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Contents

Five-day Workshop Reports

Chapter 1

Transport in Unsteady Flows: from Deterministic Structures to Stochastic Models and Back Again (17w5048)

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Background and Rationale for Workshop

Transport in unsteady turbulent flows is an issue of central importance in many geophysical and engineering problems. The problem is challenging from a mathematical perspective because of the broad range of scales of motions present in such flows. Additionally, these flows tend to self-organize into coherent structures, such as small scale filaments and large scale jets. The multiscale and nonlinear interactions across these spatial scales has a profound influence on the transport of energy, enstrophy, momentum, heat, plankton, or pollutants and chemicals in the oceans and atmosphere. In recent years, there has been considerable interest and progress in identifying larger scale time-varying structures (such as mesoscale oceanic eddies, the circumpolar Antarctic vortex, oceanic jets and fluid interfaces) from observational and experimental data using methods based on rigorous mathematics rather than heuristic arguments. Such structures, sometimes called Lagrangian Coherent Structures (LCSs [23, 38]), identify crucial dynamical barriers in the flow and characterize transport and mixing. At the same time, a parallel line of research has been pursued investigating stochastic dynamical representation of transport by small, unresolved scales of flow [16, 36]. This question is complicated by the two-way nature of the interaction: the larger, resolved scales organize the smaller, unresolved ones; but these then feed back on the resolved scales through nonlinear interactions. Processes that have been approached from this stochastic dynamical perspective include transport by mesoscale ocean eddies in ocean models [10, 50, 51, 27], stochastic backscatter of energy in numerical weather prediction models [43], and the mixing of scalar fields or chemical species in engineering devices [40, 14].

The dominant diagnostic methods for determining LCSs typically use velocity data to impute where fluid particles go using <u>deterministic</u> advection [23, 6, 17, 18, 42, 38, 32, 21]. The velocity field at each position is assumed to be known precisely for all time, permitting the numerical advection of fluid particles from one time instance to another. For real unsteady flows, however, the full velocity field at future times and across all relevant spatial scales is typically not known. Stochastic modelling approaches have been developed to accommodate the lack of information about flow structures on some scales. To date, however, these approaches have made limited use of the insights regarding transport afforded by LCS for either the development of stochastic parameterizations or the study of how these small-scale transports are modulated by larger, resolved coherent structures.

The presence of unresolved small-scale variability limits the genuinely predictive capabilities of deterministic coherent structure identification. However, coherent structures provide a low-dimensional yet meaningful description of the flow dynamics and so developing ways to use them for flow prediction would be tremendously valuable for both geophysical and engineering applications. The natural question that then emerges is how the stochastic dynamics and LCS perspectives can be brought together for a fluid dynamical analysis of the (probabilistic) predictability of transport in unsteady flows. One approach is through finite-time coherent sets for advection-diffusion equations and other stochastic processes [17]. There are also promising tools to approach this problem have recently emerged from interesting new theoretical approaches in random dynamical systems [5, 13], which for example indicate that classical deterministic bifurcation behavior of invariant entities is qualitatively different under stochastic perturbations.

In the past year or so, there has been an emerging tentative dialogue between researchers from the stochastic analysis and the fluid dynamics LCS communities. This workshop was designed to strengthen these nascent interactions and promote their development towards an active, fruitful research dialogue by bringing together researchers with complementary approaches to a topical problem. The hope was to push deterministic flow models towards being more realistic, and stochastic approaches towards being more applicable in fluid dynamics settings. This workshop therefore focussed on the interplay of stochastic effects and coherent flow structures on transport in unsteady flows, with a longer-term goal of addressing predictability, for which a probabilistic interpretation seems the most appropriate approach. By bringing together participants with expertise in theory, modeling and experiment, the workshop planned to coax theory from dynamical systems, fluid dynamics and stochastic analysis to work together. The presence of experimental and observational researchers, whose insight helped guide models and theory in directions relevant for applications, was an important aspect of the workshop. Several early career researchers in modeling, theory and experiments were also among the invitees, with the goal of ensuring that the impact of this workshop lasts into the future.

The workshop's stated goals were:

1. Assess from experiment and observation the predictive utility of LCS and transport estimates arising from deterministic models of real world systems;

2. Investigate how LCS-based perspectives on transport can inform the development of stochastic models of subgrid-scale transport;

3. Harmonize the deterministic and stochastic perspectives on transport, and discover which objects are analogous to LCS for stochastic models.

Workshop Structure

Given the diverse nature of the research areas of the participants, we designed this workshop in a very specific way to elicit interactions between the participants, in order to work towards the goals of the workshop. Thus, we structured the workshop around several distinct components:

1. **Perspectives**: Eleven participants, selected because of their broad knowledge in their subject area, their dissemination skills, and their perceived potential to reach beyond their area, were invited to present 'Perspective' talks on the first and part of the second days of the workshop. These talks were limited to 15 minutes, with 15 minutes discussion thereafter. The Perspectives were intended to introduce certain areas of research to participants from other areas, and the presenters were asked specifically to pose questions and issues that would spill over into related research areas. This highly challenging task, limited as it was to 15 minutes, was taken up with avidity by all our presenters. Illuminating and thought-provoking Perspectives were presented by Stephen Griffies (ocean mesoscales), Michael Ghil (variability in the ocean and atmosphere), Adam Monahan (atmospheric transport), Judith Berner (prediction), Jonathan Lilly (ocean observations and turbulence), Cecile Penland (limitations of models), Guido Boffetta (bio-organisms under turbulence), Nicholas Ouellette (extracting coherence from experimental data), Gary Froyland (set-oriented techniques for transport and coherence), Oliver Junge (computational methods for coherence using transfer operators), and Jason Frank (numerical discretization impacts on preserving coherence). Several presenters identified well-defined issues or potential links to other areas, which led to rich discussions, often involving many of the participants.

