of novel methods leading to new results.

In this talk we prove a Turán type theorem for the 3-graph $\mathcal{K}^3_{3,3}$ whose vertices are $\{x_i, y_i : 1 \leq i \leq 3\}$ and $\{z_{ij} : 1 \leq i, j \leq 3\}$ and whose edges are $\{x_1, x_2, x_3\}$, $\{y_1, y_2, y_3\}$ and $\{\{x_i, y_j, z_{ij}\} : 1 \leq i, j \leq 3\}$. We prove that for large n, the unique largest $\mathcal{K}^3_{3,3}$-free 3-graph on n vertices is a balanced blowup of the complete 3-graph on 5 vertices. Our proof consists of three stages. First we determine the Turán density of $\mathcal{K}^3_{3,3}$ using lagrangians of intersecting families. We then use the stability method to prove that every $\mathcal{K}^3_{3,3}$-free 3-graph with sufficiently many edges is close to a balanced blowup of the complete 3-graph on 5 vertices. Finally, we use this stability result to determine exactly the Turán number of $\mathcal{K}^3_{3,3}$ and to prove uniqueness.

Ramsey Theory

THE CRITICAL WINDOW FOR THE CLASSICAL RAMSEY-TURÁN PROBLEM
Po-Shen Loh *joint with J. Fox and Y. Zhao*

The first application of Szemerédi's powerful regularity method was the following celebrated Ramsey-Turán result proved by Szemerédi in 1972: any K_4-free graph on n vertices with independence number $o(n)$ has at most $(\frac{1}{8} + o(1))n^2$ edges. Four years later, Bollobás and Erdos gave a surprising geometric construction, utilizing the isoperimetric inequality for the high dimensional sphere, of a K_4-free graph on n vertices with independence number $o(n)$ and $(\frac{1}{8} - o(1))n^2$ edges. Starting with Bollobás and Erdos in 1976, several problems have been asked on estimating the minimum possible independence number in the critical window, when the number of edges is about $n^2/8$. These problems have received considerable attention and remained one of the main open problems in this area. In this paper, we give nearly best-possible bounds, solving the various open problems concerning this critical window.

More generally, it remains an important problem to determine if, for certain applications of the regularity method, alternative proofs exist which avoid using the regularity lemma and give better quantitative estimates.

In their survey on the regularity method, Komlós, Shokoufandeh, Simonovits, and Szemerédi surmised that the regularity method is likely unavoidable for applications where the extremal graph has densities in the regular partition bounded away from 0 and 1. In particular, they thought this should be the case for Szemerédi's result on the Ramsey-Turán problem. Contrary to this philosophy, we develop new regularity-free methods which give a linear dependence, which is tight, between the parameters in Szemerédi's result on the Ramsey-Turán problem.

A PROBLEM OF ERDOS ON THE MINIMUM NUMBER OF k-CLIQUES
Jie Ma *joint with S. Das, H. Huang, H. Naves and B. Sudakov*

Fifty years ago Erdos asked to determine the minimum number of k-cliques in a graph on n vertices with independence number less than l (we will refer this as (k, l)-problem). He conjectured that this minimum is achieved by the disjoint union of $l - 1$ complete graphs of size $\frac{n}{l-1}$. This conjecture was disproved by Nikiforov who showed that Erdos' conjecture can be true only for finite many pairs of (k, l). For $(4, 3)$-problem, Nikiforov further conjectured that the balanced blow-up of a 5-cycle, which has fewer 4-cliques than the union of 2 complete graphs of size $\frac{n}{2}$, achieves the minimum number of 4-cliques.

Using a combination of explicit and random counterexamples, we first sharpen Nikiforov's result and show that Erdos' conjecture is false whenever $k \geq 4$ or $k = 3, l \geq 2074$. After introducing tools (including Flag Algebra) used in our proofs, we state our main theorems, which characterize the precise structure of extremal examples for $(3, 4)$-problem and $(4, 3)$-problem, confirming Erdos' conjecture for $(k, l) = (3, 4)$ and Nikiforov's conjecture for $(k, l) = (4, 3)$. We then focus on $(4, 3)$-problem and sketch the proof how we use stability arguments to get the extremal graphs, the balanced blow-ups of 5-cycle.

DENSITIES OF CLIQUES AND INDEPENDENT SETS IN GRAPHS
Humberto Naves *joint with H. Huang, N. Linial, Y. Peled and B. Sudakov*

A variety of problems in extremal combinatorics can be stated as: For two given graphs H_1 and H_2, if the number of induced copies of H_1 in a n-vertex graph G is known, what is the maximum or minimum number of induced copies of H_2 in G? Numerous cases of this question were studied by Turán, Erdos, Kruskal and Katona, and several others. Turán proved that the maximal edge density in any graph with no cliques of size r is attained by an $r-1$ partite graph. Kruskal and Katona found that cliques, among all graphs, maximize the number of induced copies of K_s when $r < s$ and the number of induced copies of K_r is fixed. In this talk, we discuss the following analogue of the Kruskal-Katona theorem: If the complement of a graph has fixed induced K_r-density, when is the induced K_s-density maximized? Using the technique of shifting borrowed from extremal set theory and some powerful analytical methods, one can demonstrate that the extremal graph for our proposed question is either a clique or the complement of a clique.

RAMSEY THEORY, INTEGER PARTITIONS AND A NEW PROOF OF THE ERDOS-SZEKERES THEOREM
Asaf Shapira *joint with G. Moshkovitz*

Let H be a k-uniform hypergraph whose vertices are the integers $1, \ldots, N$. We say that H contains a monotone path of length n if there are $x_1 < x_2 < \cdots < x_{n+k-1}$ so that H contains all n edges of the form $\{x_i, x_{i+1}, \ldots, x_{i+k-1}\}$. Let $N_k(q, n)$ be the smallest integer N so that every q-coloring of the edges of the complete k-uniform hypergraph on N vertices contains a monochromatic monotone path of length n.

While the study of $N_k(q, n)$ for specific values of k and q goes back (implicitly) to the seminal 1935 paper of Erdos and Szekeres, the problem of bounding $N_k(q, n)$ for arbitrary k and q was studied by Fox, Pach, Sudakov and Suk.

Our main contribution here is a novel approach for bounding the Ramsey-type numbers $N_k(q, n)$, based on establishing a surprisingly tight connection between them and the enumerative problem of counting high-dimensional integer partitions. Some of the concrete results we obtain using this approach are the following:

- We show that for every fixed q we have $N_3(q, n) = 2^{\Theta(n^{q-1})}$, thus resolving an open problem raised by Fox et al.
- We show that for every $k \geq 3$, $N_k(2, n) = 2^{\cdot^{\cdot^{2^{(2-o(1))n}}}}$ where the height of the tower is $k-2$, thus resolving an open problem raised by Eliáš and Matoušek.
- We give a new pigeonhole proof of the Erdos-Szekeres Theorem on cups-vs-caps, similar to Seidenberg's proof of the Erdos-Szekeres Lemma on increasing/decreasing subsequences.

Probabilistic Combinatorics INDEPENDENT SETS IN HYPERGRAPHS

Joszef Balogh *joint with R. Morris and W. Samotij*

Many important theorems and conjectures in combinatorics, such as the theorem of Szemerédi on arithmetic progressions and the Erdos-Stone Theorem in extremal graph theory, can be phrased as statements about families of independent sets in certain uniform hypergraphs. In recent years, an important trend in the area has been to extend such classical results to the so-called 'sparse random setting'. This line of research has recently culminated in the breakthroughs of Conlon and Gowers and of Schacht, who developed general tools for solving problems of this type. Although these two papers solved very similar sets of longstanding open problems, the methods used are very different from one another and have different strengths and weaknesses.

In this talk, we explain a third, completely different approach to proving extremal and structural results in sparse random sets that also yields their natural 'counting' counterparts. We give a structural characterization of the independent sets in a large class of uniform hypergraphs by showing that every independent set is almost contained in one of a small number of relatively sparse sets. We then derive many interesting results as fairly straightforward consequences of this abstract theorem. In particular, we prove the well-known conjecture of Kohayakawa, Luczak, and Rodl, a probabilistic embedding lemma for sparse graphs, for all 2-balanced graphs. We also give alternative proofs of many of the results of Conlon and Gowers and of Schacht, such as sparse random versions of Szemerédi's theorem, the Erdos-Stone Theorem and the Erdos-Simonovits Stability Theorem, and obtain their natural 'counting' versions, which in some cases are considerably stronger. We also obtain new results, such as a sparse version of the Erdos-Frankl-Rödl Theorem on the number of H-free graphs and, as a consequence of the KLR conjecture, we extend a result of Rödl and Ruciński on Ramsey properties in sparse random graphs to the general, non-symmetric setting. Similar results have been discovered independently by Saxton and Thomason.

SELF CORRECTING ESTIMATES FOR THE TRIANGLE-FREE PROCESS

Tom Bohman *joint with P. Keevash*

Consider the triangle-free process. We start with $G(0)$ which is the empty graph on n vertices. Given $G(i)$, let $O(i)$ be the set of pairs xy such that xy is not an edge in $G(i)$ and $G(i) + xy$ does not contain a triangle. The edge e_{i+1} is chosen uniformly at random from $O(i)$, and we set $G(i+1) = G(i) + e_{i+1}$. This process terminates at a maximal triangle-free graph $G(M)$.

In previous work, the speaker showed that with high probability we have $M = \Theta\left(n^{3/2}\sqrt{\log n}\right)$. In this talk we present a refinement of that argument that gives a more precise result. With high probability we have

$$M = \left(\frac{1}{2\sqrt{2}} + o(1)\right) n^{3/2}\sqrt{\log n}.$$

The new argument makes use of the fact that many keep statistics of the process exhibit self-correcting behavior; that is, when these variables deviate substantially from their expected trajectory they are subject to a drift that brings them back toward the trajectory.

We also establish a high probability upper bound on the independence number of $G(M)$. It follows from this bound that we have

$$R(3,k) \geq \left(\frac{1}{4} + o(1)\right)\frac{k^2}{\log k},$$

where the Ramsey number $R(3,k)$ is the smallest n such that every graph on n vertices has a triangle or an independent set on k vertices.

Similar results have been obtained independently by Gonzalo Fiz Pontiveros, Simon Griffiths, Robert Morris and Roberto Imbuzeiro Oliveira.

ON THE KLR CONJECTURE IN RANDOM GRAPHS

David Conlon *joint with T. Gowers, W. Samotij and M. Schacht*

The KLR conjecture of Kohayakawa, Luczak, and Rödl is a statement that allows one to prove that asymptotically almost surely all subgraphs of the random graph $G_{n,p}$, for sufficiently large $p := p(n)$, satisfy an embedding lemma which complements the sparse regularity lemma of Kohayakawa and Rödl. We prove a variant of this conjecture which is sufficient for most applications to random graphs. In particular, our result implies a number of recent probabilistic versions of classical extremal combinatorial theorems due to Conlon, Gowers, and Schacht. We also discuss several further applications.

CYCLES IN RANDOM SUBGRAPHS OF GRAPHS
Choongbum Lee *joint with M. Krivelevich and B. Sudakov*

Many theorems in graph theory establishes the fact that if a graph G satisfies certain property $\mathcal{C}$, then it has some property $\mathcal{P}$. One example is Dirac's theorem, which asserts that every graph on n vertices of minimum at least $\frac{n}{2}$ is Hamiltonian. Once such a result is established, it is natural to ask, "How strongly does G satisfy $\mathcal{P}$"? Recently, there has been a series of work answering this question in various ways for several theroems. In this context, we further study the following classical theorem in more depth: "every graph G of minimum degree at least k contains a path of length at least k and a cycle of length at least $k+1$".

For a given finite graph G of minimum degree at least k, let G_p be a random subgraph of G obtained by taking each edge independently with probability p. We show that (i) if $p \geq \omega/k$ for a function $\omega = \omega(k)$ that tends to infinity as k does, then G_p asymptotically almost surely contains a cycle (and thus a path) of length at least $(1-o(1))k$, and (ii) if $p \geq (1+o(1)) \ln k/k$, then G_p asymptotically almost surely contains a path of length at least k.

The binomial random graph $G(n,p)$ is G_p for $G = K_n$. It is known that $G(n,p)$ contains a path of length $(1-o(1))n$ if $p \gg \frac{1}{n}$, and is Hamiltonian if $p \gg (1+o(1)) \ln n/n$. Our theroems can also be viewed as extensions of these theorems, by taking the graph G to be the complete graph K_k.

EIGENVALUES AND QUASIRANDOM HYPERGRAPHS
Dhruv Mubayi *joint with J. Lenz*

Chung and Graham began the systematic study of hypergraph quasirandom properties soon after the foundational results of Thomason and Chung-Graham-Wilson on quasirandom graphs. One feature that became apparent in the early work on hypergraph quasirandomness is that properties that are equivalent for graphs are not equivalent for hypergraphs, and thus hypergraphs enjoy a variety of inequivalent quasirandom properties. In the past two decades, there has been an intensive study of these disparate notions of quasirandomness for hypergraphs, and a fundamental open problem that has emerged is to determine the relationship between these quasirandom properties. Our first result completely determines the poset of implications between essentially all hypergraph quasirandom properties that have been studied in the literature. This answers a recent question of Chung, and in some sense completes the project begun by Chung and Graham in their first paper on hypergraph quasirandomness in the early 1990's.

Let $p(k)$ denote the partition function of k. For each $k \geq 2$, our second result describes a list of $p(k) - 1$ quasirandom properties that a k-uniform hypergraph can have. These new properties connect previous notions on hypergraph quasirandomness, beginning with the early work of Chung and Graham and Frankl-Rödl related to strong hypergraph regularity, the spectral approach of Friedman-Wigderson, and more recent results of Kohayakawa-Rödl-Skokan and Conlon-Hàn-Person-Schacht on weak hypergraph regularity and its relation to counting linear hypergraphs.

For each of the quasirandom properties that are described, we define a hypergraph eigenvalue analogous to the graph case (thereby answering a question of Conlon et. al.) and a hypergraph extension of a graph cycle of even length whose count determines if a hypergraph satisfies the property. Our work can therefore be viewed as an extension to hypergraphs of the seminal results of Chung-Graham-Wilson for graphs.

THE EVOLUTION OF THE ACHLIOPTAS PROCESSES
Lutz Warnke *joint with O. Riordan*

In the *Erdos–Rényi random graph process*, starting from an empty graph, in each step a new random edge is added to the evolving graph. One of its most interesting features, both mathematically and in terms of applications, is the 'percolation phase transition': as the ratio of the number of edges to vertices increases past a certain critical point, the global structure changes radically, from only small components to a single macroscopic ('giant') component plus small ones.

We study *Achlioptas processes*, which are widely studied variations of the classical Erdos–Rényi process. Starting from an empty graph these proceed as follows: in each step *two* potential edges are chosen uniformly at random, and using some rule *one* of them is selected and added to the evolving graph. These processes were introduced by Dimitris Achlioptas in 2000, and he originally asked whether there is a rule which substantially delays the appearance of the linear size 'giant' component compared to the classical case. In fact, many simulations suggested that for certain rules the percolation phase transition is particularly radical: more or less as soon as the macroscopic component appears, it is already extremely large; this phenomenon is known as 'explosive percolation'. In particular, Achlioptas, D'Souza and Spencer recently presented 'conclusive numerical evidence' for the conjecture that the largest component can grow from size at most $\sqrt{n}$ to size at least $n/2$ in at most $2n^{2/3}$ steps.

We first disprove this conjecture in a strong form: no matter which rule is used, we show it is impossible to obtain explosive percolation. Afterwards we discuss some recent progress in our mathematical understanding of Achlioptas processes, which is based on a new approach for proving convergence to the solution of a system of differential equations.

Extremal Combinatorics in Algebra

STABLE DICTATORS AND JUNTAS IN THE SYMMETRIC GROUP

Ehud Friedgut *joint with D. Ellis and Y. Filmus*

A useful theme in the application of analytical methods in combinatorics is the attempt to characterize combinatorial structures that depend on a single coordinate (a dictatorship), or on few coordinates (a junta). An important feature of this method is the ability to prove combinatorial stability results via analytical stability results. In this talk I present such a result in the symmetric group. The result here is complemented by the results that David Ellis presented in his talk.

Some definitions and background to the main result. Let V_1 denote the space of all functions on S_n that have their Fourier transform concentrated on the irreducible representations contained in the permutation representation (i.e. the trivial, and the $(n-1)$-dimensional one). A result of Ellis, Friedgut and Pilpel shows that the only Boolean functions in V_1 are the indicators of disjoint unions of cosets of the stabilizer of a point. Here I present a stability version of this.

Theorem: Let $f : S_n \to \{-1, 1\}$, and let $E[f] = 0$. Let f_1 be the orthogonal projection of f onto V_1, and let $E[(f - f_1)^2] \le \epsilon$. Then there exists a Boolean function g in V_1 such that $E[(f - g)^2] = O(\epsilon^{1/7})$.

STABLE JUNTAS IN THE SYMMETRIC GROUP, II

David Ellis *joint with Y. Filmus and E. Friedgut*

We prove that a Boolean function on S_n of expectation $O(1/n)$, whose Fourier transform is highly concentrated on the first two irreducible representations of S_n, is close in structure to a union of *cosets of point-stabilizers* (or *1-cosets*, for short).

This phenomenon is not 'stability' in the strongest sense. Indeed, a Boolean function f has its Fourier transform completely supported on the first two irreducible representations of S_n, if and only if it is a *dictatorship* (a function determined by the image or the preimage of just one element.) A union of two 1-cosets which are *not* disjoint is not close in structure to any of these, and yet its characteristic function *does* have its Fourier transform highly concentrated on the first two irreducible representations of S_n. We may call our result a 'quasi-stability' result.

We use our result to give a natural proof of a stability result on intersecting families of permutations, originally conjectured by Cameron and Ku, and first proved by the first author. We also use it to prove a 'quasi-stability' result for an edge-isoperimetric inequality in the transposition graph on S_n, namely that subsets of S_n with small edge-boundary in the transposition graph are close to being unions of cosets of point-stabilizers.

In the talk, we sketch the proof of our quasi-stability result. This relies on analysing the second and third moments of a (non-negative) affine shift of the relevant projection of the Boolean function. We choose a useful representation of this affine shift, in terms of the expectations of the function restricted to 1-cosets.

We are also able to prove a generalization of our quasi-stability result, dealing with Boolean functions on S_n whose Fourier transform is close to being of 'degree t', for fixed t. Namely, we show that prove that a Boolean function on S_n of expectation $O(1/n^t)$, whose Fourier transform is highly concentrated on irreducible representations of S_n corresponding to Young diagrams with at most t cells below the first row, must be close in structure to a union of cosets of stabilizers of ordered t-tuples. The proof of this uses some technical lemmas on the representations of the symmetric group, and is considerably more involved.

We use the general result to prove an exact edge-isoperimetric inequality for the symmetric group: namely, that if n is large depending on t, and $\mathcal{A} \subset S_n$ with $|\mathcal{A}| = (n-t)!$, then the edge-boundary of

$\mathcal{A}$ (in the transposition graph on S_n) is minimized when $\mathcal{A}$ is a coset of a stabilizer of t points. This verifies a conjecture of Ben-Efraim in these cases.

WHAT ARE HIGH-DIMENSIONAL PERMUTATIONS AND HOW MANY ARE THEY?
Nathan Linial *joint with Z. Luria*

What is the higher-dimensional analog of a permutation? If we think of a permutation as given by a permutation matrix, then the following definition suggests itself: A d-dimensional permutation of order n is an $n \times n \times \ldots n = [n]^{d+1}$ array of zeros and ones in which every *line* contains a unique 1 entry. A line here is a set of entries of the form $\{(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{d+1}) | n \geq y \geq 1\}$ for some index $d+1 \geq i \geq 1$ and some choice of $x_j \in [n]$ for all $j \neq i$. It is easy to observe that a one-dimensional permutation is simply a permutation matrix and that a two-dimensional permutation is synonymous with an order-n Latin square. We seek an estimate for the number of d-dimensional permutations. Our main result is the following upper bound on their number

$$\left((1+o(1))\frac{n}{e^d}\right)^{n^d}.$$

We tend to believe that this is actually the correct number, but the problem of proving the complementary lower bound remains open. Our main tool is an adaptation of Brègman's proof of the Minc conjecture on permanents. More concretely, our approach is very close in spirit to Schrijver's and Radhakrishnan's proofs of Brègman's theorem.

This result is part of our ongoing effort to seek higher-dimensional counterparts for many basic concepts in combinatorics.

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Chapter 31

Groups and Geometries (12w5034)

September 2 - 7, 2012

Organizer(s): Inna Capdebosq (Warwick, UK), Martin Liebeck (Imperial College London, UK), Bernhard Mühlherr (Giessen, Germany), Gernot Stroth (Halle, Germany)

Overview of the Field, and Recent Developments

As groups are just the mathematical way to investigate symmetries, it is clear that a significant number of problems from various areas of mathematics can be translated into specialized problems about finite permutation groups, linear groups, algebraic groups, and so on. In order to go about solving these problems a good understanding of the finite groups, especially the simple ones, is necessary. Examples of this procedure can be found in questions arising from algebraic geometry, in applications to the study of algebraic curves, in communication theory, in arithmetic groups, model theory, computational algebra and random walks in Markov theory. Hence it is important to improve our understanding of groups in order to be able to answer the questions raised by all these areas of application.

The research areas covered at the meeting falls into three main areas.

Fusion systems and new approaches to the classification of finite simple groups

A fusion system over a p-group S is a category whose objects form the set of all subgroups of S, whose morphisms are certain injective group homomorphisms, and which satisfies axioms that are modelled on conjugacy relations in finite groups. The definition was originally motivated by representation theory, but fusion systems also have applications to local group theory and to homotopy theory. It is on the applications to finite group theory that we focus here. In this respect the basic theory – such as how to define normal subsystems of fusion systems, and so on – has only been developed in the past few years (see for example[1], [10]). Yet there are now profound, if currently speculative, ideas of Aschbacher, Chermak and others about a possible program to use the theory of fusion systems to simplify a huge part of the classification of finite simple groups which deals with the groups of component type.

Another new approach is the Meierfrankenfeld-Stellmacher-Stroth program [11]. For a prime p, one says that a finite group G has characteristic p if $C_G(O_p(G)) \leq O_p(G)$, where $O_p(G)$ is the largest normal p-subgroup of G; and G is said to have local characteristic p if every p-local subgroup of G has characteristic p. Broadly speaking, the MSS program attempts to understand (and classify) the finite groups of local characteristic p. Success in this program would replace a huge part of the original classification of finite simple groups. The methods involved are diverse, involving amalgams, representation theory as well as local group theory.

Finally, there is of course the original Gorenstein-Lyons-Solomon program [5], which aims at giving a more or less self contained proof of the classification of finite simple groups within eleven monographs. The first six have been completed and published, and much of the material for the remaining volumes is currently in manuscript form. Two of the outstanding problems remaining to be solved are the well-known “$e(G) = 3$" and “uniqueness"cases, on which several of the conference participants are working.

Geometry and groups

There are many ways in which group theory is related to geometry. The most appropriate geometric tools for the investigation of finite simple groups are their actions on combinatorial objects, in particular buildings. The recent highlights of this interaction are the notion of complete reducibility introduced by Serre in around 2000 which yielded a proof of the center conjecture in 2009, and the investigation of Phan-presentations of finite simple groups, which constituted the starting point for the proof of the rank conjecture for arithmetic groups over function fields.

The importance of buildings for finite group theory is due to the fact that they provide an efficient tool to study exceptional groups of Lie-type. Most beautiful examples for this are the recent application of Tits' extension theorem for spherical buildings in the MSS-programme and the theory of Phan-presentations for exceptional groups. In this context, Van Maldeghem and his collaborators are currently working on a program which is designed to provide more combinatorial information about exceptional groups by a suitable interpretation of the Freudenthal-Tits magic square in the split case. Another focus of current research is affine buildings and in particular trees. Using geometric methods Struyve was able to accomplish the solution of the existence problem for affine buildings which was open since the development of Bruhat-Tits theory in the 1970. Finite group theory is most important in the investigation of locally finite affine buildings. Recent examples are provided by the work of Caprace and De Medts on Moufang sets at infinity of a tree and Grüninger's work on Moufang sets at infinity of affine Moufang twin trees.

Applications of simple group theory

There are many applications of the theory of simple groups, particularly the classification theorem of finite simple groups. Here we discuss two such recent applications.

The first application concerns the theory of expansion in finite groups. A famous result of Helfgott [7] asserts that arbitrary generating sets A in $SL(2, p)$ expand, in the sense that the product of three copies of A is either significantly larger than A, or is already equal to the whole group. This result has turned out to have quite a range of applications, from the theory of expander graphs to the development of new sieve methods in analytic number theory. Recently, Breuillard, Green and Tao [3], and independently Pyber and Szabo [12], have spectacularly generalised Helfgott's result to all families of finite groups of Lie type, using methods of additive combinatorics and model theory as well as the structure theory of the groups. These methods seem to offer scope for further applications, both within and outside group theory.

The second application concerns algebraic geometry. Guralnick and Tiep [6] have recently proved a number of results, such as the Kollar-Larsen conjecture on holonomy groups of vector bundles on a smooth projective variety. This was done via work of Kollar, Larsen and Katz which translated the conjecture to a question about subgroups of $GL(V)$ which are irreducible on some symmetric power of the vector space V, which was solved by Guralnick and Tiep using detailed structure theory of simple groups together with representation theory and some theory of algebraic groups. They have also proved results on crepant resolutions and quotients of Calabi-Yau varieties, again via translations to group theoretic problems. It seems that there is scope for further applications of group theory to algebraic geometry in this direction.

Presentation Highlights

Fusion systems and the classification

One of the main highlights of the whole meeting was Michael Aschbacher's talk *Fusion systems and groups of component type*. In the talk, Aschbacher discussed a speculative program to use the theory of fusion systems to simplify that part of the classification of the finite simple groups dealing with the groups of component type. Many participants expressed interest in developing the ideas for this program.

Complementing Aschbacher's talk was the lecture of Andy Chermak, *The normal structure of linking systems*. He discussed the category of partial groups and the subcategory of objective partial groups. Among these are the finite group-like "centric linking systems". He focussed on normal subsystems of linking systems and on the problem of how to view these as linking systems in their own right. The results outlined are in analogy with Aschbacher's theory of normal subsystems of fusion systems.

Another highlight in this area was the lecture of Ellen Henke, *Cohomology, F-Isomorphism and Fusion in Finite Groups*. This talk was about the proof of a general group theoretical result which has implications for mod p cohomology and higher chromatic cohomology theories. The result is as follows: suppose we are given a finite group G, an odd prime p, and a subgroup H of G containing a Sylow p-subgroup of G. Then H controls fusion in G if and only if it controls fusion of elementary abelian subgroups. The analogous result is true for $p = 2$ if one considers abelian subgroups of exponent at most 4 instead of elementary abelian subgroups. The most striking fact about the proof is is that it is carried out in the category of fusion systems rather than in the category of groups.

Concerning other parts of the revision program, there were talks by Ron Solomon, Kay Magaard and Chris Parker. Magaard (*Groups of even type which are not of even characteristic*) and Parker (*Groups which are almost Lie type*) talked about recent contributions to the Meirerfrankenfeld-Stellmacher-Stroth program, while Solomon (*Characterizing Lie Type Groups*) reported on a recent major theorem with Richard Lyons which forms an important part of the Gorenstein-Lyons-Solomon program.

Geometry and groups

One of the main themes in this area discussed at the meeting was the topic of buildings. A highlight was the talk of Richard Weiss (*The local structure of Bruhat-Tits buildings*). Roughly speaking, Bruhat-Tits buildings are classified by spherical buildings defined over a field K that is complete with respect to a discrete valuation. In particular, each spherical building Δ defined over a complete field K is the "building at infinity" of a unique Bruhat-Tits building X, and the residues of the building X are spherical buildings defined over the residue field $\bar{K}$. Bruhat-Tits buildings are not uniquely determined by their residues (in contrast to spherical buildings), but their residues are nevertheless an important structural feature of these buildings. Weiss discussed efforts to give a complete description of all the possibilities that arise for each family of spherical buildings Δ defined over a complete field K.

Another talk in this area was that of Koen Struyve, *Galois descent of Bruhat-Tits buildings*. Bruhat-Tits buildings are a class of affine buildings associated to certain classical, algebraic and mixed groups defined over (skew) fields with a complete valuation. An open question in Bruhat-Tits theory concerned the existence of Bruhat-Tits buildings for these groups, with the remaining open cases pertaining to certain exceptional groups of relative rank 1 and 2. In his talk Struyve announced a general solution of this problem using combinatorial and geometric methods.

Hendrik van Maldeghem, in his talk *Characterizations of groups by geometries and geometries by groups*, presented two recent characterizations of groups and geometries related to buildings. The first one characterizes the "standard orbits" of certain modular representations of the groups of the second row of the Freudenthal-Tits Magic Square by a simple extension of the Mazzocca-Melone axioms that were designed for quadric Veronesean varieties (corresponding, however, to the first cell of the second row of the FT Magic Square). The second characterization is one of the finite Hermitian unital in the spirit of the "Moufang property" for projective planes (and, more generally, twin buildings).

Another highlight on a geometric theme was the talk of Pierre-Emanuel Caprace, *Simple locally compact groups and branching*. The global structure of a Lie group is largely determined by its local structure, encoded in the Lie algebra. In the case of locally compact groups that are non-discrete and totally disconnected, the local structure is given by a commensurability class of profinite groups. Caprace's results illustrate that, when the ambient group is simple and compactly generated, there is also a local-to-global correspondence in that case, although it is not as tight as in Lie theory. This gives rise to the tantalising prospect of a possible classification program for certain types of simple, locally compact, totally disconnected groups.

Other geometric talks were given by Arjeh Cohen (*Constructing Riemann surface models from regular maps*) and Jonathan Hall (*Algebras from groups and geometry*). Cohen described computational methods for finding algebraic models of the smooth complex projective curves (Riemann surfaces) connected to regular maps, and Hall discussed recent progress on some elegant connections between the concept of triality in Lie theory and various types of alternative algebras, tracing the topic from pioneering work of Study, Cartan and Moufang to the present.

Applications

One of the highlights here was Laci Pyber's talk *Growth in linear groups* on his recent spectacular results on growth. His well-known "Product theorem" with Szabo [12] (proved independently by Breuillard, Green and Tao [3]) showed that for a generating set S of a finite group of Lie type G of bounded rank, either S "grows" (in the sense that $|S^3| > |S|^{1+\epsilon}$ for some absolute constant $\epsilon > 0$), or S^3 is the whole of G. This result has had a huge impact, for example proving Babai's conjecture on the poly-logarithmic diameter of Cayley graphs for such groups G, and leading to new families of expander graphs. Pyber's new results (also with Szabo) give precise structural information about arbitrrary (not necessarily generating) subsets S which do not grow. Let S be an inverse-closed subset of $\mathrm{GL}(n, F)$ satisfying $|S^3| < K|S|$ for some $K > 1$, where F is an arbitrary field. Then S is contained in the union of polynomially many (more precisely $K^{c(n)}$) cosets of a finite-by-soluble subgroup G normalised by S. Moreover G has a finite subgroup P normalised by S such that G/P is soluble and S^3 contains a coset of P. This includes the Product theorem mentioned above.

Another talk in this area was given by Nick Gill, *The width of a finite simple group*. He described recent work with Pyber, Short and Szabo on the "Product Decomposition Conjecture" of Liebeck, Nikolov and Shalev [9]. Given a finite simple group G and a subset A of G, the conjecture states that G can be written as a product of the order of $\log |G|/ \log |A|$ conjugates of A. Gill discussed progress on this conjecture, using the Product Theorem of Pyber's talk, and uncovered some interesting new connections between classical additive combinatorics and normal subsets of a group.

Another highlight was the talk of Pham Huu Tiep on his joint work with Guralnick [6], which uses the theory of finite and algebraic linear groups to study questions in algebraic geometry. In particular they have proved some conjectures of Kollar and Larsen concerning stability of vector bundles, quotients of Calabi-Yau varieties, and (non-)existence of crepant resolutions.

On another front, Robert Guralnick (*Maximal Subgroups of Finite Groups*) talked about some extraordinary and unexpected new results on first cohomology groups of simple groups with coefficients in an irreducible module. In many cases the dimensions of these are given by coefficients of Kazhdan-Lusztig polynomials, which are notoriously difficult to compute, but there has been a recent breakthrough. This links up with questions about counting the number of maximal subgroups and the number of conjugacy classes of maximal subgroups of finite groups. The problem naturally splits up into first looking at maximal subgroups of almost simple groups and secondly getting certain bounds on the size of first cohomology groups. Guralnick discussed results with Larsen and Tiep on both problems. In particular, the recent cohomology results imply, most unexpectedly, that Wall's conjecture – stating that the number of maximal subgroups of a finite group G is less than $|G|$ – is false.

There were also several talks on applications to permutation groups by Cheryl Praeger (*Coprime subdegrees for primitive permutation groups and completely reducible linear groups*), Tim Burness (*Bases for algebraic groups*) and Barbara Baumeister (*Permutation Groups and Applications*). Praeger duscussed a remarkable joint result with Dolfi, Guralnick and Spiga [4]: if a group H acts irreducibly on a finite vector space V, then for every pair of non-zero vectors, their orbit lengths have a nontrivial common factor. Burness talked about bases of permutation groups: if G is a permutation group on a set X, a subset B of X is a base for G if the pointwise stabilizer of B in G is trivial. The minimal size of a base is called the base size of G. Bases for finite permutation groups have been widely studied since the nineteenth century, with many new results and applications in recent years. Burness discussed his recent work with with Guralnick and Saxl on bases for primitive actions of simple algebraic groups over algebraically closed fields, computing the exact base size in almost all cases. This leads to new results on base sizes for the corresponding finite groups of Lie type. Baumeister presented examples, applications and characterizations of permutation polytopes.

Scientific Progress Made and Outcome of the Meeting

The presentations given at the meeting and the lively discussions there revealed that there are various important directions of research which are highly interrelated. The exciting developments in the area of fusion system provide new stimulation and insights for the ongoing programs which are closely related to the revision of the classification of the finite simple groups. In particular the talk of Aschbacher elucidated that there is a need to bring both fields closer together. The participants of the meeting were convinced that there is a high potential to make substantial progress in both areas by joining forces.

Concerning the Meierfrankenfeld-Stellmacher-Stroth program, the talk of Parker proved that the endgame – identifying the groups in question – is in good shape, and so we can say there is a promising roadmap for the program. Nevertheless there are still large parts remaining to be done. As Stellmacher told us during the meeting, there is work in progress on these missing parts. Moreover Magaard reported in his talk about new results, which might replace the assumption of local characteristic p with parabolic characteristic p. This would be important in both projects, the MSS and the Gorenstein-Lyons-Solomon programs. It will provide us with a bridge between them. To use this there must be some alterations in the MSS-program, and there is work in progress on this. Ron Solomon presented recent progress concerning Gorenstein-Lyons-Solomon program. The highlight of his talk was a theorem that essentially gives a characterization of large alternating groups and groups of Lie type of higher ranks. This result will be the final milestone for the completion of volume 7 of the GLS-series. His talk was followed by further discussions by the participants involved in the GLS-program about the progress of the program and futher steps to be taken.

Caprace presented in his talk an exciting new perspective by highlighting the importance of the classification of finite simple groups for the theory of totally disconnected locally compact groups. Groups acting on locally finite buildings provide many important examples of these. The participants working on geometric aspects were especially enthusiastic about this new direction of research. There is a high chance that the material in Caprace's talk marks the starting point for a new program on locally finite buildings, since it reveals questions and problems that had not yet been considered at all. Although the expectations are very high there is still a lot be explored in order to estimate the true impact of this development.

Also concerning the applications of simple group theory, we mention in particular the talk of Tiep in which he described some quite unexpected applications of linear groups to conjectures in algebraic geometry. Following on from this, there is scope for extending these developments into a whole program working towards much more general conjectures, and several participants expressed interest in joining such a program.

A geometric program which is also still in its early stages was presented in Van Maldeghem's talk on the Freudenthal-Tits magic square. Here there are already concrete contributions on the geometry of the related buildings. The remarkable application of giving a higher rank analogue of the Mazocca-Melone characterization for the quadratic Veronesian varieties provides also new geometric insights into this classical work. Several participants expressed their opinion that further work on this ambitious programme will yield much stronger results. There are indeed several problems about geometries related to exceptional groups which have been open more than 20 years – for example questions about embeddability of certain point-line spaces and existence of certain ovoids. Solutions to some of these questions seem now to be within reach. The participants working on Moufang sets expressed their confidence that this work will give a geometric – and thus more concrete – approach to Moufang sets related to exceptional groups. The absence of such an approach is a major obstacle in their attempts to get further in their efforts towards a classification.

Another major geometric theme was the structure theory of affine buildings. Here the emphasis was mainly on the applications of geometric tools which were developed earlier in the context of simple groups of Lie type. Weiss presented in his talk a local structure theory for Bruhat-Tits buildings of type $\tilde{B}_2$. For Bruhat-Tits buildings of rank at least 3, this is by far the hardest case, since there are particular phenomena which cause problems in the context of exceptional quadrangles. One important insight provided by the solution of these problems is that certain exotic examples of Lie-type groups have a natural meaning in the context of semi-simple algebraic groups. During the conference the question about the impact on the theory of semi-simple groups came up as a direction to be explored further.

Final remarks

One major goal of the conference was to bring together researchers who are working in different directions which are closely related to finite simple groups. Apart from maintaining the scientific exchange between these areas there was the intention of creating new perspectives in research and stimulating scientific collaboration. The lectures and the time schedule were designed by the organizers in accordance with these objectives. In the selection of the lectures preference was given to subjects which offered the participants the possibility to learn about new developments in an area. It was also made sure that there remained plenty of time for discussion between the talks.

The feedback of many participants to the organizers was very positive. The outstanding quality of the talks was often mentioned, as well as many new collaborations started and ongoing ones continued. Beside the outstanding scientific level, people enjoyed following the lectures because the speakers succeeded in presenting topics by explaining clearly the main ideas and avoiding technical details. It was also remarked positively, that there was a comparatively high number of young speakers and that all of them gave beautiful lectures.

Finally, we remark that of the 42 participants, 11 were female.

Participants

Aschbacher, Michael (California Institute of Technology)
Baumeister, Barbara (Universitaet Bielefeld)
Burness, Tim (University of Bristol)
Capdeboscq, Inna (University of Warwick)
Caprace, Pierre-Emmanuel (UniversitŐ catholique de Louvain)
Chermak, Andrew (Kansas State University)
Cohen, Arjeh (Eindhoven)
Decelle, Sophie (Imperial College)
Devillers, Alice (University of Western Australia)
Foote, Richard (University of Vermont)
Gill, Nick (Open University)
Giudici, Michael (The University of Western Australia)
Grimm, Mathias (UniversitŁt Halle)
Gr§ninger, Matthias (Universite Catholique de Louvain)
Guralnick, Robert (University of Southern California)
Hall, Jonathan I. (Michigan State University)
Henke, Ellen (University of Copenhagen)
Horn, Max (Justus-Liebig-UniversitŁt Gieğen)
Ivanov, Alexander (Imperial College London)
Kantor, William (Brookline, MA)
Liebeck, Martin (Imperial College London)
Magaard, Kay (University Birmingham)
Malle, Gunter (Technische UniversitŁt Kaiserslautern)
Morgan, Luke (University of Western Australia)
Muehlherr, Bernhard (University of Giessen, Germany)
Parker, Chris (University of Birmingham)
Parmeggiani, Gemma (Universita degli studi di Padova)
Praeger, Cheryl (The University of Western Australia)
Pyber, Laci (Renyi Institute of Mathematics Budapest)
Solomon, Ronald Mark (The Ohio State University)
Steinbach, Anja (Justus Liebig University Giessen)
Stellmacher, Bernd (Universitaet Kiel)
Stroth, Gernot (University of Halle, Germany)
Struyve, Koen (Ghent University)
Testerman, Donna (Ecole Polytechnique Federale de Lausanne)
Tiep, Pham (University of Arizona)
Timmesfeld, Franz (UniversitŁt Gieğen)
Toborg, Imke (Universität Halle)
van Maldeghem, Hendrik (Ghent University)
Waldecker, Rebecca (MLU Halle-Wittenberg)
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Wilson, Robert (Queen Mary London)

Bibliography

[1] M. Aschbacher, Normal subsystems of fusion systems, Proc. Lond. Math. Soc. **97** (2008), 239–271.

[2] M. Bate, B. Martin and G. Röhrle, A geometric approach to complete reducibility, Invent. Math. **161** (2005), 177–218.

[3] E. Breuillard, B. Green and T. Tao, Approximate subgroups of linear groups, Geom. and Funct. Anal. **21** (2011), 774–819.

[4] S. Dolfi, R. Guralnick, C.E. Praeger and P. Spiga, Coprime subdegrees for primitive permutation groups and completely reducible linear groups, Israel J. Math., to appear.

[5] D. Gorenstein, R. Lyons and R. Solomon, The classification of the finite simple groups., Volumes 1–6, Mathematical Surveys and Monographs, 40.1–40.6. American Mathematical Society, Providence, RI.

[6] R. Guralnick and P.H. Tiep, Symmetric powers and a problem of Kollar and Larsen, Invent. Math. **174** (2008), 505–554.

[7] H. Helfgott, Growth and generation in $SL_2(Z/pZ)$, Annals of Math. **167** (2008), 601–623.

[8] B. Leeb and C. Ramos-Cuevas, The center conjecture for spherical buildings of types F_4 and E_6, Geom. Funct. Anal. **21** (2011), 525–559.

[9] M.W. Liebeck, N. Nikolov and A. Shalev, A note on product decompositions in simple groups, Bulletin London Math. Soc., to appear.

[10] M. Linckelmann, Introduction to fusion systems, Group Representation Theory, EPFL Press, Lausanne, 2007, pp. 79–113.

[11] U. Meierfrankenfeld, B. Stellmacher and G. Stroth, Finite groups of local characteristic p : an overview, Groups, combinatorics and geometry (Durham, 2001), 155–192, World Sci. Publ., River Edge, NJ, 2003.

[12] L. Pyber and E. Szabo, Growth in finite simple groups of Lie type of bounded rank, preprint: arXiv:1005.1858.

Chapter 32

Model Reduction in Continuum Thermodynamics: Modeling, Analysis and Computation (12w5029)

September 16 - 21, 2012

Organizer(s): Eduard Feireisl (Academy of Sciences of the Czech Republic, Prague), Josef Málek (Charles University, Prague), Zdeněk Strakoš (Charles University, Prague)

Overview of the Field

There are several rather different interpretations of the field *model reduction* in fluid mechanics. It can be viewed from the point of view of *mathematical modeling* as elaborating models and their adequately reduced versions that would simplify the underlying theory and produce, at lower computational costs, the desired information on the fluid system in question. *Mathematical analysis* may see the model reduction processes as a purely theoretical task, where the formal passage from the primitive to target systems is rigorously justified by the tool of modern functional analysis. *Model reduction* at this level may also include the study of systems reduced to invariant manifolds or attractors as well as explicit solution formulas based on group symmetries and other physically relevant simplifications of a given problem. Probably the most specific use of the term *model reduction* occurs in numerical analysis and implementations of numerical schemes. Here *model reduction* or *model order reduction* is understood as an effective process of reducing the number of equations used for modeling a given system, without substantial changes in the accuracy of the expected output. Unlike researchers in the field of modeling and analysis, numerical analysts have usually very clear ideas concerning the specific methods and tools used in the model reduction process. This holds, in particular, in model reduction in signal and image processing, where the model reduction can often be linked with some classical tasks of analysis and approximation theory.

The main goal of the meeting was to bring together experts in mathematical and numerical analysis as well as mathematical modeling to examine the recently emerging problems and to share ideas in a well focused environment. Given the diversity of the topics and the variety of reduced systems and their applications, these researchers would have probably never met together at any other meeting. This posed a challenge for individual presentations as well as for guiding the discussions. At the same time it gave an opportunity to see many issues from new and unconventional perspectives.

State of the Art and Major Challenges

The principal topics discussed during the meeting were highlighted in the key note lectures delivered by leading specialists in the respective field.

Mathematical theory of (complete) fluid systems

Complete fluid systems play the role of primitive systems in the theory of model reduction. They are designed and believed to provide a complete description of the observed phenomena. Accordingly, the fluids considered must feature all relevant physical properties: they are compressible, heat conducting, viscous, chemically reacting, or enjoying other properties as the case may be. Clearly, a mathematical description of these materials becomes rather involved; the resulting system of equations reflects the basic physical principles of conservation of mass, balance of momentum and energy, among others. The apparent mathematical complexity, however, does not necessarily imply that the model is more difficult to handle by the available mathematical tools. A typical example is *viscosity*, sometimes neglected in the models of "perfect" fluids, that may provide a strong regularizing effect and give rise to mathematically tractable models.

The relevant mathematical theory can be developed either in the framework of classical description supposing that all fields are smooth functions of space and time, or using the more recent concept of "weak" solutions, where the relevant physical principles are expressed by means of integral identities. Note that the "weak" formulation is actually much closer to the original interpretation of the underlying physical principles of balance and conservation, expressed originally in terms of integral identities rather than the more common systems of partial differential equations derived under the assumption of smoothness of all physical fields. This naturally reflects the principal focus of the underlying mathematical tools. While the differential operators focus on the local behaviour of the given functions, the integrals account for the global properties of the integrands over the domain of integration.

New mathematical tools emerged quite recently to handle the problem of solvability of complete fluid systems, among which the general concept of relative entropies (energies) and its application to the study of the mutual relation between classical and "modern" weak solutions.

The state of the art of the well-posedness theory of complete fluid systems can be summarized as follows:

- Most of the physically consistent systems are well posed *locally* in time. Specifically, for a given set of (initial) data, the problem admits a classical solution existing on an undetermined and possibly very short interval of time, see Matsumura and Nishida [19], Valli [29], Valli and Zajackowski [30], among others.

- The classical solutions could exist globally, provided the initial state of the system is closed to an equilibrium. Here, the presence of viscosity or other dissipative mechanisms play a crucial role, see Matsumura and Nishida [18].

- The weak solutions exist, under a physically grounded hypotheses imposed on the constitutive relations, globally-in-time. In general, however, they are not (known to be) uniquely determined by the data, see [10].

- The weak and strong solutions emanating from the same initial state coincide as long as the latter exists, see [12].

The last statement is usually called *weak-strong* uniqueness principle. As shown quite recently, the principle remain valid even in a larger class of weak solutions called *dissipative solutions*. The fact that the weak solutions exist globally in time and for any physically admissible data makes them a perfect tool for studying the problems arising in the process of model reduction, i.e., here the well justified simplification of the model while preserving its approximation properties. It allows to exploit, in particular, the singular limits arising in the scale analysis of more complex fluid systems, see [10].

One can however take a different standpoint and consider complete models for more complicated, constrained or unconstrained, fluids. Non-newtonian purely viscous or viscoelastic, compressible or incompressible, fluids; Cahn-Hillard, Allen-Cahn or Korteweg type generalizations of Navier-Stokes fluids; or equations describing flows through porous (rigid or elastic) media; or complete models arising from the theory of interacting continua can serve as examples of complete systems that differ from the compressible Navier-Stokes-Fourier fluid model and where in its full generality the mathematical theory is rather in its pregnant state.

Mathematical theory of incompressible fluids

The so-called incompressible (Newtonian) fluids can already be viewed as an example of a model reduction, here we wish to mean the well justified modification, obtained by means of the low Mach number (or incompressible) limit of a complete fluid system. Classical solvability of the underlying Navier-Stokes system represents an outstanding open problem of the theory of partial differential equations, also very popular as one of the "millenium problems", see Fefferman [9]. Numerical experiments may to a certain extent indicate the limitations of the rigorous mathematical theory, and the newly emerging mathematical models may offer an attractive alternative to the classical systems.

On the other hand, there are many fluids or fluid-like materials that can be well modeled as incompressible, yet their behavior cannot be described by the linear relation between the Cauchy stress and the symmetric part of the velocity gradient. As there are many such non-Newtonian features there are many different systems that have been proposed for their description. To be more specific, non-Newtonian phenomena called shear-thinning/shear-thickening and pressure thickening connected with significant heat conduction can be described by a viscous heat-conducting incompressible fluid model in which the viscosity depends on the pressure, shear rate and the temperature. The large data existence of weak solution for certain class of initial and boundary value problems is established in [3], see also Diening et al. [5] and [2] for stronger results in case where the material coefficients are independent of the pressure.

Scale analysis and singular limits

As already pointed out, singular limits give rise to reduced models after performing a scale analysis and letting some characteristic numbers go to zero or become infinite. The incompressible Navier-Stokes system, the Euler equations of gas dynamics, the Oberbeck-Boussinesq and anelastic approximation may be viewed as singular limits of complete fluid systems. Singular limits are often performed formally by means of asymptotic expansion of all quantities with respect to a singular parameter, see for example the survey paper by Klein et al. [16]. Their rigorous justification is, however, usually considerably more difficult. Recent development of the mathematical theory of complete fluid systems enables to perform rigorously certain singular limits, even in the case of the so-called ill-prepared data, where the primitive system is in a state that is "far away" from the target stay, see [10]. Here, the new tools based on the concept of relative entropies applied to the primitive (full) system proved to be rather efficient, [11].

In non-Newtonian fluid mechanics or in the theory of mixtures, higher complexity of the models as well as the need to solve computationally given problems, the model reduction is frequently the only possible method of choice, see [27]. Reduction of the complete models can be due to constraints (such as incompressibility, rigid body dynamics, restriction to isothermal processes or no-slip boundary conditions) or due to geometrical setting in which the considered class of processes with given fluid model takes place (if one direction in the setting is significantly small it then leads for example to thin film, shallow water or shallow ice approximations) or due to other geometrical restrictions (to small deformation gradients, for example).

Clearly, a proper scaling cannot be decided by the theory but rather by experiments performed in the real world situation or by collecting and comparing numerous observations of practical experiments, numerical experiments included. On the other hand, the process of *filtering*, meaning omitting certain features of the complete system that are not “observed” in the experiments, should be rigorously justified by careful mathematical analysis in order to avoid, or at least understand, spurious solutions and unexpected oscillations in the numerical computations. Clearly, a concerted action of the specialists in the field of modeling, analysis, and numerics is needed.

Analysis of multiscale problems

Asymptotic analysis plays a crucial role in the design of efficient numerical methods for flows in a singular regime. These problems are characterized by multiple space and time scales, and by the fact that the standard numerical methods may either completely fail or become expensive. As the goal is to apply numerical methods to complete fluid systems, it is important to understand the qualitative changes of solutions in the singular limit regime. A typical example are rapid oscillations of acoustic waves in the low Mach number limit that can be eliminated by the method of acoustic filtering. Clearly, applying similar techniques requires a detailed mathematical analysis of the problem.

Mathematical modeling of new materials

Mathematical modeling of a large variety of new materials represents a true challenge. Liquid crystals, polymeric fluids, geological materials, biological liquids and soft tissues, "smart" materials require a substantial modification of rheological laws as well as the underlying mathematical theory. Without an appropriate understanding the physical structure of materials it is impossible to develop a meaningful mathematical model. In addition, future material models must address complicated and interconnected thermal, mechanical and chemical processes that go far beyond the classical approaches. Growth and deformation of biological tissues, deformation of composite materials and shape memory alloys, flows of polymer or metal melts, flows of mixtures and geophysical materials, liquefaction of soil, transport processes in porous media and their interaction with the substrate form a base for important real-world applications. Theory that is developed to describe the macroscopic behaviour of complex bodies should be built on a continuum mechanics approach without incorporating ad hoc state variables that do not have a clear physical meaning. An artificial combination of microscopic and macroscopic theories should also be avoided (this by no means underestimates the role of multiscale approaches to computations).

On the other hand, mathematical properties of the rheologically more complex fluids may shed some light on the nowadays unsurmountable classical problems, the difficulty of which might be attributed to the fact that they are "incomplete", so that the information provided is not sufficient for their well-posedness in the mathematical sense. Of course, these systems are more complex than the classical models of fluid mechanics and thermodynamics. Thus, the need to find appropriate approximate models is of high importance.

Discretisation, numerical analysis and computation

Numerical computation assumes a finite dimensional approximation of the mathematical model. This is typically done using some spatial meshes over the given domain and by some form of time discretisation. The unknown functions are then approximated as linear combinations of a finite number of basis functions, which leads to of a finite dimensional representation of the original model. As the mesh refines and/or the parameter(s) characterizing the quality of the discretization (such as the time step or the size of the mesh elements) goes to zero, the state-of-the-art paradigm investigates convergence of the finite dimensional solution to the solution of the original model. Proving such convergence often requires fine mathematical techniques. Here *a-priori* error analysis indicates how the error (asymptotically) decreases as the mesh is refined. Bounds of this type do not involve *computed* approximate solution and their actual value is uncomputable because it typically involves the unknown solution of the problem. When performing computations, one needs to estimate the size of the *actual error.* This is done using the so-called *a posteriori* error analysis and it allows to stop the computations when the required accuracy is reached.

The numerical solution process represents in case of difficult problems a tremendous challenge. Despite the fact that discretisation, a-priori and a-posteriori error estimation and algebraic (matrix) computations represent well-established fields, many fundamental issues remain open. They should not be studied separately *after* the mathematical model, its possible analysis and reduction has been performed. Modeling with its mathematical analysis together with discretisation, error estimation and solving the resulting finite dimensional discrete problems should be considered interdependent and closely related tasks of a single solution process. A failure in one subtask may at the end simply mean forming of a numerically unsolvable problem and therefore failure of the whole numerical solution process.

The fact that the state-of-the-art results often give rather partial answers can be documented on the prevailing approach to proving convergence of the discrete approximate solution when the mesh refines using some form of *adaptation.* The proofs are based on seeing individual mesh refinement steps as contractions for some error estimators with some fixed contraction parameter independent of the mesh. This seemingly allows reaching an arbitrary prescribed accuracy in a finite number of contraction steps, which is also claimed in literature. This is, however, impossible, simply due to the fact that the discretised algebraic problem needs to be solved numerically and, apart from trivial cases, it cannot be solved exactly; see the reviews [28] and [1]. In difficult problems we even do not wish to seek a highly accurate numerical solutions of the discretised problems since that would make the whole solution process unfeasible. The principal questions which needs to be addressed is therefore what is the size of the maximal attainable accuracy of our computations, whether the prescribed required accuracy can be reached and at which price.

Presentation Highlights

In accordance with the general idea of the meeting, the presentations were of survey character of the state of the art in the respective fields, rather than highly specialized talks on particular technical results accessible only to the specialists in the field.

Mathematical analysis

Several presentations, including the key note lecture, were devoted to the *mathematical theory of complete fluid systems*. The existence results for both evolutionary and stationary problems in the framework of weak and dissipative solutions were discussed. The method of relative entropies was exploited both in the context of the complete fluid systems (study of singular limits, weak-strong uniqueness, long-time behavior) and in the analysis of stability of the shock waves in inviscid fluids.

The problem of *regularity or conditional regularity* plays a crucial role in the analysis of the systems of partial differential equations arising in fluid mechanics. Since the weak solutions exist globally-in-time, it is of great interest to know whether these solutions are, in fact, smooth. On the other hand, if the solutions are not (or not known to be) smooth, the regularity criteria provides a useful insight in the mechanism of a possible blow up and help to eliminate the unphysical situations. Note that regularity of solutions of the inviscid incompressible (Euler) equations still remains one of the most challenging open problems, where the alternative weak solutions exhibit a number of rather pathological properties, see DeLellis and Székehylidi [4].

Rigorous analysis of *singular limits* is an undeniable part of the model reduction process. Here, the primitive system is put in a dimensionless form, where several characteristic numbers appear as new parameters of the problem. Identifying the limit system when one or several of these parameters vanish or become infinite is a mathematical challenge. At the level of analysis, "identifying" means proving convergence of the solutions of the primitive system to those of the target system. This can be done either by compactness arguments based on uniform bounds, or by means of measuring the distance from the limit system by means of a relative entropy. This procedure, rather new in the context of heat conducting fluids, was highlighted by several speakers. Mathematical analysis of the limit system, like Oberbeck-Boussinesq system or anelastic approximation was also discussed.

Mathematical modeling

The leading topic of the modeling part of the workshop was the recent development of the *implicit constitutive theory*. A simply looking basic idea of this approach, namely writing the constitutive equation interrelating two quantities A, B in the form

$$\mathcal{F}(A, B) = 0 \text{ instead of } A = \mathcal{G}(B) \text{ or } B = \mathcal{C}(A),$$

leads to completely new complex models of materials with complicated rheology, and the use of this method in the process of model reduction is one of the revolutionary leading ideas of the workshop.

The primal advantage of the implicit constitutive theory consists in the possibility to describe much larger class of material responses. In addition, as the quantity A and B appears in the classical theories (A and B can stand for the Cauchy stress and the velocity gradient, or heat flux and the temperature gradient, or the Cauchy stress and the deformation gradient) there is no apriori need to introduce new type of boundary conditions. Even more, the implicit constitutive theory brings clarity and simplicity to the theoretical foundation of continuum mechanics: it gives transparent justification to incompressible fluids with pressure and shear-rate dependent viscosity as well as to nonlinear models within the framework of linearized elasticity (small gradients of the displacement); see Rajagopal [21] and [22].

The considered models should obey the laws of thermodynamics. The development of the implicit constitutive relations is therefore combined with another recent ingredient, namely *the principle of the maximization of the rate of entropy production*, see Rajagopal and Srinivasa [23]. Such a framework allows developing thermodynamically consistent fully three-dimensional constitutive models. Here the material is characterized by the way how it stores energy and the way how it produces entropy. These storage and production mechanisms are specified by a choice of the constitutive equation for the entropy (or another suitable thermodynamic potential) and the rate of entropy production. Concepts of implicit constitutive relations allow to handle very general forms of storage and dissipation mechanisms. The entropy production is then maximized with respect to a constraint enforcing the validity of the reduced dissipation identity, and possibly also with respect to other constraints such as the incompressibility of the material. As a condition identifying the maximum one gets the required relation between the quantity A (for example the Cauchy stress) and the quantity B (for example the symmetric part of the velocity gradient).

Such approach is based on a small set of well articulated and justified fundamental (axiomatic) assumptions. So far, this approach has been successful in providing:

- appropriate thermodynamic setting for compressible heat-conducting fluids of a Korteweg type (see [14]),
- different viewpoints on Bingham and Herschel-Bulkley fluids (and other activation or deactivation criteria in general) (see [24] and [2]),
- an approach to characterize the structure of the boundary conditions (important for complex materials) as the constitutive equations on the surface (see Heida [13]).

The effective use of this new methodology was highlighted in several talks and possible applications, in particular in mathematical analysis and numerical implementations discussed.

Discretisation, numerical analysis and computation

Besides the standard numerical topics concerning the design and analysis of convergence of the new discretisation schemes applied to complex fluid systems, a substantial part of the numerical talks was devoted to *a posteriori* estimates and the problem of reliability of numerical methods. The decisive criterion is the degree of precision in which the results of computations reflect the properties of the genuine (analytical) solution of a given equation or system.

The key lectures were devoted to the topics of *adaptivity* as a form of the discretised model reduction, construction of *efficient and robust computational algorithms* and the *control of errors* of computed approximate solutions. All speakers emphasized the interplay between the infinite dimensional function representation and the reduced discrete representation of the model. In control of the discretisation and computational error, a-posteriori error analysis must consider algebraic errors and must include investigation of numerical stability; see, e.g. [15, 26, 8] an the recent survey paper by Rannacher [25] which all contain many references to other relevant works.

Construction of efficient computational algorithms requires *global communication* transferring the information obtained for different times and/or at different (and possibly distant) parts of the solution domain. It was demonstrated how this can be achieved, e.g., via incorporating *coarse space components* in domain decomposition methods; see [6]. The coarse components representation can be considered a form of the model reduction which can be used for substantial acceleration of computations. Efficient *preconditioning* represent another principal tool for achieving the same goal; see, e.g., the classical book by Elman, Silvester and Wathen [7], which is currently being revised and extended for the second edition. Preconditioning should reflect the physical nature of the problem expressed in the mathematical model. It can be motivated using a functional analytic operator description (so called "operator preconditioning"). Practical derivation of computational algorithms is, however, often much easier using a finite dimensional algebraic setting with its description via matrices. Combination of both views can lead to development of fast and robust solvers, with *Krylov subspace methods* (see the recent book [17]) as a possible basic underlying iterative scheme. Finally, construction of *fully computable a-posteriori error estimators* which allow for the *local error control* and comparison of the size of the error from different sources (discretisation, linearization, inexact algebraic computation) is a prerequisite for reliable, robust and efficient adaptive approaches [1]. This requires combination of rather diverse techniques from functional analysis through numerical analysis to analysis of iterative matrix computations including effects of rounding errors.

Scientific Progress Made

One of the main achievements of the meeting was dissemination of the new methods, sofar known only to specialists in their specific (sometimes even narrow) fields, to the representatives of the modeling, analysis, discretisation and computational communities. Several possibilities of applications of theoretical tools in the analysis of convergence of numerical schemes emerged, as well as new directions in the theoretical studies discovered in the framework of the implicit constitutive theory. Last but the least, the necessity of effective feedback and comparison of results and methods used in the three leading areas - modeling, analysis, numerics - appeared as necessary for the progress in the field of model reduction, where the last term may have many (interrelated) meanings.

It becomes very clear that a goal of reaching a substantial progress in model reduction in continuum thermodynamics, which would open new ways of research substantially beyond the current state-of-the-art, requires utilization of specific knowledge of the physical nature of a well chosen specific problems used as case studies. There is no hope for aiming at a general approach developing a universal computational framework. The hope is rather in investigating important particular examples with utilizing similarities between mathematical description of real phenomena from different fields with cautious well-justified generalizations to possible large classes of problems whenever applicable.

Outcome of the Meeting

The nature of the workshop has evoked much more questions then gave answers. For various reasons, science is getting more and more specialized, which brings, together with large benefits, also a great danger of fragmentation. Researchers working in different fields of the same scientific discipline (such as mathematics) see each other more and more rarely, and they rarely communicate across the disciplines with papers which are widely read and discussed. Sometimes we can observe growing isolation instead of tightening links between communities and scientific schools. Results are developed within one field without being communicated to, considered and used within related fields. It also becomes rather difficult to challenge common well established views and approaches. The widely adopted malign overemphasizing the publish or perish policy stimulates much more the standard mainstream production over a difficult and cross-disciplinary communication. But such communication is, in our opinion, desperately needed not only for solving difficult real-world-inspired problems, but for sake of the science itself.

This workshop tried to go in the direction of building bridges between different areas of mathematics related to the main topic, with including fundamental motivations from the corresponding parts of physics. The gaps which we need to deal with are neither narrow nor shallow, and a considerable effort will be needed just to establish a regular and fruitful communication. We consider such effort immensely important. Intensive discussions between participants proved that it can work. We believe that coming months will bring materialized outcomes in the form of joint work and papers.

Among the widely discussed questions which will be further discussed or investigated we mention:

- The prevailing paradigm in numerical solving of mathematical modeling problems is based on discrete approximation of the infinite dimensional problem via the Finite Element Method (FEM). Such approximation is constructed using *locally supported* basis functions which results in algebraic problems formulated using *sparse matrices.* This is presented as a principal advantage of FEM. The common view, that the locality of FEM bases and sparsity of the resulting matrices gives a great advantage of the FEM approach over some alternative approaches in particular when solving the discretized algebraic problems, might be worth of a second thought. As mentioned above, difficult problems can not be solved without exploiting the *global transfer* of information in time and over the domain. This requirement seems to be in some controversy with the philosophy of FEM and with the sparsity of the resulting matrices. Algebraic computations then can not be efficiently done without incorporating powerful global transfers of information which is handled by computational techniques such as preconditioning in iterative methods. It seems that here algebraic computations is getting difficult partially due to form of discretisation which prefers local approximation. The difficulty may show up, e.g., in comparison (including their spatial distribution) of the errors from different sources; see [20].

- Efficient numerical computations requires reconciling of different mathematical and computational requirements which are not always in line. Parallelism and scalability do not always go along with numerical stability and a need for global communication. Computer science tools do not always serve mathematical needs. There seems to be, in general, an insufficient communication between the computer science, mathematical modeling and applied mathematics communities. The trend seems to be rather to the worse then to the better.

- An interplay of the mathematical descriptions of problems on different levels (modeling - discretisation - computation) with tools ranging from fundamental mathematical analysis to construction and analysis of methods in matrix computations (including numerical stability analysis) represents a tremendous challenge by itself. Without a genuine *will for collaboration* of experts in all these fields, no real breakthrough in the topic of the workshop can be achieved.

- In linear model reduction known in linear dynamical systems and control there is a deep underlying mathematical background such as Padé approximation, Gauss-Christoffel quadrature, continued fractions, problem of moments, minimal partial realization etc. (just give a few examples) which also links classical topics from analysis and approximation theory to modern computational tools such as Krylov subspace methods; see [17]. *Nonlinearity* makes things from this point of view extraordinary complicated. No similar unified mathematical background essentially exists and solid mathematical foundations are yet to be built.

As appeared several times in our report, no significant progress in challenges mentioned above can be achieved, in our opinion, by a group of researchers working within a narrow field. In order to bridge the gaps, a well coordinated effort of all sides is needed. This workshop has tried to make a first step in this direction.

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Bibliography

[1] M. Arioli, A. Miedlar, J. Liesen, Z. Strakoš. Interplay between discretisation and algebraic computation in adaptive numerical solution of elliptic PDE problems. *Submitted for publication*, 2012.

[2] M. Bulíček, P. Gwiazda, J. Málek, A. Świerczewska-Gwiazda On unsteady flows of implicitly constituted incompressible fluids. *SIAM J. Math. Anal.*, **44**:2756Ű-2801, 2012.

[3] M. Bulíček, J. Málek, K. R. Rajagopal Mathematical analysis of unsteady flows of fluids with pressure, shear-rate, and temperature dependent material moduli that slip at solid boundaries. *SIAM J. Math. Anal.*, **41**:665–707, 2009.

[4] C. De Lellis and L. Székelyhidi. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, **195**:225–260, 2010.

[5] L. Diening, M. Ružička, and J. Wolf. Existence of weak solutions for unsteady motions of generalized Newtonian fluids. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, **9**:1Ű46, 2010.

[6] O. Dubois, M. Gander, S. Loisel, A. St-Cyr, and D Szyld. The optimized Schwarz method with a coarse grid correction. *SIAM J. Sci. Comput.*, **34**:421–458, 2012.

[7] H. Elman, D. Silvester, and A. Wathen. *Finite elements and fast iterative solvers with applications in incompressible fluid dynamics*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2005.

[8] A. Ern and M. Vohralík. Adaptive inexact Newton methods with a posteriori stopping criteria for nonlinear diffusion PDEs. *Submitted for publication*, 2012.

[9] C. L. Fefferman. Existence and smoothness of the Navier-Stokes equation. In *The millennium prize problems*, pages 57–67. Clay Math. Inst., Cambridge, MA, 2006.

[10] E. Feireisl and A. Novotný. *Singular limits in thermodynamics of viscous fluids*. Birkhäuser-Verlag, Basel, 2009.

[11] E. Feireisl and A. Novotný. Inviscid incompressible limits of the full Navier-Stokes-Fourier system. *Commun. Math. Phys.*, 2012. To appear.

[12] E. Feireisl and A. Novotný. Weak-strong uniqueness property for the full Navier-Stokes-Fourier system. *Arch. Rational Mech. Anal.*, **204**:683–706, 2012.

[13] M. Heida. On the derivation of thermodynamically consistent boundary conditions for the Cahn-Hilliard-Navier-Stokes system. *Inter. J. Engr. Science*, accepted for publication, 2012.

[14] M. Heida and J. Málek. On compressible Korteweg fluid-like materials. *Inter. J. Engr. Science*, **48**:1313Ű1324, 2010.

[15] P. Jiránek, Z. Strakoš, and M. Vohralík. A posteriori error estimates including algebraic error and stopping criteria for iterative solvers. *SIAM J. Sci. Comput.*, **32**:1567–1590, 2010.

[16] R. Klein, N. Botta, T. Schneider, C.D. Munz, S. Roller, A. Meister, L. Hoffmann, and T. Sonar. Asymptotic adaptive methods for multi-scale problems in fluid mechanics *J. Engrg. Math.*, **39**:261–343, 2001.

[17] J. Liesen and Z. Strakoš. *Krylov subspace methods. Principles and analysis*. Oxford University Press, Oxford, 2012.

[18] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, **20**:67–104, 1980.

[19] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of compressible and heat conductive fluids. *Comm. Math. Phys.*, **89**:445–464, 1983.

[20] J. Papež, Z. Strakoš and J. Liesen. On distribution of the discretization and algebraic error in 1D Poisson model problem. *Submitted for publication*, 2012.

[21] K. R. Rajagopal. On implicit constitutive theories. *Appl. Math.*, **48**:279-319, 2003.

[22] K. R. Rajagopal. Non-linear elastic bodies exhibiting limiting small strain. *Mathematics and Mechanics of Solids*, **16**:122-139, 2011.

[23] K. R. Rajagopal and A. R. Srinivasa. On thermomechanical restrictions of continua. *Proc. R. Soc. Lond. A*, **460**: 631Ű651, 2004.

[24] K. R. Rajagopal and A. R. Srinivasa. On the thermodynamics of fluids defined by implicit constitutive relations. *Z. Angew. Math. Phys.*, **59**: 715Ű729, 2008.

[25] R. Rannacher. A short course on numerical simulation of viscous flow: discretization, optimization and stability analysis. *Discrete and Continuous Dynamical Systems Series S*, **5**:1147–1194, 2012.

[26] D. J. Silvester and V. Simoncini. An optimal iterative solver for symmetric indefinite systems stemming from mixed approximation. *ACM Trans. Math. Software*, **37**:Art. 42, 22, 2011.

[27] O. Souček and Z. Martinec. Iterative improvement of the shallow-ice approximation. *Journal of Glaciology*, **54**:812-822, 2008.

[28] Z. Strakoš and J. Liesen. On numerical stability in large scale linear algebraic computations. *ZAMM Z. Angew. Math. Mech.*, **85**: 307–325, 2005.

[29] A. Valli. An existence theorem for compressible viscous fluids. *Ann. Mat. Pura Appl. (4)*, **130**:197–213, 1982.

[30] A. Valli and M. Zajaczkowski. Navier-Stokes equations for compressible fluids: Global existence and qualitative properties of the solutions in the general case. *Commun. Math. Phys.*, **103**:259–296, 1986.

Chapter 33

Lie Algebras, Torsors and Cohomological Invariants (12w5008)

September 30 - October 5, 2012

Organizer(s): Stefan Gille (University of Alberta), Nikita Karpenko (Paris 6), Arturo Pianzola (University of Alberta), Vera Serganova (University of California, Berkeley), Kirill Zainoulline (University of Ottawa)

Overview of the Field

The theory of Lie algebras deals with the study and classification of (in-)finite dimensional Lie algebras and has many applications in representation theory, combinatorics and theoretical physics. Many interesting infinite dimensional Lie algebras can be thought as being "finite dimensional" when viewed, not as algebras over the given base field, but rather as algebras over their centroids. From this point of view, the algebras in question look like twisted forms of simpler objects. The quintessential example of this type of behaviour is given by the celebrated affine Kac-Moody Lie algebras which have particular importance in theoretical physics, especially conformal field theory and the theory of exactly solvable models. The connection between the "forms" point of view and Extended Affine Lie Algebras (EALAs for short) – a class of infinite dimensional Lie algebras that as rough approximations can be thought as higher analogues of the affine Kac-Moody Lie algebras – was one of the central themes of the workshop.

The theory of torsors and the associated linear algebraic groups was brought to the forefront of modern algebra by two fundamental discoveries made during the end of the 90's. The first is the proof of Milnor's conjecture by V. Voevodsky [Fields Medal, 2002] which was based on the computation of motivic cohomology of the norm quadric. This inspired an intensive study of quadratic forms, e.g. torsors of an orthogonal group, their motives and cohomological invariants (Karpenko's ICM 2010 lecture). The second discovery is due to Z. Reichstein and deals with the notions of essential and canonical dimensions of linear algebraic groups (Reichstein's ICM 2010 lecture). Roughly speaking, this numerical invariant characterizes the complexity (splitting properties) of a torsor. There are several classical open conjectures in modern algebraic geometry which are closely related to torsors. These formed another central theme of the workshop.

Using cohomological language, a torsor can be identified with a twisted form of an algebraic variety corresponding to a cocycle in the first non-abelian Galois cohomology. This algebraic variety is usually described by some combinatorial data involving root systems and representations, hence, providing a strong connection between the theory of Lie algebras and torsors. The main purpose of the workshop was to exploit and to develop this connection. Namely, using the language of torsors we intended to provide new applications to the theory of Lie algebras; and, vice versa, having in hands Lie algebras to obtain new results concerning various conjectures on torsors and linear algebraic groups.

Recent Developments

The last five years can be characterized as a boom of research activity in the workshop areas. To support this observation we should mention the recent results by

(a) Garibald-Merkurjev-Rost-Serre-Totaro-Zainoulline which set up various connections between cohomological invariants and irreducible representations of Lie algebras, e.g. the Dynkin index;

(b) Karpenko-Merkurjev-Reichstein on essential and canonical dimensions of quadratic forms and linear algebraic groups which essentially used the representation theory.

(c) Petrov-Semenov-Zainoulline on motivic decompositions of projective homogeneous varieties, which are based on V. Kac computations of the Chow groups of compact Lie groups.

(d) Gille-Pianzola, where the classification of multiloop Lie algebras has been related to the classification of torsors over Laurent polynomial rings.

(e) Kac-Lau-Pianzola the remarkable fact that the torsor point of view can also be used to study conformal superalgebras, a fact that lead to the concept of differential conformal superalgebras.

Note also that one of the organizers (Karpenko) was invited to give sectional talk at the International Congress of Mathematicians (2010), one of the speakers (Merkurjev) has received the Cole Prize in Algebra (2012) for his achievements in the theory of essential dimension and another speaker (Parimala) was invited to give a plenary talk at the ICM (2010).

Open problems and directions

As the subject of the workshop consists of several emerging areas of modern algebra/algebraic geometry it contains many open questions and problems. These can be described as follows

I. Torsors and cohomological invariants

Cohomological invariants existed long before this terminology was introduced by Jean-Pierre Serre in the mid of 90's. For instance the (signed) determinant and the Clifford algebra of a quadratic form can be considered as a cohomological invariant with values in the Galois cohomology. To bring some order in the various existing invariants Serre developed a theory of cohomological invariants in the following abstract setting:

By a cohomological invariant one means a natural transformation from the first Galois cohomology with coefficients in an algebraic group G (the pointed set which describes all G-torsors) to a cohomology functor $h(-)$, where h is a Galois cohomology with torsion coefficients, a Witt group, a Chow group with coefficients in a Rost cyclic module M, etc. The ideal result here would be to construct enough invariants to classify all G-torsors.

This concept was developed further by Garibaldi-Merkurjev-Rost-Totaro, leading to the complete understanding of invariants in lower degrees. For instance, in degree 2 the group of invariants is generated by the classes of Tits algebras in the Brauer group and in degree 3 it is generated by the Rost invariant.

I.a) Applications to extended affine Lie algebras

The connection between the “forms" point of view, which is related to the theory of Reductive Group Schemes developed by Demazure and Grothendieck, and Extended Affine Lie Algebras (EALAs) is one of the central themes of the proposal. P. Gille and A. Pianzola have pioneered this approach. The language and theory of G-torsors, where G is a reductive group scheme over a Laurent polynomial ring, appears then quite naturally. This point of view brings extremely powerful tools to the study of infinite dimensional Lie theory.

On the other side the study of EALAs has been greatly simplified by work of E. Neher, which reduces the classification problem to the so called cores of the EALA, which are are in fact multiloop algebras (except for a family of fully understood cases). It is not true, however, that every multiloop algebra is the centreless core of an EALA. Finding cohomological invariants that characterize the isomorphism classes of torsors corresponding to the multiloop algebras attached to EALAs is part of the proposal. This question is connected with some deep work in progress by Chernousov, Gille and Pianzola dealing with the problem of conjugacy of Cartan subalgebras of multiloop algebras.

I.b) Applications to representations and cohomology of Lie algebras and homogeneous spaces

The cohomology theory of Lie algebras on one hand is used to compute central and abelian extensions of Lie algebras. Central extensions of Lie algebras often have richer representation theories than their centerless quotients. On the other hand, cohomology of manifolds connects representation theory with geometry and physics. An example of such an interplay is a recently established cohomological interpretation of the elliptic genus, an object originally introduced in string theory.

Another interesting direction is to study maps between cohomologies of vector bundles on homogeneous varieties arising from homomorphisms of the underlying homogeneous varieties. So far the best understood case is the case of varieties of Borel subgroups. In particular, Dimitrov-Roth have showed that the diagonal embedding leads to a natural geometric construction of extreme components of the tensor product of irreducible representations.

II. Torsors and Motives

Following the general philosophy of Grothendieck one can introduce a universal cohomological invariant which takes values in the category of motives. The link between the world of motives and torsors is provided by the celebrated Rost Nilpotence Theorem (RNT) which can be viewed as a generalized Galois descent property. In [Duke 2003] Chernousov-Gille-Merkurjev proved the RNT for arbitrary projective homogeneous varieties over semisimple algebraic groups, hence, opening the door to the study of motives of projective homogeneous varieties.

II.a) Applications to linear algebraic groups

Based on this result and computations of Victor Kac of the Chow ring of G Petrov-Semenov-Zaynullin [Ann. Sci. ENS, 2008] computed the motive of generically split projective homogeneous varieties in terms of generalized Rost motives introduced by Voevodsky. They also showed that the motivic behavior of such varieties can be described by a certain discrete numerical invariant, the J-invariant. As an application of the motivic J-invariant Petrov-Semenov [Duke, 2010] classified all generically split homogeneous varieties of linear algebraic groups. As another application, Semenov (2010) has given a construction of a cohomological invariant of degree 5 of an exceptional group of type E_8, hence, proving a conjecture by J.-P. Serre. This invariant has tremendous applications to the study of subgroups of compact Lie groups of type E_8.

II.b) Applications to quadratic forms and algebras with involutions

Quadratic forms and central simple algebras with involutions provide classical examples of torsors for a (projective) orthogonal group. Its cohomological invariants and motives have been extensively studied during the last decade. We should mention here the works of Karpenko and Vishik who use motives to investigate the splitting behavior of quadratic forms and algebras with involutions, e.g. the Vishik's construction of fields with u-invariant $2^r + 1$, $r > 3$, his motivic decomposition type theory; Karpenko's result on the first Witt indices of quadratic forms and the proof of the Hoffmann's conjecture, his recent proofs of hyperbolicity and isotropy conjectures for algebras with involutions. In the proof of the hyperbolicity conjecture Karpenko (2010) uses the theory of upper motives. Observe that for a projective homogeneous variety the dimension of its upper motive measures its canonical p-dimension, hence, providing a new approach to study the canonical and essential dimensions of algebraic groups.

II.c) Applications to Del Pezzo surfaces and representations of exceptional algebraic groups.

In 1990 Batyrev conjectured that universal torsors over Del Pezzo surfaces can be embedded into homogeneous spaces of exceptional algebraic groups. The particular cases of this conjecture were proven by Popov and Derenthal. Serganova and Skorobogatov recently suggested a universal proof of the Batyrev conjecture which covers a new case of E_8. Very recently Gille [Invent.Math. 2010] has proven the RNT for del Pezzo surfaces, hence, providing a new approach to study their geometry, motives and Chow groups.

Presentation Highlights and Scientific Progress Made

I. The first day of the workshop was devoted to general lectures on the topics related to linear algebraic groups and cohomology theories. There were two morning talks by senior researchers E. Bayer-Fluckiger (EPFL, Switzerland) and A. Vishik (Nottingham, UK). The afternoon session was started with the talk by A. Merkurjev (UCLA, USA) - recipient of the Cole prize in Algebra (2012). The last talk of the afternoon was given by the young researcher S. Baek (KAIST, South Korea).

Eva Bayer-Fluckiger *Embeddings of maximal tori of type CM in orthogonal groups*

Let k be an algebraic number fields. Embeddings of maximal tori in orthogonal groups have been studied in several papers, and occur in various arithmetic questions. The case of tori of type CM (that is, tori associated to CM etale algebras) is of special interest of some of the applications. The aim of this talk was to give necessary and sufficient criteria for such an embedding to exist under some conditions, which are fulfilled in the CM case.

Alexander Vishik *Stable and Unstable operations in Algebraic Cobordism*

In the talk the speaker described and effectively constructed (unstable) additive operations $A \to B$, where A is a theory obtained from Algebraic Cobordism of M. Levine - F. Morel by change of coefficients, and B is any Generalized Oriented Cohomology Theory. Among the applications of this technique are the following major results:

1. Description of unstable operations in Algebraic Cobordism theory. The description of stable ones comes as well
2. The Theorem claiming that multiplicative operations $A \to B$ (where A, B are as above) are in 1-to-1 correspondence with the homomorphisms of the respective formal group laws.
3. The construction of Integral (!) Adams operations in Algebraic Cobordism and all the theories obtained from it by change of coefficients (giving classical Adams operations in case of K_0).
4. The construction of Symmetric Operations for all primes p (previously known only for $p = 2$), and the construction of Tom Dieck - style Steenrod operations in Algebraic Cobordism.

Alexander Merkurjev *Generic values of quadratic forms and essential dimension*

Let $f : X \longrightarrow Y$ be dominant map between varieties over the field F. The functor $\mathcal{A}_f$ assigns to a field extension $K \supseteq F$ the image of the induced map $X(K) \to Y(K)$. In this talk the speaker introduced and discussed the essential dimension of the functor $\mathcal{A}_f$ for various f.

In particular he discussed in detail the following example. Let (V, q) be a quadratic space over the field F and V_0 the open subset of all $v \in V$, such that $q(v) \neq 0$. Then by restriction q induces a morphism $V_0 \longrightarrow \mathbb{G}_m$ (also denoted by q).

He showed then using the "general" theory for the functor $\mathcal{A}_q$ the following theorem:

- Let $q = \sum\limits_{i=1}^{n} a_i x_i^2$ be an anisotropic quadratic form over the field F, and $F \subseteq L \subseteq F(x_1, \ldots, x_n)$ be a field which contains the generic value $q = q(x_1, \ldots, x_n)$. If there are $b_1, \ldots, b_n \in L^n$, such that $q(x_1, \ldots, x_n) = q(b_1, \ldots, b_n)$ then the degree $[F(x_1, \ldots x_n) : L]$ is finite and odd.

Sanghoon Baek *On the torsion of Chow groups of Severi-Brauer varieties*

Let p be a prime and A a central simple algebra of p-power degree over a field. We denote by $SB(A)$ the corresponding Severi-Brauer variety. Consider the Grothendieck ring $K(SB(A))$ and its gamma filtration $\Gamma^d K(SB(A))$ for $d \geq 0$. By a theorem of Quillen, the gamma filtration on $K(SB(A))$ is determined by the indices of (tensor) powers of A. Based on this observation, Karpenko introduced the sequence of the exponents of distinct indices of powers of A, which is called the reduced sequence of A. Moreover, Karpenko showed that the torsion part of the 2nd quotient $\Gamma^2 K(SB(A))/\Gamma^3 K(SB(A))$ of the gamma filtration is determined by a certain index of the reduced sequence. Note that the 0th and the 1st quotients are torsion-free. Now we consider the torsion part of Chow group $CH^d(SB(A))$ of cycles modulo the rational equivalence relation. For $d = 0, 1$, they are all torsion free. However, for $d = 2$ there is torsion and it is shown that the torsion part is annihilated by the order of torsion subgroup of $\Gamma^2 K(SB(A))/\Gamma^3 K(SB(A))$. In the talk the speaker provided upper bounds for the annihilators of the torsion subgroups of Chow groups of the Severi-Brauer varieties for a large class of central simple algebras (see [1]).

II. The second day of the workshop was devoted to the interactions between the classical theory of (infinite-dimensional) Lie algebras on one hand side and torsors, linear algebraic groups on the other side. The morning session was started by the talk of V. Popov devoted to the celebrated Gelfand-Kirillov's conjecture. Then E. Neher presented recent developments in the theory of derivations of Lie algebras.

The afternoon session consisted of talks by a graduate student Z. Chang and young researchers N. Lemire and I. Dimitrov.

Vladimir Popov *Rational functions on semisimple Lie algebras and the Gelfand-Kirillov Conjecture*

The talk was aimed at describing the recent solution of the rationality problem for fields of rational functions on semisimple Lie algebras and the intimately related construction of counterexamples to the Gelfand–Kirillov conjecture on the fields of fractions of universal enveloping algebras of simple Lie algebras. This solution exploits a notion generalizing that of the usual torsor.

More precisely, a field extension E/F is called pure (or purely transcendental or rational) if E is generated over F by a finite collection of algebraically independent elements. A field extension E/F is called stably pure (or stably rational) if E is contained in a field L which is pure over both F and E. Finally, we shall say that E/F is unirational if E is contained in a field L which is pure over F.

Let k be a field of characteristic 0. Let G be a connected reductive algebraic group over k. Let V be a finite dimensional k-vector space and let $G \to GL(V)$ be an algebraic group embedding over k. Let $k(V)$ denote the field of k-rational functions on V and $k(V)^G$ the subfield of G-invariants in $k(V)$. It is natural to ask whether $k(V)/k(V)^G$ is pure (or stably pure).

This question may be viewed as a birational counterpart of the classical problem of freeness of the module of (regular) covariants, i.e., the $k[V]^G$-module $k[V]$ (Here $k[V]$ is the algebra of k-regular functions on V and $k[V]^G$ is the subalgebra of its G-invariant elements.) The question of rationality of $k(V)$ over $k(V)^G$ also comes up in connection with counterexamples to the Gelfand-Kirillov conjecture.

Recall that a connected reductive group G is called split if there exists a Borel subgroup B of G defined over k and a maximal torus in B is split. If G is split and the G-action on V is generically free, i.e., the G-stabilizers of the points of a dense open set of V are trivial, then the following conditions are equivalent:

(i) the extension $k(V)/k(V)^G$ is pure;

(ii) the extension $k(V)/k(V)^G$ is unirational;

(iii) the group G is a âĂIJspecial groupâĂİ.

Over an algebraically closed field, special groups were defined by Serre and later classified by himself and Grothendieck.

The purity problem for $k(V)/k(V)^G$ is thus primarily of interest in the case where the G-action on V is faithful but not generically free. For k algebraically closed, such actions have been extensively studied and even classified, under the assumption that either the group G or the G-module V is simple.

Let g be the Lie algebra of G. The homomorphism $Int\colon G \to Aut(G)$ sending $g \in G$ to the map $Int(g)\colon G \to G$, $x \mapsto gxg^1$, determines the conjugation action of G on itself, $G \times G \to G$, sending (g, x) to $Int(g)(x)$. The differential of $Int(g)$ at the identity is the linear map $Ad(g)\colon g \to g$. This defines an action of G on g, called the adjoint action. As usual, we will denote the fields of k-rational functions on G, respectively, g, by $k(G)$, respectively, $k(g)$, and the fields of invariant k-rational functions for the conjugation action, respectively, the adjoint action, by $k(G)^G$, respectively, $k(g)^G$.

The purpose of the talk was to address the following purity questions (see [2]):

- Is the field extension $k(g)/k(g)^G$ pure? stably pure?
- Is the field extension $k(G)/k(G)^G$ pure? stably pure?

Erhard Neher *Derivations of algebras obtained by étale descent*

Let g be a simple finite-dimensional Lie algebra over the complex numbers. The celebrated affine Kac-Moody Lie algebras are of the form $E = L \oplus kc \oplus kd$, where L is a (twisted) loop algebra of the form $L(g, \pi)$ for some diagram automorphism π of g. The element c is central and d is a degree derivation for a natural grading of L. It is thus natural to study the derivations of loop and, more generally, multiloop algebras.

The speaker described derivations of Lie algebras obtained by étale descent and discussed various applications to multiloop algebras and extended affine Lie algebras. The talk was based on joint work with Arturo Pianzola [3].

Zhihua Chang *Twisted Loop Algebras Based on Conformal Superalgebras*

Superconformal algebras are infinite dimensional Lie superalgebras of interest in theoretical physics. They are closely related to twisted loop algebras based on conformal superalgebras. To classify twisted loop algebras based on a given conformal superalgebra, differential conformal superalgebras were introduced by V. Kac, M. Lau, and A. Pianzola in 2009. In this talk, the speaker gave first a brief introduction to the general theory of twisted forms of differential conformal superalgebras. After that he discussed the classification of twisted loop algebras based on the $N = 1, 2, 3$, small $N = 4$ and large $N = 4$ conformal superalgebras.

Ivan Dimitrov *Constructing subrepresentations via the Borel-Weil-Bott theorem*

An embedding $G \subset G'$ of reductive algebraic groups gives rise to an embedding $G/B \subset G'/B'$ of the corresponding homogeneous varieties. For any line bundle L' on G'/B' one has the natural map of cohomologies $\pi : H^q(G'/B'; L) \to H^q(G/B, L)$, where L is the restriction of L' to G/B. The Borel-Weil-Bott theorem implies that the dual map π, when nonzero, is a GâĂŞmodule homomorphism $\pi : V \to V'$, where V and V' are irreducible modules respectively over G and G'. Varying L' (and respectively q) so that $H^q(G'/B'; L) = (V')^*$ we obtain a purely geometric construction of certain irreducible GâĂŞsubmodules of V' which we call cohomological components. In this talk the speaker discussed several types of embeddings $G \subset G'$ and the corresponding cohomological components âĂŤ their properties, relationship to other interesting problems as well as necessary and sufficient conditions for non vanishing of π. In the case when G is embedded diagonally into $G' = G \times G$, the cohomological components lie on faces of the Littlewood-Richardson cone of codimension equal to the rank of G. With an appropriate choice of an embedding of G/B, one can also obtain generators of the algebra of invariant polynomials on the Lie algebra of G as cohomological components. The talk was based on the joint work with Mike Roth as well as results of Valdemar [4].

Nicole Lemire *Stably Cayley Groups over Fields of Characteristic 0.*

Let k be a field of characteristic 0 and G be a connected linear algebraic k-group. We say that a birational isomorphism $\phi\colon G \to Lie(G)$ is a Cayley map if it is equivariant with respect to the conjugation action of G on itself and the adjoint action of G on its Lie algebra $Lie(G)$, respectively. A Cayley map can be thought of as a (partial) algebraic analogue of the exponential map. A prototypical example is the classical Cayley map for the special orthogonal group SO_n defined by A. Cayley in 1846. We say that G is a Cayley group if it admits a Cayley map. We say that G is stably Cayley if $G \times_k \mathbb{G}_m^r$ is Cayley for some $r \geq 0$, where $\mathbb{G}_m$ denotes the multiplicative group. In the case where k is algebraically closed, Cayley and stably Cayley groups were studied by the speaker, Popov and Reichstein.

In this talk the speaker focused on the case where k is an arbitrary field of characteristic 0. The following âĂIJtoyâĂİ example illustrates how much more intricate the notions of Cayley and stably Cayley group become in this situation.

Let T be a k-torus of dimension d. By definition, T is Cayley (respectively, stably Cayley) over k if and only if T is k-rational (respectively, stably k-rational). If k is algebraically closed, then $T = \mathbb{G}_m^d$, hence T is always rational, and thus always Cayley.

Observe that there is a well-known criterion for stable rationality of T in terms of its character lattice $X(T)$: T is stably rational if and only if the character lattice $X(T)$ is quasi-permutation. Note that the term âĂIJcharacter latticeâĂİ here is the lattice of characters of T with the natural action of the absolute Galois group $Gal(k)$. It has been conjectured that every stably rational torus is rational. To the best of our knowledge, this conjecture is still open, and there is no simple lattice-theoretic criterion for the rationality of T.

The talk was based on the recent joint work with Blunk, Borovoi, Kunyavskii and Reichstein (see [5]).

III. The third day of the workshop was devoted to recent trends in the theory of the u-invariant of quadratic forms – an important invariant of torsors for orthogonal groups. This consisted of talks by R. Parimala and D. Saltman.

Raman Parimala *Bounding symbol lengths in Galois cohomology*

Bounding symbol lengths in Galois cohomology has had important implications to bounding the u-invariant of fields. This approach leads to finiteness of the u-invariant of function fields in one variable over a totally imaginary number field, provided a conjecture of Colliot-Thélène on the Brauer-Manin obstruction and the existence of zero cycles of degree one on smooth projective varieties over number fields holds.

David Saltman *Finite u Invariant and Bounds on Cohomology Symbol Lengths*

At a AIM workshop in January 2011, Parimala asked whether in a field with finite u invariant there was a bound on the "symbol length" of any element of μ_2 cohomology in any degree. The speaker answered this question in the affirmative for fields of characteristic 0, and at the same time obtained bounds on the Galois groups that realize all the properties of these cohomology elements and showed that his results extend to finite field extensions.

IV. The morning session of the fourth day of the workshop was devoted to the weak commensurability problem that gives connection between algebraic groups and differential geometry. There were two talks on this subject by S. Garibaldi and V. Chernousov. The afternoon session was devoted to various topics – from the theory of central simple algebras to motives and Schubert calculus in algebraic cobordism. All talks where given by young researchers (M. Florence, O. Haution and V. Kiritchenko).

Skip Garibaldi *Algebraic groups and weak commensurability*

The notion of weak commensurability introduced by Gopal Prasad and Andrei Rapinchuk gives deep connections between algebraic groups and differential geometry, as well as new tools for studying algebraic groups over number fields. The speaker discussed recent applications to the questions:

- If two simple linear algebraic groups over a number field have the same isogeny classes of maximal tori, must the groups be isogenous?
- If two locally symmetric spaces M_1 and M_2 are weakly commensurable – i.e., if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$ where L denotes the set of lengths of closed geodesics—must M_1 and M_2 have a common finite-sheeted cover?

Although there are well-known examples where the answer to each of these questions is "no", the answer is nonetheless frequently "yes".

Vladimir Chernousov *On the genus of a division algebra*

Let K be a field, $Br(K)$ be its Brauer group, and for any integer $n > 1$ let $nBr(K)$ be the subgroup of $Br(K)$ annihilated by n. For a finite-dimensional central simple algebra A over K, we let $[A]$ denote the corresponding class in $Br(K)$, and we then define the genus $gen(D)$ of a central division K-algebra D of degree n to be the set of classes $[D]Br(K)$ where D is a central division K-algebra having the same maximal subfields as D (in more precise terms, this means that D has the same degree n, and a field extension P/K of degree n admits a K-embedding $P \to D$ if and only if it admits a K-embedding $P \to D'$. The speaker addressed the following two questions:

- When does $gen(D)$ consist of a single class ?
- When is $gen(D)$ finite ?

The key result discussed at the talk can be formulated as follows:

Let K be a field of characteristic different from 2. (1) If K satisfies the following property:

(*) if D and D are central division K-algebras of exponent 2 having the same maximal subfields then $D = D$ (in other words, for any D of exponent 2, $|gen(D) \cap 2Br(K)| = 1$),

then the field of rational functions $K(x)$ also satisfies (*). (2) If $|gen(D)| = 1$ for any central division K-algebra D of exponent 2, then the same is true for any central division $K(x)$-algebra of exponent 2.

The talk was based on the joint work with A. Rapinchuk and I. Rapinchuk [6].

Mathieu Florence *Central simple algebras of index p^n in characteristic p*

Let k be a field of characteristic $p > 0$, and let A/k be a central simple algebra of index $d = p^n$ and exponent p^e. Using a result of Hochschild, of which we provide a new proof, we show that A is Brauer equivalent to the tensor product of at most $d - 1$ cyclic algebras of degree p^e. This improves drastically the previously known upper bounds, mainly due to TeichmÃijller, Mammone and Merkurjev.

Olivier Haution *Invariants of upper motives*

The canonical dimension of a smooth complete algebraic variety measures to which extent it can be rationally compressed. In order to compute it, one often rather study the p-local version of this notion, called canonical p-dimension (p is a prime number). In this paper, we consider the relation of p-equivalence between complete varieties, constructed so that p-equivalent varieties have the same canonical p-dimension. In particular, two complete varieties X and Y are p-equivalent, for any p, as soon as there are rational maps $X \to Y$ and $Y \to X$. In order to obtain restrictions on the possible values of the canonical p-dimension of a variety, one is naturally led to study invariants of p-equivalence. In the talk the speaker introduced a systematic way to produce such invariants (and in particular, birational invariants), starting from a homology theory. He provides examples related to K-theory and cycle modules. Then he describes the relation between two such invariants of a complete variety X: its index n_X, and the integer d_X defined as the g.c.d. of the Euler characteristics of the coherent sheaves of $\mathcal{O}_X$-modules. The latter invariant contains both arithmetic and geometric information; this can be used to give bounds on the possible values of the index n_X (an arithmetic invariant) in terms of the geometry of X. For instance, a smooth, complete, geometrically rational (or merely geometrically rationally connected, when k has characteristic zero) variety of dimension $< p1$ always has a closed point of degree prime to p. The talk is based on the paper [7].

Valentina Kiritchenko *Schubert calculus for equivariant algebraic cobordism*

Let k be a field of characteristic zero, and G a connected reductive group split over k. Recall that a smooth spherical variety is a smooth k-scheme X with an action of G and a dense orbit of a Borel subgroup of G. Well-known examples of spherical varieties include flag varieties, toric varieties and wonderful compactifications of symmetric spaces. In her talk, the speaker discussed the equivariant cobordism rings of the following two classes of spherical varieties: the flag varieties and the wonderful symmetric varieties of minimal rank (the latter include wonderful compactifications of semisimple groups of adjoint type). The equivariant cohomology and the equivariant Chow groups of these two classes of spherical varieties have been extensively studied before. Based on the theory of algebraic cobordism by Levine and Morel, and the construction of equivariant Chow groups by Totaro and Edidin-Graham, the equivariant cobordism was initially introduced by D. Desphande for smooth varieties. It was subsequently developed into a complete theory of equivariant oriented Borel-Moore homology for all k-schemes by A. Krishna. Similarly to equivariant cohomology, equivariant cobordism is a powerful tool for computing ordinary cobordism of the varieties with a group action. The techniques of equivariant cobordism have been recently exploited to give explicit descriptions of the ordinary cobordism rings of smooth toric varieties, and that of the flag bundles over smooth schemes. The talk was based on the joint work with Amalendu Krishna [8].

V. The last day of the workshop started with talks on essential dimension by Shane Cernele (graduate student) and Roland Lotscher (young researcher) and was finished by the talk of Philippe Gille devoted to the topological properties of torsors.

Shane Cernele *Essential dimension and error-correcting codes*

Let p be a prime, $r \geq 3$, and $n_i = p^{a_i}$ for positive integers $a_1, \ldots, a_r$. Define $G = GL_{n_1} \times \ldots \times GL_{n_r}$ and let μ be a central subgroup of G, over a field of characteristic zero. The Galois cohomology set $H^1(K, G/\mu)$ classifies r-tuple of central simple algebras satisfying linear equations in the Brauer group $Br(K)$. The key object of the talk is the essential dimension and essential p-dimension of G/μ. To any central subgroup μ, one associates a finite module C called the code associated to μ, and define a weight function w from C to the positive integers. In the talk the speaker showed that the essential dimension of G/μ depends only on C. Using general cohomological methods as well as results of Karpenko and Merkurjev, and Popov, he gave lower and upper bounds on the essential dimension of G/μ in terms of w and C. For some subgroups μ he found matching lower and upper bounds.

Roland Lötscher *Essential p-dimension of algebraic groups, whose connected component is a torus.*

Let p be a prime integer and k a base field of characteristic not p. In this talk the speaker was studying the essential dimension of linear algebraic k-groups G whose connected component G_0 is an algebraic torus. This is a natural class of groups; for example, normalizers of maximal tori in reductive linear algebraic groups are of this form.

To state the main result, recall that a linear representation $\rho\colon G \to GL(V)$ is called generically free if there exists a G-invariant dense open subset UV such that the scheme-theoretic stabilizer of every point of U is trivial. A generically free representation is clearly faithful but the converse does not always hold; see below. We will say that ρ is p-generically free (respectively, p-faithful) if $ker\rho$ is finite of order prime to p, and ρ descends to a generically free (respectively, faithful) representation of $G/ker\rho$.

Then the main result says the following

Let G be an extension of a (possibly twisted) finite p-group F by an algebraic torus T defined over a p-special field k of characteristic not p. Then $\min\dim(\rho) - \dim G \leq ed(G;p) \leq \min\dim(\mu) - \dim G$, where the minima are taken over all p-faithful linear representations ρ of G and p-generically free representations μ of G, respectively.

The talk was based on the joint work with Mark MacDonald, Aurel Meyer and Zinovy Reichstein [9].

Philippe Gille *Topological properties of torsors and homogeneous spaces over valued fields*

This was a report on work in progress with Laurent Moret-Bailly [10]. Let K be the fraction field of a henselian valuation ring R of positive characteristic p. Let Y be a K-variety, H an algebraic group over K, and $f : X \to Y$ an H-torsor over Y. The speaker considered the induced map $X(K) \to Y(K)$, which is continuous for the topologies deduced from the valuation. If Z denotes the image of this map, he investigated the following questions:

(a) Is Z locally closed (resp. closed) in $Y(K)$?

(b) Is the continuous bijection $X(K)/H(K) \to Z$ a homeomorphism ?

Outcome of the Meeting

The workshop attracted 42 leading experts and young researchers from Belgium, Canada, France, South Korea, Germany, Russia, Switzerland, USA. There were 19 speakers in total: 10 talks were given by senior speakers, 7 talks by young researches and postdocs and 2 talks by PhD students.

The morning lectures given by senior speakers provided an excellent overview on the current stage of research in the theory of Lie algebras, torsors and cohomological invariants. There were several new results announced, e.g. Merkurjev (on essential dimension), Saltman (on the u-invariant), Vishik (on the cohomological operations). The afternoon sessions provided a unique opportunity for young speakers to present their achievements. Numerous discussions between the participants after the talks have already lead to several joint projects, e.g. Neher-Pianzola, Calmès-Zainoulline.

The organizers consider the workshop to be a great success. The quantity and quality of the students, young researchers and the speakers was exceptional. The enthusiasm of the participants was evidenced by the frequent occurrence of a long line of participants waiting to ask questions to the speakers after each lecture. The organizers feel that the material these participants learned during their time in BIRS will prove to be very valuable in their research and will undoubtedly have a positive impact on the research activity in the area.

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Bibliography

[1] Sanghoon Baek, On the torsion of Chow groups of Severi-Brauer varieties, Preprint 2012, *arXiv:1206.2704*.

[2] Jean-Louis Colliot-Thélène, Boris Kunyavskii, Vladimir L. Popov, Zinovy Reichstein, Is the function field of a reductive Lie algebra purely transcendental over the field of invariants for the adjoint action, *Compositio Math.* **147** (2011), n.2, 428–466.

[3] Erhard Neher, Arturo Pianzola, Ëtale descent of derivations, Preprint 2012.

[4] Ivan Dimitrov, Mike Roth, Geometric realization of PRV components and the Littlewood-Richardson cone, Contemporary math. 490, 2009.

[5] Marc Blunk, Michail Borovoi, Boris Kunyavskii, Nicole Lemire, Zinovy Reichstein, Stably Cayley groups over fields of characteristic 0, Preprint 2012, *arXiv:1207.1329*.

[6] Vladimir Chernousov, Andrei Rapinchuk, Igor Rapinchuk, On the genus of a division algebra. C. R. Acad. Sci. Paris (2012).

[7] Olivier Haution, Invariants of upper motives, Preprint 2012, *arXiv:1208.2586*.

[8] Valentina Kiritchenko, Amalendu Krishna, Equivariant Cobordism of Flag Varieties and of Symmetric Varieties, Preprint 2011, *arXiv:1104.1089*.

[9] Roland Lötscher, Mark MacDonald, Aurel Meyer, Zinovy Reichstein, Essential p-dimension of algebraic tori, Preprint 2009, *arXiv:0910.5574*.

[10] Philippe, Gille, Laurent Moret Bailly, Actions algébriques de groupes arithmétiques, à paraÃőtre dans les actes de la conférence “Torsors, theory and applications", Edimbourg (2011), Proceedings of the London Mathematical Society, édité par V. Batyrev et A. Skorobogatov.

Chapter 34

Topological Data Analysis and Machine Learning Theory (12w5081)

October 14 - 19, 2012

Organizer(s): Gunnar Carlsson (Stanford University), Rick Jardine (University of Western Ontario), Dmitry Feichtner-Kozlov (University of Bremen), Dmitriy Morozov (Lawrence Berkeley National Laboratory)

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Persistent homology

Perhaps the most important idea in applied algebraic topology is persistence. It is a response to the first difficulty that one encounters in attempting to assign topological invariants to statistical data sets: that the topology is not robust and has a sensitive dependence on the length scale at which the data set is being considered. The solution is to calculate the topology (specifically the homology) at all scales simultaneously, and to encode the relationship between the different scales in an algebraic invariant called the persistence diagram. The effective algorithm for doing so was published in 2000 by Edelsbrunner, Letscher and Zomorodian [2]. Topological data analysis would not be possible without this tool.

Since then, persistence has been developed and understood quite extensively. Cohen-Steiner, Edelsbrunner and Harer [3] proved the important (and nontrivial) theorem that the persistence diagram is stable under perturbations of the initial data. Zomorodian and Carlsson [7] studied persistence algebraically, identifying the points of the persistence diagram with indecomposable summands of a module over the polynomial ring $k[t]$, the monomial t representing a change of scale by a fixed increment. Generalising this approach using polynomial rings with two or more variables, they showed that the corresponding situation with two or more independent length scales is in some sense algebraically intractable, with no complete descriptive invariants available. Carlsson and de Silva [1] showed that persistence can be made to work over "non-monotone" parameters, in contrast to "monotone" parameters such as length scale; this is known as zigzag persistence. Bubenik and Scott [3] have described and studied persistence in terms of category theory.

Measured view. In becoming familiar with the literature on persistence, one quickly realises that the existing mathematical foundations are dependent on certain strong finiteness assumptions. For instance, one can do persistence on the sublevelsets of a function on a compact manifold, but it is usually assumed that the function be a Morse function. This ensures that the resulting persistent data is finite: each sublevelset has finite-dimensional homology, and there are only finitely many essentially distinct levels, separated by the critical points. In computational terms, this is a perfectly natural assumption (a computer can only handle finite data), but for some of the theoretical work this assumption is limiting. This was perhaps first seen in the proof of the stability theorem by Cohen-Steiner et al, which has to navigate around this assumption.

Vin de Silva presented new work [5] with co-authors Chazal, Glisse and Oudot, which addresses these problems through a deep analysis of the structure of so-called "persistence modules": 1-parameter families of vector spaces and maps between them. The basic challenge is to construct the persistence diagram and show that it is stable, without the usual strong finiteness assumptions about the persistence module. They achieve this by establishing an equivalence between persistence diagrams and a certain kind of measure defined on rectangles in the plane. The equivalence is proved under a weaker finiteness condition, called "q-tameness", which is seen to hold quite widely: for any continuous function (not just Morse) on a finite simplicial complex, for the Vietoris–Rips complex on a compact metric space, etc. It turns out that all of the standard results can be proved much more easily, and in greater generality, when one works with these measures. The authors introduce a new notational system, a sort of "quiver calculus", which gives short transparent proofs of the linear algebra lemmas that show up in the persistence literature, by interpreting those results as statements about the indecomposable summands of certain quiver representations.

Statistics of diagrams. An important workshop theme emerged during John Harer's talk: the need to introduce statistical techniques to topological data analysis. The overarching idea is that the input to most analysis techniques is generated stochastically. Therefore, it is interesting to talk not only about a single persistence diagram, but about an entire collection of them, so that we can try to calculate means, variances and apply statistical inference techniques.

Harer described his work [8] with Mileyko and Mukherjee showing that the space of persistence diagrams allows for the definition of probability measures, which support expectations and variances, among other properties. The authors showed that the space of persistence diagrams with the Wasserstein metric is complete and separable, and described when it is compact. Harer also sketched an algorithm [9] to compute Fréchet means in this space.

Inference using landscapes. Peter Bubenik gave an alternative approach [10] to this problem. He mapped persistence diagrams (also known as barcodes) to certain functions $\lambda : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, called persistence landscapes. Let B be a barcode. Call half the length of an interval its radius. Then $\lambda_k(t)$ is the largest radius such that the interval of radius r centered at t is a subinterval of k intervals in B. This space inherits nice structure from $L^p(\mathbb{N} \times \mathbb{R})$. In L^2, the Fréchet means and variances are the pointwise means and integral of the pointwise variances, respectively. One has a strong law of large numbers and a central limit theorem. One can apply statistical inference using the permutation. Bubenik applied these ideas to synthetic data drawn from a torus and a sphere and to brain MRI data. In the latter he showed that the persistence landscapes of the triangulation of the outer surface filtered by angle defect seemed to discriminate between high-functioning autistic subjects and controls.

Computational advances. The rapid growth in the range of applications of algebraic topology suggests the need for efficient algorithms for computing homology groups, persistent homology, and induced maps on homology. There are a variety of strategies that have been adopted. The most fundamental is to view Smith diagonalization as a purely algebraic problem and then to seek an optimal algorithm. The worst case analysis of such algorithms suggests a supercubical complexity with respect to the size of the complex, which is prohibitive for large datasets. An alternative strategy is to develop efficient algorithms for restricted problems, for example by restricting the dimension of the complex or restricting the computation to that of Betti numbers. Konstantin Mischaikow described an approach [11] that uses discrete Morse theory to preprocess the data, thereby producing a smaller complex on which an algebraic Smith normal form computation can be performed.

Mischaikow's talk centered around three topics:

1. A review of the notion of complex as introduced by Tucker and Lefschetz where the focus is on incidence numbers that relate neighboring cells. In the contexts of data analysis and computational dynamics which motivate this work, this local information is often natural to the input whereas the

associated boundary operator must be derived from this information. The explicit construction of this boundary operator, which can be costly for large complexes, is unnecessary in the approach.

2. Algorithms for efficiently computing Morse complexes which are based on the co-reduction algorithm.
3. The use of Morse complexes to efficiently compute the induced maps on homology.

Generalization: Étalage. Persistent homology quantifies topological information in data. Very importantly, this quantification is stable to measurable perturbations of the data. With this in mind, Amit Patel presented abstractions of various notions in the theory of persistent homology to multi-parameter families of spaces.

A fundamental notion is the persistent homology group. Let $f : \mathbb{X} \to \mathbb{R}$ be a continuous map from a topological space to the real line. The map f defines a 1-parameter family of spaces $\mathbb{X}_t = f^{-1}(t)$. The *persistent homology group* over an interval $(a, b) \subseteq \mathbb{R}$ is the intersection of the images of the two homomorphisms $i : \mathsf{H}_*(f^{-1}(a)) \to \mathsf{H}_*(f^{-1}[a, b])$ and $j : \mathsf{H}_*(f^{-1}(b)) \to \mathsf{H}_*(f^{-1}[a, b])$ induced by inclusion of spaces. Roughly speaking, the persistent homology group is the homology that is common to all fibers $\mathbb{X}_t$ over the interval (a, b). Now let $g : \mathbb{X} \to \mathbb{M}$ be a map to an oriented m-manifold. This map defines an m-parameter family of spaces $\mathbb{X}_p = g^{-1}(p)$. One would like an abstraction of the notion of the persistent homology group to path-connected open sets $U \subseteq \mathbb{M}$. Patel introduced the well group, which serves this purpose. The *well group* is the image of a homomorphism $\Phi : \mathsf{H}^{\mathrm{c}}_{*+m}(f^{-1}(U)) \to \mathsf{H}_*(f^{-1}(U))$ from the homology of $f^{-1}(U)$ with closed support to the homology of $f^{-1}(U)$ with compact support shifted down by m dimensions. This homomorphism is the cap product with the pullback of the orientation on $\mathbb{M}$. The well group is the homology that is common to all fibers above U, and, furthermore, it is stable to homotopic perturbations of the map g.

The theory of persistent homology assembles the local information of the persistent homology groups into a global structure called the *persistence diagram*. Patel introduced the *étalage* of $\mathbb{M}$ which abstracts the notion of a persistence diagram to the higher dimensional setting. An étalage of $\mathbb{M}$ is a Hausdorff topological space $\mathbb{E}$ along with a continuous map $\pi : \mathbb{E} \to \mathbb{M}$ that is locally a homeomorphism. In addition, assigned to each connected component of $\mathbb{E}$ is an integer. As with persistence diagrams, one can read off the rank of the stable homology of each fiber $\mathbb{X}_p$ by summing the integers assigned to each point in $\pi^{-1}(p)$.

Persistence stability for geometric complexes. The classical theory of persistent homology, restricted to tame functions and filtrations of finite simplicial complexes, does not directly address certain questions in topological data analysis: e.g., multiscale homology inference for metric spaces or scalar fields analysis on discrete data. To overcome this issue, Frédéric Chazal and collaborators extended and generalized persistent homology and its stability results. As an application they have proven the robustness of the persistent homology of various families of geometric filtered complexes built on top of compact metric spaces or spaces endowed with a similarity measure.

Theorem 9 ([6]). *Let X be a pre-compact metric space. Then the Vietoris–Rips and Čech filtrations built on top of X induce, at the homology level, persistence modules that are q-tame. In particular, they have well-defined persistence diagrams. Moreover, if X and Y are two compact metric spaces then the bottleneck distance between the persistence diagrams of the Vietoris–Rips (resp. Čech) filtrations built on top of X and Y is upper bounded by the Gromov–Hausdorff distance between X and Y.*

This result is an ingredient for mathematically well-founded statistical developments for topological data analysis using persistence theory.

Persistent homology and metric geometry. Facundo Mémoli presented ongoing work on importing constructions from metric geometry into persistent topology. His aim is to define a notion of distance d_F between filtered finite spaces compatible with the standard stability result for persistence diagrams of filtrations. More precisely, $d_F(X, Y)$ is defined as the infimal $\varepsilon > 0$ for which one can find a finite set Z and surjective maps $\varphi_X : Z \to X$ and $\varphi_Y : Z \to Y$ such that the L^∞ norm of the difference of the pullback filtrations $\varphi_X^* F_X, \varphi_Y^* F_Y : \mathrm{pow}(Z) \to \mathbb{R}$ is bounded by ε.

Now, one has a generalization of the combinatorial stability theorem of [21] that does not assume that the two filtrations are defined on the same space. Below, d_B is the bottleneck distance between persistence diagrams.

Theorem 10 ([23])**.** *For all (X, F_X) and (Y, F_Y), $d_B\big(D_*(F_X), D_*(F_Y)\big) \le d_F(X, Y)$.*

Mémoli also described different ways in which metric spaces and metric measure spaces induce filtrations, and how these induced filtrations are stable with respect to suitable notions of distance. Each of these "ways" gives rise to a *filtration functor*. In this manner, a measure-dependent notion of Vietoris-Rips filtration arises, which is quantitatively stable in a precise sense.

To be clear, given a finite mm-space (X, d_X, μ_X), the *weighted* Vietoris-Rips filtration $F_X^{\omega R} : \mathrm{pow}(X) \to \mathbb{R}$ is given for $p \ge 1$ by

$$\sigma \mapsto \left(\sum_{x,x' \in \sigma} \big(d_X(x,x')\big)^p \mu_X(x)\mu_X(x') \right)^{1/p}.$$

The hope is that with such constructions one would be able to capture topological features of a given dataset in a manner which is robust to noise or outliers. In order to express stability, in the case of metric spaces the natural notion of distance is the Gromov-Hausdorff distance, whereas in the case of mm-spaces, the notion is the Gromov-Wasserstein distance $d_{GW,\infty}$ [20]. From Theorem 10 above one obtains:

Corollary 11. *[23] For all finite mm-spaces X and Y one has $d_B\big(D_*(F_X^{\omega R}), D_*(F_Y^{\omega R})\big) \le d_{GW,\infty}(X, Y)$.*

Mémoli also revisited a theme, which was discussed in other talks in the workshop, namely, the issue of pinning down a suitable notion of what it might mean to do statistics on persistence diagrams. He described a measured construction that can be regarded as a step that takes place *before* the computation or definition of the Fréchet mean of a collection of persistence diagrams.

Topological connections

Topological statistical mechanics. Configuration spaces of points are well-studied in several branches of mathematics, including algebraic topology, geometric group theory, and combinatorics. Give the particles thickness, and you have what physicists might describe as phase space for a hard spheres gas. When the points are points, the topology of the configuration space is well understood. But hardly anything is known when points have thickness. The changes in the topology as the thickness varies could be thought of as topological phase transitions.

In joint work [12] with Baryshnikov and Bubenik, Matthew Kahle has started to develop a Morse theory for these configuration spaces. In particular, they have proved a theorem that "critical points", where the topology changes, correspond to mechanically balanced configurations of spheres.

With a similar point of view, Carlsson, Gorham, Mason, and Kahle [13] implemented a computational approach to study these spaces. They find complicated behavior, even for a small number of particles. With only five disks in a square, it seems that the topology of the configuration space changes a few dozen times as the radius of the particles varies.

Finally, MacPherson and Kahle have a work in progress where they consider the asymptotic behavior of Betti numbers for n disks in an infinite strip as $n \to \infty$. They find that that there is a regime where the Betti numbers grow polynomially, and a regime where they grow exponentially. Understanding these kinds of asymptotics for any bounded region seems to be an attractive and wide-open problem.

Topological dimensionality reduction. Data in high dimensional spaces is ubiquitous across a variety of domains. A major problem in data analysis is how to create effective schemes to reduce the dimensionality while maintaining or improving the ability to do geometric and statistical inference. Anthony Bak showed how ideas from topological data analysis can guide intelligent dimension reduction choices.

Many data analysis problems consist of a collection of objects, each of which has associated features. For the case that the features are real-valued we organize this information into a matrix $S = (s_{ij})$, where s_{ij} is the value of the j^{th} feature on the i^{th} object. The goal is to understand the objects, or row space of this matrix. As a preprocessing step, we analyze the geometry of the column space, or “dual space”, with the goal of reducing the number of columns.

Bak applied these ideas in two very different situations:

1. A simulation in the spirit of “evasion” and sensor network problems. We have a series of sensors in the plane observing a different set of moving particles. Each sensor is “dumb” in that it only records aggregate information on all of the particles within a certain radius. In simulations Bak laid the sensors out in a series of circles with varying density. The goal was to reduce the number of sensors while maintaining the topological coverage so that one could, for example, detect when an object passes in or out of a circle.

2. Real world microarray data consisting of gene activation levels for E. coli from a collection of 600 experiments. E. coli genes are physically laid out in a circle (unlike the human genome which is laid out in a line on each chromosome) and genes near each other have a tendency to activate together in what is called an “operon” group. The goal is to reconstruct the circular ordering of the genome from the microarray data.

The central idea in both cases was to build a metric space using correlation distance between sensors and to remove sensors until the topology, as measured with zigzag persistence, changed. In both examples he reported a dramatic dimensionality reduction, going from hundreds of features to 50 in the first case and only five in the case of E coli. For E. coli the ability to reconstruct the ordering improved over both using all the experiments and using PCA as the dimension reduction step.

Novel techniques for clustering. The clustering problem in data analysis is to decide how to split a large data set into a number of smaller sets based on the geometry of the point cloud. The goal is for points in each subset to be closer to each other than to points in the other subsets. Jesse Johnson discussed a new approach [14] to this problem based on ideas from three-dimensional geometric topology called thin position. In the pure topology setting, these techniques are very effective at finding two dimensional surfaces that efficiently cut three-dimensional spaces into smaller pieces. Because of the close link between topology and geometry in dimension three, thin position also tends to find geometrically efficient partitions of these spaces, so it is very natural to apply them to finding efficient geometric partitions of data sets.

This topic was also a natural complement to the workshop’s theme of homological techniques because in the pure topology setting, thin position tends to find structure that homology misses, while missing the structure that homology measures. To get a complete understanding of a three-dimensional manifold, one must understand both its homology groups and its thin position structure. It appears that the same is true for data analysis: To best understand a large, high dimensional data set, one should consider information that comes both from persistence homology and from the algorithms based on thin position.

Biological applications. The past decade has witnessed developments in the field of biology that have brought about profound changes in understanding the dynamic of disease and of biological systems in general. New technology has given biologists an unprecedented wealth of information, but it has generated data that is hard to analyze mathematically, thereby making its biological interpretation difficult. These challenges, stemming in part from the very high dimensionality of the data, have given rise to a myriad novel exciting mathematical problems and have provided an impetus to modify and adapt traditional mathematics tools, as well as develop novel techniques to tackle the data analysis problems raised in biology.

Monica Nicolau discussed data transformations and topological methods for solving biology-driven problems. The general approach of her work was to address some of the computational challenges of these large data types by combining data transformations and topological methods. Through the definition of high-dimensional mathematical models for different biological states, for example healthy vs. disease, or various developmental stages of cells, data can be transformed to mod out all characteristics but those relevant or statistically significant to the problem under study. Once data has been transformed, analysis of significance involves analysis of the shape of the data. Nicolau described how adaptation of topological methods in the context of discrete point clouds has proved to be a powerful tool for identifying statistical significance as well as providing methods for visualizing data.

Scale selection. One of the keywords in topological data analysis, besides "shape", is "scale". This is not an obvious fact, given that a great part of algebraic topology deals with utmost flexibility in the sense of homotopy invariance and could thus be considered scale-free. On the other hand, it exemplifies what we must give up as soon as ideal shapes are approximated by point clouds, and how topology, at the interface with applied statistics, can use its inherent flexibility and global point of view to find out at which scale features are present in a data set and to also detect and describe multi-scale phenomena.

Daniel Müllner focused on a situation where several scale choices are made on overlapping parts of a data set. This happens at the core of the "Mapper" algorithm, a tool for visualization, exploration and data analysis, which has been successful both academically and commercially. The original Mapper algorithm divides a data set into overlapping slices and chooses a scale independently for each, in order to cluster the data within each slice. Müllner demonstrated how this can lead to inconsistencies and false positives of topological features and in general makes it hard to determine which representations of a data set among many different possible outputs are appropriate. However, one can leverage spatial coherence and link scale choices for neighboring regions together in a "scale graph". By choosing optimal paths through the scale graph, those shortcomings can be resolved successfully. Müllner and co-authors developed heuristics for the scale graph method with the goal of making it work well for a broad range of data sets without optimizing too much for a particular context.

One problem in this process, which is still a big challenge in all of applied topology, is that noise in sampled data quickly smudges or destroys features. As an independent idea, which works very well together with the scale graph, Müllner proposed a new way of looking at dendrograms from hierarchical clustering. The basic idea is that stability intervals for the number of clusters in a dendrogram are not necessarily disjoint, but it is beneficial to consider more than one choice as feasible for a given scale if two clusterings differ for example only by inclusion or exclusion of noise points. This leads to a new rating of stability intervals which is conservative in the sense that it changes very little for low noise but improves the clustering considerably for higher noise levels. Since this new approach is not restricted to the Mapper context of multiple scale choices, it also has potential outside the application it was invented for, and it seems worthwhile to explore where it can improve clustering choices in other situations.

Geometric connections

Shape reconstruction. Dominique Attali reported on reconstructing shape of data points distributed in low-dimensional subspaces of a high-dimensional ambient space. This problem arises in many fields, including computer graphics and machine learning. Typical shape reconstruction methods start by building the Delaunay complex and then select a set of simplices such as the α-complex. Such approaches work well for point clouds in two- and three-dimensional spaces, which have Delaunay triangulations of affordable size. But, as the dimension of the ambient space increases, the size of the Delaunay triangulation explodes and other strategies must be found. If the data points lie on a low-dimensional submanifold, it seems reasonable to ask that the result of the reconstruction depends only upon the intrinsic dimension of the data. This motivated de Silva [19] to introduce *weak Delaunay complexes* and Boissonnat and Ghosh [18] to define *tangential Delaunay complexes*. For medium dimensions, Boissonnat et al. [17] have modified the data structure representing the Delaunay complex and are able to manage complexes of reasonable size up to dimension 6 in practice. In particular, they avoid the explicit representation of all Delaunay simplices by storing only edges in what they call the *Delaunay graph*, an idea close to that of using Rips complexes that Attali described.

Given a point set P and a scale parameter α, the *Vietoris-Rips complex* is the simplicial complex whose simplices are subsets of points in P with diameter at most 2α. Rips complexes are examples of *flag complexes*, and as such enjoy the property that a subset of P belongs to the complex if and only if all its edges belong to the complex. In other words, Rips complexes are completely determined by the graph of their edges. This compressed form of storage makes Rips complexes very appealing for computations, at least in high dimensions.

Attali (with André Lieutier and David Salinas) obtained two results. First, they established that Rips complexes can capture the homotopy type when distances are measured using the ℓ_∞ norm [15]. Unfortunately, the sampling condition was not as weak as they were hoping for, especially as the dimension of the ambiant space increases. Encouraged by experiments that were indicating this result should also hold when measuring distances using the Euclidean norm instead of the ℓ_∞ norm, they revisited the question and found conditions under which Rips complexes reflect the homotopy type of shapes when measuring distances using the Euclidean norm [16].

Mesh generation. A common approach to Topological Data Analysis (TDA) starts with a point cloud, proceeds to a function induced by the points, builds a simplicial complex, and then analyzes the persistent homology of the sublevel sets of that function as a filtration on the complex. The most common case of this is to explore the distance function to a set of points in Euclidean space. If we generalize to consider Lipschitz smooth functions on low-dimensional Euclidean space, there are many other choices of functions that apply. From this perspective, it is natural to consider the well-established field of mesh generation; it aims to efficiently build small simplicial complexes that give good approximations to Lipschitz functions. Don Sheehy showed how to apply results from mesh generation, some standard and some new, to give guaranteed approximate persistence diagrams for a wide class of functions.

His main results [24, 25, 26, 27] state that given a point set and any t-Lipschitz function f bounded from below by c times the second-nearest neighbor distance function to the point set, there exists a filtered mesh that has approximately the same persistence diagram, i.e. the diagrams drawn on the log-scale are close in bottleneck distance. Moreover, this mesh is independent of the function. It depends only on the point set, the constants c, t, and the desired approximation guarantee. Specifically, for a so-called ε-refined mesh, the resulting persistence diagram will be a $(1+ct\varepsilon/(1-\varepsilon))$-approximation to the persistence diagram of f itself. Moreover, for most reasonable point sets, one can guarantee that the mesh will have only linear size, though the constant factors depend exponentially on the ambient dimension.

The number of vertices will depend on the input and the amount of extra refinement. The dependence is a simple exponential in $1/\varepsilon$. The number of simplices incident to any vertex will be $2^{O(d^2)}$. It is not clear if it is possible to fill space with significantly fewer simplices per vertex. Volume arguments immediately imply a $2^{\Omega(d \log d)}$ lower bound for quality meshes, ruling out the possibility of a simple exponential dependence on the dimension.

Distance to measure. The notion of distance to a measure was introduced [28] in order to extend existing geometric and topological inference results from the usual Hausdorff sampling condition to a more probabilistic model of noise. This function can be rewritten as a minimum of a finite number of quadratic functions, one per isobarycenter of k distinct points of the point set, thus allowing to compute the topology of its sublevel sets using weighted alpha-complexes, weighted Rips complexes, etc.

However, as the number of isobarycenters grows exponentially with the number of points, it is necessary in practice to approximate this function by another function that can be written as a minimum of quadratic functions. A natural problem is then the following: given a target error ε, determine the minimum number of quadratic functions needed to approximate the distance to the measure with error ε. Quentin Mérigot presented recent probabilistic lower bounds on this number.

Machine Learning.

Learning mixtures of Gaussians. In recent years there has been an increase of interest in using algebraic-geometric methods to analyze data by recovering structure in probability distributions, in the field of âĂIJalgebraic statisticsâĂİ. This development has been somewhat parallel to using algebraic-topological methods to understand the shape of the data through recovering the homology groups or other topological and geometric invariants of the data.

Mikhail Belkin discussed recent work [29] on using real algebraic geometry to recover the parameters of mixtures of high-dimensional Gaussian distributions as well as other parametric families. Unlike most of the existing work in algebraic statistics, these parametric families are typically not âĂIJexponential familiesâĂİ. Specifically, his main result is that a mixture of Gaussians with a fixed number of components can be learned using the number of samples and operations polynomial in the dimension of the space and other relevant parameters. Moreover, a version of this statement for fixed dimension holds for a much broader class of distributions, called "polynomial families", i.e. families whose moments are polynomial functions of the parameters.

The overarching point of Belkin's talk was that there may be interesting connections between the fields of algebraic statistics, where the object of study is the space of parameters, and topological data analysis, where the geometry of a space is analyzed directly.

Modes in the mixtures of Gaussians. Brittany Fasy presented a recent result [30] on Gaussian mixtures. The mixture analyzed was the sum of $n+1$ identical isotropic Gaussians, where each Gaussian is centered at the vertex of a regular n-simplex. All critical points of this mixture are located on one-dimensional lines (axes) connecting barycenters of complementary faces of the simplex. Fixing the width of the Gaussians and varying the diameter of the simplex from zero to infinity by increasing a parameter called the scale factor, gives the window of scale factors for which the Gaussian mixture has more modes, or local maxima, than components. Using the one-dimensional axes containing the critical points, the interval of scale factors can be computed. Furthermore, the extra mode created is subtle, but becomes more pronounced as the dimension increases.

A natural open question in the area is: Restricting our attention to some class of kernels (for example, unimodal continuous kernels), which kernel observes the most (or the fewest) ghost modes?

Robus PCA. Consider a dataset of vector-valued observations that consists of a modest number of noisy inliers, which are explained well by a low-dimensional subspace, along with a large number of outliers, which have no linear structure. Lerman, McCoy, Tropp and Zhang [32] have suggested a convex optimization problem that can reliably fit a low-dimensional model to this type of data. They first minimize the function $F(\mathbf{Q}) := \sum_{i=1}^{N} \|\mathbf{Q}\mathbf{x}_i\|$ over $\{\mathbf{Q} \in \mathbb{R}^{D\times D} : \mathbf{Q} = \mathbf{Q}^T, \mathrm{Tr}(\mathbf{Q}) = D - d$ and $\mathbf{Q} \preccurlyeq \mathbf{I}\}$. The subspace is then defined as the span of the bottom d eigenvectors of this minimizer. They referred to this subspace recovery optimization as the REAPER optimization problem.

When the inliers are contained in a low-dimensional subspace they provided a rigorous theory describing when this optimization can recover the subspace exactly. The theory (based on an earlier work of Zhang and Lerman [33]) establishes exact recovery under some combinatorial conditions (which ask for sufficient spread of inliers throughout the underlying subspace and non-concentration of outliers per directions as well as some control on the magnitude of outliers). It also shows that under some probabilistic settings (e.g., Gaussian distributions of inliers and outliers) these combinatorial conditions, and thus the subspace recovery, are guaranteed. An example of such a probabilistic guarantee is formulated as follows, where $\mathbf{P}_L$ denotes the orthogonal projector onto the subspace L and $\mathcal{N}(\mathbf{0}, \mathbf{V})$ denotes a normal distribution with mean $\mathbf{0}$ and covariance matrix $\mathbf{V}$.

Theorem 12. *Fix a number $\beta > 0$, and assume that $1 \le d \le (D-1)/2$. Let L_* be an arbitrary d-dimensional subspace of $\mathbb{R}^D$, and draw N_{in} inliers i.i.d. $\mathcal{N}(\mathbf{0},\ (\sigma_{\mathrm{in}}^2/d)\,\mathbf{P}_{L_*})$ and N_{out} outliers i.i.d. $\mathcal{N}(\mathbf{0},\ (\sigma_{\mathrm{out}}^2/D)\,\mathbf{I}_D)$. Let C_1, C_2 and C_3 be positive universal constants. If the sampling sizes and the variances satisfy the relation*

$$\frac{N_{\mathrm{in}}}{d} \ge \mathrm{C}_1 + \mathrm{C}_2\,\beta + \mathrm{C}_3 \cdot \frac{\sigma_{\mathrm{out}}}{\sigma_{\mathrm{in}}} \cdot \left(\frac{N_{\mathrm{out}}}{D} + 1 + 4\beta\right), \tag{34.0.0.1}$$

then L_* is the unique solution to the REAPER optimization problem except with probability $4\,\mathrm{e}^{-\beta d}$.

Furthermore, Lerman, McCoy, Tropp and Zhang [32] presented an efficient iterative algorithm for solving this optimization problem, which converges linearly and its computational cost is comparable to that of the non-truncated SVD. Coudron and Lerman [31] established probabilistic convergence rates for the REAPER optimization problem (of an i.i.d. sampled data from a continuous setting); it is of the same order as that of full PCA. These imply some nontrivial robustness to noise (similar to the one of PCA) as well as sample complexity estimates for robust PCA (like those of PCA).

Towards understanding the Gaussian-weighted graph Laplacian. The Gaussian-weighted graph Laplacian, as a special form of graph Laplacians with general weights, has been a popular empirical operator for data analysis applications, including semi-supervised learning, clustering, and denoising. There have been various studies of the properties and behaviors of this empirical operator; most notably, its convergence behavior as the number of points sampled from a hidden manifold goes to infinity. Yusu Wang presented two new results on the theoretical properties of the Gaussian-weighted graph Laplacian. The first one is about its behavior as the input points are sampled from a *singular manifold*; while previous theoretical study of the Gaussian-weighted graph Laplacian typically assumes that the hidden domain is a compact smooth manifold. A singular manifold can consist of a collection of potentially intersecting manifolds with boundaries; it represents one step towards modeling more complex hidden domains. The second result is about the stability of the Gaussian-weighted Graph Laplacian if the hidden manifold has certain small perturbation. The goal is to understand how the spectrum of Gaussian-weighted Graph Laplacian changes with respect to perturbations of the domain.

Both of these problems are connected to topological data analysis. In particular, the work on singular manifolds is closely related to learning stratified spaces from point samples, although approached from a different direction from current topological methods for stratification inference. It would be interesting to investigate how these two somewhat complimentary lines of work can be combined to produce efficient algorithms for stratification learning. It is likely that one should focus on constraint families of stratified spaces so that theoretical guarantees can be obtained when they are inferred from point samples.

For the work on the stability of Gaussian-weighted graph Laplacian, a key question is a general perturbation model that allows small topological changes of the underlying domain. The presented perturbation model allows certain topological changes, but is still rather special. It would be interesting study under what generalized perturbation models one can still obtain bounded perturbations in the spectrum of graph Laplacian.

Manifold learning via Lie groups. The Cheeger inequality [37, 36] is a classic result that relates the isoperimetric constant of a manifold (with or without boundary) to the spectral gap of the Laplace-Beltrami operator. An analog of the manifold result was also found to hold on graphs [35, 34, 47] and is a prominent result in spectral graph theory. Given a graph G with vertex set V, the Cheeger number is the following isoperimetric constant

$$h := \min_{\emptyset \subsetneq S \subsetneq V} \frac{|\delta S|}{\min\left\{|S|, |\overline{S}|\right\}}$$

where δS is the set of edges connecting a vertex in S with a vertex in $\overline{S} = V \setminus S$. The Cheeger inequality on the graph relates the Cheeger number h to the algebraic connectivity λ [42] which is the the second eigenvalue of the graph Laplacian. It states that

$$2h \geq \lambda \geq \frac{h^2}{2\max_{v \in V} d_v}$$

where d_v is the number of edges connected to vertex v (also called the degree of the vertex). For more background on the Cheeger inequality see [39].

A key motivation for studying the Cheeger inequality has been understanding expander graphs [44] — sparse graphs with strong connectivity properties. The edge expansion of a graph is the Cheeger number in these studies and expanders are families of regular graphs $\mathcal{G}$ of increasing size with the property $h(G) > \varepsilon$ for some fixed $\varepsilon > 0$ and all $G \in \mathcal{G}$. A generalization of the Cheeger number to higher dimensions on simplicial complexes, based on ideas in [45, 46], was defined and expansion properties studied in [40] via cochain complexes. In addition, it has long been known [41] that the graph Laplacian generalizes to higher dimensions on simplicial complexes. In particular, one can generalize the notion of algebraic connectivity to higher dimensions using the cochain complex and relate an eigenvalue of the k-dimensional Laplacian to the k-dimensional Cheeger number. This raises the question of whether the Cheeger inequality has a higher-dimensional analog.

Sayan Mukherjee examined the combinatorial Laplacian which is derived from a chain complex and a cochain complex. First, the negative result: for the cochain complex a natural Cheeger inequality does not hold. For an m-dimensional simplicial complex we denote λ^{m-1} as the analog of the spectral gap for dimension $m-1$ on the cochain complex, and we denote h^{m-1} as the $(m-1)$-dimensional coboundary Cheeger number. In addition, let S_k be the set of k-dimensional simplexes, and for any $s \in S_k$ let d_s be the number of $(k+1)$-simplices incident to s. The following result implies that there exists no Cheeger inequality of the following form for the cochain complex. Specifically, there are no constants p_1, p_2, C such that either of the inequalities

$$C(h^{m-1})^{p_1} \geq \lambda^{m-1} \qquad \text{or} \qquad \lambda^{m-1} \geq \frac{C(h^{m-1})^{p_2}}{\max_{s \in S_{m-1}} d_s}$$

hold in general for an m-dimensional simplicial complex X with $m > 1$. The case of h^0 and λ^0 with $p_1 = 1$ and $p_2 = 2$ reduces to the Cheeger inequality on the graph and the Cheeger inequality holds.

For the chain complex Mukherjee and co-authors obtain a positive result: there is a direct analogue for the Cheeger inequality in certain well-behaved cases. Whereas the cochain complex is defined using the coboundary map, the chain complex is defined using the boundary map. Denote γ_m as the analog of the spectral gap for dimension m on the chain complex and h_m as the m-dimensional Cheeger number defined using the boundary map. If the m-dimensional simplicial complex X is an orientable pseudomanifold or satisfies certain more general conditions, then

$$h_m \geq \gamma_m \geq \frac{h_m^2}{2(m+1)}.$$

This inequality can be considered a discrete analog of the Cheeger inequality for manifolds with Dirichlet boundary condition [37, 36].

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Bibliography

[1] G. Carlsson and V. de Silva. Zigzag Persistence. *Found. Comput. Math.* (2010).

[2] H. Edelsbrunner, D. Letscher and A. Zomorodian. Topological persistence and simplification. *Discrete Comput. Geom.* **28** (2002), 511-533.

[3] David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. Stability of Persistence Diagrams. *Discrete and Computational Geometry,* **37**:103–120, 2007.

[4] P. Bubenik and J. A. Scott. Categorification of persistent homology. Preprint, arXiv:1205.3669, 2012.

[5] V. de Silva, F. Chazal, M. Glisse, and S. Oudot. The structure and stability of persistence modules. Manuscript, 2012.

[6] V. de Silva, F. Chazal, and S. Oudot. Persistence stability for geometric complexes. Manuscript, 2012.

[7] A. Zomorodian and G. Carlsson. Computing Persistent Homology. *Discrete and Computational Geometry,* **33**(2):249–274, 2005.

[8] Yuriy Mileyko, Sayan Mukherjee, and John Harer. Probability measures on the space of persistence diagrams. Inverse Problems, **27**, 2011.

[9] Katharine Turner, Yuriy Mileyko, Sayan Mukherjee, and John Harer. Fréchet Means for Distributions of Persistence Diagrams. Preprint, 2012.

[10] Peter Bubenik. Statistical topology using persistence landscapes. Manuscript, arXiv:1207.6437, 2012.

[11] S. Harker, K. Mischaikow, M. Mrozek, and V. Nanda. Discrete Morse Theoretic Algorithms for Computing Homology of Complexes and Maps. Preprint, 2012.

[12] Yuliy Baryshnikov, Peter Bubenik, and Matthew Kahle. Min-type Morse theory for configuration spaces of hard spheres. Preprint, 2011.

[13] Gunnar Carlsson, Jackson Gorham, Jeremy Mason, and Matthew Kahle. Computational topology for configuration spaces of hard disks. Physical Review E, 2012.

[14] Jesse Johnson. Topological graph clustering with thin position. Manuscript, arXiv:1206.0771, 2012.

[15] D. Attali and A. Lieutier. Reconstructing shapes with guarantees by unions of convex sets. In *Proc. 26th Ann. Sympos. Comput. Geom.*, pages 344–353, Snowbird, Utah, June 13-16 2010. [download], [hal-00468610].

[16] D. Attali, A. Lieutier, and D. Salinas. Vietoris-Rips complexes also provide topologically correct reconstructions of sampled shapes. *Computational Geometry: Theory and Applications (CGTA)*, 2012. In Press. [download].

[17] J. Boissonnat, O. Devillers, and S. Hornus. Incremental construction of the Delaunay triangulation and the Delaunay graph in medium dimension. In *Proceedings of the 25th annual symposium on Computational geometry*, pages 208–216. ACM, 2009.

[18] J. Boissonnat and A. Ghosh. Manifold reconstruction using tangential Delaunay complexes. In *Proceedings of the 2010 annual symposium on Computational geometry*, pages 324–333. ACM, 2010.

[19] V. De Silva. A weak characterisation of the Delaunay triangulation. *Geometriae Dedicata*, 135(1):39–64, 2008.

[20] Facundo Mémoli. Gromov-Wasserstein distances and the metric approach to object matching. *Foundations of Computational Mathematics*, pages 1–71, 2011. 10.1007/s10208-011-9093-5.

[21] David Cohen-Steiner, Herbert Edelsbrunner, and Dmitriy Morozov. Vines and vineyards by updating persistence in linear time. In *SCG '06: Proceedings of the twenty-second annual symposium on Computational geometry*, pages 119–126, New York, NY, USA, 2006. ACM.

[22] Facundo Mémoli. Some properties of GromovâĂŞ-Hausdorff distances. *Discrete & Computational Geometry*, pages 1–25, 2012. 10.1007/s00454-012-9406-8.

[23] Facundo Mémoli. Persistent Homology and Metric Geometry. In preparation. May 2012.

[24] B. Hudson, G. L. Miller, S. Y. Oudot, and D. R. Sheehy. Topological Inference via Meshing. Proceeding of Annual ACM Symposium on Computational Geometry, pages 277–286, 2010.

[25] D. R. Sheehy. Mesh Generation and Geometric Persistent Homology. PhD Thesis, 2011.

[26] D. R. Sheehy. New Bounds on the Size of Optimal Meshes. Computer Graphics Forum, 2012.

[27] G. L. Miller, T. Phillips, and D. R. Sheehy. Beating the Spread: Time-Optimal Point Meshing. Proceeding of Annual ACM Symposium on Computational Geometry, 2012.

[28] F. Chazal, D. Cohen-Steiner, and Q. Mérigot. Geometric inference for probability measures. *Foundations of Computational Mathematics*, 11:733–751, 2011.

[29] M. Belkin and K. Sinha. Polynomial Learning of Distribution Families. Proceeding of Annual IEEE Symposium on Foundations of Computer Science, 2010.

[30] Herbert Edelsbrunner, Brittany Terese Fasy, and Günter Rote. Add Isotropic Gaussian Mixtures at Own Risk: More and More Resilient Modes in Higher Dimensions. Proceedings of Annual ACM Symposium on Computational Geometry, 2012.

[31] M. Coudron and G. Lerman. On the sample complexity of robust pca. NIPS, 2012.

[32] G. Lerman, M. McCoy, J. A. Tropp, and T. Zhang. Robust computation of linear models, or How to find a needle in a haystack. *ArXiv e-prints 1202.4044*, Feb. 2012.

[33] T. Zhang and G. Lerman. A novel M-estimator for robust pca. Submitted, available at arXiv:1112.4863.

[34] N. Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986.

[35] N. Alon and V.D. Milman. λ_1, Isoperimetric Inequalities for Graphs, and Superconcentrators. *Journal of Combinatorial Theory, Series B*, 38(1):73–88, 1985.

[36] P. Buser. On Cheeger's Inequality $\lambda_1 \geq h^2/4$. In *Proc. Sympos. Pure Math*, volume 36, pages 29–77, 1980.

[37] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. *Problems in analysis*, pages 195–199, 1970.

[38] F. Chung. Random walks and local cuts in graphs. *Linear Algebra and its applications*, 423(1):22–32, 2007.

[39] F.R.K. Chung. *Spectral graph theory*. Amer. Mathematical Society, 1997.

[40] D. Dotterrer and M. Kahle. Coboundary expanders. *Arxiv preprint arXiv:1012.5316*, 2010.

[41] B. Eckmann. Harmonische Funktionen und Randwertaufgaben in einem Komplex. *Comm. Math. Helv.*, 17(1):240–255, 1944.

[42] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23(2):298–305, 1973.

[43] A. Gundert and U. Wagner. On Laplacians of random complexes. In *Proceedings of the 2012 Symposium on Computational Geometry*, pages 151–160. ACM, 2012.

[44] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bulletin of the American Mathematical Society*, 43(4):439, 2006.

[45] N. Linial and R. Meshulam. Homological connectivity of random 2-complexes. *Combinatorica*, 26(4):475–487, 2006.

[46] R. Meshulam and N. Wallach. Homological connectivity of random k-dimensional complexes. *Random Structures & Algorithms*, 34(3):408–417, 2009.

[47] B. Mohar. Isoperimetric numbers of graphs. *Journal of Combinatorial Theory, Series B*, 47(3):274–291, 1989.

[48] O. Parzanchevski, R. Rosenthal, and R.J. Tessler. Isoperimetric inequalities in simplicial complexes. *arXiv preprint arXiv:1207.0638*, 2012.

Chapter 35

Recent Advances in Transversal afnd Helly-type Theorems in Geometry, Combinatorics and Topology (12w5020)

October 21 - 26, 2012

Organizer(s): Imre Barany (Renyi Institute and University College London), Ted Bisztriczky (University of Calgary), Luis Montejano (National University of Mexico), Deborah Oliveros (Universidad Nacional Aut—noma de MŽxico), Richard Pollack (Courant Institute of Mathematics)

A prominent role in combinatorial geometry is played by Helly's theorem, which states the following: Theorem: [22] Let $\mathcal{A}$ be a finite family of at least $d+1$ convex sets in the d-dimensional euclidean space $\mathbb{R}^d$. If every $d+1$ members of $\mathcal{A}$ have a point in common, then there is a point common to all members of $\mathcal{A}$.

Helly's theorem has stimulated numerous generalization and variants. There are many interesting connections between Helly's theorem and its relatives, the theorems of Radon, of Caratheodory and of Tverberg, theorems that have been the object of active research, and inspired many problems in the field. To see a sample of numerous problems associated to Helly's theorems, see the paper that now a days is one of the most cited papers in discrete geometry: "Helly's Theorem and Its Relatives" [9].

In the past ten years, there has been a significant increase in research activity and productivity in the area. (For an excellent survey in the area, see [42].) Notable advances have been made in several subareas including the development of the theory of transversals (see [24]); topological versions of Helly Theorem; the proofs of interesting colorful theorems generalizing classical results (see [43]); and many others such as the problem of finding a line transversal to a family of mutually disjoint congruent disks in the plane.

This workshop brought together senior and junior researchers in the area with the objective of interchanging ideas and assessing recent advances, of fostering awareness of the inter-disciplinary aspects of the field such as geometry, topology, combinatorics, and computer science, and of mapping future directions of research.

The workshop combined and interesting mixture of talks, problem sessions and many time for discussions in groups. During the week the academic interest was mainly centerer about 4 topics:

- Helly-type Theorems, Piercing and (p, q)-Theorems,
- Variations and Generalizations of Tverberg's theorem,
- Transversal Theorems,

  and

- Finite sets of points and finite sets of convex sets.

Although many of the talks at this workshop on transversals and Helly-type theorems deal essentially with classical subjects related to this area, many of the talks also had deep relationships with other areas of discrete and non-discrete mathematics, such as algebraic topology, algebraic geometry and probability.

Applications of algebraic topology to discrete geometry was an especially interesting topic. The common theme of several of the talks given in this workshop relate algebraic topology to discrete geometry, transversals and Helly-type theorems using the topology of the space of Grassmannians and its canonical vector bundle together with the structure of the cohomology ring of spaces, was used several times during the workshop. The relations with the Algebraic Geometry were very interesting, specially those concerning linear space transversals to secant linear spaces in $\mathbb{R}^d$ and tangent lines to four balls in $\mathbb{R}^3$.

Helly-type Theorems, Piercing and (p,q)-Theorems

Given a universe (a set) $\mathcal{U}$ and a property $\mathcal{P}$, (closed under inclusions, for subsets of $\mathcal{U}$). Results of the type "if every subset of cardinality μ of a finite family $\mathcal{F} \subset \mathcal{U}$ has property $\mathcal{P}$, then the entire family $\mathcal{F}$ has property $\mathcal{P}$ " are called Helly type theorems The minimum number μ for which the result is true is called the Helly number of the Helly type theorem $(\mathcal{U}, \mathcal{P}, \mu)$.

If the Universe $\mathcal{U}$ consists of a special family of sets and the property $\mathcal{P}$ is to be pierced with k elements, or equivalently, to have a transversal of cardinality k, then we have a Helly-Gallai type theorem. These theorems have been widely studied for different settings (see for instance surveys such as [9] ,[E]). In fact such theorems are in general not easy to find, see, for instance, Danzer and Grünbaum [11] where they show that even for the case of families of Boxes in $\mathbb{R}^d$ such theorems does not always exist. During theworkshop several discussions about this subject where obtained. For example, L. Montejano and D. Oliveros obtained a Helly-Gallai theorem when $\mathcal{U}$ is the family of closed intervals in $\mathbb{R}^d$ showing that μ can be bounded by a function of k which is a polynomial of degree 5.

In 1930 Helly realized that a finite family of sets in $\mathbb{R}^d$ has nonempty intersection if for any subfamily of size at most $d+1$, its intersection is homeomorphic to a ball in $\mathbb{R}^d$. In fact, the result is true if we replace the notion of topological ball by the notion of acyclic set, see [7] and [26]. In 1970, Debrunner [12] proved that a finite family of open sets in $\mathbb{R}^d$ has nonempty intersection if for any subfamily of size j, $1 \leq j \leq d+1$, its intersection is $(d-j)$-acyclic.

During the workshop, L. Montejano spoke about a new Topological Helly theorem that follows the same spirit, except that instead of $\mathbb{R}^d$, it is require a topological space X in which $H_*(U) = 0$ for $* \geq d$ and every open subset U of X. Moreover, instead of the hypothesis $(d-j)$-acyclic, he just require that the $(d-j)$-dimensional reduced homology group is zero.

That is, for a topological space X with the property that $H_*(U) = 0$, for $* \geq d$ and every open subset U of X, a finite family of open sets in X has nonempty intersection if for any subfamily of size j, $1 \leq j \leq d+1$, the $(d-j)$-dimensional reduced homology group of its intersection is zero, where $H_{-1}(U) = 0$ if and only if U is nonempty.

The fact that this is a non-expensive topological Helly theorem —in the sense that it does not require the open sets to be simple— from the homotopy point of view (we only require its $(d-1)$-dimensional homology group to be zero), allows Montejano to prove some new results concerning transversal planes to families of convex sets like the following:
Let F be a pairwise disjoint family of at least 6 smooth, convex bodies in $\mathbb{R}^3$ with the property that for any subfamily $F' \subset F$ of cardinality 5, F' admits a transversal line and for any subfamily $F' \subset F$ of cardinality 4, the space of transversal lines is connected. Then, F' admits a transversal line.

During this workshop, it was discussed the following conjecture stated by Xavier Goaoc: Let $F = \{A_1, ..., A_n\}$ be a pairwise disjoint collection of convex sets in $\mathbb{R}^d$, $n \geq 2$. Then space of transversal lines consists of a finite collection of acyclic sets. Known true for $n = 2$ and 3, $d = 3$. During the workshop the conjecture was proved for $d = 4$, $n = 4$. If the conjecture turns out to be true, this will give interesting connections and generalizations with the Montejano's transversal results stated in the above paragraph.

About Helly type theorems and Piercing (p, q) problems, Deborah Oliveros spoke about About piercing numbers of families of planes, lines and intervals. She presented some bounds for the (p, q) problem and for piercing numbers of some families of affine hyperplanes, lines and intervals following the spirit of Erdös-Gallai, joint work with M. Huicochea, J. Jeronimo and L. Montejano.

About colourful and fractional (p,q)-problems, Ferenc Fodor considered colourful and fractional versions of the classical (p, q)-problem for systems of intervals in the real line. This was a preliminary report of ongoing research with I. Bárány, L. Montejano, and D. Oliveros. In fact, several discussions during the workshop allow the authors to finish this project.

Concerning piercing number, Juergen Eckhoff in [14] stated the following problem: let $N(p, q)$ be the piercing number of axis-parallel boxes in the plane having the (p, q)-property. Wegner (1965) conjectured that $N(p, 2) < 2(p-1)$ for all $p > 1$. This is true (and best possible) for $p = 2, 3, 4$ and would imply, among other things, that $N(p, q) = p - q + 1$ for all $q > 3$.

In his talk, Jürgen Eckhoff spoke about *The teasing strip problem*. The τ-strip problem consists in proving (or disproving) the following conjecture: If a finite set of points in the plane is such that every three of the points lie in some strip of width 1, then all points lie in some strip of width τ;. (Here $\tau = 1.6180...$ is the golden number.) The conjecture is more than 40 years old and, despite considerable progress, still unsolved. Jürgen Eckhoff talk described a new approach, based on numerical evidence, which may help to tackle the conjecture.

Helge Tverberg proposed the following problem on $(1, k)$-separation (H.T.1979). That is, let k be a positive integer. Then there is a positive integer $f(k)$ so that for every family of $f(k)$ pairwise disjoint compact convex sets in the plane there is a line separating at least one of the sets from at least k of the other sets. The best upper estimate for $f(k)$ so far is ca. $7.2(k - 1)$, obtained by M.Novick in [36] while the best lower estimate $3k - 1$ is given by K.Hope and M.Katchalski in Math.Scand.66 (1990),44-46.

In statistics, there are several measures of the depth of a point p relative to a fixed set S of sample points in dimension d. One of the most intuitive is the simplicial depth of p introduced by Liu (1990), which is the number of simplices generated by points in S that contain p. In general obtaining a lower bound for the simplicial depth is a challenging problem. In fact, in terms of simplicial depth, Carathéodory Theorem can be restated as follows: If p belongs to the convex hull of S then the simplicial depth is at least 1.

In 1982 Bárány showed that the simplicial depth is a least a fraction of all possible simplices generated from S. Gromov (2010) improved the fraction via a topological approach. Bárány's result uses a colourful version of Carathéodory Theorem leading to the associated colourful simplicial depth.

During the work shop we also have an interesting combinatorial, computational, and geometric approaches to the colourful simplicial depth by A. Deza. where he provide a new lower bound for the colourful simplicial depth improving the earlier bounds of Bárány and Matoušek and of Stephen and Thomas. Computational approaches for small dimension and the colourful linear programming feasibility problem introduced by Bárány and Onn were discussed. All these results based on join works with Frédéric Meunier (ENPC Paris), Tamon Stephen (Simon Fraser), Pauline Sarrabezolles (ENPC Paris), and Feng Xie (Microsoft)

Generalizations and Variations of Tverbergs Theorem

The workshop include several very interesting developments about generalizations and variations of Tververg Theorem, one of the most beautiful theorems in combinatorial convexity is Tverberg's theorem, which is the r-partite version of Radon's theorem, and it is very closely connected with the multiplied, or colorful versions of the theorems of Helly, Hadwiger and Caratheodory. The first of these colorful versions was discovered by Barany and Lovasz and has many applications (see [3]).

First, Pablo Soberón spoke about Equal coefficients in coloured Tverberg partitions. He analyze a variant of the coloured Tverberg partitions where the convex hulls of the colourful sets are required to intersect using the same coefficients. He give a theorem of this kind with an optimal number of colour classes and points, and extend it to intersections with tolerance.

Ricardo Strausz spoke about A generalization of Tverberg's theorem. In his lecture the following generalization of Tverberg's theorem was presented: every set of $(t+1)(k-1)(d+1)+1$ points in the euclidian

d-space admits a k-Tverberg partition with tolerance t. That is, there is a k-Tverberg partition such that, whenever t points are removed from the configuration, the partition of the remaining points is still intersecting. This is a joint work with Pablo Soberon and answers positively a conjecture of Natalia Garcia-Colin.

In fact, Juergen Eckhoff proposed the following open Problem that is variation on Tverberg's Theorem:

Let A be a set of at least $(k-1)(d+1)+2$ points in d–dimensional space. Is it true that A admits a "balanced" Tverberg k–partition, that is, a partition into k subsets whose convex hulls intersect and whose cardinalities differ by at most 1? This is false if A has $(k-1)(d+1)+1$ points. See Discrete Math. 221 (2000) 61-78.

Concerning Radon partitions Ricardo Strausz stated the following problem. How many minimal Radon Partitions determines a configuration? Wold be possible to be $n-d-1$?.

Given $d \geq 1$, $r \geq 1$, $k \geq 2$, let $\nu(r,k,d)$ be the maximum number such that there is a set $X \subset \mathbb{R}^d$ of size ν such that for every partition $A_1, A_2, \ldots, A_k$ of X into k parts, there is a subset $C \subset X$ of size r such that

$$\bigcap_{i=1}^{k} \langle A_i \backslash C \rangle_{conv} = \emptyset.$$

The following values of ν are known $\nu(1,2,d) = 2d+2$ for $d = 1,2,3$ done by Larman, $\nu(1,2,4) = 10$, by Las Vergnas, Forge, Schuchert, $\nu(1,2,d) \geq \lceil \frac{5d}{3} \rceil + 3$ for all d by Ramirez-Alfonsín, $\nu(r,2,d) \geq 2d+r+3$ for $r > 1$ of García-Colín, and $\nu(r,k,d) \geq k(\lfloor \frac{d}{2} \rfloor + r + 1) - 1$ by Soberón, and $\nu(r,k,d) \leq (r+1)(k-1)(d+1)$ by Soberón and Strausz.

Then Pablo Soberon conjecture that $\nu(r,k,d) = (r+1)(k-1)(d+1)$.

The discrete center point theorem states that for any finite set $X \subset \mathbb{R}^d$ there exists a *center point* $c \in \mathbb{R}^d$ such that any closed half-space $H \ni c$ contains at least $\left\lceil \frac{|X|}{d+1} \right\rceil$ points of X. where as the dual center point theorem states that for any family of n hyperplanes in general position in $\mathbb{R}^d$ there exists a point c such that any ray starting at c intersects at least $\left\lceil \frac{n}{d+1} \right\rceil$ hyperplanes.

In this set up, Roman Karasev spoke about Projective center point and Tverberg theorems, were he present projective versions of the center point theorem and Tverberg's theorem, interpolating between the original and the so-called "dual" center point and Tverberg theorems. Furthermore he give a common generalization of these and many other known (transversal, constraint, dual, and colorful) Tverberg type results in a single theorem, as well as some essentially new results about partitioning measures in projective space, and focusing on two classical topics in discrete geometry: the center point theorem from Neumann and Rado and Tverberg's theorem.

Many deep generalizations of these classical results have been made in the last three decades, starting from the topological generalization by Bárány, Shlosman, and Szucz. A good review on this topic and numerous references are given in Matoušek's book. After this book was published, new achievements were made by Hell, Engström and Engström–Norén, K., and Blagojević–M.–Ziegler , establishing "constrained", "dual", and "optimal colorful" Tverberg type theorems.

Here the use of the adjective "dual" is rather descriptive, it does not refer to projective duality. Thus it is interesting to dualize it once more projectively and compare it with the original center point theorem.

Then, the projective dual of the "dual center point theorem" can be stated as follows. Assume that X is a family of n points in $\mathbb{RP}^d$ and $c \in \mathbb{RP}^d$ is another point such that the family $X \cup c$ is in general position. Then there exists a hyperplane $W \subseteq \mathbb{RP}^d$ such that together with any hyperplane $H_1 \ni c$ it partitions $\mathbb{RP}^d$ into two parts each containing at least $\left\lceil \frac{n}{d+1} \right\rceil$ points of X.

From the proof of this theorem Roman Karasev can assure that W does not contain c; however if we omit the general position assumption then the theorem remains true by a compactness argument but W may happen to contain c.

Now he is going to interpolate between the original center point theorem and the latter "dual to dual" version (they appear as special cases when V is the hyperplane at infinity or when V is a point):

He also interpolate between Tverberg's theorem and its dual, and generalize further and state a very general theorem incorporating almost all is know about (dual, transversal, constrained, colorful) Tverberg type theorems.

Transversal Theorems

In 1955 Hadwiger [20] posed the problem of determining the smallest number k with the property that if every collection of k members of the family of $n \leq k$ pairwise disjoint unit disks in the plane are met by a line, then all the disks are met by a line; that is, he proposed to find a Helly number for the problem of finding a line transversal to a family of disjoint unit disks in the plane. There is an example proposed in the same paper by Hadwiger, consisting of 5 almost touching disks centered at the vertices of a regular pentagon in such a way that every four of them have a line transversal but the set of all of them does not. Hadwiger's problem was solved by Danzer [10], showing that a Helly number does exist for $k = 5$. In 1989 Tverberg [40] gave a generalization of Danzer's theorem on unit disks for disjoint translations of a compact convex set in the plane.

Denote by $\mathcal{F}$ a family of *ovals* (compact convex sets with non-empty interior) in the euclidean plane and let say that $\mathcal{F}$ has property T if there is a line that intersects all members of $\mathcal{F}$. If there is a line that meets not all but at most k members of $\mathcal{F}$, then $\mathcal{F}$ has the property $T - k$. Finally, if each k-element subfamily of $\mathcal{F}$ has a transversal line, then $\mathcal{F}$ has property $T(k)$. With this notation Danzer theorem cited above, says that $T(5)$ imply T for families of disjoint unit disks, however, it is known that for congruent ovals satisfying $T(5)$ does not imply T in general.

In this workshop Aladar Heppes spoke about an extension of Danzer's theorem for families of moderately overlapping unit discs. Particularly he spoke about finite family of at least five $\frac{2}{3}$-disjoint unit discs. If any 5-tuple if the discs has a line transversal then there is a line meeting all discs. (Joint work with T. Bisztriczky and K. Böröczky).

On this same order of ideas D. Oliveros propose the investigation of the existence of (p, q)–theorem or $T(p, q)$–theorem for transversals, that ensures a property T with some tolerance, That is $T(p, q)$–property will imply, that if out of p discs q of them have a transversal line if this implies that all but k of them have a transversal, the firs natural number to ask is 5 due to Danzer theorem, problem that has been answer negative for A. Holmsen in the case $(5, q)$ for every $1 < q < 5$, but still open for the more general cases, in fact to to D. Oliveros and L. Montejano [34] show the existence of a constant p such that $T(p, p-1)$ implies $T - k$ (for some k).

The workshop broth also the opportunity to discuss interesting problems of F. Sottile about of linear space transversals to secant linear spaces in $\mathbb{R}^d$ that has deep connections with algebraic geometry.

Fix positive integers $k < d$. For $t \in \mathbb{R}$, let $\gamma(t) := (t, t^2, \ldots, t^d) \in \mathbb{R}^d$. Then $\gamma := \gamma(\mathbb{R})$ is the *moment curve* in $\mathbb{R}^d$.

Let $I \subset \gamma$ be an interval (image of an interval in $\mathbb{R}$). (An affine) linear space L is secant to γ along I if L is affinely spanned by its intersections (necessarily exactly $\dim L + 1$) with I. Such a linear space does not meet $\gamma \setminus I$.

Algebraic geometry (Schubert calculus) together with a result of Mukhin, Tarasov, and Varchenko [35] tells us that if we take general $(d-k)$-planes $L_1, \ldots, L_{(k+1)(d-k)}$ secant to γ, then there are finitely many complex k-planes that are transversal to (meet) each L_i. The actual number $\delta_{k,d}$, is huge, it is

$$\delta_{k,d} \ := \ [(k+1)(d-k)]! \cdot \frac{0!1!2! \ldots k!}{(d-k)!(d-k+1)! \ldots d!} .$$

Then F. Sottile problem is concern about conditions on the L_i which force these common transversals to be real. The following conjecture was made in [18], based on extreme (more than 1 tera-Hertz year of computing) experimental evidence and some theoretical justifications.

Secant Conjecture:If the linear spaces $3L_1, \ldots, L_{(k+1)(d-k)}$ are secant to γ along disjoint intervals, then there are exactly $\delta_{k,d}$ real k-planes transversal to each of the L_i.

There are some special cases of this that are known.

First of all, if an interval I of secancy shrinks to a point $\gamma(t)$, then the secant plane becomes an osculating plane. If we replace secant by osculating in this conjecture, we recover the conjecture of Shapiro and Shapiro, which has been proven.

The case $k = d-2$ was proven by Eremenko and Gabrielov in a paper in the Annals of Mathematics [16]. It is equivalent to the following statement: A rational function, all of whose critical points lie on a circle, maps that circle to a circle. Their proof used the Uniformization Theorem in complex analysis and was quite difficult. They later gave a proof that used only elementary complex analysis and some algebraic geometry [17]. Neither proof is elementary in any sense.