2. Issue identification: The Perspectives and subsequent discussions brought into focus many issues for future discussion, and potential multidisciplinary collaborations between the participants. A listing of these issues was maintained, and on day two of the workshop, participants were asked to vote on which of these issues they would like to focus. By a process of elimination, several subject areas were identified. After much discussion, issues were amalgamated into three specific areas:

1. Lagrangian Coherent Structures: What Good Are They?
2. Stochastic Parameterization.
3. Prediction.

Thus, the issues to be discussed in detail emerged organically from the structure of the workshop.

3. Problem-solving groups: Significant time was devoted to addressing specific issues under the three groups that were identified above. The time allocated was roughly 1.5 hours on the second day, 5 hours on the fourth day, and 1 hour on the fifth day. Outlines of the discussions from each of these groups appears in later sections of this report. On the whole, the discussions in these groups was lively and constructive, with the participants taking considerable effort to reach across disciplinary barriers. The organizers wish to thank Shane Keating and Jonathan Lilly in particular for help in coordinating the discussions, and setting up online resources to help document them. The discussions in general helped in bridging the terminology divide, in helping understand why certain issues were considered important or difficult in various research areas, in seeking overlapping interests, and in formulating issues that could lead to future collaborations.

4. Standard talks: The workshop also had ongoing talks that were of the more standard variety. In keeping with the goals of the workshop, efforts were made to give as much exposure as possible to early-career researchers to present their research. One way in which this was achieved was by limiting standard talks also to 15 minutes. This resulted in well-thought out and interesting talks from both junior and senior participants, which also elicited much discussion. Talks were presented by Cecilia González-Tokman, Hussein Aluie, Alexis Tantet, Thomas Peacock, Valerio Lucarini, Jeroen Wouters, Kathrin Padberg-Gehle, Jacques Vanneste, Daniel Karrasch, Philippe Miron, Nikki Vercauteren, Ryan Abernathey, Irina Rypina, Georg Gottwald, Péter Koltai and Shane Keating. The talks cut across many of the research areas represented in the workshop, and included aspects of ergodic theory, fluid experiments, turbulence approaches, transfer operators in the presence of noise, coral reef connectivity, multistability in climate, subgrid-scale parameterization for finite time-scale separation, coherence obtained from trajectories, dispersion in large deviation regimes, coherence as thought of in terms of a heat equation, Gulf of Mexico geometry analysis from float data, atmospheric boundary layer flow structures, Eulerian eddy fluxes and Lagrangian eddies, encounter number as a new diagnostic of mixing, statistical consistency of numerical integrators, space-time characterization of coherence, and stochastic turbulent modeling based on satellite observations of the ocean.

5. Poster session: A poster session was held in the evening of the second day. In contrast possibly with many other workshops or conferences, the majority of the poster presenters were senior participants. Posters were presented by Michael Allshouse, Sanjeeva Balasuriya, Daan Crommelin, Amber Holdsworth, Douglas Kelley, Björn Schmalfuß and Marek Stastna.

Group on “Lagrangian Coherent Structures: What Good Are They?”

As the central topic of the workshop, this group was the largest of the three and included many participants from the oceanographic community and the mathematics/dynamics community. Several participants from the atmospheric science community also attended some of these sessions. Because of the open question asked of the group (“What good are Lagrangian coherent structures”), the first two sessions were mainly wide-ranging discussions about which questions are of scientific significance.

These sessions were also crucially an open and highly informative learning and knowledge-transfer environment, where the mathematicians/dynamicists were able to communicate directly with mathematically savvy oceanographers. In one direction, this served to explain the features and limitations of current mathematical and computational approaches for analysing dynamics with coherent structures, for example, sampling issues [28]. In the other direction, among several aspects, this provided an explanation of key aspects of ocean models, how the physics of fluid flow might better connect with the relevant mathematics, and an appreciation of the important, unknown aspects of the lifecycle of coherent structures. Several participants remarked that these open discussion sessions provided a truly unique opportunity to engage with different communities in a very collegial atmosphere, and were in many ways the highlight and most valuable part of the workshop.

The question "What good are coherent structures?" was accompanied by the related questions "What is the role of coherent structures in dynamics?" and "What sort of transport is useful in ocean and atmospheric science? (at which scales, of what quantities, etc...)." Physical manifestations of coherent structures include hurricanes, western boundary currents, and jets. Eddies/vortices at all scales are another type of coherent structure occurring frequently in both the atmosphere [45] and the oceans [48]. Many aspects related to the mixing and transport associated with all of these coherent entities were highlighted in the discussions [28, 44, 45, 2, 48]. In addition to the usual impacts on transport in the ocean, these features also play a role in coral reef connectivity and search and rescue efforts. Submesoscale coherent vortices are very important to fisheries. There is a hypothesis within the fisheries community, for example, that they act as "nurseries" for larval fish for periods of time that are relevant for larval development; that is, the time it takes for a fish larva to grow large enough to swim. Thus, it would be interesting to know how much exchange occurs across barriers. Coherent structures may also be useful for dimension reduction, as touched on by the "Predictability" group, and identifying regions across which it would make sense to compute fluxes more generally. The dimension reduction in this case might, for example, be in extracting "modes" corresponding to each coherent structure.