The general case of the Conjecture of Shapiro and Shapiro has been given two proofs by Mukhin, Tarasov, and Varchenko [35]. These proofs used differential equations, representation theory of quantum groups, hyperplane arrangements, and mathematical physics, and are extremely sophisticated. A consequence of this result is the statement that general secant planes have the expected number of complex transversal k-planes.

The Secant Conjecture is also true if $k = d-2$. In this case it is a statement about rational functions which take the same value at each of $2d-2$ pairs of real points. This proof relies on the results of [16, 17] and uses a fixed point theorem from topology. The Secant Conjecture is also true if the points of secancy of each linear space form an arithmetic sequence (in the domain $\mathbb{R}$ of the moment curve γ), with the same step size for each linear space [35]. That result of Mukhin, Tarasov, and Varchenko used a similar mix of methods from mathematical physics.

There ample scope for new, elementary ideas. Here are some questions to focus.

Problem 3. In the case $k = 1$ and $d = 3$, the Secant Conjecture asserts that given four lines secant to the moment curve in $\mathbb{R}^3$ along disjoint (think consecutive) intervals, then the two (a priori complex) lines that meet all four are in fact real. Can one find an elementary proof of this fact?

Of the 17 combinatorial configurations of quadruples of secant lines along the projective closure of γ, four can have non-real transversals while the other 13 can only have real transversals. For twelve of the thirteen with only real transversals there is an elementary argument for this reality, and the only one which does not yet have an elementary proof is the configuration of the Secant Conjecture. See 4 of [18].

Problem 4. Give an elementary proof that there is one real secant k-plane when the linear spaces $L_1, \ldots, L_{(k+1)(d-k)}$ are secant along disjoint intervals? Is there an elementary proof of the Secant Conjecture in any family of subcases?

The Secant Conjecture is much wider than described above. Another class of problems are as follows. Let $a_1, \ldots, a_n$ be positive integers with $a_1 + \cdots + a_n = (k+1)(n-k)$. Then we have planes $L_1, \ldots, L_n$ secant to γ along disjoint intervals where $\dim L_i = d-k+1-a_i$, for each i. (The conjecture given above has each $a_i = 1$.) For example, when $k = 2$ and $d = 2a+1$, we set $n = 4$ and each $a_i = a$. Then there are $a + 1$ real lines meeting four a-planes that are secant along disjoint intervals of γ. It is possible to show this with an elementary argument?

Jorge Ramirez Alfonsin spoke on a problem closely related with the Kneser Theorem about transversals to the convex hull of all subsets of size k of a finite collection of points in $\mathbb{R}^d$. The Kneser Conjecture proved by Lovasz concerns the computation of the chromatic number of the Kneser graphs, which is a purely combinatorial problem. In this talk there is a relation of this problem with the following geometric problem: What is the maximum number n such that any finite set $N \subset \mathbb{R}^d$ of size n has a hyperplane transversal to the family of all convex hulls of k-set of N? It turns out that this number is related to the chromatic number of the Kneser graph $G^2(n,k)$.

In his talk, Ramirez-Alfonsin defined $M(k,d,\lambda) =$ the maximum positive integer n such that every set of n points in $\mathbb{R}^d$ has the property that the convex hull of all k-sets have a transversal $(d-\lambda)$-plane, and he introduced a special *Kneser hypergraph* establishing a close connection between its *chromatic number* and $M(k,d,\lambda)$. In fact, he defined the *Kneser hypergraph* $KG^{\lambda+1}(n,k)$ as the hypergraph whose vertices are $\binom{[n]}{k}$ and a collection of vertices $\{S_1, \ldots, S_\rho\}$ is a hyperedge of $KG^{\lambda+1}(n,k)$ if and only if $2 \leq \rho \leq \lambda+1$ and $S_1 \cap \cdots \cap S_\rho = \phi$. He remarked that $KG^{\lambda+1}(n,k)$ is the Kneser graph when $\lambda = 1$. Furthermore he noted that the Kneser hypergraph defined by him is different from that defined in [1] and using the cohomology structure of the space of Grasmannians and following the spirit of Dolnikov [13]. It is possible to prove that

$$\chi(KG^{\lambda+1}(n,k)) \leq d - \lambda + 1, \text{ then } n \leq M(k,d,\lambda).$$

Finally, he conjectured that $M(k,d,\lambda) = (d-\lambda) + k + \lceil \frac{k}{\lambda} \rceil - 1$.

In his talk he also discussed recent progress toward the validity of this conjecture in the case when $k = 4$. During the workshop important discussions concerning the valitidity of this conjecture took place. Several important new ideas were developed which hopefully will give rise to the solution of the conjecture. This is a join work with J. Arocha, J. Bracho and L. Montejano.

In this same order of ideas J. Eckhoff proposed a another problem about fractional transversals. Let $\mathcal{F}$ stand for a finite family of convex sets in the plane. What is the smallest number $\alpha > 0$ such that, if $\mathcal{F} \in T(3)$, then some subfamily $\mathcal{G}$ of $\mathcal{F}$ with $\mathcal{F} \geq \alpha|\mathcal{F}|$ has a common transversal?.

Katchalski conjetured that $\alpha \sim \frac{2}{3}$ but Holmsen (2010) showed that $\frac{1}{3} \leq \frac{1}{2}$ and believes that $\alpha = \frac{1}{2}$.

Furthermore, if $N(m, k)$ denote the smallest number n such that, if $|\mathcal{F}| = n$ and $\mathcal{F} \in T(k)$, then some m members of $\mathcal{F}$ have a common transversal. Wegner (unpublished) showed that $N(4, 3) = 6$, and Eckhoff (2008) conjetured that $N(k + 1, k) = k + 2$ if $k \geq 4$, that was probed by Novick in (2012) for $k \geq 8$. What about the cases $4 \leq k < 8$.

Alfredo Hubard proposed the following problem: Given K and L smooth convex bodies, with the property that $bd(K)$ and $bd(L)$ intersect transversally and assume you know $bd(K) \cap bd(L)$ what can you say about $\tau(K) \cap \tau(L)$? Where $\tau(K)$ is the space of tangent hyerplanes to K.

In 2001, Macdonald, Pach, and Theobald [31] proved that four spheres in $\mathbb{R}^3$ in general position have 12 common complex tangent lines, four unit spheres centered at the vertices of a regular tetrahedron with edge length e satisfying $\sqrt{3} < e < 2$ will have exactly 12 common real tangent lines. And Megyesi considered when the four spheres have coplanar centers [33]. That four unit spheres can have at most 8 common real tangents.

It is not hard to find four unequal spheres with coplanar centers having 12 common tangents. Three spheres of radius 4/5 centered at the vertices of an equilateral triangle with side length $\sqrt{3}$ and one of radius 1/3 at the triangles center have 12 common real tangents.

At the workshop, F. Sottile stated a very interesting set of problems concerning tangent lines to four spheres with exactly 12 common real tangents. Let $\mathcal{C}$ be the set of configurations of four spheres with 12 common real tangents.

Problem 1. Determine the topology of the configuration space $\mathcal{C}$. Is $\mathcal{C}$ connected? Is it possible to continuously transform the tetrahedral configuration into the one with coplanar centers, staying within $\mathcal{C}$? Are there any other (essentially different) configurations of four spheres with 12 common tangents?

All known examples of unit spheres in $\mathcal{C}$ have at least one pair overlapping. Fulton asked if it were possible to find four disjoint unit spheres with 12 common tangents. Theobald and I [39] gave an example of four disjoint spheres with 12 common tangents:

Problem 2. Do there exist four disjoint unit spheres with 12 common tangents? What is the maximum number of isolated real tangent lines to four disjoint unit spheres?

Sottile believe that the answer to the first question is no, but that it would be extremely hard to show that. There are also examples of four disjoint unit spheres with 8 common isolated real tangent lines. For more on this problem of line transversals to spheres, see the survey [39]

Finite sets of points and finite sets of convex sets

The Erdös-Szekeres theorem states that every sufficiently large set of points in general position in the plane contains a large subset which is convexly independent. During this workshop there were a bast number of talks focusing in generalizations of Erdös- Szekeres Theorem, for instance, there are several results and conjectures on possible extensions to pseudo-line arrangements or convex sets, and at this respect, Andreas Holmsen presented his joint work with Michael Dobbins and Alfredo Hubard, about a several generalizations of the Erdos-Szekeres theorem, and presented a unified viewpoint and report of their progress on some of these questions.

Alfredo Hubard, spoke about the topology and geometry of the realization spaces by families of convex bodies. He say that two families of convex bodies have the same combinatorial type if there is a selfhomeomorphism of the cylinder $S^{d-1}xR$ that maps the graphs of the support functions of one family to the the graphs of the support functions of the other one. He metrize the space of families of convex bodies with the Hausdorff metric. This talk was about results on the topology and geometry of all families with a fixed combinatorial type.

In fact A. Holmsen proposed the following topological problem: Consider 5 pairs of points in $\mathbf{R}^4$ and let S denote the union of the 32 distinct 4-dimensional simplices obtained by choosing a point from each pair and taking their convex hull. It is known that S is *simply connected*. Show that S is *contractible*.

Furthermore, Xavier Goaoc stated the following problem of finite sets of points

Two ordered n-point sets in the plane are χ-equivalent if for any $1 \leq i, j, k \leq n$, the orientations of the triples points with indices i, j and k are the same in both sets. A chirotope of size n is an equivalence class for that equivalence relation. What is the probability that in a chirotope chosen uniformly at random, the first four points are in convex position?

The following problem on squares (T.Rado 1928) was stated by H. Tverberg. T.Rado asked for the best constant c such that given a finite set of closed axis-parallel squares in the plane,one can find a subset ,consisting of disjoint squares, such that its area is at least c times the original area. He conjectured that $c = 1/4$. It is known that 1/4 works (and is best possible) if the squares are congruent,but M.Ajtai showed by an example (1973) that 1/4 does not work in general. L.Mirsky asked about the special case when the squares have sidelenghts 1 and 2. In that case a fairly simple argument shows that one may reduce the problem to the case when each small square is a square on a (generalized) chessboard while each large one is formed by 2 white and 2 black squares on the board. Does c=1/4 work then? A good set of references is found in a paper by S.Bereg et al. in Algorithmica 57 (2010),538-561.

Edgardo Roldan stated a problem on Partitions related with the Yao-Yao Theorem. Consider the smallest number $N(d, k)$ such that the following holds: For any “nice" measure in $\mathbf{R}^d$ there is a partition of $\mathbf{R}^d$ into $N(d, k)$ convex pieces of equal μ-measure such that every hyperplane avoids at least k of these pieces.

In [41], A. C. Yao and F. F. E. Yao showed that $N(d, 1) \leq 2^d$. This is known as the Yao-Yao Theorem.

B. Bukh asked if $N(d, 1) = O(d)$. A construction was given in [38] that implies $N(d, 1) \geq C2^{d/2}$ for some fixed constant C, however there is still no better upper bound.

Another question is what happens when $k > 1$. One can split $\mathbb{R}^d$ into two pieces of equal measure and construct a Yao-Yao partition in each, this gives a total of 2^{d+1} pieces. Since every hyperplane avoids 2 of them, then $N(d, 2) \leq 2^{d+1}$. Another bound is obtained by iterating the Yao-Yao partition method, after m steps we obtain 2^{md} pieces and every hyperplane avoids $2^{md} - (2^d - 1)^m$. This gives $N(d, 2^{md} - (2^d - 1)^m) \leq 2^{md}$.

These bounds on $N(d, k)$ are rather rough. In [38] it is shown that $N(d, 2) \leq 3 \cdot 2^{d-1}$, but the method used fails for $k > 2$. It would be interesting to find better bounds for $N(d, k)$ than those obtained from simple iterations of this kind.

Remark. The polynomial ham sandwich theorem gives another way to partition a measure in $\mathbb{R}^d$ (see [27] for example). The number of pieces a hyperplane intersects is well controlled but the convexity of the pieces is lost.

Conclusions

The workshop was successful in many ways, bringing together old and new colleagues from all over the world. We had participants from many countries including Russia, Germany, France, USA, Mexico, Korea, Canada, Hungary, and Denmark, among others. The talks were far from being the only academic activity of the workshop. We had many formal and informal mathematical discussions and all these activities have given rise to many new research projects and new collaboration.

We appreciate and would like to thank the support we have received from BIRS. The excellent facilities and environment that it provides are perfect for creative interaction and the exchange of ideas, knowledge, and methods within the Mathematical Sciences. We would like to thank programme coordinator Wynne Fong and Station Manager Brenda Williams for all their support in the organization of the conference. We would like to thank as well all the participants of the Recent Advance in Transversal and Helly-type Theorems in Geometry, Combinatorics and Topology Workshop for all their enthusiasm and the productive, enjoyable environment that was created.

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Bibliography

[1] N. Alon, P. Frankl and L. Lovász, The chromatic number of Kneser hypergraphs, *Trans. Amer. Math. Soc.*, **298 (1)** (1986), 359–370.

[2] J. L. Arocha, J. Bracho and L. Montejano, Transversal lines to lines and intervals, *Lecture Notes in Pure and Applied Mathematics Series*, **Vol. 253** (2003), p 1–17.

[3] I. Bárány, A generalization of Carathéodory's theorem, *Discrete Math.*, **40** (1982), 141–152.

[4] I. Bárány and S. Onn, Carathéodory's Theorem, colorful and applicable, *In Intuitive Geometry*, Budapest, (1995) **Vol. 6** of Bolyai Soc. Math. Stud., Janos Bolyai Math. Soc., Budapest, (1997), 11–21.

[5] J. Bracho, L. Montejano, D. Oliveros., The topology of the space of transversals through the space of configurations, *Topology and its Applications*, **120(1–2)** ,(2002), 92–103.

[6] F. Breuer. Uneven splitting of ham sandwiches. *Discrete and Computational Geometry*, 2009.

[7] S.A. Bogatyi, Topological Helly theorem, *Fundam. Prikl. Mat.*,**8(2)** , (2002), 365–405.

[8] S. S. Chern, On the multiplication in the characteristic ring of a sphere bundle, *Annals of Math*. **49** (1948), 362–372.

[9] L. Danzer, B. Grunbaum and V. Klee, Helly's Theorem and Its Relatives, *Convexity, Proc. Symp. Pure Math.* **Vol. 7**, A.M.S. (1963) 101–180.

[10] L. Danzer, Über ein Problem aus der Kombinatorischen Geometrie. *Arch. Math. (Basel)* **8** (1957), 347–351.

[11] L. Danzer and B. Grünbaum. *Intersection properties of boxes in* $\mathbb{R}^d$. Combinatorica, 2:237–246, 1982.

[12] H.E. Debrunner, Helly type theorems derived from basic singular homology, *The American Mathematical Montly*, **17(4)** (1970), 375–380.

[13] V.L. Dol'nikov, Transversals of families of sets in R^n and a connection between the Helly and Borsuk theorems, *Russian Acad. Sci. Sb. Math.* **79(1)** (1994), 93–107,

[14] Jürgen Eckhoff, Discrete and Computational Geometry, The Goodman-Pollack Festschrift, *Israel J. Math.* **3** ,(1965), 187–198; 2) , *Algorithms and Combinatorics* , Springer-Verlag, Berlin Heidelberg New York, **Vol. 25** (2003), 347–377

[15] J. Eckhoff, Helly, Radon and Carathéodory type theorems, *in Handbook of Convex Geometry* (P.M. Gruber and J.M. Wills Eds.), North-Holland, Amsterdam, (1993).

[16] Eremenko, A. and Gabrielov, A., Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry. *Ann. of Math.* **(2), 155** (2002), 105–129.

[17] Eremenko, A. and Gabrielov, A. Elementary proof of the B. and M. Shapiro conjecture for rational functions. *Notions of Positivity and the Geometry of Polynomials*, Springer Base (2011) 167–178.

[18] García-Puente, Luis and Hein, Nickolas and Hillar, Chris and Mart in del Campo, Abraham and Ruffo, James and Sottile, Frank and Teitler, Zachariah, The Secant Conjecture in the real Schubert Calculus. *Exper. Math.*,**21**, (2012), 252–265.

[19] L. Habert and M. Pacchiola, Arrangements of double pseudolines. Proccedings of the 25th AnnṠympos. Comput. Geom. (2009), 314–323. Aahrus, Denmark.

[20] H. Hadwiger, Über Eibereiche mit gemeinsamer Treffgeraden. *Portugal Math.* **16** (1957), 23–29.

[21] H. Hadwiger, H. Debrunner and V. Klee, *Combinatorial Geometry in the Plane*, Holt, Rinehart and Winston, New York (1964).

[22] E. Helly, Über Mengent konvex Kóper mit gemeinschaftlichen Punkten. *Monatsh. Math.* **37** (1930), 281–302.

[23] A. Holmsen. Recent progress on line transversals to families of translated ovals. *Contemporary Mathematics*, **Vol. 453**, American Mathematical Society (2008), 283–298.

[24] [J. E. Goodman, R. Pollack, and R. Wenger. Geometric transversal theory. In J. Pach, editor, *New Trends in Discrete and Computational Geometry,* **Vol. 10** of Algorithms Combin., Springer-Verlag, Berlin, (1995), 163–198.

[25] J.E. Goodman and R. Pollack, Hadwiger's transversal theorem in higher dimensions, *Journal of the Amer. Math. Soc*. **1** (1988), 301–309.

[26] G.Kalai and R. Meshulam, Leray numbers of projections and a topological Helly type theorem, *Journal of Topology* **1**(2008), 551-556

[27] H. Kaplan, J. MatouË‡sek, M. Sharir. Simple proofs of classical theorems in discrete geometry via the Guthâ "Katz polynomial partitioning technique. *D*iscrete & Computational Geometry, **48(3)**:499â 517, 2012.

[28] M. Katchalski and T. Lewis, Cutting families of convex sets, *Proc. Amer. Math. Soc.* **79** (1980), 457–461.

[29] M. Kneser, Aufgabe 300, *Jahresbericht der Deutschen Mathematiker-Vereinigung, 2. Abeteilung* **58** (1955), 27.

[30] L. Lovász, Kneser's conjecture, chromatic number and homotopy *J. Combinatorial Theory Ser. A* **25** (1978), 319–324.

[31] Macdonald, I. G. and Pach, J. and Theobald, T., Common tangents to four unit balls in $\{R$ (94)3, *Discrete Comput. Geom.*,**26**, (2001), 1–17.

[32] J. Matousek, Using the Borsuk-Ulam Theorem. Lectures on topological methods in combinatorics and geometry. Written in cooperation with Anders Björner and Günter M. Ziegler. Universitext. Springer-Verlag, Berlin, (2003).

[33] Megyesi, G., Lines tangent to four unit spheres with coplanar centres. **Discrete Comput. Geom**. **26**, (2001), 493–497.

[34] Montejano L., Oliveros D., Tolerance for Helly type theorems *Discrete and Computational Geometry.* **Vol. 45**, No. 2, (2011). 348–357.

[35] Mukhin, E. and Tarasov, V. and Varchenko, The B.and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz *Ann. of math.* **(2) 1970** (2009), 863–888.

[36] M. Novick Discrete and Computational Geometry **47**(2012),378–392.

[37] L. S. Pontryagin, Characteristic cycles on differential manifolds, *Transactions of the Amer. Math. Soc.* ,**No. 32** (1950), 149–218.

[38] E. Roldán-Pensado and P. Soberón. An extension of a theorem by Yao & Yao. *Arxiv preprint arXiv:1112.5737*, (2011).

[39] Sottile, Frank and THeobald, Thorsten, Surveys on Discrete and Computational Geometry - Twenty Years Later. Line problems in nonlinear computational geometry, *AMS Contemporary Mathematics* **453**.,(2008).

[40] H. Tverberg, Proof of Grünbaum's conjecture on common transversals for translates, *Discrete Comput. Geom.* **4** (1989), 191–203.

[41] A. C. Yao and F. F. E. Yao. A general approach to d-dimensional geometric queries. *Proceedings of the seventeenth annual ACM symposium on Theory of computing*, ACM, (1985), 163–168.

[42] R. Wenger, Helly-type Theorems and Geometric Transversal,*Handbook of Discrete and Computational Geometry.* Second Edition, Edited by J. E. Goodman., J. O'Rourke. Discrete Mathematics and its Applications (Boca Raton) Chapman and Hall/CRC, Boca Raton, RFL, (2004).lll

[43] R. T. Živaljevic, In pursuit of colored carathéodory, Bárány theorems, *Publications del L'Institut Math ématique* **57(71)** (1995), 91–100.

Chapter 36

New Trends in Noncommutative Algebra and Algebraic Geometry (12w5049)

October 28 - November 2, 2012

Organizer(s): Michael Artin (Massachusetts Institute of Technology), Jason Bell (Simon Fraser University), Colin Ingalls (University of New Brunswick), Lance Small (University of California, San Diego), James Zhang (University of Washington)

Overview of the Field

Noncommutative algebra is a rich field that has influences rooted in geometry, physics, algebraic combinatorics, and representation theory. Today, the field has many overlapping facets that spill over into other disciplines, but are nevertheless all connected by the common use of algebraic methods in their study. We briefly give an overview of some of the main trends that shape the field today, putting particular emphasis on those trends that were most represented during the workshop.

1. **Quantum Groups and Quantum Cluster Algebras**

   Quantum groups are a class of Hopf algebras that can be regarded, in a natural sense, as noncommutative deformations of classical objects (enveloping algebras of finite-dimensional Lie algebras, coordinate rings of affine algebraic groups, etc.). Recently, there has been a lot of work on Quantum cluster algebras, which were introduced by Berenstein and Zelevinsky [4]. In the non-quantum setting there has been much work showing that many classical objects are in fact cluster algebras and a common theme during the meeting was an investigation into the extent to which quantum analogues of these results hold (see for example, the papers [19, 23, 37]). In addition to this, there are many other directions of study in this area. As an example of some of the other areas of research in quantum groups, we note that these objects often support a rational action by an affine algebraic group, and this action often stratifies the prime spectrum into finitely many disjoint pieces, each of which is homeomorphic to an affine scheme of finite type. This then gives rise to many combinatorial questions: namely, how many strata are there for a given quantum algebra admitting a finite stratification and what are the dimensions of the associated affine schemes. There has been a lot of recent work inspired by these problems [6, 7, 10, 34, 35].

2. **Cherednik algebras**

   Cherednik algebras are, like quantum groups, noncommutative deformations of more classical objects—in this case, skew group rings of polynomial rings by a finite group. In fact, much of the machinery developed in the study of quantum groups applies in this situation. The development of this facet of noncommutative algebra has been very rapid since being introduced by Etingof and Ginzburg [18] in 2002. Many conjectures in algebraic combinatorics have in fact been answered using the Cherednik algebra machinery. See, for example, Gordon [20] to answer a conjecture of Haiman, by Berest, Etingof and Ginzburg [9] to answer questions about rings of quasi-invariants that arise in integrable systems. The immense recent interest in this area and rapid development has led to a large number of open problems in this area.

3. **Noncommutative surfaces** Since curves, having one parameter, are commutative, (or at least finitely generated modules over their centres), as exhibited in [2], [3], it is natural to study noncommutative surfaces. In commutative algebraic geometry there is a classification of surfaces with the most important Theorems being Castelnouvo's and Enriques' which characterize rational and ruled surfaces respectively. Artin's conjecture predicts that noncommutative surfaces are either rational, ruled or module finite over their centres. The study of noncommutative rational surfaces produced seminal works in this area [4], [5], and new techniques and objects such as twisted homogeneous coordinate rings. Chan and Nyman have shown a version of Enriques' Theorem in the noncommutative setting [10].

4. **Hopf Algebra Actions**

   In the noncommutative world, Hopf algebra actions on Artin-Schelter regular algebras forms a natural setting. The classification of finite quantum subgroups of quantum SL_2, namely, so-called quantum binary polyhedral groups, has been worked out recently following the classification of Artin-Schelter regular algebras of dimension 2 [17]. This leads to the study of Kleinian or DuVal singularities of Hopf actions on noncommutative surfaces. Invariant theory of finite group actions (as well as finite dimensional Hopf algebra actions) on Artin-Schelter regular algebras of higher global dimension has been developed, see [28, 29, 30], which includes noncommutative versions of several classical results such as Watanabe theorem (a criterion for the fixed subrings being Gorenstein), Shephard-Todd-Chevalley theorem (a criterion for the fixed subring being regular) and Kac-Watanabe theorerm (a criterion for the fixed subrings being a complete intersection). These studies are closely connected to the representation theory of finite group/Hopf algebras, homological aspects of noncommutative algebra and noncommutative algebraic geometry.

5. **Calabi-Yau algebras and skew (or twisted) Calabi-Yau algebras**

   Calabi-Yau algebras were introduced by Ginzburg in 2006 as a noncommutative version of coordinate rings of Calabi-Yau varieties, and Calabi-Yau triangulated categories were introduced by Kontsevich in 1998. Since then, the study of Calabi-Yau algebras/categories has been related to a large number of research areas such as quivers with superpotentials, differential graded algebras, cluster algebras/categories, string theory and conformal field theory, noncommutative crepant resolutions, and so on. In addition to work of Kontsevich and Ginzburg, some foundations of the topic have been established by Keller and Van den Bergh recently. Skew (or twisted) Calabi-Yau algebras are generalization and a companion of Calabi-Yau algebras. For example, all Artin-Schelter regular algebras are skew Calabi-Yau and only a subclass of Artin-Schelter regular algebras are Calabi-Yau. On the other hand, Calabi-Yau algebra can be produced by using skew Calab-Yau algebras. One invariant attached to a skew Calabi-Yau algebra is its Nakayama automorphism. Several identities about the Nakayama automorphism have been proved. These identities helps to understand the questions such as when a smash product of Calabi-Yau algebra with Hopf algebra is Calabi-Yau. Associated to Calabi-Yau algebras there are various combinatroial data (such as super-potential) which might be calculated by computer algorithms.

Recent Developments and Open Problems

We summarize some of the main open problems that exist in the area today. Again, we place emphasis on the problems that were viewed as most important by speakers at the conference and we do not claim to give a complete list of all of the most important open problems in the area.

1. *Artin's proposed birational classification of noncommutative projective surfaces*

Artin [1] conjectured that every division algebra of transcendence degree two that comes from a noncommutative projective surface is either finite-dimensional over its center or is isomorphic to a Skylanin division ring or to a division ring of the form $K(x;\sigma,\delta)$, where K is the function field of a smooth projective curve, σ is an automorphism of K, and δ is a σ-derivation of K. By looking at rank one discrete valuations of these division algebras, it can be shown that they are non-isomorphic. This conjecture is very natural after the landmark work of Artin and Stafford [2, 3] that classified all noncommutative projective curves (or noncommutative graded domains of Gelfand-Kirillov dimension two). It is generally believed that the conjecture is currently beyond available methods, but there is a large amount of work devoted to understanding this conjecture. As some examples, we note the work of Chan, Hacking, Ingalls, Kulkarni [11, 12, 13], the manuscript by Artin and de Jong, and the series of papers by Keeler, Rogalski, Stafford and Sierra [27, 33], some of which look at classifying projective surfaces in a given birational class on Artin's list.

2. *Understanding which quantum groups are quantum cluster algebras*

   The notion of quantum cluster algebras was introduced by Berenstein and Zelevinsky [4]. These algebras are noncommutative deformations of cluster algebras and over the past few years there has been a large amount of work devoted to understanding their basic properties. In the classical setting, there is now a wealth of examples of classical objects that are cluster algebras. In the quantum setting, however, there are many questions which ask if quantum analogues of these classical results hold. As a few of the more important examples, we note that Berenstein and Zelevinsky conjectured that the quantized coordinate ring of a double Bruhat cell is in fact a quantum cluster algebra, which was proved by Berenstein, Fomin, and Zelevinsky in the classical setting. Another interesting case is the quantum Grassmannian, which has been shown to be a quantum cluster algebra in only a few cases [23].

3. *Problems on Hopf actions*

   One influential program is the quantum McKay correspondence. There are several proposals in the literature, but there is not one that is currently sufficiently broad to include many noncommutative algebras/schemes. This program involves several pieces: (a) Hopf algebra actions on noncommutative algebras/schemes, (b) quotient singularities of noncommutative schemes, (c) representation theory of Hopf algebras and fixed subrings, (d) homological algebra aspects (such that various derived categories are related), (e) combinatorial information coming from algebra, geometry and representation theory, and the most important of all, (f) the connection between these pieces. There is some work in dimension 2 which had been mentioned during the workshop. As the classification of Artin-Schelter regular algebras of dimension 3 is now known, thanks to the work of Artin, Schelter, Tate and Van den Bergh [5], this question could be tractable in dimension 3, and would lead to significant new mathematics.

Presentation Highlights

Many participants commented that the level of quality of the lectures at this conference was very high and there were many great presentations during the course of the week.

Many of the talks dealt with the burgeoning area of Hopf algebra actions on algebras. There is a rich history of studying group actions on algebras and the notion of Hopf algebra actions is a natural extension. Daniel Chan, Kenneth Chan, Chelsea Walton and others discussed new developments in this area and gave many open problems that have arisen during the course of their investigations. In particular, Daniel Chan's talk considered the action of Hopf algebras on projective varieties and on twisted homogeneous coordinate rings of projective varieties. Their work also involved results with Ellen Kirkman, Yanhua Wang, and James Zhang [15, 16, 17].

Ellen Kirkman and Ken Brown gave closely related talks to those of Daniel Chan, Kenneth Chan, and Chelsea Walton. Ken Brown looked at the problem of determining the Hopf algebras over a field of characteristic zero that can be viewed as deformations of the commutative polynomial algebra in n variables over k.

This is a problem that generated some interest at the conference and Ken Brown initiated related work with other participants on this problem while at the meeting.

Ellen Kirkman spoke on a closely related topic of the ring of invariants A of a graded ring R under the action of a finite group G of graded automorphisms. Such algebras have been extensively studied in the commutative case and in particular one has a beautiful characterization, due to Gulliksen [24] of when such algebras are complete intersections. This holds when any of the following equivalent conditions holds:

(a) the Ext-algebra $E(A) := \bigoplus_{n=0}^{\infty} \mathrm{Ext}_A^n(k,k)$ of A has finite GK-dimension;

(b) the Ext-algebra $E(A)$ is noetherian.

This naturally leads one to the notion of a noncommutative complete intersection. Kirkman described her joint work with James Kuzmanovich and James Zhang [30] on the relationship between these two notions in the noncommutative case—specifically when the algebra A is of the form R^G for some Artin-Schelter regular algebra R and some finite group G.

Birge Huisgen-Zimmerman gave an excellent survey on the so-called Finitistic Dimension Conjectures for finite-dimensional algebras, which are usually attributed to Bass. Although the conjectures themselves are roughly fifty years old, a large amount of progress has been made in the past ten years. In particular, the lecture examined the representation theory and homology of biserial algebras, a class of algebras that encode the representation theory of the Lorentz group.

Some of the presentations dealt with quantum groups and the connections between noncommutative algebra and algebraic combinatorics and geometry.

Ken Goodearl described his recent work with Milen Yakimov [19]. This work extends many classical results on unique factorization in coordinate rings. Due to the pioneering work of Fomin and Zelevinsky [25, 26], there has been considerable interest in the study of cluster algebras. In particular, many coordinate rings of affine varieties have been shown to be cluster algebras and, moreover, many of these families of cluster algebras have also been shown to be UFDs. The noncommutative analogue of being a unique factorization domain involves a condition on height one prime ideals being principal and generated by a normal element—this definition is equivalent to being a UFD in the commutative case. The notion of a quantum cluster algebra is a relatively recent concept and not as much work has been done regarding whether certain quantized coordinate rings are quantum cluster algebras and whether they are noncommutative UFDs. Ken Goodearl's presentation discussed how for a large class of quantum algebras one has that they are both quantum cluster algebras and noncommutative UFDs.

Stéphane Launois discussed minimal conditions on positivity of minors in a matrix needed to verify that it is either totally positive or totally nonnegative [31]. This is a problem of some interest within the combinatorial community. Launois showed how one could give algorithms inspired by techniques from the study of quantum groups. In particular, Cauchon's deleting derivations algorithm can be used to give an algorithm that requires $\mathrm{O}(n^4)$ operations to determine if a matrix is a totally nonnegative matrix in a given totally nonnegative cell. (The complexity of the "naïve" algorithm grows exponentially with n.)

Some of the lectures gave overviews of recent solutions and counter-examples to previously open problems from the literature. David Saltman spoke on joint work with Louis Rowen [32], in which they give a surprising counter-example to the question of whether $D_1 \otimes_F D_2$ is necessarily a domain when D_1 and D_2 are division algebras and F is an algebraically closed field that is contained in the centres of both D_1 and D_2. It is well-known that when D_1 or D_2 is a field then the tensor product does not have zero divisors, but the general problem had been open for many years. By using a combination of geometric techniques and results from the theory of Brauer groups, Saltman was able to produce explicit examples of tensor products that meet the conditions described above and which fail to be domains.

Milen Yakimov's presentation gave an overview of his recent proof of two conjectures in the field of quantum groups [36]. Namely, he gave proofs of both the Andruskiewitsch-Dumas and Launois-Lenagan conjectures, which give a conjectural classification of the automorphism groups of certain families of noncommutative associative algebras. It should be pointed out that the method of his proof in fact applies to a very large class of noncommutative algebras that contains the families examined by both Andruskiewitsch-Dumas and Launois-Lenagan. In particular, he gave a general rigidity theorem that can be applied in many situations to find the automorphism group of an associative algebra.

Toby Stafford spoke on joint work with Iain Gordon [21, 22] on equidimensionality of charactersitic varieties over Cherednik algebras and $\mathbb{Z}$-algebras. Gordon and Stafford have developed a great deal of machinery during the course of their study of Cherednik algebras and it has resulted in many successes, including Gordon's proof of Haiman's deep n!-Theorem. Stafford's lecture showed that as an application of the homological machinery he developed with Gordon, one can generalize work of Gabber's on unitary algebras, thus providing an answer to another open problem.

There was also some work that dealt with topics that are of some interest in physics, in particular in string theory: Calabi-Yau algebras and Mirror symmetry. In particular, Raf Bocklandt discussed connections between noncommutative projective geometry and Artin-Schelter regular algebras and mirror symmetry for Riemann surfaces. These connections have not been fully explored at this point, but they will undoubtedly lead to new results in noncommutative geometry.

Daniel Rogalski spoke on twisted Calabi-Yau algebras—his talk concerned forthcoming joint work with James Zhang and Manuel Reyes. These are algebras with some especially nice homological properties. In the connected graded setting, they are essentially just the Artin-Schelter regular algebras, which have a dualizing complex equal to a complex shift of the algebra, possibly twisted on one side by an automorphism (the Nakayama automorphism). He discussed some nontrivial formulas for what happens to the Nakayama automorphism when one takes a smash product with a Hopf algebra action or does a graded twist. Our methods rely on graded local cohomology and the formulas involve the notion of the homological determinant of an automorphism. We also note that in wide generality, the homological determinant of the Nakayama automorphism is equal to 1.

Quanshui Wu also spoke on Calabi-Yau algebras and focused on the problem of how to compute the Hochschild and cyclic cohomology for three-dimensional graded Calabi-Yau algebras via the Poisson cohomology for Poisson algebras.

There were some talks related to noncommutative projective geometry and to representations of Clifford algebras. Representations of a Clifford algebras of homogeneous forms correspond to the class of Ulrich bundles on the associated hypersurface. Rajesh Kulkarni spoke about how the connection can be used to study the vector bundles on cubic and quartic surfaces, and in particular can be used to show that a smooth quartic surface is a linear Pfaffian. He also described how one can construct stable Ulrich bundles on a class of three-folds by using the so-called twisted tensor product construction in Clifford algebras.

Susan Sierra spoke on forthcoming joint work with Daniel Rogalski and Toby Stafford on maximal orders in the Sklyanin algebra. Her talk dealt with understanding the maximal orders in the three-Veronese subalgebra of a generic Skylanin algebra, and showed that each subalgebra could be interpreted, in a natural way, as a blow-up of the ambient algebra at a divisor on an associated elliptic curve E. As a result, they are able to classify all subalgebras in the three-Veronese subalgebra of a generic Sklyanin algebra that are maximal orders. This provides an complete study of noncommutative Del Pezzo surfaces and represents a large first step in trying to understand all connected graded algebras of GK-dimension three that are birationally isomorphic to a Sklyanin algebra.

Finally, Louis Rowen gave a survey of the polynomial identities of algebras with involution, with a particular focus on Specht's problem.

Scientific Progress Made

As a result of this meeting, many people looking at different facets of noncommutative algebra were able to have discussions and start projects with people working on related areas. In fact, there are many examples of projects that were begun at this meeting (we give a few testimonials to this effect in the following section). We note that this meeting had many younger researchers, postdocs, and graduate students and this meeting was particularly useful in allowing them to network and create joint projects with more senior researchers. One of the often repeated remarks made by different participants was an interest in Hopf algebra actions—some of these participants had not previously been introduced to this thriving area and many expressed interest in working on some of the problems posed by younger researchers during the meeting.

Outcome of the Meeting

This workshop brought together 38 researchers, which included a mix of Ph.D. students, several postdocs, and researchers from Europe, Australia, China, and North and South America, working in different areas of noncommutative algebra and geometry, including ring theory, noncommutative algebraic geometry, representation theory, the study of Hopf algebra actions, and quantum groups.

Many of the participants privately remarked that this was one of the best conferences to which they had been. There were many great talks and several of the participants gave testimonials about research programs that they started at Banff—we include a few examples below.

Sarah Witherspoon said: “The workshop was great, and I am very happy that I went. The lectures were largely very good, with several on topics very close to my own research, and others further away but on topics that will benefit my research program as I learn more. I particularly liked the variety of speakers, including young mathematicians working on very current problems. The workshop also gave me a very valuable opportunity to meet with collaborators who were there: I am starting to work on a project with Chelsea Walton, and it was helpful to touch base with her in Banff. I have been discussing with Nicolas Andruskiewitsch a plan to (at least partially) prove a conjecture on cohomology of Hopf algebras, and it was important that we were able to talk in person in Banff over a period of several days.”

Ulrich Kraehmer said: “Many talks raised more questions than giving answers and equipped the audience with some ideas that the speaker had. This really is the most inspiring way to hold workshops, and together with another participant I have started thinking seriously about one of the questions that have been asked.”

Ken Brown said: “Discussions with James Zhang and his student Guangbin Zhuang following my seminar led to significant developments in our ongoing research programme on connected Hopf algebras, which will certainly greatly improve the paper I am currently preparing on this, and may well lead to James and Guangbin becoming co-authors.”

Tomasz Brzenski said: “My own research is in differential aspects of Noncommutative Geometry. This means that I am studying algebras of the kind similar to those discussed during the workshop, but from a different point of view. Most of the lectures were very useful for me, allowing me to become more familiar with both mathematical concepts and literature as well as to learn new techniques and approaches. I am convinced that I will use these while working on my current project that involves studying of quantum orbifolds obtained by actions (of groups and Hopf algebras) on quantum spaces. Definitely, the meeting in Banff was one of the most exciting meetings I attended in the past few years.”

Stéphane Launois wrote: “This meeting has been a great opportunity for me to present my latest work, and to hear about the latest developments in the field. It allowed me to meet my collaborators from America. In particular, I had the chance to discuss with Karel Casteels his recent work (posted on the arXiv the week before the workshop took place), so he could explain me the general strategy of his proof of the Goodearl-Lenagan Conjecture. This is especially appreciated as his paper is more than 50 pages long, and so having the insights of the author is crucial in order to fully understand the details of his proof. Finally, the workshop certainly allowed me identify new directions of research which I am now planning to pursue with my PhD students. Overall this has been a great workshop which will certainly influence my future research!"

Martin Lorenz wrote: "The conference was outstanding; the lectures presented a comprehensive panorama of a large portion of current noncommutative algebra. I also very much appreciated the thorough coverage that quantum invariant theory received during the conference. This will be most useful to my own work and that of my students."

Milen Yakimov said the following: "The BIRS workshop was a great opportunity for me to learn about the current developments in the area of noncommutative algebraic geometry and its relations to Hopf algebras and representation theory. It will have a great influence on my future research, both in terms of working on problems related to the talks and using methods described in lectures. The workshop was also very stimulating in relation to working on ongoing research projects with other participants."

Following this workshop, there will be a closely related half-year program at MSRI that will be held from January to May, 2013. Many participants have pointed out that the BIRS workshop has really provided a focus to their research and will have some role in shaping the research conducted at MSRI.

Participants

Andruskiewitsch, NicolĞs (Universidad Nacional de Cordoba)
Bell, Jason (Simon Fraser University)
Bocklandt, Raf (University of Amsterdam)
Brown, Ken (University of Glasgow)
Brzezinski, Tomasz (Swansea University)
Casteels, Karel (University of California, Santa Barbara)
Chan, Daniel (Univesity of New South Wales)
Chan, Kenneth (University of Washington)
Goodearl, Kenneth (University of California, Santa Barbara)
Guangbin Zhuang, Guangbin (University of Washington)
Huisgen-Zimmermann, Birge (University of California, Santa Barbara)
Ingalls, Colin (University of New Brunswick)
Kirkman, Ellen (Wake Forest University)
KrŁhmer, Ulrich (University of Glasgow)
Kulkarni, Rajesh (Michigan State University)
Launois, StŐphane (University of Kent)
Leuschke, Graham (Syracuse University)
Lorenz, Martin (Temple University)
Montgomery, Susan (University of Southern California)
Negron, Cris (University of Washington)
Rogalski, Daniel (University of California at San Diego)
Rowen, Louis (Bar Ilan University)
Saltman, David J (Center for Communications Research - Princeton)
Sierra, Susan (University of Edinburgh)
Sisodia, Gautam (University of Washington)
Small, Lance (University of California, San Diego)
Smith, Paul (University of Washington)
Stafford, J. Toby (University of Manchester)
Vancliff, Michaela (University of Texas at Arlington)
Vonessen, Nikolaus (University of Montana)
Walton, Chelsea (Temple University)
Wang, Xingting (University of Washington)
Wang, Linhong (Southeastern Louisiana University)

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Yekutieli, Amnon (Ben Gurion University)
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Bibliography

[1] M. Artin, Geometry of quantum planes, *Azumaya algebras, actions, and modules* (Bloomington, IN, 1990), 1–15, Contemp. Math., 124, AMS, Providence, RI, 1992.

[2] M. Artin and J. T. Stafford, Noncommutative graded domains with quadratic growth, *Invent. Math.,* **122** (1995) 231-276.

[3] M. Artin and J. T. Stafford, Semiprime graded algebras of dimension two, *J. Algebra* **227** (2000), no. 1, 68–123.

[4] M. Artin, J. Tate, and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, *The Grothendieck Festschrift,* Vol. I, 33–85, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990.

[5] M. Artin, J. Tate, and M. Van den Bergh, Modules over regular algebras of dimension 3, *Invent. Math.* **106** (1991), no. 2, 335–388.

[6] J. Bell, K. Casteels, S. Launois, Enumeration of torus-invariant strata with respect to dimension in the big cell of the quantum minuscule Grassmannian of type Bn. New trends in noncommutative algebra, 27–40, *Contemp. Math.,* **562**, Amer. Math. Soc., Providence, RI, 2012.

[7] J. Bell, K. Casteels, S. Launois, Enumeration of H-strata in quantum matrices with respect to dimension. *J. Combin. Theory Ser. A* **119** (2012), no. 1, 83–98.

[8] A. Berenstein and A. Zelevinsky, Quantum cluster algebras, Adv. Math. **195** (2005), no. 2, 405–455.

[9] Y. Berest, P. Etingof, V. Ginzburg, Cherednik algebras and differential operators on quasi-invariants. *Duke Math. J* **.118** (2003), no. 2, 279–337..

[10] K. Casteels, Quantum Matrices by Paths, arXiv:1210.6399.

[11] D. Chan, P. Hacking, C. Ingalls, Canonical singularities of orders over surfaces, *Proc. Lond. Math. Soc.* (3) **98** (2009), no. 1, 83–115.

[12] D. Chan and C. Ingalls, Noncommutative coordinate rings and stacks, *Proc. London Math. Soc.* (3) **88** (2004), no. 1, 63–88.

[13] D. Chan, R. Kulkarni, Numerically Calabi-Yau orders on surfaces, *J. London Math. Soc.* (2) **72** (2005), no. 3, 571–584.

[14] D. Chan, A. Nyman, Noncommutative Mori Contractions and P1 Bundles, arXiv:0904.1717.

[15] K. Chan, C. Walton, J. J. Zhang, Hopf actions and Nakayama automorphisms, arXiv:1210.6432.

[16] K. Chan, C. Walton, Y.-H. Wang, J. J. Zhang, Hopf actions on filtered regular algebras, arXiv:1211.6513.

[17] K. Chan, E. Kirkman, C. Walton, J. J. Zhang, Quantum binary polyhedral groups and their actions on quantum planes, in preparation.

[18] P. Etingof, V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. *Invent. Math.* **147** (2002), no. 2, 243–348.

[19] K. R. Goodearl, M. T. Yakimov, From quantum Ore extensions to quantum tori via noncommutative UFDs, arXiv:1208.6267.

[20] I. Gordon, On the quotient ring by diagonal invariants. *Invent. Math.* **153** (2003), no. 3, 503–518.

[21] I. Gordon, J. T. Stafford, Rational Cherednik algebras and Hilbert schemes. II. Representations and sheaves, *Duke Math. J.* **132** (2006), no. 1, 73–135.

[22] I. Gordon, J. T. Stafford, Rational Cherednik algebras and Hilbert schemes, *Adv. Math.* **198** (2005), no. 1, 222–274.

[23] J. Grabowski and S. Launois, Quantum cluster algebra structures on quantum Grassmannians and their quantum Schubert cells: the finite-type cases, *Int. Math. Res. Not.* (2011), no. 10, 2230–2262.

[24] T. Gulliksen, A homological characterization of local complete intersections. *Compositio Math.* **23** (1971), 251–255.

[25] S. Fomin and A. Zelevinsky, Cluster algebras I. Foundations, *J. Amer. Math. Soc.* **15** (2) (2002) 497–529.

[26] S. Fomin and A. Zelevinsky, Cluster algebras. II. Finite type classification, *Invent. Math.* **154** (2003), no. 1, 63–121.

[27] D. Keeler, D. Rogalski, J. T. Stafford, Naïve noncommutative blowing up. *Duke Math. J.* **126** (2005), no. 3, 491–546.

[28] E. Kirkman, J. Kuzmanovich, J.J. Zhang, Shephard-Todd-Chevalley Theorem for skew polynomial rings, *Algebr. Represent. Theory*, **13** (2010), no. 2, 127–158.

[29] E. Kirkman, J. Kuzmanovich, J.J. Zhang, Gorenstein subrings of invariants under Hopf algebra actions, *J. Algebra* **322** (2009), no. 10, 3640ÍC3669.

[30] E. Kirkman, J. Kuzmanovich, J.J. Zhang, Noncommutative complete intersections, in preparation.

[31] S. Launois, T. H. Lenagan, Efficient recognition of totally nonnegative matrix cells, arXiv:1207.3613.

[32] L. Rowen, D. Saltman, Tensor Products of Division Algebras and Fields, arXiv:1106.5517.

[33] S. Sierra, Classifying birationally commutative projective surfaces, *Proc. Lond. Math. Soc.* (3) **103** (2011), no. 1, 139–196.

[34] M. Yakimov, On the spectra of quantum groups, arXiv:1106.3821.

[35] M. Yakimov, Strata of prime ideals of De Concini-Kac-Procesi algebras and Poisson geometry, arXiv:1103.3451.

[36] M. Yakimov, The Launois-Lenagan conjecture, arXiv:1204.5440.

[37] A. Zelevinsky, Cluster algebras: origins, results and conjectures, *Advances in algebra towards millennium problems*, 85–105, SAS Int. Publ., Delhi, 2005.

Chapter 37

Spectral Analysis, Stability and Bifurcation in Modern Nonlinear Physical Systems (12w5073)

November 4 - 9, 2012

Organizer(s): Paul Binding (University of Calgary), Thomas Bridges (University of Surrey), Yasuhide Fukumoto (Kyushu University), Igor Hoveijn (University of Groningen), Oleg Kirillov (Helmholtz-Zentrum Dresden-Rossendorf), Dmitry Pelinovsky (McMaster University)

Overview of the Field

Linearised stability analysis of stationary and periodic solutions of both finite and infinite dimensional dynamical systems is a central issue in many (physical) applications. Such systems usually depend on parameters, so an important question is what happens to stability when the parameters are varied. This implies that one has to study the spectrum of a linear operator and its dependence on parameters. Moreover, systems arising in physics and other applications often possess special structure, for example Hamiltonian systems. Therefore spectrum and Jordan structure no longer suffice to characterize equivalent systems (under smooth coordinate transformations) but additional invariants are needed. Identifying and interpreting these in infinite dimensional systems seems more involved than in finite dimensional situations. For example, one may consider the symplectic or Krein signature for imaginary eigenvalues in linear finite dimensional Hamiltonian systems. When such eigenvalues meet as the parameters vary, the existence of additional invariants causes non-generic behaviour. In particular, a collision of eigenvalues on the imaginary axis may have dynamical consequences since the stability may change, depending on the additional invariants. At such a collision the boundary of the so called stability domain in parameter space may have singularities. This phenomenon occurs in numerous applications and it may have various physical consequences and interpretations. On the other hand stability questions can also be studied by index theory (Morse index, Maslov index). These approaches are not unrelated; for example the symplectic or Krein signature is connected to the Morse index.

Recent Developments and Open Problems

Already established (e.g., by Floquet, Lyapunov and Poincare by the end of the XIXth century) and greatly benefitting from the mathematical achievements of the XXth century, classical stability analysis of stationary and periodic solutions of finite- and infinite-dimensional dynamical systems has experienced a rebirth during the last two decades. One of the reasons for this new active phase is the significant progress in nonlinear PDEs where solitary waves were discovered for a wide class of equations related to modern physical applications. Another motivating source is the remarkable success of the methods of geometrical optics in the analysis of three-dimensional destabilization of two-dimensional flows of ideal and viscous fluid. This provides a universal mechanism whereby complex three-dimensional motion can arise directly from large-scale two-dimensional coherent structures, which is important for the theory of turbulence. The geometrical optics stability analysis was extended by Eckhoff to general systems of symmetric hyperbolic PDEs including the case of multiple roots of the dispersion relation that determines bi-characteristics along which a localized perturbation evolves according to the amplitude equation that serves for stability analysis. The third research area that stimulates development of such stability theory nowadays is dissipation-induced instabilities and their relation to non-Hermitian degeneracies of the spectrum. Between these three subject areas there exist numerous connections that we shall discuss briefly.

Geometrical optics stability analysis is an example of separation of fast and slow variables in the adiabatic approximation. The WKB solution may contain an additional term that expresses the cumulative change of the complex phase as the wave propagates along a ray (geometric phase). There exists a link between geometric phase and strong stability of Hamiltonian systems.

On the other hand multiple roots of the dispersion relation are connected to non-trivial physical effects such as conical refraction discovered by Hamilton in crystal optics and found in hydromagnetics by Ludwig in 1960s. The presence of a double semi-simple eigenvalue determines a singularity of the dispersion surface - Hamilton's diabolical point (DP) - that yields a conical ray surface, which is observable in experiments with birefringent crystals. In the presence of absorption and optical activity the conical singularities of the dispersion surface can transform into branch points that correspond to double eigenvalues with the Jordan block (exceptional points, EPs). This happens because the matrix determining the dispersion relation becomes a non-Hermitian one, for which an EP has a lower codimension than for DP.

Dispersion surfaces with the same Hermitian and non-Hermitian singularities are characteristic of travelling waves in rotating fluids and structures with frictional contact such as a rotating glass bowl of a glass harmonica that is touched by the fingers of a musician. The glass harmonica is a gyroscopic system perturbed by dissipative and non-conservative positional forces acting at the contact. Its industrial counterpart is rotating machinery like brake disks, paper calendars, and turbine shafts. The vibrations of a glass harmonica provide an audible example of dissipation-induced instability in a Hamiltonian system.

The effect of dissipation on Hamiltonian systems can be visualized by its action on pure imaginary eigenvalues. Landahl discovered that in an incompressible laminar boundary layer over a flexible surface the wall damping destabilizes waves of the Tollmien-Schlichting type that have a negative energy sign. MacKay studied movement of eigenvalues of Hamiltonian equilibria under non-Hamiltonian perturbation and found that the way in which simple pure imaginary eigenvalues of an equilibrium of a Hamiltonian system migrate under non-Hamiltonian perturbation is related to the energy and dissipation rate for the corresponding modes. Maddocks and Overton initiated the study of multiple pure imaginary eigenvalues and showed that for an appropriate class of dissipatively perturbed Hamiltonian systems, the number of unstable modes of the dynamics linearized at a nondegenerate equilibrium is determined solely by the index of the equilibrium regarded as a critical point of the Hamiltonian.

In the general case of non-Hamiltonian vector fields, the occurrence of double imaginary eigenvalues (1:1 resonant Hopf-Hopf bifurcation) has codimension three, whereas the codimension is one for Hamiltonian-Hopf bifurcation. Langford had shown that the interface between these two cases possesses the Whitney umbrella singularity in the parameter space; the Hamiltonian systems lie on its handle (for periodic systems this has been shown by Hoveijn and Ruijgrok). The Whitney umbrella singularity on the stability boundary of a near-Hamiltonian system corresponds to a double pure imaginary eigenvalue with a Jordan block (EP). Bottema discovered that this singularity explains a discrepancy between the stability domain of an undamped system and that in the limit of vanishing dissipation - Ziegler's destabilization paradox - which is a typical phenomenon in structural and contact mechanics, atmospheric physics and fluid-structure interactions.

The role of the energy sign of the mode associated with an eigenvalue of the Hamiltonian system is visible not only in case of dissipative perturbations. The Hamiltonian-Hopf bifurcation in which two pairs of complex conjugate eigenvalues approach the imaginary axis symmetrically from the left and right, then merge in double purely imaginary eigenvalues and separate along the imaginary axis (or the reverse) is caused by the interaction of eigenvalues with the opposite sign of the energy.

The concepts of signature and positive and negative energy modes were influenced by work of Weierstrass, Rayleigh and Thompson and Tait. Precise formulations in finite dimensions were available by the 1930s and were later developed by Sobolev (1943), Pontryagin (1944), Krein (1950) and Burgoyne & Cushman (1977). In Hamiltonian systems, a sign of the energy which is associated with purely imaginary eigenvalues (in the linearization about an equilibrium) or Floquet multipliers (in the linearization about a periodic orbit) is called symplectic (or Krein) signature whereas the Hamiltonian-Hopf bifurcation is frequently called the Krein collision.

Krein defines the signature as the sign of the square norm induced by the indefinite inner product where the metric operator J is symplectic in case of Hamiltonian systems. In general, the indefinite metric is given by an indefinite bilinear form on the underlying Hilbert space, and selfadjointness is defined by analogy with the Euclidean inner product. J-selfadjoint operators were first introduced by Sobolev (1943) in connection with rotating shallow water. This line of inquiry was continued by Pontryagin (1944) in his pioneering article. Pontryagin's theorem on invariant subspaces started a new branch of Functional Analysis, dedicated to the theory of linear operators in indefinite metric space. The spectral properties and the geometry of sign-definite invariant subspaces of dissipative and contractive operators acting on indefinite metric spaces were studied by Azizov and Iohvidov (1980-1985). In many cases, linearization of a nonlinear wave equation at a spatially localized solution such as a bound state or a solitary wave results in a generalized eigenvalue problem that can be studied using the spectral theory of a J-self-adjoint operator acting in some indefinite metric space. A mean-field α^2-dynamo of magnetohydrodynamics provides another example of a J-self-adjoint operator, where the metric operator J is not symplectic.

Krein (1950-1970) took an axiomatic approach to the spectral theory of unitary and self-adjoint operators acting in Pontryagin space. In addition to establishing the connection between signature and instability (e.g., collision of modes of opposite signature is a necessary condition for complex instability), an important consequence of Krein's work was a signature for linear Hamiltonian systems with periodic coefficients, where the energy sign is time dependent. In other words, the use of energy sign is of limited use whereas the concept of signature generalizes to the case of non-constant coefficients.

Many interesting non-autonomous periodic and non-periodic ODEs arise in the geometrical optics stability analysis. For example, the amplitude equation for elliptic instability reduces to the Schrödinger equation with periodic potential. Another source is linearization of nonlinear PDEs about solitary wave solutions. Indeed, Hamiltonian evolution PDEs in one space dimension, such as the nonlinear Schrödinger equation, reaction-diffusion equation, long-wave short-wave resonance equations, the fifth-order Korteweg de Vries equation (the latter arises, e.g., in beam buckling, pattern formation, thin-film flows and in the theory of capillary-gravity water waves), have the property that their steady part is a finite dimensional Hamiltonian system. For such systems, solitary wave solutions can be characterized as homoclinic orbits of the Hamiltonian ordinary differential equation (ODE).

The spectral problem associated with the linearization about a given homoclinic orbit, in the time dependent equations, yields a parameter-dependent family of linear Hamiltonian systems where the spatial variable plays the role of time. The Hamiltonian matrix depends on the spatial variable as well as on the parameter which can be interpreted both as the control parameter and as the spectral parameter of the linearization that determines the stability of the solitary wave. Since the latter vanishes when the spatial variable tends to infinity, the limit yields an autonomous Hamiltonian system "at infinity" whose eigenvalues depend on the control parameter. Note that this is a situation of the theory of multiparameter eigenvalue problems due to Atkinson, Volkmer and Binding where eigencurves in the plane of the two spectral parameters is a natural object of investigation.

Solutions of the system at infinity form a fixed reference stable Lagrangian subspace. The signed count of its non-trivial intersections with an image of a path of unstable Lagrangian subspaces of the non-autonomous Hamiltonian system gives the Maslov index of this path which is a function of the control parameter. The system at infinity is also used in the construction of the Evans function of the control parameter whose zeros determine the discrete spectrum of the linearization. The Evans function is related to an exterior product of the paths of stable and unstable Lagrangian subspaces. With the variation of the control parameter the Maslov index jumps by one at each eigenvalue of the linearization and thus counts the number of discrete eigenvalues. This intriguing and non-obvious property is used both for calculating the Maslov index of a homoclinic orbit and for determining the number of eigenvalues in the right half of the complex plane, i.e., the instability index of the solitary wave.

The Maslov index applies to a wide range of other physical applications: semi-classical quantization, quantum chaology, classical mechanics, etc. Providing a count of eigenvalues of some self-adjoint operators, it is also used in the stability analysis of traveling waves. When a solitary wave is a limit of periodic waves one can obtain its Maslov index by using the classical definition of the index for periodic orbits and increasing the period to infinity. In this procedure the formula for the Maslov index requires knowledge of the Krein indices of the Floquet multipliers on the unit circle.

The count of eigenvalues in generalized eigenvalue problems and operator polynomials by means of the eigenvalues of the operators at powers of the spectral parameter goes back to results of Kelvin and Tait (1869). This is related both to the classical topics of gyroscopic stabilization and its destruction in the presence of dissipation and non-conservative positional forces (dissipation-induced instabilities), and to the modern question of instabilities of traveling and solitary waves. Pontryagin and Krein space decomposition for establishing sharp bounds on the number of unstable eigenvalues of the original operator recently led to a significant generalization of the classical Kelvin-Tait-Chetaev theorem on gyroscopic stabilization and clarification of its topological meaning in terms of the Euler characteristic of a surface constructed by means of the first integral of the dynamical system. Combination of these results with singularity theory, group-theoretic and perturbation approaches is expected to result in a rather complete and unified constructive theory of dissipation-induced instabilities.

Presentation Highlights

Index theorems and count of eigenvalues

- In the opening talk **Peter Lancaster** discussed new algebraic arguments for a classical Kelvin-Tait-Chetaev theorem and its generalizations, emphasizing the links between linear algebra and mechanics on the example of gyroscopic stabilization and its behavior under dissipative perturbations.

- **Richard Kollar** presented a graphical Krein signature theory that combines a graphical interpretation of the Krein signature well-known in the spectral theory of polynomial operator pencils as well as in the theory of multiparameter eigenvalue problems with the generalization of the Evans function, the Evans-Krein function, that allows the calculation of Krein signatures in a way that is easy to incorporate into existing Evans function evaluation codes at virtually no additional computational cost. The graphical Krein signature makes extremely elegant the proofs of index theorems for linearized Hamiltonians in the finite dimensional setting: a general result implying as a corollary Vakhitov-Kolokolov criterion (or Grillakis-Shatah-Strauss criterion) generalized to problems with arbitrary kernels, and a count of real eigenvalues for linearized Hamiltonian systems in canonical form. Finally it was demonstrated how the graphical approach can be used to derive new types of criteria prohibiting Hamiltonian-Hopf bifurcations under collisions of two eigenvalues of opposite signature. The talk was a unique and comprehensive survey of the index theorems motivated by very different physical, algebraic, and control theory applications.

- **Jussi Behrndt** discussed the spectral properties of a class of ordinary and partial differential operators with indefinite weight functions. These operators are not symmetric or self-adjoint with respect to a Hilbert space scalar product but they can still be viewed to be symmetric with respect to a suitably chosen Krein space inner product. A general approach via decomposition and perturbation methods were presented to obtain results on the structure of the real and non-real spectrum, as well as quantitative bounds on the non-real spectrum.

- The root radius and root abscissa of a monic polynomial are respectively the maximum modulus and the maximum real part of its roots; both these functions are nonconvex and are non-Lipschitz near polynomials with multiple roots. **Michael Overton** presented constructive methods for efficient minimization of these nonconvex functions in the case that there is just one affine constraint on the polynomial's coefficients. Then he turned to the spectral radius and spectral abscissa functions of a matrix, which are analogously defined in terms of eigenvalues. He explained how to use nonsmooth optimization methods to find local minimizers of these quantities for parameterized matrices and how to use nonsmooth analysis to study local optimality conditions for these nonconvex, non-Lipschitz functions. The pseudospectral radius and abscissa of a matrix A that are respectively the maximum modulus or maximum real part of elements of its pseudospectrum (the union of eigenvalues of all matrices within a specified distance of A), are also nonconvex functions but locally Lipschitz, although the pseudospectrum itself is not a Lipschitz set-valued map. A new method to compute these quantities efficiently for a large sparse matrix A was discussed at the end of the talk.

Challenging stability and instability problems in physical applications

- **Davide Bigoni** demonstrated counterintuitive examples of structures buckling in tension, where no compressed elements are present. These simple structures exhibit interesting postcritical behaviors, for instance, multiple configurations of vanishing external force. An experimental realization of the flutter instability in the Ziegler pendulum induced by dry friction was demonstrated with the destabilizing effect of dissipation.

- **Edgar Knobloch** reviewed some of the essential properties of the magnetorotational instability, i.e. a magnetic field induced instability of differential rotation that is likely to be of fundamental importance in astrophysics because of its angular momentum transport properties, both in the dissipationless regime and in the dissipative regime, emphasizing the role played by magnetic cross-helicity in determining the nature of this instability. Applications to transport of angular momentum require an understanding of the amplitude of the instability. Its evolution is complex, however, because it involves three radically different timescales: the rotation frequency, the inverse Alfven travel time and the dissipation rate. An asymptotically reduced model was presented that sheds light on the equilibration process both in an intermediate, nominally dissipationless regime, and in the ultimate regime where dissipation takes over, showing how phase mixing can saturate Maxwell and Reynolds stresses even when the instability is still evolving.

- **Emmanuele Tassi** discussed the negative energy modes that are an important issue for the stability properties of continuous media, such as plasmas or fluids. These are spectrally stable modes, possessing negative energy. Their identification is important, because negative energy modes can be destabilized by small perturbations, induced, for instance, by dissipation. A general and effective framework for the study of negative energy modes is the Hamiltonian one. The knowledge of the Hamiltonian structure of a system, allows to unambiguously identify the presence of negative energy modes, through the reduction of the Hamiltonian for the linearized system to its normal form. Two examples of Hamiltonian (in particular Lie-Poisson) systems of interest for plasma physics, were presented, that possess negative energy modes, when linearized about homogeneous equilibria. The two models describe the phenomena of magnetic reconnection and of electron temperature gradient driven turbulence, respectively. Both systems exhibit Krein bifurcations when negative energy modes merge with positive energy modes for critical values of the wavelength of the perturbations.

- **Panayotis Kevrekidis** presented an overview of recent theoretical, numerical and experimental work concerning the static, stability, bifurcation and dynamic properties of coherent structures that can emerge in one- and higher-dimensional settings within Bose-Einstein condensates at the coldest temperatures in the universe (i.e. at the nanoKelvin scale). It was discussed how this ultracold quantum mechanical setting can be approximated at a mean-field level by a deterministic PDE of the nonlinear Schrödinger type and what the fundamental nonlinear waves of the latter are, such as dark solitons and vortices. A further layer of simplified description via nonlinear ODEs encompassing the dynamics of the waves within the traps that confine them, and the interactions between them, was then presented. Finally, an attempt was taken to compare the analytical and numerical implementation of these reduced descriptions to recent experimental results and speculate towards a number of interesting future directions within this field.

- **William Langford** presented a mathematical model of convection in a rotating hemispherical shell of fluid, with radial gravity and a pole-to-equator temperature gradient on the inner boundary. The fluid in the model satisfies the Navier-Stokes Boussinesq PDE and the heat equation. For moderately strong values of the temperature gradient, convection cells appear that resemble the Hadley, Ferrel and polar cells of the present day climate of the Earth. The model reproduces the trade winds, westerlies, jet stream and polar easterlies of today's climate. As the temperature gradient is decreased, the Hadley cell slows in circulation velocity and expands poleward; also the jet stream moves poleward. All these changes have been observed recently in the atmosphere of Earth. Eventually, for still smaller values of the temperature gradient in the model, the Ferrel and polar cells disappear. Furthermore, the model exhibits bistability and hysteresis. One of these two stable states resembles today's climate; the other is more like the "equable" paleoclimate that existed on Earth for much of geological time.

- **Stephane Le Dizes** analyzed the characteristics of the linear waves living on a vortex in an incompressible inviscid homogeneous fluid, showing that a large axial wavenumber asymptotic analysis can be used to provide information on their spatial structure and dispersion relation. The stabilizing role of critical point singularities was discussed and analysed in this framework. Asymptotic results were illustrated and compared to numerical results for a family of vortices ranging from the Rankine vortex (disk of uniform vorticity) to the Lamb-Oseen vortex (gaussian vorticity profile). Then, the waves on similar vortices but in a fluid uniformly stratified in the direction of the vortex axis were considered. It was shown that stratification is a source of instability. Using the large axial wavenumber asymptotic analysis, it was demonstrated that the instability mechanism is associated with the radiative character of the waves. Connections with similar instability in shallow water or in a compressible fluid was made. Experimental evidence of the radiative instability was also provided.

- High-Reynolds number flows are dominated by vortical structures. Vortex filaments are unstable to a number of instabilities: the long wavelength Crow instability, the short wavelength Moore-Saffman-Tsai- Widnall (MSTW) instability and the ultra-short wavelength elliptical instability. The MSTW instability concerns a vortex in strain and was first examined by Moore and Saffman in a general context but with asymptotically small strain. Most of the actual studies since have concentrated on the case of a piecewise continuous profile of vorticity (a Rankine vortex) which supports discrete normal modes. **Stefan Llewellyn Smith** considered in his talk the more general case of non-infinitesimal strain using exact solutions of the Euler equations called hollow vortices and smooth vorticity profiles by looking at an initial-value problem.

- **Paolo Luzzatto-Fegiz** presented the conditions for the development of a Hamiltonian-Hopf instability in vortex arrays. By building on the theory of Krein signatures for Hamiltonian systems, and considering constraints owing to impulse conservation, it was demonstrated that a resonant instability (developing through coalescence of two eigenvalues) cannot occur for one or two vortices. This deduction was illustrated by examining available linear stability results for one or two vortices. It was indicated that a resonant instability may, however, occur for three or more vortices. For these more complex flows, a simple model was proposed, based on an elliptical vortex representation, to detect the onset of a resonant instability. An example was given in support of the theory by examining three co-rotating vortices, for which a linear stability analysis has been performed. The stability boundary in this model is in a good agreement with the full stability calculation. In addition, it was shown that eigenmodes associated with an overall rotation or an overall displacement of the vortices always have eigenvalues equal to zero and $\pm i\Omega$, respectively, where Ω is the angular velocity of the array.

- **Dmitry Pelinovsky** presented a sharp criterion of transverse stability and instability of line solitons in the discrete nonlinear Schrodinger (dNLS) equation on a square two-dimensional lattice near the anti-continuum limit. The fundamental (single-site) line soliton is proved to be transversely stable (unstable) when it bifurcates from the hyperbolic (elliptic) point of the dispersion surface. The results hold for both focusing and defocusing dNLS equation via a staggering transformation. The one-dimensional dNLS equation with the continuous diffraction term was also considered and it was proven that the fundamental line soliton is transversely unstable in both cases when it bifurcates from the hyperbolic and elliptic points of the dispersion surface. In the former case, the instability is caused by the resonance between eigenvalues of negative energy (Krein signature) and the continuous spectrum of positive energy. Analytical results were illustrated numerically.

- In Hamiltonian systems, the Hamiltonian Hopf (HH) bifurcation occurs when two pairs of stable eigenvalues collide at some parameter value and bifurcate to the quartet. According to the Krein-Moser theorem, this bifurcation can only happen if the colliding eigenvalue pairs have opposite signature, which can be determined by evaluating the energy on the eigenfunction. Such a transition to instability (overstability) is seen in the discrete spectrum of PDEs that describe many physical systems, such as the fluid plasma two-stream instability, top-hat distribution description of Jean's instability, and the contour dynamics description of shear flow or Kelvin-Helmholtz instability. The continuum Hamiltonian Hopf (CHH) bifurcation is a similar, but mathematically more challenging, bifurcation that occurs in Hamiltonian PDEs with a continuous spectrum. Examples include the Vlasov equation, Euler's fluid equation, MHD, etc. To understand this bifurcation it is necessary to first attach a signature to the continuous spectrum of Hamiltonian PDEs, a nontrivial task since eigenfunctions of the continuous spectrum are non-normalizable. Having the signature, a version of Krein-Moser theorem is possible provided an appropriate definition of structural stability and parameter variation are given. Thus, in the CHH bifurcation, the continuous spectrum plays the role of one of the eigenvalue pairs of the HH bifurcation. In his talk **Phil Morrison** reviewed the HH bifurcation in the PDE context, gave examples, and described how the CHH bifurcation appears in a variety of physical systems. Rigorous aspects of CHH were described in a companion talk by **George Hagstrom**.

Dissipation-induced instabilities

- **Gianne Derks** considered the infinite time behaviour of a family of stationary solutions of Euler's equation, which can be described as constrained minima of energy on level sets of enstrophy. For free boundary conditions, this family shadows solutions of 2D Navier-Stokes equations. However, under the no-slip and under the Navier-slip boundary conditions and in a circular domain, the infinite time Navier-Stokes evolution orbit of a starting point on the family of constrained minima has order 1 distance to the family, however small the viscosity is. The viscosity in the Navier-Stokes equations is a singular perturbation for Euler's equation and one might suspect that the viscosity-induced instability is related to this singularity. This is not the case: we show that the same phenomenon can be observed for the averaged Euler equations and second grade fluids with Navier-slip boundary conditions in a circular domain.

- In 1952 Ziegler observed (I) that viscous dissipation can move pure imaginary eigenvalues of a Lyapunov stable time-reversible non-conservative mechanical system (Ziegler's pendulum loaded by a follower force) to the right half of the complex plane and (II) that the threshold of asymptotic stability generically does not converge to the threshold of the Lyapunov stability of the non-damped system when dissipation coefficient tends to zero. In 1956 Bottema related the structurally unstable situation (II) to the Whitney umbrella singularity of the stability boundary. **Oleg Kirillov** has shown the examples of Hamiltonian, reversible and $\mathcal{PT}$-symmetric systems of physics and mechanics with the similar effects of dissipation-induced instabilities and non-commuting limits of vanishing dissipation. The relation of these effects to the multiple non-derogatory eigenvalues occurring both on the stability boundary and inside the domain of asymptotic stability was discussed, the connection to the spectral abscissa minimization was shown and in the Hamiltonian case it was demonstrated that a suitable combination of damping and nonconservative positional forces can destabilize the eigenvalues with both positive and negative Krein (symplectic) signature of the unperturbed system.

- The talk of **Olivier Doare** was devoted to the influence of dissipation on local and global instabilities in the media of infinite length. The waves that are neutral in absence of dissipation become temporally amplified when damping terms are added in the wave equation. The concept of wave energy, introduced in plasma physics, represents a considerable value to the discussion of this effect. The energy of a wave is defined as the work done on the system to generate the neutral wave from $t = -\infty$ to $t = 0$. Consequently, a wave is of negative energy if its establishment lowers the total energy of the system. It was then found that negative energy waves are destabilized by addition of damping. Studies considering waves propagating in an infinite medium are referred to as *local*. Additionally to plasma physics, negative energy waves have been studied in mechanics, in the context of compliant panels interaction with inviscid flows and the instabilities of the surface between two non-miscible fluids. It was found that the presence of gyroscopic terms in the wave equation is necessary to have negative energy waves in the system and that the existence of negative energy waves is a necessary condition to observe destabilization by dissipation in the finite length system. Some simple local criteria, based on characteristic length of rigidity and damping forces were used to develop simple criteria that predict global instability. Finally, destabilization by damping in the context of recent works on energy harvesting using fluttering piezoelectric flexible plates was discussed.

Integrable systems and bifurcations

- In families of isoperimetrically constrained variational principles the signs of the eigenvalues of the Hessian of the energy with respect to the Lagrange multipliers enters into the second order necessary conditions. **John Maddocks** explained in his talk how the computation of looping probability for an elastic polymer, such as DNA, can be cast in terms of path integrals where the leading order approximation involves an isoperimetric variational principle, and the first, or semi-classical, correction involves the determinant of the same Hessian of the energy with respect to the Lagrange multipliers.

- **Pietro-Luciano Buono** discussed recent results about the Hip-Hop orbit of the Newtonian 2N-body problem. The Hip-Hop orbit (in reduced space) is a periodic solution with time-reversing and spatio-temporal symmetries and in fact, it is a brake orbit. The analytical proof of linear instability of the Hip-Hop orbit was presented using Maslov index methods. Numerical simulations were shown of the Hip-Hop orbit as the energy is varied which exhibits a sequence of symmetry-breaking bifurcations and avenues for classifying those bifurcations were discussed.

- **Richard Cushman** in his talk treated in detail an example of a one parameter family of Hamiltonian systems, which exhibits an S^1-equivariant sign exchange bifurcation in its linearization about an equilibrium point.

- The uncovering of the role of monodromy in integrable Hamiltonian fibrations has been one of the major advances in the study of integrable Hamiltonian systems in the past few decades: on one hand monodromy turned out to be the most fundamental obstruction to the existence of global action-angle coordinates while, on the other hand, it provided the correct classical analogue for the interpretation of the structure of quantum joint spectra. Fractional monodromy is a generalization of the concept of monodromy: instead of restricting our attention to the toric part of the fibration we extend our scope to also consider singular fibres. In his talk **Konstantinos Efstathiou** analyzed fractional monodromy for n_1:$(-n_2)$ resonant Hamiltonian systems with n_1, n_2 coprime natural numbers. In particular, systems that for $n_1, n_2 > 1$ contain one-parameter families of singular fibres which are 'curled tori', were considered. The geometry of the fibration was simplified by passing to an appropriate branched covering. In the branched covering the curled tori and their neighborhood become untwisted thus simplifying the geometry of the fibration: essentially the same type of generalized monodromy was obtained independently of n_1, n_2. Fractional monodromy was then recovered by pushing the results obtained in the branched covering back to the original system.

- **Jeroen Lamb** discussed a system with a deterministic pitchfork bifurcation with the additive noise. It was shown that there is qualitative change in the random dynamics at the bifurcation point in the sense that after the bifurcation, the Lyapunov exponent cannot be observed almost surely in finite time. This bifurcation was associated with a breakdown of both uniform attraction and equivalence under uniformly continuous topological conjugacies, and with non-hyperbolicity of the dichotomy spectrum at the bifurcation point.

- **Zensho Yoshida** considered bifurcation of equilibrium points in fluids or plasmas using the notion of Casimir foliation that occurs in a noncanonical Hamiltonian formulation of an ideal fluid or plasma. The nonlinearity of the system makes the Poisson operator inhomogeneous on phase space (the function space of state variables), resulting in a nontrivial center of the Poisson algebra; the center elements are called Casimirs. Orbits are constrained on level-sets of Casimirs, i.e. Casimir leaves. Even if a Hamiltonian is simple (typically a fluid/plasma Hamiltonian is just the "norm" of phase space, unlike bumpy

Hamiltonians modeling strongly coupled systems), energy contours on a Casimir leaf may have considerably complicated shapes. Invoking a simple model of plasma, it was shown that the equilibrium points on Casimir leaves bifurcate as Casimir parameters change. The energies of bifurcated equilibrium points can be compared to estimate the stability. In ideal dynamics, however, a higher-energy state may sustain stably by other Casimir constraints; in fact “resonant singularities" generate infinite number of “singular Casimir elements" which foliate the phase space and separate different equilibrium points. A singular perturbation (introduced by finite dissipation) destroys the Casimir leaves, removing the topological constraint and allowing the state vector to move towards lower-energy state in unconstrained phase space. An extended Hamiltonian mechanical representation of such an instability caused by a singular perturbation was proposed.

- **Yasuhide Fukumoto** considered a steady Euler flow of an inviscid incompressible fluid characterized as an extremum of the total kinetic energy (=the Hamiltonian) with respect to perturbations constrained to an isovortical sheet (=coadjoint orbits). The criticality in the Hamiltonian was used to calculate the energy of three-dimensional waves on a steady vortical flow, and, as a by-product, to calculate the mean flow, induced by nonlinear interaction of waves with themselves. These formulas were applied to study the linear and weakly nonlinear stability of a rotating flow confined in a cylinder of elliptic cross-section. The linear instability, parametric resonance between a pair of Kelvin waves, is known as the Moore-Saffman-Tsai-Widnall (MSTW) instability. The linear stability characteristics is well captured from the viewpoint of Krein's theory of Hamiltonian spectra. Furthermore, with the mean flow induced by the Kelvin waves, a hybrid method of combining the Eulerian and the Lagrangian approaches was shown to be effective to deduce the amplitude equations to third order.

- Stability problems of various fluid models (PDEs) are widely addressed in the studies of complex fluids, geophysical fluids, astrophysical and laboratory plasmas and so on. If the model is physically well-posed, it is promising to find its Hamiltonian structure and Casimir invariants in the dissipationless limit, which can yield a priori estimates for Lyapunov stability. More detailed stability analysis is often facilitated by restoring the Lagrangian description of fluid, especially when there are many Lagrangian invariants (i.e., frozen-in fields). The Lagrangian viewpoint is advantageous in that it enjoys an effective use of variational principle. For example, in linear stability analysis, the variational principle renders the eigenvalue problem being composed of Hermitian and anti-Hermitian operators. The concepts of action-angle variables and adiabatic invariance can be formulated for not only discrete spectrum but also continuous one. By invoking the Lie series expansion, weakly nonlinear analysis is also performed systematically, and the normal forms for mode-mode couplings are extracted by least algebraic manipulations. Even for strongly nonlinear problem such as explosive instability, the variational principle enables us to infer its mechanism in a heuristic manner. **Makoto Hirota** in his talk gave an overview of recent advancements in this Lagrangian approach.

Semi-classical approximation

- **Michael Berry** presented a study of the evolution of optical polarization in a stratified nontransparent dielectric medium twisted cyclically along the propagation direction. The twist is chosen to encircle a degeneracy (branch-point) in the plane of parameters describing the medium. Polarization evolutions are determined analytically and illustrated as tracks on the Poincaré sphere and the stereographic plane. Even when the twist is slow, the exact evolutions differ sharply from those of the local eigenpolarizations and can display extreme sensitivity to initial conditions with the tracks exhibiting elaborate coilings and loopings that would be very interesting to explore experimentally. Underlying these dramatic violations of adiabatic intuition are the disparity of exponentials and the Stokes phenomenon of asymptotics.

- **Sergey Dobrokhotov** discussed a multi-dimensional analogue of the problem of the level splitting for a Schrödinger-particle placed into a symmetric double-well potential. The splitting is described by the formula $\triangle E = A\exp(-\frac{J}{h})$ with the phase J based on a certain classical trajectories known as instanton. The constructive but complicated formula of the amplitude A is also connected with the instanton. In the talk it was shown that the splitting formula takes more natural and simple form if one changes the instanton by so-called libration (unstable closed trajectories) and use the normal forms coming from classical mechanics. Finally, a non-trivial question on the level splitting in the presence of magnetic field was discussed. It was demonstrated that in 2-D case using the partial Fourier transform and mixed momentum-position coordinates one can reduce the quantum double-well problem with magnetic field to the standard quantum double-well problem and to study the splitting in this situation too.

- **Setsuro Fujiié** considered the semi-classical Schrödinger operator $P := -h^2\Delta + V(x)$ in R^n, where h is a small positive (semi-classical) parameter and $V(x)$ is a real-valued smooth potential decaying at infinity. If the classical dynamics for the corresponding classical Hamiltonian $p(x,\xi) := |\xi|^2 + V(x)$ has *trapped* trajectories on $p^{-1}(z_0)$ for a real positive energy z_0, it is expected that there exist the so-called *resonances* close to z_0 in the lower half complex plane. The imaginary part of resonances, called *width*, means the reciprocal of the exponential decay rate of the corresponding states for the evolution as time tends to $+\infty$. In the talk the width of resonances associated with an unstable equilibrium of the potential was discussed. If $x = x_0$ is an unstable equilibrium, i.e. a local maximum of the potential, then the point $(x, \xi) = (x_0, 0)$ in the phase space is a hyperbolic fixed point of the Hamilton vector field, and it is itself a trapped trajectory. Hence resonances may appear near the energy $E_0 := V(x_0)$. Contrary to the case of a stable equilibrium, the trap by an unstable equilibrium is much weaker, and, as consequence, the resonance width should be large. Additional higher bumps of potential create *homoclinic* trajectories converging to the hyperbolic fixed point as time tends to $+\infty$ and $-\infty$. Assuming that the trapped set at the energy level E_0 consists of the hyperbolic fixed point and these homoclinic trajectories, lower bounds for the resonance width were obtained.

Scientific Progress Made

Many participants commented during the meeting as well as after it that they benefited a lot from the cross-disciplinary nature of the talks. In particular,

- **Michael Berry** found that the workshop was well conceived, bringing together people in fields that are different yet similar enough for meaningful and indeed useful connections to develop.
- **Almut Burchard** and **Marina Chugunova** continued their collaboration and exchanged recent ideas obtained after working apart from each other for a while.
- **Sergey Dobrokhotov** said that the workshop turned out to be very useful because many results impact his and his colleagues research; the new contacts he made could lead to joint works in the future.
- **Panos Kevrekidis** and **Dmitry Pelinovsky** have used the workshop space to enhance their collaboration and to complete a manuscript, which is now published in Europhysics Letters [4].
- **Oleg Kirillov** was included in almost every discussion. As a feedback during the workshop, he improved his manuscript, which is now accepted for Phil. Trans. R. Soc. A [3].
- **Richard Kollar** benefited from many discussions with other speakers and said that the main highlight of this workshop is bringing together people from different groups and research background but specializing on the same range of stability and bifurcation problems. These discussions contributed to his recent review paper for SIAM Reviews [1].
- **Michael Overton** received very good feedback on his talk and got to know many colleagues whom he had not met before.
- **Carsten Trunk** had admitted that his personal research activities were greatly influenced by the workshop, so much that even his PhD students will feel it.
- **Charles Williamson** made new contacts and said the meeting had helped to one of his students in developing his own future and scientific contacts.

- **Zensho Yoshida** had benefited by the workshop very much and started new interactions.

About twenty specialized articles are in preparation for the volume [2] that will be published by Wiley-ISTE with the aim to collect the highlights of the meeting.

Outcome of the Meeting

The BIRS Workshop on Spectral Analysis, Stability and Bifurcations in Modern Nonlinear Physical Systems collected together a unique combination of experts in modern dynamical systems, mathematical physics, PDEs, numerical analysis, operator theory, and applications.

One of the immediate outcomes of the meeting that makes its materials available for a broader audience is a post-conference volume of papers [2] from the participants of the workshop. This project is aimed to collect unique viewpoints of the participants on the history, current state of the art, and prospects of research in their fields contributing to the progress of stability theory. Our vision of this book is a collection of essays - mathematical, physical, and mechanical. The contributions will show connections between different approaches, applications, and ideas. We believe that such a book could set up the benchmarks and goals for the next generation of researchers and be a true event in modern stability theory.

The other outcomes will be seen over a long range of time, when the ideas formulated and discussed during the workshop, as well as new collaborations made, will lead to new scientific publications and new research discoveries.

Participants

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Bibliography

[1] R. Kollar, P. D. Miller, Graphical Krein signature theory and Evans-Krein functions, *SIAM Review* (subm.) 2013, arXiv:1209.3185.

[2] *Nonlinear Physical Systems: Spectral Analysis, Stability and Bifurcations*, (O.N. Kirillov and D. Pelinovsky, eds.), Wiley-ISTE, London, 2014.

[3] O.N. Kirillov, Stabilizing and destabilizing perturbations of $\mathcal{PT}$-symmetric indefinitely damped systems, *Phil. Trans. R. Soc. A* **373** (2013).

[4] D.E. Pelinovsky, P.G. Kevrekidis, D.J. Frantzeskakis, PT-symmetric lattices with spatially extended gain/loss are generically unstable, Eur. Phys. Lett. **101** (2013), 11002 (6 pages).

Chapter 38

Nonequilibrium Statistical Mechanics: Mathematical Understanding and Numerical Simulation (12w5013)

November 11 - 16, 2012

Organizer(s): Joel Lebowitz (Rutgers University), Stefano Olla (Université Paris Dauphine), Gabriel Stoltz (Ecole des Ponts and INRIA Rocquencourt)

Overview of the Field

Aims of statistical mechanics

Statistical mechanics is a theory allowing to infer the macroscopic behavior of physical systems from the knowledge of their microscopic description (namely the interaction laws between the elementary constituents of matter, such as atoms in a classical framework). The foundations of this theory date back to the nineteenth century. Major contributors include Maxwell, Boltzmann, Gibbs, Kelvin and Einstein.

The orders of magnitude used in the microscopic description of matter are very different from those of the macroscopic quantities we are used to. For instance, the number of particles under consideration in a macroscopic sample of material is of the order of the Avogadro number $\mathcal{N}_A \sim 10^{23}$, while typical distances are expressed in Å (10^{-10} m), energies are of the order of $k_{\rm B}T \simeq 4 \times 10^{-21}$ J at room temperature, and the typical times are of the order of 10^{-15} s when the proton mass is the reference mass. For practical numerical computations of matter at the microscopic level, following the dynamics of every atom would require simulating $\mathcal{N}_A$ atoms and performing $\mathrm{O}(10^{15})$ time integration steps, which is of course impossible!

Despite its intrinsic limitations on spatial and time-scales, molecular simulations have been used and developed over the past 50 years, and its number of users keeps increasing. It nowadays has two major aims. First, it can be used as a *numerical microscope* to perform "computer" experiments. This was the initial motivation for simulations at the microscopic level: physical theories were tested on computers. This use of molecular simulation is particularly clear in its historic development, which was triggered and sustained by the physics of simple liquids. It is still very helpful to observe numerical trajectories to guide intuition about what is happening in the system. Another major aim of molecular simulation, maybe even more important than the previous one, is to compute macroscopic quantities or thermodynamic properties, typically through averages of some functionals of the system. In this case, molecular simulation is a way to obtain *quantitative* information on a system, instead of resorting to approximate theories, constructed for simplified models, and giving only qualitative answers.

Nonequilibrium statistical mechanics

Equilibrium statistical mechanics is nowadays a fairly well-understood scientific field, both from theoretical and numerical viewpoints. The situation is dramatically different for nonequilibrium systems, which are subjected flows of energy, momentum, particles, etc. It is often the case that a stationary state is attained thanks to thermal baths at the boundaries of the system, which model a coupling with an infinite equilibrium system. In this case, the thermodynamic state of the system is described by some probability measure, whose analytical expression is usually unknown.

From a theoretical viewpoint, nonequilibrium systems pose many challenges. First of all, showing the very existence and uniqueness of a stationary measure for the stochastic or deterministic dynamics under consideration turns out to be non-trivial in many cases, *e.g.* [14] and references therein. Once the existence and uniqueness of the invariant measure is obtained, it still remains to connect results with those obtained from linear response theory in order to obtain formulas such as the Green-Kubo formula, which relate time-correlations of some observables to a given transport coefficient such as thermal conductivity, viscosity, etc. From a technical viewpoint, this task is easier when the generator of the dynamics has a spectral gap. The dynamics of many interesting systems, especially infinite systems obtained as the thermodynamic limit of finite size systems, however turn out not to have a spectral gap even when some stochastic perturbation is added to the deterministic Hamiltonian dynamics.

One difficulty for the analysis of nonequilibrium systems is that the stationary measure a priori depends on particular features of the evolution under consideration. In contrast, for equilibrium systems, there are (infinitely) many different dynamics which can be used to sample this measure. For instance, the equilibrium canonical ensemble can be sampled using a Langevin dynamics (with various values of the friction parameter) or Metropolis-Hastings algorithms.

Another difficulty of nonequilibrium systems is the non-locality of the stationary measure. A local perturbation in the force-field of the dynamics will in general globally affect the invariant measure, even far away from the perturbation.

Simulating nonequilibrium systems

The efficient simulation of nonequilibrium systems is an important issue from a numerical viewpoint.

Transport properties can be understood for systems close to equilibrium, using results from linear response theory. There are several general numerical approaches to estimate transport coefficients (shear viscosity, thermal conductivity, autodiffusion constant, etc), based either on (i) equilibrium simulations, where the transport coefficient is obtained by the Green-Kubo formula (integrated equilibrium correlation function) or Einstein formula; (ii) nonequilibrium simulations in a steady-state, with either boundary-driven systems or field-driven dynamics where the response of the system to some mechanical or thermal forcing is measured as a function of the forcing strength; and (iii) the study of transient states, where a system initially at a equilibrium is perturbed, and the relaxation of the perturbation is monitored as a function of time.

Recent Developments and Open Problems

Understanding general properties of nonequilibrium systems

Even when the invariant measure of nonequilibrium systems is unknown it is relevant to attempt to unravel steady-state properties such as the existence of long-range space or time correlations, or appearance of metastable states when a forcing is introduced into the system. Even at a qualitative level a clear picture has not emerged yet, even for the simplest systems on one-dimensional lattices. One of the first theoretical studies of long-range correlations in exclusion processes is [6].

Thermal transport in general systems

Thermal transport properties can be studied in systems subjected to a temperature gradient (fixed for instance by setting the temperatures at the boundaries) by considering the energy current flowing from the hot to the cold interface. When Fourier's law is applicable, linear temperature profiles are obtained when the temperature difference is small, and the current is proportional to the temperature gradient.

Despite many efforts in the past, our understanding of the microscopic ingredients necessary for the validity of Fourier's law is still scarce (see for instance the review article [4], which is still of contemporary interest). A first distinction to be made is whether the system under consideration is on a lattice or not, and whether its dynamics is deterministic or involves some stochastic perturbation. Of course, studying systems on a lattice is much easier from a theoretical viewpoint than considering fluid systems without any internal structure. Adding stochastic perturbations helps as well. On the other hand, it may be depressing (or stimulating, depending on one's inclination) to realize that our understanding of thermal transport processes in very simple fluids described by a deterministic mechanical model of hard spheres is almost still as incomplete as it was in 1978 when Joel Lebowitz reviewed the problem in [11]. Even for three dimensional lattices systems, there is no proof at the moment that the conductivity is finite in systems with generic (nonlinear) interaction potentials.

Thermal transport in one-dimensional systems

The topic of thermal transport in one-dimensional systems has been a very active field of research in the last fifteen years – motivated in part by the carbon nanotube technology, and also by the desire to understand the origin of Fourier's law (or its failure!) in one of the simplest atomic systems. There are many research groups working on this topic around the world, both from theoretical and numerical viewpoints (see for instance the review articles [12, 7]).

Despite the simplicity of the model, it is still not completely clear at the moment which ingredients of the interaction potentials between the particles on the lattice, of externally applied pinning potentials or forces, and of added stochastic perturbations are necessary or sufficient to obtain normal thermal transport. It is however believed that the conservation of invariants in the system, such as the energy, the volume and the momentum, plays an important role.

From a technical viewpoint, the rigorous mathematical results known to this date are mostly upper bounds on the divergence of the thermal conductivity (through upper bounds on the decay of the autocorrelation function of the currents). Obtaining lower bounds is a current important theoretical challenge in this domain.

Derivation of macroscopic evolution equations for microscopic systems

The thermodynamic limit of transport processes at the microscopic scales can be understood by some limiting process, using appropriate space-time scalings. The system should first be understood in the hyperbolic time-scale, where the system is studied over long times Nt with space renormalized by a factor N^{-1}. Locally, the state of the system is expected to be close to some equilibrium Gibbs measure consistent with the local invariants of the system, such as the energy, the momentum and the volume for Hamiltonian dynamics. In the hyperbolic scaling, a reduced description of the evolution of the microscopic system is given by coupled partial differential equations involving the invariants of the dynamics, known as the hydrodynamic limit (see [10]). The limiting set of equations is often not difficult to obtain formally. To rigorously prove the convergence, the method of choice is the relative entropy method of Yau [16]. Some appropriate ergodicity properties are however required to apply it. It is currently a major open problem to rigorously derive hydrodynamic limits for deterministic systems [4]. The situation is more favorable when the Hamiltonian dynamics is perturbed by appropriate stochastic processes.

Once the behavior of the system is understood in the hyperbolic space-time scaling, it is possible to study the diffusion of the energy on a longer time-scale – which amounts to understanding the energy fluctuations. If the process has a diffusive behavior consistent with Fourier's law, then the relevant time scale is indeed the diffusive one. However, the energy often "superdiffuses" in one dimensional chains, so that the correct time scaling for time should rather be $N^{\alpha}t$ for $1 < \alpha < 2$ instead of $N^2 t$ (see for instance [13] for a review of the cases when such statements can be made mathematically rigorous). The macroscopic evolution of the fluctuation fields obtained with this process typically are a coupled system of nonlinear stochastic partial differential equations. Needless to say, giving a mathematical meaning to these limiting objects is already quite a challenge, not to mention rigorously proving the convergence in itself.

Numerical analysis of simulation methods

In contrast to equilibrium sampling, where many strategies exist to enhance the convergence of longtime averages of appropriate dynamics, only few if any works have considered some kind of importance sampling method for nonequilibrium systems: Numerical studies are often performed by a straightforward longtime integration of the dynamics. In many situations however, these averages are not converging very fast because of the large variance of the estimator compared to the average value to be computed (see for instance the discussion in [9]). Moreover, in some situations such as thermal transport computations in very long atom chains, long transient regimes have to be removed from the computation of the average.

Variance reduction techniques are used on a daily basis in equilibrium simulations. The use of such techniques is often based on the explicit expression of the invariant measure, and its modification under appropriate perturbations. There are three main strategies: importance sampling, stratification, and the control variate method. None of them can be used as such for nonequilibrium systems. Indeed, the control variate method is an interesting technique for very specific, low dimensional dynamics for which there is a strong coupling (see for instance [8]); while the non trivial modifications of the invariant measure of nonequilibrium systems under perturbations somehow prevent the use of importance sampling and stratification by constrained dynamics.

Let us also mention that the numerical analysis of nonequilibrium sampling methods is rather scarce: we are not aware of a study of the time-step error arising in the numerical discretization of the Green-Kubo formula or in the definition of the transport coefficients as given by linear response results.

Presentation Highlights

The speakers were given 40 minutes to deliver their lectures, including 5 minutes question. It was most often the case that questions were asked (and answered!) as they arose during the talk – a very familial and informal way of proceeding, and a nice feature of smaller scale workshops with participants hosted on site and living together for a week.

Thermal transport

A substantial fraction of the talks were devoted to the study of thermal transport, both from theoretical and numerical angles.

More theoretically oriented studies

Cédric Bernardin (Ecole Normale Supérieure de Lyon) discussed the anomalous diffusion in Hamiltonian systems perturbed by a conservative noise. He actually considered one of the simplest Hamiltonian models, with only two conserved quantities (instead of three in standard Hamiltonian systems). He treated in particular the case of exponential interaction potentials à la Toda, for which refined results can be stated.

Jani Lukkarinen (University of Helsinki) reviewed results for two models satisfying Fourier's law in chains of harmonic oscillators: systems subjected to self-consistent bulk thermostatting mechanisms, and systems perturbed by a non-momentum conserving bulk stochastic noise. In both cases, some new results were given about the hydrodynamical equations for typical macroscopic energy and displacement profiles, as well as their fluctuations and large deviations.

Carlangelo Liverani (Universita Roma Tor Vergata) discussed the statistical properties of a very simple (if not the simplest possible) fast-slow deterministic system of dimension 2. One variable, which should be thought of as the energy, evolves on a much slower time scale than the other, while the other variable evolves according to a mixing dynamics. The motivation for considering this system is to understand how microscopically deterministic system may be described by mesoscopic stochastic evolutions. This would be an important step in rigorously deriving hydrodynamic equations from a weakly interacting microscopic Hamiltonian system.

François Huveneers (Université Paris-Dauphine) presented results on the asymptotic localization of energy in some Hamiltonian chains. Energy transfer in close to integrable Hamiltonian systems can sometimes be much slowed down or even suppressed. Anderson localization, breathers, KAM tori or Nekhoroshev estimates can in some cases be invoked to justify this claim. However, given a chain at positive temperature in the infinite volume limit, it is generally hard to infer any clear picture on heat transfer out of such mathematical results. This can however be studied for nearly integrable Hamiltonian chains in a weak coupling regime, and allows to write rigorous asymptotic estimates suggesting a very rapid fall-off of the thermal conductivity with the coupling strength. Both disorder and strong anharmonicity play a similar role in the considered regime.

More numerically oriented presentations

Abhishek Dhar (International Centre for theoretical sciences, Bangalore) suggested a Levy walk description of anomalous heat transport. His talk was based on some recent work suggesting that a good description of heat transport in low dimensional systems is obtained by modeling heat carriers as Levy walkers rather than simple random walkers.

Alessandra Iacobucci (Université Paris-Dauphine) presented numerical results obtained for chains of rotors subjected to both thermal and mechanical forcings, in a nonequilibrium steady-state. Unusual nonlinear profiles of temperature and velocities are observed in the system. In particular, the temperature is maximal in the center, which is an indication of the nonlocal behavior of the system. Despite this uncommon behavior, local equilibrium holds for long enough chains. The numerical results also show that, when the mechanical forcing is strong enough, the energy current can be increased by an inverse temperature gradient. This counterintuitive result again reveals the complexity of nonequilibrium states.

Gary Morris (University of New South Wales) was the only speaker this week to consider heat conduction problems in fluid systems, namely quasi-one-dimensional hard disks confined in some narrow channel, using periodic boundary conditions in the orthogonal direction. In his numerical studies, he considered both low and large density systems.

Non-thermally driven driven systems

Federico Bonetto (Georgia Institute of Technology) presented some results for a simple model of electric conduction in a non equilibrium steady state. The models consists of N particles moving in a periodic array of scatterers under the influence of an electric field and of a Gaussian thermostat keeping the energy fixed. Analytic result for the behavior of the steady state of the system at small electric field can be obtained. In this regime, the velocity distribution becomes independent of the geometry of the scatterers. For a large number of particles, the system can be described by a linear Boltzmann type equation.

Christian Maes (University of Leuven) discussed the extension of Clausius heat theorem to driven systems. Clausius theorem asserts that the heat divided by the temperature is an exact differential for reversible processes. Christian Maes suggested an extension of this thermodynamic fact to simple systems of nonequilibrium statistical mechanics using dynamical large deviations and the excess heat produced when switching from one nonequilibrium state to another.

David Mukamel (Weizmann Institute) presented theoretical and numerical results on long-range correlations in some driven lattice systems, such as the existence of long range-order, spontaneous symmetry breaking and non-local response to local perturbations. He focuses on two cases: particle diffusion processes with localized perturbations of the transitions rates (the aim being then to understand the long-range perturbations of the density), and interface dynamics for spins systems with a line defect.

Martin Evans (University of Edinburgh) reported on the peculiar condensation in one-dimensional particle systems where the hopping between sites of a one dimensional lattice occurs with a rate increasing with the number of particles. Some clusters of particles spontaneously form in the system, and then move at increasing speed as they gain particles. Ultimately, they produce a moving condensate which comprises a finite fraction of the mass in the system. Surprisingly, the relaxation time to steady state decreases with system size.

A focus on discrete nonlinear Schrödinger systems

Stefano Lepri (Istituto dei Sistemi Complessi) introduced a nonequilibrium forcing in the discrete nonlinear one-dimensional Schrödinger equation. This system can be regarded as a minimal model for stationary transport of bosonic particles like photons in layered media or cold atoms in deep optical traps. Due to the presence of two conserved quantities, energy and norm (or number of particles), the model displays coupled transport in the sense of linear irreversible thermodynamics. Numerical studies show that the Onsager coefficients are finite in the thermodynamic limit, *i.e.* transport is normal. Depending on the position in the parameter space, the "Seebeck coefficient" may be either positive or negative. For large differences between the thermostat parameters, density and temperature profiles may display an unusual nonmonotonic shape. This is due to the strong dependence of the Onsager coefficients on the state variables.

Roberto Livi (Universita di Firenze) reported results complementary to the ones presented by Stefano Lepri. More precisely, he focused on another parameter regime leading to the formation and persistence of breathers (stable nonlinear excitations), which may explain anomalous transport properties. These excitations are also responsible for the appearance of metastable states living over exceedingly long time scales, thus inhibiting any appreciable signal of relaxation to equilibrium, while yielding the formation of "negative" temperature conditions.

Hydrodynamic limits and kinetic models

Derivation of hydrodynamic limits

In connection with the lectures by Jani Lukkarinen and Cédric Bernardin, Marielle Simon (Ecole Normale Supérieure de Lyon) derived hydrodynamic equations of diffusion type for atom chains subjected to velocity-flipping processes. The proof of convergence uses the relative entropy of the law of the process with respect to the local equilibrium Gibbs state appropriately modified by a second order correction term. A crucial technical step to use the relative entropy method is to control the energy moments. This can fortunately be done thanks to the very specific structure of the invariant measure of the system.

On a related topic, Milton Jara (IMPA) presented results beyond the hydrodynamic limit, on the fluctuations of one-dimensional chains of oscillators perturbed by a noise that conserves the energy and momentum. When the strength of the noise is tuned properly (the so-called weakly asymmetric scaling), the scaling limit of the fluctuations of the conserved quantities is given by a system of stochastic Burgers equations, which corresponds to a generalization of the celebrated KPZ equation.

Makiko Sasada (Keio University) reviewed stochastic energy exchange systems of locally confined particles in interaction. These systems have been extensively studied recently since they can be seen as accessible models for the rigorous study of the derivation of Fourier's law from microscopic dynamics of mechanical origin. As a generalization of these dynamics, Grigo et al. introduced a class of pure jump Markov processes of energies and studied the spectral gap of their generator under the assumption that the rate function of the energy exchange is uniformly positive. The results presented by Makiko Sasada are an extension to the case where the rate function does not have a uniform lower bound. Obtaining a spectral gap is a first mandatory step in the rigorous derivation the hydrodynamic behavior.

Johannes Zimmer (University of Bath) proposed a formulation of hydrodynamic limits in a more analytical framework. He first recalled that the diffusion equation, obtained as an appropriate scaling limit of Brownian motion, can be reformulated as gradient flow of the entropy in the Wasserstein metric. The latter formulation is physically very appealing, since it reveals in a mathematically rigorous way that the entropy can be seen as a driving force out of equilibrium. The connection with the original particle system is however quite cumbersome. Johannes Zimmer proposed to combine a Large Deviation principle with Gamma-convergence techniques to make the connection more straightforward.

Boltzmann-type approaches

Herbert Spohn (Technische Universität München) introduced the Hubbard Hamiltonian, which describes electrons on a lattice with on-site interactions. He showed that for small interactions the dynamics is well approximated by a kinetic equation, the matrix-valued Hubbard-Boltzmann equation. He discussed general features of this equation and, for one dimensional chains with nearest neighbor hopping, presented simulation results of the spatially homogeneous equation and its approach to the steady state.

Chanwoo Kim (Cambridge University) reported his results on thermally forced systems described by a Boltzmann equation. The first issue is to construct steady-state solutions in a general bounded domain with diffuse reflection boundary conditions corresponding to a non-isothermal temperature of the wall. This is done in a perturbative regime. Further investigation shows that Fourier's law does not hold in the kinetic regime corresponding to rarefied gases.

Numerical methods

Efficient simulation

Tony Lelièvre (Ecole des Ponts) presented the mathematical analysis of two numerical methods to accelerate the sampling of metastable dynamics where the system remains for very longtime in a region of the configuration space before hopping to another one. The first method consists in adding a non-gradient force to an equilibrium overdamped Langevin dynamics, so that the invariant measure is unchanged, and to optimize the non-gradient field to maximize the spectral gap. The analysis can be rigorously performed for harmonic systems. The second method is the parallel replica algorithm proposed by A. Voter, whose properties can be studied using properties of quasi-stationary distributions.

Carsten Hartmann (Freie Universität Berlin) reviewed some ideas of risk-sensitive optimal control theory, and showed how to apply them to efficiently bias a system. A typical application is the computation of exit times out of metastable states. Using appropriate biasing forces (which amount in simple one dimensional cases to a tilting of double well potentials), the transitions from one metastable to the other are dramatically enhanced. It turns out that the non-gradient forces applied to the system to bias it are in fact gradient forces at optimum.

David Sanders (National University of Mexico) reported on the development of efficient algorithms for simulating deterministic dynamics in a quenched (fixed) random environment of obstacles on a lattice, such as a Lorentz lattice gas or mirror model. The focus is on the low densities of obstacles for which a straightforward simulation of the dynamics becomes computationally prohibitive.

Nonequilibrium techniques for computing equilibrium properties

Rémi Joubaud (Imperial College London) showed how to use Langevin dynamics to compute the shear viscosity of a system. Such computations are often performed with deterministic dynamics using Nosé-Hoover or Gaussian thermostats. The interest of the Langevin dynamics is that its generator has a spectral gap, which allows to rigorously prove linear response results. In particular, following the seminal work of Irving and Kirkwood, it is possible to prove a conservation equation relating the variations of the stress tensor and of the average velocity. An important issue is the dependence of the transport coefficient on the parameters of the dynamics. Rémi Joubaud exposed some new results on the asymptotic behavior of the shear viscosity coefficient for large frictions, based on asymptotic analysis of an appropriate Poisson equation. These theoretical results were illustrated by numerical simulations of a bi-dimensional Lennard-Jones system.

The Langevin dynamics used by Rémi Joubaud can actually be seen as the limit of some mechanical model, as suggested by Frédéric Legoll (Ecole des Ponts). In fact, the proof is done for a single large particle, placed in an ideal gas heat bath composed of point particles that are distributed consistently with the background flow field and interact with the large particle through elastic collisions. In the limit of small bath atom masses, the large particle dynamics converges to a Langevin-type stochastic dynamics, which is parametrized by the background flow field. This derivation follows the ideas of D. Dürr, S. Goldstein and J. Lebowitz.

Mathias Rousset (INRIA Rocquencourt) showed how to use constrained, time-inhomogeneous Langevin dynamics to compute equilibrium free energy differences from nonequilibrium switching processes where the system is driven from one state to another. The method first requires a proper definition of the work function, and then a fluctuation identity (known as the Jarzynski-Crooks relation) relating the statistics of forward and backward switchings. Consistent numerical schemes allowing to evaluate the work without time-step error were also presented.

Using theoretical concepts of nonequilibrium statistical mechanics for numerical analysis

In his talk, Yannis Pantazis (University of Massachussetts) presented recent results allowing to quantify the degree of non-reversibility induced by the discretization of reversible Markov processes. Despite an extensive literature on the numerical analysis for SDE's, their stability properties, strong and/or weak error estimates, large deviations and infinite-time estimates, no quantitative results were known on the lack of reversibility of the discrete-time approximation process. Such quantitative estimates can be provided by the relative entropy production rate of the process, where forward and time-reversed trajectories are compared. Crucially, from a numerical point of view, the entropy production rate is an *a posteriori* quantity, hence it can be computed in the course of a simulation as the ergodic average of a certain functional of the process (the so-called Gallavotti-Cohen action functional). The method was illustrated for various numerical schemes for the overdamped and underdamped Langevin dynamics.

Scientific Progress Made

General knowledge dissemination to younger and more isolated researchers

Let us first mention that the workshop was an outstanding opportunity for many researchers to meet and work together: the community of researchers interested in fundamental issues of nonequilibrium systems is indeed a rather small one, and every occasion to efficiently interact with fellow colleagues is welcome. One of the participants (who has a position in Mexico) for instance mentioned that this was the first conference in two years where he really felt his topics fit in. Needless to say, he benefited from many inspiring discussions.

As organizers we also took care of inviting younger researchers, so that several PhD students and post-docs were present. We believe that they gained awareness on the issues and interesting open problems in the field, and that this meeting helped them to obtain a broader view on how their work fits in the general scientific stage.

Specific progress arising from discussions held at the workshop

Most of the lectures gave rise to passionate discussions during the coffee breaks or lunches and dinners. On top of these informal discussions whose long term impact is difficult to estimate, let us mention some illustrative examples of situations in which the diversity of the backgrounds of the researchers in the audience was beneficial.

Space-time scalings for thermal transport

During the presentation, or rather, the discussion session led by Lebowitz on thermal transport, an important issue was raised by Stefano Olla about the importance of space-time scalings in thermal transport studies. Indeed, many studies (especially numerical ones) focus on the study of properties in the nonequilibrium steady-state where the time scales have disappeared. A notable exception are the recent results by Ahbishek Dhar, who presented numerical simulations backing up an interpretation of the anomalous space-time scalings of energy transport using a simplified model based on Levy walks of energy carriers. The relevance of the study of space-time scalings arose several times during the conference, and is an important concern of more theoretically oriented researchers who have the feeling that the numerical validation of the prediction based on hydrodynamic limits is not as extensive as it should be.

Normal conductivity of one-dimensional atom chains?

On tuesday evening, several physicists and mathematicians gathered to discuss the recent numerical results obtained by Zhao et al. [17], which tend to show that the thermal conductivity of some one dimensional atom chains are finite provided the interaction potential is of FPU type, with a sufficient level of anharmonicity (parameter α sufficiently large). The discussion first focused on technical details of the simulations in order to critically assess the reliability of the result. The computations are currently being checked both by the italian group featuring Roberto Livi and Stefano Lepri, and by Ahbishek Dhar in India. Preliminary results however seem to confirm the surprising results of [17].

The second part of the discussion attempted to provide more convincing physical explanations than the one proposed in [17]. Unfortunately, no satisfactory understanding emerged in spite of the stimulating interactions among the participants – this problem nonetheless remains as a stimulating counter-example to some of the currently well-accepted theories accounting for the divergent thermal conductivity of one dimensional chains. The discussion was a perfect advertisement tribune for many people who were not aware of the results [17] before the workshop. There is no doubt that future work in this direction will be strongly influenced by the findings in this article.

Are some previous simulation data in contradiction with new theoretical results?

The presentation by Cédric Bernardin led to a discussion on the relationship between the new theoretical results [2], which seem to indicate that the divergence of the thermal conductivity in one dimensional systems perturbed by a stochastic exchange process preserving energy and momentum should not depend on the intensity of the exchange process; whereas previous related numerical studies [1, 9, 3] documented such a dependence.

One important point is that the theoretical study [2] is performed for an infinite system at equilibrium, at variance with numerical simulations performed on finite systems, either at equilibrium [1] or in a nonequilibrium setting [9, 3]. It is common folklore that there should be a simple relationship between the slow long-time tail decay of the autocorrelation of the current in the Green-Kubo formula (described by some power law decay) and the divergence of the thermal conductivity of open systems in their steady states. The argument is that the autocorrelation should be integrated over times of order N. There is however no clear mathematical result backing up this belief. Even worse, it even not clear from a physical viewpoint whether the scaling for the cut-off time in the Green-Kubo formula indeed is N in the presence of a strong noise (which tends to reduce the mean free path of non-interacting phonons) rather than a more general dependence N^{α}, with α depending on the noise strength.

During this discussion, a question arose: can it even be shown that the current vanishes in the thermodynamic limit in one-dimensional chains, as we expect? (except for the trivial case of harmonic chains with mass disorder for which it is known that the conductivity decreases exponentially with the system size) It seems that, although the steps in the proof of this statement are quite clear, there are still subtle mathematical obstructions to answering this quite natural and simple question. Needless to say, even less is known at the theoretical level about a possible scaling the thermal conductivity with the system size.

Outcome of the Meeting

Interactions among subgroups of participants

The feedback we received from workshop participants is overwhelmingly positive. The workshop aimed at gathering a mixed audience, composed of mathematicians, physicists and computer scientists, studying nonequilibrium systems from a theoretical viewpoint or through numerical simulations. Many researchers appreciated the diversity of the viewpoints of the audience. Answering questions about the numerical counterpart of your theoretical results is always refreshing for those more interested in theoretical aspects, while the researchers focusing on applications benefited both from the feedback from mathematicians developing a fundamental understanding of the various phenomena arising in nonequilibrium systems, and from applied mathematicians caring about the numerical accuracy and efficiency of the simulation techniques.

It is indeed our belief that there is a strong interplay between theoretical considerations, which may trigger numerical validations or extensions, and numerical simulations. It is often the case that some ad-hoc dynamics invented for a given application turns out to be of much broader interest; also, numerical experiments may motivate new theoretical results or investigations. We believe that gathering practitioners (physicists and mathematicians) from different fields and theoreticians (again, physicists and mathematicians) fostered new insights and cross-fertilization.

We mentioned in the proposal that one of the aims of this workshop was to motivate more work on the mathematical understanding of simulation techniques, for instance by suggesting some appropriate importance sampling or variance reduction methods for nonequilibrium dynamics. Although this very challenging topic has not been addressed during the presentations in the conference, there were many discussions between applied mathematicians and practitioners about possible work tracks in this direction, especially for the more demanding thermal transport simulations. These discussions seemed to motivate several applied mathematicians to carefully consider the development of more efficient numerical methods for the computation of properties of nonequilibrium systems, in particular transport coefficients. In addition, several PhD students or post-docs, working on theoretical aspects of statistical mechanics, expressed interest in learning techniques of numerical simulation in order to perform computations on their own. This would no doubt be a very efficient way of focusing more mathematical attention on the proper design of efficient numerical techniques rather than seeing simulations as some black-box tool to confirm theoretical results.

On a more ethical note

Let us finally mention the "concerned scientist" session led by Joel Lebowitz, who took opportunity of the presence of researchers from various backgrounds to spread awareness about human right violations in general (using a current brochure from Amnesty international), and situations involving scientists in particular. He presented more precisely two organizations aiming at defending the rights of scientists and supporting their families, namely the Committee of concerned scientists [5] (whom he is a co-chair of) and the Scholars at Risk Network [15].

Participants

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Bibliography

[1] G. Basile, L. Delfini, S. Lepri, R. Livi, S. Olla, and A. Politi, Anomalous transport and relaxation in classical one-dimensional models, *Eur. Phys. J. Special Topics* **151** (2007), 85–93.

[2] C. Bernardin and P. Gonçalves, Anomalous fluctuations for a perturbed Hamiltonian system with exponential interactions, *arXiv preprint* **1205.1879** (2012).

[3] C. Bernardin and G. Stoltz, Anomalous diffusion for a class of systems with two conserved quantities, *Nonlinearity* **25** (2012), 1099- 1133.

[4] F. Bonetto, J. L. Lebowitz, and L. Rey-Bellet, Fourier's law: a challenge for theorists. In *Mathematical Physics 2000* (A. Fokas, A. Grigoryan, T. Kibble, and B. Zegarlinsky, eds.), 128–151, Imperial College Press, 2000.

[5] Committee of Concerned Scientists, http://concernedscientists.org/

[6] B. Derrida, J. L. Lebowitz, and E. R. Speer, Large deviation of the density profile in the steady state of the open symmetric simple exclusion process, *J. Stat. Phys.* **107**(3/4) (2002), 599–634.

[7] A. Dhar, Heat transport in low-dimensional systems, *Adv. Phys.* **57** (2008), 457–537.

[8] J. B. Goodman and K. K. Lin, Coupling control variates for Markov chain Monte Carlo, *J. Comput. Phys.* **228** (2009), 7127–7136.

[9] A. Iacobucci, F. Legoll, S. Olla and G. Stoltz, Thermal conductivity of the Toda lattice with conservative noise, *J. Stat. Phys.* **140**(2) (2010), 336-348.

[10] C. Kipnis and C. Landim, *Scaling limits of interacting particle systems*, Springer, 1999.

[11] J. Lebowitz, Exact results in nonequilibrium statistical mechanics: Where do we stand?, *Suppl. Prog. Theor. Phys* **64** (1978), 35–49.

[12] S. Lepri, R. Livi, and A. Politi, Thermal conduction in classical low-dimensional lattices, *Phys. Rep.* **377** (2003), 1–80.

[13] S. Olla, Energy diffusion and superdiffusion in oscillators lattice networks. In *New Trends in Mathematical Physics* (V. Sidoravicius, ed.), 539–547, Springer, 2009.

[14] L. Rey-Bellet, Open classical systems. In *Open Quantum Systems II* (S. Attal, A. Joye, and C.-A. Pillet, eds.), *Lecture Notes in Mathematics* **1881**, 41–78, Springer, 2006.

[15] Scholars at Risk Network, http://scholarsatrisk.nyu.edu/

[16] H. T. Yau, Relative entropy and hydrodynamics of Ginzburg-Landau models, *Lett. Math. Phys.* **22**(1) (1991), 63–80.

[17] Y. Zhong, Y. Zhang, J. Wang, and H. Zhao, Normal heat conduction in one-dimensional momentum conserving lattices with asymmetric interactions, *Phys. Rev. E* **85**(6) (2012), 060102.

Chapter 39

Thin Liquid Films and Fluid Interfaces: Models, Experiments and Applications (12w5035)

December 9 - 14, 2012

Organizer(s): Robert Behringer (Duke University), Karen Daniels (North Carolina State University), Rachel Levy (Harvey Mudd College), Omar K. Matar (Imperial College London), Michael Shearer (North Carolina State University), Thomas Witelski (Duke University)

Overview of the Field

Broadly considered, fluid dynamics is one of the most longstanding and fundamental areas in the physical sciences. The very intricate and diverse phenomena observed in fluid systems have motivated a vast body of mathematical research seeking to analyze and predict these behaviors. Building on the classical Navier-Stokes equations, mathematical formulations of problems in fluid dynamics are comprised of statements for the conservation of mass and force balances on all of the components in the system. These mathematical models take the form of systems of coupled nonlinear partial differential equations. In problems where the masses of fluids being considered have a finite extent, appropriate boundary conditions must be imposed at the edges of the fluid volume. The primary focus of this workshop was problems where the fluid is free to move within the system and the domain occupied by the fluid must be determined as part of the solution of the problem. In such *free-surface* problems the governing system can often be reduced to equations for the evolution of the liquid-gas interface, as in the study of water waves on the surface of rivers or ponds described by the shallow water equations, in which slenderness of the fluid layer is employed to obtain an asymptotic model. For problems at small lengthscales, as in many naturally occurring phenomena (e.g. rain drops on lotus leaves) and the design of engineering materials and microfluidic devices, surface tension can play a strong role in determining the dynamics. The inclusion of surface tension (and related surface tension gradient effects) yields challenging models given by higher-order nonlinear partial differential equations.

This workshop brought together a broad array of researchers working on problems in fluids dynamics spanning experimental studies, materials science, basic physics, engineering design, biological applications, mathematical modeling and computations. The diverse and interdisciplinary participants of the workshop created a very open and interactive forum that addressed issues stemming from novel physical problems and engineering designs. These directions generally motivate improved mathematical formulations for the observed delicate fluid behaviors.

Recent Developments and Open Problems

While the foundations of lubrication theory for thin liquid films date back to the work of Reynolds in the late 1800's, the 1985 survey paper *"Wetting: statics and dynamics"* by P.G. de Gennes [26] was an important recent milestone in the study of the fluid dynamics of thin films. From a very physically oriented perspective, it reviewed the state of the experimental and theoretical understanding of the small-scale behaviors in spreading of liquid layers over solid substrates. De Gennes' paper highlighted results showing that the presence of capillary forces in simple models for advancing fluid fronts produced unphysical singularities in the stress at the moving contact line. Among the physical effects mentioned for consideration in resolving this open question were physical properties (topography and surface roughness) and chemical properties (van der Waals and other intermolecular forces) of the solid substrate, microscopic precursor fluid films, and slip effects. Among the further areas and open questions that de Gennes pointed to for further work were the spreading behaviors of complex fluids (mixtures and long-chain polymer melts) and the role of surfactant chemicals in dynamically modifying surface tension. Many questions in these directions are still open problems.

These fundamental physical questions regarding fluid dynamics have become even more pressing in recent times with the development of technologies that can precisely control surface properties of materials at micro- and nano-scales – these issues underpin problems like the design of water repellent materials, microfluidic devices for manipulating of tiny volumes of liquids, and better medical treatments for disorders involving physiological fluid layers (on the eye, or in the lung, for example). Careful mathematical analysis of these complicated systems has driven extensive studies of stability, dynamics, and asymptotics of partial differential equations, which in turn can identify important operating regimes for further experimental studies.

Following de Gennes, the next generation of research produced many advances on mathematical modeling and formulations of computationally tractable systems of equations. An extensive review was given in Oron [60] of models and results for lubrication models of flows of thin films of viscous fluids with special attention to van der Waals forces, surface tension, thermocapillary effects. In the same 1997 issue of *Reviews of Modern Physics*, Eggers [28] gave a comprehensive review of research on the break-up of free-surface fluid flows in free-space (i.e. rupture of liquid sheets and jets). This included a review of the classical stability theory given by Rayleigh as well as recent experimental and computational studies showing finite-time break-up of fluid filaments into droplets. Later studies showed many relations between break-up of fluids in free-space with the finite-time rupture of thin films coating solid substrates. The broad array of applications of lubrication models with surface tension effects was presented in [58]. Many of the fundamental issues of rigorous mathematical analysis of contact line dynamics and the associated weak solutions of the nonlinear thin film PDE were given in [11], where computational issues for contact lines and related regularized forms of the PDE were also examined.

The year 2009 brought more notable review articles of the field. Craster and Matar [23] surveyed the state of the art in mathematical models of thin film flows including thermal effects, surfactants, and models including evaporation or inertial effects. Special attention was given to intermolecular effects and other physical properties of the solid substrate (topography, elasticity, porosity). Connecting back to de Gennes' article, [12] gave an overview of modern developments on models and experimental studies of contact line dynamics. [30] surveyed the progress in the study of stability and break-up of fluid jets and sprays, and a closely related survey highlighted the role of self-similar solutions in representing the dynamics leading to finite-time singularities of nonlinear PDEs in fluid dynamics and other applications [29].

The above survey articles and the body of research overviewed by them form the background and general scope of research that the workshop contributed to. The 2012 workshop also builds on other BIRS workshops that have been run on thin film dynamics [1, 22], free-surface flows [39] and dynamics of complex fluids [33, 9].

Presentation Highlights

The format of the workshop was designed to provide for extensive time for questions following-up on research talks. These questions often led to very lively discussions on a wide range of mathematical, modeling, and experimental aspects. Further research connecting to various workshop themes was also presented in evening poster sessions. Below we briefly review some of the results covered in the workshop, grouped together by different aspects of the research along with useful related references.

Contact line dynamics

As described above, studies of moving contact line behavior in various system configurations are active areas of interest. David Sibley [64] presented a comprehensive comparison of different mathematical models for motion of contact lines, including slip effects, precursor films, diffusive interface equations and others. In the context of coating flows, Satish Kumar [70] illustrated the considerations needed to accurately model contact line dynamics that can allow for wetting failure and the entrainment of air between the film and the solid substrate. Gunter Auernhammer's work made use of an experimental system with a rotating cylindrical drum to examine the dynamics of the contact line and the influence of the drum's surface roughness and the presence of surfactant in the fluid [32, 31]. Daniel Herde described his work on numerical simulations of contact line dynamics and measurements of the contact angle in shear-driven flows over substrates with varying chemical (wetting) properties. Joshua Dijksman described his work on tracking the radially-collapsing contact line in a circular spinning bucket in a poster presentation [57, 56].

Influence of substrate properties

Work relating to the influence of substrate properties on the spreading of fluid layers spanned a wide range of directions from experimental physics to PDE analysis. Lou Kondic presented molecular dynamics computations that showed dramatic behaviors of nanoscale drops of liquid metals on dewetting substrates. Break-up of the drop was illustrated in different geometries – finite line segments and rings [59, 45, 37]. Karin Jacobs reviewed research on nanoscale polymer films on substrates with different surface properties. Good agreement on dewetting characteristics of the films in experiments has been obtained from analytic models that include slip effects [5, 8, 7, 4, 6, 34, 35]. Ralf Seemann carried out experimental studies of fluids filling grooves on solid substrates. The geometry of the fluid free-surface depends on the fluid volume and the groove shape (triangular or rectangular) and the fluid-solid contact angle [62, 42, 38, 43, 41, 10]. Electowetting was used to control the contact angle. This facilitated an exploration of whether the spreading is reversible. In some cases, break up into droplets was observed. When the substrate was an elastic solid, fluid capillary forces could deform groove walls and yield coupling to behavior in adjacent grooves. Joao Cabral described how elastic instabilities of film coatings could be used to control desired patterns down to very small lengthscales for improved microfabrication of structured substrates [18, 71]. Elastic deformation of the substrate played an even stronger role in the research of Joshua Bostwick, who considered the spreading of liquid droplets on soft gels. Capillary forces at the liquid-gel contact line can drive fracture of the gel. Elasticity theory applied to the gel can predict the rate of spreading of the drop and the instabilities that had been seen in earlier experiments [25].

Longer-scale substrate topography effects, namely flows on curved surfaces, were also considered. Yvonne Stokes talked about her work modeling and computing flows in helical channels [50]. The geometry of the channel was used to separate out particles in slurries. Stephen Wilson presented joint work with Brian Duffy on fluid rings on the outside of rotating horizontal cylinders. Colin Paterson presented further related work in a poster presentation [51, 27, 67]. Marina Chugunova described analytical techniques she has used for studying the thin films on rotating and fixed horizontal cylinders and how these approaches can also be applied to the fourth order Mullins equation for thermal grooving under surface diffusion [19, 16].

Complex fluids and surfactants

"Complex fluids" broadly refers to flows where more properties must be determined besides the local velocity field at each point. This can apply to non-Newtonian flows where a separate evolution equation is given for the stress field. But in the context of this workshop it describes: (i) nematic fluids, where a "director" molecular orientation vector field is present, and (ii) multi-component immiscible fluids. Linda Cummings presented a lubrication model for spreading nematic liquid crystal films with particular attention to the influence of the boundary conditions on the director field [52, 24]. Dimitri Papageorgiou described new results on the stability of the interfaces between three immiscible phases confined and flowing in a finite-width inclined channel. Dirk Peschka presented models (full Stokes flow and lubrication models with sharp contact lines vs. precursor films) and results from numerical simulations to capture the coupled spreading/draining of a drop of one fluid on a thin layer of another fluid [46]. Stephen Garoff described experimental studies on the spreading of surfactants over fluid mixtures having similar properties to biofluids in the lungs. While results imply that capillary effects are dominant, some aspects suggest that models must include effects like autophobing (finite spreading) [21, 44]. Further related results on autophobing of surfactants were presented in a poster by Ellen Swanson [63]. Rachel Levy presented a poster on intriguing experimental results in the complementary exterior problem on the spreading of a layer of surfactant spreading into a finite region of "clean" fluid film surface.

Uwe Thiele described an intriguing mathematical formulation for obtaining the evolution equations for complex fluids including two-layer flows [36, 61] and surfactant- or particle-ladened thin film flows as gradient dynamics based on an appropriate energy functional and mobility coefficient [68]

Fluid-structure interactions

While many of the studies described in the other sections involve the influence of contact with solids on the motion of a mass of fluid, we use the term "fluid-structure interaction" to describe systems where there is a significant coupling or feedback of the fluid dynamics to the structure or unsteady motion of the solid. Ofer Manor presented a poster on the instabilities that occur in fluid droplets subjected to acoustic forcing on vibrating solid substrates [54, 55, 20]. Stephen Wilson discussed the steady states for tilted rigid plates floating on thin film flows [17]. In a geophysical context, John Lister described the dynamics of *laccoliths* – intrusions of fluids (volcanic magma) between layers of deformable solids (sedimentary rock). Lister compared and contrasted the spreading behavior for the model, a lubrication flow coupled to an elastic membrane (yielding a sixth order PDE), against the contact line behaviors for the thin film equation and the second order porous medium equation for gravity currents. On a much smaller scale, Kara Maki described her work on modeling the elastic deformations of soft contact lenses and the influence of the tear film.

Jets, fluid sheets in free space, and draining flows

Several talks in the workshop considered the dynamics of interfacial flows in free-space. Burt Tilley presented results from continuing collaborative work with Mark Bowen on the rupture of a thin sheet of viscous fluid due to thermocapillary effects [13, 69]. Linda Smolka talked about her approach to the transient stability analysis of inertially driven thinning/expanding circular fluid sheets [65]. Both of these talks had interesting connections (through aspects of draining flows) to the very engaging talk by Stephen Davis on the complex dynamics of foams. Davis described the interaction of lamella sheets connected at Plateau border junctions forming dynamic networks that exhibit coarsening [66, 2, 3, 15]. Rouslan Krechetnikov talked about his continuing studies of the very unusual dynamics of droplet formation coming about as tip streaming from a larger pendant drop. Analysis of the Marangoni stresses and surface tension singularities accompanying droplet pinch-off incorporated fluid flow, chemical, and electrical effects [48, 47, 49].

Diffusion, evaporation, thermal, and other effects

Finally, other presentations had strong links to many of the previous areas (including complex fluids and contact lines) but also brought up different important fundamental effects to help model observed behaviors in different systems. Aaron Persad described very interesting results on thermal patterns and waves arising the the evaporation of sessile drops of volatile liquids [40]. Kara Maki presented other results on the influence of particle suspensions in evaporating droplets in a poster presentation [53]. While the influence of evaporation has received detailed attention in Richard Braun's extensive work on tear films coating the eye [14], at the workshop he considered the need to also include further analysis of heat transfer and osmolality from the eye to better describe the tear film [14]. Shilpa Khatri described her experimental studies of fluid droplets rising through layers of density-stratified fluids. While speeds of drops moving through homogeneous layers are well-understood, the influence of stratification interfaces can produce unexpected behaviors for rising drops and settling particles. Effects including fluid entrainment, convective mixing and time-scales for density diffusion can play important roles.

Outcome of the Meeting

The workshop was widely regarded as being successful in fostering a stimulating interaction across the broad range of research approaches being used in this area of fluid dynamics. The arrangement of talks given in the schedule could not predict or fully capture the interesting connections and interactions between the topics, participants, and problems that evolved during the meeting. While generally being strongly grounded in physical systems, all of the presentations were very suggestive of challenging problems for further mathematical modeling and analysis.

In addition to the notable talks and poster presentations mentioned in the previous sections, the workshop also included discussion periods drawing on the shared interests and expertise of the participants. Some of the topics considered in these discussions included: contact line hysteresis, contact lines for mixtures and with evaporation, modeling of wetting on rough surfaces, liquid lenses and instabilities of liquid bilayers, and formulating more general models for surfactant driven flows.

Participants

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Bibliography

[1] R. Almgren, R. P. Behringer, A. L. Bertozzi, M. C. Pugh, M. Shearer, and T. P. Witelski. Nonlinear dynamics of thin films and fluid interfaces. BIRS workshop, 03w5021, 2003.

[2] A. M. Anderson, L. N. Brush, and S. H. Davis. Foam mechanics: spontaneous rupture of thinning liquid films with Plateau borders. Journal of Fluid Mechanics, 658:63Ð88, 2010.

[3] A. M. Anderson and S. H. Davis. Solidification of free liquid films. Journal of Fluid Mechanics, 617:87Ð106, 2008.

[4] O. Baeumchen, R. Fetzer, and K. Jacobs. Reduced Interfacial Entanglement Density Affects the Boundary Conditions of Polymer Flow. Physical Review Letters, 103(24):247801, 2009.

[5] O. Baeumchen, R. Fetzer, M. Klos, M. Lessel, L. Marquant, H. Haehl, and K. Jacobs. Slippage and nanorheology of thin liquid polymer films. Journal of Physics-Condensed Matter, 24(32):325102, 2012.

[6] O.Baeumchen,R.Fetzer,A.Muench,B.Wagner,andK.Jacobs.ComprehensiveAnalysisofDewetting Profiles to Quantify Hydrodynamic Slip. In Ellero, M, editor, IUTAM Symposium on Advances in Micro- and Nanofluidics, volume 15, pages 51Ð65. IUTAM, 2009.

[7] O. Baeumchen and K. Jacobs. Can liquids slide? Linking stability and dynamics of thin liquid films to microscopic material properties. Soft Matter, 6(24):6028Ð6035, 2010.

[8] O. Baeumchen and K. Jacobs. Slip effects in polymer thin films. Journal of Physics-Condensed Matter, 22(3):033102, 2010.

[9] N. Balmforth and I. Frigaard. Visco-plastic fluids: from theory to application. BIRS workshop, 05w5028, 2005.

[10] J.-C.Baret,M.M.J.Decre,S.Herminghaus, and R.Seemann.Transport dynamics in open microfluidic grooves. Langmuir, 23(9):5200Ð5204, 2007.

[11] A. L. Bertozzi. The mathematics of moving contact lines in thin liquid films. Notices Amer. Math. Soc., 45(6):689Ð697, 1998.

[12] D. Bonn, J. Eggers, J. Indekeu, J. Meunier, and E. Rolley. Wetting and spreading. Reviews of Modern Physics, 81(2):739Ð805, 2009.

[13] M. Bowen and B. S. Tilley. Thermally induced van der Waals rupture of thin viscous fluid sheets. Physics of Fluids, 24(3):032106, 2012.

[14] R. J. Braun. Dynamics of the Tear Film. Annual Review of Fluid Mechanics, 44:267Ð297, 2012.

[15] L. N. Brush and S. H. Davis. A new law of thinning in foam dynamics. Journal of Fluid Mechanics, 534:227Ð236, 2005.

[16] A.Burchard, M.Chugunova, and B.K.Stephens. Convergence to Equilibrium for aThin-Film Equation on a Cylindrical Surface. Communications in Partial Differential Equations, 37(4):585Ð609, 2012.

[17] J. Q. Carou, S. K. Wilson, N. J. Mottram, and B. R. Duffy. Asymptotic and numerical analysis of a simple model for blade coating. Journal of Engineering Mathematics, 63(2-4):155Ð176, 2009.

[18] A. Chiche, C. M. Stafford, and J. T. Cabral. Complex micropatterning of periodic structures on elastomeric surfaces. Soft Matter, 4(12):2360Ð2364, 2008.

[19] M. Chugunova and R. Taranets. Existence and long-time behaviour of weak solutions for the Mullins equation. Nonlinear Analysis-Theory Methods & Applications, 75(18):6550Ð6561, 2012.

[20] D. J. Collins, O. Manor, A. Winkler, H. Schmidt, J. R. Friend, and L. Y. Yeo. Atomization off thin water films generated by high-frequency substrate wave vibrations. Physical Review E, 86(5):056312, 2012.

[21] T. E. Corcoran, K. M. Thomas, S. Garoff, R. D. Tilton, T. M. Przybycien, and J. M. Pilewski. Imaging the Postdeposition Dispersion of an Inhaled Surfactant Aerosol. Journal of Aerosol Medicine and Pulmonary Drug Delivery, 25(5):290Ð296, 2012.

[22] R. Craster, G. M. Homsy, and D. T. Papageorgiou. Small scale hydrodynamics: microfluidics and thin films. BIRS workshop, 10w5035, 2010.

[23] R. V. Craster and O. K. Matar. Dynamics and stability of thin liquid films. Reviews of Modern Physics, 81(3):1131Ð1198, 2009.

[24] L.J.Cummings,T.S.Lin,andL.Kondic.Modelingandsimulationsofthespreadinganddestabilization of nematic droplets. Physics of Fluids, 23(4):043102, 2011.

[25] K. E. Daniels, S. Mukhopadhyay, P. J. Houseworth, and R. P. Behringer. Instabilities in droplets spreading on gels. Physical Review Letters, 99(12):124501, 2007.

[26] P. G. deGennes. Wetting - statics and dynamics. Reviews of Modern Physics, 57(3):827Ð863, 1985.

[27] B. R. Duffy and S. K. Wilson. Thin-film and curtain flows on the outside of a rotating horizontal cylinder. Journal of Fluid Mechanics, 394:29Ð49, 1999.

[28] J Eggers. Nonlinear dynamics and breakup of free-surface flows. Reviews of Modern Physics, 69(3):865Ð929, 1997.

[29] J. Eggers and M. A. Fontelos. The role of self-similarity in singularities of partial differential equations. Nonlinearity, 22(1):R1ÐR44, 2009.

[30] J. Eggers and E. Villermaux. Physics of liquid jets. Reports on Progress in Physics, 71(3):036601, 2008.

[31] D. Fell, G. Auernhammer, E. Bonaccurso, C.-J. Liu, R. Sokuler, and H.-J. Butt. Influence of Surfactant Concentration and Background Salt on Forced Dynamic Wetting and Dewetting. Langmuir, 27(6):2112Ð2117, 2011.

[32] D. Fell, N. Pawanrat, E. Bonaccurso, H.-J. Butt, and G. K. Auernhammer. Influence of surfactant transport suppression on dynamic contact angle hysteresis. Colloid and Polymer Science, 291(2):361Ð 366, 2013.

[33] J. F. Feng and C. Liu. Interfacial dynamics in complex fluids. BIRS workshop, 06w5047, 2006.

[34] R. Fetzer and K. Jacobs. Slippage of newtonian liquids: Influence on the dynamics of dewetting thin films. Langmuir, 23(23):11617Ð11622, 2007.

[35] R. Fetzer, A. Muench, B. Wagner, M. Rauscher, and K. Jacobs. Hydrodynamic slip: A comprehensive analysis of dewetting profiles. Langmuir, 23(21):10559Ð10566, 2007.

[36] M. Geoghegan and G. Krausch. Wetting at polymer surfaces and interfaces. Progress in Polymer Science, 28(2):261Ð302, 2003.

[37] A. G. Gonzalez, J. A. Diez, and L. Kondic. Stability of a liquid ring on a substrate. Journal of Fluid Mechanics, 718:246Ð279, 2013.

[38] S. Herminghaus, M. Brinkmann, and R. Seemann. Wetting and dewetting of complex surface geometries. Annual Review of Materials Research, 38:101Ð121, 2008.

[39] H. Huang, R. M. Miura, D. T. Papageorgiou, and M. Siegel. Analysis, computations, and experiments on pinch-off in liquid jets. BIRS focused research group, 05frg060, 2005.

[40] G. Karapetsas, O. K. Matar, P. Valluri, and K. Sefiane. Convective Rolls and Hydrothermal Waves in Evaporating Sessile Drops. Langmuir, 28(31):11433Ð11439, 2012.

[41] K. Khare, M. Brinkmann, B. M. Law, E. L. Gurevich, S. Herminghaus, and R. Seemann. Dewetting of liquid filaments in wedge-shaped grooves. Langmuir, 23(24):12138Ð12141, 2007.

[42] K. Khare, M. Brinkmann, B. M. Law, S. Herminghaus, and R. Seemann. Switching wetting morphologies in triangular grooves. European Physical Journal-Special Topics, 166:151Ð154, 2009. 7th Biannual European Coating Symposium 2007, Paris, France.

[43] K. Khare, S. Herminghaus, J.-C. Baret, B. M. Law, M. Brinkmann, and R. Seemann. Switching liquid morphologies on linear grooves. Langmuir, 23(26):12997Ð13006, 2007.

[44] K. Koch, B. Dew, T. E. Corcoran, T. M. Przybycien, R. D. Tilton, and S. Garoff. Surface Tension Gradient Driven Spreading on Aqueous Mucin Solutions: A Possible Route to Enhanced Pulmonary Drug Delivery. Molecular Pharmaceutics, 8(2):387Ð394, 2011.

[45] L. Kondic, J. A. Diez, P. D. Rack, Y.-F. Guan, and J. D. Fowlkes. Nanoparticle assembly via the dewetting of patterned thin metal lines: Understanding the instability mechanisms. Physical Review E, 79:026302, 2009.

[46] K. Kostourou, D. Peschka, A. Muench, B. Wagner, S. Herminghaus, and R. Seemann. Interface morphologies in liquid/liquid dewetting. Chemical Engineering and Processing, 50(5-6, SI):531Ð536, 2011.

[47] R. Krechetnikov. Marangoni-driven singularities via mean-curvature flow. Journal of Physics A, 43(24):242001, 2010.

[48] R. Krechetnikov. Structure of Marangoni-driven singularities. Physics of Fluids, 24(2):022111, 2012.

[49] R. Krechetnikov and G. M. Homsy. On physical mechanisms in chemical reaction-driven tip-streaming. Physics of Fluids, 16(7):2556Ð2566, 2004.

[50] S.Y. Lee, Y. Stokes, and A. L. Bertozzi. A model for particle laden flow in a spiral concentrator. ICTAM XIII proceedings, 2012.

[51] G.A.Leslie,S.K.Wilson,and B.R.Duffy. Non-isothermal flow of a thin film of fluid with temperature-dependent viscosity on a stationary horizontal cylinder. Physics of Fluids, 23(6):062101, 2011.

[52] T. S. Lin, L. Kondic, and L. J. Cummings. Defect modeling in spreading nematic droplets. Physical Review E, 85(1):012702, 2012.

[53] K. L. Maki and S. Kumar. Fast Evaporation of Spreading Droplets of Colloidal Suspensions. Langmuir, 27(18):11347Ð11363, 2011.

[54] O. Manor, M. Dentry, J. R. Friend, and L. Y. Yeo. Substrate dependent drop deformation and wetting under high frequency vibration. Soft Matter, 7(18):7976Ð7979, 2011.

[55] O. Manor, L. Y. Yeo, and J. R. Friend. The appearance of boundary layers and drift flows due to high-frequency surface waves. Journal of Fluid Mechanics, 707:482Ð495, 2012.

[56] S.Mukhopadhyay and R.P.Behringer.Wetting dynamics of thin liquid films and drops under marangoni and centrifugal forces. Journal of Physics-Condensed Matter, 21(464123):1Ð9, 2009.

[57] S. Mukhopadhyay, N. Murisic, R. P. Behringer, and L. Kondic. Evolution of droplets of perfectly wetting liquid under the influence of thermocapillary forces. Physical Review E, 83(046302):1Ð8, 2011.

[58] T. G. Myers. Thin films with high surface tension. SIAM review, 40(3):441Ð462, 1998.

[59] T. D. Nguyen, M. Fuentes-Cabrera, J. D. Fowlkes, J. A. Diez, A. G. Gonzalez, L. Kondic, and P. D. Rack. Competition between Collapse and Breakup in Nanometer-Sized Thin Rings Using Molecular Dynamics and Continuum Modeling. Langmuir, 28(39):13960Ð13967, 2012.

[60] A. Oron, S. H. Davis, and S. G. Bankoff. Long-scale evolution of thin liquid films. Reviews of Modern Physics, 69(3):931Ð980, 1997.

[61] A. Pototsky, M. Bestehorn, D. Merkt, and U. Thiele. Alternative pathways of dewetting for a thin liquid two-layer film. Physical Review E, 70:025201, 2004.

[62] R.Seemann,M.Brinkmann,S.Herminghaus,K.Khare,B.M.Law,S.McBride,K.Kostourou,E.Gure- vich, S. Bommer, C. Herrmann, and D. Michler. Wetting morphologies and their transitions in grooved substrates. Journal of Physics-Condensed Matter, 23(18, SI):184108, 2011.

[63] R. Sharma, R. Kalita, E. R. Swanson, T. E. Corcoran, S. Garoff, T. M. Przybycien, and R. D. Tilton. Autophobing on Liquid Subphases Driven by the Interfacial Transport of Amphiphilic Molecules. Langmuir, 28(43):15212Ð15221, 2012.

[64] D. N. Sibley, N. Savva, and S. Kalliadasis. Slip or not slip? A methodical examination of the interface formation model using two-dimensional droplet spreading on a horizontal planar substrate as a prototype system. Physics of Fluids, 24(8):082105, 2012.

[65] L. B. Smolka and T. P. Witelski. On the planar extensional motion of an inertially driven liquid sheet. Physics of Fluids, 21(4):042101, 2009.

[66] P. S. Stewart and S. H. Davis. Dynamics and stability of metallic foams: Network modeling. Journal of Rheology, 56(3):543Ð574, 2012.

[67] J. M. Sullivan, C. Paterson, S. K. Wilson, and B. R. Duffy. A thin rivulet or ridge subject to a uniform transverse shear stress at its free surface due to an external airflow. Physics of Fluids, 24(8):082109, 2012.

[68] U. Thiele, A. J. Archer, and M. Plapp. Thermodynamically consistent description of the hydrodynamics of free surfaces covered by insoluble surfactants of high concentration. Physics of Fluids, 24(10):102107, 2012.

[69] B. S. Tilley and M Bowen. Thermocapillary control of rupture in thin viscous fluid sheets. Journal of Fluid Mechanics, 541:399Ð408, 2005.

[70] E. Vandre, M. S. Carvalho, and S. Kumar. Delaying the onset of dynamic wetting failure through meniscus confinement. Journal of Fluid Mechanics, 707:496Ð520, 2012.

[71] J. A. Warren, J. T. Cabral, and J. F. Douglas. Solution of a field theory model of frontal photopolymerization. Physical Review E, 72(2):021801, 2005.

Two-day Workshop Reports

Chapter 40

North West Functional Analysis Seminar (12w2163)

March 30 - April 1, 2012

Organizer(s): Dr. Martin Argerami (University of Regina), Dr. Marcelo Laca (University of Victoria), Dr. Michael Lamoureux (University of Calgary), Dr. Volker Runde (University of Alberta)

Overview of the Workshop

The Northwest Functional Analysis Seminar (NWFAS) is a bi-annual regional scientific meeting of academic researchers and graduate students in functional analysis. Participants are drawn from universities in Western Canada and the American Northwest. The two main goals of the meeting are to enable researchers from a large geographical area to stay in touch with developments in the general field but outside their respective areas of specialization, and to provide a forum for junior researchers (junior faculty, postdoctoral fellows, and graduate students) to present their results to a wider audience and to form contacts with other functional analysts within the region. The majority of talks are given by junior researchers, with a few presentations by senior researchers on topics currently attracting high levels of interest.

This two-day workshop is the fifth edition of the seminar, following very successful earlier meetings held in 2003, 2005, 2007 and 2009. In addition to presenting a strong scientific program, the workshop provides the only venue by which the region's researchers in functional analysis come into contact regularly. The NWFAS also has formative role in developing the profiles of early-career researchers. The session is organized around a dozen forty-minute talks, allowing time for informal discussion.

A PIMS Collaborative Research Group in Operator Algebras and Noncommutative Geometry led by researchers in Victoria, Edmonton, and Regina wrapped up its activities by the spring of 2012, and this session was an opportunity to discuss the results of the CRG.

Overview of the Field

Functional analysis is a very broad field, encompassing the analysis of general families of functions on spaces (topological, or measurable) and the algebras of operators that act on those functions. It is the basis for foundational results in a wide variety of fields of mathematics and its applications, including the existence and uniqueness of solutions to differential equations, the analysis of dynamical systems, harmonic analysis, mathematical physics, quantum mechanics, numerical methods, among others. Hilbert spaces, Banach spaces, Fourier spaces, group algebras, Banach algebras, C*-algebras, von Neumann algebras, are some of the main objects of study.

Topics of interest include the embedding and approximation properties for families of functions on topological or measurable spaces and groups, properties of individual and algebras of (linear) operators on these families of functions, classification and identification problems for these algebras, quantum extensions of classical notions in functional analysis.

Recent Developments and Open Problems

As a broad area, there are many open problems in functional analysis and its sub-disciplines. Some are very specific – for instance, when does a function come about as a restriction of some differentiable function (the Whitney problem), or when does a linear operator have an invariant subspace (the invariant subspace problem). Some are very general – for instance, what is a useful classification scheme for C*-algebras (e.g. the Elliott program). Some bring together diverse areas into some commonality – for instance, how do the invariants of a dynamical system correlate with the invariants of similar C*-algebras that encode the dynamics. A long standing problem has been to generalize the beautiful results of Fourier analysis on locally compact abelian groups to a wide class of groups and their quantum analogues.

Presentation Highlights

The highlight of the presentations was the wide diversity of talks delivered and the success of the young researchers to show their stuff. This meeting brought together researchers in harmonic analysis (who study the functional analysis of groups) with researchers in C*-algebras, others in quantum physics, and more in dynamical systems and even tilings. For the first time with the North West Functional Analysis Seminar, we were able to post videos of the talks online, through the BIRS web server, which allowed these talks to reach a much wider audience. The speakers were very accommodating in allowing their presentations to be recorded and broadcast.

Scientific Progress Made

The progress made here was in the communication between the diverse groups, helping to identify what expertise there is within Western Canada to collaborate on some of these challenges. For instance, Dr. Gilad Gour, a quantum physicist at the University of Calgary, gave his very first talk to a functional analysis crowd, and discovered that many of his problems in QM can be formulated, and perhaps solved, in the framework of C*-algebra, and in particular in the language of completely positive maps that represent quantum channels and other constructions in QM. He went on to reformulate his talk for a national mathematical meeting in the summer, based on his exposure to the audience at NWFAS. Dr. Antoine Julien gave an excellent presentation on substitution tilings, a geometric construction that leads to aperiodic structures such as Penrose tiles, which it turns out results in a spectral triple that is a important notion in linear operator theory. A large group of presentations focused on the harmonic analysis of groups: compactifications, factorization, amenability, trace formulas, among other. A very interesting presentation by a postdoctoral fellow, David Alonso-Gutierrez focused on the convex geometry of high dimensional bodies, which has important connections to the properties of linear operators in high, and infinite, dimensional linear spaces.

Outcome of the Meeting

Some of the discussions at the meeting have led to ongoing collaborations between research teams – for instance, many of the questions coming up in quantum communications and quantum cryptography are nicely phrased in the language of states, positive operators, completely positive maps, in C*-algebras. The Quantum group of physicists at Calgary that Dr. Gour works with is now interacting with the C*-algebraists to extend this language into mathematical tools that can attack some of these problems.

The communications also made clear the strengths of these groups of researchers in Western Canada – the strong groups in harmonic analysis and geometry at Alberta and Saskatchewan, quantum research at Alberta and Calgary, dynamics at Victoria, are some of the examples.

We left encouraged with the revealed strengths and looking forward to further interactions.

Participants

Alonso, David (University of Alberta)
Alzulaibani, Alaa (University of Regina)
Argerami, Martin (University of Regina)
Bhattacharya, Angshuman (University of Regina)
Brenken, Berndt (University of Calgary)
Brudnyi, Alex (University of Calgary)
Choi, Yemon (University of Saskatchewan)
Emerson, Heath (University of Victoria)
Erlijman, Juliana (University of Regina)
Floricel, Remus (University of Regina)
Gao, Niushan (University of Alberta)
Georgescu, Magdalena (University of Victoria)
Gheorghiu, Vlad (Institute for Quantum Computing)
Gour, Gilad (University of Calgary)
Guex, Sebastien (University of Alberta)
H£ynes, Siri-Malen (Norwegian University of Science and Technology)
Julien, Antoine (University of Victoria)
Kinzebulatov, Damir (University of Toronto)
Laca, Marcelo (University of Victoria)
Lamoureux, Michael (University of British Columbia)
Lau, Anthony To-Ming (University of Alberta)
Mazowita, Matthew (University of Alberta)
Phillips, John (University of Victoria)
Poulin, Denis (University of Alberta)
Prochno, Joscha (University of Alberta)
Putnam, Ian (University of Victoria)
Rivasplata, Omar (University of Alberta)
Runde, Volker (University of Alberta)
Samei, Ebrahim (University of Saskatchewan)
Sourour, Ahmed R. (University of Victoria)
Spektor, Susanna (University of Alberta)
Tahmasebi, Nazanin (University of Alberta)
Troitsky, Vladimir (University of Alberta)
Viselter, Ami (University of Alberta)
Zabeti, Omid (University of Alberta)

Chapter 41

Ted Lewis Workshop on SNAP Math Fairs in 2012 (12w2168)

April 27 - 29, 2012

Organizer(s): Tiina Hohn (MacEwan University), Ted Lewis (SNAP Mathematics Foundation), Andy Liu (University of Alberta)

This was the tenth year that a spring math fair workshop has been held at BIRS. The participants came from elementary schools, junior-high and high schools, from independent organizations, and from universities and colleges.

The purpose of the workshop was to bring together educators who are interested in using our particular type of math fair, called a SNAP math fair, to enhance the mathematics curriculum and introduce a culture of active mathematics in the school community. (The name SNAP is an acronym for the guiding principles of this unconventional type of math fair: It is student-centered, non-competitive, all-inclusive, and problem-based.) Unlike many "poster-session" science fairs, a SNAP math fair is highly interactive and the student presenters are very engaged with the visitors to the math fair. For more information, see http://www.mathfair.com

The purpose of these workshops is to bring together persons who wish to enhance mathematics in the K-12 classroom. The participants fall into several categories: teachers from K-12, university and college instructors, and puzzle makers and collectors. The BIRS math fair workshop included presentations by persons who have organized math fairs, puzzle sessions where teachers have an opportunity to try math fair puzzles, and suggestions about how to use games and math puzzles in the class room. Also we had an opportunity to discuss some helpful activities to accommodate the demands of the new curriculum. The workshop provided a rich interaction between the various groups.

The BIRS math fair workshops contribute greatly to the enriching the curriculum, and the workshops have helped encourage the use of SNAP math fairs around the world. The information about various resources available locally as well as access to information and research results on line are valuable tools for our schoolteachers.

Participants

Beltaos, Elaine (Grant MacEwan University)
Beltaos, Lillian (Teslacentral Enterprises)
Desaulniers, Shawn (University of British Columbia)
Finn, Carleen (Rocky View Schools)
Francis, Krista (University of Calgary)

Graves, Sean (University of Alberta)
Hoffman, Janice (Edmonton Public Schools)
Hohn, Tiina (MacEwan University)
Jones, Carolyn (Centre for Education)
Laporte, Cathy (Edmonton Catholic School Division)
Lewis, Ted (SNAP Mathematics Foundation)
Liu, Andy (University of Alberta)
Morrill, Ryan (University of Alberta)
Pasanen, Trevor (University of Alberta)
Taylor, Carla (St.Edmund School)
Thompson, Tanya (Mastermind Toys)

Chapter 42

Differential Schemes and Differential Cohomology (12w2151)

June 22 - 24, 2012

Organizer(s): R.C. Churchill (Hunter College and Graduate Center, City University of New York, and the University of Calgary), Y. Zhang, (University of Manitoba)

Overview

From a historical perspective Algebraic Geometry was initially concerned with the study of zero sets of polynomials. It was realized from the outset that the subject was closely related to Number Theory, and work during the middle third of the last century, e.g. by Zariski and Weil, suggested that a reformulation of the subject, so as to incorporate such related fields, was in order. Such a reformulation was soon accomplished by Grothendieck, and subsequently refined by Deligne and others. It seems somewhat of an understatement to assert that Grothendieck's ideas revolutionized the way one views Algebraic Geometry.

Differential Algebraic Geometry began, somewhat more recently, as the study of zero sets of differential polynomials, e.g. if K denotes the field $\mathbf{R}(\cos t, \sin t)$ with derivation $' = d/dt,$ then $(\cos t, \sin t) \in K^2$ is a zero of the differential polynomial $x_1^2 - x_1' x_2 - 1$ [11,12]. Although there have been significant attempts to reformulate the subject in the style of Grothendieck (e.g. see [4,5,6,10,12,13,15] and references therein), in particular so as to make use of those recently-developed techniques, these innovations have not been widely accepted. One reason for this, in the opinion of the organizers, is that many individuals who work in algebraic geometry and/or number theory are simply not familiar with differential algebra.

This meeting was conceived as an attempt to maintain a focus on, if not to alleviate, this problem. Several of the participants were selected specifically because they knew nothing about differential algebra, but were familiar with at least one of these two other areas.

Presentation Highlights

There were four talks. We give a summary of each, in the order presented.

1. Jim Freitag "Local Problems in Differential Algebra"

   The first part of the talk covered basic notions in differential algebraic geometry by extending the basics of algebraic geometry to the differential setting. Notions of dimension were then introduced, illustrated by several important examples and the end of the talk designed to indicate some of the power of model theory in differential algebraic geometry (e.g. see [14]).

2. Henri Gillet " Differential Algebra, Functors, and Arithmetic Geometry"

   The first part of the talk dealt with an application to deformations of a family of varieties that enables one to lift a derivation from the base space to the total space. This was done in characteristic 0 by means on an exponential map. The second part established an analogous result in characteristic p using formal group actions in place of derivations.

3. Andy Magid "Grothendieck Topology"

   The talk was an introduction, in outline form for a general audience, to Grothendieck topologies, abelian sheaves on such entities, and the Cech and derived functor cohomologies of the latter. The 0 and 1 Cech cohomology sets on non-abelian sheaves were also explained, as well as the connection to principal homogeneous spaces.

4. Ray Hoobler "Cohomology in a Differential Algebra Setting"

   The talk examined how a differential Azumaya algebra over an ordinary differential ring could be made locally isomorphic to a matrix ring with coordinate-wise differentiation. The basic idea was to use a Grothendieck topology to set up and locally solve a differential equation whose solutions provide a constant basis for the matrix ring. In this way the differential Azumaya algebras become local principal homogeneous spaces in the Grothendieck topology; well-known cohomological tools could then be used to identify the differential Brauer group with the ordinary Brauer group [7]. The partial case follows a similar pattern using results of Andre [1].

Outcome of the Workshop

There were several ongoing discussions outside of the main talks. They centered around questions such as:

- What kind of differential ring extension is analogous to a separable ring extension and can it be used to generate a useful Grothendieck topology?
- What do the "points" in the $\Delta-$flat topology look like?
- What kind of differential ring extensions have lifting properties analogous to Grothendieck's definition of a formally smooth extension?
- How would a differential algebraic space be defined?

Some of these issues continue to be discussed by email and may ultimately lead to full understanding and, possibly, publication.

Participants

Arreche, Carlos (CUNY-Graduate Center)
Bauer, Mark (University of Calgary)
Chipalkatti, Japdeep (University of Manitoba)
Churchill, Richard (The City Univesity of New York and Univesity of Calgary)
Cockett, Robin (University of Calgary)
Freitag, Jim (UIC)
Gillet, Henri (University of Illinois at Chicago)
Hoobler, Ray (CUNY)
Jardine, Rick (University of Western Ontario)
Juan, Lourdes (Texas Tech University)
Keigher, William (Rutgers-Newark)
Magid, Andy (University of Oklahoma)
Padmanabhan, R. (The University of Manitoba)
Sanabria, Camilo (Universidad de los Andes)

Scheidler, Renate (University of Calgary)
Sit, William (The City College of The City University of New York)
Sun, Yao (Key Laboratory of Mathematics Mechanization, Academy of Mathematics & System Science)
Wang, Dingkang (Acad Math/Sys)
Wechter, Matt (UIC)
Zhang, Yang (Univesity of Manitoba)

Bibliography

[1] Y. André, Différentielles non commutatives et théorie de Galois difféntielle ou aux différences, *Ann. Sci. École Norm. Sup.* (4) **34** (2001), no. 5, pp. 685-739.

[2] A. Buium, *Differential Function Fields and Moduli of Algebraic Varieties*, LNM 1226, Springer, New York, 1986.

[3] A. Buium, *Differential Algebraic Groups of Finite Dimension*, LNM 1506, Springer, New York, 1992.

[4] A. Buium, *Differential Algebra and Diophantine Geometry*, Hermann, Paris, 1994.

[5] G. Carrà Ferro, The ring of global sections of the structure sheaf on the differential spectrum, *Matematiche* (Catania) **33** (1978) no. 1, pp. 1-17 (1980) (Italian).

[6] H. Gillet, Differential Algebra - a scheme theory approach, in *Differential Algebra and Related Topics*, Proceedings of the International Workshop (Newark, November 2000), eds. L. Guo, P.J. Cassidy, W.F. Keigher & W.Y. Sit, World Scientific 2002, pp. 95-123.

[7] R. Hoobler, *The Differential Brauer Group* (revised version), posted on the website of the Kolchin Seminar on Differential Algebra, 12/09.

[8] L. Juan and A.R. Magid, Differential central simple algebras and Picard-Vessiot representations, *Proc. Amer. Math. Soc.* **136** (2008) no. 6, pp. 1911-1918.

[9] I. Kaplansky, *An Introduction to Differential Algebra*, Second Edition, Hermann, Paris, 1976.

[10] W.R. Keigher, Differential schemes and premodels of differential fields, *J. Algebra* **79** (1981), n. 2, pp. 37-50.

[11] E.R. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press, New York, 1973.

[12] J.J. Kovacic, Differential Schemes, in *Differential Algebra and Related Topics*, Proceedings of the International Workshop (Newark, November 2000), eds. L. Guo, P.J. Cassidy, W.F. Keigher & W.Y. Sit, World Scientific 2002, pp. 71-94.

[13] J.J. Kovacic, Global sections of diffspec, *J. Pure Appl. Algebra* **171** (2002), no. 2-3, pp. 265-288.

[14] T. Scanlon, Model theory and differential algebra, in *Differential Algebra and Related Topics*, Proceedings of the International Workshop (Newark, November 2000), eds. L. Guo, P.J. Cassidy, W.F. Keigher & W.Y. Sit, World Scientific 2002, pp. 125-50.

[15] D. Trushin, Global sections of structure sheaves of Keigher rings, *Proc. Amer. Math. Soc.* **140**, (2012), 505-511.

Chapter 43

Women's Workshop on Communications and Signal Processing (12w2147)

July 13 - 15, 2012

Organizer(s): Sheila Hemami (Cornell University), Urbashi Mitra (University of Southern California), Milica Stojanovic (Northeastern University), Sarah Kate Wilson (Santa Clara University)

Background

Women are underrepresented in Electrical Engineering at all levels. This fact has been documented in several places including [1, 2]. In particular, women are underrepresened at the tenured and full professor level in engineering [3]. This is also true for the subfields of Communications and Signal Processing where women make up slightly less than 7.5 % of the total membership. However, of the few women who are in these fields many are very successful.

While there is no single way to increase the percentage of women in these research areas; we believe targeted mentoring and networking can help retain and increase the number of women in leadership positions. The IEEE Signal Processing and Communications societies have long recognized the need to integrate junior members of these societies into society leadership. A common strategy employed to achieve this goal is the modestly sized workshop where junior researchers can interact and learn from more senior researchers. Such workshops foster the mentoring and networking that can make a difference in a career. Our ultimate goal is to create a more diverse and vibrant research community in signal processing and communications.

Overview of the Workshop

There have been many workshops devoted to women in engineering, but they have mainly focused on the mechanics of being a woman in engineering. For example, they concentrate on gender differences, how women can advance, work-life balance, children, etc.

The organization Networking Networking Women (N2Women) [4] has several workshops targeting women in communications networks. They also have a workshop that is technical in nature and co-located with an existing IEEE/ACM conference, InfoCom. Our goal was also a technical workshop but one where women could bond outside the technical discussions. As such, we envisioned a stand-alone workshop.

Structure of the Workshop

As one of the goals of the workshop was to promote networking among junior women researchers and with more senior women researchers, we wanted to ensure that we had a close, friendly environment. We also wanted to focus on the technical rather than the social issues involved with women in communications and signal processing. Technical talks were a key part of the workshop for several reasons:

- The participants are technical people, not sociologists.
- We wanted to emphasize that women can be successful technical contributors.
- We wanted a forum where women, both junior and senior, can find new avenues for research and collaboration.

Funding

To make this first workshop attractive and affordable we solicited and received travel funds from the United States Office of Naval Research, IEEE Communications Society and IEEE Signal Processing Society.