There was a substantial discussion concerning coherent structures and the physical dynamics that must be obeyed in the oceanographic and atmospheric contexts. One could consider coherent structures as emergent behavior, and it was noted that they arise in "maximal entropy" solutions. There is for example a strong sense in the atmospheric community that coherent structures emerge due to baroclinic instabilities, and their geometry is associated with wavenumbers that are preferentially chosen by the dynamics. Most current work with coherent structures does not take into account such dynamics, but instead works directly with a given velocity field. Is it possible to incorporate the governing dynamics into standard coherent structure analysis? This broad theme was identified as requiring additional investigation, and an area in which there could be further interaction between the (mathematical) coherent structures analysts, and the (physical law abiding) geophysical scientists.

Another aspect that arose in the discussions was that coherent structure identification could also be used as a criterion of "fitness" of global circulation models. The quality of global circulation models could be assessed by the coherent structures they give rise to, and comparing these to observations.

This was strong interest from the oceanographic community in identifying coherent structures directly from observations, and it was noted that recent methods have been developed to do this, particularly in the case where the data is sparse, scattered, and possibly incomplete [3, 20, 22, 8, 19, 41]. Over the past few years, the coherent structure community has gradually been developing tools that are targeted for explicitly this situation, and it is hoped that an ongoing development—consistent with the limitations of the observational data—will continue to occur. The dialogue established at this workshop is expected to help in this endeavor, as the needs of the practitioners are becoming more apparent to the theorists, and the theoretical limitations of conclusions more obvious to the geophysicists.

In addition to coherent structures acting as efficient transporters of heat, salt, nutrients, etc., there is also a modeling interest in better understanding their formation, interaction with surrounding fluid and each other, and death. A brief discussion on this issue did not lead to any obvious conclusions, but could potentially be an aspect for future investigation.

The group was conscious of having to formulate some concrete problems for work in the immediate future. The following were identified as having the potential for such work, possibly among the participants of the workshop:

- Downscale GCM (Global Climate Model) model/drifter data and quantify sensitivity and uncertainty in diagnosis of coherent structures.

- Look at the dynamical role or physical significance of coherent structures in simple (but realistic) turbulence models, e.g., the recent data set [1] generated from the quasigeostrophic model, and others [44, 2, 39].

- Consider the lifecycle of coherent structures, the role of geometry in interacting / merging structures [4, 49], and the role of "entrained periphery" of the coherent core.
- Work on theoretical and applied approaches to the issue of incorporating stochasticity into deterministic coherent structure viewpoints. (There is current progress in this direction [7] as a result of discussions at the workshop, with several follow-up analyses currently in preparation.)

Group on "Stochastic Parameterization"

The discussion sessions on stochastic parameterization were attended by participants from the atmospheric science community, mathematicians involved in stochastic modeling, and oceanographers.

Initially, much of the discussion focused on past experiences and results from stochastic parameterizations for atmospheric processes. This was perhaps not surprising, as stochastic methods for parameterizing atmospheric processes (for example convection) have been under development for over 15 years [37, 11], whereas exploration of these methods for ocean modeling is a more recent development [39, 29, 51, 27, 50]. However, later on topics more specific to oceanography were also addressed in the discussions.

A marked difference in what is expected from stochastic parameterization in ocean and atmosphere modeling is that in current atmospheric GCMs, most of the spectrum of eddies are explicitly resolved, whereas in ocean GCMs, they are not. This is directly related to the difference in Rossby radius between atmosphere and ocean. In the atmosphere, it is on the order of 1000 km, a scale that is well resolved in current atmospheric GCMs. As a consequence, atmospheric eddies exist primarily on synoptic scales and these are well resolved. By contrast, the oceanic Rossby radius becomes as small as $\mathcal{O}(10\ \mathrm{km})$ at high latitudes, so that oceanic eddies are not well resolved in most ocean GCMs. For ocean modeling, an important driver for the interest in stochastic parameterization comes from the intent to use these as a way of representing mesoscale eddies. These can be seen as part of the adiabatic processes of ocean flow, hence it is of interest that parameterizations do as little as possible to disturb the energy-conserving nature of the adiabatic processes. In the atmosphere, on the other hand, much of the research on stochastic parameterization is targeting the representation of diabatic processes (notably, convection).

A few of the issues and questions that were brought up during the discussions were:

- Are there guidelines for knowing when a stochastic parameterization will be useful or effective in the atmosphere/ocean/GCMs? Before engaging in a search for the optimal method or approach for stochastic parameterization, one would like to be able to assess in what situations and under what conditions a stochastic parameterization can be useful at all. Two important aspects were discussed in this context: (i) Is there temporal scale separation between the processes that are resolved and the processes that must be parameterized? This is to be distinguished from the question of whether or not there is spatial scale separation. (ii) If the resolved scales of a system receive a large amount of random "kicks" (perturbations) from the unresolved scales, so that the system is approaching (but has not reached) a kind of thermodynamic limit, a stochastic parameterization of these kicks can be appropriate.
- It was discussed that there are different goals that researchers have in mind for stochastic parameterizations. Representing sub-gridscale coherent structures and partially resolved coherent structures is one such goal. A relevant distinction here relating to coherent structures and transport is between transport <u>of</u> coherent structures and transport <u>by</u> coherent structures. Other aims that were stated in the discussion were (i) capturing non-diffusive behavior (neglected in traditional diffusive eddy parameterizations), (ii) increasing variability on small scales, and (iii) improving the numerical stability of models.
- An issue that was raised several times during discussions is the absence of broadly agreed-on performance measures for stochastic parameterization. Should there be a "skill-score" for parameterizations, similar in spirit to the skill scores that are used to evaluate performance of numerical weather prediction models? This was considered useful, while it was also acknowledged that the different aims and expectations that researchers have for including stochastic parameterizations in their models make it difficult to identify performance measures that will be widely regarded as useful and important.
- Two further topics addressed in the discussion were (i) methodology and best-practices, and the advantages and disadvantages of methodologies based on first principles versus more empirical approaches; and (ii)

the usefulness of intermediate-complexity models for developing and testing stochastic parameterizations methods. This is a class of models that is no longer state of the art. However, testing methods in these models can be very useful as an intermediate step between highly idealized models on the one hand and state of the art models on the other.

During the week, a subgroup formed to discuss stochastic parameterization issues specifically regarding the atmospheric boundary layer (ABL). The primary question of interest was the simulation of the observed multiple regimes of the stably stratified ABL (SBL) and transitions between them [31, 33, 46, 47]. Under conditions of stable stratification, the ABL is observed to exist in two states: the weakly stable boundary layer (WSBL) characterized by strong flow, a weak temperature inversion, and sustained turbulence; and the very stable boundary layer (VSBL) characterized by weak flow, a strong temperature inversion, and weak and intermittent turbulence. These features are quite shallow (typically within a few tens of metres of the surface) and are therefore poorly resolved by standard regional and global models of the atmosphere. Transitions from the WSBL to the VSBL typically occur shortly after sunset during periods of strong radiative surface cooling; this process is expected to be captured reasonably well by models. In contrast, the VSBL to WSBL transition is not well understood. It appears to be related to a collection of intermittent turbulent processes of diverse origin (collectively denoted as submesoscale, many of which processes are coherent structures) that has been described as a "stochastic mix" [31]. Some of these processes, such as instabilities associated with the resolved shear, may be simulated by regional and global models. Others, such as density currents and breaking gravity waves, will not be. Representation of the SBL has been a persistent bias of weather and climate models, with consequent biases in simulation and prediction of near-surface temperature and wind fields of interest in applications ranging from agriculture to renewable energy.

The subgroup first discussed the problem in broad terms, considering which phenomena physically relevant to SBL dynamics can potentially be captured by existing parameterizations and which likely cannot. Initial attention was focused on the results of [9], which showed that even in high-resolution (1/3 km horizontal) mesoscale models such as Weather Research and Forecasting (WRF), parameterized eddy diffusion strongly suppresses submesoscale motion. Attention then turned to the fact that even those processes that can in principle be captured require finer than standard vertical resolution. The following research plan was developed: targeted case studies will be carried out using WRF for the time period and location corresponding to the well-instrumented CASES-99 field campaign. The initial simulations will vary vertical resolution, horizontal resolution, and the strength of parameterized diffusion to assess the effect of these factors on the simulation of SBL regimes and transitions between them. Having assessed this deterministic "baseline" of simulation quality, a new state-dependent stochastic representation of submesoscale vertical turbulent transport will be introduced. The starting point for this parameterization will be the statistical analyses of [47] and the proof-of-concept stochastic parameterization of [24]. By driving the local WRF simulation with the large-scale forecast ensemble produced for that time, we will also be able to assess how the factors of increased resolution and stochastic physics improve ensemble spread in boundary layer quantities.

Some ideas were discussed for the next-generation of ocean parameterizations. Stochastic instead of deterministic methods were regarded as an interesting and promising approach, however other promising ideas were considered as well. Anisotropic diffusion is drawing quite some interest in the oceanography community [15]. Another promising approach is to adopt methods from non-Newtonian fluid mechanics [39, 4, 25].

Group on "Prediction"

For much of their history of study, there has been a recognized link between coherent structures and prediction in complex flows. By construction, coherent structures capture parts of the flow that maintain their structure for long times relative to the rest of the field. Thus, their dynamics may be in some sense "simpler" than the rest of the flow, which could potentially be modeled in a purely stochastic way [35]. Lagrangian structures, like those that were the focus of this workshop, typically contain much more information and are much more stringently coherent than their Eulerian counterparts, since they explicitly account for fluid advection. Thus, when they were first introduced, there was significant excitement about using them as new predictive tools. Unfortunately, Lagrangian structures have not yet lived up to their promise in this regard, in part because methods for computing them typically require knowledge of the future evolution of the flow. To discuss whether this and other issues can be surmounted or whether Lagrangian structures will only ever be useful as a post-hoc analysis tool, one of the working groups at this workshop focused on potential strategies for using Lagrangian structures for flow prediction.

One of the first issues the group faced is the question of what precisely is meant by 'prediction'; that is, what do we want to predict, and over what time horizon? In many cases of practical interest, one wants to know where some transported quantity will go; in the Deepwater Horizon oil spill, for example, the most important quantity to be predicted is the future distribution of the oil. But in some other situations, such as in predicting the evolution of a turbulent flow, one wants to know about the evolution of the flow properties themselves. It is possible that different strategies may have to be employed to handle these different kinds of predictions. It was also pointed out that, given the finite-time and aperiodic nature of the flows of interest, prediction is never going to be exact or mathematically rigorous, in part because the coherent structures themselves do not last forever in these kinds of flows. The best we may be able to do is to assign likelihoods to possible future scenarios based on the current coherent structures.