Recruitment

We had two categories of participants: women with more established careers who would be the speakers and women just beginning their academic careers. For the more established women speakers, we asked several prominent researchers to participate. The selected speakers provided a comprehensive range of topics on communications and signal processing. For the more junior participants, we advertised via flyers at conferences, mailings to targeted research groups and advertisements on society websites and social media sites. We requested from each applicant a two-page abstract of their poster, a curriculum vita and the names of two references. We received 29 applications and accepted 24 applicants. Five of the accepted applicants could not attend due to visa or other issues yielding 19 untenured women participants in the fields of communications and signal processing. They ranged in experience from near the completion of their Ph.D. program to a few years into a position as an assistant professor.

This group's makeup was international. We had one participant each from India, the United Kingdom, Spain, Tunisia and Colombia. In addition, we had two from Canada and the rest from the United States. A group photo is shown in Figure 43.1.

Social aspect

Fostering networking involves making good personal as well as technical connections. The workshop had the following social networking aspects.

Friday evening we held an informal session in the Corbett Hall residence lounge. The focus of the session was open-ended and the participants, both junior and senior researchers, were sitting in a round table format. Discussions included:

- The challenges of having and raising children at different stages of one's career.
- Choosing an academic versus an industrial career.
- Methods and issues associated with moving from a career as a Ph.D. in industry to an academic career.
- How a Ph.D. can both broaden and limit career choices.

A workshop registration fee was charged to each participant of 150 U.S. Dollars. This helped offset the cost of meals at the Banff Center as well as a banquet dinner Saturday night.

We also planned a group hike Saturday afternoon which was well-attended.

On Sunday afternoon, the participants met in the TransCanada Pipeline Pavilion to discuss the workshop and what we should plan for the future.

Figure 43.1: Group photo from the Women's Workshop

Technical Activities

The workshop began with a morning poster session; the remainder of the workshop comprised 35 minute presentations from the more senior participants.

The poster session had 17 presenters. The topics covered ranged from video compression to using communications system to model Parkinson's disease. Before the poster session, each presenter gave a one-minute 'elevator' pitch about their work. The goal was for each presenter to crystalize what the work was about, why it was important and why we should go see the work.

A listing of the posters is presented in Appendix 43 and the workshop talks in Appendix 43.

Outcomes of the Meeting and Lessons Learned

One of the initial goals of the meeting was to introduce younger pre-tenured women to post-tenured women to initiate mentoring and networking. An additional benefit of the all-women workshop, was that participants felt more relaxed presenting their results than they would in a mixed environment. The perception was perhaps that they did not have to prove themselves in the same way they would in front of a more predominantly-male workshop.

We found also that the meeting stimulated new research avenues for both the senior and junior attendees and created new research networks among all the participants. Several of the participants have identified paths for future joint research.

At the wrap-up discussion, one of the participants said that she came to the workshop to meet more established women researchers and to learn from them; but she realized that an equally important network was the younger women she met at the workshop.

Because the talks and posters were of a technical nature, many of the participants came away with new research ideas. Three of the more senior researchers discussed a future research project focusing on signal processing and whale sounds. Comments about the workshop included:

- "[My postdoc] was extremely happy about [the workshop], and came back with a lot of very interesting ideas for research and collaborations. So, at least from her perspective (and mine!), it was a great success."

- "This indeed was a unique event in that we are such a small group and yet we managed to cover such a wide range of topics. Being a small all-women group also allowed to create an informal atmosphere which made it very easy to approach professors and talk with them. To me it was truly inspiring to meet all these successful women. "
- "It's such a great pleasure attending the first Women's workshop in Banff, and I certainly gained a lot experience in working in academic field as a female!"
- "My participation in the BIRS workshop allowed me to make important contacts with people from my research area, as well as to be able to get significant insights of their current areas of research which provides me new interesting ideas for my job. Also the contacts I made during the workshop may help me through my career with postdoc opportunities or joint projects."

The general consensus from the group is that they would like to have another women's workshop on communications and signal processing in two years. Every year would be too often, but a workshop held every two years would be soon enough to provide some continuity.

For future workshops, the following suggestions were made:

- Rather than have a single time slot for a poster session, have two poster sessions with half the presenters at each. This allows more freedom for people to see the work of other presenters.
- Continue to have the workshop in a nature-oriented location like Banff where participants stay focused on the workshop without the distractions of a larger venue.
- Extend the workshop an additional day.
- Have break-out sessions where participants can brainstorm possible collaborations.
- Have presentations that are more tutorial in nature rather than new results.
- Advertise in more venues and earlier to increase recruitment.

Posters

- **Zahra Ahmadian** (University of British Columbia)
  Pre-Rake DS-UWB System Design - an Overview
- **Nasim Arianpoo** (University of British Columbia)
  Network Coding in Wireless Mesh Networks
- **Yue Chen** (Queen Mary University of London)
  Cooperative User Relay Assisted Load Balancing in LTE Networks
- **Arsenia Chorti** (Princeton University)
  Physical Layer Security in Wireless Networks with Active Eavesdroppers
- **Raja Ghozi** (ENIT, Tunis)
  Elderly Altered Auditory Perception in Urban Spaces
- **Cristina Gomez Santamaria** (Universita Pontifica Bolivariana-Colombia)
  Combining Eigenbeamforming and OSTBC in a MU Macrocell scenario with Partial CSITx
- **Sumana Gupta** (IIT Kanpur)
  A Novel Technique for Color Video Compression

- **Julie Jackson** (AFIT)
  Exploitation of OFDM Communications for Passive Radar Imaging

- **Victoria Kostina** (Princeton University)
  Lossy joint source-channel coding in the finite blocklength regime

- **Abbie Kressner** (George Tech)
  Causal Locally Competitive Algorithm for the sparse decomposition of audio signals

- **Yao Li** (Rutgers University)
  Enhancing Throughput-Complexity Tradeoff in Coded Content Distribution

- **Sandra Roger** (Technical University of Valencia)
  Rapid Prototyping of MIMO Detectors Using Graphic Processing Units

- **Neveen Shlayaan** (University of Nevada, Las Vegas)
  The Ill-posed Inverse Radon Problemin Neutron Tomography

- **Samantha Summerson** (Rice University)
  ParkinsonâĂŹs Disease: Interference in the Neural Communications Channel

- **Vanessa Testoni** (University of California, San Diego)
  The Hierarchical Signal Depedent Transform: A Framework for Creating Orthonormal Basis Matching the Local Signal Characteristics

- **Preetha Thulasiraman** (US Naval Postgraduate School)
  Interference Aware Resource Allocation Using Multiobjective Optimization for Mobile Wireless Networks

- **Laura Toni** (University of California, San Diego)
  Channel Coding Optimization Based on Slice Visibility for Transmission of Compressed Video over OFDM Channels

- **Hongmie Xie** (Lehigh University)
  Distributed Storage Codes Based on Evaluation of Linearized Polynomials

Workshop Talks

- S. Aissa, (University of Quebec) *Is cooperation a Must in Future Cognitive radio networks?*

- P. Cosman, (University of California, San Diego) *Subcarrier Mapping Based on Slice Visibility for Video Transmission over OFDM Channels*

- M. Effros (California Institute of Technology) *Reduction as a Route to a Computational Information Theory*

- S. Kishore, (Lehigh University) *Smart Electricity Systems and the Role of Communications Engineering*

- M. Ostendorf, (University of Washington) *Human Language: a Signal Processing Perspective*

- M. Stojanovic, (Northeastern University) *OFDM over Rapidly Varying Channels Partial FFT Demodulation and its Application to Underwater Acoustic Channels*

- S.K. Wilson, (Santa Clara University) *Blinded by the Light: OFDM and Optical Wireless Communications*

- S. Wood, (Santa Clara University)*Computational Imaging Challenges*

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Chen, Yue (Queen Mary University of London)
Chorti, Arsenia (Princeton University)
Cosman, Pam (University of California- San Diego)
Effros, Michelle (California Institue of Technology)
Ghozi, Raja (ENIT- Tunis)
Gomez Santamaria, Cristina (Universita Pontifica Bolivariana-Colombia)
Gupta, Sumana (IIT Kanpur)
Jackson, Julie (Air Force Institute of Technology)
Kishore, Shalinee (Lehigh University)
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Bibliography

[1] B.L Yoder, Engineering by the Numbers, *Profiles of Engineering and Engineering Technology Colleges*, American Society of Engineering Education (ASEE), June 2012

www.asee.org/papers-and-publications/publications/college-profiles/2011-profile-engineer

[2] E. Guizzo, The EE Gender Gap is Widening, In *IEEE Spectrum*, December 2008

http://spectrum.ieee.org/computing/software/the-ee-gender-gap-is-widening

[3] Committee on Gender Differences in Careers of Science, Engineering and Mathematics Faculty, In *Gender Differences at Critical Transitions in the Careers of Science, Engineering and Mathematics Faculty*, National Academies Press, 2010.

[4] http://cms.comsoc.org/eprise/main/SiteGen/n2women/Content/Home.html

Chapter 44

Inductive Constructions for Rigidity Theory (12w2181)

July 20 - 22, 2012

Organizer(s): Walter Whiteley, York University Toronto, Bill Jackson, Queen Mary, University of London, England, Tibor Jordán, Eötvös University, Budapest, Hungary, Brigitte Servatius, Worcester Polytechnic Institute, Mass, USA, Meera Sitharam, University of Florida, Gainesville, FL, USA

Overview of the Field

The rigidity and flexibility of a structure, either man-made in buildings, linkages, and lightweight deployable forms, or found in nature ranging from crystals to proteins, is critical to the form, function, and stability of the structure. The mathematical theory of 'rigidity and flexibility' is developing methods for the analysis and design of man-made structures, as well as predictions of the behavior of natural structures such as proteins. We live in 3-dimensions, and a fundamental problem is to develop results for 3-dimensions which are as good and as efficient as the recently developed theory for structures in 2-dimensions.

One of the key ways to build examples and prove general results is an inductive construction: a sequence of local steps that build all possible structures from a few simple starting examples. Since at least the classic book of Henneberg [**?**], inductive constructions for infinitesimal rigidity of structures have played a key role in combinatorial characterizations of graphs supporting infinitesimal rigidity and independence of structures [24], [**?**]. More recently key results in global rigidity of structures were proved using key inductive constructions [**?**, **?**].

Recent Developments and Open Problems

Recent work from a new generation of contributors to rigidity theory have developed new, refined inductive constructions for rigidity and independence of new classes of frameworks, under symmetry and periodicity conditions, for specialized families of examples, as well as for broader problems in CAD constraints.

Some examples are:

- the recent Ph.D. Thesis of Tony Nixon on inductive construction of frameworks on surfaces [**?**] and the ongoing work on inductive classes (including current investigations of Nixon [**?**, **?**], Ross, Theran and Malestin);

- the thesis of Elissa Ross on inductive construction of fixed lattice isostatic periodic frameworks in the plane [?],
- Ross's more recent complete characterization of which gains yield generically isostatic bar-body periodic frameworks on any fixed lattice in 3-space;
- some recent work of Schulze on classes of symmetric frameworks [?, ?];
- the recent work of Lee-St.John on CAD Constraints in 3-D [?];
- the paper of Connelly, Jordán and Whiteley on global rigidity of redundant body-bar frameworks in all dimensions [?];
- ongoing work of Connelly, Whiteley, and others, on when vertex splitting in 3-D (and higher) preserves global rigidity;
- the inductive proof of the molecular conjecture and extensions by Katoh and Tanigawa [?];
- work of Finbow-Singh and Whiteley on inductive constructions of Block and Hole polyhedra, and possible extensions [?];
- recent and ongoing work of Cheng and Sitharam related to 3-D bar-and-joint rigidity; on constructing graphs (or locating maximal subgraphs) satisfying weaker notions of independence [?];
- paper and ongoing work of Cheng, Sitharam and Streinu related to 3D bar-and-joint rigidity; on constructions of specialized families of independent graphs [?]
- work of Berg, Jackson and Jordán on global rigidity of frameworks in 2-D [?, ?, ?];
- work of Jackson, Jordán, Whiteley, Servatius and Nguyen on global rigidity of mixed direction/length frameworks in 2-D [?, ?, ?, ?, ?].

As mentioned above, recent results for symmetry generic frameworks, and periodically generic frameworks have generated extended inductive techniques, and some of these results have potential extensions and applications to the study of protein structures with symmetry (such as dimers) or repetitive structures such as beta sheets or crystals.

For these generic results (and some geometric results), inductive constructions have provided full combinatorial characterizations, sometimes as alternatives to non-inductive techniques, and sometimes as the initial proofs which may later find non-inductive proofs. Recently, inductive proofs have been particularly prominent in the study of 3D body-bar frameworks, and the solution of the long standing molecular conjecture, proving key results in all dimensions.

This leaves the central, 100 year old problem of extensions to the bar and joint frameworks in 3-D. Some of the above results are steps in this direction.

Presentation Highlights

There was a survey talk by Tony Nixon `http://www.birs.ca/events/2012/2-day-workshops/12w2181/videos/watch/201207201407-Nixon.mp4` There was another long talk by Bill Jackson about the obstacles to proving that X-replacement is an inductive construction step in the 3-D bar-joint case. In addition, there were 5-7 minute talks by all participants, effectively moderated by one of the organizers, Tibor Jord'an. These talks offered a glimpse of the upcoming presentations, or posed an open problem. These can be found at: `http://www.birs.ca/events/2012/2-day-workshops/12w2181/videos/watch/201207201535-Whiteley.mp4`

- Oleg Karpenkov posed questions about inductive constructions for tensegrity structures, useful since the number of geometric conditions for the different regions of the configuration space explode.

- Meera Sitharam asked for inductive construction of $(3,6)$-sparse graphs, which can be shown to be equivalent to construction of 3D generalized body-hingestructures of appropriate sparsity, where many bodies could share a hinge and hinges could share points. It has been shown that for maintaining sparsity, in d = 2,3 dimensions, the average number of hinges per body is strictly smaller than $d+1$. This was used to show that all maximal $(3,6)$-sparse subgraphs provide an upper bound on the rank of the bar-joint rigidity matroid. For $d = 2$ this bound is tight, but the conjecture is that it is far from tight for higher dimensions, i.e, the average number of hinges is no more than 3, in 3D (Sitharam) and there is at least 1 body with at most 2 hinges in any dimension (Jackson). Inductive constructions will help.
- Bill Jackson talked about free submechanisms of mechanisms. Specifically he conjectured that for any submechanism K of a mechanism G, there is a generic framework (G,p) where K is free.
- John Owen talked about inductive operations for graphs that ensure that the Galois group of the original graph is a subgroup of the Galois group of the augmented graph (after applying the inductive operation). Specifically, he argued that this does not seem to hold for the Henneberg 2 and X-replacement moves.
- Jialong Cheng gave a new inductive construction maintaining independence (and isostaticity) of 3D bar-joint graphs, and also for constructing 3D bar-joint circuits. These inductive constructions have the additional feature that they maintain "nucleation-free" property, which is the second obstacle in obtaining a combinatorial characterization of 3D-rigidity: this method helps prove, understand and extend some of the construction schemes given by Tay [**?**].
- Bernd Schulze talked about the need for inductive constructions to control the number of cases for proving characterizations of isostatic incidentally symmetric frameworks. Specifically he gave the example of his theorem on C_S (single reflection) incidentally symmetric isostatic frameworks in 2D.
- Audrey Lee-St John asked for inductive constructions on bi-colored graphs that are expressed as specific types of unions of spanning trees.
- Steve Power talked about Generalized Periodic Rigidity Matrices with function entries. Starting with motif edges, he was interested in inductive constructions to control the determinant and the RUM spectrum.
- Viktoria Kazanitzky asked about characterizing absolutely 2-rigid graphs after defining them and showing some basic properties and counterexamples to various attempts at characterization.
- Herman Servatius talked about 2-sums of matroids, frameworks and circuits. This led to matroid decompositions of rigidity matroids into non-graphic, non-rigidity matroids. He also talked about such decompositons for universally rigid graphs.
- Csaba Király talked about balanced generic circuits without long paths and gave interesting ways of constructing them, together with many examples.
- Wendy Finbow-Singh and Walter Whiteley described inductive constructions on modified triangulated surfaces [**?**].
- Laura Chavez Lomeli talked about tree partictions for 2D circuits and inductive constructions of circuits by splitting and gluing.
- Walter Whiteley talked about inductive constructions for periodic structures on fixed lattices, specifically "coatings" on "substrates." He talked about inductive constructions both for the plane case and for body bar.
- Shinichi Tanigawa talked about symmetry-forced generic rigidity for the dihedral group. He had a characterization using inductive constructions and one case that was the minimal counterexample to the inductive constructions, namely the double cycle. He asked for the correct sparsity condition.
- Viet Han Nguyen talked about operations preserving rigidity and global rigidity of direction-length frameworks. She gave results and posed open questions concerning specific operations such 1 extensions on direction edges.

Scientific Progress Made and Outcome of the Meeting

Many collaborative groups were formed during the workshop. Some groups have reported progress on projects whose origin, direction or momentum can be traced back to the workshop.

- Steve Power reports: "the workshop facilitated collaboration with John Owen and Tony Nixon and we have just submitted a joint paper to a research journal, with improved results and inductive techniques. The paper was recently put on the ArXiv:

  A. Nixon, J.C. Owen and S.C. Power, A Laman theorem for frameworks on surfaces of revolution, `http://arxiv.org/abs/1210.7073`, (2012).

  Also I learnt of a number of directions from the international researchers which I expect to impact on my future work (such as global rigidity and universal rigidity) and my joint work with my new postdoc, Dr. Derek Kitson."

- Louis Theran reports: "Tony Nixon and I have been working on the question of generic rigidity of frameworks supported by surfaces with no isometries. An inductive approach seems promising.

  Also, Audrey Lee-St John and I have been working on some things relating to body-CAD and matroids. Bernd Schulze and I have also been looking a bit at generic incidental symmetry in the plane."

- Bill Jackson reports: " Viet Hang Nguyen and myself began discussing the rigidity of d-dimensional body-direction-length frameworks in Banff. Hang is currently visiting me in london to continue this research. we have used a recursive construction to characterise rigidity in the cases when the bodies are either rigid or direction rigid and are now working on the case when the bodies are length rigid."

- Brigitte Servatius reports: "Bill Jackson asked a question related to this one: It is true that a 1-extension preserves the degree of freedom of a bar-and joint framework, infinitesimally and generically, but is it true that for a generic realization of a mechanism the operation of 1-extension may be performed without restricting the motion?

  Brigitte and Herman Servatius worked out a counterexample to this question. This example is mentioned in a recent article of Jackson and Jordán [**?**]. Brigitte and Herman Servatius are writing up a short paper giving not just a counterexample but a more general answer to the question."

- Walter Whiteley reports: "Wendy Finbow-Singh and I have applied the inductive constructions (see Presentation Highlights) to extend results resolving Kuiper's Conjecture [83] to give the proof that any triangulated sphere with one added edge which forms a 4-connected graph, then the graph is a generic circuit in 3-space."

Participants

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Chapter 45

Theoretical and Applied Aspects of Nonnnegative Matrices (12w2156)

July 27 - 29, 2012

Organizer(s): Shaun Fallat (University of Regina), Daniel Szyld (Temple University), Michael Tsatsomeros (Washington State University), Pauline van den Driessche (University of Victoria)

The workshop brought together young and experienced researchers who study nonnegative matrix theory and its applications. The speakers at the workshop presented recent progress, open problems and challenges involving nonnegative matrices and their generalizations. Specifically, discussed were eventually nonnegative matrices; combinatorial aspects of nonnegative matrix theory and its interplay with graph theory; numerical issues and applications in optimization and the solution of matrix equations. As desired and expected, new ideas, suggestions and alternative points of view were raised by the participants.

Presentation Highlights

The workshop opened with three overview talks, covering theory, numerical aspects and applications of nonnegative matrices. The subjects of interest were matrices possessing various types of nonnegativity properties: entrywise nonnegative matrices, eventually nonnegative matrices, completely positive matrices, and totally nonnegative matrices. Also discussed were co-positive matrices, P-matrices and positive semi-definite matrices.

Bryan Shader opened the workshop with an Olympics inspired talk. He presented a forward-looking survey of some of the ways combinatorics is using nonnegative matrix theory, including the chip-firing and rotor-routing models on directed graphs [5, 13]. A summary of relations between analytic and algebraic properties of a nonnnegative matrix with concepts of the digraph associated with the matrix was presented. He also linked various generalizations of the notion of primitivity in nonnegative matrix theory, which provide an avenue for the combinatorial analysis of products of nonnegative matrices [10, 11].

In **Leslie Hogben's**[1] and in **Craig Erickson's** talks, past and recent results on eventual nonnegativity and positivity were surveyed and discussed; see [6, 8, 9, 12]. The biggest challenge in this area remains to be the lack of a characterization of eventually nonnegative matrices that would lend itself to an applicable test. Indeed, unlike eventual positivity, the fulfilment of the so called weak Perron-Frobenius conditions is necessary but not sufficient.

Chun-Hua Guo introduced us to Algebraic Riccati Equations under the condition that the coefficients form a block matrix that is an M-matrix. These types of equations are inspired by transport theory and Markov

[1]Leslie Hogben dedicated her talk to the late Uri Rothblum for his many and important contributions to nonnegative matrix theory.

models. There are challenges in this area pertaining to the design and analysis of solution algorithms, which will surely require the use of nonnegative matrix theory.

There were also participants who brought a different perspective to the workshop through their particular interests. **Pierre Maréchal** presented some of his recent work on minimizing the condition number among all matrices in a compact subset of the positive semi-definite matrices. He showed that many classical applied problems admit such a formulation. The results can be applied under additional nonnegativity constraints, which made this presentation intriguing to those working in the theory and numerics of nonnegative matrices. **Shawn Wang** [1] applied nonnegative matrix theory to provide a rigorous proof for the convergence of a Gauss-Seidel type fixed point method that relies on nonnegative matrix products.

Colin Garnett told us all about ISBN as an error correcting code. Treating the ISBN numbers as a quotient lattice and using the Smith Normal Form, he described (and calculated the probability of solving accurately) the problem of generating a basis for the lattice from a given number of lattice points.

More open problems

The following open problems and ideas for future research were also identified and discussed.

Wayne Barrett presented and proved some fundamental and very elegant upper and lower bounds for the spectral radius of an adjacency matrix. These bounds linked the spectral radius directly to graphical constants such as the number of edges, the clique number, etc. Interestingly, the comparison of such lower and upper bounds provide new and non-trivial (proofs of) inequalities among these graphical quantities. The speaker then suggested the use of this technique for the discovery of more such relations, in the general case or in special cases of graphs. This idea exemplifies the interplay between matrix and graph theory which was prevalent throughout the workshop.

Naomi Shaked-Monderer talked about completely positive matrices. She explained why the maximum cp-rank is obtained not only in the interior of the cone of such matrices but also on the boundary. It is hoped that this new discovery will assist in obtaining a sharp upper bound for the cp-rank of $n \times n$ completely positive matrices and possibly resolve the conjecture that $\lfloor \frac{n^2}{4} \rfloor$ is the desired bound [2]. It was noted that the new result was recently used in the proof of this conjecture for $n = 5$.

Daniel Szyld [3, 4] connected positive and eventually positive matrices to properties of their spectral projections, and posed the challenge of characterizing an eventually nonnegative matrix A in terms of its spectral projection P and a decomposition of the type $A = \rho(A)P + Q$. During his talk on eventual nonnegativity and on matrices satisfying the Perron-Frobenius property, Daniel Szyld mentioned two auxiliary results regarding the mapping of one or two nonnegative vectors into the interior of the nonnegative orthant by an orthogonal matrix that is arbitrarily close to the identity. He asked whether these results can be generalized e.g., to three vectors in $\mathbb{R}^3$. That is, given nonnegative vectors u, v, w in $\mathbb{R}^3$, is there always an orthogonal matrix Q so that, for any given $\epsilon > 0$,

1. $Qu, Qv, Qw \in \operatorname{int} \mathbb{R}^3$
2. $\|Q - I\| < \epsilon$?

The following day Bryan Shader described a viable idea for producing a counterexample, taking into account that such an orthogonal matrix cannot be a reflection and thus can only be a rotation in $\mathbb{R}^3$. Finally, in Daniel Szyld's talk, questions regarding the shape of the cone of eventually nonnegative matrices came up and were discussed by the participants.

Shahla Nasserasr presented Garloff's conjecture on the checker-board interval of invertible totally nonnegative matrices (see [7]), and suggested a particular approach to resolving the conjecture. It was noted that results in interval regularity and interval P-matrices may be relevant.

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Participants

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Bibliography

[1] Heinz H. Bauschke, Joshua Sarada, and Xianfu Wang, On moving averages, 2012 (available at http://arxiv.org/pdf/1206.3610).

[2] J.H. Drew, C.R. Johnson, and R. Loewy, Completely positive matrices associated with M-matrices, *Linear and Multilinear Algebra*, **37** (1994), 303–310.

[3] Abed Elhashash and Daniel B. Szyld, On general matrices having the Perron-Frobenius property, *Electronic Journal of Linear Algebra*, **17** (2008), 389–413.

[4] Abed Elhashash and Daniel B. Szyld, Two characterizations of matrices with the Perron-Frobenius property, *Numerical Linear Algebra with Applications*, **16** (2009), 863–869.

[5] Alexander E. Holroyd, Lionel Levine, Karola Meszaros, Yuval Peres, James Propp, David B. Wilson, Chip-Firing and Rotor-Routing on Directed Graphs, *Progress in Probability*, **60** (2008), 331–364.

[6] L. Hogben. Eventually cyclic matrices and a test for strong eventual nonnegativity. *Electronic Journal of Linear Algebra*, **19** (2010), 129–140.

[7] Shaun M. Fallat and Charles R. Johnson, Totally Nonnegative Matrices, Princeton University Press, 2011.

[8] S. Carnochan Naqvi and J.J. McDonald, Eventually nonnegative matrices are similar to seminonnegative matrices, *Linear Algebra and its Applications*, **381** (2004), 245–258.

[9] D. Noutsos and M. Tsatsomeros, Reachability and holdability of nonnegative states, *SIAM Journal on Matrix Analysis and Applications*, **30** (2008), 700–712.

[10] D.D. Olesky, B. L. Shader, and P. van den Driessche, Exponents of tuples of nonnegative matrices, *Linear Algebra and its Applications*, **356** (2002), 123–134.

[11] V. Yu. Protasov and A.S. Voynov, Sets of nonnegative matrices without positive products, *Linear Algebra and its Applications*, **437** (2012), 749–765.

[12] U. G. Rothblum, Nonnegative matrices and stochastic matrices. In *Handbook of Linear Algebra*, L. Hogben, Editor, Chapman & Hall/CRC Press, Boca Raton, 2007.

[13] Mikhail Volkov, Synchronizing Automata and the Cerny Conjecture, *Proc. 2nd Int'l Conf. Language and Automata Theory and Applications*, LNCS, **5196** (2008), 11–27.

Chapter 46

Math Kangaroo Contest Workshop (12w2184)

August 3 - 5, 2012

Organizer(s): Rossitza Marinova (Concordia University College of Alberta), Valeria Pandelieva (Statistics Canada)

The Second Math Kangaroo Contest workshop event was generously supported by BIRS and PIMS. BIRS provided the venue for the workshop, PIMS supported the Math Kangaroo Contest with $2000 for travel and meals of participants. Contest representatives and volunteers from the Greater Toronto Area, Ottawa, Halifax, Sudbury, Calgary, Edmonton, and New Orleans attended the workshop. Also, there were participants from IEEE Northern Canada and Southern Alberta sections, CMS, Math Pickle and the University of Lethbridge. In total, the number of in-person participants was 21. In addition, three participants attended sessions via skype conference calls.

Overview of the Field

Math Kangaroo is an annual international math competition for school children. This is the world's largest math competition, with more than six million participants worldwide. The main purpose of Math Kangaroo is to introduce participants to math challenges in an enjoyable way, thus, inspiring their further interest and advancement in mathematics. It has the potential to provide each student with a great and valuable experience in competitive math.

The competition consists of multiple-choice questions, arranged in increasing difficulty. In Canada, the 2012 competition took place on March 25 and was offered to students in grades 2-12. The involvement of the youngest group of students in second grade was a pilot project for this year. The 2012 Canadian edition of the competition was administered in Ottawa, the Greater Toronto Area, Edmonton, Calgary, Montreal, Halifax, St. John's, Winnipeg, Sudbury, North Bay, and Langley. Almost 1800 students participated in the contest, and hundreds were involved in various training and learning activities prior to the contest day.

The Canadian Math Kangaroo program contributes to the science, engineering and education communities through its activities that revolve around the contest but go far beyond its organization.

Recent Developments and Open Problems

The objectives of the workshop were for the participants to share experiences and have discussions on issues related to the Math Kangaroo program, namely:

1. Organizing training sessions and the use of appropriate training material

   Since training is a part of the contest preparation, Math Kangaroo representatives have organized practice classes. The development of appropriate materials assists regions with offering training sessions.

2. Running math clubs and groups

   Training sessions provide opportunities for school children to explore and expand their math and logical skills in a non-competitive environment. Running math clubs/circles became popular during the last years. The math outreach educational materials developed through Canadian Math Kangaroo help educators and parents to reach each student and address individual strengths and learning styles. There is an increased interest in training materials, consequently, an increased need to make these materials more accessible.

3. How to financially support activities

   Financially supporting activities is important for centres if they wish to deliver more extensive training and organize long award ceremonies.

4. Pre-contest activities, contest day, and post-contest activities

   Sharing experience on running the above activities always help to improve the organization of the competition.

5. Expanding the competition locally and nationally

   Expanding the competition locally and nationally is a continuous process. The partnership with the Canadian Mathematical Society plays an important role in the expansion. There is a mutual interest to continue the collaboration between CMS and the Canadian Math Kangaroo contest in future. The effort put into the registration system and the contest-related support activities from the CMS staff exceeded the preliminary expectations and it was worth it. The system is functioning, the updates and improvements will require significantly less effort, and it provides a good base for handling the expected increased numbers of centres and participants in future.

6. Informing the community about the Math Kangaroo program by hosting workshops and seminars for teachers and educators

   It is necessary to inform the community about the Math Kangaroo program by hosting workshops and seminars for teachers and educators. Some universities offer public lectures for parents and relatives during the competition. Math Kangaroo volunteers make presentations at conferences and meetings on the program activities.

7. Get together national and local coordinators as well as meet new coordinators and help them in the organization of the contest in their city/school

   The workshop in BIRS offered a great opportunity to get together national and local coordinators as well as meet new coordinators in order to help them in the organization of the contest in their city/school.

8. Sending winners in Europe to participate in math camps

   Sending winners in Europe to participate in math camps or/and organizing math camps in USA and Canada are objectives of Math Kangaroo that need solutions.

9. Organizing math camps in USA and Canada for winners from USA, Canada, Mexico, France, Romania, Bulgaria, Poland, and others

   This is a long-term goal of CMKC, which requires planning and cooperation with other organizations.

The Math Kangaroo contest is unique to Canada. Almost all other competitions run through schools. Students can participate in Math Kangaroo independently of their home school's involvement, typically at universities. It is still one of the very few math contests available for Canadian elementary students. While the reputation, the merit, and the quality of inspired learning are at a very high level, the atmosphere on the contest day is unique compared to most of the other contests.

Presentation Highlights

The workshop consisted of several presentations with discussions on topics of interest to the organization and the workshop participants.

- **Valeria Pandelieva**: *Priorities of the Math Kangaroo contest in Canada*. The presentation started with a brief history of the Canadian Math Kangaroo and its relationship with other similar organizations including the international organization "Kangaroo without Borders." The workshop participants learned about the international organization and the ways the competition is organized in other countries. It was pointed out that the 2012 edition of the contest was successful with more then a 35% increase in participants compared to 2011 including more than 1000 students receiving training in 2012. Also, it highlighted the priorities of the organization approved by the Board of Directors in June 2012.

- **Josey Hitesman and Kathy Huyng**: *Administration of Math Kangaroo: legal matters associated with children; training and contest events; awards presentation; advertising; fund-raising; website; reporting; feedback*. Although relevant materials have been provided to all representatives through the administrator portal, they need to be updated and new materials should be developed.

- **Rossitza Marinova**: *Math enrichment activities: contest training; clubs; circles; camps*.

  The purpose of the contest training classes is to introduce students to the contest environment and the type of Math Kangaroo problems. It is a problem solving class, where students also may learn new mathematical concepts; circles / clubs have been offered in some Math Kangaroo centres and the material can go beyond the standard curriculum. The atmosphere is relaxed rather than formal and students are encouraged to share ideas and solve problems together. International Math Kangaroo camps provide a unique experience in an international environment and there have been several attempts for organizing (or at least participating) in regional summer math camps.

- **Gordon Hamilton** from Math Pickle [1] talked about math enrichment programs and demonstrated a sample lesson on unsolved problems for third grade students.

- **Anis Haque, M. Sherman**: *IEEE and Mathematics*. This presentation consisted of an overview of the organization IEEE and its Teachers In-Service Program (TISP). Two lesson plans were discussed. A demonstration on how binary system can be used to build circuits was made.

- **Tchavdar Marinov**: *Introduction to LaTeX*. This presentation introduced participants to LaTeX, including how to install and start with the editing system. Examples were provided for illustration.

- **Sophie Chrysostomou** from Toronto and **Mariya Svishchuk** from Calgary each gave 15 minutes presentations regarding the competition in their city.

- **Johan Rudnick**, the Executive Director of the Canadian Mathematical Society (CMS) gave an hour talk on the role of CMS and its partnership with Canadian Math Kangaroo Contest (CMK). CMS and CMK have worked together since 2011 and intend to continue their collaboration.

Meeting Progress Made

As part of the workshop, the Canadian Math Kangaroo Contest held its Annual General Meeting on Sunday, August 5th. The 2012 Annual report was presented by Valeria Pandelieva and it stressed again on the priorities for the next two years. The 2012 Financial report was presented by Rossitza Marinova. A discussion on the two reports took place and the reports were approved. A new Board of Directors was elected, as follows: Valeria Pandelieva, Rossitza Marinova, Pamela Brittain.

The two-days BIRS workshop facilitated discussions and decisions on how to further improve the organization of the Math Kangaroo program. In particular, this includes contributions from all members and interested workshop participants to the priority areas and activities as follows: transition and future collaboration with CMS; website improvements and automated marking; development of forms, templates, promotional materials; corporate policies; translation in French.

Extensive discussion took place on how to do *marking and calculating the contest results*. Indeed, the Website and its functionality are important for assisting administrators.

Outcome of the Meeting

The workshop is another significant milestone for the Canadian Math Kangaroo Contest organization. Representatives from various cities and provinces exchanged ideas and discussed issues. The major meeting outcomes include:

- Increased the visibility of the contest and its accompanying activities.
- Connected volunteers to inspire and help them show K-12 students how fascinating mathematics is.
- Stated, discussed and approved the priorities of the program for next few years.
- Elected the new Board of Directors during the Annual General Meeting of CMKC.
- Expanding the partnership with the CMS.
- Formed groups to create forms for volunteers, photograph consent, liability issues involving children on campus, centre reports, etc.

Sharing information and ideas is crucial for maintaining a program of such scope, diversity, quality and continuity. The BIRS workshop facilitated efficient collaboration, coordination, and knowledge transfer among the Math Kangaroo volunteers and partners. New coordinators learned how to run the Math Kangaroo contest and related activities.

Participants

Bhandari, Ganesh (Mount Royal University)
Chrysostomou, Sophie (University of Toronto Scarborough)
Halvorsen, John (Concordia University College of Alberta)
Hamilton, Gordon (MathPickle)
Haque, Anis (University of Calgary)
Hitesman, Josey (Concordia University College of Alberta)
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Kharaghani, Hadi (University of Lethbridge)
Krishnamurthy, Ashok (Mount Royal University)
Liu, Claire (University of Calgary)
Marinov, Tchavdar (Southern University at New Orleans)
Marinova, Rossitza (Concordia University of Edmonton)
Melcher, Jonathan (University of Alberta)
Murray, Lois (Dalhousie University)
Pandeliev, Todor (AVG Technologies)
Pandelieva, Valeria (Canadian Math Kangaroo Contest)
Petterson, Keelan (Silvercrest Contracting Inc.)
Rudnick, Johan (Canadian Mathematical Society)
Sherman, Mooney (IEEE Northern Canada)
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Bibliography

[1] http://mathpickle.com

Focused Research Group Reports

Chapter 47

Novel Approaches to the Finite Simple Groups (12frg158)

April 22 - April 29, 2012

Organizer(s): John McKay (Concordia University), Roland Friedrich (Humboldt University Berlin)

Motivation

It is now accepted that the classification of finite simple groups (CFSG) is complete. Our purpose is in constructions which we re-examine in the light of observations that suggest that a substantial amount of insight is to be gained if we pursue novel connections to other, apparently disparate, areas of mathematics. This may lead to a natural home for the 26 sporadic simple groups which today are considered the complement of the Lie-Chevalley groups.

Overview of the Field

In order to understand the importance of the new ideas and conjectures linked with Monstrous Moonshine (MM) that we set out to develop, we have to understand its history, see [15, 19, 20]. Although the Monster is the most spectacular of the sporadic groups, it is just one of 26 groups under our wing. We discern three major evolutionary periods linked with the Monster group, the early and foundational one [5, 6, 7, 9, 10, 11, 17, 18, 22, 23], the time when the fruits of previous labour could be picked, culminating in Borcherds' Fields medal rewarded proof of the conjecture [2], and finally the post Moonshine one, where it is becoming evident that we are not dealing with one particular isolated phenomenon but rather with a (possible) new class of objects of which MM is the most prominent representative so far, and its new mathematical connections [16]. An electronic collection of relevant articles for the subject has been posted at: http://www.fields.utoronto.ca/~jplazas/MoonshineRoadmap.html

Recent Developments and Open Problems

During the focused research group various parts of this picture became clearer. In particular the following lines of work connect our approach with some of the topics discussed from the perspective of other fields:

We give a list of open problems:

1. What is the relation between $M, 2B, 3F'_{24}$ and $E8, E7$, and $E6$ respectively?
2. What does adjacency signify when the affine Dynkin nodes are identified with moonshine functions?
3. What is the significance of $27AB$, two classes which restrict to the same MM function?
4. The (column) rank of the MM functions is 163 $(= 194 - 22 - 9)$. $h(\sqrt{-163}) = 1$. Is this significant?
5. Characterise the sign pattern of replicable functions when q-coefficients are replaced by their signs in $\{0, \pm 1\}$.
6. From Conway-Norton we note there are 360 cusps arising from the genus zero Riemann surfaces attached to the (classes of) cyclic subgroups of M. Are these identifiable with the 120 tritangent planes of $W(E8)$?

Presentation Highlights

Prof. John McKay from Concordia University initiated the meeting by reporting on recent developments in the field, presenting a series of open problems. Then he formulated several conjectures and explained how they would give rise to major new research directions (see the latter parts of this report). As part of the preparation of the meeting, some of McKay's ideas and problems were circulated amongst the participants before the meeting, and some of the subsequent talks referred directly to this program.

Prof. Matilde Marcolli from Caltech gave two presentations entitled: "Multiplicative genera for noncommutative manifolds?", http://www.its.caltech.edu/~matilde/NCManifoldsModular.pdf
and "From CFT to CFT", http://www.its.caltech.edu/~matilde/CFTtoCFT.pdf

In his first lecture, Prof. Doran, from the University of Alberta, explained "the connection between $K3$ surfaces of high Picard rank and (moonshine) modular functions, starting from the "mirror moonshine conjecture" of Lian-Yau, through more recent results on modular parametrisation in several variables. Special attention was paid to the master equation for modular parametrisation, a differential equation whose algebraic solutions are the classical modular equations for modular curves."

In his second lecture, Prof. Doran described "recent work relating these same $K3$ surfaces of high Picard rank to compactified fibres in Landau-Ginzburg models mirror to smooth Fano threefolds. The modular curves related to moonshine are closely linked to Fano threefolds of Picard rank one."

Prof. Jack Morava from Johns Hopkins University spoke on the elliptic cohomology approach, in particular as presented by N. Ganter [16], to generalised Moonshine and Hecke operators [21].

Dr. Jorge Plazas, from the Fields Institute in Toronto, gave a talk entitled "Non-commutative spaces as a vessel for moonshine phenomena", which he summarised as: "The study of finite simple groups has led recently to considerations touching upon a wide spectrum of areas. The underlying richness of the theory is clear from fascinating and puzzling phenomena at the core of which lies monstrous moonshine.

Many of the prevalent structures arising in this context find a natural common framework in non-commutative geometry. Non-commutative spaces of Q-lattices and modular Hecke algebras encode various of these structures. We need to understand the connection between these spaces and finite simple groups."

Scientific Progress Made

In order to best utilise the time in Banff, the organisers collected relevant articles and related questions for the workshop and sent them to the participants beforehand. In the style of a French "groupe de travail", they were asked to prepare a didactic presentation connecting the reading material with their personal mathematical expertise and perspective and to comment on the linked questions.

As an effect of the early preparation of all attendees, several conjectures previously made, in particular by J. McKay, could in fact be turned into concrete research plans during the time in Banff, one example being the application of non-commutative geometry and the Bost-Connes system to Monstrous Moonshine and replicability.

Also, it became clear that Faber polynomials, which are also linked to replicability [1, 8, 14, 19, 20, 22], have a connection with Witt vectors, λ-rings and their associated Adams operations and Hopf algebras [12, 13]. These new results strongly support earlier ideas made about the role Q-lattices should have for moonshine, as replicability can be understood in terms of Adams operations arising in the context of the structure of λ-rings. On the other hand, the understanding of λ-ring structures associated to non-commutative spaces of Q-lattices is fundamental [3].

According to ideas of Marcolli and Plazas, the action of the monster Lie algebra on the above spaces might be feasible once the position of the Virasoro algebra in relation to the Hopf algebra of codimension one foliations of Connes and Moscovici [4] is better understood.

By using the Hopf algebra of codimension one foliations they intend to express differential identities for principal moduli in a manner reminiscent of related work in mirror symmetry, which could be connected with moduli of $K3$ surfaces.

Outcome of the Meeting

The first success of the meeting was to bring together different mathematicians whose research interests are connected with the sporadic groups, and in particular the Monster, working separately, in order to create an initial network, which other people are invited to join, and to set-up a shared knowledge base. The list of participants included the organisers, J. McKay and R. Friedrich, further in alphabetic order, Ch. Doran, L. Hesselholt, M. Marcolli, J. Morava, M. Laca and J. Plazas. R. Donagi could not attend but was in contact.

Second, as a result of the gathering, three concrete research plans emerged, which are now further pursued. Namely:

Elliptic cohomology, integrable systems and replicability J. Morava, N. Ganter, L. Hesselholt

Univalent functions, Sato-Segal-Wilson Grassmannians and conformal field theory R. Friedrich, J. McKay

Non-commutative geometry and Bost-Connes systems Ch. Doran, M. Marcolli and J. Plazas.

During the BIRS workshop, Ch. Doran, J. Plazas and M. Marcoli began a collaboration on the application of methods of A. Connes et al. to the master equation and its generalisations.

Finally, during the week a thematically related article was completed by R. Friedrich and J. McKay [13].

Disclaimer

This report is partially based on the written résumés of the participants, with their permission granted to use and modify their texts by the organisers.

Participants

Doran, Charles (University of Alberta, Canada)
Friedrich, Roland (Humboldt-UniversitŁt zu Berlin)
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Laca, Marcelo (University of Victoria)
Marcolli, Matilde (California Institute of Technology)
McKay, John (Concordia University)
Morava, Jack (Johns Hopkins University)
Plazas, Jorge (Concordia University)

Bibliography

[1] D. Alexander, C. Cummins, J. McKay, C. Simons, *Completely replicable functions* Groups, combinatorics & geometry (Durham, 1990), 87-98, London Math. Soc. Lecture Note Ser., **165**, Cambridge Univ. Press, Cambridge, 1992.

[2] R. E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992), no. 2, 405-444.

[3] A. Connes, M. Marcolli, *Q-lattices: quantum statistical mechanics and Galois theory*, J. Geom. Phys. **56** (2006), no. 1, 2Ð23.

[4] A. Connes, H. Moscovici, *Modular Hecke algebras and their Hopf symmetry*, Mosc. Math. J. 4 (2004), no. **1**, 67Ð109, 310.

[5] J. Conway, *A simple construction for the Fischer-Griess monster group*, Invent. Math. **79** (1985), no. 3, 513-540.

[6] J. Conway, *Three lectures on exceptional groups. In Sphere packings, Lattices and Groups (Conway and Sloane)*, Third edition. Springer-Verlag, New York, 1999.

[7] J. H. Conway, *Understanding groups like $\Gamma_0(N)$. Groups, difference sets, and the Monster* (Columbus, OH, 1993), 327-343, Ohio State Univ. Math. Res. Inst. Publ., 4, de Gruyter, Berlin, 1996.

[8] J. Conway, J. McKay, A. Sebbar, Abdellah, *On the discrete groups of Moonshine*, Proc. Amer. Math. Soc., **132**, 2004, 8, 2233-2240.

[9] J. Conway, S. Norton, *Monstrous moonshine* Bull. London Math. Soc. **11** (1979), no. 3, 308-339.

[10] C. J. Cummins, S. P. Norton, *Rational Hauptmoduls are replicable*, Canad. J. Math. **47** (1995), no. 6, 1201-1218.

[11] I. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics, **134**, Academic Press, Inc., Boston, MA, 1988.

[12] R. Friedrich, J. McKay, *Free Probability and Complex Cobordism*, C. R. Math. Rep. Acad. Sci. Canada Vol. **33** (4), pp. 116-122 (2011)

[13] R. Friedrich, J. McKay, *Formal Groups, Witt vectors and Free Probability*, arXiv-math, (2012)

[14] D. Ford, J. McKay, S. Norton, *More on replicable functions*, Comm. Algebra **22** (1994), no. 13, 5175-5193.

[15] T. Gannon, *Moonshine beyond the Monster: The Bridge Connecting Algebra, Modular Forms and Physics*, (Cambridge Monographs on Mathematical Physics), Cambridge University Press (2006)

[16] N. Ganter, *Hecke operators in equivariant elliptic cohomology and generalized Moonshine*, Groups and symmetries, 173, **209**, CRM Proc. Lecture Notes, 47, Amer. Math. Soc., Providence, RI, 2009.

[17] R. Griess, *The friendly giant*, Invent. Math. **69**, (1982), no. 1, 1-102.

[18] R. Griess, *Twelve sporadic groups*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[19] J. McKay, A. Sebbar, *Replicable functions: an introduction*, Frontiers in number theory, physics, and geometry. II, 373-386, Springer, Berlin, 2007.

[20] J. McKay, A. Sebbar, *Fuchsian groups, automorphic functions and Schwarzians*, Math. Ann. 318 (2000), no. 2, 255-275.

[21] J. Morava, *Moonshine Elements in Elliptic Cohomology*, in Groups and Symmetries: From Neolithic Scots to John McKay (CRM Proceedings & Lecture Notes), Vol. **47**, AMS (2009), 247-257.

[22] S. P. Norton, *Non-Monstrous Moonshine. Groups, difference sets, and the Monster* (Columbus, OH, 1993), 433-441, Ohio State Univ. Math. Res. Inst. Publ., **4**, de Gruyter, Berlin, 1996

[23] J. Tits, *On R. Griess' "friendly giant"*, Invent. Math. **78** (1984), no. 3, 491-499.

Chapter 48

Geometrization of Smooth Characters (12frg163)

May 6 - 13, 2012

Organizer(s): Pramod Achar (Louisiana State University), Clifton Cunningham (University of Calgary), Masoud Kamgarpour (Hausdorff Institute, Bonn), David Roe (PIMS Calgary and Washington), Hadi Salmasian (University of Ottawa)

During the first two weeks in July 2011, at the Mathematisches Forschungsinstitut Oberwolfach (MFO), a group consisting of Pramod Achar, Clifton Cunningham, Masoud Kamgarpour and Hadi Salmasian worked on a geometric approach to the local Langlands correspondence as it pertains to algebraic groups over p-adic fields. The same research group, with one addition, David Roe, met for one week in May 2012 at the Banff International Research Station (BIRS) to continue this project. This report describes the work done by this group at both the MFO in July 2011 and at BIRS in May 2012.

Introduction

We seek to replace the basic ingredients of both sides of the local Langlands correspondence with geometric avatars (in this case, perverse sheaves) and then bring techniques from algebraic geometry to bear on the correspondence itself. We hope, in the process, to see how to make local Langlands correspondence more categorical. The main results we have established thus far are explained (but not proved) in this report, in sections corresponding to the four points below.

Throughout this report, F denotes a local non-Archimedean field with residue field $\mathbb{F}_q$ and $\bar{F}$ denotes a fixed separable closure of F. Let p be the characteristic of $\mathbb{F}_q$. Although we assume nothing regarding the characteristic of F, we are particularly interested in the case when F has characteristic 0. We write Γ for $\mathrm{Gal}(\bar{F}/F)$ and W_F (resp. W'_F) for the Weil (resp. Weil-Deligne) group of F. Let G be a connected reductive group over F, and T a torus over F.

(48) We have found a category of perverse sheaves whose simple objects naturally correspond to complete Langlands parameters for $G(F)$; we refer to the simple objects in this category as geometric parameters for G. See Section 48.

(48) We have sketched an argument that the category of geometric parameters is Koszul. See Section 48.

(48) When T is an unramified induced torus, we have found a category of perverse sheaves whose simple objects naturally correspond to characters of admissible representations of $T(F)$; we refer to simple objects in this category as geometric characters for T. See Section 48.

(48) We have found a function from geometric characters for $F^\times$ to geometric parameters for $F^\times$; this function is a bijection (on isomorphism classes) by class field theory. See Section 48

Since 12frg163 met, in related work with Aaron Christie and Anne-Marie Aubert, we have also found how to geometrize certain cusp forms appearing in the part of the local Langlands correspondence proved by Lusztig. However, no details of that progress will appear in this report.

Contents

The Local Langlands Correspondence

In order to give some context for our work, we give a brief description of the current status of the the local Langlands correspondence. A *complete Langlands parameter for G* is a pair (ϕ, ϵ), where $\phi : W'_F \to {}^L G$ is an admissible L-homomorphism and ϵ is an irreducible representation of the finite group $\mathcal{S}_\phi := Z_{\check{G}}(\phi)/Z_{\check{G}}(\phi)^\circ Z(\check{G})^{W_F}$. The local Langlands correspondence promises a bijection between complete parameters and characters Θ_π of admissible irreducible complex representations π of $G(F)$. The bijection $\Theta_\pi \leftrightarrow (\phi, \epsilon)$ must satisfy certain natural conditions, notably compatibility with the principle of functoriality and local class field theory.

The local Langlands correspondence has been proved for certain families of groups, including general linear (Harris-Taylor and Henniart), symplectic and odd-orthogonal groups (Arthur, building on recent work by Ngo and forthcoming work by Waldspurger). A slightly weaker statement is known for even-orthogonal groups (Arthur) and the proof for some other classes, including unitary groups, is currently under construction following Arthur's ideas. Besides these, the local Langlands correspondence has also been proved for a few low-rank groups, such as the rank-2 group of symplectic similitudes (Gan-Takeda). From a completely different perspective, the local Langlands correspondence is also fairly well understood for certain families of representations of quasi-split groups (recent work by Debacker, Reeder, Gross, and Yu), including some (but not all) depth-zero supercuspidal representations. From a different perspective again, the local Langlands correspondence was proved more than 15 years ago by Lusztig for cuspidal unipotent representations of connected algebraic groups over non-Archimedean local fields. The general case of the local Langlands correspondence remains open.

Geometric parameters

In this section we explain how to geometrize Langlands parameters of p-adic groups. There is considerable overlap between the ideas presented here and those appearing in [14] as they pertain to p-adic fields; a discussion of this overlap can be found at the end of Section 48.

Let G be a connected, reductive linear algebraic group over F; for simplicity, we assume here that G is also quasi-split (so all L-parameters are admissible L-parameters). Write F_G for the splitting field for G in $\bar{F}$ and Γ_G for the Galois group $\mathrm{Gal}(F_G/F)$. We use the finite model for the Langlands group: ${}^L G = \check{G} \rtimes \Gamma_G$ is a quasisplit reductive linear algebraic group over $\bar{\mathbb{Q}}_\ell$ (or $\mathcal{C}$, according to taste).

Cocycles

Let I_F be the inertia group for F; thus, $I_F = \mathrm{Gal}(\bar{F}/F^{\mathrm{nr}})$, where F^{nr} is the maximal unramified extension of F in $\bar{F}$. We being by explaining how to view $Z^1(I_F, \check{G})$ (cocycles continuous for the discrete topology on $\check{G}$) as an ind-variety. For every finite extension F' of F_G, set $I_{F'/F} = \mathrm{Gal}(F' \cdot F^{\mathrm{nr}}/F^{\mathrm{nr}})$ and let $I_{F'/F} \to \Gamma_G$ be the composition

$$I_{F'/F} = \mathrm{Gal}(F' \cdot F^{\mathrm{nr}}/F^{\mathrm{nr}}) \cong \mathrm{Gal}(F'/F' \cap F^{\mathrm{nr}}) \hookrightarrow \mathrm{Gal}(F'/F) \twoheadrightarrow \mathrm{Gal}(F_G/F) = \Gamma_G.$$

The finite group Γ_G acts algebraically on $\check{G}$ in the sense that, for every $\gamma \in \Gamma_G$, the function $\gamma : g \mapsto {}^\gamma g$ is a morphism of algebraic groups. It follows that we can interpret $Z^1(I_{F'/F}, \check{G})$ as an algebraic variety; indeed, it is a closed subvariety of the product of $|I_{F'/F}|$-copies of $\check{G}$:

$$Z_{F'} := \left\{ z = (z(\sigma))_{\sigma \in I_{F'/F}} \in \prod_{\sigma \in I_{F'/F}} \check{G} \,|\, z(\sigma\sigma') = z(\sigma)\, {}^{\sigma}z(\sigma'), \quad \forall \sigma, \sigma' \in I_{F'/F} \right\}.$$

It is clear that there is a canonical bijection between the $\bar{\mathbb{Q}}_\ell$-rational points on $Z_{F'}$ and the set $Z^1(I_{F'/F}, \check{G})$. If F'' is a finite extension of F', itself a finite extension of F_G, then restriction defines a morphism of algebraic varieties $Z_{F'} \to Z_{F''}$. With this in mind, it is easy to see how to view $Z^1(I_F, \check{G})$ as an ind-variety:

$$Z := \varinjlim_{F'/F_G} Z_{F'}.$$

It is now clear that there is a canonical bijection between the $\bar{\mathbb{Q}}_\ell$-rational points on Z and the set $Z^1(I_F, \check{G})$.

$$Z^1(I_F, \check{G}) \cong Z(\bar{\mathbb{Q}}_\ell) \tag{48.0.0.1}$$

During our programme we proved:

Lemma 13. *For every finite extension F' of F_G, the group $\check{G}$ (resp. $\check{G}_{\Gamma\text{-}ad} := \check{G}/Z(\check{G})^\Gamma$) is reductive and acts on $Z_{F'}$ in the category of algebraic varieties; moreover, $\check{G}$ (resp. $\check{G}_{\Gamma\text{-}ad}$) acts on Z in the category of ind-varieties.*

Langlands parameters with trivial monodromy

In order to recognize $\mathcal{H}om_{\Gamma_G}(W_F, {}^LG)$ as an ind-variety, it seems necessary to make a slightly disagreeable choice: we fix a lift $\mathrm{Fr} \in W_F$ of arithmetic Frobenius for $\mathbb{F}_q$; equivalently, we fix a splitting of the short exact sequence

$$1 \longrightarrow I_F \longrightarrow W_F \longrightarrow W_{\mathbb{F}_q} \longrightarrow 1.$$

$$\text{(splitting } \mathrm{Fr}_F : W_{\mathbb{F}_q} \dashrightarrow W_F)$$

Using this choice we may identify elements $\phi \in \mathcal{H}om_{\Gamma_G}(W_F, {}^LG)$ with pairs (z, s) tied together by the condition

$$z({}^{\mathrm{Fr}}\sigma) = s\ {}^{\mathrm{Fr}}z(\sigma)\ {}^{{}^{\mathrm{Fr}}\sigma}(s^{-1}), \quad \forall \sigma \in I_F;$$

to recover such a pair from ϕ let s be the image of Fr in $\check{G}$ and let $z \in Z^1(I_F, \check{G})$ be the restriction of ϕ to I_F.

To pass from $\mathcal{H}om_{\Gamma_G}(W_F, {}^LG)$ to L-parameters with trivial monodromy we need one more condition. Let $\check{G}^{\text{Fr-ss}}$ be the subvariety (neither open nor closed, in general) of $s \in \check{G}$ such that $s \rtimes \mathrm{Fr}$ lies in the variety ${}^LG^{\text{ss}}$ of semisimple elements in ${}^LG$. For every finite extension F' of F_G, define

$$Y_{F'} := \{(z, s) \in Z_{F'} \times \check{G}^{\text{Fr-ss}} \,|\, z({}^{\mathrm{Fr}}\sigma) = s\ {}^{\mathrm{Fr}}z(\sigma)\ {}^{{}^{\mathrm{Fr}}\sigma}(s^{-1}), \quad \forall \sigma \in I_{F'/F}\}$$

and set $Y := \varinjlim_{F'/F_G} Y_{F'}$. We may now write

$$Y = \{(z, s) \in Z \times \check{G}^{\text{Fr-ss}} \,|\, z({}^{\mathrm{Fr}}\sigma) = s\ {}^{\mathrm{Fr}}z(\sigma)\ {}^{{}^{\mathrm{Fr}}\sigma}(s^{-1}), \quad \forall \sigma \in I_F\}.$$

Lemma 14. *For each lift* Fr *of arithmetic Frobenius for $\mathbb{F}_q$, making use of Equation 48.0.0.1 and Lemma 13, there is a canonical, $\check{G}$-equivariant bijection between the $\bar{\mathbb{Q}}_\ell$-rational points on $Y = Y_{\mathrm{Fr}}({}^LG)$ and the set of Langlands parameters for G with trivial monodromy.*

$$\mathcal{H}om_{\Gamma_G}(W_F, {}^LG^{ss}) \longrightarrow Y(\bar{\mathbb{Q}}_\ell)$$

defined by $\lambda \mapsto (\lambda|_{I_F}, \lambda(\mathrm{Fr}))$.

The geometric parameter ind-variety

Let $X = X_{\mathrm{Fr}}({}^L G)$ be the ind-variety defined by $X = \varinjlim_{F'/F_G} X_{F'}$, where $X_{F'}$ is the variety of triples $(z, s, N) \in Z_{F'} \times \check{G}^{\text{Fr-ss}} \times \check{\mathfrak{g}}^{\text{nilp}}$ such that, for each $\sigma \in I_{F'/F}$,

$$
\begin{aligned}
s\ {}^{\mathrm{Fr}}z(\sigma)\ {}^{\mathrm{Fr}\sigma}(s^{-1}) &= z({}^{\mathrm{Fr}}\sigma) && (48.0.0.2)\\
s\ {}^{\mathrm{Fr}}N\ s^{-1} &= qN, && (48.0.0.3)\\
z(\sigma)\ {}^{\sigma}N\ z(\sigma)^{-1} &= N. && (48.0.0.4)
\end{aligned}
$$

Lemma 15. *Although the ind-variety $X = X_{\mathrm{Fr}}({}^L G)$ does depend on the choice* Fr *made above, the isomorphism class of X does not. For each lift* Fr *of arithmetic Frobenius for $\mathbb{F}_q$, there is a canonical, $\check{G}$-equivariant bijection between the $\bar{\mathbb{Q}}_\ell$-rational points on X and the set of Langlands parameters for G.*

For reasons that will be apparent later (looking ahead to Theorem 17), we refer to $X_{\mathrm{Fr}}({}^L G)$ as the *geometric parameter ind-variety* for G.

Vogan varieties and a stratification of the geometric parameter ind-variety

Notice that the geometric parameter ind-variety comes equipped with various $\check{G}$-equivariant projections, given below.

$$
\begin{tikzcd}
 & & X \arrow[dl, "(z,s,N)\mapsto(z,s)"', "\pi"] \arrow[drr, "(z,s,N)\mapsto N"] & & \\
 & Y \arrow[dl, "(z,s)\mapsto z"'] \arrow[dr, "(z,s)\mapsto s"] & & & \check{\mathfrak{g}}^{\text{nilp}} \\
Z & & \check{G}^{\text{Fr-ss}} & &
\end{tikzcd}
$$

Lemma 16. *The $\check{G}$-equivariant morphism $\pi : X \to Y$ given by $(z, s, N) \mapsto (z, s)$ determines a stratification of X into disjoint, connected $\check{G}$-stable (resp. $\check{G}_{\Gamma\text{-}ad}$-stable) subvarieties of X:*

$$X = \coprod_{\mathcal{O} \subseteq Y} \pi^{-1}(\mathcal{O}), \tag{48.0.0.5}$$

where the disjoint union is taken over all $\check{G}$-orbits in Y. Moreover, the group $\check{G}$ (resp. $\check{G}_{\Gamma\text{-}ad}$) acts on $\pi^{-1}(\mathcal{O})$ with only finitely many orbits, which are locally closed subvarieties of X. Thus, X admits a stratification into locally closed, $\check{G}$-stable (resp. $\check{G}_{\Gamma\text{-}ad}$-stable) subvarieties.

As with all the results in this report, the proof of this lemma will appear elsewhere. However, it is worth taking a moment to discuss the construction of this stratification, since it is will play an important role in the main result of this section, Theorem 17.

Fix $y = (z, s) \in Y$. By construction, there is some finite extension F' of F_G such that $y \in Y_{F'}$. Observe that $\pi^{-1}(Y_{F'}) = X_{F'}$ and that the projection $\pi_{F'} : X_{F'} \to Y_{F'}$, given by $(z, s, N) \mapsto (z, s)$, is a morphism of algebraic varieties. Thus, $\pi^{-1}(y)$ is a closed subvariety in X. The proof of Lemma 16 shows that

$$\pi^{-1}(z, s) = \{(z, s, N) \in X_{F'} \mid N \in Z_{\check{\mathfrak{g}}}(z)_{s,q}\}$$

where $Z_{\check{\mathfrak{g}}}(z)_{s,q}$ is the q-eigenspace of the semisimple automorphism of the Lie algebra of

$$Z_{\check{G}}(z) = \{g \in \check{G} \mid\ z(\sigma)\ ({}^{\sigma}g)\ z(\sigma)^{-1} = g,\ \forall \sigma \in I_{F'/F}\}$$

given by $s \times \mathrm{Fr} : N \mapsto s\ ({}^{\mathrm{Fr}}N)\ s^{-1}$. Observe that $Y_{F'}$ is a subvariety of $|I_{F'/F}|$-copies of $\check{G}$, on which $\check{G}$ acts, component-wise, by conjugation. The proof of Lemma 16 also shows that $\check{G}$-orbit $\mathcal{O}_{\check{G}}(y)$ of $y = (z, s) \in Y$ is a subvariety in Y, and provides an isomorphism

$$\pi^{-1}(\mathcal{O}_{\check{G}}(y)) \cong \check{G} \times^{Z_{\check{G}}(y)} \pi^{-1}(y)$$

where

$$Z_{\check{G}}(y) = Z_{\check{G}}(z,s) = \{g \in Z_{\check{G}}(z) \mid\ s\ ({}^{\mathrm{Fr}}g)\ s^{-1} = g\}.$$

Since the varieties $\pi^{-1}(\mathcal{O}_{\check{G}}(y))$ appear in [14] (though without situating them in the geometric parameter ind-variety X) we refer to them as *Vogan varieties*. Since some of the arguments used to prove Lemma 16 also appear in Vogan's work. As shown in [14], each Vogan variety is stratified into finitely-many locally closed subvarieties formed by $\check{G}$-orbits. As the proof of Lemma 16 shows, each Vogan variety is also $\check{G}_{\Gamma\text{-ad}}$-stable and each Vogan variety is stratified into finitely-many locally closed subvarieties formed by $\check{G}_{\Gamma\text{-ad}}$-orbits. In summary, the geometric parameter ind-variety X is stratified by the $\check{G}$-orbits (resp. $\check{G}_{\Gamma\text{-ad}}$-orbits) in the Vogan varieties appearing in X. That is the content of Lemma 16.

Geometric parameters

The geometrization of Langlands parameters is achieved by introducing the categories

$$\mathsf{Perv}_{\check{G}}(X) := \bigoplus_{\mathcal{O}} \mathsf{Perv}_{\check{G}}(\pi^{-1}(\mathcal{O})) \tag{48.0.0.6}$$

and

$$\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X) := \bigoplus_{\mathcal{O}} \mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(\pi^{-1}(\mathcal{O})), \tag{48.0.0.7}$$

where the categorical sum of abelian categories is taken over $\check{G}$-orbits in Y. Objects in this category are *finite* direct sums of perverse sheaves on Vogan varieties. Note that any finite union of Vogan varieties in X form a variety in the geometric parameter ind-variety $X = X_{\mathrm{Fr}}({}^{L}G)$.

Theorem 17. *There is a canonical bijection between isomorphism classes of simple objects in the abelian category*

$$\mathsf{Perv}_{\check{G}_{\Gamma\text{-}ad}}(X)$$

and equivalence classes of pairs (ϕ, ϵ) where ϕ is a Langlands parameter and ϵ is an irreducible representation of the finite group

$$\mathcal{S}_\phi := Z_{\check{G}}(\phi)/Z_{\check{G}}(\phi)^0\ Z(\check{G})^\Gamma.$$

Likewise, there is a canonical bijection between isomorphism classes of simple objects in the abelian category

$$\mathsf{Perv}_{\check{G}}(X),$$

and equivalence classes of pairs (ϕ, τ) where ϕ is a Langlands parameter and τ is an irreducible representation of the finite group

$$Z_{\check{G}}(\phi)/Z_{\check{G}}(\phi)^0.$$

Theorem 17 (and Lemma 16, upon which the theorem depends) is a variation on results due to Vogan; see [14, Cor. 4.6]. Because of this theorem, we refer to $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X)$ as the *category of geometric parameters* and refer to simple objects in $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X)$ as *geometric parameters*. We also refer to $\mathsf{Perv}_{\check{G}}(X)$ as the category of *geometric pure parameters* and refer to simple objects in $\mathsf{Perv}_{\check{G}}(X)$ as *geometric pure parameters*; the use of the term 'pure' in this context will be justified elsewhere. For a simple example of category $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X)$, see Section 48.

As mentioned above, there is considerable overlap between the ideas presented in this section and [14]. While the geometric parameter ind-scheme X does not appear in [14], the Vogan varieties do, and the idea of interpreting complete Langlands parameters as equivariant perverse sheaves on Vogan varieties is one of the key ideas in [14], although our group arrived at this idea independently. In particular, if, in the second part of Theorem 17, one replaces the ind-variety X by a single Vogan variety and if one also replaces equivalence classes of complete pure Langlands parameters by equivalence classes of complete pure Langlands parameters with given infinitesimal character, then one recovers a result that can also be found in [14].

Geometric parameters and Koszul duality

In Section 48 we saw how to geometrize (and categorify) complete Langlands parameters (resp. complete pure Langlands parameters): by Theorem 17, isomorphism classes of simple objects in the category $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X_{\mathrm{Fr}}({}^LG))$ (resp. in the category $\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG))$ correspond to equivalence classes of complete Langlands parameters (resp. complete pure Langlands parameters).

In this section we sketch an argument, developed during our programme, showing that these categories are Koszul, in a sense made precise below. Here we consider only the category of geometric pure parameters, leaving a treatment of the category of geometric parameters for another time. As mentioned in Section 48, the Vogan varieties $\pi^{-1}(\mathcal{O}_{\check{G}}(y))$ appearing in the parameter ind-variety $X_{\mathrm{Fr}}({}^LG)$ lie in distinct components of $X_{\mathrm{Fr}}({}^LG)$. In this section we use Lemma 14 to match y with an L-homomorphism $\lambda : W_F \to {}^LG$ and use the notation

$$X^\lambda = \pi^{-1}(\mathcal{O}_{\check{G}}(y))$$

for the Vogan variety determined by the orbit $\mathcal{O}_{\check{G}}(y)$. Consequently, Equation (48.0.0.5) yields a categorical direct sum decomposition

$$\mathsf{Perv}_{\check{G}}(X) = \bigoplus_{\lambda} \mathsf{Perv}_{\check{G}}(X^\lambda)$$

where the sum is taken over all equivalence classes (for the action of $\check{G}$) of L-homomorphisms $\lambda : W_F \to {}^LG$. In our study of Koszulness, it is therefore enough to treat each summand category, $\mathsf{Perv}_{\check{G}}(X^\lambda)$, separately.

In this section we also wish to emphasise the fact the category under consideration, $\mathsf{Perv}_{\check{G}}(X^\lambda)$, is completely determined by the quasi-split reductive algebraic group $\check{G}$, equipped with an action of $\Gamma = \mathrm{Gal}(\bar{F}/F)$. For this reason we break from the notation of Section 48 and write G for any connected complex reductive group equipped with an action of Γ.

Overview of Koszul duality

Consider a nonnegatively graded ring $A = \bigoplus_{i\geq 0} A^i$. Given a graded module $M = \bigoplus_{i\in\mathbb{Z}} M^i$, let $M(j)$ be the graded module whose i-th component is given by $M(j)^i = M^{i-j}$. The ring A is said to be *Koszul* if the following conditions hold:

- A^0 is a semisimple ring.
- Regarding A^0 as a graded A-module, we have $\mathcal{E}\S\sqcup^i(A^0, A^0(j))$ vanishes unless $i = j$.

Under certain finiteness conditions, there is a duality phenomenon that occurs: the graded ring $A^\dagger = \bigoplus_{i\geq 0} \mathcal{E}\S\sqcup^i(A_0, A_0(i))$ is again Koszul, and there is a natural isomorphism $(A^\dagger)^\dagger \cong A$.

The importance of this notion in representation theory was established by the breakthrough discovery by Beilinson–Ginzburg–Soergel [3] that certain rings related to Lie algebra representations in category $\mathcal{O}$ are Koszul, and, moreover, that the Koszul duals of these rings also admit descriptions in terms of category $\mathcal{O}$. Since then, a number of additional examples of Koszul duality have been established: see, for instance, [1, 6, 13].

Geometric Koszul duality

Geometric examples of Koszul duality have been particularly important. In the seminal work [3], the authors considered the flag variety X for a reductive group G, stratified by orbits of a Borel subgroup B. They show that the category $\mathsf{Perv}_{B,c}(X)$ of perverse sheaves that are constructible with respect to this stratification is Koszul.

More precisely, they show that $\mathsf{Perv}_{B,c}(X)$ is equivalent to a category of *ungraded* modules over a Koszul ring (whose grading has been forgotten). In order to bring graded phenomena into the geometric setting, one must make use of the richer structure of "mixed geometry": either mixed ℓ-adic perverse sheaves on a variety over a finite field, or mixed Hodge modules on a complex variety. This step is quite delicate: the category of *all* mixed perverse sheaves or mixed Hodge modules is too large, and has unwanted $\mathcal{E}\S\sqcup$-groups that contradict Koszulity. But suitable modified categories can sometimes play the role of graded modules. In this report, for simplicity, we denote these modified categories (in either the mixed ℓ-adic or Hodge setting) with notation such as "$\mathsf{Perv}^{\mathrm{mix}}_{B,c}(X)$," suppressing technical issues in their definition.

In the case of the flag variety, the category $\mathsf{Perv}^{\mathrm{mix}}_{B,c}(X)$ turns out to be equivalent to its Koszul dual. A far-reaching generalization of this result in the setting of Kač–Moody groups has recently been established by Bezrukavnikov–Yun [5].

A key property of $\mathsf{Perv}^{\mathrm{mix}}_{B,c}(X)$ is that it is equipped with a *de-grading functor* $\varkappa : \mathsf{Perv}^{\mathrm{mix}}_{B,c}(X) \to \mathsf{Perv}_{B,c}(X)$ (see [3, 4.3]) that allows one to make a comparison of $\mathcal{E}\S\sqcup$-groups between the two categories. Further ingredients in the proof of Koszulity are discussed in Section 48 below.

Aim of the project

We hope to show that the category $\mathsf{Perv}_G(X^\lambda)$ of G-equivariant perverse sheaves on X^λ is "Koszul." As above, this means that a certain "mixed" (ℓ-adic or Hodge) category $\mathsf{Perv}^{\mathrm{mix}}_G(X^\lambda)$ is Koszul, and that there is a de-grading functor $\varkappa : \mathsf{Perv}^{\mathrm{mix}}_G(X^\lambda) \to \mathsf{Perv}_G(X^\lambda)$.

(Note that this is, in general, a smaller category that the category $\mathsf{Perv}_{G,c}(X^\lambda)$ of all perverse sheaves that are constructible with respect to the stratification by G-orbits. In contrast with the flag variety case considered above, the category $\mathsf{Perv}_{G,c}(X^\lambda)$ contains unwanted objects of no representation-theoretic significance.)

Outline for the proposed research project

Some of the themes that have arisen in previous work on Koszul duality in geometric settings include: pointwise purity and parity vanishing; quasi-hereditary categories; and derived equivalences for the perverse t-structure. Below, we consider these themes in the context of Vogan varieties.

Pointwise purity and parity vanishing

A simple object $L \in \mathsf{MHM}^\diamond(X)$ is said to be *pointwise pure* if, for every orbit $S \subset X$, the restriction $L|_C$ is a pure object of $D^{\mathrm{b}}\mathsf{MHM}(C)$. The close relationship between pointwise purity and Koszul duality has been observed by a number of authors; see, for instance, [4, Remark 4]. It plays a prominent role in [3, 5]. Another key feature is *parity vanishing*: this is the requirement that the cohomology sheaves $H^i(L|_C)$ vanish for all odd i (or perhaps all even i, depending on the dimensions of C and of the support of L). This type of condition holds on the flag variety [8] and on the nilpotent cone [12].

For Vogan varieties, it seems that both properties can be deduced from the work of Lusztig on perverse sheaves on graded Lie algebras [9]. Indeed, Lusztig's motivation seems to have been the study of Vogan varieties, and the precise link between his work and these varieties is likely well understood by experts, but we have been unable to find a thorough account of this link in the literature. Thus, this aspect of the project will be mainly expository; we nevertheless believe it will be useful contribution.

Quasi-hereditary property

Vogan varieties shares the property with the flag variety that the push-forward functors attached to orbits are t-exact. In other words, for an orbit $C \subset X^\lambda$ and a local system E on C, the objects

$$j_! E[\dim C] \qquad \text{and} \qquad j_* E[\dim C]$$

(where $j : C \to X^\lambda$ is the inclusion map) are perverse. These objects, called *standard* and *costandard* perverse sheaves, respectively, satisfy at least the first five of the six axioms in [3, 3.2]. By an argument of Ringel explained in *loc. cit.*, one can then deduce that the categories $\mathsf{Perv}_G(X)$ and $\mathsf{Perv}^{\mathrm{mix}}_G(X)$ have enough projectives and injectives.

For the flag variety, the next step is to establish a derived equivalence $D^{\mathrm{b}}\mathsf{Perv}(X) \to D^{\mathrm{b}}(X)$, using a key $\mathcal{E}\S\sqcup^2$-vanishing property for standard and costandard objects. (This is the sixth axiom in [3, 3.2].) Unfortunately, the relevant $\mathcal{E}\S\sqcup^2$-group can be nonzero on the Vogan variety, and indeed, the derived category $D^{\mathrm{b}}\mathsf{Perv}_G(X^\lambda)$ is not, in general, equivalent either to $D^{\mathrm{b}}(X^\lambda)$ or to the G-equivariant derived category $D^{\mathrm{b}}_G(X)$.

Realization functor

To rephrase the last observation: the $\mathcal{E}\S\sqcup$-groups in $\mathsf{Perv}^{\mathrm{mix}}_G(X)$ cannot directly be identified with $\mathcal{H}om$-groups in any "geometric" derived category. Thus, a study of these $\mathcal{E}\S\sqcup$-groups is the most difficult aspect of the project.

A rather general construction [2] gives us a t-exact functor $\rho : D^{\mathrm{b}}\mathsf{Perv}^{\mathrm{mix}}_G(X) \to D^{\mathrm{b}}_{G,\mathrm{m}}(X)$, called a *realization functor*. This functor induces an isomorphism on $\mathcal{E}\S\sqcup^1$-groups and an injective map on $\mathcal{E}\S\sqcup^2$-groups, but beyond that, little can be said in general. In our setting, we hope to use parity-vanishing phenomena in $\mathsf{Perv}^{\mathrm{mix}}_G(X)$ to establish a tighter relationship between the two triangulated categories, and ultimately to deduce the Koszulity of $\mathsf{Perv}^{\mathrm{mix}}_G(X)$ from known $\mathcal{E}\S\sqcup$-vanishing facts in $D^{\mathrm{b}}_{G,\mathrm{m}}(X)$.

Identifying the Koszul dual

As noted above, many of the most celebrated results on the theme of Koszul duality have two parts: they establish the Koszulity of some ring arising in representation theory, and they identify the Koszul dual ring as an object having representation-theoretic significance on its own. Unfortunately, for the moment, we do not know of a suitable candidate category that might be the Koszul dual of the (putatively) Koszul category $\mathsf{Perv}_G(X^\lambda)$. We hope to study this question through examples in the future.

Geometric characters for p-adic tori

During our programme we understood how to geometrize admissible characters of unramified, induced p-adic tori, generalising earlier work on geometrization of admissible characters of $F^\times$.

Classical geometrization

Let us begin by recalling classical geometrization. For the moment, let G be a connected, commutative algebraic group over $\mathbb{F}_q$. In this context, geometrization is well-understood: use the Lang morphism for G to define $\pi_1(G,\bar{e}) \to G(\mathbb{F}_q)$ and thus convert each character $\chi : G(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$ into a character of the fundamental group $\pi_1(G,\bar{e}) \to \bar{\mathbb{Q}}_\ell^\times$. In this way we define an (isomorphism class of an) ℓ-adic local system $\mathcal{L}_\chi$ on the etale site of G, from the character χ. By base change, the local system $\mathcal{L}_\chi$ defines a local system $\bar{\mathcal{L}}_\chi$ on $\bar{G} := G \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ equipped with an isomorphism $\phi_\chi : \mathrm{Fr}^*\bar{\mathcal{L}}_\chi \to \bar{\mathcal{L}}_\chi$ such that the trace of Frobenius, $t^{\mathcal{L}_\chi}_{\mathrm{Fr}} : G(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell^\times$, defined by the diagramme

$$\begin{tikzcd} (\mathrm{Fr}^*\bar{\mathcal{L}}_\chi)_{\bar g} \arrow[r, "(\phi_\chi)_{\bar g}"] \arrow[d, dash, "\text{canonical isomorphism}"'] & (\bar{\mathcal{L}}_\chi)_{\bar g} \\ (\bar{\mathcal{L}}_\chi)_{\bar g}, \arrow[ur, "t^{\mathcal{L}_\chi}_{\mathrm{Fr}}(g)"'] & \end{tikzcd}$$

recovers the character $\chi : G(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$. It is easy to characterise the local systems on G that arise in this manner: if $\mathcal{L}$ is a local system on G and if there is an isomorphism

$$\mathcal{L} \boxtimes \mathcal{L} \cong m^*\mathcal{L}, \tag{48.0.0.8}$$

where $m : G \times G \to G$ is the multiplication map for G, then $t^{\mathcal{L}}_{\mathrm{Fr}} : G(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell^\times$ is a character, and all ℓ-adic characters of $G(\mathbb{F}_q)$ are produced in this way. The final miracle is this: if $\mathcal{L}$ and $\mathcal{L}'$ both admit isomorphisms as in (48.0.0.8) and if $t^{\mathcal{L}}_{\mathrm{Fr}} = t^{\mathcal{L}'}_{\mathrm{Fr}}$, then $\mathcal{L} \cong \mathcal{L}$. Consequently, the trace of Frobenius $\mathcal{L} \mapsto t^{\mathcal{L}}_{\mathrm{Fr}}$ defines an isomorphism of groups

$$\left\{ \begin{matrix} \text{local systems } \mathcal{L} \text{ on } G \\ \exists\, \mathcal{L} \boxtimes \mathcal{L} \cong m^*\mathcal{L} \end{matrix} \right\}_{/\mathrm{iso}} \longrightarrow \mathcal{H}om_{\mathrm{grp}}(G(\mathbb{F}_q), \bar{\mathbb{Q}}_\ell)$$

These facts are well-known. Since isomorphism classes of local systems appearing on the left-hand side above correspond to characters of $G(\mathbb{F}_q)$, it is common to refer to such local systems as *character sheaves* on G. We will revisit this definition in the next two sections.

Geometric characters for commutative groups

In order to justify the claims made above, one must make crucial use of the fact that G is connected and finitely generated over $\mathbb{F}_q$, in that section. But we wish to loosen these conditions on G to admit non-connected, commutative group schemes over $\mathbb{F}_q$. As we understood during our programme, for that we require a new definition, given here.

Let G be a commutative group scheme over $\mathbb{F}_q$. A *geometric character* on G is an ℓ-adic local system $\bar{\mathcal{L}}$ on $\bar{G} := G \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}\left(\bar{\mathbb{F}}_q\right)$, with three supplementary structures:

1. an isomorphism $\mu : m^*\bar{\mathcal{L}} \longrightarrow \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}$;
2. a rigidification $r : \bar{\mathcal{L}}_{\bar{e}} \xrightarrow{\cong} \bar{\mathbb{Q}}_\ell$ at the geometric point $\bar{e}$ of $\bar{G}$ lying above the origin e of G;
3. an isomorphism $\phi : \mathrm{Fr}^*\bar{\mathcal{L}} \to \bar{\mathcal{L}}$.

The quartuple $\mathcal{L} = (\bar{\mathcal{L}}, \mu, r, \phi)$ must also satisfy some natural compatability conditions which we omit from this report. It is a consequence of this definition that if $\mathcal{L} = (\bar{\mathcal{L}}, \mu, r, \phi)$ is a geometric character then $\bar{\mathcal{L}}$ is an irreducible local system on $\bar{G}$. We write $\mathrm{GC}(G)$ for the additive category generated by geometric characters on G, with obvious definition for morphisms. (This category will be treated carefully in one of the papers based on our programme.) Simple objects in $\mathrm{GC}(G)$ are geometric characters on G.

Comparison with character sheaves

If we return to the case when G is a connected, commutative algebraic group over $\mathbb{F}_q$, then the forgetful functor $(\bar{\mathcal{L}}, \mu, r, \phi) \mapsto (\bar{\mathcal{L}}, \phi)$ takes geometric characters on G to character sheaves on G, as defined in Section 48. While this functor is full and essentially surjective, it is not faithful. If $G = T$ is also an algebraic torus, and $(\bar{\mathcal{L}}, \mu, r, \phi)$ is a geometric character such that $\bar{\mathcal{L}}^n = \bar{\mathbb{Q}}_\ell$ for some positive integer n then $\bar{\mathcal{L}}$ is a character sheaf on $\bar{T}$, as defined by Lusztig, and all Frobenius-stable character sheaves on $\bar{T}$ arise in this manner.