To do so, the group recognized that an important gap in our current understanding of coherent structures is that we do not know much about their life cycle. That is, given the locations and some properties of coherent structures now, and potentially at times in the past, we have little sense of how they will evolve in the future. Part of the reason for this lack of understanding is that current methods for locating Lagrangian structures have been developed simply by considering the structure of the Lagrangian evolution equation $\dot{\mathbf{x}} = \mathbf{u}$, where $\mathbf{x}$ is position, the dot signifies the time-derivative along a Lagrangian trajectory, and $\mathbf{u}$ is the velocity field. Typically, the velocity field is assumed to be given, and no characteristics of its dynamics are assumed. In reality, however, the evolution of the velocity field, and therefore the coherent structures themselves, is constrained by physics: its dynamics must satisfy conservation of mass, momentum, and energy, the laws of thermodynamics, and the like. Often, the velocity field will be a solution of the Navier–Stokes equations, which bring their own complications; but in some cases, such as in data-assimilated large-scale ocean models, it will come from a somewhat simpler system. Thus, one potential path forward for making coherent structures more predictive is to learn how to constrain their possible future evolution by the properties of the velocity field that they arise from.

A second strategy the group discussed for making coherent structures more predictive is to model their evolution by an ordinary differential equation. The (approximate) flow evolution could then be captured by a finite (and hopefully small) set of such ODEs, one per structure, that are coupled, but that would be easier to solve than the full Navier-Stokes equations. This idea is reminiscent of how the utility of coherent structures was originally conceived of by the turbulence community [35]. Developing such an ODE model, however, may be quite challenging, since Lagrangian structures are potentially complex, spatially extended objects. It may thus be necessary to simplify the structures themselves to be able to model them appropriately. It was noted that this approach is now being taken in the context of Koopman operators, by approximating them in terms of only a few eigenmodes. Something similar may be possible to do with transfer operators and almost-invariant sets, and then their evolution may potentially be modeled by an ODE (which may potentially have to be stochastic to compensate for the information removed by the spectral truncation). Much work remains to be done to make these ideas viable in practice for the unsteady flows of interest in many applications, since these eigenmode decompositions have been only been studied so far for steady or periodic cases. But they do represent a promising strategy going forward.

The final topic discussed by the group was what appropriate test cases are for developing strategies for prediction using coherent structures. Clearly, steady or time-periodic flows are not sufficient; in those cases, prediction is anomalously easy, since the time dependence is simple. Thus, some of the classic test cases for studying Lagrangian structures, such as the double gyre, may not be appropriate for looking at their connection to prediction. Looking to observational data from, for example, the ocean is likely not the best choice either, due to inherent finite spatiotemporal resolution and numerous physical processes that will affect the evolution of structures outside of pure fluid advection. The group suggested that an appropriate case of intermediate difficulty might be a well controlled laboratory experiment or direct numerical simulation of Navier–Stokes, where the flow can be forced to be statistically stationary and will have known length, time, and velocity scales.

Thus, although much work remains to be done, the discussion group on prediction via coherent structures was able to identify concrete paths forward that should be explored in future research.

Outcomes and the Future

The topic of this workshop was highly multidisciplinary, and its structure was designed to initiate dialogues between researchers from many areas. The general sentiment from participants and organizers was that this purpose had been achieved to a large degree. The highly collegial atmosphere in which all discussions were held led to a greater understanding of each others' research areas, and bodes well for future interactions between the participants.

The groups discussed many different ways in which statistical models, coherent structures, and applications in oceanography, atmospheric sciences and turbulence overlapped. Many nascent intersections were identified, as have been detailed in the separate reports from each of the groups. (These will not be separately enumerated here.) It is expected that some of these will coalesce into coherent transdisciplinary research projects in the near future.

Additionally, there were many areas/topics that saw lively debates but that did not necessarily result in a well-defined identification of a problem or an approach. It is hoped that the seeds planted at this workshop will in the medium term result in the crystallization of these into cogent research questions and/or grant proposals. A number of the participants discussed initial plans to organize a follow-up meeting within a couple of years.

Participants

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Chapter 2

Combinatorial Reconfiguration (17w5066)

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Organizer(s): Naomi Nishimura (University of Waterloo), Takehiro Ito (Tohoku University), Amer Mouawad (University of Bergen)

Overview of the Field

Reconfiguration is the study of relationships among solutions to an instance of a problem, and in particular the step-by-step transformation from one solution into another such that the intermediate result of each step is also a solution. As a concrete example of a reconfiguration problem, we consider the assignment of customers to power stations such that each customer obtains as much power as is required without exceeding the capacity of what each station can produce. Each such assignment can be viewed as a solution. When a station needs to be repaired, it may be necessary to change to another assignment, moving a single customer at each step in order to minimize disruptions. Commonly-asked questions in the area of reconfiguration include both structural questions concerning the solution space and algorithmic questions about the complexity of solving various related problems.