Greenberg of Neron

In this section we introduce a geometric space needed to geometrize admissible characters of p-adic tori. Let us set some notation and briefly recall the filtration of admissible characters by depth. Let F be a non-Archimedean local field with residual field $\mathbb{F}_q$ and let T be an algbraic torus over F. Let $\chi : T(F) \to \bar{\mathbb{Q}}_\ell^\times$ be an admissible character. Then the depth of χ is given by

$$\inf\{r \geq 0 \,|\, \forall s > r,\ T(F)_s \subset \ker(\chi)\},$$

where the filtration

$$T(F) \supseteq T(F)_0 \supseteq \cdots \supseteq T(F)_s \supseteq \cdots,$$

is defined in [10] or equally in [11]. Let $\mathcal{H}om_d(T(F), \bar{\mathbb{Q}}_\ell^\times)$ be the group of ℓ-adic characters of $T(F)$ with depth less than or equal to d. In the next few sections we will explain how to geometrize elements of the group $\mathcal{H}om_d(T(F), \bar{\mathbb{Q}}_\ell^\times)$, along the lines of Section 48

Neron models

The Neron model for T is a smooth group scheme T_R locally of finite type over R with generic fibre T, such that for every smooth group scheme Y over R, the canonical function

$$\mathcal{H}om_R(Y, T_R) \xrightarrow[\substack{\text{base change along} \\ \mathrm{Spec}(F)\to\mathrm{Spec}(R)}]{\substack{\text{restriction to} \\ \text{generic fibres}}} \mathcal{H}om_F(Y \times_S \mathrm{Spec}(F), T)$$

is bijective; in particular, $T_R(R) \cong T(F)$. Neron models exist for all p-adic tori, and are unique up to isomorphism.

Greenberg transform

Let A be an Artin local ring; let k be its residual field. Marvin Greenberg [7] has defined a functor

$$\left(\mathrm{Sch}_{/A}\right)_{\mathrm{lft}} \xrightarrow{\substack{\text{Greenberg}\\ \text{transform}}} \left(\mathrm{Sch}_{/k}\right)_{\mathrm{lft}} \qquad X \longmapsto \mathrm{Gr}(X)$$

with a number of agreeable properties, including, for every X and Y, locally of finite type over A: a canonical bijection $X(A) \cong \mathrm{Gr}(X)(k)$; if X if affine (resp. smooth, finite etale) then so is $\mathrm{Gr}(X)$; if $X \to Y$ is an open subscheme (resp. a closed subscheme) then so is $\mathrm{Gr}(X) \to \mathrm{Gr}(Y)$.

Geometrization of characters of bounded depth

During our programme we put together a proof of the following result.

Theorem 18. *Let T be an induced, unramified torus over F. For each $d \in \mathcal{N}$, let T_d be the Greenberg transform of $T_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}\left(R/\mathfrak{p}^{d+1}\right)$, where T_R is a Neron model for T. The trace of Frobenius defines an isomorphism of groups from isomorphism classes of simple objects in $\mathrm{GC}(T_d)$ to $\mathcal{H}om_d(T(F), \bar{\mathbb{Q}}_\ell^\times)$.*

Admissible geometric characters

Consider the commutative pro-algebraic group $T_{\mathbb{F}_q} := \varprojlim_{d\in\mathcal{N}} T_d$. Amazingly, this limit exists in the category of groups schemes over $\mathbb{F}_q$. It comes equipped with a canonical isomorphism

$$T_{\mathbb{F}_q}(\mathbb{F}_q) \cong T(F).$$

A geometric character on $T_{\mathbb{F}_q}$ is *admissible* if there is an integer $d \in \mathcal{N}$ and a geometric character on T_d such that $\mathcal{L} = f^*\mathcal{L}_d$ where $f : T_{\mathbb{F}_q} \to T_d$ is the obvious map. Let $\mathrm{GC}_{\mathrm{ad}}(T_{\mathbb{F}_q})$ be the category of admissible geometric characters on $T_{\mathbb{F}_q}$. For a simple example of category $\mathrm{GC}_{\mathrm{ad}}(T_{\mathbb{F}_q})$, see Section 48.

The main result of Section 48 is the following theorem, which follows from Theorem 18, the definition above, and a small amount of extra work.

Theorem 19. *Let T be an induced, unramified torus over F. The trace of Frobenius defines an isomorphism of groups, compatible with the fitration by depth on both sides, from isomorphism classes of simple objects in $\mathrm{GC}_{ad}(T_{\mathbb{F}_q})$ to $\mathcal{H}om_{ad}(T(F), \bar{\mathbb{Q}}_\ell^\times)$.*

Geometric reciprocity

In Section 48 we saw how to geometrize Langlands parameters for quasisplit groups G over F by introducing the category $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X_{\mathrm{Fr}}({}^LG))$ and studying isomorphism classes of simple objects in this category. In Section 48 we saw how to geometrize admissible characters of unramified, induced T tori over F by introducing the category $\mathrm{GC}_{\mathrm{ad}}(T_{\mathbb{F}_q})$ and studying its simple objects. This raises the question: supposing $G = T$, is there a functor from $\mathrm{GC}_{\mathrm{ad}}(T_{\mathbb{F}_q})$ to $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X_{\mathrm{Fr}}({}^LT))$ that defines the reciprocity map for T by restriction to (isomorphism classes of) simple objects? In this section we answer that question when $G = T = \mathbb{G}_{\mathbf{m},F}$.

Geometric parameters for $\mathbb{G}_{\mathbf{m},F}$

Set $G = \mathbb{G}_{\mathbf{m},F}$. Then $\check{G} = \mathbb{G}_{\mathbf{m},\bar{\mathbb{Q}}_\ell}$ and $\Gamma_G = 1$ so ${}^LG = \mathbb{G}_{\mathbf{m},\bar{\mathbb{Q}}_\ell}$. Recall that the definition of Y (see Section 48) and X (see Section 48) require the choice of a lift Fr of Fr_q. We will revisit this choice in Section 48. Observe that $\check{G}$ acts trivially on Z (see Section 48) and Y and X. The ind-variety Y is a totally disconnected space. For each admissible $\lambda : W_F \to \bar{\mathbb{Q}}_\ell^\times$, the corresponding Vogan variety $X^\lambda = \pi^{-1}(\mathcal{O}_{\check{G}}(y))$ is $\{y\}$, where $y \in Y$ is the point corresponding to λ under Lemma 14. So $X = Y$ is a totally disconnected space with trivial $\mathbb{G}_{\mathbf{m},\bar{\mathbb{Q}}_\ell}$-action. With reference to Section 48, $\mathsf{Perv}_{\check{G}}(X) = \mathsf{Perv}(Y)$; thus,

$$\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG)) = \bigoplus_{y\in Y} \mathsf{Perv}(\{y\}) \cong \bigoplus_{\lambda\in\mathcal{H}om_{\mathrm{ad}}(W_F,\bar{\mathbb{Q}}_\ell^\times)} \mathsf{Perv}(\mathrm{Spec}\left(\bar{\mathbb{Q}}_\ell\right)).$$

It follows that $\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG))$ is equivalent to the category of finite-dimensional admissible ℓ-adic representations of W_F^{ab}:

$$\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG)) \cong \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell,\mathrm{ad}}(W_F^{\mathrm{ab}}). \tag{48.0.0.9}$$

Under this equivalence, simple objects in $\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG))$ correspond to one-dimensional representations of W_F. The equivalence above induces a bijection between isomorphism classes of simple objects in $\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG))$ and admissible characters of W_F:

$$\mathcal{H}om_{\mathrm{ad}}(W_F, \bar{\mathbb{Q}}_\ell^\times) \longrightarrow \mathrm{simp.\,obj}\,\mathsf{Perv}_{\check{G}}(X_{\mathrm{Fr}}({}^LG))_{/\mathrm{iso}}$$

defined by $\lambda \mapsto (\bar{\mathbb{Q}}_\ell)_{\{y\}}$, where y corresponds to λ under Lemma 14 and $(\bar{\mathbb{Q}}_\ell)_{\{y\}}$ is the sheaf on $X = Y$ supported at $\{y\}$ where it is the constant sheaf $\bar{\mathbb{Q}}_\ell$. This is a special case of Theorem 17.

Geometric characters for $\mathbb{G}_{\mathbf{m},F}$

Set $T = \mathbb{G}_{\mathbf{m},F}$, so $T_{\mathbb{F}_q}$ is the Greenberg transform of the Neron model of $\mathbb{G}_{\mathbf{m},F}$. Theorem 19 can be strengthened to an equivalence of categories

$$\mathrm{GC}_{\mathrm{ad}}(T_{\mathbb{F}_q}) \cong \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell,\mathrm{ad}}(T(F)) \tag{48.0.0.10}$$

between the category of admissible geometric parameters on $T_{\mathbb{F}_q}$ and the category of finite-dimensional, admissible ℓ-adic representations of $T(F)$. This equivalence (and the proof of Theorem 19) is too complicated to describe in this report, but is currently being prepared for publication.

Geometric reciprocity for non-Archimedean local fields

Class field theory provides an isomorphism $W_F^{\mathrm{ab}} \cong F^\times$ and thus an equivalence of categories between $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell,\mathrm{ad}}(W_F^{\mathrm{ab}})$ and $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell,\mathrm{ad}}(F^\times)$. In light of Sections 48 and 48, this determines an equivalence between $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X_{\mathrm{Fr}}({}^LT))$ and $\mathrm{GC}_{\mathrm{ad}}(T_{\mathbb{F}_q})$, when $T = \mathbb{G}_{\mathbf{m},F}$.

$$\begin{tikzcd}
\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X_{\mathrm{Fr}}({}^LT)) \arrow[r, dotted, dash, "\text{geometric reciprocity functor}"] \arrow[d, dash, "\text{equivalence}"', "\text{Section 48}"] & \mathrm{GC}_{\mathrm{ad}}(T_{\mathbb{F}_q}) \arrow[d, dash, "\text{Section 48}"', "\text{equivalence}"] \\
\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell,\mathrm{ad}}(W_F^{\mathrm{ab}}) \arrow[r, dash, "\text{class field theory}", "\text{equivalence}"'] & \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell,\mathrm{ad}}(F^\times)
\end{tikzcd}$$

During the last day of our programme we discussed a geometric construction which leads to a functor directly from $\mathsf{Perv}_{\check{G}_{\Gamma\text{-ad}}}(X_{\mathrm{Fr}}({}^LT))$ to $\mathrm{GC}_{\mathrm{ad}}(T_{\mathbb{F}_q})$ without recourse to class field theory. We (presumptuously) call this a geometric reciprocity functor. If it agrees with the equivalence given by class field theory, such a functor would actually recover the isomorphism $W_F^{\mathrm{ab}} \cong F^\times$ from our geometric reciprocity functor. This is now a topic of research in progress.

Participants

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Bibliography

[1] E. Backelin, *Koszul duality for parabolic and singular category $\mathcal{O}$*, Represent. Theory **3** (1999), 139–152.

[2] A. Beĭlinson, *On the derived category of perverse sheaves*, F-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Mathematics, vol. 1289, Springer-Verlag, Berlin, 1987, pp. 27–41.

[3] A. Beĭlinson, V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), 473–527.

[4] R. Bezrukavnikov, *Cohomology of tilting modules over quantum groups and t-structures on derived categories of coherent sheaves*, Invent. Math. **166** (2006), 327–357.

[5] R. Bezrukavnikov and Z. Yun, *On Koszul duality for Kac–Moody groups*, arXiv:1101.1253.

[6] T. Braden, *Koszul duality for toric varieties*, Trans. Amer. Math. Soc. **359** (2007), 385–415.

[7] M. Greenberg, *Schemata over local rings. II*, Ann. of Math. (2) **78** (1963), 256-266.

[8] D. Kazhdan and G. Lusztig, *Schubert varieties and Poincaré duality*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, RI, 1980, pp. 185–203.

[9] G. Lusztig, *Study of perverse sheaves arising from graded Lie algebras*, Adv. Math. **112** (1995), 147–217.

[10] A. Moy and G. Prasad, *Jacquet functors and unrefined minimal F-types*, Comment. Math. Helv. **71** No. 1 (1996), 98–121.

[11] P. Schneider and U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. **85** (1997), 97–191.

[12] T. A. Springer, *A purity result for fixed point varieties in flag manifolds*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), 271–282.

[13] S. Riche, *Koszul duality and modular representations of semisimple Lie algebras*, Duke Math. J. **154** (2010), 31–134.

[14] D. A. Vogan, *The local Langlands conjecture*, Representation theory of groups and algebras, Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI, 1993, pp. 305–379.

Chapter 49

The $\overline{\partial}$-Method: Inverse Scattering, Nonlinear Waves, and Random Matrices (12frg176)

July 29 - August 5, 2012

Organizer(s): Ken McLaughlin (University of Arizona), Peter Perry (University of Kentucky)

Overview of the Field

The $\overline{\partial}$-method in inverse scattering enables, in principle at least, an explicit solution to certain completely integrable, dispersive nonlinear equations in two space dimensions. The $\overline{\partial}$-method also has potential application to problems arising in the study of random matrix models and orthogonal polynomials in the plane. The following two examples indicate the directions open to exploration and the nature of the common, underlying mathematical problem.

The Davey-Stewartson (DS) II equation. The DS II equation is a completely integrable model that describes monochromatic, weakly nonlinear waves in shallow water. The solution u gives the (complex) amplitude $u(x, y, t)$ of such a wave. The defocussing DS II equation is the system ($\epsilon > 0$ is a parameter)

$$i\epsilon q_t + 2\epsilon^2 \left(\partial^2 + \overline{\partial}^2\right) q + (g + \overline{g})q = 0 \quad \text{and} \quad \overline{\partial} g = -\partial\left(|q|^2\right), \quad \partial := \frac{\partial}{\partial z}, \quad \overline{\partial} := \frac{\partial}{\partial \overline{z}}, \quad z := x + iy. \tag{49.0.0.1}$$

For the initial-value problem we fix an initial condition: $q(x, y, 0) = q_0(x, y)$. The elliptic equation for g is to be solved subject to the condition that $g \to 0$ as $x, y \to \infty$.

To solve by inverse scattering, suppose that $q_0 \in \mathcal{S}(\mathbb{R}^2)$. There is a nonlinear map $\mathcal{R}$ taking q_0 to a function $r_0 \in \mathcal{S}(\mathbb{R}^2)$, the scattering transform. The solution is constructed by solving the $\overline{\partial}$-problem

$$\overline{\partial}_k \nu(\kappa, \sigma) = \frac{1}{2}\overline{r_0(\kappa, \sigma)} e^{-2iS(\kappa,\sigma)/\epsilon} \sigma_1 \overline{\nu(\kappa, \sigma)}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \overline{\partial}_k := \frac{\partial}{\partial \overline{k}}, \quad k = \kappa + i\sigma,$$

where ν is a 2-component vector that tends to $(1, 0)^{\mathsf{T}}$ as $k \to \infty$, and where $S(\kappa, \sigma) := \Im\{kz\} + 2t\Re\{k^2\}$ is a real-valued phase function. The solution $u(x, y, t)$ is then computed from the reconstruction formula

$$q(x, y, t) = 2 \lim_{k \to \infty} \overline{k}\, \overline{\nu_2(\kappa, \sigma)} = \frac{1}{\pi\epsilon} \iint_{\mathbb{R}^2} e^{2iS(\kappa,\sigma)/\epsilon} r_0(\kappa, \sigma) \nu_1(\kappa, \sigma)\, d\kappa\, d\sigma.$$

Moreover, the problem of computing the scattering transform map $\mathcal{R}$ can be formulated also as a quite similar $\overline{\partial}$-problem, but this time set in the complex z-plane. Thus, the technical core of this problem is the analysis of a $\overline{\partial}$-problem involving parameters z, t, and ϵ that enter in a singular fashion.

Normal Matrix Models. The joint probability measure of complex eigenvalues $z_1, \ldots, z_N$ for a unitary-invariant ensemble of normal random $N \times N$ matrices can be taken in the form

$$P(z_1, \ldots, z_N) dm(z_1) \cdots dm(z_N) := \frac{1}{Z_N} \prod_{j \neq k} |z_j - z_k|^2 \prod_{n=1}^{N} e^{-NV(z_n)} \, dm(z_n) \tag{49.0.0.2}$$

where $V : \mathbb{C} \to \mathbb{R}$ is a (confining) potential, Z_N is the normalization constant (partition function), and $dm(z)$ is Lebesgue measure in the z-plane. As in Hermitian matrix models, the statistics of eigenvalues may be studied through the associated monic orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ defined by:

$$\int_{\mathbb{C}} P_n(z) \overline{P_m(z)} e^{-NV(z)} \, dm(z) = h_n \delta_{mn}, \quad h_n > 0, \quad P_n(z) = z^n + \cdots . \tag{49.0.0.3}$$

These orthogonal polynomials can be equivalently obtained via the solution of a matrix $\overline{\partial}$-problem in which the matrix size N and degree n enter as parameters. Indeed, if $Y_n(z, \overline{z})$ denotes the 2×2 matrix that satisfies

$$\overline{\partial} Y_n(z, \overline{z}) = \overline{Y_n(z, \overline{z})} \begin{pmatrix} 0 & -e^{-NV(z)} \\ 0 & 0 \end{pmatrix}$$

then $P_n(z) = Y_{n,11}(z, \overline{z})$. To analyze the asymptotic distribution of eigenvalues as $N \to \infty$ along with the fine structure of local correlations, one needs information about $P_n(z)$ for n as large as N, and in this setting the $\overline{\partial}$-problem is *in principle* well-suited to asymptotic analysis because the large parameters N and n appear *explicitly* in the conditions on Y_n. Thus, the details of eigenvalue statistics can be worked out if the $\overline{\partial}$-problem can be analyzed accurately in the limit $N, n \to \infty$. Again, the technical core of this problem is the accurate asymptotic analysis of a $\overline{\partial}$-problem involving large parameters.

The $\overline{\partial}$ method is, potentially, as powerful a tool in these sets of related problems as the Riemann-Hilbert method has proven to be in the study of completely integrable systems "in one space dimension" such as the KdV, mKdV and NLS equations, random matrix distributions for symmetric, orthogonal, and unitary matrices, and orthogonal polynomials on the circle or the line. The purpose of this Focussed Research Group was to bring together researchers in completely integrable systems and dispersive equations, together with experts in harmonic analysis and PDE, to better develop the $\overline{\partial}$-methods.

Recent Developments and Open Problems

Dispersive nonlinear partial differential equations in two dimensions have been extensively studied in recent years, both by PDE methods and inverse scattering methods. The former methods yield much stronger local existence and well-posedness results than can be expected from inverse scattering methods, but the latter promise to yield much more detailed behavior on semi-classical asymptotics and long-time behavior if parameter dependence of solutions to the underlying $\overline{\partial}$ problems can be controlled. Model equations include the Davey-Stewartson (DS), Kadomtsev-Petviashvilli (KP), and Novikov-Veselov (NV) equations. The $\overline{\partial}$-method was developed by Fokas-Ablowitz [1, 2, 3] and Beals-Coifman [5, 6, 7] Its application to inverse scattering has been studied by many authors including Ablowitz, Fokas and their collaborators (see the monograph [4] for references up to 1990), and Grinevich, Grinevich-Manakov, and Grinevich-Novikov [15, 16, 17, 18, 19, 20, 21, 22]. Mathematically rigorous treatments of the scattering maps for the DS and NV equation include those of Brown [9], Sung [31], Lassas-Mueller-Siltanen [24], Nachman [28], Perry [29, 30]. A major challenge involves the classification and analysis of singularities of scattering maps that lead to soliton solutions, blow-up in finite time, and other dynamical phenomena. Another major challenge involves understanding semiclassical limits of two-dimensional dispersive equations, as described in greater detail below. The semiclassical method has yielded insights into the dynamics of dispersive equations in one dimension: see, for example, the recent work of Buckingham-Miller [11, 12]. We expect that similar insights will be gained from the study of semiclassical limits, for example, in the DS II equation.

Two-dimensional random matrix models and orthogonal polynomials in the plane have been the subject of intensive investigation in recent years. Its and Takhtajan [23] outlined a program for studying large-N asymptotics of orthogonal polynomials and random matrix models by $\overline{\partial}$ methods. Elbau and Felder [14] studied certain perturbations of the Gaussian case $V(z) = z^2$ (cf. (49.0.0.2)) and showed that the density of eigenvalues converges, in the limit $N \to \infty$, to the characteristic function of an explicit bounded region in the complex plane. Bagh, Bertola, Lee, and McLaughlin [8] carried out a complete analysis of certain random matrix models by reducing the underlying $\overline{\partial}$ problem for the orthogonal polynomials to a Riemann-Hilbert problem. An understanding of the full $\overline{\partial}$ problem remains elusive.

Presentation Highlights

The FRG began with presentations by Peter Perry on inverse scattering for the Davey-Stewartson equation, based on [29] and by Ken McLaughlin introducing random matrix models. The purpose of these lectures was to establish a knowledge base among all participants in the two key areas of research considered. Samuli Siltanen lectured on electrical impedance imaging, the inverse scattering transform, and $\overline{\partial}$ methods. Michael Christ gave an illuminating lecture on the Brascamp-Lieb-type inequalities which underlie much of the progress in analysis of scattering maps in [9] and [29] (see the Appendix to [29], written by Michael Christ, for details and references to the literature).

McLaughlin and Miller led an ongoing discussion on semi-classical analysis for the defocussing DS II equation (49.0.0.1) with $q = q(x, y, t, \varepsilon)$ a complex-valued function having initial data of the form

$$q(x, y, 0) = A(x, y) \exp\left(iS(x, y)/\varepsilon\right)$$

The problem is to study solutions in the limit $\varepsilon \to 0$. Passing to the inverse scattering method, one sees that the solution to the semiclassical DS II problem is obtained in two steps.

First, one solves the following $\overline{\partial}$-problem for $\mu = (\mu_1, \mu_2)^T$ to compute the scattering transform $r$of the initial data:

$$\begin{aligned} \varepsilon\overline{\partial}\mu &= \frac{q}{2} \exp\frac{1}{\varepsilon}\left(\overline{kz} - kz\right)\sigma_1\overline{\mu} \\ \lim_{|z|\to\infty} \mu(z, k) &= (1, 0)^T. \end{aligned} \tag{49.0.0.4}$$

and recovers the scattering transform r from the formula

$$r(k) = 2 \lim_{|z|\to\infty} z\mu_2(z, k)$$

The scattering transform of the full solution then evolves according to

$$r(k, t) = r(k, 0) \exp\frac{2it}{\varepsilon}\left(k^2 + \overline{k}^2\right).$$

Second, to recover $q(x, y, t, \varepsilon)$, one solves the $\overline{\partial}$ problem for $\nu = (\nu_1, \nu_2)^T$:

$$\begin{aligned} \varepsilon\overline{\partial}\nu &= \frac{\overline{r}}{2} \exp\frac{1}{\varepsilon}\left(\overline{kz} - kz\right)\sigma_1\overline{\nu} \\ \lim_{|k|\to\infty} \nu(z, k) &= (1, 0)^T. \end{aligned} \tag{49.0.0.5}$$

One recovers the potential from the formula

$$q(x, y, t) = 2 \lim_{|k|\to\infty} \overline{k}\overline{\nu_2}.$$

Thus, analytically, one needs to understand the small-ε limit of the $\overline{\partial}$-problems (49.0.0.4) and (49.0.0.5).

Scientific Progress Made

- Peter Miller and Ken McLaughlin initiated a study of semiclassical limits for the defocussing Davey-Stewartson II equation. Subsequently, Sarah Hamilton, a postdoctoral research fellow working with Samuli Siltanen, carried out numerical computations which show some interesting features of the semiclassical limits. One of the themes of the upcoming conference and workshop at the University of Kentucky will be the analytical study of semiclassical limits, as a direct outgrowth of these discussions.
- Ken McLaughlin led discussions on the analysis of the $\overline{\partial}$ probem for 2D orthogonal polynomials. By summarizing known results obtained via reductions to Riemann-Hilbert methods for special examples, team members developed a colection of approximations which should be valid for a large family of orthogonal polynomials, and attempted to arrive at a small-norm $\overline{\partial}$ probem amendable to known analytical methods.
- Peter Perry, in discussions with Peter Miller, Ken McLaughlin, and Samuli Siltanen, studied soliton solutions to the focussing DS II equation and the more general problem of so called *exceptional sets* where the scattering transforms have singularities. He initiated a study of determinants and soliton solutions for the Davey-Stewartson II equation. This led to a collaborative paper with colleague Russell Brown and graduate student Michael Music on determinants in inverse scattering [10]. One of the key insights that led to this paper–that the determinant itself satisfies a $\overline{\partial}$-equation that allows the determinant to be computed in terms of scattering data–originated in discussions at the 2012 FRG.

Outcome of the Meeting

The following publications have already resulted in part from discussions at this meeting: [10, 13].

The following subsequent meetings have been organized in part by participants in the FRG (participants in the FRG in boldface type):

- Exceptional Circle Helsinki Workshop, University of Helsinkii,[1] August 12-16, 2013 . Organized by **Samuli Siltanen** and Sarah Hamilton. This meeting involved eleven researchers from the United States and Europe.
- Conference and Workshop on Scattering and Inverse Scattering in Multi-Dimensions, University of Kentucky, [2] May 15-23, 2014, co-sponsored by the National Science Foundation, the Institute for Mathematics and its Applications, and the University of Kentucky. Organized by **Ken McLaughlin**, **Peter Miller**, and **Peter Perry**. This conference and workshop will include lectures by **Kari Astala** (Helsinki), James Colliander (Toronto), **Ken McLaughlin** (Arizona), **Peter Miller** (Michigan), **Peter Perry** (Kentucky), Andreas Stahel (Bern University), Paolo Santini (Rome), and Jean-Claude Saut (Paris-Sud). Approximately 40 participants,including 20 graduate and postdoctoral students, are expected.

The May 2014 meeting in the University of Kentucky will have the following research foci: (1) Semiclassical limits of $\overline{\partial}$-problems and dispersive equations, (2) Eigenvalue distributions of random normal matrices, (3) Direct scattering and exceptional sets, (4) Inverse scattering and exceptional sets, (5) One-dimensional limits of two-dimensional inverse problems, All of these foci grow out of discussions at the FRG and its successor meeting in Helsinki.

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[1] See conference webpage, `https://wiki.helsinki.fi/display/mathstatHenkilokunta/Exceptional+Circle+Workshop+2013`

[2] See conference webpage, `http://math.as.uky.edu/scattering-and-inverse-scattering-multidimensions`

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Bibliography

[1] M. Ablowitz, A. S. Fokas, The inverse scattering problem for multidimensional $2+1$ problems. In *Nonlinear Phenomena (Oaxtepec, 1982)*, pp. 137-183 Lecture Notes in Physics **189**, 1983.

[2] M. J. Ablowitz, A. S. Fokas, Method of solution for a class of multidimensional nonlinear evolution equations. *Phys. Rev. Lett.* **51** (1983), no. 1, 7–10,

[3] M. J. Ablowitz, A. S. Fokas, On the inverse scattering transform of multidimensional nonlinear equations related to first-order systems in the plane. *J. Math. Phys.* **25** (1984), no. 8, 2494–2505.

[4] M. J. Ablowitz, P. A. Clarkson, *Solitons, Evolution Equations, Inverse Scattering.* London Mathematical Society Lecture Note Series, **149**. Cambridge University Press, Cambridge, 1991.

[5] R. Beals, R. R. Coifman, Multidimensional inverse scatterings and nonlinear partial differential equations. *Pseudodifferential Operators and Applications* (Notre Dame, Ind., 1984), 45–70, *Proc. Sympos. Pure Math.*, **43**, Amer. Math. Soc., Providence, RI, 1985.

[6] R. Beals, R. R. Coifman, Linear spectral problems, nonlinear equations and the $\overline{\partial}$-method. *Inverse Problems* **5** (1989), no. 2, 87–130.

[7] R. Beals, R. R. Coifman, The spectral problem for the Davey-Stewartson and Ishimori hierarchies. In: A. Degasperies, Allan P. Fordy, and M. Lakshmanan, eds., *Nonlinear Evolution Equations: Integrability and Spectral Methods*. Manchester University Press, 1990.

[8] F. Balogh, M. Bertola, S. Y. Lee, K. D. T-R McLaughlin. Strong asymptotics of the orthogonal polynomials with respect to a measure supported on the plane. Preprint, `arXiv:1209.6366`.

[9] R. M. Brown, Estimates for the scattering map associated with a two-dimensional first-order system. *J. Nonlinear Sci.* **11** (2001), no. 6, 459–471.

[10] R. Brown, M. Music, P. Perry. Determinants in two-dimensional inverse scattering. *In preparation*.

[11] R. Buckingham, P. D. Miller. The sine-Gordon equation in the semiclassical limit: critical behavior near a separatrix. *J. Anal. Math.* **118** (2012), no. 2, 397âĂŞ492.

[12] R. Buckingham, P. D. Miller. The sine-Gordon equation in the semiclassical limit: dynamics of fluxon condensates. *Mem. Amer. Math. Soc.* **225** (2013), no. 1059.

[13] R. Croke, J. Mueller, M. Music, P. Perry, S. Siltanen, A. Stahel. The Novikov-Veselov equation: theory and computation. To appear in *Contemporary Mathematics*.

[14] P. Elbau, G. Felder. Density of eigenvalues of random normal matrices. *Comm. Math. Phys.* **259** (2005), no. 2, 433âĂŞ450.

[15] Grinevich, P. G. Rational solitons of the Veselov-Novikov equations are reflectionless two-dimensional potentials at fixed energy. *Teoret. Mat. Fiz.* **69** (1986), no. 2, 307–310.

[16] Grinevich, P. G. The scattering transform for the two-dimensional Schrödinger operator with a potential that decreases at infinity at fixed nonzero energy. *Uspekhi Mat. Nauk* **55** (2000), no. 6(336), 3–70; translation in *Russian Math. Surveys* **55** (2000), no. 6, 1015–1083.

[17] Grinevich, P. G., Manakov, S. V. Inverse problem of scattering theory for the two-dimensional Schrödinger operator, the $\overline{\partial}$-method and nonlinear equations. *Funktsional. Anal. i Prilozhen.* **20** (1986), no. 2, 14–24, 96. English translation: *Functional Anal. Appl.* **20** (1986), no. 2, 94–103.

[18] Grinevich, P. G., Novikov, R. G. Analogues of multisoliton potentials for the two-dimensional Schrödinger operator. *Funktsional. Anal. i Prilozhen.* **19** (1985), no. 4, 32–42, 95. English translation: *Functional Anal. Appl.* **19** (1985), no. 4, 276–285.

[19] Grinevich, P. G., Novikov, R. G. Analogues of multisoliton potentials for the two-dimensional Schrödinger operator, and a nonlocal Riemann problem. *Dokl. Akad. Nauk SSSR* **286** (1986), no. 1, 19–22. English translation: *Soviet Math. Dokl.* **33** (1986), no. 1, 9–12.

[20] Grinevich, P. G., Novikov, S. P. Inverse scattering problem for the two-dimensional Schrödinger operator at a fixed negative energy and generalized analytic functions. *Plasma theory and nonlinear and turbulent processes in physics, Vol. 1, 2* (Kiev, 1987), 58–85, World Sci. Publishing, Singapore, 1988.

[21] Grinevich, P. G., Novikov, S. P. A two-dimensional "inverse scattering problem" for negative energies, and generalized-analytic functions. I. Energies lower than the ground state. *Funktsional. Anal. i Prilozhen.* **22** (1988), no. 1, 23–33, 96; translation in *Funct. Anal. Appl.* **22** (1988), no. 1, 19–27

[22] Grinevich, P. G., Novikov, R. G. Transparent potentials at fixed energy in dimension two. Fixed-energy dispersion relations for the fast decaying potentials. *Comm. Math. Phys.* **174** (1995), no. 2, 409–446.

[23] A. R. Its., L. Takhtajan. Normal matrix models, dbar-problem, and orthogonal polynomials on the complex plane. Preprint, `arXiv:0708.3867`, 2007.

[24] Lassas, M., Mueller, J. L., Siltanen, S. Mapping properties of the nonlinear Fourier transform in dimension two. *Comm. Partial Differential Equations* **32** (2007), no. 4-6, 591–610.

[25] Lassas, M., Mueller, J. L., Siltanen, S., Stahel, A. The Novikov-Veselov equation and the inverse scattering method, Part I: Analysis. *Phys. D* **241** (2012), no. 16, 1322–1335.

[26] M. Music.

[27] M. Music, P. Perry, and S. Siltanen. Exceptional circles of radial potentials. *Inverse Problems* **29** (2013), no. 4, 045004.

[28] Nachman, A. I. Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. of Math.* (2) **143** (1996), no. 1, 71–96.

[29] P. Perry. Global well-posedness and large-time asymptotics for the defocussing Davey-Stewartson II equation in $H^{1,1}(\mathbb{R}^2)$. To appear in *J. Spectral Theory*.

[30] P. Perry. Miura maps and inverse scattering for the Novikov-Veselov equation. To appear in *Analysis and Partial Differential Equations*.

[31] L.-Y. Sung, An inverse scattering transform for the Davey-Stewartson II equations. I, II,III. *J. Math. Anal. Appl.* 183 (1994), no. 1, 121–154; no. 2, 289–325; no. 3, 477–494.

Chapter 50

The p-adic Langlands Program for Non-Split Groups (12frg159)

August 19 - 26, 2012

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Overview of the Field

The Langlands program was originally formulated as a link between number theory and analysis, but over the last 40 years it has grown to link together much of pure mathematics and parts of theoretical physics. Results on the Langlands program have been at the heart of many of the most spectacular developments in number theory in the last 20 years, including the proofs of Fermat's Last Theorem, Serre's Conjecture and the Sato–Tate conjecture. The p-adic Langlands program is an exciting recent generalisation of the Langlands program, which has already led to major results in number theory, in particular the proofs of the two-dimensional Fontaine–Mazur conjecture by Emerton and Kisin ([2],[4]).

Recent Developments and Open Problems

The p-adic Langlands program is still at a nascent stage, and at present the local correspondence only exists for the group $\mathrm{GL}_2/\mathbb{Q}_p$, and the global correspondence only for $\mathrm{GL}_2/\mathbb{Q}$. Experience with the classical Langlands program has shown that the full strength of the correspondence is only apparent when one works with multiple groups at once, at which point Langlands' functoriality principle becomes one of the most powerful tools available to number theorists. In particular, it is frequently vital to be able to work with other forms of a given group; for example, when making arguments with modular forms, it is frequently helpful to be able to pass to quaternion algebras, which are non-split forms of GL_2, and the proof of the Sato–Tate conjecture heavily relies on the use of unitary groups, which are non-split forms of GL_n. It is therefore of great interest to explore the p-adic Langlands correspondence for non-split groups, as one anticipates that similar advantages will be gained here, just as in the classical case.

Scientific Progress Made

Much of our progress was centered around understanding an inertial version of the existing p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, and understanding the interaction of the p-adic local Langlands correspondence with the Taylor–Wiles–Kisin patching method, with a view towards generalisations to non-split quaternion algebras, and extensions of $\mathbb{Q}_p$.

The classical local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ gives a bijection between Frobenius semisimple representations of the Weil–Deligne group of $\mathbb{Q}_p$ over any algebraically closed field of characteristic zero, and irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over the same field. Here we will restrict our attention to Frobenius semisimple representations of the Weil group $W_{\mathbb{Q}_p}$ itself (equivalently, Weil–Deligne representations with trivial monodromy operator); this corresponds to omitting the Steinberg representation and its twists from the $\mathrm{GL}_2(\mathbb{Q}_p)$ side of the correspondence.

The Weil group representations are naturally organized into continuous families, by placing two representations into the same component of the family if their restrictions to inertia coincide. (The point is that the image of inertia under a representation of the Weil group is stipulated to be finite, and it cannot vary continuously; but the Frobenius eigenvalues can be made to vary in a family.) Similarly, the irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ can be arranged in families (this is part of the theory of the *Bernstein centre*), with two representations lying in the same family if they share a common minimal type. (If π is any irreducible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, it decomposes into a direct sum of irreducible subrepresentations of $\mathrm{GL}_2(\mathbb{Z}_p)$, and there is a unique such subrepresentation that is minimally ramified, which we call the *minimal type* of π.)

Henniart [3] has shown that there is an *inertial local Langlands correspondence* between those two-dimensional representations of the inertia group $I_{\mathbb{Q}_p}$ which extend to representations of $W_{\mathbb{Q}_p}$, and those representations of $\mathrm{GL}_2(\mathbb{Z}_p)$ which arise as the minimal K-type of a smooth irreducible $\mathrm{GL}_2(\mathbb{Q}_p)$-representation, so that the family of Weil group representations with a fixed restriction to inertia matches via local Langlands with the family of smooth irreducible $\mathrm{GL}_2(\mathbb{Q}_p)$-representations with the corresponding minimal K-type.

Let D_p denote the unique ramified quaternion algebra over $\mathbb{Q}_p$. The local Jacquet–Langlands correspondence induces a bijection between the irreducible smooth representations of $D_p^\times$ (which are necessarily finite-dimensional, since $D_p^\times$ is compact modulo its centre) and those irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ which are *not* principal series representations. It may seem surprising that a finite-dimensional representation of the group $D_p^\times$, which is essentially compact, can carry the same amount of information as an infinite-dimensional representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, but one can note that the $\mathrm{GL}_2(\mathbb{Q}_p)$-representations in the image of the Jacquet–Langlands correspondence are determined up to a twist by their minimal type (since in the non-principal series case, the family of representations with a fixed minimal type is simply a family of twists), and so one can essentially regard the local Jacquet–Langlands correspondence as matching representations of two compact groups (namely $D_p^\times$ modulo its centre, and $\mathrm{GL}_2(\mathbb{Z}_p)$ modulo its centre).

One project that we emphasised during the workshop is to study analogues of the inertial local Langlands correspondence and of the local Jacquet–Langlands correspondence in the context of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$; recall that this correspondence has been constructed by Colmez and Paškūnas [1, 5], and associates to any continuous representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ (with E a finite extension of $\mathbb{Q}_p$) a corresponding admissible continuous unitary Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E.

Whereas a smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E will decompose as a direct sum of finite-dimensional irreducible $\mathrm{GL}_2(\mathbb{Z}_p)$-subrepresentations, this need not be true of a continuous representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on an E-Banach space. Indeed, if $\Pi(\rho)$ is the Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to some continuous $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ via p-adic local Langlands, then $\Pi(\rho)$ does not contain any non-zero finite-dimensional $\mathrm{GL}_2(\mathbb{Z}_p)$-subrepresentation unless ρ is de Rham (up to a twist), and even in this case $\Pi(\rho)$ will not be semisimple as a $\mathrm{GL}_2(\mathbb{Z}_p)$-representation. In most cases we actually expect $\Pi(\rho)$ to be topologically irreducible as a $\mathrm{GL}_2(\mathbb{Z}_p)$-representation (even though it is infinite-dimensional!). Thus we cannot expect to define a notion of minimal type in the context of the p-adic local Langlands correspondence.

However, this non-semisimplicity suggests the following alternative approach to phrasing the inertial local Langlands correspondence in the p-adic context. Namely, during the workshop we formulated the following conjecture.

Conjecture 1 If ρ and ρ' are two continuous representations of $G_{\mathbb{Q}_p}$ over E, then there is a natural isomorphism $\mathrm{Hom}_{I_p}(\rho, \rho') \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\Pi(\rho), \Pi(\rho'))$.

Note that this would simply be false in the context of the classical local Langlands correspondence, already in the case when $\rho = \rho'$, since an infinite-dimensional smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is a direct sum of an infinite number of irreducible $\mathrm{GL}_2(\mathbb{Z}_p)$-representations. The reason that it has a chance to be true in the p-adic case is the non-semisimple nature of the $\mathrm{GL}_2(\mathbb{Z}_p)$-action on $\Pi(\rho)$ and $\Pi(\rho')$ that was noted above.

We expect to prove this conjecture by using the description of the p-adic local Langlands correspondence in terms of (φ, Γ)-modules [1].

Establishing a p-adic local Jacquet–Langlands correspondence will be much more difficult then establishing the inertial correspondence, since currently very little is known about the p-adic representation theory of the group $D_p^\times$. Furthermore, the classical local Jacquet–Langlands correspondence is characterized by character identities, and we don't have character theory available in the p-adic context.

Nevertheless, we are hopeful that we can obtain a correspondence. Since so little is known about the p-adic representation theory of $D_p^\times$, it seems safest to use the relationship with Galois representations as an anchor, and so during the workshop we formulated the following conjecture.

Conjecture 2 There is an injection $\rho \hookrightarrow \Pi^{\mathrm{JL}}(\rho)$ from the isomorphism classes of continuous representations $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ to the isomorphism classes of admissible unitary continuous E-Banach space representations $\Pi^{\mathrm{JL}}(\rho)$ of $D_p^\times$.

Recalling that $\Pi(\rho)$ denotes the E-Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to ρ as in the preceding definition via the p-adic local Langlands correspondence, we would then declare $\Pi(\rho)$ and $\Pi^{\mathrm{JL}}(\rho)$ to be related by the p-adic Jacquet–Langlands correspondence.

Just an in the classical case, it seems strange at first that one might hope to match representations of the essentially compact group $D_p^\times$ with representations of the non-compact group $\mathrm{GL}_2(\mathbb{Q}_p)$, and one of the points of establishing Conjecture 1 is to allay this concern: this conjecture shows that little information about $\Pi(\rho)$ is lost by restricting to the compact group $\mathrm{GL}_2(\mathbb{Z}_p)$. Just as we explained above that in the classical local Jacquet–Langlands correspondence one is more-or-less matching representations of the essentially compact group $D_p^\times$ with representations of the compact group $\mathrm{GL}_2(\mathbb{Z}_p)$, the same will be true in the p-adic setting — except that now the representations will be infinite-dimensional!

A second point to note is that, unlike in the classical local Jacquet–Langlands correspondence, we do not restrict the ρ that we consider. (In the classical Jacquet–Langlands correspondence, omitting principal series representations from the correspondence corresponds to omitting reducible Weil group representations on the other side of the local Langlands correspondence.) Our reason for believing that no such restriction is necessary is as follows: the p-adic local Langlands correspondence is compatible with p-adic interpolation, and we expect that the p-adic Jacquet–Langlands correspondence should be similarly compatible. But on the Galois side, those ρ which are de Rham with distinct Hodge–Tate weights, and whose associated Weil–Deligne representations are irreducible, are Zariski dense in the space of all two-dimensional ρ. Thus if we imagine that there is some way to interpolate the classical Jacquet–Langlands correspondence into a p-adic correspondence, there should be no restriction on the ρ that we consider.

We expect the p-adic Jacquet–Langlands correspondence to be compatible with the classical Jacquet–Langlands correspondence in the following manner, namely that $\Pi^{\mathrm{JL}}(\rho)$ will contain a finite-dimensional subrepresentation of $D_p^\times$ if and only if ρ is de Rham with distinct Hodge–Tate weights (up to a twist), and this finite-dimensional subrepresentation will match with the Weil–Deligne representation associated to ρ (up to a twist by an algebraic representation depending on the Hodge–Tate weights of ρ) via the composition of local Langlands and classical Jacquet–Langlands.

One approach to constructing the p-adic Jacquet–Langlands correspondence that we intend to pursue in future work is global, making use of techniques related to Taylor–Wiles–Kisin patching; note that Taylor–Wiles–Kisin patching is applicable in this context because the group $D_p^\times$ is compact modulo its centre — so this brings out the importance for our strategy of working with (essentially) compact groups, and lends additional importance to proving Conjecture 1. With this in mind, we spent some of the workshop investigating Taylor–Wiles–Kisin patching in relation to the p-adic Langlands correspondence. We now explain what we discovered.

We assume that p is an odd prime, and that k is a finite field of characteristic p. Let F be a totally real field, and $\overline{\rho} : G_F \to \mathrm{GL}_2(k)$ a continuous representation which is irreducible and modular, and satisfies the additional assumptions necessary to apply the Taylor–Wiles–Kisin method, namely that $\overline{\rho}_{|G_{F(\zeta_p)}}$ is absolutely irreducible, with a further technical assumption if $p = 5$.

We fix a quaternion algebra D over F, split at exactly one archimedean prime and unramified at the primes above p, so that D determines a family of Shimura curves over F. For an ideal $\mathfrak{n}$ in the ring of integers $\mathcal{O}_F$, prime to p and to the discriminant of D, we let $X_1(\mathfrak{n})$ denote the corresponding Shimura curve of level $\mathfrak{n}$. We assume that $\mathfrak{n}$ is chosen so that $X_1(\mathfrak{n})$ has no elliptic points (i.e. the congruence subgroup that determines it is torsion free). We fix a prime v of F above p, and let $\mathcal{O}_{F_v}$ denote the completion of $\mathcal{O}_F$ at v.

We fix a finite extension E of $\mathbb{Q}_p$ (which will serve as our coefficient field), with ring of integers $\mathcal{O}_E$, uniformizer ϖ_E, and residue field k_E, which we assume contains k (so that we may regard ρ as being defined over k_E). If L is any finitely generated $\mathcal{O}_E$-module equipped with a smooth representation of $\mathrm{GL}_2(\mathcal{O}_{F_v})$, then L determines a local system $\widetilde{L}$ on $X_1(\mathfrak{n})$ (actually, one has to take a little care with central characters in order for this to be true, but we suppress that detail in this discussion), and we may consider the cohomology group $H^1(X_1(\mathfrak{n}), \widetilde{L})_{\overline{\rho}}$, where the subscript $\overline{\rho}$ indicates that we complete at the ideal corresponding to $\overline{\rho}$ in the Hecke algebra generated by Hecke operators at primes away from $\mathfrak{n}$, p, and the discriminant of D. (This ideal is either maximal, or else the unit ideal.)

The Taylor–Wiles–Kisin method allows us, by adding carefully chosen auxiliary primes to the level $\mathfrak{n}$, to pass to a limit and "patch" these cohomology groups into a coherent sheaf $M_\infty(L)$ over the local deformation ring $R_v(\overline{\rho})$ which parameterizes framed deformations of $\overline{\rho}_{|G_{F_v}}$ over complete local $\mathcal{O}_E$-algebras, with some auxiliary formal variables adjoined (the *patching variables*). In fact a careful application of the method shows that we may perform this patching compatibly for all choices of L, and so regard $M_\infty(L)$ as a functor from the category of $\mathrm{GL}_2(\mathcal{O}_{F_v})$-modules to the category of coherent sheaves on $\operatorname{Spec} R_v(\overline{\rho})[[x_1, \dots, x_n]]$ (where $x_1, \dots, x_n$ are the patching variables).

Suppose for a moment that $F = \mathbb{Q}$, so that p-adic local Langlands and local-global compatibility are available. One can then give a different construction of the patched modules $M_\infty(L)$ which shows that, despite the global nature of their construction, they are in fact of purely local nature. Indeed, one of us [5] has constructed a universal representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\operatorname{Spec} R_p(\overline{\rho})$, which we denote by $\mathcal{P}$, and which realizes p-adic local Langlands, in the sense that the fibre of $\mathcal{P}$ over a point corresponding to a continuous lifting $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ of $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is the dual to the Banach space representation $\Pi(\rho)$ attached to ρ via p-adic local Langlands. (The appearance of a dual here is a purely technical point.) During the workshop we showed, using the local-global compatibility result of [2], together with the compatibility of classical and p-adic local Langlands, that the patched module $M_\infty(L)$ is equal to the basechange from $R_p(\overline{\rho})$ to $R_p(\overline{\rho})[[x_1, \dots, x_n]]$ of the $R_p(\overline{\rho})$-module $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\mathcal{P}, L^\vee)^\vee$ (where again, the appearance of the various duals is purely technical). Heuristically, this expresses the fact that $M_\infty(L)$ is supported at exactly those Galois representations ρ whose associated $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\Pi(\rho)$ contains a non-zero quotient of L as a $\mathrm{GL}_2(\mathbb{Z}_p)$-subrepresentation.

Outcome of the Meeting

We are now writing a joint paper that will prove Conjecture 1 and explain the connections between the Taylor–Wiles–Kisin method and the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

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Bibliography

[1] Pierre Colmez Représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et (φ, Γ)-modules. *Astérisque* 330 (2010), 283–512.

[2] Matthew Emerton. Local-global compatibility in the p-adic Langlands programme for $\mathrm{GL}_2/\mathbb{Q}$.

[3] Guy Henniart Appendix to Breuill–Mézard's paper. *Duke Math. J.*, 115 (2002), no. 2, 205âĂŞ310.

[4] Mark Kisin. The Fontaine-Mazur conjecture for GL_2. *J. Amer. Math. Soc.*, 22(3):641–690, 2009.

[5] Vytautas Paškūnas. The image of Colmez's Montreal functor.

Chapter 51

A t-Pieri rule for Hall-Littlewood P-functions and $QS(t)$-functions (12frg164)

August 26 - September 2, 2012

Organizer(s): Soojin Cho (Ajou University), Sarah Mason (Wake Forest University), Jim Haglund (University of Pennsylvania), Vasu Tewari (University of British Columbia), Steph van Willigenburg (University of British Columbia), Meesue Yoo (Korea Institute for Advanced Study)

Overview of the Field

One of the central areas of research in algebraic combinatorics is the study of Macdonald polynomials. These were introduced in the late 1980s by Macdonald as symmetric polynomials with two parameters q, t, which reduce to certain well-known polynomials such as Hall-Littlewood P-functions, Jack polynomials and zonal polynomials when q and t are set to certain values. Despite their relatively straightforward definition in terms of a scalar product, they were notoriously difficult to manipulate until a combinatorial formula for them was discovered [2].

Another vibrant area of research in algebraic combinatorics is that of quasisymmetric functions. The Hopf algebra of quasisymmetric functions was introduced by Gessel in the 1980s as a source of generating functions for P-partitions. However, since then, like Macdonald polynomials, they have arisen in other areas of mathematics such as the representation theory of the 0-Hecke algebra, encoding the flag f-vector of posets in discrete geometry, and the study of riffle shuffles in probability.

At the natural intersection of these two areas are the symmetric functions known as Schur functions, which are refined by quasisymmetric functions and are specializations of Macdonald polynomials at $q = t = 0$. Schur functions are themselves a hub of investigation due to their appearance as the irreducible characters of the symmetric group in representation theory, connections to Schubert classes in algebraic geometry and as generating functions for tableaux in enumerative combinatorics.

Recent Developments and Open Problems

In [4] a new basis for quasisymmetric functions, termed quasisymemtric Schur functions, was discovered. These functions were founded on the combinatorics of Macdonald polynomials, were further developed in

[1, 3], and furthermore showed that many of the beautiful properties of Schur functions can be refined to the Hopf algebra of quasisymmetric functions. Amongst the many properties that refine from Schur functions to quasisymmetric Schur functions are the expansion in terms of fundamental quasisymmetric functions, Kostka numbers, two formulations of the Littlewood-Richardson rule, and the Pieri rules. The Pieri rules for Schur functions are rules for expressing the product of a Schur function indexed by a diagram of row or column shape with a generic Schur function indexed by a diagram of partition shape as a sum of Schur functions whose indices are determined by operations on the diagram of partition shape. Similarly, the Pieri rules for quasisymmetric Schur functions are rules for expressing the product of a Schur function indexed by a diagram of row or column shape with a generic quasisymmetric Schur function indexed by a diagram of composition shape as a sum of quasisymmetric Schur functions whose indices are determined by operations on the diagram of composition shape. In all of these cases classical proofs for the Schur function scenario were able to be adapted in order to yield proofs for the quasisymmetric Schur function case.

In [4] it was shown that some of the combinatorial structure of the quasisymmetric Schur functions can be extended to include an extra parameter t. In particular they showed that these $QS(t)$-functions refine a well-known Hall-Littlewood symmetric function basis in the same way that quasisymmetric Schur functions refine Schur functions, that is, that the sum over all compositions, whose parts rearrange the parts of a given partition, of the $QS(t)$-function corresponding to the composition, equals the Hall-Littlewood P-function corresponding to the partition. Therefore, what other properties of Hall-Littlewood P-functions refine to $QS(t)$-functions?

Recently Haglund noted empirically using Maple that the expansion of the product of a Schur function and a Hall-Littlewood P-function, when expanded in terms of the P-function basis, has coefficients which are polynomials in t with nonnegative integral coefficients. In the special case when we multiply by a Schur function indexed by a diagram of row shape, Yoo refined this conjecture and obtained an elegant combinatorial description of these coefficients, called a t-Pieri rule. When $t = 0$ this reduces to the classical Pieri rule for Schur functions. Additionally, Yoo developed an elegant combinatorial conjecture for expressing the product of a Schur function indexed by a diagram of row or column shape with a $QS(t)$-function indexed by a diagram of composition shape, as a sum of $QS(t)$-functions whose indices are determined by operations on the diagram of composition shape that also generate powers of t and $(1-t)$.

Therefore, our major goal for the FRG was to prove these latter two conjecures as a means to understanding the t-Pieri rule conjecture of Yoo and eventually the conjecture of Haglund.

Scientific Progress Made

A key step in proving both the Pieri rules for Schur functions and the Pieri rules for quasisymmetric Schur functions is to prove the special case of a product with a Schur function indexed by the diagram consisting of one cell, called Monk's rule. Therefore our first step was to prove the conjecture for expressing the product of a Schur function indexed by the diagram consisting of one cell, s_1, with a generic $QS(t)$-function indexed by a diagram of composition shape α, QH_α.

Although we were not able to complete the proof of $QH_\alpha s_1$ during our FRG we were successful in developing a number of avenues necessary to adapt classical proofs for this case. More precisely, we

- reformulated Yoo's conjecture into a combinatorial identity equivalent to the original conjecture;
- extended the insertion algorithm of Mason, which, when fully verified, would be a fundamental ingredient to proving the above combinatorial identity bijectively;
- established that this new algorithm is reversible;
- conjectured that the change in power of t associated to a tableau before and after insertion is equal to the number of skipped steps that occur during insertion;
- extended all these facets in terms of superfillings, which generate an equivalent definition for QH_α and hence give a second approach to proving the conjecture;
- produced a plethora of data to support the algorithm.

Additionally, Yoo's conjecture was verified for the cases $QH_{(n)}s_1$, $QH_{(1^n)}s_1$, $QH_{(m1^{n-m})}s_1$ and when $t = 0$. Lastly, during an investigation of the dual Hopf algebra basis to $QS(t)$-functions, we conjectured that the image under the forgetful map is a modified Hall-Littlewood function, and that those indexed by a diagram of a partition or reverse partition shape when expressed as a sum of Schur functions have coefficients which are polynomials in t with nonnegative integral coefficients that can be explicitly computed combinatorially.

Outcome of the Meeting

Further to making substantial progress on our project from which a journal article will result, we also learned a range of techniques and relationships between our areas of expertise, through informal lectures we gave to each other.

We would like to thank BIRS for this invaluable Focussed Research Group opportunity.

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Bibliography

[1] C. Bessenrodt, K. Luoto, and S. van Willigenburg, Skew quasisymmetric Schur functions and noncommutative Schur functions, *Adv. Math.* **226** (2011), 4492Ð-4532.

[2] J. Haglund, M. Haiman, and N. Loehr, A combinatorial formula for Macdonald polynomials, *J. Amer. Math. Soc.* **18** (2005), 735Ð-761.

[3] J. Haglund, K. Luoto, S. Mason, and S. van Willigenburg, Refinements of the Littlewood-Richardson rule, *Trans. Amer. Math. Soc.* **363** (2011), 1665Ð-1686.

[4] J. Haglund, K. Luoto, S. Mason, and S. van Willigenburg, Quasisymmetric Schur functions, *J. Combin. Theory Ser. A* **118** (2011), 463Ð-490.

Chapter 52

The advent of Quark-Novae: Modeling a new paradigm in Nuclear Astrophysics (12frg183)

September 2 - 9, 2012

Organizer(s): Rachid Ouyed (University of Calgary)

Different aspects of the Quark-Nova model are being developed at universities around the world. For this workshop, contributors in this emerging field met to share and stimulate research in the astrophysics of the Quark-Nova.

First, we presented an overview of the progress made in the fundamental aspects of the quark-nova model and the quantum chromodynamics (QCD) phase transition at high density. Second, we examined the observational successes of the Quark-Nova model pertaining to nucleosynthesis and the chemical evolution of the Galaxy. Finally, we discussed new and general directions for the development and refinement of the Quark-Nova model. It was the intention to complete a thorough overview of the model, combining all aspects and making it accessible for an audience at the level of a physics or astrophysics graduate student.

What is a Quark-Nova?

A Quark-Nova (also referred to as a Quark Nova explosion; Ouyed et al. 2002) is the violent explosion resulting from the conversion of a neutron star core to quark matter. The result is a star made entirely of quarks; the quark star. The neutron star itself is formed from a SuperNova preceding the Quark-Nova.

The supernova explosion leaves behind a neutron star (the compact remnant) and the expanding stellar envelope (the supernova remnant). The fast rotation and misaligned magnetic field of the neutron star create an electromagnetic light-house effect. The iron crust encases a neutron-rich interior of the neutron star. The spin-down of the neutron star increases its core density to extreme densities (Staff et al. 2006), liberating quarks from neutrons through a process known as quark deconfinement. This phase transition releases neutrinos and photons that build up a fireball as the of the neutron star core collapses (Keränen et al. 2005). The outer layers are blown off during the Quark-Nova (Ouyed&Leahy 2009). If the delay between the super nova and Quark-Nova is on the order of weeks, the supernova remnant is re-energized by the Quark-Nova (Leahy&Ouyed 2008). This process would release immense amounts of energy, perhaps explaining the most energetic explosions in the universe (Ouyed et al. 2012); rough calculations have estimated that as much as 10^{47} joules of energy could be released from the phase transition inside a neutron star. Quark-novae could explain a multitude of astrophysical phenomena from gamma ray bursts through super-luminous super novae.

The Quark-Nova Model

Quark Deconfinement and Detonation

In the case of a neutron star burning into quark matter made up of Up (u), Down (d) and Strange (s) quarks, there is a radial density gradient in the interior core of the neutron star that the burning interface progresses through. As the interface burns outwards it reaches lower densities, making the interface move slower, until it reaches the critical point where the interface halts due to the deleptonization process (Niebergal et al. 2010). The critical point depends on a number of factors including the equation-of-state of neutron matter and quark matter, neutrino cooling efficiency, and most importantly the dynamics of the combustion in multiple dimensions (e.g. the wrinkling of the combustion front, sensitivity to instabilities, etc.; see Niebergal et al. 2010). The situation is then an under-pressured *(u,d,s)*-quark core, with the outer layers of the neutron star lying on top. The under-pressured core will eventually collapse - similar to the core collapse phase during a supernova - causing the outer lying material to fall onto the core, leading to a second explosion; *a Quark-Nova explosion*.

The QN neutron-rich Ejecta

Following a QN explosion, the neutron star's metal-rich outer layers are ejected. There are four possible scenarios for the state of the ejecta: first, if it is very light then it will become gravitationally unbound from the quark star (Ouyed&Leahy 2009); second, if the ejecta is too heavy it will fall back into the quark star releasing tremendous amounts of energy; third, the ejecta velocity can be low enough such that it remains bound to the QS and mass low enough that it can be suspended by the quark star's magnetic pressure; and in the fourth case, the ejecta velocity is also low enough to remain bound but its angular momentum is large enough (e.g. by the propellor mechanism) that the ejecta remains in a Keplerian orbit (Ouyed et al. 2007a; 2007b).

In this workshop we investigate the thermal and dynamic evolution of a relativistically expanding iron-rich shell from a QN explosion. The QN produces a photon fireball (Ouyed et al. 2005a) which acts as piston to eject and accelerate the crust of the parent neutron star to relativistic speeds. Although the presented model is based on physical arguments, the physics of the ejection is in reality more complicated and so would require more detailed studies and the help of numerical simulations. For example, the process of clumping, crystallization, and breakup of the ejecta, would require better knowledge of the ambient conditions surrounding the ejecta. The astrophysical implications will be discussed elsewhere.

The Compact Remnant (the Quark Star)

The Quark-Nova leaves behind a quark star in a superconducting quark matter phase (Vogt et al. 2004). Once the neutron star has made the transition from hadronic to superfluid-superconducting quark matter, and through a Meissner effect, the quark star's interior magnetic field is forced inside rotationally induced vortices of the superfluid quark matter. The vortices and interior magnetic field are aligned with the rotation axis of the star. The exterior dipole field is forced to align with the rotation axis immediately after the neutron star has made the transition to superconducting quark matter, as simulated in Ouyed et al. (2006); see also Ouyed et al. (2004).

Scientific Progress Made

The different aspects brought by each participant were combined to allow a discussion of what impact the QN Ejecta and Compact Remnant would have in the environment where a Quark Nova occurs. The discussion led to predictable observations that would lend support for the existence of Quark Novae.

Impact of the QN Ejecta in the environment

A promising source for the production of ultra high energy cosmic rays (UHECRs) is the relativistic shocks resulting from the interaction of the QN ejecta with its environment. Such a connection has been reported and investigated in Ouyed et al. (2005b). QNe as possible UHECR sources seem to account for the observed extragalactic flux and can contribute at least partially to the galactic cosmic rays. The QN model for the acceleration of UHECRs seems to possess features that can be tested in future cosmic ray detectors. We thus predict UHECRs to be seen in connection with some events involving QNe.

Impact of the Compact Remnant in the environment

The fate of the ejected neutron star crust is determined by the initial conditions of the remnant (i.e. rotation period, magnetic field, and shell mass). One example of such an outcome is the propeller mechanism. This results in a degenerate Keplerian torus (Ouyed et al. 2007b) forming from the ejected matter rather than a co-rotating shell (Ouyed et al. 2007a).

Outcome of the Meeting

Aside from reviewing the progress made on each aspect of the model, the meeting was also used to solidify the general and future directions of research into the Quark Nova. During the workshop we started the process of writing a review paper on Quark-Novae. Currently we are in the late stages of finishing the paper.

Predictions and Observation

Surveys that search for Type-II Supernovae such as the Palomar Transient Factory (PTF) have the potential to observe a Quark Nova. These surveys search for transient events and follow up using larger telescopes around the world. If a Quark Nova were to be observed, a characteristic double-hump would be observed as the luminosity peaked twice (see Ouyed et al. 2009). Once for the supernova explosion (preceding the Quark-Nova), and again for the Quark-Nova proper. This would constitute a unique photometric signature of a Quark-Nova (Ouyed et al. 2012; Ouyed&Leahy 2012).

Which astrophysical phenomenon will produce measurable gravitational waves is still an open question. Once gravitational wave detectors begin to collect data we may learn that core collapse Supernovae produce a distinct gravitational wave signature. If this is the case then we would expect an event where a Supernova is followed by a Quark-Nova to produce the unique gravitational wave signature of two successive pulses separated in times (by a few hours to a few weeks; see Staff et al. 2012).

Spatial Simulations Needed

From the meeting, it became apparent that simulations of the hadronic-to-quark-matter phase transition inside the neutron star need to be extended to 2D and 3D in order to capture the QN explosion in details.

Mathematical Model

Another outcome of the meeting was an answer to the need to build a more robust mathematical model to describe in more detail the interaction between the Quark-Nova ejecta and the proceeding Supernova ejecta/envelope. We were able to devise an approach to make the mathematical model used for the Quark-Nova more mathematically sound and go beyond current approximations.

Our current Superluminous Supernovae calculations assume a completely uniform Supernova envelope and do not account for an inhomogeneous density profile or element stratification (distribution of different elements in the envelope) Aside from leaving these assumptions, a more realistic model would take into account opacities and shock physics.

The nuclear spallation calculations for the Quark-Nova model (Ouyed et al. 2011; Ouyed 2012) will be extended with a more accurate mathematical framework (beyond the layer approximation) and keeping track of individual isotopes. This involves working will more refined interaction cross-sections, and using up-to-date spallation cross-sections from literature. We will then make the calculations taking into account all of the heating mechanism used by spallation reaction. The production of heavy elements during the expansion of the QN ejecta (Jaikumar et al. 2007) will be explored in more detail in the future.

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Bibliography

[1] Jaikumar, P., Meyer, B. S., Otsuki, K., & Ouyed, R. 2007, Astronomy&Astrophysics, 471, 227

[2] Keränen, P., Ouyed, R., & Jaikumar, P. 2005, The Astrophysical Journal, 618, 485

[3] Leahy, D., & Ouyed, R. 2008, Monthly Notices of the Royal Astronomical Society, 387, 1193

[4] Niebergal, B., Ouyed, R., & Jaikumar, P. 2010, Physical Review C, 82, 062801

[5] Ouyed, R., Dey, J., & Dey, M. 2002, Astronomy&Astrophysics , 390, L39

[6] Ouyed, R., Elgaroy, O., Dahle, H., & Keränen, P. 2004, Astronomy&Astrophysics, 420, 1025

[7] Ouyed, R., Rapp, R., & Vogt, C. 2005a, The Astrophysical Journal, 632, 1001

[8] Ouyed, R., Keränen, P., & Maalampi, J. 2005b, The Astrophysical Journal, 626, 389

[9] Ouyed, R., Niebergal, B., Dobler, W., & Leahy, D. 2006, The Astrophysical Journal, 653, 558

[10] Ouyed, R., Leahy, D., & Niebergal, B. 2007a, Astronomy&Astrophysics, 473, 357

[11] Ouyed, R., Leahy, D., & Niebergal, B. 2007b, Astronomy&Astrophysics, 475, 63

[12] Ouyed, R., & Leahy, D. 2009, The Astrophysical Journal, 696, 562

[13] Ouyed, R., Leahy, D., & Jaikumar, P. 2009, Proceedings for "Compact stars in the QCD phase diagram II (CSQCD II)", May 20-24, 2009, KIAA at Peking University, Beijing- P. R. China, [http://vega.bac.pku.edu.cn/rxxu/csqcd.htm], [arXiv:0911.5424]

[14] Ouyed, R., Leahy, D., Ouyed, A., & Jaikumar, P. 2011, Physical Review Letters, 107, 151103

[15] Ouyed, R., Kostka, M., Koning, N., Leahy, D. A., & Steffen, W. 2012, Monthly Notices of the Royal Astronomical Society, 423, 1652

[16] Ouyed, R. 2012, Monthly Notices of the Royal Astronomical Society, 9, In Press [arXiv:1203.2703]

[17] Ouyed, R., & Leahy, D. 2012, [arXiv:1202.2400]

[18] Staff, J. E., Ouyed, R., & Jaikumar, P. 2006, The Astrophysical Journal, 645, L145

[19] Staff, J. E., Jaikumar, P., Chan, V., & Ouyed, R. 2012, The Astrophysical Journal, 751, 24

[20] Vogt, C., Rapp, R., & Ouyed, R. 2004, Nuclear Physics A, 735, 543

Chapter 53

Generalized Gauss Maps and Farey Statistics (12frg172)

September 16 - 23, 2012

Organizer(s): J.S.Athreya, F.-P.Boca, J.Chaika, Y.Cheung, N.Chevalier, S.Lelievre, J.Tseng, A.Quas

Abstract

This is a short summary of the work done during our Banff FRG Workshop (September 16-23, 2012). We focus on the issue of the mixing of the BCZ map, as this was our primary subject of discussion.

Background

The workshop focused on recent developments in homogeneous dynamics, as related to Diophantine approximation. Boca, Cobeli and Zaharescu, in the early 2000s, made a study of the fine statistics of Farey fractions. To a first approximation, the Farey fractions with denominator up to Q are uniformly distributed in the unit interval. The distribution of gaps between successive Farey fractions, however, was found to be far from the exponential distribution that one might naively expect. In fact, with suitable normalization, the distribution is seen to be piecewise analytic. More recently, Athreya and Cheung unified the study of Boca, Cobeli and Zaharescu with the study of horocycle flow on the space of unimodular lattices. The purpose of the workshop was to further develop this circle of ideas, and set out future research directions. The participants had a fairly wide array of research specialties: interval exchange transformations, homogeneous dynamics, C^* algebras, Diophantine approximation, translation surfaces and ergodic theory.

The BCZ map

Motivated by the study of Farey fractions, Boca-Cobeli-Zaharescu [5] introduced the *BCZ map* $T : \Omega \to \Omega$ on the triangle $\Omega := \{(a, b) : a, b \in (0, 1], a + b > 1)\}$. T acts via

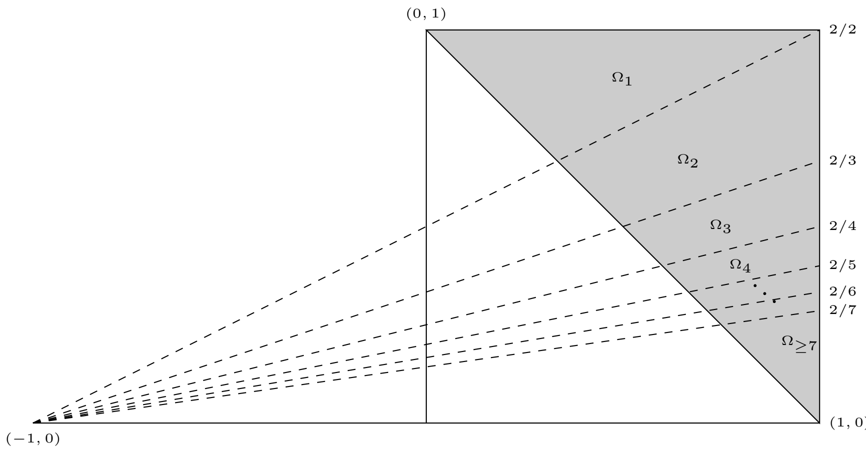
$$T(a, b) = (a, b)A_k^T,$$

where

$$A_k = \begin{pmatrix} 0 & 1 \\ -1 & k \end{pmatrix}$$

on the region $\Omega_k := \{(a,b) \in \Omega : \kappa(a,b) = k\}$, where $\kappa(a,b) = \left\lfloor \frac{1+a}{b} \right\rfloor$. Boca-Zaharescu [4] posed several questions about the ergodic properties of this map:

Figure 53.1: The Farey triangle $\Omega = \bigcup_{k \geq 1} \Omega_k$



Question 1. *Is T ergodic? Mixing? What is the entropy of T?*

Several questions were answered by Athreya-Cheung [3], who showed:

Theorem 1. *T is a first return map of the horocycle flow on the space of lattices $SL(2,\mathcal{R})/SL(2,\mathbb{Z})$. Let $h_s = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}$, the first return map to the transversal $\tilde{\Omega} := \left\{ \Lambda_{a,b} := \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} SL(2,\mathbb{Z}) : (a,b) \in \Omega \right\}$ is given by T. The return time function is $R(a,b) = \frac{1}{ab}$. That is, $h_{R(a,b)}\Lambda_{a,b} = \Lambda_{T(a,b)}$.*

Using the ergodicity, zero-entropy, and measure rigidity of the horocycle flow, this result shows that the BCZ map is ergodic, zero-entropy, and that in fact Lebesgue measure is the unique ergodic invariant measure not supported on a periodic orbit. However, mixing (and weak mixing) are not properties that pass from a flow to a section. Mixing of the BCZ map would also have some interesting number theoretic applications. Thus, a motivating question of the workshop was:

Question 2. *Is the BCZ map mixing? Is it weak mixing?*

Gauss Map Dreams

Another motivating question of the workshop was to place the BCZ map in the general context of naturally ocurring, geometrically meaningful cross-sections to various homogeneous flows. The most classical example of such is the (natural extension) of the Gauss Map, which links geodesic flow on the modular surface $SL(2,\mathcal{R})/SL(2,\mathbb{Z})$ to the study of continued fractions. More generally, the study of these maps gives number theoretic and statistical applications. Thus, we had a general goal:

Idea 1. *The BCZ is a 'Gauss Map' for horocycle flow'. Try and find more Gauss maps explicitly computable transversals and return maps for well-known flows, especially so that number theoretic quantities are computable. That is, can we fill more entries in the table below?*

Interesting Invariants	Farey Statistics	Gauss Map	Renormalization Dynamics	'Resident' Dynamics
	Levy Constant $\frac{\pi^2}{12\log 2}$	$G(x) = \{\frac{1}{x}\}$	geodesic flow on $SL(2,\mathcal{R})/SL(2,\mathbb{Z})$	circle rotations
	Levy Constants	Cheung-Chevalier Map	diagonal flow on $SL(n,\mathcal{R})/SL(n,\mathbb{Z})$	toral translation
		Rauzy Induction	Teichmüller Flow on Strata of Hodge Bundle Ω_g	IETs/Translation Surfaces
Index	Farey Sequences	BCZ	horocycle flow on $SL(2,\mathcal{R})/SL(2,\mathbb{Z})$	
	Generalized Farey Sequences		Horospherical Flows on $SL(n,\mathcal{R})/SL(n,\mathbb{Z})$	

Table 53.1: Gauss Map Dreams

While there was significant discussion on these topics (particularly following the work of Cheung-Chevallier [6] on higher-order Levy constants and the work of Athreya-Chaika [1] and Athreya-Chaika-Lelievre [2] on translation surfaces, the primary focus of the workshop became the question of mixing of the BCZ map.

Mixing

There are well-known dynamical arguments to prove and disprove mixing in various examples. As our initial intuitions on the properties of BCZ varied, we decided to discuss two possible approaches; one to disprove mixing, using an argument of Katok [7] on Interval Exchange Transformations (IETs), and one to prove mixing, using an argument of Marcus [8] for horocycle flows.

Katok

J. Chaika, an expert on IETs, described Katok's argument on non-mixing of IETs, which relies on a double-inducing construction and controlling the sizes of various pieces of the associated towers. After some discussion, it became clear that this argument could not be used in this setting directly, and that there were some serious obstacles to employing even the general strategy. Namely, controlling the height and size of towers in the double-inducing construction was non-trivial.

Marcus

A. Quas, an expert in ergodic theory, described Marcus's theorem on mixing of horocycle flows, which relies crucially on proving equidistribution of geodesic segments when pushed forward under the horocycle flow. In our setting, geodesic segments in $SL(2,\mathcal{R})/SL(2,\mathbb{Z})$ translate to radial segments in the transversal, and the equidistribution desired becomes the following: for any suitably nice function v on Ω_1, we want, for any $z_0 \in \Omega$,

$$\int_0^b v(T^N z_{-t})dt \to b\int_{\Omega_1} vdm,$$

where $z_{-t} := g_{-t}z_0 := e^{-\frac{t}{2}}z_0$. Here, $g_{-t} = \begin{pmatrix} e^{-\frac{t}{2}} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix}$. acts via the action on the lattice

$$\Lambda_{z_0} = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}\mathbb{Z}^2.$$

The main thread of our discussions was developing a strategy to prove this equidistribution. Since if we were to replace the map T by the flow h_s, we know that this is true by Marcus, it became crucial to understand the time change created by inducing. Below, we discuss how this time change can be interpreted in many different ways, and in particular as a lattice point problem.

Time Changes and Lattice Points

Let $z_0 = (x, y) \in \Omega_1$, our standard BCZ transversal, and let $t \in [0, b]$ be a small parameter, and consider z_{-t} for $t \leq -2 \ln \frac{1}{x}$, which guarantees $z_{-t} \in \Omega_1$. Let $N > 0$, and let $s = s_{N,t}$ be such that

$$T^N(z_{-t}) = h_s g_{-t} z = g_{-t} h_{e^t s} z.$$

Then, since for all $v \in \Omega_1$, $g_t v \in \Omega_1$ for all $t > 0$, we have $h_{e^t s} z = g_t(g_{-t} h_{e^t s} z) \in \Omega_1$, so there is a $M_t > 0$ so that

$$h_{e^t s} z = T^{M_t}(z).$$

That is, we apply T^N to the entire radial segment $\{e^{\frac{t}{2}} z_0 : t \in [0, b]\}$, and want to express the point $T^N(e^{\frac{t}{2}} x, e^{\frac{t}{2}} y)$ as a point along the radial segment determined by $T^{M_t}(x, y)$, for some M_t. We want to understand the distribution of the quantity M_t along the segment $[0, b]$. The possible values of M_t range (roughly) between N and $e^b N$. Can we show that for an $M \in [N, e^b N]$ (a range of size $\approx bN$), that

$$|\{t \in [0, b] : M_t = M\}| \sim \frac{b}{bN} = \frac{1}{N}?$$

Roof Functions

The value $s = s_{N,t}$ is a partial sum of the roof function $R(x, y) = \frac{1}{xy}$ along the orbit of BCZ. Precisely, we have

$$s_{N,t} = \sum_{i=0}^{N} R(T^i(z_{-t})).$$

We also have

$$e^t s_{N,t} = \sum_{i=0}^{M_t - 1} R(T^i(z_0)).$$

This gives a heuristic for the expected size of M_t, using the ergodic theorem, since we expect the first sum to be $\approx N \int_\Omega R$ and the second to be $\approx M_t \int_\Omega R$, so we expect M_t to have size $e^t N_t$.

Lattice Points

We can interpret M_t and N in terms of (primitive) lattice points. $s_{N,t}$ is the slope of the N^{th} vector (where we order vectors by slope) in the lattice Λ_{z_0} in the vertical strip

$$V_{e^{-\frac{t}{2}}} = \{y \geq 0, 0 < x \leq e^{-\frac{t}{2}}\},$$

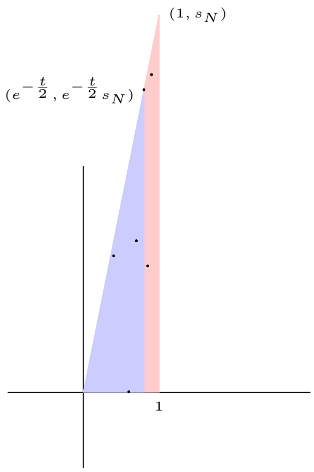
since this will be the N^{th} slope for the lattice $\Lambda_{z_{-t}}$ in the standard vertical strip $V_1 = \{y \geq 0, 0 < x \leq 1\}$. Then M_t is the total number of points in Λ_{z_0} in the triangle which is the standard trip V_1 bounded above by the line of slope $s_{N,t}$. That is, defining the triangles

$$T_t = \Delta((0,0), (1,0), (1, s_{N,t})) \text{ and } \tilde{T}_t = \Delta((0,0), (e^{-\frac{t}{2}}, 0), (e^{-\frac{t}{2}}, e^{-\frac{t}{2}} s_{N,t})),$$

we have (we are always counting only primitive points)

$$N = |\Lambda_{z_0} \cap \tilde{T}_t| \text{ and } M_t = |\Lambda_{z_0} \cap T_t|.$$

Figure 53.2: The triangle $\tilde{T}_t$ is in blue, and T_t is the union of the blue triangle and the red trapezoid. Marked points are primitive lattice points, so in this picture $N = 3$ and $M_t = 5$.



M_t and Mixing

Recall that to prove mixing, we want to prove equidistribution of geodesic segments, that is, we want to show that for any nice function v on Ω_1,

$$\int_0^b v(T^N z_{-t})dt \to b \int_{\Omega_1} vdm.$$

Using our definition of M_t, we have

$$\int_0^b v(T^N z_{-t})dt = \int_0^b v(T^{M_t} z_0)dt = \sum_{r=0}^{bN} v(T^{N+r} z_0)|\{t \in [0,b] : M_t = t + r\}|.$$

If we could show that

$$m_{t+r} = |\{t \in [0,b] : M_t = t + r\}| \sim 1/N,$$

then our integral would be an integral over the horocycle flow, and we could use unique ergodicity to get the equidistribution of this integral.

A natural interpretation of $|\{t \in [0,b] : M_t = t + r\}|$ is the *gap* between the r^{th} and $(r+1)^{st}$ x-coordinate (written in increasing order of x-coordinate) of primitive points in our lattice Λ_{z_0} in the trapezoidal strip with vertices at $(e^{-\frac{t}{2}}, 0), (1, 0), (e^{-\frac{t}{2}}, e^{-\frac{t}{2}} s_{N,t}), (1, s_{N,t})$, that is, the symmetric difference between the triangles T_t and $\tilde{T}_t$.

Independence

It is not clear that the values $|\{t \in [0,b] : M_t = t + r\}|$ equidistribute. Another idea is to show that $T^k(z_0)$ (which yields, in particular, the horizontal component of the k^{th} vector that we see in the vertical strip V_1) becomes *independent* of the length $m_k = |\{t \in [0,b] : M_t = k\}|$. A precise version version of this is the following: fix $b > 0$, let $N \gg 0$, and consider the collection of points

$$\{(T^k(z_0), Nm_k : N \leq k \leq e^b N\} \subset \Omega \times \mathcal{R}^+.$$

Then we would like the uniform measure on this set to converge to some measure on $\Omega \times \mathcal{R}^+$ which looks like a product measure, of say, dm on Ω and an $\exp(-1)$ distribution on $\mathcal{R}^+$. We also note that

$$m_k = |\{t : R^{(N)}_{e^{-\frac{t}{2}}}(z_0) = k\}|,$$

where

$$R^{(N)}_{e^{-\frac{t}{2}}}(z_0) = \sum_{i=0}^{N-1} N_t(T^i_{e^{-\frac{t}{2}}}(z_0))$$

is the number of times the orbit of the point z_0 hits Ω_1 when it has hit the smaller transversal $\Omega_{e^{-\frac{t}{2}}}$ N times.

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Bibliography

[1] J. S. Athreya and J. Chaika, *On the distribution of gaps for saddle connection directions*, to appear, Geometric and Functional Analysis.

[2] J. S. Athreya, J. Chaika, and S. Lelievre, *The distribution of gaps for saddle connections on the golden L*, in preparation.

[3] J. S. Athreya and Y. Cheung *A Poincare section for horocycle flow on the space of lattices*, preprint.

[4] F. P. Boca and A. Zaharescu, *Farey fractions and two-dimensional tori*, in *Noncommutative Geometry and Number Theory* (C. Consani, M. Marcolli, eds.), Aspects of Mathematics E37, Vieweg Verlag, Wiesbaden, 2006, pp. 57-77.

[5] F. Boca, C. Cobeli, and A. Zaharescu, *A conjecture of R. R. Hall on Farey points.* J. Reine Angew. Math. **535** (2001), 207 - 236.

[6] Y. Cheung and N. Chevallier, in preparation.

[7] A. Katok, *Interval exchange transformations and some special flows are not mixing.* Isr. Math. J., 35, (1980), 301-310.

[8] B. Marcus, *Ergodic properties of horocycle flows for surfaces of negative curvature.* Invent. Math. 46, 201-209 (1978).

Chapter 54

Supercharacters and Hopf Monoids (12frg166)

October 7 - 14, 2012

Organizer(s): M. Aguiar (Texas A&M), C. Benedetti (York U.), N. Bergeron (York U.), A. Lauve (Loyola University), F. Saliola (UQAM), N.Thiem (U. of Colorado, Boulder)

Before the workshop, we suggested that the following reading list [1, 2, 3, 4, 5, 6] would be a good refreshment for all of us. At the start of the workshop, we identified the following problems that were worth considering given the recent progress in the area.

1. Higman's conjecture.
2. Compute the antipode of the monoid $\mathbf{scf}(U)$ in the supercharacter basis.
3. Explicitly describe the multiplication in the supercharacter basis of type D.
4. Find a Hopf theoretic interpretation for the inner product in $\mathbf{scf}(U)$.
5. Find a categorification of NCQSym and of other Hopf monoid structures (like $\mathbf{\Pi}$).
6. Construct a Hopf monoid of Young tableaux.

For each question we considered what we knew and what we could do. We worked on some questions more than others. Below, we recount the week's progress on each question.

Higman's conjecture (see [2] for more background).

Consider the group $UT_n(q)$ of unipotent upper triangular matrices over the finite field $\mathbb{F}_q$. Higman conjectured that the number $k_n(q)$ of conjugacy classes of $UT_n(q)$ is a polynomial in q. This problem has been open for more than 50 years. A refinement of this conjecture is given in [2] as follows. Define the constants $c_n(q)$ satisfying the generating series

$$\sum_{n\geq 1} c_n(q)x^n = 1 - \frac{1}{\sum_{n\geq 0} k_n(q)x^n}.$$

It is conjectured in [2] that the $c_n(q)$ should also be a polynomial in q, and moreover, if we let $t = q - 1$, the polynomial $c_n(t)$ should have positive integer coefficients. At BIRS, we proved the following propositions:

Proposition 54.0.1. *The Hopf monoid* $\mathbf{cf}(U)$ *is free. In particular, the integers* $c_n(q)$ *are nonnegative for every* $n \geq 1$ *and prime power* q*. They count the number of free generators of* $\mathbf{cf}(U)$ *of degree* n*.*

We remark that in the case of $\mathbf{f}(U)$ and $\mathbf{scf}(U)$ it is straightforward to show that these monoids are free. In these cases we are able to show that the numbers $c_n^f(q)$ and $c_n^s(q)$ of free generators of degree n in each monoid, respectively, are polynomials. Moreover we understand the coefficients of the polynomials $c_n^f(t)$ and $c_n^s(t)$ when $t = q-1$ as counting certain free generators with given properties. We have some conjectures about the coefficients of the (alleged) polynomial $c_n(q)$. Denoting by $[t^r]P(t)$ the coefficient of t^r in a polynomial $P(t)$ we state the following conjecture.

Conjecture 54.0.2. *For any* n, q, r*, we expect* $[t^r]c_n^f(t) \geq [t^r]c_n(t) \geq [t^r]c_n^s(t)$*. Moreover*
$[t]c_n(t) = 1, \quad [t^2]c_n(t) = (n-2)^2, \quad [t^3]c_n(t) = \frac{1}{12}(n-3)(n-2)(5n^2-29n+48)$

At this time we do not have a concrete plan to prove these conjectures. A remark about Proposition (54.0.1) has been included in [2].

Compute the antipode of the monoid $\mathrm{scf}(U)$ in the supercharacter basis.

There are two results known for the antipode on the supercharacter basis in $\mathbf{scf}(U)$. Using some partial order on the indexing set, we know the leading term of the image under the antipode map using this ordering. This gives us a triangularity relation. Also, we understand explicitly the antipode for the supercharacters indexed by a single arc. These results have not been published yet but are in the process of being written by Nat and Nantel. It was suggested that computing the antipode for supercharacters indexed by two intersecting arcs would be a good problem for students, so we did not work on this problem any further.

Explicitly describe the multiplication in the supercharacter basis of type D.

In type A, the Hopf algebra constructed in [1] is defined in terms of representation-theoretic operations. In particular, the multiplication is defined in terms of inflation maps between special subgroups of $UT_n(q)$ and the group $UT_n(q)$ itself. For the Hopf algebra of type D, constructed in [4], the multiplication is defined in a way that mimics the type A definition even though no inflation maps are known to exist for type D. One can multiply supercharacters of type D with this new multiplication, but nothing guarantees that the result would be positive linear combination of supercharacters. During her visit at BIRS, Carolina showed the following

Proposition 54.0.3. *The multiplication (as defined in [4]) of two supercharacters of type* D *is positive when expanded in terms of supercharacters of type* D*. Moreover, a dimension counting argument suggests that it should be coming from some sort of type* D *inflation map.*

This is added in her paper [4] and opens the door to a new (super)representation problem:

Problem 54.0.4. *Find a representation-theoretic map from* $UT_n^D(q)$ *to* $UT_k^D(q) \times UT_{n-k}^D(q)$ *that can be used to construct the multiplication as defined in [4].*

Find a Hopf theoretic interpretation for the inner product in $\mathrm{scf}(U)$.

The Hopf algebra of symmetric functions is known to be a Hopf ring if we take into account the "internal multiplication" induced by the inner product of characters. It is natural to ask if a similar structure exists for the Hopf algebra of supercharacters. A representation-theoretic inner product certainly exists, but the compatibility relation with the other two operations is not fully satisfied (the distributivity law fails). We are interested in knowing whether these three operations exhibit any additional structural relations in a broader context. We explored some avenues, including a possible module structure on Hopf monoids, but there is nothing very concrete to report at this time.

Find a categorification of NCQSym and of other Hopf monoid structures (like Π).

This is a very vast question. In [1], we categorified $NCSym$ showing that it is isomorphic to the Grothendieck group of the category of super-representations of $UT_n(q)$ with inflation and restriction functors defining the Hopf structure. In [2], we lifted these notions to the Hopf monoid level (constructing $\mathbf{scf}(U)$ which maps to NCSym under the $\overline{\mathcal{K}}$ functor of [3] when $q = 2$). It is still open to find a category of representations of groups (or even algebras) that would be linked to NCQSym. We still have no candidate to propose.

A second question is to find other family of groups with (super)representations and functors that defines a Hopf monoid. For example we would like to have this for the Hopf monoid $\mathbf{\Pi}$. We did not work on this general question, but we agreed to have it in the back of our minds going forward. Section 54 below is our attempt to find a Hopf monoid more suitable for the family of symmetric groups.

Construct a Hopf monoid of Young tableaux.

Our general motivation is to better understand the interaction between the representation theory of towers of groups (or algebras) and Hopf monoid structures. For the tower of symmetric groups, the Grothendieck group of the category of S_n-representations (for $n \geq 0$) is isomorphic to the Hopf algebra Sym of symmetric functions. We presently do not have a Hopf monoid equivalent of this statement. To better understand the general theory, we felt that the following question is central to our program: what is a Hopf monoid constructed from the representation theory of the symmetric groups that projects to Sym under the $\overline{\mathcal{K}}$ functor of [3]? The monoid $\mathbf{\Pi}$ of [3] does project to Sym under $\overline{\mathcal{K}}$ but we do not have a categorification of $\mathbf{\Pi}$ from a tower of groups (or algebras). In this direction, we constructed the following Hopf monoid of Young tableaux with the hope that a certain quotient of this monoid has the desired structure (see [3] for most of the notation used here). Given a finite set J, let $L[J]$ denote the set of all linear orders on J. Given $\sigma \in L[J]$, we say that T is a standard tableau on J with respect to σ if $T\colon \lambda \to J$ is a bijection from a Ferrer diagram λ of a partition of the integer $|J|$ to the set J which is increasing along row and column with respect to the linear order σ. Let $SYT_\sigma[J]$ denote the set of all standard tableaux with respect to the linear order σ. Inspired by the work of [7] we defined a multiplication (μ) and a comultiplication (δ) on such objects. The following was proven during the Banff week.

Theorem 54.0.5. *Let $\mathcal{T}$ be the species such that for any finite set J, we have $\mathcal{T}[J] = Span\{(T, \sigma) | \sigma \in L[J], T \in SYT_\sigma[J]\}$. With the operations (μ, δ), $\mathcal{T}$ is a Hopf monoid. Moreover we have an embedding of Hopf monoids $\Psi\colon \mathcal{T} \hookrightarrow (L^* \times L)$.*

Given this, we have the following commutative diagram of Hopf monoids

$$\begin{tikzcd}
\overrightarrow{\Sigma} \arrow[r, hook, "\Theta"] \arrow[d, two heads] & \mathcal{T} \arrow[r, hook, "\Psi"] \arrow[d, two heads] & (L^* \times L) \arrow[d, two heads, "\Psi^*"] \\
C \arrow[r, hook] \arrow[d, two heads] & A \arrow[r, hook] \arrow[d, two heads] & \mathcal{T}^* \arrow[d, two heads, "\Theta^*"] \\
B \arrow[r, hook] & D \arrow[r, hook] & \overrightarrow{\Sigma}^*
\end{tikzcd}$$

where $A = \mathrm{Im}(\Psi^* \circ \Psi)$ and $B = \mathrm{Im}(\Theta^* \circ \Psi^* \circ \Psi \circ \Theta)$. We constructed explicitly all the maps above and showed that under the functor $\overline{K}$ all four Hopf monoids A, B, C and D map to Sym. Furthermore, A and B are self-dual by construction. Both A and B are interesting candidates for our quest, but we need to understand the structure better. To this end, we studied $\ker(\Psi^* \circ \Psi)$ and $\ker(\Theta^* \circ \Psi^* \circ \Psi \circ \Theta)$. We will write our findings in an upcoming paper. Our preliminary exploration suggests the following.

Speculation 54.0.6. *The Hopf monoid $\mathcal{T}$ seems to be isomorphic to the Hadamard product $\mathcal{I} \times L$, where $\mathcal{I}$ is the species of involutions.*

Speculation 54.0.7. *Generators of the kernel of $\Theta^* \circ \Psi^* \circ \Psi \circ \Theta$ seems to be related to flows on n-spheres.*

Item (54.0.7) is fascinating and, if true, the space B will have a structure related to a very interesting geometry on n-spheres.

Final remarks: In the course of the workshop, we had some extensive computation programmed in SAGE for parts Section 54 and Section 54. The Hopf monoid of Young tableaux $\mathcal{T}$ has been implemented by Franco and may find its way to SAGE/Combinat if the community shows some interest. Algorithms for computing $\ker(\Psi^* \circ \Psi)$ and $\ker(\Theta^* \circ \Psi^* \circ \Psi \circ \Theta)$ were also implemented, which allowed us to generate interesting data and discover the speculative geometry in (6.3).

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Bibliography

[1] M. Aguiar, C. André, C. Benedetti, N. Bergeron, Z. Chen, P. Diaconis, A. Hendrickson, S. Hsiao, I.M. Isaacs, A. Jedwab, K. Johnson, G. Karaali, A. Lauve, T. Le, S. Lewis, H. Li, K. Magaard, E. Marberg, J-C. Novelli, A. Pang, F. Saliola, L. Tevlin, J-Y. Thibon, N. Thiem, V. Venkateswaran, C.R. Vinroot, N. Yan, M. Zabrocki, Supercharacters, symmetric functions in noncommuting variables, and related Hopf algebras, *Adv. Math.* **229** (2012), no. 4, 2310–2337.

[2] M. Aguiar , N. Bergeron and N. Thiem, Hopf monoids from functions on unitriangular matrices Accepted in Alg. and Numb. Theory (2012). (arXiv:1203.1572).

[3] M. Aguiar and S. Mahajan, *Monoidal functors, species and Hopf algebras.*, CRM Monograph Series, **29** American Mathematical Society, 2010.

[4] C. Benedetti, Combinatorial Hopf algebra of superclass functions of type D, (2011) ArXiv:1111.6185.

[5] N. Bergeron and N. Thiem, A supercharacter table decomposition via power-sum symmetric functions, To appear in *int. J. of Algebra and Computation* (2012). arXiv:1112.4901.

[6] A. Lauve and M. Mastnak The primitives and antipode in the Hopf algebra of symmetric functions in noncommuting variables *Adv App Math* **47-3** (2011) 536–544 .

[7] S. Poirier and C. Reutenauer, Algèbres de Hopf de tableaux, *Ann. Sci. Math. Québec*, **19** (1995), no. 1, 79–90.

[8] F. Saliola, http://garsia.math.yorku.ca/˜saliola/supercharacters/.

Chapter 55

Spectral and asymptotic stability of nonlinear Dirac equation (12frg188)

December 2 - 12, 2012

Organizer(s): Nabile Boussaid (Université de Franche-Compté), Andrew Comech (Texas A&M University), Stephen Gustafson (University of British Columbia), Slim Ibrahim (University of Victoria), Tetsu Mizumachi (Kyushu University), Kenji Nakanishi (Kyoto University), Dmitry Pelinovsky (McMaster University), Atanas Stefanov (Kentucky University)

Overview of the field and recent developments

The Dirac equation remains of utmost importance in High Energy Physics and Solid State Physics since its invention in 1928 [Dir28], and became the main building block of Quantum Electrodynamics. The idea to consider a self-interacting spinor field has appeared in Physics for a long time [Iva38, FLR51, FFK56, Hei57, Wak66].

The two most widely known models of self-interacting spinor fields are the massive Thirring model [Thi58], which is a model with the vector-vector (or *current-current*) self-interaction (in 1D, this model is completely integrable [Kor79]) and the model with the scalar-scalar self-interaction introduced by Soler [Sol70]:

$$i\frac{\partial}{\partial t}\psi = -i\sum_{j=1}^{n}\alpha_j\frac{\partial}{\partial x_j}\psi + m\beta\psi - (\psi^*\beta\psi)\beta\psi, \qquad \psi(x,t)\in\mathcal{C}^4, \quad x\in\mathcal{R}^3,$$

where α_j and β are the 4×4 Dirac matrices. The one-dimensional version of the Soler model is also known as the massive Gross-Neveu model [GN74, LG75]. Let us also point out that the nonlinear Dirac equation is a representative of a wider class of models, known as *coupled mode equations*. Such equations appear in Solid State Physics [OKL$^+$00], in the theory of graphene [NGM$^+$05], and in Nonlinear Optics [dSS94, dSSS96, GWH01].

Dirac-type equations with the self-interaction of local type have been widely considered in Physics since the seventies. There were many results on the existence of localized solutions (*solitary waves*, or *gap solitons* of the form $\phi(x)e^{-i\omega t}$, with $\phi\in L^2(\mathcal{R}^n)$. The existence of solitary waves in the Dirac equation was already justified numerically in [Sol70]. For the Dirac-Maxwell system, the existence of solitary waves was indicated in [Wak66, Lis95]. The stability properties of these localized solutions, which are of crucial importance for understanding the role they could play (and also for applications e.g. in nonlinear optics), have remained a complete mystery, and were accessed either by numerics [RnRnSV74, AS83, AS86, BC12a, MQC$^+$12, XST12] or by heuristic arguments [Sol70, Bog79, MM86, SV86, BSV87, CKMS10]; essentially, no definite results were ever produced.

In the recent time, properties of the nonlinear Dirac equation and related models are being addressed by the mathematical community. Existence of solitary waves of the nonlinear Dirac equation was rigorously justified in [CV86, Mer88, ES95]; for the Dirac-Maxwell system, it was rigorously proved in [EGS96, Abe98].

Local and global well-posedness was addressed in [EV97] (semilinear equation), [Bou00] (interacting Dirac and Klein-Gordon equations in 1D), and in [MNNO05] (global well-posedness for the nonlinear Dirac equation in 3D, for small initial data in the energy class with additional regularity assumption for the angular variables). More recent results include [DFS10] (almost optimal well-posedness for Dirac-Maxwell) and [DS11] (global well-posedness for the nonlinear Dirac-Maxwell in 2D). Uniqueness of finite energy solutions to Dirac-Maxwell was studied in [MN03].

We are interested in developing the analytic approach to the stability properties of solitary wave solutions to the nonlinear Dirac equation and related systems. The pivotal papers by Zakharov [Zak67], Vakhitov and Kolokolov [VK73], Cazenave and Lions [CL82], Weinstein [Wei85], and Grillakis, Shatah, and Strauss [GSS87] gave the exhaustive analysis of spectral and orbital stability in the nonlinear Schrödinger equation and many similar $\mathbf{U}(1)$-invariant models; The nonlinear Dirac equation remained off-limits. Earlier heuristic approaches which we mentioned above were based on considering whether the energy is minimized or not under certain families of perturbations on the hypersurface of constant charge. These criteria were of doubtful validity, though; Let us point out that even the well-known semiformal instability result known as Derrick's theorem [Der64], which is based on the analysis of the values of the energy functional under dilations, while correctly predicting instability of stationary localized solutions in 3D (and in higher dimensions), fails to show such an instability in 1D and 2D. On the other hand, this instability easily follows by spectral theory methods [KS07], suggesting that the spectral stability approach is not only more rigorous mathematically, but also more precise.

In the last three years, the participants of the proposed focused research group made several breakthroughs which allow us to rigorously study the spectral stability and the asymptotic stability. We hope that combining our efforts will make the nonlinear Dirac equation as clearly accessible to stability analysis as the nonlinear Schrödinger equation and similar equations.

Presentation highlights

Here are some of the presentations which took place.

- Andrew Comech described the results on spectral instability on the nonlinear Dirac equation in 3D and higher-order nonlinear Dirac equation in 1D and 2D [CGG12].
- Kenji Nakanishi described his work on the Dirac-Maxwell system [MN03], focussing on the stability, spectral and nonlinear, of such a system. The related recent works of Piero d'Ancona on the wellposedness are [DFS05, DS11].
- Nabile Boussaid gave a presentation on a recent work [BC12b], discussing the recent progress on spectra stability of solitary waves in the Soler model.
- Tetsu Mizumachi described the related results in the context of the Kadomtsev-Petviashvili equation basing on his recent work [MT12].
- Slim Ibrahim gave a presentation on finite-time blow-up results for hyperbolic equations [IL12].
- Atanas Stefanov presented latest developments on asymptotic stability results for the nonlinear Dirac equation in the external potential [PS12].

The main conclusion is that the nonlinear Dirac equation and related systems have solitary wave solutions which are spectrally stable, and that the local and global well-posedness is known, at least to some extent. We expect that the problems on asymptotic stability of solitary waves in Dirac-Maxwell system are well within the reach of today's methods of nonlinear Analysis. This may be counter-intuitive to many researchers because the energy functional is unbounded from below, so that the energy conservation does not provide bounds which are routinely available to people who work on nonlinear Schrödinger and Klein-Gordon type equations.

Nonlinear Dirac equation and related models: Open problems

During discussions, the following promising directions of research were pointed out.

Maxwell-Dirac system (3D, also 2D, 1D)

1. in 3D, well-posedness in the critical space (L^2 for the spinor). The well-posedness in H^ϵ is due to d'Ancona et. al [DFS10].
2. In 2D, growth of H^1 norm (global well-posedness in L^2 is known)
3. Orbital and asymptotic stability of ground states coming from the nonrelativistic limit
4. Well-posedness/existence of ground states away from non-relativistic limit (eg. via Min-Max of [EGS96])
5. Orbital stability of such ground states
6. Instability of sign-changing standing waves (*excited states*) bifurcating from the nonrelativistic limit. This question is still open even for Schrödinger-Poisson system and for NLS.

Schrodinger-Poisson system (with indefinite Hamiltonian)

1. Non-degeneracy of sign-changing standing waves (to allow for the analysis of nonrelativistic-limit-type bifurcations of standing waves of Maxwell-Dirac)
2. Orbital/asymptotic stability of ground states (0, Q_w)
3. Related: dispersive estimates for linearization of the Hartree equation about ground states
4. Linear Schrodinger with $1/|x|$ potential

Dirac Models in 1D

1. L^2 ill-posedness in general ($H^{1/2}$ well-posedness is known; Thirring model is globally well-posed in L^2)
2. Well/ill-posedness in $H^{1/2}$
3. Asymptotic stability of ground states: cubic & quintic NLD; cubic NLS (cf. Cuccagna, via the approach of Deift); NLKG
4. Small-data scattering
5. Asymptotic stability of traveling waves of massless NLS (nonlinear transport)

Dirac Models in higher dimensions

1. In any dimension: NLD with potential supporting several eigenvalues: which corresponding nonlinear bound states are stable? unstable? Is it always possible to uniquely indicate the "ground state"?

2. Related question: instability of "nonlinear excited states" in NLS

3. 2D: stability/instability (spectral/orbital/asymptotic) of ground states for cubic NLD coming from the nonrelativistic limit, where stability is "marginal": the pair of purely imaginary eigenvalues is located asymptotically close to the non-degenerate zero eigenvalue.

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Miscellaneous

1. Cubic Klein-Gordon: small-data scattering for nonlinearities below L^2-critical, down to "Strauss exponent"

2. (Non-)unconditional uniqueness for any of the equations above

3. To formulate the "Dirac Map" problem

Bibliography

[Abe98] S. Abenda, Solitary waves for Maxwell-Dirac and Coulomb-Dirac models, *Ann. Inst. H. Poincaré Phys. Théor.* **68** (1998), 229–244.

[AS83] A. Alvarez and M. Soler, Energetic stability criterion for a nonlinear spinorial model, *Phys. Rev. Lett.* **50** (1983), 1230–1233.

[AS86] A. Alvarez and M. Soler, Stability of the minimum solitary wave of a nonlinear spinorial model, *Phys. Rev. D* **34** (1986), 644–645.

[BC12a] G. Berkolaiko and A. Comech, On spectral stability of solitary waves of nonlinear Dirac equation in 1D, *Math. Model. Nat. Phenom.* **7** (2012), 13–31.

[BC12b] N. Boussaid and A. Comech, On spectral stability of nonlinear Dirac equation, *ArXiv e-prints* **1211.3336** (2012).

[Bog79] I. L. Bogolubsky, On spinor soliton stability, *Phys. Lett. A* **73** (1979), 87–90.

[Bou00] N. Bournaveas, A new proof of global existence for the Dirac Klein-Gordon equations in one space dimension, *J. Funct. Anal.* **173** (2000), 203–213.

[BSV87] P. Blanchard, J. Stubbe, and L. Vàzquez, Stability of nonlinear spinor fields with application to the gross-neveu model, *Phys. Rev. D* **36** (1987), 2422–2428.

[CGG12] A. Comech, M. Guan, and S. Gustafson, On linear instability of solitary waves for the nonlinear Dirac equation, *ArXiv e-prints* **1209.1146** (2012).

[CKMS10] F. Cooper, A. Khare, B. Mihaila, and A. Saxena, Solitary waves in the nonlinear Dirac equation with arbitrary nonlinearity, *Phys. Rev. E* **82** (2010), 036604.

[CL82] T. Cazenave and P.-L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, *Comm. Math. Phys.* **85** (1982), 549–561.

[CV86] T. Cazenave and L. Vázquez, Existence of localized solutions for a classical nonlinear Dirac field, *Comm. Math. Phys.* **105** (1986), 35–47.

[Der64] G. H. Derrick, Comments on nonlinear wave equations as models for elementary particles, *J. Mathematical Phys.* **5** (1964), 1252–1254.

[DFS05] P. d'Ancona, D. Foschi, and S. Selberg, Null structure and almost optimal local well-posedness o f the Dirac-Klein-Gordon system, *ArXiv Mathematics e-prints* (2005).

[DFS10] P. d'Ancona, D. Foschi, and S. Selberg, Null structure and almost optimal local well-posedness of the Maxwell-Dirac system, *Amer. J. Math.* **132** (2010), 771–839.

[Dir28] P. Dirac, The quantum theory of the electron, *Proc. R. Soc. A* **117** (1928), 610–624.

[DS11] P. d'Ancona and S. Selberg, Global well-posedness of the Maxwell-Dirac system in two space dimensions, *J. Funct. Anal.* **260** (2011), 2300–2365.

[dSS94] C. de Sterke and J. Sipe, Gap solitons, *Progress in Optics* **33** (1994), 203–260.

[dSSS96] C. de Sterke, D. Salinas, and J. Sipe, Coupled-mode theory for light propagation through deep nonlinear gratings, *Phys. Rev. E* **54** (1996), 1969–1989.

[EGS96] M. J. Esteban, V. Georgiev, and E. Séré, Stationary solutions of the Maxwell-Dirac and the Klein-Gordon-Dirac equations, *Calc. Var. Partial Differential Equations* **4** (1996), 265–281.

[ES95] M. J. Esteban and É. Séré, Stationary states of the nonlinear Dirac equation: a variational approach, *Comm. Math. Phys.* **171** (1995), 323–350.

[EV97] M. Escobedo and L. Vega, A semilinear Dirac equation in $H^s(\mathbf{R}^3)$ for $s \geq 1$, *SIAM J. Math. Anal.* **28** (1997), 338–362.

[FFK56] R. Finkelstein, C. Fronsdal, and P. Kaus, Nonlinear spinor field, *Phys. Rev.* **103** (1956), 1571–1579.

[FLR51] R. Finkelstein, R. LeLevier, and M. Ruderman, Nonlinear spinor fields, *Phys. Rev.* **83** (1951), 326–332.

[GN74] D. J. Gross and A. Neveu, Dynamical symmetry breaking in asymptotically free field theories, *Phys. Rev. D* **10** (1974), 3235–3253.

[GSS87] M. Grillakis, J. Shatah, and W. Strauss, Stability theory of solitary waves in the presence of symmetry. I, *J. Funct. Anal.* **74** (1987), 160–197.

[GWH01] R. H. Goodman, M. I. Weinstein, and P. J. Holmes, Nonlinear propagation of light in one-dimensional periodic structures, *J. Nonlinear Sci.* **11** (2001), 123–168.

[Hei57] W. Heisenberg, Quantum theory of fields and elementary particles, *Rev. Mod. Phys.* **29** (1957), 269–278.

[IL12] S. Ibrahim and A. Lyaghfouri, Blow-up solutions of quasilinear hyperbolic equations with critical Sobolev exponent, *Math. Model. Nat. Phenom.* **7** (2012), 66–76.

[Iva38] D. D. Ivanenko, Notes to the theory of interaction via particles, *Zh. Éksp. Teor. Fiz* **8** (1938), 260–266.

[Kor79] V. Korepin, Direct calculation of the s matrix in the massive thirring model, *Theoretical and Mathematical Physics* **41** (1979), 953–967.

[KS07] P. Karageorgis and W. A. Strauss, Instability of steady states for nonlinear wave and heat equations, *J. Differential Equations* **241** (2007), 184–205.

[LG75] S. Y. Lee and A. Gavrielides, Quantization of the localized solutions in two-dimensional field theories of massive fermions, *Phys. Rev. D* **12** (1975), 3880–3886.

[Lis95] A. G. Lisi, A solitary wave solution of the Maxwell-Dirac equations, *J. Phys. A* **28** (1995), 5385–5392.

[Mer88] F. Merle, Existence of stationary states for nonlinear Dirac equations, *J. Differential Equations* **74** (1988), 50–68.

[MM86] P. Mathieu and T. F. Morris, Charged spinor solitons, *Canad. J. Phys.* **64** (1986), 232–238.

[MN03] N. Masmoudi and K. Nakanishi, Uniqueness of finite energy solutions for Maxwell-Dirac and Maxwell-Klein-Gordon equations, *Comm. Math. Phys.* **243** (2003), 123–136.

[MNNO05] S. Machihara, M. Nakamura, K. Nakanishi, and T. Ozawa, Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation, *J. Funct. Anal.* **219** (2005), 1–20.

[MQC+12] F. G. Mertens, N. R. Quintero, F. Cooper, A. Khare, and A. Saxena, Nonlinear Dirac equation solitary waves in external fields, *Phys. Rev. E* **86** (2012), 046602.

[MT12] T. Mizumachi and N. Tzvetkov, Stability of the line soliton of the KP-II equation under periodic transverse perturbations, *Math. Ann.* **352** (2012), 659–690.

[NGM+05] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, Two-dimensional gas of massless Dirac fermions in graphene, *Nature* **438** (2005), 197–200.

[OKL+00] E. A. Ostrovskaya, Y. S. Kivshar, M. Lisak, B. Hall, F. Cattani, and D. Anderson, Coupled-mode theory for Bose-Einstein condensates, *Phys. Rev. A* **61** (2000), 031601.

[PS12] D. E. Pelinovsky and A. Stefanov, Asymptotic stability of small gap solitons in nonlinear Dirac equations, *J. Math. Phys.* **53** (2012), 073–705.

[RnRnSV74] A. F. Rañada, M. F. Rañada, M. Soler, and L. Vázquez, Classical electrodynamics of a nonlinear Dirac field with anomalous magnetic moment, *Phys. Rev. D* **10** (1974), 517–525.

[Sol70] M. Soler, Classical, stable, nonlinear spinor field with positive rest energy, *Phys. Rev. D* **1** (1970), 2766–2769.

[SV86] W. A. Strauss and L. Vázquez, Stability under dilations of nonlinear spinor fields, *Phys. Rev. D (3)* **34** (1986), 641–643.

[Thi58] W. E. Thirring, A soluble relativistic field theory, *Ann. Physics* **3** (1958), 91–112.

[VK73] N. G. Vakhitov and A. A. Kolokolov, Stationary solutions of the wave equation in the medium with nonlinearity saturation, *Radiophys. Quantum Electron.* **16** (1973), 783–789.

[Wak66] M. Wakano, Intensely localized solutions of the classical Dirac-Maxwell field equations, *Progr. Theoret. Phys.* **35** (1966), 1117–1141.

[Wei85] M. I. Weinstein, Modulational stability of ground states of nonlinear Schrödinger equations, *SIAM J. Math. Anal.* **16** (1985), 472–491.

[XST12] J. Xu, S. Shao, and H. Tang, Numerical methods for nonlinear Dirac equation, *ArXiv e-prints* (2012).

[Zak67] V. Zakharov, Instability of self-focusing of light, *Zh. Éksp. Teor. Fiz* **53** (1967), 1735–1743.

Research in Teams Reports

Chapter 56

Dialgebras, Leibniz Algebras, and Quasi-Jordan Algebras (12rit142)

April 29 - May 6, 2012

Organizer(s): Murray Bremner (University of Saskatchewan, Canada), Raúl Felipe (Centro de Investigación en Matemáticas, Guanajuato, Mexico), Michael Kinyon (University of Denver, Colorado, USA), Juana Sánchez-Ortega (Universidad de Málaga, Spain)

Background

The primary motivation for this workshop is the recent discovery of many new varieties of nonassociative structures that can be regarded as "noncommutative" analogues of classical structures. This development originated in the work of Loday [6, 7] in the early 1990's on Leibniz algebras (which satisfy the Jacobi identity but are not necessarily anticommutative) and associative dialgebras (with two associative operations); the connection between these two structures closely resembles that between Lie algebras and associative algebras, but also presents many new features. Ten years later, Liu [5] introduced alternative dialgebras, the natural analogue of alternative algebras in the setting of dialgebras. Shortly after that, Raúl Felipe (one of the participants in this workshop) and one of his Ph.D. students [10] initiated the study of quasi-Jordan algebras (also known as Jordan dialgebras), which are related to Jordan algebras as Leibniz algebras are to Lie algebras. Around the same time, Kolesnikov [4] developed a general method for passing from a variety of nonassociative algebras defined by polynomial identities to the corresponding variety of dialgebras. This method has recently been simplified and formalized by three of the participants in this workshop [2] in the so-called KP algorithm.

The Cayley-Dickson Process

An important topic in classical algebra is the theory of composition algebras and their close connection with quadratic forms and the eight-square theorem. This leads to the construction, starting from the real numbers, of the complex numbers, quaternions, and octonions, through a doubling process which originated in the works of Cayley and Dickson. For the early history of these developments, see Dickson [3], and for the completion of the classical theory, see Albert [1] and Schafer [9]. The original idea of Raúl Felipe and his son, Raúl Felipe-Sosa, was to create an analogue of this theory in the setting of dialgebras. Their initial work on this project was done in 2011, and was extended during the visit of Murray Bremner to Havana in December 2011, who also used the computer algebra system Maple to construct examples of dialgebras analogous to the classical quaternions and octonions. The main difficulty at this point was to find the appropriate conditions under which the double of an associative dialgebra would necessarily be an alternative dialgebra. This topic is also related to ongoing work, by Juana Sánchez-Ortega and her colleagues in Málaga, on dialgebra versions of the Moufang identities and a generalization of Artin's theorem (every alternative algebra on two generators is associative) to the setting of dialgebras.

The Results of Our Workshop

We began with a review of our earlier calculations and a discussion of possible approaches to resolving the difficulty mentioned above. The fundamental problem seemed to be to determine an appropriate analogue in the setting of dialgebras of the classical conditions that xx^* and $x + x^*$ belong to the base field for any element x of an algebra with involution $x \mapsto x^*$. After a number of false starts, we have now found the correct hypotheses, which involve conditions relating the symmetric elements in a dialgebra with involution to the Leibniz bracket on the dialgebra. We are confident that this approach will also allow us to prove that the double of a flexible dialgebra is again flexible. A closely related problem, although not strictly related to the Cayley-Dickson process, is to prove that every alternative dialgebra becomes a Jordan dialgebra by means of the quasi-Jordan product (also called the antidicommutator or Jordan diproduct).

We also discussed the little-known paper by Loday and Pirashvili [8], which was brought to our attention by Michael Kinyon. This work provides a different interpretation of associative dialgebras using the so-called infinitesimal tensor product on the category of linear maps. We expect that this will provide another approach to the problem of generalizing the Cayley-Dickson process to dialgebras, and will also lead to a much simpler justification of the KP algorithm for converting varieties of algebras into varieties of dialgebras (the current proof of its correctness relies heavily on a great deal of machinery from the theory of operads).

Future Publications and Collaborations

The purpose of this Research in Teams workshop was to develop existing collaborations and to lay the foundations for future collaborations among the participating researchers in the area of nonassociative algebra. In this regard it has already been very successful. We expect to obtain at least three journal articles resulting from our collaboration at BIRS:

1. *The Cayley-Dickson process for dialgebras*.
2. *Varieties of dialgebras and the tensor category of linear maps*.
3. *Artin's theorem for alternative dialgebras*.

In closing, we mention that three of us are co-organizers of a Research School to be held in February 2013 on topics closely related to this BIRS workshop; for details, see

`http://www.cimat.mx/Eventos/associative_and_nonassociative/`

This event is supported financially by CIMPA, the International Center for Pure and Applied Mathematics,

`http://www.cimpa-icpam.org/`

and will take place at the CIMAT (Centro de Investigación en Matemáticas) in Guanajuato, Mexico. This school will introduce graduate students and postdoctoral researchers, primarily from the developing countries of Latin America, to contemporary developments in associative and nonassociative algebras and dialgebras, in both theoretical and algorithmic aspects.

Participants

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Bibliography

[1] A. A. Albert, Quadratic forms permitting composition, *Annals of Mathematics* (2) **43** (1942), 161–177.

[2] M. R. Bremner, R. Felipe and J. Sánchez-Ortega, Jordan triple disystems, *Computers and Mathematics with Applications* **63** (2012), 1039–1055.

[3] L. E. Dickson, On quaternions and their generalization and the history of the eight-square theorem, *Annals of Mathematics* (2) **20** (1919), 155–171.

[4] P. S. Kolesnikov, Varieties of dialgebras, and conformal algebras, *Siberian Mathematical Journal* **49** (2008), 257–272.

[5] D. Liu, Steinberg-Leibniz algebras and superalgebras, *Journal of Algebra* **283** (2005), 199–221.

[6] J.-L. Loday, Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Enseignement Mathématique* (2) **39** (1993), 269–293.

[7] J.-L. Loday, Algèbres ayant deux opérations associatives (digèbres), *Comptes Rendus de l'Académie des Sciences, Paris, Série I, Mathématiques* **321** (1995), 141–146.

[8] J.-L. Loday and T. Pirashvili, The tensor category of linear maps and Leibniz algebras, *Georgian Mathematical Journal* **5** (1998), 263–276.

[9] R. D. Schafer, On the algebras formed by the Cayley-Dickson process, *American Journal of Mathematics* **76** (1954), 435–446.

[10] R. Velásquez and R. Felipe, Quasi-Jordan algebras, *Communications in Algebra* **36** (2008), 1580–1602.

Chapter 57

Isometries and isomorphisms of spaces of continuous functions (12rit146)

May 13 - 20, 2012

Organizer(s): H. G. Dales (Lancaster, UK), F. Dashiell (Los Angeles, California, USA), A. T.-M. Lau (Edmonton, Alberta, Canada), D. Strauss (Leeds, UK)

Overview of the Field

This is a report of the work of four colleagues as part of 'Research in Teams' at BIRS; the reference is 12rit146.

Our work has split into two rather distinct, but related, parts, and we summarize them separately.

The first part is directly concerned with isometries and isomorphisms of spaces of continuous functions.

Let K be a locally compact space. Then $C_0(K)$ denotes the collection of all the continuous, complex-valued functions on K that vanish at infinity, so that $C_0(K)$ is a linear space with respect to the pointwise operations. The *uniform norm* on K is denoted by $|\cdot|_K$, so that

$$|f|_K = \sup\{|f(x)| : x \in K\} \quad (f \in C_0(K))\,.$$

Then $(C_0(K), |\cdot|_K)$ is a Banach space. Further, with respect to the product given by pointwise multiplication of functions, $C_0(K)$ is a Banach algebra; it is a commutative C^*-algebra.

We shall be concerned in particular with the space $C(K)$ when K is totally disconnected, or Stonean; the latter means that K is compact and extremely disconnected, in the sense that pairs of disjoint open sets in K have disjoint closures.

We recall that the real Banach lattice $C(K)_{\mathcal{R}}$ is Dedekind complete if and only if the space K is Stonean.

Much of our work is formulated in the language of Boolean algebras. It is standard that the Stone space of a Boolean algebra B is the collection of ultrafilters on B, taken with the Stone topology; this gives the Stone space $St(B)$ of B. This space is always compact and totally disconnected, and it is Stonean if and only if B is complete. Important examples are the σ-field of Borel sets of a compact space and the space of regular-open subsets of a topological space.

We recall that two Banach spaces E and F are isomorphic, written $E \sim F$, if there is a continuous linear bijection from E onto F, and are isometric, written $E \cong F$, if there is an isometry from E onto F. We are concerned with the question when a Banach space E is isomorphic or isometric to a space of the form $C(K)$, and when two spaces of the form $C(K)$ and $C(L)$ are isomorphic. There is a substantial difference between the isomorphic and isometric theories. The classical Banach–Stone theorem tells us that $C(K) \cong C(L)$ if and only if K and L are homeomorphic, and Milutin's theorem states that $C(K) \sim C(L)$ when both K

and L are compact, uncountable, metric spaces. However many basic questions remain open. For example, is it true that, for each compact space K, the Banach space $C(K)$ is isomorphic to $C(L)$ for some totally disconnected space L?

We are also concerned with the question when a space of the form $C(K)$ is isomorphically or isometrically the dual of another Banach space, and when it is the bidual of such a space. Many classical theorems discuss this situation. For example, it is well-known that a separable Banach space which is isomorphically a dual space has the Krein–Milman and Radon–Nikodym properties.

Indeed, it is very classical that $C(K)$ is isometrically a dual space (so that $C(K)$ is a von Neumann algebra) if and only if K is hyper-Stonean, in the sense that the union of the supports of the normal measures on K is dense in K; this notion goes back to Dixmier [2]. The normal measures on K are the measures on K that are order-continuous on $(C(K)_{\mathcal{R}}, \leq)$, and it is standard that the positive normal measures are those positive measures μ on K such that $\mu(L) = 0$ whenever L is a compact subspace of K such that $\mathrm{int}_K L = \emptyset$. The space $N(K)$ of normal measures is a closed subspace of $M(K)$, the Banach space of all measures on K. In the case where K is Stonean, $N(K)$ is the unique isometric predual of $C(K)$.

Important examples of spaces of the form $C(K)$ are the Baire functions. The Baire functions of order 0 are the continuous functions on $[0, 1]$, now denoted by $B_0(K)$. Let α be an ordinal with $\alpha > 0$. Given a definition of the Baire class of order β for each $\beta < \alpha$, we define B_α, the Baire class of order α, to be the space of bounded functions on $[0, 1]$ which are pointwise limits of sequences of functions in the union of the earlier classes; the recursive construction terminates at $\alpha = \omega_1$. The Baire functions on $[0, 1]$ are the members of the final class, B_{ω_1}. The spaces B_α are commutative C^*-algebras for the pointwise product, and so have the form $C(\Phi_\alpha)$ for a compact space Φ_α.

Let K be a locally compact space. Then the commutative C^*-algebra $C_0(K)$ has a second dual which is also a commutative C^*-algebra, and so this second dual has the form $C(\widetilde{K})$ for a certain compact space $\widetilde{K}$, called the hyper-Stonean envelope of K.

The second part of our work is concerned with weakly almost periodic functions on algebras of measures.

Let A be a Banach algebra. An element $\lambda \in A'$ is *weakly almost periodic* if the map $a \mapsto a \cdot \lambda, \;\; A \to A'$, is weakly compact. The space of these elements is denoted by $WAP(A)$.

Let G be a locally compact group, and let $(M(G), \star)$ be the measure algebra on G. Then we have the commutative C^*-algebra $C(\widetilde{G})$, as above. It is easy to see that we can identify $WAP(M(G))$ as a closed subspace of $C(\widetilde{G})$. It is a very striking theorem of Daws [1] that, in fact, $WAP(M(G))$ is a C^*-subalgebra of $C(\widetilde{G})$. However it is not easy to identify which elements of $C(\widetilde{G})$ belong to this algebra. Some partial results will be discussed below.

Recent Developments and Open Problems

The subject of isometries and isomorphisms of spaces of continuous functions was intensively studied some time ago; many strong results were obtained, but apparently basic open questions were left unresolved. There have been few recent advances, save perhaps in our own work. Here are some open questions.

Question 1 *Is it true that, for each compact space K, the Banach space $C(K)$ is isomorphic to $C(L)$ for some totally disconnected space L?*

Question 2 *Are any or all of the Banach spaces B_α and B_β pairwise isomorphic in the cases where $2 \leq \alpha, \beta < \omega_1$ and $\alpha \neq \beta$?*

Question 3 *Let X be a compact space. Suppose that $C(X)$ is isometrically isomorphic to the second dual of some Banach space. Does there always exist a locally compact space K such that $C(X)$ is isometrically isomorphic to $C(\widetilde{K}) = C_0(K)''$?*

Question 4 *Let E be an injective Banach space. Is E isomorphic to a 1-injective space? Let K be a locally compact space such that $C_0(K)$ is an injective Banach space. Is $C_0(K)$ isomorphic to $C(L)$ for some Stonean space L?*

Question 5 *Let K be a compact space such that $C(K)$ is isomorphically a dual space. Is K necessarily totally disconnected? Does there exist a Stonean space L such that $C(K)$ is isomorphic to $C(L)$?*

Question 6 *Let K be a compact space such that $C(K)$ is isomorphically a dual space, and suppose that L is a clopen subspace of K. Is $C(L)$ also always isomorphically a dual space?*

The main questions associated with the theory of weakly almost periodic functions on algebras of measures is the following.

Question 7 *Let $\mathcal{T}$ be the unit circle, a compact group, and let χ be the characteristic function of the spectrum Φ of $L^\infty(\mathcal{T})$, where Φ is regarded as a clopen subset of the hyper-Stonean envelope of $\mathcal{T}$. Does χ belong to $WAP(M(\mathcal{T}))$?*

Question 8 *Is there always a topological invariant mean on $WAP(M(G))$. If so, is it unique?*

Presentation Highlights

Since this was a workshop for four people assembled for 'Research in teams', there were no formal presentations.

Scientific Progress Made

We made progress in several areas, but we did not fully resolve any of the above open questions.

1) The standard discussion of properties defining hyper-Stonean spaces assume in advance constructed gave several new examples of spaces K for which $N(K) = \{0\}$ and $N(K) \neq \{0\}$. Some constructions involve the Gleason space G_K of K; some involve the Stone–Cech compactification of certain spaces. Further, we established the following theorem. Let K be a compact space. Then the union of the supports of the normal measures on K is dense in K if and only if the Gleason cover of K is hyper-Stonean.

2) We gave some new, more explicit, constructions of the hyper-Stonean envelope of a compact space.

3) We gave some new examples showing when $C(K)$ is isomorphically a dual space. For example, one can have this property even when $N(K) = \{0\}$.

4) Question 3, above, was already resolved by Lacey in an old paper [4] in the case where $C(X)$ is isometrically isomorphic to the second dual of a separable Banach space; see also [5]. The analogous question in the isomorphic theory of Banach spaces was resolved in a similar way by Stegall [8]; for related work, see [3]. However, all these works are rather complicated and difficult to follow. We have developed a different approach to this theorem, using background from topology and Boolean algebras: it provides a simpler proof of known results and some new results. We have hopes that it will resolve the general question.

5) We have compared the C^*-algebra $WAP(M(G))$ with the space $B^b(G)$ of bounded Borel functions on G, regarded as a subspace of $C(\widetilde{G})$.

6) Question 8, above, asks for an extension of an old result of Ryll-Nardzewski [7] given in the case where G is discrete. We obtained a partial result by showing that $WAP(M(G))$ always has a left-invariant mean; this used a fixed-point theorem from [6].

Our work is being written in two separate articles. The first, 'Isometries and isomorphisms of spaces of continuous functions' currently contains around 60 pages; the second, 'Weakly almost periodic functions on algebras of measures' currently contains around 14 pages.

Outcome of the Meeting

The four participants are continuing to work on preparing our work for publication. In particular, Dales, Dashiell, and Lau will meet again in Los Angeles in October, 2012, to attempt to make further progress

Lau will attend a conference on *Harmonic analysis* in Luminy, in October, 2012, and will discuss the second paper mentioned.

A proposal has been made to the *Fields Institute* in Toronto for a Thematic Program on *Banach algebras and harmonic analysis* in the first half of 2014; Dales and Lau are among the organisers of this programme. It is expected that questions related to our work, and related matters, will be discussed during this semester.

We are very grateful to BIRS for their continuing support of our work.

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Bibliography

[1] M. Daws, Weakly almost periodic functionals on the measure algebra, *Math. Zeit.*, 265 (2010), 285–296.

[2] J. Dixmier, Sur certains espaces considérés par M. H. Stone, *Summa Brasiliensis Math.*, 2 (1951), 151–182.

[3] J. Hagler, Complemented isometric copies of L_1 in dual Banach spaces, *Proc. American Math. Soc.*, 130 (2002), 3313–3324.

[4] H. Elton Lacey, A note concerning $A^* = L_1(\mu)$, *Proc. American Math. Soc.*, 29 (1971), 525–528.

[5] H. Elton Lacey, *Isometric theory of classical Banach spaces*, Springer, Berlin, 1974.

[6] A. T.-M. Lau and Y. Zhang, Fixed point properties of semigroups of non-expansive mappings, *J. Functional Analysis*, 254 (2008), 2534–2554.

[7] C. Ryll-Nardzewski, On fixed points of semigroups of endomorphisms of linear spaces, *Proc. Fifth Berkeley Symposium on Maths. and Staistics*, Volume II, Contributions to Probability Theory. Part I, 55–61, University of California Press, 1966.

[8] C. Stegall, Banach spaces whose duals contain $\ell_1(\Gamma)$ with applications to the study of dual $L_1(\mu)$ spaces, *Trans. American Math. Soc.*, 176 (1973), 463–477.

Chapter 58

Toric Boij–Söderberg Theory (12rit171)

May 20 - 27, 2012

Organizer(s): Christine Berkesch (Duke University), Daniel Erman (University of Michigan), Gregory G. Smith (Queen's University)

Overview of the Field

In commutative algebra, homological methods have traditionally centered around minimal free resolutions for modules over a local ring or for graded modules over a standard graded polynomial ring $S := \Bbbk[x_0, \ldots, x_n]$. The key foundational results include the following:

- the Hilbert–Burch Theorem [E, Theorem 20.15], which characterizes the Cohen–Macaulay codimension 2 quotients of S in terms of their minimal graded free resolution;
- the Hilbert Syzygy Theorem [E, Theorem 1.13], which establishes that, for every finitely generated graded S-module, the graded minimal free resolution has finite length (at most $n + 1$); and
- the Auslander–Buchsbaum Formula [E, Theorem 19.9], which relates the depth of a graded S-module to the length of its minimal graded free resolution.

Through the fundamental relation between graded modules over S and sheaves over $\mathbb{P}^n$, all of these results lead to significant insights into the geometry of projective space. Where are the analogous homological tools from commutative algebra for studying the geometry of a smooth toric variety X other than $\mathbb{P}^n$?

The work of David Cox [C] unquestionably provides the correct context. The Cox ring R of X (also known as the total coordinate ring of X) is a positively $\mathrm{Pic}(X)$-graded polynomial ring. The fundamental relation extends to a beautiful correspondence between $\mathrm{Pic}(X)$-graded modules over R and sheaves on X. Unfortunately, the minimal $\mathrm{Pic}(X)$-graded free resolutions for R-modules do *not* provide similar geometric insights on X; see [MS]. For example, the length of a minimal $\mathrm{Pic}(X)$-graded free resolution can be significantly larger than the dimension of X. Roughly speaking, the geometric information is concealed by extraneous algebraic strands in the resolution. How should one extract the significant data from a minimal $\mathrm{Pic}(X)$-graded free resolution?

Recent Developments and Open Problems

The most spectacular recent development in the homological methods for S is known as Boij–Söderberg Theory. Born out of the 2006 conjectures of Mats Boij and Jonas Söderberg [1], this theory describes an unexpectedly simple polyhedral structure on the Betti numbers of graded S-modules. While proving these conjectures, David Eisenbud and Frank-Olaf Schreyer [ES] also discovered a duality with the cohomology tables of algebraic vector bundles on $\mathbb{P}^n$.

The ambitious goal for this workshop was to create a version of Boij–Söderberg theory relating sheaf cohomology on a toric variety X to appropriate free complexes over the associated Cox ring R. The recent preprint of David Eisenbud and Daniel Erman [E^2], which provides a more general construction for the duality pairing in Boij–Söderberg Theory on $\mathbb{P}^n$, served as the starting point. In particular, we focused on developing the foundational homological results for free complexes with irrelevant homology.

Scientific Progress Made

During our week at BIRS, we outlined a broad framework for homological commutative algebra over the Cox ring R. The central objects, which we baptized ***splendid complexes***, are finite free complexes over R with irrelevant homology. By creating and implementing new routines in *Macaulay2* [M2], we explored numerous examples of "short" splendid complexes. Our most important advance was likely the invention of a method for shrinking a splendid complex — we create a smaller free complex by identifying and then removing an irrelevant strand of the complex. By exhaustively iterating this process, we obtained minimal splendid complexes in many different examples.

Building on this method, we formulated precise conjectural analogues for both the Hilbert–Burch Theorem and the Hilbert Syzygy Theorem, and we identified some of the crucial ingredients for a toric variant of the Auslander–Buchsbaum Formula. In addition, we generated a detailed sketch for a proof of our splendid version of the Hilbert–Burch Theorem. We also produced several splendid complexes which we think are new candidates to appear as the extremal rays in a multigraded Boij–Söderberg Theory.

Outcome of the Meeting

We now have a clear program for developing the necessary homological techniques in commutative algebra for studying smooth toric varieties. We are actively pursuing the conjectures arising from this workshop. We also expect to transform our *Macaulay2* code into a package that can be used by other researchers and distributed with a future version of this software system. Once we are armed with the required homological tools, we will return to the search for a multigraded decomposition result that parallels that main theorems in the Boij–Söderberg theory for $\mathbb{P}^n$.

Participants

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Bibliography

[BS] Mats Boij and Jonas Söderberg, Graded Betti numbers of Cohen-Macaulay modules and the multiplicity conjecture, *J. Lond. Math. Soc. (2)* **78** (2008), 85–106.

[C] David A. Cox, The homogeneous coordinate ring of a toric variety, *J. Algebraic Geom.* **4** (1995), 17–50.

[E] David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics 150, Springer-Verlag, New York, 1995.

[E^2] David Eisenbud and Daniel Erman, Categorified duality in Boij-SÃűderberg Theory and invariants of free complexes, available at `arXiv:1205.0449v1 [math.AC]`.

[ES] David Eisenbud and Frank-Olaf Schreyer, Betti numbers of graded modules and cohomology of vector bundles, *J. Amer. Math. Soc.* **22** (2009), 859–888.

[M2] Daniel R. Grayson and Michael E. Stillman, *Macaulay2*, a software system for research in algebraic geometry, available at `http://www.math.uiuc.edu/Macaulay2/`

[MS] Diane Maclagan and Gregory G. Smith, Multigraded Castelnuovo-Mumford regularity, *J. Reine Angew. Math.* **571** (2004), 179–212.

Chapter 59

A New Implementation of Fletcher's Exact Merit Function for Nonlinear Optimization (12rit180)

May 27 - June 3, 2012

Organizer(s): Michael P. Friedlander (Vancouver, Canada), Dominique Orban (Montréal, Canada)

Overview

Consider the general nonlinear optimization problem

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad c(x) = 0, \tag{59.0.0.1}$$

where $f : \Re^n \to \Re$ and $c : \Re^n \to \Re^m$ are twice-continuously differentiable functions. A fundamental challenge in developing iterative algorithms for this solution of this problem is the inherent tradeoff between minimizing the objective function f, and satisfying the constraint $c(x) = 0$. Penalty functions encapsulate these competing demands by providing a measure of progress towards the solution, and provide for a way of transforming the constrained (and difficult) problem into an unconstrained (and easier problem). A penalty function may be either *exact*—i.e., its unconstrained minimizer coincides with a solution of (59.0.0.1)—or *inexact*—i.e., its unconstrained minimizer is only an approximate solution, and an infinite sequence of unconstrained problems must be solved. Exact-penalty functions, however, are generally nonsmooth, which entail a host of complicating factors.

The method that we consider is rooted in a smooth and exact penalty function first proposed by [2] for equality-constrained problems. There has been a long-held view that that Fletcher's penalty function is not practicable because it is costly to compute; see comments by Bertsekas (1976), [1], and [3]. Our aim in this project is to challenge that notion, and to demonstrate that the computational kernels are no more expensive than other well-accepted methods for nonlinear optimization, such as sequential quadratic programming.

The penalty function that we consider for (59.0.0.1)

$$\phi_\sigma(x) := f(x) - c(x)^T y_\sigma(x), \tag{59.0.0.2}$$

where $y_\sigma(x)$ are Lagrange multiplier estimates defined as the solution of the least-squares problem

$$\text{minimize}_y \ \frac{1}{2}\|A(x)y - g(x)\|_2^2 + \sigma c(x)^T y, \tag{59.0.0.3}$$

where we used the notation

$$g(x) := \nabla f(x), \quad A(x) := \nabla c(x), \quad \text{and} \quad Y_\sigma(x) := \nabla y_\sigma(x). \tag{59.0.0.4}$$

Note that A and Y_σ are n-by-m matrices. In our initial exploration, we make the simplifying assumption that $A(x)$ is full rank for all x. Hence, the solution $y_\sigma(x)$ and its gradient $Y_\sigma(x)$ are uniquely defined.

Notation

Let $H(x) = \nabla^2 f(x)$ and $H_i(x) = \nabla^2 c_i(x)$. We also define

$$g_\sigma(x) := g(x) - A(x)y_\sigma(x), \tag{59.0.0.5a}$$

$$H_\sigma(x) := H(x) - \sum_{i=1}^m [y_\sigma(x)]_i H_i(x), \tag{59.0.0.5b}$$

which we recognize as the gradient and Hessian, respectively, of the usual Lagrangian function $L(x, y)$ evaluated at x and $y(x)$. Also, define the matrix operators

$$S(x, v) := \nabla_x[A(x)^T v] = \nabla_x \begin{bmatrix} \nabla c_1(x)^T v \\ \vdots \\ \nabla c_m(x)^T v \end{bmatrix} = \begin{bmatrix} v^T H_1(x) \\ \vdots \\ v^T H_m(x) \end{bmatrix};$$

$$T(x, w) := \nabla_x[A(x)w] = \nabla_x \left[\sum_{i=1}^m \nabla c_i(x) w_i \right] = \sum_{i=1}^m w_i H_i(x),$$

for all $v \in \Re^n$ and $w \in \Re^m$. In particular, note that for all $u \in \Re^m$, all $v \in \Re^n$ and all $w \in \Re^m$,

$$S(x, v)^T u = \left[\sum_{i=1}^m u_i H_i(x) \right] v = T(x, u)v\,,$$

$$S(x, v)w = \begin{bmatrix} v^T H_1(x)w \\ \vdots \\ v^T H_m(x)w \end{bmatrix},$$

and

$$T(x, w)^T v = \left[\sum_{i=1}^m w_i H_i(x) \right] v = T(x, w)v\,.$$

If A has full rank at some feasible x^*, the operators

$$P = A(x^*)\big(A(x^*)^T A(x^*)\big)^{-1} A(x^*)^T \quad \text{and} \quad \bar{P} := I - P \tag{59.0.0.6}$$

define orthogonal projectors onto range$(A(x^*))$ and its complement, respectively.

The gradient and Hessian of ϕ_σ may be written as

$$\nabla \phi_\sigma(x) = g_\sigma(x) - Y_\sigma(x)c(x), \tag{59.0.0.7a}$$

$$\nabla^2 \phi_\sigma(x) = H_\sigma(x) - A(x)Y_\sigma(x)^T - Y_\sigma(x)A(x)^T - \nabla\left[Y_\sigma(x)c\right], \tag{59.0.0.7b}$$

where $H(x) = \nabla^2 f(x)$ and $H_i(x) = \nabla^2 c_i(x)$ are the Hessians of the objective and each constraint function, respectively. The last term $\nabla_x[Y_\sigma(x)c]$ in the expression for $\nabla^2 \phi_\sigma$ purposefully drops the argument on c to emphasize that this gradient is made on the product $Y(x)c$, with $c := c(x)$ held fixed. This term involves third derivatives of f and c, and as Fletcher shows, it is both convenient and computationally efficient to ignore this term; we leave this term unexpanded.

Scientific Progress Made

During our workshop we established a better understanding of how an algorithm might dynamically update the penalty parameter. In this section we give explicit expressions for threshold values of the penalty parameter.

It follows directly from the gradient and Hessian expressions (59.0.0.7) of ϕ_σ and the definition (59.0.0.3) of y_σ that the following definitions are equivalent to the usual optimality conditions defined via the Lagrangian function; see, e.g., [3, Ch. 12].

First-order KKT point A point x^* is a first-order KKT point of (59.0.0.1) if for any $\sigma \geq 0$ the following hold:

$$c(x^*) = 0, \tag{59.0.0.8a}$$
$$\nabla\phi_\sigma(x^*) = 0. \tag{59.0.0.8b}$$

The elements of $y^* := y_\sigma(x^*)$ comprise the vector of Lagrange multipliers of (59.0.0.1) associated to x^*.

We can similarly derive second-order optimality conditions based on the Hessian of ϕ_σ.

Second-order KKT point The first-order KKT point x^* satisfies the second-order necessary KKT condition for (59.0.0.1) if for any $\sigma \geq 0$

$$p^T\nabla^2\phi_\sigma(x^*)p \geq 0 \quad \text{for all } p \text{ such that } A(x^*)^Tp = 0. \tag{59.0.0.9}$$

The condition is sufficient if

$$p^T\nabla^2\phi_\sigma(x^*)p > 0 \quad \text{for all } p \neq 0 \text{ such that } A(x^*)^Tp = 0. \tag{59.0.0.10}$$

The second-order KKT condition holds for all $\sigma \geq 0$, and only requires the correct curvature of ϕ_σ in directions in the tangent space of the constraints. However, we can explicitly derive a threshold value of σ that causes a stationary point of ϕ_σ to be feasible, or causes ϕ_σ to be locally convex at a second-order KKT point x^*. For a given first or second-order KKT point x^* for (59.0.0.1), we define

$$\sigma^* := \frac{1}{2}\|PH_\sigma(x^*)P\|. \tag{59.0.0.11}$$

Theorem 20. *If $\nabla\phi_\sigma(\bar{x}) = 0$ for some $\bar{x}$, then*

$$\sigma > \|A(\bar{x})^TY_\sigma(\bar{x})\| \quad \Longrightarrow \quad g(\bar{x}) = A(\bar{x})y_\sigma(\bar{x}), \quad c(\bar{x}) = 0. \tag{59.0.0.12a}$$

If x^* is a first-order KKT point for (59.0.0.1)*, then*

$$\sigma \geq \|A(x^*)Y_\sigma(x^*)^T\| \quad \Longrightarrow \quad \sigma \geq \sigma^*. \tag{59.0.0.12b}$$

If x^* is a second-order necessary KKT point for (59.0.0.1)*, then*

$$\nabla^2\phi_\sigma(x^*) \succeq 0 \quad \Longleftrightarrow \quad \sigma \geq \sigma^*, \tag{59.0.0.12c}$$

If x^* is second-order sufficient, then the inequalities in (59.0.0.12c) *hold strictly.*

Proof. We prove, in order, (59.0.0.12a), (59.0.0.12c), and (59.0.0.12b). First note that for any x, the vector of Lagrange multiplier estimates $y_\sigma(x)$ satisfies the linear system

$$A(x)^TA(x)y_\sigma(x) = A(x)^Tg(x) - \sigma c(x), \tag{59.0.0.13}$$

which define necessary and sufficient optimality conditions for (59.0.0.3).

Proof of (59.0.0.12a). The condition $\nabla\phi_\sigma(\bar{x}) = 0$ implies that

$$g(\bar{x}) = A(\bar{x})y_\sigma(\bar{x}) + Y_\sigma(\bar{x})c(\bar{x}).$$

Using this equation in (59.0.0.13) evaluated at $\bar{x}$, yields, after simplifying,

$$A(\bar{x})^T Y_\sigma(\bar{x}) c(\bar{x}) = \sigma c(\bar{x}).$$

Taking norms of both sides and using the triangle inequality gives $\sigma\|c(\bar{x})\| \le \|A(\bar{x})^T Y_\sigma(\bar{x})\|\,\|c(\bar{x})\|$, which immediately implies that $c(\bar{x}) = 0$. The condition $\nabla\phi_\sigma(\bar{x}) = 0$ then becomes $g_\sigma(\bar{x}) = 0$, which completes the proof of (59.0.0.12a).

Proof of (59.0.0.12c). We first obtain an expression for $A(x)Y_\sigma(x)^T$ in terms of $H_\sigma(x)$ by differentiating both sides of (59.0.0.13), which yields

$$S(x, A(x)y_\sigma(x)) + A(x)^T\left[T(x, y_\sigma(x)) + A(x)Y_\sigma(x)^T\right] = S(x, g(x)) + A(x)^T[H(x) - \sigma I].$$

Isolating the term $A(x)^T A(x) Y_\sigma(x)$ on the left-hand side, and using the linearity of S in its second argument, we rearrange terms to arrive at

$$A(x)^T A(x) Y(x)^T = S(x, g(x) - A(x)y_\sigma(x)) + A(x)^T[H(x) - T(x, y_\sigma(x)) - \sigma I].$$

Using the definitions (59.0.0.5), this can be expressed as

$$A(x)^T A(x) Y(x)^T = A(x)^T[H_\sigma(x) - \sigma I] + S(x, g_\sigma(x)). \tag{59.0.0.14}$$

Because x^* satisifes the first-order conditions (59.0.0.8), $g_\sigma(x^*) = 0$, and it follows from the above equation and the definition of P that

$$A(x^*)Y_\sigma(x^*)^T = P(H_\sigma(x^*) - \sigma I). \tag{59.0.0.15}$$

We substitute this equation into (59.0.0.7b) and use the relation $P + \bar{P} = I$ to obtain the expression

$$\nabla^2\phi_\sigma(x^*) = \bar{P}H_\sigma(x^*)\bar{P} - PH_\sigma(x^*)P + 2\sigma P. \tag{59.0.0.16}$$

Because $\|P\| \le 1$, the relationship (59.0.0.12c) follows.

Proof of (59.0.0.12b). Again using properties of the projector P, it follows from (59.0.0.15) that

$$\begin{aligned}\sigma &\ge \|A(x^*)Y_\sigma(x^*)^T\| \\ &= \|P(H_\sigma(x^*) - \sigma I)\| \\ &\ge \|P(H_\sigma(x^*) - \sigma I)P\| \\ &\ge \|PH_\sigma(x^*)P\| - \sigma\|P\| \\ &\ge 2\sigma^* - \sigma.\end{aligned}$$

Thus, $\sigma \ge \sigma^*$, as required. □

Outcome of the Meeting

Theorem 20 gives us a concrete method for testing if a candidiate threshold parameter is sufficiently large. Other crucial items completed during this workshop included

- A method for computing the gradient and Hessians in (59.0.0.7) that has a cost of factorizing only a single projection matrix;
- Extensions of the penalty function to handle more general constraints, including affine and bound constraints.

Participants

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Bibliography

[1] A. R. Conn, N. I. M. Gould, and Ph. L. Toint. *Trust-Region Methods*. MPS-SIAM Series on Optimization. Society of Industrial and Applied Mathematics, Philadelphia, 2000.

[2] R. Fletcher. A class of methods for nonlinear programming with termination and convergence properties. In J. Abadie, editor, *Integer and nonlinear programming*, pages 157–175. North-Holland, Amsterdam, 1970.

[3] J. Nocedal and S. J. Wright. *Numerical Optimization*. Springer, New York, second edition, 2006.

Chapter 60

Strong Asymptotics for Cauchy Biorthogonal Polynomials (12rit173)

June 10 - 17, 2012

Organizer(s): Marco Bertola (CRM,Université de Montréal and Department of Mathematics and Statistics, Concordia University), Misha Gekhtman (Department of Mathematics, University of Notre Dame), Jacek Szmigielski (Department of Mathematics and Statistics, University of Saskatchewan)

Overview of the Field

The main objective of the workshop was to complete a project on the strong asymptotics of Cauchy biorthogonal polynomials [1] and an associated Cauchy two-matrix model introduced in [2].

The model consists of two random Hermitean positive-definite matrices M_1, M_2 of size $n \times n$ equipped with the probability measure

$$d\mu(M_1, M_2) = \frac{1}{\mathcal{Z}_n} \frac{dM_1 dM_2}{\det(M_1 + M_2)^n} \mathrm{e}^{-N\mathrm{Tr}(U(M_1))} \mathrm{e}^{-N\mathrm{Tr}(V(M_2))}$$

where U, V are scalar functions defined on $\mathbb{R}$. The model was termed the *Cauchy matrix model* because of the shape of the coupling term. Similarly to the case of the Hermitean one-matrix models for which the spectral statistics is expressible in terms of appropriate orthogonal polynomials [3], this two-matrix model is solvable with the help of a new family of biorthogonal polynomials named *the Cauchy biorthogonal polynomials* [4]. The Cauchy biorthogonal polynomials are two sequences of monic polynomials $(p_j(x))_{j=0}^{\infty}, (q_j(y))_{j=0}^{\infty}$ with deg p_j =deg $q_j = j$ that satisfy

$$\iint_{\mathbb{R}_+ \times \mathbb{R}_+} p_j(x) q_k(y) \frac{\mathrm{e}^{-N(U(x)+V(y))}}{x+y} d\,x d\,y = h_k \delta_{jk}\ ,\ \ \forall j, k \geq 0\ , h_k > 0\ .$$

These polynomials appeared initially in an inverse problem for the nonlinear dispersion model (the Degasperis-Procesi equation) [5] and were further developed in [6] in relation with the spectral theory of the cubic string as well as applied to other nonlinear partial differential equations [7].

Recent Developments

One of the problems we set out to solve was the large $N = n + r$ asymptotics of the Cauchy biorthogonal polynomials $(p_j(x))_{j=0}^{\infty}, (q_j(y))_{j=0}^{\infty}$. This required first the formulation of an appropriate Riemann-Hilbert

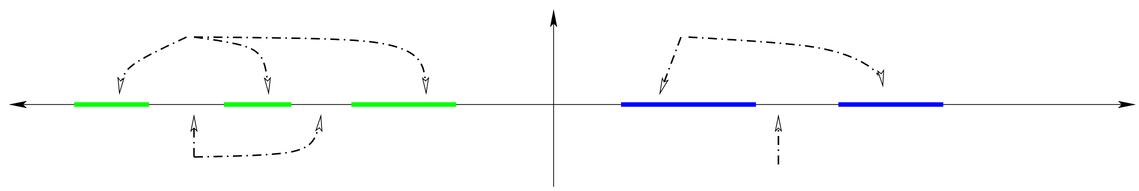

Figure 60.1: The RHP for the outer parametrix

problem and that was accomplished in [1]. With the help of suitable equilibrium potentials, called below ρ_1 and ρ_2 and a sequence of deformation we formulated the outer and inner parametrix problems essentially following the machinery laid out by Deift and Zhou [8] with an important modification that our RH problem deals with 3×3 matrices, rather than 2×2, and it exhibits a different asymptotic behaviour at infinity. For example

Problem 60.0.1 (Outer parametrix)**.** *Find a 3×3 matrix $\Psi(z)$, analytic in $\mathcal{D}_0 := \mathbb{C} \setminus ((-\infty, b_0] \cup [a_0, \infty))$ and with the following properties*

(i) the jumps indicated in Figure 60.1 with specific values

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \text{ on the left (green) cuts} \qquad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ on the right (blue) cuts}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2\pi i N \sigma_l} & \\ 0 & 0 & e^{2\pi i N \sigma_l} \end{pmatrix} \text{ on the left } l\text{-th gap} \qquad \begin{pmatrix} e^{2\pi i N \epsilon_l} & 0 & 0 \\ 0 & e^{-2\pi i N \epsilon_l} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ on the right } l\text{-th gap}$$

(ii) the growth conditions at $z = \infty$ and near an endpoint are, respectively

$$\Psi(z) \sim \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^r & & \\ & 1 & \\ & & z^{-r} \end{pmatrix}, \ \Psi(z) = \mathcal{O}\left((z-a)^{-\frac{1}{4}}\right), \ a \in \{a_i, b_i\}_{i=1,\dots}$$

We solved completely the RH problem 60.0.1 by giving an explicit solution in terms of the Riemann Theta functions associated to certain Riemann surface $\mathcal{L}$ which can be realized as a double of the bordered Riemann surface obtained by gluing together three Riemann spheres slit along the support of one of the equilibrium measures ρ_1 and glued there with the middle Riemann sphere which is subsequently slit also along the support of the second equilibrium measure ρ_2 and glued across it with the third Riemann sphere, cut along the support of ρ_2. More details can be found in [9].

Outcome of the Meeting and Open Problems

The original project was completed and in fact during the workshop the paper reporting the results was submitted as well as posted on the archives [9]. As expected, all spectral statistics, including gap probabilities, were proved to follow the standard universality results for the one-matrix Hermitean model. However, it is expected that new universal behavior will appear at the zero eigenvalue where two interacting matrices (a positive-definite M_1 and a negative-definite $-M_2$) in some sense come "close" one to another. We started investigating this scaling regime by looking at the concrete example, which is of independent interest as it involves certain classical special functions. More concretely, during the workshop we made a significant progress on the explicit construction of both Cauchy biorthogonal polynomials as well as the accompanying two-matrix model for the case of two Laguerre-type measures $d\mu = x^a e^{-x}, d\nu = x^b e^{-x}, x > 0$.

Theorem 21. *Given two Laguerre-type measures $d\mu, d\nu$ specified above let us set $\alpha := a + b$. Then the associated Cauchy biorthogonal polynomials p_n, q_n are expressed as (they are normalized so that the leading coefficient is the same)*

$$p_n(z) = (-1)^n \sqrt{2n+\alpha+1} \sqrt{\frac{\Gamma(a+n+1)}{\Gamma(b+n+1)}} \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)\Gamma(a+1)} {}_2F_2(-n, \alpha+n+1; a+1, \alpha+1; z)$$
$$q_n(z) = (-1)^n \sqrt{2n+\alpha+1} \sqrt{\frac{\Gamma(b+n+1)}{\Gamma(a+n+1)}} \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)\Gamma(b+1)} {}_2F_2(-n, \alpha+n+1; b+1, \alpha+1; z)$$

and they satisfied 3*-rd order differential equations:*

$$[z(\Delta-n)(\Delta+\alpha+n+1) - \Delta(\Delta+a)(\Delta+\alpha)]\, p_n = 0,$$
$$[z(\Delta-n)(\Delta+\alpha+n+1) - \Delta(\Delta+b)(\Delta+\alpha)]\, q_n = 0,$$

where $\Delta = z\frac{d}{dz}$.

The fact that the Cauchy biorthogonal polynomials are expressed in this case by hypergeometric functions of type ${}_2F_2$ is a welcoming sign that the theory of these polynomials is an extension of "classical function theory". Moreover, the scaling limit points to a new universal behaviour:

Theorem 22. *The orthonormal $p_n(z), q_n(z)$ in the scaling regime behave as*

$$q_{n+r}(zn^{-2}) = (-1)^{n+r+1} \left(1 + \frac{r(\alpha+2\Delta) + \frac{(\alpha+1)(\alpha+2\Delta)}{2}}{n} + \frac{1}{n^2}\left(\frac{(\alpha+2\Delta)(\alpha+2\Delta-1)r^2}{2} + rC_1 + C_2\right)\right) G(z)$$

where C_1, C_2 are two operators that are independent of r (and n) while $G(z)$ is the Meijer G function

$$G(z) \equiv G^{1,0}_{0,3}\left(z \middle| 0, -\alpha, -a\right) = \frac{1}{2\pi i} \int_\gamma \frac{\Gamma(u)}{\Gamma(1+\alpha-u)\Gamma(1+a-u)} z^{-u} du.$$

Conjecture 3. *All correlation functions in the scaling regime $\frac{z}{n^2}, n \to \infty$, can be expressed as differential expressions in the the scale invariant Δ applied to the Meijer G function.*

Participants

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Bibliography

[1] M. Bertola, M. Gekhtman, and J. Szmigielski. Cauchy biorthogonal polynomials. *J. Approx. Theory*, 162(4): 832–867, 2010.

[2] M. Bertola, M. Gekhtman, and J. Szmigielski. The Cauchy two–matrix model. *Comm. Math. Phys.*, 287(3):983–1014, 2009.

[3] M. L. Mehta. *Random matrices*, volume 142 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, third edition, 2004.

[4] M. Bertola, M. Gekhtman, and J. Szmigielski. Cauchy Biorthogonal Polynomials. *arXiv:0904.2602*, 2009.

[5] H. Lundmark and J. Szmigielski. Degasperis-Procesi peakons and the discrete cubic string. *IMRP Int. Math. Res. Pap.*, (2):53–116, 2005.

[6] M. Bertola, J. Szmigielski, and M. Gekhtman. Cubic String Boundary Value Problems and Cauchy Biorthogonal Polynomials. *arXiv:0904.2188*, 2009.

[7] K. Mohajer and J. Szmigielski. On an inverse problem associated with an integrable equation of Camassa-Holm type: explicit formulas on the real axis. *Inverse Problems*, 28(1):015002, 13, 2012.

[8] P. A. Deift. *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, volume 3 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 1999.

[9] M. Bertola, M. Gekhtman, and J. Szmigielski. Strong asymptotics for Cauchy biorthogonal polynomials with application to the Cauchy two–matrix model. *arXiv:1206.3199*, 2012.

Chapter 61

Renormalization Group Methods for Polymer and Last Passage Percolation Models (12rit174)

July 22 - 29, 2012

Organizer(s): Tom Alberts (California Institute of Technology), Yuri Bakhtin (Georgia Institute of Technology), Eric Cator (TU Delft), Konstantin Khanin (University of Toronto)

Overview of the Field

Our workshop is mostly concentrated on the fields of statistical mechanics and probability theory. Here the classical objects are simple random walk and Brownian motion, and these are well studied and commonly used processes. We will be considering directed polymers [4] and last passage percolation paths [2], which in the appropriate light can be seen as simple random walks moving through non-homogeneous environments with random features. Their study overlaps with and provides insight into many different areas of mathematics, such as random matrix theory [7], random Schrodinger operators [5], tropical geometry [6], stochastic PDE [3], and more.

Recent Developments and Open Problems

In the last few years a series of papers [1] have started revealing a conjectural universal structure for 1+1-dimensional polymer and LPP models. The former is described by the continuum random polymer, and the latter by the so-called Airy sheet. The connection between the two is not yet fully understood as it is for their discrete analogues, but it is expected that enlightenment will come shortly. Exploring this structure was the main focus of our workshop.

Current understanding of the universal structure follows exclusively from asymptotic analysis of exact formulas from discrete statistical mechanics models. These formulas are very involved and the asymptotic analysis is difficult. Moreover they are generally not robust to even small perturbations in the type of discrete models under consideration, and even discovering them is not easy. At present one of the major obstacles to more progress is a lack of these exact formulas.

Scientific Progress Made

The main goal of our project is to understand connections between continuum directed polymer models and continuum last passage percolation. In discrete settings the relationship between these two models is relatively clear, but proper notions of their continuum versions/scaling limits are now just beginning to emerge. A full understanding of the continuum objects is important for understanding the universal features and statistics of these models, unencumbered from the non-universal and arbitrary features imposed in consideration of the discrete models. The classical analogy here is the relationship between discrete simple random walk and continuum Brownian motion. Studying the latter reveals the universal properties of this type of random spatial motion without having to work around the peculiarities (lattice type, jump distribution) that the discrete walk can impose.

The focus of our workshop was to study the universality structure in the continuum without relying on exact formulas, and instead try to proceed using renormalization group arguments. This moves the analysis more into the framework of dynamical systems, albeit on a complicated space. The advantage is that the renormalization group ideas are simple, intuitive, and useful for enhancing the conceptual understanding of the problem, and with the current progress in this field we believe there is a high likelihood they can be rigorously applied here.

We discovered a particular example where this program can likely be carried out. Consider the quenched endpoint distribution of a continuous random polymer moving through a continuum space-time random environment. Assuming the polymer has time length one, the endpoint distribution can be represented as a random function of x (its density with respect to Lebesgue measure):

$$f_\beta^W(x) := \frac{1}{\mathcal{Z}_\beta^W}\mathbb{E}_{0\to x}\left[:\!\exp\!:\left\{\beta\int_0^1 W(t,B_t)\,dt\right\}\right]\varrho(x).$$

Here W is a space-time white noise on $[0,1]\times\mathbb{R}$ (the first coordinate is time, the second space), and the expectation $\mathbb{E}_{0\to x}$ is over Brownian bridge paths from 0 to x. Further ϱ is the standard Gaussian density

$$\varrho(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

The random variable $\mathcal{Z}_\beta^W$ is a normalization constant so that f_β^W integrates to 1 on the line. We will mostly be interested in the unnormalized quantity

$$\varrho_\beta^W(x) := \mathcal{Z}_\beta^W f_\beta^W(x).$$

Due to the fact that W is a random field whose distribution is invariant under shifts, it follows that the endpoint distribution is the same as that for regular Brownian motion (i.e. $\varrho(x)$) but perturbed by a random process that is necessarily stationary with respect to shifts. This implies that we may rewrite ϱ_β^W as

$$\varrho_\beta^W(x) = \varrho(x)e^{A_\beta^W(x)}$$

where $A_\beta^W(x)$ is a stationary process on the line whose exponential has mean one. Now since a polymer of length 2 can be written as a scaling of a concatenation of two independent polymers of length 1, it follows that

$$\left(\varrho_\beta^W * \varrho_\beta^{\tilde{W}}\right)(x) \equiv \sqrt{2}\varrho^W_{\sqrt{2}\beta}(\sqrt{2}x),$$

where $\equiv$ indicates equality in law, $\tilde{W}$ is an independent copy of the white noise, and $*$ indicates convolution. The various factors of $\sqrt{2}$ on the right hand side are because of scaling properties of the Brownian motion and the white noise.

The latter identity in law holds for the particular choice of the stationary process given by $e^{A_\beta^W(x)}$ inherited from the given f_β^W. But it also can be generalized. Let $A(x)$ be a stationary process on the line whose exponential has mean one. Then it is a straightforward calculation to show that

$$\left(\varrho e^A\right) * \left(\varrho e^{\tilde{A}}\right)(x) = \sqrt{2}\varrho(\sqrt{2}x)e^{C(\sqrt{2}x)},$$

where $\tilde{A}$ is an independent copy of A, and on the right hand side C is a new stationary process whose exponential has mean one. Thus this class of stationary processes is closed under the operation of multiplication by ϱ and convolution, after an appropriate scaling of the spatial variable. This gives a simple set of dynamics on the stationary processes which we believe can be studied using renormalization group arguments. The main goal is to prove the existence of a fixed point. Other work to be carried out is to prove that these dynamics have only finitely many unstable directions (hopefully just one) in the space of processes, and every other direction is stable (a stable fixed point is clearly $A = 0$).

Outcome of the Meeting

Using the framework described above we were able to make some progress towards understanding the structure of the stable manifold of stationary distributions under the given dynamics. In a very rough sense we were able to understand part of the negative spectrum of the operator described in the last section, and we continue to work towards understanding the positive spectrum. As part of our progress we were able to develop some seemingly new techniques for studying stationary distributions on the line that may ultimately be of interest in other applications.

Participants

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Bibliography

[1] Gideon Amir, Ivan Corwin, and Jeremy Quastel. Probability distribution of the free energy of the continuum directed random polymer in $1 + 1$ dimensions. *Comm. Pure Appl. Math.*, 64(4):466–537, 2011.

[2] Eric A. Cator, Leandro P. R. Pimentel, and Marcio W. A. Souza. Influence of the initial condition in equilibrium last-passage percolation models. *Electron. Commun. Probab.*, 17:no. 7, 7, 2012.

[3] Robert Dalang, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao. *A minicourse on stochastic partial differential equations*, volume 1962 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006, Edited by Khoshnevisan and Firas Rassoul-Agha.

[4] Giambattista Giacomin. *Random polymer models*. Imperial College Press, London, 2007.

[5] Eugene Kritchevski, Benedek Valkó, and Bálint Virág. The Scaling Limit of the Critical One-Dimensional Random Schrödinger Operator. *Comm. Math. Phys.*, 314(3):775–806, 2012.

[6] Jürgen Richter-Gebert, Bernd Sturmfels, and Thorsten Theobald. First steps in tropical geometry. In *Idempotent mathematics and mathematical physics*, volume 377 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 2005.

[7] Benedek Valkó and Bálint Virág. Continuum limits of random matrices and the Brownian carousel. *Invent. Math.*, 177(3):463–508, 2009.