For any combinatorial problem, there are multiple ways of defining the solution space, and for each such definition, there are a variety of related problems. A reconfiguration graph for an instance of a problem can be defined as a set of nodes, one for each solution to the problem (where there may be constraints with respect to solution size), and a set of edges, where there is an edge between any two nodes that are adjacent (for some definition of adjacency). The definition of adjacency gives rise to the notion of a reconfiguration step, an operation by which a solution is transformed into an adjacent solution. The sequence of solutions traversed in a sequence of reconfiguration steps, or equivalently, a path in the reconfiguration graph through the nodes associated with those solutions, is called a reconfiguration sequence.

Typical questions that arise include the following:

- Given two solutions, does there exist a reconfiguration sequence transforming one into the other? This is known as the reachability problem.
- Given two solutions, what is the length of the shortest reconfiguration sequence transforming one into the other? This is the decision version of the shortest transformation problem.
- Given two solutions, produce a reconfiguration sequence (or a shortest reconfiguration sequence). This is the search version of the shortest transformation problem.
- Given a problem, under what additional constraints (if any) is it guaranteed that for any instance the reconfiguration graph is connected? This is the connectivity problem.
- Given a problem, under what additional constraints (if any) is it guaranteed that for any instance the reconfiguration graph has a bound on the diameter?

For a given problem, there may be more than one natural definition of adjacency (and, corresponding to each such definition, the notion of a reconfiguration step). For the problem of independent set, where the solution is a subset of the vertices such that there is no edge between any of the vertices in the set, we can view a reconfiguration step as changing the position of a token, where there is a token on each vertex in the independent set. This immediately implies at least three possible ways of moving tokens: token jumping (TJ) [14], in which a token is moved from a vertex to any other vertex in the graph, token sliding (TS) [8, 14], in which a token is moved from a vertex along an edge to a neighbouring vertex, and token addition/removal (TAR) [11, 14], where in each step a token is added or removed. For the problem of k-colouring, where each vertex in a graph is assigned a colour such that for each edge the endpoints have different colours, a reconfiguration step could consist of changing the colour of a single vertex or swapping two colour classes. In yet another example of adjacency, for the problem of satisfiability of Boolean formulas, a reconfiguration step might consist of the flipping of a single variable in the truth assignment (from true to false or from false to true).

For the reconfiguration steps of token jumping and token sliding, all feasible solutions have the same number of tokens. In other situations, it may be necessary to give a bound on the size of solutions under consideration. As an example problem, we consider the problem of vertex cover, where the goal is to determine a subset of the vertices of a graph such that each edge in the graph has at least one endpoint in the set. If we take as our feasible solutions all vertex covers, of any size, then it is easy to see that we can find a reconfiguration sequence from any vertex cover to any other: at each step we add one of the vertices in the target solution (adding a vertex to a solution gives another solution) and after all have been added, removing a vertex that is not in the target solution (which is still a solution, as it contains all the vertices in the target solution). A more interesting problem results from restricting the size of solutions to be at most k or $k + 1$, where k is the size of a minimum solution.

In addition to characterizing the solution spaces of classical computational problems (as well as solutions to puzzles), reconfiguration is related to problems in a variety of areas. In the area of network security, a network can be represented as a graph where each vertex corresponds to a server and each edge to a communication link; a step-by-step change in firewalls corresponds to the reconfiguration of a cut. In the Frequency Assignment Problem, the goal is to assign frequencies to users of a wireless network such that the interference between them is minimized and the range of used frequencies is as small as possible. Frequencies can be viewed as colours, giving rise to a colouring problem. Due to the frequent changes in demand and addition of new transmitters as well as the difficulty of finding optimal assignments, there is a need to change between assignments. Reconfiguration gives a method of doing so without interrupting service for all customers. As a final example, the moving of objects on a plane (modeled by moving tokens on a graph) has applications in the 3D-printing industry, where a "head" follows a path in laying down a layer of material. The use of multiple heads to reduce printing time necessitates ways to plan movement without collisions and, ideally, minimizing the distance traveled.

Recent Developments and Open Problems

As the field is quite young, all progress in the field qualifies as recent developments. Below are listed various types of results along with related open problems.

Defining Adjacency

Relations among different types of reconfiguration steps have been demonstrated; for independent set reconfiguration, TJ and TAR are equivalent [14], and for clique reconfiguration, TJ, TAR, and TS are all equivalent [12]. In this context, problems are equivalent if an instance is solvable using one type of reconfiguration step if and only if it is solvable using another type of reconfiguration step.

Open problems in this area include defining adjacency for problems with solutions that are, for example, given as sequences rather than sets, or for areas such as computational geometry.

Structural Results

Structural results focus on properties of the reconfiguration graph; most results in the category specify when the reconfiguration graph is connected. Most of the results of this type focus on the problems of k-colouring and dominating set. In the latter problem, the goal is to find a set of vertices such that each vertex outside the set has a neighbour in the set.

There are many open problems with respect to not only connectivity of the reconfiguration graph, but also other properties, such as k-connectivity, properties of isolated vertices, and characteristics of various connected components.

Hardness Results

In many cases, there is a correspondence between the complexity of a classical problem and the complexity of the related reachability reconfiguration problem: often both can be solved in polynomial time, or both are intractable (NP-complete and PSPACE-complete, respectively). Many of the PSPACE-hardness proofs follow the chain of reductions used for NP-hardness reductions, starting with the PSPACE-completeness of 3SAT reconfiguration [5].

There are, however, a few exceptions to this general pattern. Although the problem of 3-colouring is NP-complete, its reachability reconfiguration problem can be solved in polynomial time [13]. Conversely, the shortest path problem can be solved in polynomial time, but its reachability reconfiguration problem is PSPACE-complete [1]. Although exceptions to the general rules are known, currently there are no results explaining the reasons for such exceptions.