Chapter 62

Models for minimal Cantor $\mathbb{Z}^2$-systems (12rit170)

August 5 - 12, 2012

Organizer(s): Thierry Giordano (University of Ottawa), Ian F. Putnam (University of Victoria), Christian F. Skau (Norwegian University of Science and Technology)

Overview of the Field

The work of Herman, Putnam and Skau [6] used ideas from operator algebras to present a complete model for minimal actions of the group $\mathbb{Z}$ on a compact, totally disconnect metrizable space having no isolated points, i.e. a Cantor set. The data (a Bratteli diagram, with extra structure) is basically combinatorial and the two great features of the model were that it contained, in a reasonably accessible form, the orbit structure of the resulting dynamical system and also cohomological data provided either from the K-theory of the associated C^*-algebra or more directly from the dynamics via group cohomology. This led to a complete classification of such systems up to orbit equivalence [5]. This was the first extension of a famous program initiated by Henry Dye [2] in the study of orbit equivalence in ergodic theory to the topological situation. (See also [7, 1].)

The classification in [5] was extended to include minimal actions of $\mathbb{Z}^2$ in [3] and minimal actions of finitely generated abelian groups in [4]. However, what was not extended was the original model and this has handicapped the general understanding of these actions. The higher dimensional case is an important one, since, in particular, it has applications to the study of quasicrystals.

Recent Developments and Open Problems

A couple of years ago, the three of us had some new ideas on how to produce a model for actions of $\mathbb{Z}^2$. The two essential new insights were to start from the cohomological data, rather than the combinatorial data and secondly, to make some simplifying assumptions on the properties of the cohomology. The second point in particular is key; our knowledge of the general properties of the cohomology of minimal actions of $\mathbb{Z}^d$ is scant, when d exceeds one. (It is even conceivable that the problems involve some rather deep issues of decidability.) Of course, this means that the model can only produce dynamics whose cohomology satisfies these properties, but with the profound lack of success in discerning whether or not they hold in general, it seems a reasonable special case.

Scientific Progress Made

We have been able to establish most of the construction rigorously. During the stay at BIRS, we were able to show that the first cohomology group for the model was correct. We also made progress in computing the second. Rather remarkably, it seems that the second actually agrees with the data from (some) Bratteli diagram, which is more than we had expected. One of the crucial technical steps involves the structure of Bratteli diagrams having associated dimension groups with exactly two states. Here, we established some explicit estimates. Still remaining are some subtle estimates regarding the geometry of the finite approximations, but it seems unlikely that this problem is serious.

We also had plans to extend the model to admit more than one invariant measure and also to allow for torsion in the cohomology. On the former issue, initial investigations looked promising, but we obtained few hard results. No real progress was made on the latter.

Outcome of the Meeting

We made considerable progress in obtaining complete proofs of many of the technical estimates. We started writing these, which will become part of the final paper.

Anticipated Impact

We anticipate the result will have impact in a number of ways.

1. To provide a much broader class of examples of actions of $\mathbb{Z}^2$.

   While the model as we have it is not complete (it does not produce every minimal Cantor action of $\mathbb{Z}^2$), it should significantly extend the class of known examples, which is surprisingly small.

2. To clarify the structure of minimal actions of $\mathbb{Z}^2$.

   This should be an immediate consequence of having a larger class of examples and a systematic way of producing them.

3. Understanding the range of the cohomological invariants for $\mathbb{Z}^2$-actions

   While in the case of $\mathbb{Z}$, the range of the cohomology invariant is completely understood, this is still a complete mystery for higher rank actions. In particular, as a consequence of the orbit equivalence classification, it may be possible to have minimal actions of $\mathbb{Z}$ which are not orbit equivalent to any action of $\mathbb{Z}^2$. This situation does not occur in ergodic theory.

4. To emphasize the importance of the ring structure in cohomology.

   This ring structure is trivial for $\mathbb{Z}$-actions and while many computations of cohomology groups have been made in higher rank examples, the ring structure has mostly been ignored. It plays a crucial part in our model which should increase its profile in general.

5. To focus attention on the hypotheses used on the cohomology.

   As mentioned above, the structure of the cohomology groups for higher rank actions is not well-understood. Our construction makes some serious hypotheses on these groups as its starting point. It therefore draws attention to this lack of understanding and may encourage investigations into these issues.

Participants

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Bibliography

[1] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory Dynam. Systems **1** (1981), 431-450.

[2] H.A. Dye, *On groups of measure preserving transformations I*, Amer. J. Math. **81**(1959), 119-159.

[3] T. Giordano, H. Matui, I.F. Putnam and C.F. Skau, *Orbit equivalence for Cantor minimal $\mathbb{Z}^2$-systems*, J. Amer. Math. Soc. 21 (2008), 863-892.

[4] T. Giordano, H. Matui, I.F. Putnam and C.F. Skau, *Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems*, Inventiones Mathematicae, 179 (2010), 119-158.

[5] T. Giordano, I.F. Putnam and C.F. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. **469** (1995), 51-111.

[6] R.H. Herman, I.F. Putnam and C.F. Skau, *Ordered Bratteli diagrams, dimension groups and topological dynamics*, Internat. J. Math. **3** (1992), 827-864.

[7] D.S. Ornstein and B. Weiss, *Ergodic theory of amenable group actions I: The Rohlin lemma*, Bull. Amer. Math. Soc. **2** (1980), 161-164.

Chapter 63

Moduli spaces in conformal field theory and Teichmüller theory (12rit139)

August 12 - 19, 2012

Organizer(s): Yi-Zhi Huang (Rutgers University, New Brunswick, NJ, USA), David Radnell (American University of Sharjah, Sharjah, UAE), Eric Schippers (University of Manitoba, Winnipeg, MB, Canada), Wolfgang Staubach* (Uppsala University, Uppsala, Sweden)

* Unfortunately Wolfgang Staubach was unable to attend the meeting due to unexpected work commitments.

Overview of the Field

The definition of conformal field theory originated around 25 years ago with work of Segal ([11]) and others. Constructing mathematically rigorous examples that satisfy this definition has proved extremely difficult. One successful approach has been to develop and use vertex operator algebra theory. Due to the work of Huang (see for example [2, 3]) and Huang and Kong [4] and others, there are now a wide class of examples in genus 0 and 1. Constructing the higher-genus theory will involve the interplay of algebra, geometry and analysis.

- Algebra: Vertex operator algebras and their representation theory, category theory.
- Geometry: Riemann surfaces and their moduli spaces, infinite-dimensional Teichmüller theory.
- Analysis: Geometric function theory, quasiconformal mappings, infinite-dimensional holomorphy.

Before any construction of higher-genus conformal field theory can be carried out, there must first be a precise and mathematically rigorous definition of conformal field theory. This requires new results on the rigged moduli space of Riemann surfaces with parameterized boundaries and the associated determinant line bundle. Well developed tools from Teichmüller theory and geometric function theory can be brought to bear. Results of Radnell, Schippers and Staubach [5, 6, 7, 8, 9] over the past 9 years have solved a number of these problems. Most recently in [9] we have defined a refined Teichmüller space using a class of refined quasiconformal mappings introduced by Takhtajan and Teo [12] in the plane case. The new Teichmüller space appears to be the natural setting for conformal field theory.

Recent Developments and Open Problems

The focus of the team meeting was the following problems which are central to the definition and construction of conformal field theory from vertex operator algebras and furthering the connections with Teichmüller theory.

1. Define the determinant line of a Riemann surface whose boundary is parameterized by a refined quasisymmetry.

2. Construct local trivializations of the determinant line bundle over the rigged moduli space using "Faber polynomials", a canonical basis for the set of holomorphic functions on a domain.

3. Construct projectively flat connections on the determinant line bundle.

4. Give a rigorous and complete definition of conformal field theory and a modular functor.

5. Establish a conjectured "sewing property" for suitable "meromorphic functions" on the rigged moduli space.

6. Using the "sewing property" for these "meromorphic functions" on the rigged moduli space to show that traces of products or iterates of intertwining operators for suitable vertex operator algebras satisfy differential equations with regular singular points and thus are absolutely convergent.

Presentation Highlights

Being a "research in teams" meeting we did not give any formal presentations. However, because each of the participants works in a different area of mathematics, informal presentations were given at the beginning of the workshop to quickly bring each other up to date on the most recent developments and open problems. These were: (1) Schippers, "Faber Polynomials", (2) Huang, "meromorphic function on rigged moduli space", and (3) Radnell, "The definition of conformal field theory".

Scientific Progress Made

The main aim of the meeting was to bring together the recent work of the participants and use this to precisely formulate problems and the method of solution. We were very successful in this regard. These long standing problems have now been made very explicit and we have immediate approaches to solving them.

Refer to the list above for the statement of the problems.

1. Analysis problems involving curve regularity and the solution to the jump problem have recently been solved by two of us (Schippers and Staubach). The approach used by Huang in [1] to construct the determinant line can now be emulated in our setting of refined quasisymmetries. In genus-zero this problem has thus essentially been completed.

2. In the genus-zero case the classical Faber polynomials give a canonical basis for the space of holomorphic functions on a given domain. Three of us (Radnell, Schippers, Staubach [10]) shortly after the meeting used these together with the classical Grunsky matrices to construct global trivializations of the bundle of holomorphic maps over the rigged moduli space. This results in an alternate description of the determinant line bundle in genus zero, which can be generalized to higher genus (see next item).

   For higher-genus Riemann surfaces, analogues of these Faber polynomials exist in the literature (Tietz, [14]). During the meeting we completed most of the details involved in using these results in our setting. This has given us an explicit description of cokernel of the operator used to define the determinant line bundle, which was previously not possible to obtain. The determinant line bundle for higher-genus Riemann surfaces can therefore also constructed in a direct way.

3. We discussed the precise meaning of the conjecture/folk theorem on the existence of a projectively flat connections on the determinant line bundle and modular functor. Now that we have a precise formulation of the underlying spaces, there are many tools from Teichmüller theory available to tackle this problem.

4. The geometric category underlying the definition of conformal field theory is the moduli space of the Riemann surfaces with parameterized boundaries. Using the refined quasisymmetric mappings of Takhtajan and Teo [12] and our recently defined refined Teichmüller space [9] we now have the precise moduli space on which the rest of the definition of conformal field theory can be built. We now have a complete and rigorous definition of conformal theory and also a holomorphic modular functor.

5. During the meeting we discovered that the Faber polynomials, and their higher-genus analogues, have precisely the properties needed to define the class of "meromorphic functions" of interest. At first it seems coincidental that Faber polynomials appear in this seemly unrelated problem. However, the underlying reason is that the definition of the Faber polynomials and their generalization use the data of a boundary parametrization of the Riemann surface. Formulating things this ways enabled us to interpret these function not just as function on the Riemann surface but actually as functions on the rigged moduli space. The sewing property for these "meromorphic functions" remains to be proved. We discussed a number of approaches which will be explored in the immediate future.

6. Showing the convergence of the traces requires the solution of the previous problem. Long before the meeting, Huang obtained a clear approach based on the genus-one result of Huang [2, 3]. In order to apply this method to higher-genus, the solution of the previous problem is required. Our newly discovered approach in the above item may make it possible to prove the convergence of the traces.

Outcome of the Meeting

Due to the work of the team members over the past years we are now at the stage to be able to complete the main problems as listed above. Due to the amount of work to be done this will take some time. At the meeting, we solved problems 1, 2 and 5, and have some new insight as to how to proceed on problems 3, 4 and 6. So far, the meeting has resulted in one publication [10] to be submitted shortly and several others are in preparation. We have planned to meet again in summer 2013.

Participants

Huang, Yi-Zhi (Rutgers University)
Radnell, David (American University of Sharjah)
Schippers, Eric (University of Manitoba)

Bibliography

[1] Huang, Y.-Z. *Two-dimensional Conformal Geometry and Vertex Operator Algebras*. Progress in Mathematics, **148**. Birkhäuser Boston, Inc., Boston, MA, 1997.

[2] Huang, Y.-Z., *Differential equations, duality and modular invariance*, Commun. Contemp. Math. **7** (2005), no. 5, 649–706.

[3] Huang, Y.-Z., *Differential equations and intertwining operators*, Commun. Contemp. Math. **7** (2005), no. 3, 375–400

[4] Huang, Y.-Z. and Kong, L. *Modular invariance for conformal full field algebras*, Trans. Amer. Math. Soc. **362** no.6, (2010), 3027–3067.

[5] Radnell, D. *Schiffer Variation in Teichmüller Space, Determinant Line Bundles and Modular Functors*, PhD Thesis, Rutgers University, New Brunswick, NJ, 2003.

[6] Radnell, D. and Schippers, E. *Quasisymmetric sewing in rigged Teichmueller space*, Commun. Contemp. Math. **8** no 4 (2006), 481–534.

[7] Radnell, D. and Schippers, E., *A complex structure on the set of quasiconformally extendible non-overlapping mappings into a Riemann surface*, J. Anal. Math. **108** (2009), 277–291

[8] Radnell, D. and Schippers, E., *Fiber structure and local coordinates for the Teichmüller space of a bordered Riemann surface*, Conform. Geom. Dyn. **14** (2010), 14–34.

[9] Radnell, D., Schippers, E. and Staubach, W., *A Hilbert manifold structure on the refined Teichmueller space of bordered Riemann surfaces*, arXiv:1207.0973 [math.CV], 1–43

[10] Radnell, D., Schippers, E. and Staubach, W., *Representation of the genus zero rigged moduli space by Grunsky matrices*, preprint (18 pages, title tentative).

[11] Segal, G. *The definition of conformal field theory*, Topology, Geometry and Quantum Field Theory (U. Tillmann, ed.), London Mathematical Society Lecture Note Series, vol. 308, Cambridge University Press, 2004, Orginial preprint 1988, 421–576.

[12] Tatkhajan, L. A. and Teo, L.-P. *Weil-Petersson Metric on the Universal Teichmüller Space*. Memoirs of the American Mathematical Society. **183** no 861 (2006).

[13] Teo, L.-P. *Velling-Kirillov metric on the universal Teichmüller curve*, J. Anal. Math. **93** (2003), 271–307.

[14] Tietz, H. *Fabersche Entwicklungen auf geschlossenen Riemannschen Flächen*, Journal für die Reine und Angewandte Mathematik **190**, 22–33.

[15] Vafa, C. *Conformal theories and punctured surfaces*, Phys. Lett. B, **199** no. 2 (1987), 195–202.

Chapter 64

Positive Semidefinite Zero Forcing and Applications (12rit167)

September 23 - 30, 2012

Organizer(s): Craig Larson (Virginia Commonwealth University), Brian Lins (Hampden-Sydney College), Lon Mitchell (American Mathematical Society)

Overview

Mathematical structures consisting of nodes and connections between them, called graphs, can be used to model real-world networks and situations, such as the spread of a disease over time. Graph infection, also called zero forcing because of its algebraic properties, is such a model that also has connections to quantum physics. This Research in Teams strived to better understand zero forcing for a particular type of network that all graphs can be related to. Results can be applied to general graph infection and help provide answers to related questions in linear algebra and the control of quantum systems. In particular, we were interested in the relationship between zero forcing and the possible zero/nonzero patterns of unitary matrices.

Mathematical Background

A graph $G = (V, E)$ consists of a set V of vertices and a set E of unordered pairs of vertices called edges. We assume all graphs to be simple in that there are no multiple edges or loops (edges from a vertex to itself). A bipartite graph is a graph whose vertex set can be partitioned into two sets $V = M \cup N$ where no edge has both vertices in M or both vertices in N. The neighborhood of a vertex v in a graph G, denoted by $N(v)$, is the set of vertices adjacent to v. Consider a bipartite graph $G = (M \cup N, E)$. We will say that a subset S of either M or N has Property A if for all $v \in S$, there exists $w \neq v \in S$ such that $N(v) \cap N(w) \neq \emptyset$. A bipartite graph $G = (M \cup N, E)$ is called *strongly quadrangular* if S having Property A implies that

$$\left| \bigcup_{v,w \in S; v \neq w} N(v) \cap N(w) \right| \geq |S|,$$

when S is a subset of either M or N.

Suppose that the vertices of a graph G are colored either white or black. The positive semidefinite color-change rule is the following: If there exists a black vertex v that has exactly one white neighbor u in a connected component of the graph obtained from G by removing all of the black vertices, then change the color of u to black. A *positive semidefinite zero forcing set* for a graph G is a subset of vertices Z such that given a coloring of the vertices of G where all the vertices of Z are black, repeated application of the color-change rule can result in all of the vertices being colored black. The positive semidefinite zero forcing number $Z_+(G)$ is the size of a smallest zero forcing set.

The adjacency graph of a real symmetric positive semidefinite n-by-n matrix H has vertex set $\{1,\dots,n\}$ and edge set $\{ij : i \neq j, a_{ij} \neq 0\}$. Note that the diagonal entries of the matrix do not affect the graph. Given a graph G on n vertices, let $\mathcal{P}(G)$ be the set of positive semidefinite matrices whose graph is isomorphic to G. The *real positive semidefinite minimum rank* of G, $\mathrm{mr}_+(G)$, is the smallest possible rank among matrices in $\mathcal{P}(G)$. Setting $M_+(G) = n - \mathrm{mr}_+(G)$ gives the corresponding *maximum nullity* of G. The connection with zero forcing is that $Z_+(G) \geq M_+(G)$ for all graphs G.

A matrix U is real orthogonal if $UU^T = U^TU = I$, where U^T is the transpose of T and I is the appropriate-size identity matrix. A zero/nonzero pattern is a matrix with entries from $\{0, *\}$. The support of a matrix is the zero/nonzero pattern obtained by replacing all nonzero entries with "$*$". The bipartite graph $B(P)$ of a zero/nonzero pattern P is obtained from P by letting the rows and columns be the vertices and placing an edge between row i and column j if and only if the ij entry of U is $*$.

An n-by-n matrix A is *fully indecomposable* if it does not have a p by q zero submatrix with $p + q = n$. An orthogonal matrix is fully indecomposable if and only if its bipartite graph is connected [1].

A Givens rotation is an orthogonal matrix of the form

$$P \begin{bmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & O^T \\ & O & I \end{bmatrix} P^T$$

for some permutation matrix P, $0 \leq \theta < 2\pi$, and appropriately sized zero matrix O and identity matrix I.

Known Results

We assume all graphs are connected. For a bipartite graph $G = (M \cup N, E)$, $M_+(G) \leq Z_+(G) \min\{|M|, |N|\}$. Given an n-by-n zero/nonzero pattern P, P is the support of a real orthogonal matrix if and only if $M_+(B(P)) = n$ [2]. It was conjectured that P is the support of a real orthogonal matrix if and only if $B(P)$ is strongly quadrangular [5], but examples have been found of patterns that are strongly quadrangular but do not support a real orthogonal matrix [3].

Theorem 23 ([1])**.** *Suppose A is an n-by-n real orthogonal matrix with zero/nonzero pattern P. Let i and j be rows (or columns) of A whose supports do not intersect. There exists a Givens rotation G such that the zero/nonzero pattern of GA (AG) has the support of i and j replaced by the union of the supports of i and j. Further, if A is rational, then G can be chosen to be rational as well.*

Theorem 24 ([6])**.** *A zero/nonzero pattern of degree at most four supports an orthogonal matrix if and only if it is strongly quadrangular.*

Theorem 25 ([4])**.** *There are three strongly quadrangular 5-by-5 patterns that do not support orthogonal matrices:*

$$\begin{bmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ * & * & 0 & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ * & * & 0 & * & * \\ * & * & * & 0 & * \\ * & * & * & * & * \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ * & * & 0 & * & * \\ * & * & * & 0 & * \\ * & * & * & * & 0 \end{bmatrix}$$

Theorem 26 ([6])**.** *Let P (or its transpose) be a zero/nonzero pattern equivalent to a pattern with the following form:*

$$\begin{bmatrix} Q & J_{3\times k} & X \\ Y & Z & * \end{bmatrix}$$

where $k \geq 1$, J is the matrix of all $*$s, and

$$Q = \begin{bmatrix} * & 0 \\ 0 & * \\ * & * \end{bmatrix}$$

Further suppose that the rows of X have disjoint support and every column of Y has support disjoint from every column of Z. Then P does not support an orthogonal matrix. Further, these patterns are strongly quadrangular.

New Results

Throughout, let $G = (V = M \cup N, E)$ be a bipartite graph with $|M| = |N| = n$. Our first result motivates what follows:

Theorem 27. *If $Z_+(G) = n$ then G is strongly quadrangular.*

Since it was originally conjectured that a pattern would support a unitary matrix if and only if it was strongly quadrangular, this result shows that Z_+ gives a stronger criterion than strong quadrangularity. As a result, we wonder whether the following is true:

Conjecture 4. *An m-by-m zero/nonzero pattern P is the support of a real orthogonal matrix if and only if $Z_+(B(P)) = m$.*

If a pattern P does support a unitary matrix, then $Z_+(B(P)) = m$, so a pattern for which $Z_+(B(P)) < m$ automatically satisfies the conjecture.

We showed that the published examples where an m-by-m pattern P is strongly quadrangular but P is not the support of a real orthogonal matrix have $Z_+(B(P)) < m$.

Theorem 28. *Suppose P is a pattern from either Theorem 25 or Theorem 26. Then $Z_+(B(P)) = m$.*

In particular, the conjecture is true for all 5-by-5 or smaller patterns:

Theorem 29. *For $m \leq 5$, an m-by-m zero/nonzero pattern P is the support of a real orthogonal matrix if and only if $Z_+(B(P)) = m$.*

We next considered pattern reduction techniques that can aid in the search for a possible counterexample to the conjecture.

Theorem 30. *If P is a pattern that contains a row (or column) with exactly two nonzero entries, let P' be the pattern obtained by deleting that row (column) and one of the necessarily identical columns (rows) corresponding to the two nonzero entries. Then P supports a unitary if and only if P' does and $Z_+(B(P)) = m$ if and only if $Z_+(B(P')) = m - 1$.*

Define the union of two patterns of the same size to be the pattern with zeros only where both patterns had zeros.

Theorem 31. *Let P' be the pattern obtained from a pattern P by replacing two rows (or two columns) with the union of those rows (columns). If P supports a unitary then P' does and if $Z_+(B(P)) = m$ then $Z_+(B(P')) = m$.*

This work considerably narrowed the computations required to check the conjecture for 6-by-6 and 7-by-7 patterns. For example, in the 6-by-6 case, the reductions provided by the above results leave 147 exceptional patterns with $Z_+(B(P)) = 6$, and each of these patterns will need to be checked in order to determine if they admit a unitary matrix. If each pattern admits such a matrix, then we will have established the conjecture above for 6-by-6 matrices.

Participants

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Bibliography

[1] Gi-Sang Cheon, Suk-Geun Hwang, Seog-Hoon Rim, Bryan L. Shader, and Seok-Zun Song. Sparse orthogonal matrices. *Linear Algebra Appl.*, 373:211–222, 2003. Special issue on the Combinatorial Matrix Theory Conference (Pohang, 2002).

[2] Yunjiang Jiang, Lon H. Mitchell, and Sivaram K. Narayan. Unitary matrix digraphs and minimum semidefinite rank. *Linear Algebra Appl.*, 428(7):1685–1695, 2008.

[3] J. Richard Lundgren, K. B. Reid, Simone Severini, and Dustin J. Stewart. Quadrangularity and strong quadrangularity in tournaments. *Australas. J. Combin.*, 34:247–260, 2006.

[4] Dragomir Ž. Doković, Simone Severini, and Ferenc Szöllosi. Rational orthogonal versus real orthogonal. *Electron. J. Linear Algebra*, 18:649–673, 2009.

[5] Simone Severini. On the digraph of a unitary matrix. *SIAM J. Matrix Anal. Appl.*, 25(1):295–300 (electronic), 2003.

[6] Simone Severini and Ferenc Sz"ollosi. A further look into combinatorial orthogonality. *Electron. J. Linear Algebra*, 17:376–388, 2008.

Chapter 65

Sarason Conjecture and the Composition of Paraproducts (12rit186)

November 4 - 11, 2012

Organizer(s): S. Pott (Lund University), M. Reguera (Universitat Autonoma of Barcelona), E. T. Sawyer (McMaster University), and B. D. Wick (Georgia Institute of Technology)

Overview of the Research in Teams Project

The project that we worked on during the Research in Teams at Banff was a dyadic version of the Sarason Conjecture. This discrete problem is already very challenging and captures much of the difficulty associated with the conjecture. In particular, we were concerned with dyadic Haar paraproducts, and obtaining necessary and sufficient conditions for the boundedness of the composition of two such paraproducts. The conditions characterizing the boundedness of the composition will be much more general than the boundedness of each individual paraproduct.

Let $\mathcal{D}$ be the dyadic intervals and let $\{h_I \mid I \in \mathcal{D}\}$ be the $L^2(\mathbb{R})$ normalized Haar basis. Set $h_I^0 := h_I$, with the superscript 0 for the mean zero, and set $h_I^1 := |h_I|$, with the superscript 1 for the non-zero mean. Given a function b, and choices $\epsilon, \delta \in \{0,1\}$ define the paraproduct

$$f \in L^2(\mathbb{R}) \mapsto \mathrm{P}_b^{\epsilon,\delta} f(x) := \sum_{I \in \mathcal{D}} \frac{\langle b, h_I \rangle_{L^2}}{\sqrt{|I|}} \left\langle h_I^\delta, f \right\rangle_{L^2} h_I^\epsilon(x).$$

These discrete paraproduct operators are fundamental in harmonic analysis since they serve as dyadic examples of Calderón-Zygmund operators. The boundedness of $\mathrm{P}_b^{\epsilon,\delta}$ is well-known and the result depends on the choice of ϵ, δ and the symbol b. We were interested in the question of the boundedness of the composition

$$\mathrm{P}_b^{\epsilon,\delta}\,\mathrm{P}_d^{\epsilon',\delta'} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

when the individual paraproducts are *not* necessarily bounded. In particular, we wanted to obtain a characterization of the boundedness of the above composition for all possible choices of $\epsilon, \epsilon', \delta, \delta' \in \{0,1\}$. Additionally, there is a formulation of the question that removes the function b and d, replacing them with sequence $\{b_I\}_{I\in\mathcal{D}}$ and $\{d\}_{I\in\mathcal{D}}$ and seeks a characterization in terms of sequence information on $\{b_I\}$ and $\{d_I\}$.

Scientific Progress Made

During the workshop we were able to make substantial progress on characterizing the various paraproduct compositions that exist.

The Simple Compositions

We now note that the composition $\mathsf{P}_b^{(\alpha,0)} \circ \mathsf{P}_d^{(0,\beta)}$ coincides with the paraproduct $\mathsf{P}_{b\circ d}^{(\alpha,\beta)}$ where $b \circ d$ is the Schur product

$$b \circ d \equiv \{b_I d_I\}_{I\in\mathcal{D}} \ .$$

Thus the boundedness of a composition operator of the form $\mathsf{P}_b^{(\alpha,0)} \circ \mathsf{P}_d^{(0,\beta)}$ reduces to that of a single paraproduct $\mathsf{P}_{b\circ d}^{(\alpha,\beta)}$. There are three cases in which a single paraproduct is easily characterized, namely $\mathsf{P}_a^{(0,0)}$, $\mathsf{P}_a^{(0,1)}$ and $\mathsf{P}_a^{(1,0)}$. Define

$$\begin{aligned}
\|a\|_{\ell^\infty} &\equiv \sup_{I\in\mathcal{D}} |a_I| \, ; \\
\|a\|_{CM} &\equiv \sqrt{\sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{J\subset I} a_J^2} \, .
\end{aligned}$$

For these simple paraproducts we have the characterizations

$$\begin{aligned}
\left\| \mathsf{P}_a^{(0,0)} \right\|_{L^2\to L^2} &= \|a\|_{\ell^\infty} \, ; \\
\left\| \mathsf{P}_a^{(0,1)} \right\|_{L^2\to L^2} &= \left\| \mathsf{P}_a^{(1,0)} \right\|_{L^2\to L^2} \approx \|a\|_{CM} \ .
\end{aligned}$$

The next most difficult one is a paraproduct of type $(1,1)$. Using existing results from the literature, we were able to show that the single paraproduct $\mathsf{P}_a^{(1,1)}$ reduces to two types already characterized plus a diagonal operator $D_{(b,d)}$ that is easily characterized. Indeed, we have the norm of the bilinear form

$$\left\langle \mathsf{P}_a^{(1,1)} f, g \right\rangle = \sum_I a_I \left\langle f, A_I \right\rangle \left\langle g, A_I \right\rangle$$

on $L^2 \times L^2$ is comparable to

$$\left\| \sum_{I\in\mathcal{D}} a_I A_I \right\|_{BMO} + \sup_{I\in\mathcal{D}} \frac{1}{|I|} \left| \sum_{J\subset I} a_J \right| .$$

The Difficult Compositions

These cases presented a challenge because of the lack of cancellation present in the composition of two different paraproducts. The main idea we exploited was to rephrase the boundedness of the composition, as the boundedness of a related operator, but on a *different* Hilbert space. This translation then allowed the problem to be recast as a certain two weight inequality for an operator that could then be studied via the techniques of weighted theory from harmonic analysis.

The Hilbert space we found useful for studying the composition was the dyadic Bergman space

$$\mathcal{B} \equiv \left\{ f \in L^2(\mathcal{D}) : \sum_{I\in\mathcal{D}} |f(I)|^2 |I|^2 < \infty \right\},$$

with norm $\|f\|_{\mathcal{B}} = \sqrt{\sum_{I\in\mathcal{D}} |f(I)|^2 |I|^2}$. This can also be viewed as a weighted L^2 space on the tree $\mathcal{D}$.

Composing a Paraproduct of type $(0,1)$ with a Paraproduct of the $(1,0)$

Using the idea of recasting the problem into a different Hilbert space, we have shown that

$$\left\|\mathsf{P}_b^{(0,1)}\circ\mathsf{P}_d^{(1,0)}\right\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\approx\left\|\mathsf{T}_{b,d}^{(0,1,1,0)}\right\|_{\mathcal{B}\to\mathcal{B}}+\left\|\mathsf{T}_{d,b}^{(0,1,1,0)}\right\|_{\mathcal{B}\to\mathcal{B}}$$

with the operator $\mathsf{T}_{b,d}^{(0,1,1,0)}$ is an explicit operator acting on $\mathcal{B}$. Furthermore, the operator norm $\left\|\mathsf{T}_{b,d}^{(0,1,1,0)}\right\|_{\mathcal{B}\to\mathcal{B}}$ equals the best constant in a certain two weight inequality for the operator Q on $\mathcal{B}$, where

$$\mathsf{Q}\equiv\sum_{K\in\mathcal{D}}\widetilde{\mathbf{1}}_{T(K)}\otimes\widetilde{\mathbf{1}}_{Q(K)}.$$

Indeed, the inequality is

$$\mathsf{Q}\left(\widetilde{d^2}\cdot\right):L^2\left(\widetilde{d^2}\right)\to L^2\left(b^2\right). \tag{65.0.0.1}$$

This best constant in this inequality is in turn comparable to the best constants in the associated testing conditions. These testing conditions thus give a characterization of the boundedness of the paraproduct composition $\mathsf{P}_b^{(0,1)}\circ\mathsf{P}_d^{(1,0)}$ on $L^2(\mathbb{R})$. Indeed, the two weight norm inequality (65.0.0.1) holds if and only if the following testing condition holds:

$$\begin{aligned}\left\|\mathbf{1}_{Q_I}\mathsf{Q}\left(\sum_{J\in\mathcal{D}}\widetilde{d^2}\mathbf{1}_{Q_I}\right)\right\|_{L^2(b^2)}^2 &\lesssim \left\|\mathbf{1}_{Q_I}\right\|_{L^2\left(\widetilde{d^2}\right)}^2;\\ \text{i.e. }\sum_{J\subset I}b_J^2\frac{1}{|J|^2}\left(\sum_{L\subset J}d_L^2\right)^2 &\lesssim \sum_{L\subset J}d_L^2\ .\end{aligned}$$

Thus combining this the above discussion with the corresponding condition with b and d interchanged, we find that the composition $\mathsf{P}_b^{(0,1)}\circ\mathsf{P}_d^{(1,0)}$ is bounded on $L^2(\mathbb{R})$ if and only if both

$$\begin{aligned}\sum_{J\subset I}b_J^2\frac{1}{|J|^2}\left(\sum_{L\subset J}d_L^2\right)^2 &\lesssim \sum_{L\subset J}d_L^2;\\ \sum_{J\subset I}d_J^2\frac{1}{|J|^2}\left(\sum_{L\subset J}b_L^2\right)^2 &\lesssim \sum_{L\subset J}b_L^2,\end{aligned}$$

for all $I\in\mathcal{D}$.

Composing a Paraproduct of type $(0,1)$ with a Paraproduct of the $(0,0)$

Again, recasting the problem as an equivalent question on a discrete Bergman space, we have

$$\left\|\mathsf{P}_b^{(0,1)}\circ\mathsf{P}_d^{(0,0)}\right\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\approx\left\|\mathsf{T}_{b,d}^{(0,1,0,0)}\right\|_{\mathcal{B}\to\mathcal{B}}\ .$$

Now the operator norm $\left\|\mathsf{T}_{b,d}^{(0,1,0,0)}\right\|_{\mathcal{B}\to\mathcal{B}}$ equals the best constant in a certain two weight inequality for the operator $\widetilde{\mathsf{Q}}$ on $\mathcal{B}$ defined by

$$\widetilde{\mathsf{Q}}\equiv\sum_{K\in\mathcal{D}}\widetilde{\mathbf{1}}_{Q_\pm(K)}\otimes\widetilde{\mathbf{1}}_{T(K)}.$$

This operator is not positive, but its singular character is well-behaved, and the best constant in the two weight inequality is in turn comparable to the best constants in the associated testing conditions. These testing conditions thus give a characterization of the boundedness of the paraproduct composition $\mathsf{P}_b^{(0,1)}\circ\mathsf{P}_d^{(0,0)}$ on $L^2(\mathbb{R})$.

Using ideas for well-localized dyadic singular integral type operators we were able to show that

$$\widetilde{\mathsf{Q}}_{\mu} : L^2(\mu) \to L^2(\nu)$$

if and only if both

$$\begin{aligned}\left\| \widetilde{\mathsf{Q}}_{\mu}\left(\mathbf{1}_{T(I)}\right) \right\|_{L^2(\nu)} &\lesssim \left\| \mathbf{1}_{T(I)} \right\|_{L^2(\mu)} = \sqrt{\mu\left(T\left(I\right)\right)}, \\ \left\| \widetilde{\mathsf{Q}}_{\nu}^{*}\left(\mathbf{1}_{Q(I)}\right) \right\|_{L^2(\mu)} &\lesssim \left\| \mathbf{1}_{Q(I)} \right\|_{L^2(\nu)} = \sqrt{\nu\left(Q\left(I\right)\right)},\end{aligned}$$

hold for all $I \in \mathcal{D}$. The weights μ and ν are given by explicit formulas in terms of the sequences b and d. These conditions can in turn be rephrased back in terms of testing conditions on the paraproducts in question.

The Remaining Cases

We still are working to finish up the remaining cases of the compositions that remain.

Outcome of the Research in Teams

We are currently writing up the results from the research in teams. We are optimistic that we will be able to resolve the final cases that remain.

Participants

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Summer Schools Reports

Chapter 66

Contemporary methods for solving Diophantine equations (12ss131)

June 10 - 17, 2012

Organizer(s): Michael Bennett (University of British Columbia), Nils Bruin (Simon Fraser University), Yann Bugeaud (Université de Strasbourg), Bjorn Poonen (Massachusetts Institute of Technology), Samir Siksek (University of Warwick)

Overview of the Field

The topic of this summer school was Diophantine equations, which are among the oldest studied mathematical objects. A *Diophantine equation* is an equation where admissible solutions are restricted to the rationals or the integers, or appropriate mathematical generalizations of such objects. The equations themselves tend to be polynomial, exponential, or a mixture of both, where variables in the exponents are usually restricted to (positive) integers. A characteristic example is the equation that is central to Fermat's Last Theorem,

$$x^n + y^n = z^n, \text{ with } x, y, z \in \mathbb{Z} \text{ and } n \in \{3, 4, \ldots\}.$$

Because Diophantine equations concern themselves with objects so fundamental to mathematics, they tend to arise whenever one uses the mathematical language to formulate problems or theories. This supplies a dual motivation to the field.

On the one hand, there is an interest to understand theoretically the set of solutions to the equations and its relationship to the geometric objects defined by the equations.

On the other hand, there is a demand for practical methods that, given an explicit equation, provide a complete and explicit description of the set of solutions. In recent years, a combination of development of general theory, computational tools and computational techniques has greatly improved our ability to explicitly solve Diophantine equations. Some of the methods that have proved to be particularly relevant recently are the following.

- *The modular method (see Section 66).* The application of the ideas of Frey, Ribet, Wiles, and others, that led to the proof of Fermat's last theorem. The method only applies to a limited class of equations, but it is currently the only method that is able to handle statements about *rational* solutions to families of equations with varying exponents.

- *Linear Forms in Logarithms (See Section 66).* This method applies to a wide class of equations for which one wants to determine *integral* solutions. It provides explicit versions of results along the lines of Roth's theorem, which has been used to prove finiteness of integral points on affine hyperbolic curves.

- *Hypergeometric method (See Section 66).* This method applies to a subset of problems where linear forms in logarithms apply. When it does apply, it usually yields sharp results.

- *Cohomological obstructions (See Section 66).* One of the oldest and simplest methods to show that a Diophantine equation has no solutions is by showing that there is some *local obstruction* to having solutions. Equations that have no local obstructions to having solutions are said to have *solutions everywhere locally*. It is well known that local obstructions are not the only obstructions to having solutions. Various other obstructions have been identified, many of which were eventually shown to be part of the *Brauer-Manin obstruction*. This obstruction is, at least in theory, largely computable and is most conveniently formulated in the language of étale cohomology. The closely related *descent obstruction* is also most conveniently studied in a cohomological setting and translates the original question into one about solutions to finitely many other equations, which may have local obstructions even if the original equation does not.

- *Mordell-Weil Sieving and Chabauty's method (See Section 66).* Many of the cohomological ideas lead in principle to computable criteria, but not necessarily in a practical way. At least for determining the rational points on curves, two methods have proven to be particularly practical and successful in recent years. Both methods rely on embedding the curve into an *abelian variety* and obtaining quite detailed information on the rational points on the latter. The first method then uses combinatorial arguments to arrive at detailed information on where rational points can lie. Under standard conjectures, one can show that the obstructions derived from this are also part of the Brauer-Manin obstruction, but the method is does not need to be formulated in that language and is much easier to use in practice.

  The second method uses p-adic analysis to prove that rational points need to be *isolated* in some sense. The method does not always apply, but if it does, it complements the first method almost perfectly and often allows the explicit determination of the rational points on curves.

Objective and format of the summer school

The objective of the workshop was to give the participants (primarily graduate students, ranging from starting master's students to finishing doctoral students) a working knowledge of current methods of solving Diophantine equations. Of course, one week is too short to expect that all participants become experts in all of the methods. So our goal was instead to bring them to the point where they would be aware of the general ideas, the range of application, and references for learning more, for each method.

We chose five main relevant methods and arranged for lecture series covering each method. We distributed these lecture over the days, such that every day started with 3 lectures.

In order for the participants to properly internalize the methods, we thought it would be important for them to see the methods in action. For that reason we arranged a rich set of assignments accompanying each lecture and ample time for the participants to work on these problems in groups. We closed every day with presentations of solutions to the problems, presented by participants. Not everybody would work on all problems, but at the end of the day they at least would see a solution presented by one of their peers, thus confirming to them that these problems are indeed doable.

Since this was a summer school, not a research meeting, we thought we would collect the best group of participants not by constructing our own invitation list but by having an open application and selection process. We sent out an announcement and collected letters of interest from candidates plus letters of recommendation from supervisors. We received more applications than we had places, so we did have to make a selection.

The timing of the summer school was kindly coordinated by BIRS to be adjacent to the twelfth meeting of the Canadian Number Theory Association in nearby Lethbridge. The CNTA conferences are important international meetings and is one of the most prominent number theory conferences in North America. Therefore, the opportunity for the participants to combine this relatively specialized summer school with a large number theory conference that allowed them to place the material just learned in the larger context of current number theoretic research was an exceptional opportunity.

Outcome of the meeting

We were extremely happy with the selection of participants. They formed an exceptionally motivated and talented group. We had no problem getting volunteers to present solutions and the participants worked very hard and successfully on the set problems.

The participants have provided feedback on the workshop, see

`http://www.birs.ca/events/2012/summer-schools/12ss131/testimonials`.

Many report not only on the relevance of the material they learned, but also on the contacts they made with other young researchers.

Another noteworthy resource was the availability of online recording of all lectures. One participant was unable to attend the first few days of the workshop due to outside circumstances. However, he was able to download recordings of all the lectures he missed and to watch them on the way to BIRS. By the time he arrived, he was fully up to date.

The Modular Method

The modular approach is a method for associating Galois representations having very little ramification to (non-trivial) solutions of certain Diophantine equations via Frey curves. Occassionaly, the method proves the non-existence of such solutions—this was the case with Wiles' proof of Fermat's Last Theorem. More commonly, it provides a vast amount of local data about the solutions; to completely solve the equations it is often necessary to combine this data with global information obtained via other methods such as linear forms in logarithms.

The approach rests of three major theoretical pillars. Theses are the Modularity Theorem (due to Wiles and others), the Level-Lowering Theorem (Ribet) and Mazur's criteria for the irreducibility of Galois representations (introduced here as statements about the non-existence of isogenies).

Modularity

Theorem 66.0.1. *(The Modularity Theorem for Elliptic Curves) Associated to any* **rational** *newform* $f = q+\sum_{n\geq 2} c_n q^n$ *of level* N *and weight* 2 *is an elliptic curve* $E_f/\mathbb{Q}$ *of conductor* N *so that for all primes* $l \nmid N$

$$c_l = a_l(E_f)$$

where c_l *is the* l*-th coefficient in the* q*-expansion of* f *and* $a_l(E_f) = l+1-\#E_f(\mathfrak{F}_l)$*. For any given positive integer* N*, the association* $f \mapsto E_f$ *is a bijection between rational newforms of level* N *and isogeny classes of elliptic curves of conductor* N*.*

The association $f \mapsto E_f$ is due to Shimura. The fact that this association is surjective was previously known as the Modularity Conjecture, and first proved for squarefree N (the semi-stable case) by Wiles [25], [24]. The proof was completed in a series of papers by Diamond [9], Conrad, Diamond and Taylor [7], and finally Breuil, Conrad, Diamond and Taylor [4].

Level-Lowering

'Arises From'

Definition 1. *Let* E *be an elliptic curve over the rationals of conductor* N*, and suppose that* f *is a newform of weight 2 and level* N' *with* q*-expansion* $f = q+\sum_{i\geq 2} c_i q^i$*, and coefficients* c_i *generating the number field* $K/\mathbb{Q}$*. We shall say that the curve* E *arises modulo* p *from the newform* f *(and write* $E \sim_p f$*) if there is some prime ideal* $\mathfrak{P} \mid p$ *of* K *such that for almost all primes* l*, we have* $a_l(E) \equiv c_l \pmod{\mathfrak{P}}$*.*

In fact we can be a little more precise.

Proposition 66.0.2. *Suppose* $E \sim_p f$*. Then there is some prime ideal* $\mathfrak{P} \mid p$ *of* K *such that for all primes* l

(i) if $l \nmid pNN'$ *then* $a_l(E) \equiv c_l \pmod{\mathfrak{P}}$*, and*

(ii) if $l \nmid pN'$ *and* $l \mid\mid N$ *then* $l+1 \equiv \pm c_l \pmod{\mathfrak{P}}$.

If f is a rational newform, then we know that f corresponds to some elliptic curve F say (this is E_f in the notation of Theorem 66.0.1). If E arises modulo p from f then we shall also say that E arises modulo p from F (and write $E \sim_p F$).

Proposition 66.0.3. *(Kraus and Oesterlé [14]) Suppose that* E, F *are elliptic curves over* $\mathbb{Q}$ *with conductors* N *and* N' *respectively. Suppose that* E *arises modulo* p *from* F*. Then for all primes* l

(i) if $l \nmid NN'$ *then* $a_l(E) \equiv a_l(F) \pmod{p}$*, and*

(ii) if $l \nmid N'$ *and* $l \mid\mid N$ *then* $l+1 \equiv \pm a_l(F) \pmod{p}$.

Ribet's Level-Lowering Theorem

Let E be an elliptic curve over $\mathbb{Q}$. Let $\Delta = \Delta_{\min}$ be the discriminant for a minimal model of E, and N be the conductor of E. Suppose p is a prime, and let

$$N_p = N \Bigg/ \prod_{\substack{q \mid\mid N, \\ p \mid \mathrm{ord}_q(\Delta)}} q. \tag{66.0.0.1}$$

Theorem 66.0.4. *(A simplified special case of Ribet's Level-Lowering Theorem) Suppose* E *is an elliptic curve over* $\mathbb{Q}$ *and* $p \geq 5$ *is prime. Suppose further that* E *does not have any* p*-isogenies. Let* N_p *be as defined above. Then there exists a newform* f *of level* N_p *such that* $E \sim_p f$.

Absence of Isogenies

To be able to apply Ribet's Theorem we must know that our elliptic curve E does not have a p-isogeny.

Theorem 66.0.5. *(Mazur [18]) Suppose* $E/\mathbb{Q}$ *is an elliptic curve and that* **at least one** *of the following conditions holds.*

- $p \geq 17$ *and* $j(E) \notin \mathbb{Z}[\frac{1}{2}]$,
- *or* $p \geq 11$ *and* E *is a semi-stable elliptic curve,*
- *or* $p \geq 5$, $\#E(\mathbb{Q})[2] = 4$*, and* E *is a semi-stable elliptic curve,*

Then E *does not have any* p*-isogenies.*

Fermat's Last Theorem

Theorem 66.0.6. *(Wiles) Suppose* $p \geq 5$ *is prime. The equation*

$$x^p + y^p + z^p = 0 \tag{66.0.0.2}$$

has no solutions with $xyz \neq 0$.

Proof. Suppose $xyz \neq 0$. Without loss of generality: x, y, z are coprime, and

$$2 \mid y, \qquad x^p \equiv -1 \pmod{4}, \qquad z^p \equiv 1 \pmod{4}.$$

Associate to this solution the elliptic curve (called a Frey curve)

$$E: \quad Y^2 = X(X - x^p)(X + y^p).$$

So

$$\Delta = 16x^{2p}y^{2p}(x^p + y^p)^2 = 16x^{2p}y^{2p}z^{2p}$$

using $x^p + y^p + z^p = 0$. Also

$$c_4 = 16(z^{2p} - x^p y^p), \qquad \gcd(c_4, \Delta) = 16.$$

Applying Tate's algorithm to compute the minimal discriminant and conductor:

$$\Delta_{\min} = 2^{-8}(xyz)^{2p}, \qquad N = \prod_{\ell | xyz} \ell.$$

Moreover, $N_2 = 2$. Since $E(\mathbb{Q})[2] = 4$ and N squarefree, we know by Mazur's Theorem that E has no p-isogenies.

By Ribet, there is a newform f of level $N_p = 2$ and weight 2 such that $E \sim_p f$. However, there are no newforms of level 2 and weight 2. This gives a contradiction. $\square$

The bibliography below provides some references for further study.

Bibliography

[1] M. A. Bennett and C. M. Skinner, *Ternary Diophantine equations via Galois representations and modular forms*, Canad. J. Math. **56** (2004), no. 1, 23–54.

[2] M. A. Bennett, V. Vatsal and S. Yazdani, *Ternary Diophantine Equations of Signature* $(p,p,3)$, Compositio Mathematica **140** (2004), 1399-1416.

[3] M. A. Bennett, *Recipes for ternary Diophantine equations of signature* (p,p,k), Proc. RIMS Kokyuroku (Kyoto) **1319** (2003), 51–55

[4] C. Breuil, B. Conrad, F. Diamond and R. Taylor, *On the modularity of elliptic curves over* $\mathbb{Q}$*: wild* 3*-adic exercises*, J. Amer. Math. Soc. **14 No.4** (2001), 843–939.

[5] Y. Bugeaud, M. Mignotte and S. Siksek, *Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers*, Annals of Mathematics **163** (2006), 969–1018.

[6] Y. Bugeaud, M. Mignotte and S. Siksek, *Classical and modular approaches to exponential Diophantine equations II. The Lebesgue–Nagell Equation*, Compositio Mathematica **142** (2006), 31–62.

[7] B. Conrad, F. Diamond, R. Taylor, *Modularity of certain potentially Barsotti-Tate Galois representations*, J. Amer. Math. Soc. **12** (1999), no. 2, 521–567.

[8] H. Darmon and L. Merel, *Winding quotients and some variants of Fermat's Last Theorem*, J. reine angew. Math. **490** (1997), 81–100.

[9] F. Diamond, *On deformation rings and Hecke rings*, Ann. of Math. **144** (1996), no. 1, 137–166.

[10] E. Halberstadt and A. Kraus, *Sur les modules de torsion des courbes elliptiques*, Math. Ann. **310** (1998), 47–54.

[11] E. Halberstadt and A. Kraus, *Courbes de Fermat: résultats et problèmes*, J. reine angew. Math. **548** (2002), 167âĂŤ-234.

[12] W. Ivorra, *Sur les équations* $x^p + 2^\beta y^p = z^2$ *et* $x^p + 2^\beta y^p = 2z^2$, Acta Arith. **108** (2003), 327–338.

[13] W. Ivorra and I. Kraus, *Quelques résultats sur les équations* $ax^p + by^p = cz^2$, Canad. J. Math. **58** (2006), 115–153.

[14] A. Kraus and J. Oesterlé, *Sur une question de B. Mazur*, Math. Ann. **293** (1992), 259–275.

[15] A. Kraus, *Majorations effectives pour l'équation de Fermat généralisée*, Can. J. Math. **49** (1997), 1139–1161.

[16] A. Kraus, *Sur l'équation* $a^3 + b^3 = c^p$, Experimental Mathematics **7** (1998), No. 1, 1–13.

[17] A. Kraus, *On the Equation* $x^p + y^q = z^r$*: A Survey*, The Ramanujan Journal **3** (1999), 315–333.

[18] B. Mazur, *Rational isogenies of prime degree*, Invent. Math. **44** (1978), 129–162.

[19] K. Ribet, *On modular representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *arising from modular forms*, Invent. Math. **100** (1990), 431–476.

[20] K. Ribet, *On the equation* $a^p + 2b^p + c^p = 0$, Acta Arith. **LXXIX.1** (1997), 7–15.

[21] J.-P. Serre, *Abelian* l*-adic representations and elliptic curves*, W. A. Benjamin, New York, 1968.

[22] J.-P. Serre, *Sur les représentations modulaires de degré 2 de* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Duke Math. J. **54** (1987), no. 1, 179–230.

[23] S. Siksek and J. E. Cremona, *On the Diophantine equation* $x^2 + 7 = y^m$, Acta Arith. **109.2** (2003), 143–149.

[24] R. L. Taylor and A. Wiles, *Ring theoretic properties of certain Hecke algebras*, Annals of Math. **141** (1995), 553–572.

[25] A. Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Annals of Math. **141** (1995), 443–551.

Linear forms in logarithms and applications

A Diophantine equation (E) being given, our aim is to determine the complete list of its solutions, say, in rational integers. The first question is to decide whether (E) has only finitely many solutions or infinitely many. In the former situation, a refined information would be a bound for the total number N of solutions. Note that such a result cannot always be obtained: for instance, if one is able to show that the largest solution of (E) has no more than twice as many decimal digits as the smallest one, this gives no information on N. Furthermore, the knowledge of N is in general far from being sufficient for solving completely (E), since it is unlikely that (E) has exactly N solutions. In order to determine all the solutions to (E), we thus need an explicit upper bound B for the size (the absolute value) of the largest one. Then, at least in principle, it is possible to complete the resolution of (E) by simply checking which integers between $-B$ and B are solutions.

The first family of Diophantine equations on which a general result has been proved are the Thue equations, named after the Norwegian mathematician Axel Thue, who established in 1909 the following result.

Theorem 32. *Let $F(X,Y)$ be an irreducible, homogeneous, integer polynomial of degree at least 3. Let b be a non-zero integer. Then, the equation*

$$F(x,y) = b \tag{T}$$

has only finitely many solutions in integers x and y.

Unfortunately, the method developed by Thue does not enable him to explicitly bound from above the absolute values of the solutions x and y to (T).

Such a result was obtained more than half a century after the publication of Thue's paper, by means of the theory of linear forms in the logarithms of algebraic numbers developed by Alan Baker at the end of the 60's and which can be presented as follows.

Let $n \ge 2$ be an integer. For $1 \le i \le n$, let x_i/y_i be a non-zero rational number and b_i a positive integer. Set

$$B := \max\{3, b_1, \dots, b_n\}$$

and, for $1 \le i \le n$, set

$$A_i := \max\{3, |x_i|, |y_i|\}.$$

We assume that the rational number

$$\Lambda := \left(\frac{x_1}{y_1}\right)^{b_1} \cdots \left(\frac{x_n}{y_n}\right)^{b_n} - 1 \tag{1}$$

is non-zero. We wish to bound $|\Lambda|$ from below, thus we may assume that $|\Lambda| \le 1/2$ and we get the *linear form in logarithms*

$$|\Lambda| \ge \frac{|\log(1+\Lambda)|}{2} = \frac{1}{2}\left|b_1 \log \frac{x_1}{y_1} + \cdots + b_n \log \frac{x_n}{y_n}\right|.$$

A trivial estimate of the denominator of (1) gives

$$\log|\Lambda| \geq -\sum_{i=1}^{n} b_i \log|y_i| \geq -B\sum_{i=1}^{n} \log A_i.$$

The dependence on the A_i's is very satisfactory, unlike the dependence on B. However, for applications to Diophantine problems, a better estimate in terms of B is needed, even if it comes with a weaker dependence in terms of the A_i's.

Alan Baker [1, 2] was the first to prove such a result, and we are now able to show that, under the above assumptions, there exists an effectively computable constant $c(n)$, depending only on the number n of rational numbers involved, such that the lower estimate

$$\log|\Lambda| \geq -c(n)\ \log A_1 \dots \log A_n \log B$$

holds.

More generally, one can get analogous lower bounds if the rational numbers x_i/y_i are replaced by algebraic numbers α_i, the quantity $\log A_i$ being then essentially the absolute logarithmic height of α_i. Further information, including precise estimates and detailled proofs, can be found in the textbook of Waldschmidt [5].

Quantities like (1) occur naturally when one studies certain families of Diophantine equations, like the Thue equations or the superelliptic and hyperelliptic equations $f(x) = y^m$, where $f(X)$ is an integer polynomial and $m \geq 2$ is a fixed integer. Baker's theory can then be applied to get upper bounds for the size of the solutions to these equations (under some necessary assumptions: one must e.g. exclude the Pell equations like $x^2 - dy^2 = 1$). It also applies to the more general equation $f(x) = y^z$ in the three unknowns x, y, z (again, under some necessary assumptions).

The most striking application of Baker's theory is Tijdeman's theorem on Catalan's problem, which was posed in 1844 and is the following: do there exist consecutive positive integers other than 8 and 9 which are both pure powers? This corresponds to the exponential Diophantine equation

$$x^m - y^n = 1,$$

which has been solved completely only in 2002, by Mihăilescu. In 1976, Tijdeman used Baker's theory to prove the finiteness of the number of pairs of consecutive integers which are both perfect powers.

Theorem 33. *Let x, y, $m \geq 2$, and $n \geq 2$ be strictly positive integers such that $x^m - y^n = 1$. There exists an effectively computable, absolute constant C such that $\max\{x, y, m, n\} < C$.*

We direct the reader to the monographs [3, 4] for many applications of Baker's theory to Diophantine equations.

Bibliography

[1] A. Baker, *Linear forms in the logarithms of algebraic numbers*, Mathematika 12 (1966), 204–216.

[2] A. Baker, *A sharpening of the bounds for linear forms in logarithms I–III*, Acta Arith. 21 (1972), 117–129; 24 (1973), 33–36; 27 (1975), 247–252.

[3] T. N. Shorey and R. Tijdeman, *Exponential Diophantine equations*, Cambridge Tracts in Mathematics 87, Cambridge University Press, Cambridge, 1986.

[4] V. G. Sprindžuk, Classical Diophantine Equations, Lecture Notes in Math. 1559, Springer-Verlag, Berlin, 1993.

[5] M. Waldschmidt, Diophantine Approximation on Linear Algebraic Groups. Transcendence Properties of the Exponential Function in Several Variables, Grundlehren Math. Wiss. 326, Springer, Berlin, 2000.

The hypergeometric method

Bennett's lecture was an overview of the most classical version of the hypergeometric method and its applications to Diophantine equations. In this context, this method behaves in a similar fashion to lower bounds for linear forms in two logarithms (archimedean or otherwise), with the (substantial) drawback that it is not generally applicable, but with the advantage that, when it can be applied, the results are typically extremely sharp. Combined with gap principles (usually of an elementary nature), this technique can often be applied to bound the number of rational of integral points on certain specific curves or surfaces.

Rational approximation

While the hypergeometric method in general relates to approximation to special values of hypergeometric series, typically via special values of Padé approximations, in its most classical sense (at least from the viewpoint of Diophantine equations), it concerns rational approximation to irrational numbers. Bennett discussed the case of rational approximations to π and to algebraic number of the form $(a/b)^{m/n}$, via specialization of Padé approximants (in the latter case, to the binomial function $(1-z)^{m/n}$). By extending these ideas to working over imaginary quadratic fields, one is able to give an alternative proof of an old theorem of Ljunggren, to the effect that the positive integral solutions to the Diophantine equation $x^2 - 2y^4 = -1$ are given by $(x, y) = (1, 1)$ and $(239, 13)$. This apparently unmotivated equation arises in a surprising number of places, including in Machin's formula for computing digits of π.

Some details

The underlying principle of the hypergeometric method, as it applies to rational approximation, is the following.

Suppose that we are given a real number θ that we wish to prove to be irrational. One way to do this is to find a sequence of distinct rational approximations p_n/q_n to θ (here, p_n and q_n are integers) with the property that there exist positive real numbers α, β, a and b with $\alpha, \beta > 1$,

$$|q_n| < a \cdot \alpha^n, \quad \text{and} \quad |q_n\theta - p_n| < b \cdot \beta^{-n},$$

for all $n \in \mathbb{N}$. If we can construct such a sequence, we in fact get rather more. Namely, we obtain the inequality

$$\left|\theta - \frac{p}{q}\right| > \left(2\,a\,\alpha\,(2\,b\,\beta)^{\lambda}\right)^{-1}|q|^{-1-\lambda} \quad \text{for} \quad \lambda = \frac{\log\alpha}{\log\beta}, \tag{66.0.0.3}$$

valid for *all* integers p and $q \neq 0$ (at least provided $|q| > 1/2b$). To see this, note that

$$\left|\theta - \frac{p}{q}\right| \geq \left|\frac{p_n}{q_n} - \frac{p}{q}\right| - \left|\theta - \frac{p_n}{q_n}\right|,$$

and hence if $p/q \neq p_n/q_n$, we have

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{|q_n|}\left(\frac{1}{|q|} - \frac{b}{\beta^n}\right).$$

Now we choose n minimal such that $\beta^n \geq 2\,b\,|q|$ (since we assume $|q| > 1/2b$, n is a positive integer). Then

$$\beta^{n-1} < 2\,b\,|q| \leq \beta^n$$

and so

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{2\,|q\,q_n|} > \frac{1}{2\,|q|\,a\,\alpha^n} = \frac{1}{2\,|q|\,a\,\beta^{\lambda n}} < \frac{1}{2\,|q|\,a\,(2|q|\,b\,\beta)^{\lambda}}.$$

If instead we have $p/q = p_n/q_n$ for our desired choice of n, we argue similarly, only with n replaced by $n+1$ (whereby the fact that our approximations are distinct guarantees that $p/q \neq p_{n+1}/q_{n+1}$). The slightly weaker constant in (66.0.0.3) results from this case.

An inequality of the shape

$$\left|\theta - \frac{p}{q}\right| > |q|^{-\kappa},$$

valid for suitably large integers p and q is termed an *irrationality measure*. For real transcendental θ, any such measure is in some sense nontrivial. For algebraic θ, say of degree n, however, Liouville's theorem provides a "trivial" lower bound of n for κ.

Padé approximants to $(1-z)^{1/m}$

Given a formal power series $f(z)$ and positive integers r and s, it is an exercise in linear algebra to deduce, for fixed integer n, the existence of nonzero polynomials $P_{r,s}(z)$ and $Q_{r,s}(z)$ with rational integer coefficients and degrees r and s, respectively, such that

$$P_{r,s}(z) - f(z)\,Q_{r,s}(z) = z^{r+s+1}E_{r,s}(z)$$

where $E_{r,s}(z)$ is a power series in z. In certain situations, these *Padé approximants* (which are unique up to scaling) can be written down in explicit fashion. Such is the case for $f(z) = (1-z)^{1/m}$. Indeed, if we define, taking $r = s = n$ for simplicity,

$$P_n(z) = \sum_{k=0}^{n}\binom{n+1/m}{k}\binom{2n-k}{n}(-z)^k$$

and

$$Q_n(z) = \sum_{k=0}^{n}\binom{n-1/m}{k}\binom{2n-k}{n}(-z)^k,$$

then there exists a power series $E_n(z)$ such that for all complex z with $|z| < 1$,

$$P_n(z) - (1-z)^{1/m}\,Q_n(z) = z^{2n+1}E_n(z). \tag{66.0.0.4}$$

How could we go about discovering these polynomials for ourselves? Let us write

$$I_n(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{(1-wz)^{n+1/m}}{(w(w-1))^{n+1}}dw,$$

where γ is a closed counter-clockwise contour enclosing 0 and 1. Here, by $(1+t)^{1/m}$ for t a complex number with $|t|<1$, we mean

$$(1+t)^{1/m} = \sum_{k=0}^{\infty} \binom{1/m}{k} t^k.$$

In fact, expanding the binomial series, we may write

$$I_n(z) = \sum_{h=0}^{\infty} \binom{n+1/m}{h} (-z)^h J_h,$$

where

$$J_h = \frac{1}{2\pi i} \int_\gamma \frac{w^{h-n-1}}{(w-1)^{n+1}} dw.$$

As is well-known, the integral of a rational function $P(w)/Q(w)$ over a closed contour enclosing its poles vanishes, provided the degree of the polynomial Q exceeds that of P by at least 2. We thus have $J_h = 0$ for $0 \leq h \leq 2n$. To find the shape of the coefficients of $P_n(z)$ and $Q_n(z)$ involves a residue calculation.

To actually apply the hypergeometric method, we need some inequalities.

Lemma 66.0.7. *Let n be a positive integer and suppose that z is a complex number with $|1-z| \leq 1$. Then*

(i) We have

$$|P_n(z)| < 4^n, \ |Q_n(z)| < 4^n \ \textit{ and } \ |E_n(z)| < 4^{-n}(1-|z|)^{-\frac{1}{2}(2n+1)}.$$

(ii) For all complex numbers $z \neq 0$, we have

$$P_n(z)Q_{n+1}(z) \neq P_{n+1}(z)Q_n(z).$$

(iii) If we define

$$\sigma_{k,m} = \prod_{p|m} p^{[k/(p-1)]},$$

then

$$\sigma_{k,m} m^k \binom{n \pm 1/m}{k} \in \mathbb{Z}.$$

(iv) If we define $G_{n,m}$ to be the largest positive integer such that

$$\frac{\sigma_{n,m} m^n P_n(z)}{G_{n,m}} \ \textit{ and } \ \frac{\sigma_{n,m} m^n Q_n(z)}{G_{n,m}}$$

are both polynomials with integer coefficients, then

$$G_{n,3} > \frac{1}{42} 2^n \textit{ and } G_{n,4} > (3/2)^n,$$

for all $n \in \mathbb{N}$.

Consequences

Arguing carefully with the results of the preceding subsection, we can prove, by way of example, inequalities of the shape

$$\left|x^3 - 2y^3\right| \geq \sqrt{|x|},$$

valid for all integers x and y.

Combining this machinery with various gap principles, leads one (after, it must be confessed, a certain amount of work) to conclude that the general equation $|ax^n - by^n| = 1$ has, for fixed nonzero integers a, b and $n \geq 3$, at most one solution in positive integers x and y.

More generally, we can apply these methods to bound solutions to the number of solutions to S-unit equation, Thue equations and Thue-Mahler equations, as well as to wide classes of exponential equations.

Bibliography

[1] A. BAKER, "Rational approximations to certain algebraic numbers", *Proc. London Math. Soc. (3)* **14** (1964), 385–398.

[2] A. BAKER, "Rational approximations to $\sqrt[3]{2}$ and other algebraic numbers", *Quart. J. Math. Oxford Ser. (2)* **15** (1964), 375–383.

[3] M. BENNETT, "Simultaneous rational approximation to binomial functions", *Trans. Amer. Math. Soc.* **348** (1996), 1717–1738.

[4] M. BENNETT, "Effective measures of irrationality for certain algebraic numbers", *J. Austral. Math. Soc.* **62** (1997), 329–344.

[5] M. BENNETT, "Explicit lower bounds for rational approximation to algebraic numbers", *Proc. London Math. Soc.* **75** (1997), 63–78.

[6] F. BEUKERS, " A note on the irrationality of $\zeta(2)$ and $\zeta(3)$", *Bull. London Math. Soc.* **11** (1979), no. 3, 268–272.

[7] J.H. CHEN, "A new solution of the Diophantine equation $X^2+1=2Y^4$", *J. Number Theory* **48** (1994), no. 1, 62–74.

[8] G.V. CHUDNOVSKY, "On the method of Thue-Siegel", *Ann. Math. II Ser.* **117** (1983), 325–382.

[9] C.L. SIEGEL, "Die Gleichung $ax^n - by^n = c$", *Math. Ann.* **114** (1937), 57–68.

[10] A. THUE, "Berechnung aller Lösungen gewisser Gleichungen von der form", *Vid. Skrifter I Mat.-Naturv. Klasse* (1918), 1–9.

Cohomological obstructions to rational points

Poonen's four lectures focused on the problem of deciding whether a variety has a rational point, in particular on the use of cohomological methods to prove that a variety has no rational point. He presented introductions to the Brauer–Manin obstruction and the descent obstruction, starting from the basic definitions, and working up to the statements of recent results comparing their relative strengths.

Testing for local points

Let k be a number field. For each place v of k, let k_v be the completion. For nonarchimedean v, let $\mathcal{O}_v$ be the valuation ring of k_v. The adele ring $\mathbf{A}$ of k is the restricted direct product $\prod'_v(k_v, \mathcal{O}_v)$. Let X be a nice k-variety, by which we mean a smooth projective geometrically integral variety over k.

If X is over $\mathbb{Q}$, for instance, sometimes one can prove that X has no $\mathbb{Q}$-point by proving that X has no $\mathcal{R}$-point. More generally, one can test all the completions. This can be summarized by the observation that $X(k)$ embeds diagonally into $X(\mathbf{A}) = \prod_v X(k_v)$ (equality holds since X is projective), so if $X(\mathbf{A}) = \emptyset$, then $X(k) = \emptyset$. Moreover, it is usually easy to decide whether $X(\mathbf{A})$ is empty or not, since one can compute in advance a finite set S of places such that $X(k_v)$ is nonempty for $v \notin S$, and then using real algebraic geometry or Hensel's lemma to treat the finitely many $v \in S$.

Since the 1940s, however, it has been known that there exist examples of nice varieties X such that $X(\mathbf{A}) \neq \emptyset$ but $X(k) = \emptyset$ [10, 13]. Many such examples can be explained in a systematic way by defining subsets $X(\mathbf{A})^{\mathrm{Br}}$ and $X(\mathbf{A})^{\mathrm{descent}}$ of $X(\mathbf{A})$ known to contain $X(k)$. The emptiness of such a subset is an obstruction to the existence of a k-point.

Brauer groups of fields

The first of the obstructions is defined by using the Brauer group. For a field k, with separable closure k^{sep}, one defines an Azumaya algebra over k to be a twist of a matrix algebra, i.e., a k-algebra A (associative with 1) such that $A \otimes_k k^{\mathrm{sep}} \simeq M_n(k^{\mathrm{sep}})$ for some $n > 0$. An example is the quaternion algebra (a, b) over a field k of characteristic not 2 defined by $a, b \in k^{\times}$: it is the associative k-algebra generated by i and j modulo the 2-sided ideal generated by the relations $i^2 = a$, $j^2 = b$, $ji = -ij$. Two Azumaya algebras A and B are called equivalent if $M_n(A) \simeq M_m(B)$ for some $m, n > 0$. The Brauer group $\operatorname{Br} k$ is the set of equivalence classes of Azumaya algebras over k, with group law given by tensor product. Alternatively, $\operatorname{Br} k$ can be defined as the Galois cohomology group $H^2(k, \mathbb{G}_m)$. For example, if k is a local field, then there is an injection $\operatorname{Br} k \stackrel{\mathrm{inv}}{\hookrightarrow} \mathbb{Q}/\mathbb{Z}$ that is an isomorphism if k is nonarchimedean.

Brauer groups of schemes

Either definition can be extended to an arbitrary scheme X, by replacing the extension k^{sep} of k by an étale cover of X, and by replacing Galois cohomology by étale cohomology. In this way one obtains an Azumaya Brauer group and a cohomological Brauer group of X. A theorem of Gabber and de Jong [6] shows that they coincide for reasonable schemes X, e.g., any scheme X that is quasi-projective over a noetherian ring.

If X is a regular integral variety with function field K, then $\operatorname{Br} X$ can be understood as a subgroup of $\operatorname{Br} K$. More precisely, if X is a regular integral variety over a field k of characteristic 0, then there is an exact sequence

$$0 \to \operatorname{Br} X \to \operatorname{Br} K \stackrel{respectively}{\to} \prod_D H^1(k(D), \mathbb{Q}/\mathbb{Z}),$$

where D ranges over irreducible divisors on X, and $k(D)$ is its function field. Another way to try to understand $\operatorname{Br} X$ concretely is to use the Hochschild–Serre spectral sequence in étale cohomology, which yields the exact sequence

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} X^{\mathrm{sep}})^G \to \operatorname{Br} k \to \ker(\operatorname{Br} X \to \operatorname{Br} X^{\mathrm{sep}}) \to H^1(k, \operatorname{Pic} X^{\mathrm{sep}}) \to H^3(k, \mathbb{G}_m).$$

The Brauer–Manin obstruction

Suppose that X is a nice variety over a global field k. Given $A \in \operatorname{Br} X$, one constructs a diagram

$$\begin{tikzcd}
& X(k) \arrow[r, hook] \arrow[d, "\mathrm{ev}_A"'] & X(\mathbf{A}_k) \arrow[d, "\mathrm{ev}_A"'] \arrow[dr, "\phi_A"] & & \\
0 \arrow[r] & \operatorname{Br} k \arrow[r] & \bigoplus_v \operatorname{Br} k_v \arrow[r, "\sum \mathrm{inv}_v"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0,
\end{tikzcd}$$

which shows that the set $X(\mathbf{A})^A := \phi_A^{-1}(0)$ contains $X(k)$. Define the Brauer set $X(\mathbf{A})^{\mathrm{Br}} := \bigcap_{A \in \operatorname{Br} X} X(\mathbf{A})^A$; it too contains $X(k)$.

Example: Birch and Swinnerton-Dyer [1] proved that the nice variety X defined by

$$\begin{aligned} uv &= x^2 - 5y^2 \\ (u+v)(u+2v) &= x^2 - 5z^2 \end{aligned}$$

in $\mathbb{P}^4_{\mathbb{Q}}$ has a $\mathbb{Q}_p$-point for all $p \leq \infty$ but no $\mathbb{Q}$-point. The nonexistence of a $\mathbb{Q}$-point can be explained by using the quaternion algebra $(5, (u+v)/u)$ to show that $X(\mathbf{A})^{\mathrm{Br}} = \emptyset$.

The Brauer–Manin obstruction for abelian varieties and curves

If A is an abelian variety over a global field, let $A(\mathbf{A})_\bullet$ be the quotient of $A(\mathbf{A})$ by its connected component; one may then define $A(\mathbf{A})_\bullet^{\mathrm{Br}}$. Work of Manin [11] implies that if the Shafarevich-Tate group $\text{Ш}(A)$ is finite, then $A(\mathbf{A})_\bullet^{\mathrm{Br}}$ equals the closure of $A(k)$ in $A(\mathbf{A})_\bullet$.

Scharaschkin [14] used this to show that if X is a nice curve of genus $g \geq 2$ over a number field with a Galois-stable divisor class of degree 1, which lets one embed X in its Jacobian A, and if $\text{Ш}(A)$ is finite, then $X(\mathbf{A})_\bullet^{\mathrm{Br}} = X(\mathbf{A})_\bullet \cap \overline{A(k)}$; conjecturally this equals the closure of $X(k)$ in $A(\mathbf{A})_\bullet$. This result suggests the Mordell-Weil sieve, a practical method for proving $X(k)$ empty.

Torsors

Let G be a linear algebraic group (smooth affine group scheme of finite type over a field k). The trivial G-torsor is G equipped with the right translation action of G. A G-torsor is a twist of this: a k-scheme Y equipped with a right action of G such that the base extension Y^{sep} is isomorphic to G^{sep} as k^{sep}-scheme with right G^{sep}-action. Examples: $Y\colon x^2 + y^2 = -3$ with the action of $G\colon x^2 + y^2 = 1$, or any genus 1 curve with the action of its Jacobian. One can show that a G-torsor is trivial if and only if it has a k-point. The set of isomorphism classes of G-torsors is in bijection with $H^1(k, G)$.

One can generalize and define a G-torsor over a base variety X instead of just $\operatorname{Spec}(k)$. This is a scheme $Y \to X$ with right G-action such that its pullback $Y' \to X'$ by some étale surjective morphism $X' \to X$ isomorphic to $X' \times_k G$ with the obvious right G-action. Example: $Y = X = E$ and $G = E[2]$ for some elliptic curve E over a field k of characteristic not 2, and $Y \to X$ is the multiplication-by-2 map.

Descent obstruction

We present the theory of [9], generalizing [2, 3, 4]. Let k be a number field. Let $f\colon Y \to X$ be a G-torsor over X as above. There is a map $X(k) \to H^1(k, G)$ sending x to the class of the G-torsor $f^{-1}(x)$ over k. For each 1-cocycle $\sigma \in Z^1(k, G)$, one defines a twisted torsor $f^\sigma\colon Y^\sigma \to X$. Then

$$\begin{aligned} X(k) &= \coprod_{[\sigma]\in H^1(k,G)} \{x \in X(k) : (\text{class of } f^{-1}(x)) = [\sigma]\} \\ &= \coprod_{[\sigma]} f^\sigma(Y^\sigma(k)) \\ &\subseteq \bigcup_{[\sigma]} f^\sigma(Y^\sigma(\mathbf{A})) \\ &=: X(\mathbf{A})^f. \end{aligned}$$

Define $X(\mathbf{A})^{\mathrm{descent}} = \bigcap X(\mathbf{A})^f$ where f ranges over all G-torsors for all linear algebraic groups G over k. If one restricts the possibilities for G to connected algebraic groups, one obtains $X(\mathbf{A})^{\mathrm{connected}}$, which Harari [8] proved was equal to $X(\mathbf{A})^{\mathrm{Br}}$. In particular, $X(\mathbf{A})^{\mathrm{descent}} \subseteq X(\mathbf{A})^{\mathrm{Br}}$.

One can also define variants, by replacing $Y^\sigma(\mathbf{A})$ in the definition of $X(\mathbf{A})^f$ by a subset such as $Y^\sigma(\mathbf{A})^{\mathrm{Br}}$. Doing this and letting G range over finite étale groups leads to the étale-Brauer set $X(\mathbf{A})^{\mathrm{et,Br}}$, which was proved by Demarche [7] and Skorobogatov [15] to equal $X(\mathbf{A})^{\mathrm{descent}}$.

Example: Using ideas of Darmon [5], the proof of Fermat's last theorem can be reinterpreted using the descent obstruction.

All these obstructions are insufficient to answer the question of whether a $\mathbb{Q}$-variety has a rational point: there is a nice $\mathbb{Q}$-variety for which $X(\mathbf{A})^{\mathrm{et,Br}} \neq \emptyset$ but nevertheless $X(\mathbb{Q}) = \emptyset$ [12].

Bibliography

[1] B. J. Birch and H. P. F. Swinnerton-Dyer. The Hasse problem for rational surfaces. *J. Reine Angew. Math.*, 274/275:164–174, 1975. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, III.

[2] C. Chevalley and A. Weil. Un théorème d'arithmétique sur les courbes algébriques. *Comptes Rendus Hebdomadaires des Séances de l'Acad. des Sci., Paris*, 195:570– 572, 1930.

[3] J.-L. Colliot-Thélène and J.-J. Sansuc. *La descente sur les variétés rationnelles*. 1980.

[4] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. La descente sur les variétés rationnelles. ii. *Duke Math. J.*, 54(2):375–492, 1987.

[5] H. Darmon. Faltings plus epsilon, Wiles plus epsilon, and the generalized Fermat equation. *C. R. Math. Rep. Acad. Sci. Canada*, 19(1):3– 14, 1997.

[6] A. J. De Jong. A result of Gabber. Preprint.

[7] Cyril Demarche. Obstruction de descente et obstruction de brauer-manin étale. *Algebra Number Theory*, 3(2):237–254, 2009.

[8] David Harari. Groupes algébriques et points rationnels. *Math. Ann.*, 322(4):811–826, 2002.

[9] David Harari and Alexei N. Skorobogatov. Non-abelian cohomology and rational points. *Compositio Math.*, 130(3):241–273, 2002.

[10] Carl-Erik Lind. Untersuchungen über die rationalen Punkte der ebenen kubischen Kurven vom geschlecht Eins. *Thesis, University of Uppsala,*, 1940:97, 1940.

[11] Y. I. Manin. *Le groupe de Brauer-Grothendieck en géométrie diophantienne*. Gauthier-Villars, 1971.

[12] Bjorn Poonen. Insufficiency of the Brauer-Manin obstruction applied to étale covers. *Ann. of Math. (2)*, 171(3):2157–2169, 2010.

[13] Hans Reichardt. Einige im kleinen überall lösbare, im grossen unlösbare diophantische gleichungen. *J. reine angew. Math.*, 184:12–18, 1942.

[14] Victor Scharaschkin. *Local-global problems and the Brauer-Manin obstruction*. 1999. Ph.D. thesis, University of Michigan.

[15] Alexei Skorobogatov. Descent obstruction is equivalent to étale Brauer-Manin obstruction. *Math. Ann.*, 344(3):501–510, 2009.

Mordell-Weil sieving and Chabauty's method

Bruin's lectures mainly covered methods that in practice often allow the explicit determination of integral and rational points on curves. Heuristically one expects that these methods should work always work eventually. This is in stark contrast with the negative results proved by Matyasevitch and Davis-Putnam-Robinson, that there is no general method to decide whether a polynomial equation has any integral solutions. The key difference here is that the class of equations describing curves is restricted and does not include the type of equations that provide the counterexamples.

The method of Mordell-Weil sieving considers a curve C embedded in a group variety J. For integer points, J can be a semiabelian variety; for rational points J is an abelian variety. The group of integral points on J in either case forms a finitely generated abelian group (and in the case where J is an abelian variety, this is the same as the group of rational points, because J is projective). Mordell-Weil sieving (which, in the case when J is a multiplicative group is more appropriately named Dirichlet sieving), is based on considering the following commutative diagram for an appropriate constant $B \in \mathbb{Z}_{>1}$ and an appropriate finite set of primes S.

$$\begin{tikzcd}
C(\mathbb{Z}) \arrow[r, "\iota"] \arrow[d] & J(\mathbb{Z})/BJ(\mathbb{Z}) \arrow[d, "\prod \rho_p"] \\
\prod_{p\in S} C(\mathfrak{F}_p) \arrow[r, "\iota_S"] & \prod_{p\in S} \dfrac{J(\mathfrak{F}_p)}{B \operatorname{im} \rho_p}
\end{tikzcd}$$

If C is hyperbolic, then all sets in the above diagram are finite and if B is appropriately chosen, then ι is injective. With appropriately chosen B, S, we have that $J(\mathbb{Z})/BJ(\mathbb{Z})$ and $\prod_{p\in S} C(\mathfrak{F}_p)$ are so small compared to the set they are mapped into, that one expects the intersection of the images of $\prod \rho_p$ and ι_S to be extremely small [5]. In particular, for appropriate B, S one expects that $(\operatorname{im} \prod \rho_p) \cap (\operatorname{im} \iota_S)$ is in bijection with $C(\mathbb{Z})$. See [3] for details. Subject to standard conjectures, obstructions arising from this construction can be interpreted as part of the Brauer-Manin obstruction [6].

A method almost perfectly complementing sieving is the observation that if $J(\mathbb{Z})$ is of sufficiently low rank then one can construct a p-adic analytic function

$$\Theta_p : C(\mathbb{Z}_p) \to \mathbb{Q}_p$$

that vanishes on $C(\mathbb{Z})$. This gives us a way to prove that integral points cannot lie p-adically too close to one another. In particular, this usually is capable of proving that there can be at most one integer point in a fiber of the composition $C(\mathbb{Z}) \to J(\mathfrak{F}_p)$. Together with Mordell-Weil sieving, this can usually determine a sharp bound on the number of integral or rational points. See [1] for evidence. This method is originally due to observations by Chabauty [4], building on ideas of Skolem for determining integral solutions.

In all cases, one needs to find an embedding of C into some appropriate group variety. This is always possible if $C(\mathbb{Z})$ is non-empty. Obstructions to this can be determined via *descent obstructions* (see Section 66). These can also be made relatively efficiently computable [2].

Bibliography

[1] Nils Bruin and Michael Stoll. Deciding existence of rational points on curves: an experiment. *Experiment. Math.*, 17(2):181–189, 2008.

[2] Nils Bruin and Michael Stoll. Two-cover descent on hyperelliptic curves. *Math. Comp.*, 78(268):2347–2370, 2009.

[3] Nils Bruin and Michael Stoll. The Mordell-Weil sieve: proving non-existence of rational points on curves. *LMS J. Comput. Math.*, 13:272–306, 2010.

[4] Claude Chabauty. Sur les points rationnels des courbes algébriques de genre supérieur à l'unité. *C. R. Acad. Sci. Paris*, 212:882–885, 1941.

[5] Bjorn Poonen. Heuristics for the Brauer-Manin obstruction for curves. *Experiment. Math.*, 15:415–420, 2006.

[6] Victor Scharaschkin. *Local-global problems and the Brauer-Manin obstruction*. 1999. Ph.D. thesis, University of Michigan.

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