Algorithms

Greedy algorithms have been developed for matching, independent set, and minimum spanning tree, and dynamic programming algorithms for list colouring, shortest path, and k-colouring. In addition, the complexity of problems has been considered using the framework of parameterized complexity, where the goal is to find algorithms with running times that are polynomial in the size of the input but possibly exponential (or worse) in terms of one or more parameters of the problem. Various results consider as parameters the sizes of solutions, the lengths of reconfiguration sequences, and other attributes of inputs and problems [16].

One area with many open problems is that of finding shortest transformations between solutions. Among the few known results are algorithms for satisfiability [15] and independent set using token-sliding on caterpillars [19].

Presentation Highlights

In order to facilitate as much collaboration as possible, the workshop time was divided among technical talks, open problem presentations, and follow-up discussions. The first two types of activities are summarized here, with a selection of completed work and work in progress detailed in Section 2.

Both the high quality of the contributions and the engagement of the participants led to exciting new collaborations.

Technical Talks

Takehiro Ito (Tokohu University, Japan): Invitation to Combinatorial Reconfiguration The workshop started with this introductory talk. It is possible to view the field of reconfiguration as middle ground between standard search problems, which ask for one solution to an instance of a problem, and enumeration problems, which ask for all solutions to an instance of a problem. It is important to note that since it is a decision problem, a reachability problem does not require the generation of an actual reconfiguration sequence, which may be super-polynomial in length.

The history of combinatorial reconfiguration was presented, showing both how the field was developed and in what new directions it might go. Most of the content can be found in Sections 2 and 2.

Nicolas Bousquet (Institut Polytechnique de Grenoble, France): Token Sliding on Chordal Graphs

(Joint work with Bonamy)

The reconfiguration of independent sets using token sliding was first considered by Hearn and Demaine [8], who showed that the reachability problem is PSPACE-complete on planar graphs. In later results, polynomial-time algorithms have been found for trees, cographs, claw-free graphs, and bipartite permutation graphs.

In new work, the authors show that both the reachability and connectivity problems can be solved in polynomial-time on interval graphs. The algorithm makes use of the geometric representation of an interval graph as an intersection graph of intervals on the line, which can be obtained in polynomial time. Using this representation, one can define the leftmost independent set. Although a naive approach may not yield a polynomial-time algorithm, dynamic programming can be employed to obtain algorithms for both problems.

In addition, the authors show that on split graphs (where the vertex set can be partitioned in two sets, one inducing a clique and the other inducing a set of isolated vertices), the connectivity problem is co-NP-hard and co-W[2]-hard. In particular, they use the fact that k-Dominating Set is NP-hard and W[2]-hard and then show that the result still holds when there is no blocking set of size at most $k + 1$, where in a blocking set no vertex has a private neighbour. Using a similar construction, the same hardness results can be derived for the connectivity problem on bipartite graphs.

Akira Suzuki (Tohoku University, Japan): Reduction Tools on NCL

Nondeterministic Constraint Logic was first introduced by Hearn and Demaine [8] to show the hardness of sliding block puzzles. In a constraint graph, the directed edges have weights of 1 or 2 and are either legal or not. In the legal direction, each vertex is the head of at least two weight 2 edges. The NCL problem is to determine whether a target edge can be reversed by a sequence of legal moves, where in a legal move an edge is reversed such that the resulting direction is legal.

The problem NCL is difficult even if each vertex is either an AND vertex (the output edge is outgoing if both input edges are incoming) or an OR vertex (the output edge is outgoing if at least one input edge is incoming). Using this fact, PSPACE reductions are possible using NCL with just AND and OR gadgets. Demonstrations were given for sliding blocks and for edge colouring.

In addition, NCL can be generalized by introducing the notion of a neutral, undirected edge. This new version can accomplish all that can be accomplished using NCL, but often results in fewer gadgets. One of the new types of gadgets (in addition to AND and OR gadgets) is a link gadget.

In new work [18], it can be shown that reconfiguration of list edge-colouring and edge-colouring are PSPACE-complete using AND, OR, and link gadgets. This result completes the classification of the complexity of the reconfiguration of list edge-colouring by showing that the problem is in P for $k \leq 3$ and PSPACE-complete otherwise. For edge-colouring, the complexity is still open for $k = 4$, though this result proves PSPACE-completeness for $k \geq 5$.

The talk ended with some open questions about the complexity of more restricted constraint graphs.

Karen Seyffarth (University of Calgary, Canada): Reconfiguring Vertex Colourings of 2-Trees

(Joint work with Cavers)

Recent work was presented in the area of the connectivity problem for k-colouring for various classes of graphs. In 2008, Cereceda, van den Heuvel, and Johnson [3] proved that the reconfiguration graph for k-colouring is connected for all $k \geq col(H) + 1$, where $col(H) = \max\{\delta(G) \mid G \subseteq H\} + 1$ is the colouring number of H.

More recently, Choo and MacGillivray [4] proved a connection between colouring numbers and Hamiltonicity, namely, that $k_0(H) \leq col(H) + 2$, for $k_0(H)$ the Gray code number of H, the least integer such that the reconfiguration graph for k-colouring has a Hamiltonian cycle for all $k \geq k_0(H)$. In the same paper, values of k_0 were shown for complete graphs, trees, and cycles. Later, Gray code numbers were found for complete bipartite graphs [2].

The presentation gave a proof of the Gray code numbers for 2-trees. Except for certain cases in which $k_0(H) = 5$, the Gray code number for 2-trees is 4.

Matthew Johnson (Durham University, UK): Kempe Equivalence in Regular Graphs

(Joint work with Bonamy, Bousquet, Feghali, and Paulusma)

For a k-colouring, an (a, b)-component is a maximal connected subgraph whose vertices are coloured a or b; such components are called Kempe chains. A Kempe change is the exchanging of the colours a and b of the vertices in an (a, b)-component, resulting in another proper colouring. A Kempe class consists of k-colourings that are Kempe equivalent (one can be obtained from another by a sequence of Kempe changes).

In 1981, Las Vergnas and Meyniel showed that the set of k-colourings of a d-degenerate graph form a Kempe class, $k > d$, where a graph is d-degenerate if every induced subgraph has a vertex of degree at most d. Mohar conjectured in 2007 that for $k \geq 3$, the k-colourings of a k-regular non-complete graph form a Kempe class, but in 2013 van den Heuvel found a counterexample for $k = 3$ [10].

Bonamy, Bousquet, Feghali, Johnson, and Paulusma have proved that for $k \geq 3$, if G is a connected k-regular graph that is neither complete nor the triangular prism, then the k-colourings of G form a Kempe class. In their proof, they make use of the clique cutset lemma of Las Vergnas and Meyniel, and consider separately k-regular graphs that are not 3-connected and k-regular graphs that are 3-connected. In the latter case, they make use of a matching lemma, and consider k-regular 3-connected graphs of diameter 2.

The talk ended with several open problems, including the question of whether the 5-colourings of a toroidal triangular lattice form a Kempe class (which would have an application in physics, proving the validity of the WSK algorithm for simulating the antiferromagnetic Potts model) and what can be determined about the number of Kempe changes needed to transform one colouring into another.

Jan van den Heuvel (London School of Economics and Political Science, UK): Token-Sliding Problems

(Joint work with Brightwell and Trakultraipruk)

The classic 15-puzzle can be interpreted as a problem of moving tokens on a graph, which can then be generalized to consider different graphs. A result by Wilson in 1974 shows that the reconfiguration graph for a sliding block puzzle is connected except if the puzzle graph is a cycle on $n \geq 4$ vertices, is bipartite and not a cycle, or the exceptional graph Θ_0 on 7 vertices.

To further generalize the problem, one can ask what would happen if there were more than a single uncovered vertex and/or if not all tokens were considered to be identical. The authors show that the reconfiguration graph is connected except if the puzzle graph is not connected, a path with at least two token labels, a cycle with at least two token labels, a cycle with two token labels, with at least two of the same label, a 2-connected bipartite graph with $n-1$ distinct tokens, the graph Θ_0 with one of four possible labelings of tokens, or has connectivity 1 with at least two token labels and a separating path preventing tokens from moving between blocks.

A related problem is that of finding the shortest reconfiguration sequence, which Goldreich showed to be NP-complete for cases with $n-1$ different tokens and van den Heuvel and Trakultriapruk (2014) showed to be in P when all tokens are the same, but NP-complete when at least one token is special and all others are identical. The proof is based on a result for robot motion by Papadimitriou, Raghavan, Sudan, and Tamaki (1994).

Moritz Mühlenthaler (Technische Universität Dortmund, Germany): Reconfiguration in Matroids Revisited

Reconfiguration properties of matroids have been studied, in different guises, at least since the 1970s. The talk touched very briefly on several interesting results from the past 40 years in this area and put them in the current unified setting of reconfiguration problems [11, 10]. Furthermore, some new aspects of reconfiguration in matroids were investigated, namely the complexity of reconfiguring common independent sets of two or more partition matroids. In particular, it was shown that reachability of two common independent sets of two partition matroids can be decided in polynomial time, while for three or more matroids, the task becomes PSPACE-complete. Since many combinatorial optimization problems can be phrased in terms of finding maximum common independent sets of partition matroids, there are many interesting applications of these results.

Ruth Haas (University of Hawaii, USA): Reconfiguration of Dominating Sets

(Joint work with Seyffarth)

A subset S of the vertices of a graph is a dominating set if and only if every other vertex is adjacent to a vertex of S. The domination number, $\gamma(G)$, is the minimum cardinality of a dominating set of G, and the upper domination number, $\Gamma(G)$, is the maximum cardinality of a minimal dominating set of G. Various models of domination reconfiguration have been studied, e.g. by Subramaniam, Sridharan, and Fricke and by Hedetniemi, Hedetniemi, and Hutson; in addition, the γ-graph uses only γ sets and token jumping.

The goal is to find $d_0(G)$, the smallest value of k such that the reconfiguration graph is connected for all $k \geq d_0(G)$. The authors showed in 2014 that $d_o(G) = \Gamma(G) + 1$ for bipartite graphs and chordal graphs [6], with further results obtained by Suzuki, Mouawad, and Nishimura. Alikhani, Fatehi, and Klavzar considered which graphs might be reconfiguration graphs.

In the talk, the following new results were presented:

- All independent dominating sets are in the same connected component of the reconfiguration graph for $k = \Gamma(G) + 1$, where an independent dominating set is a maximal independent set.
- If G is both perfect and irredundant perfect, then $d_0(G) = \Gamma(G) + 1$, where a set is irredundant if every vertex in the set has a private neighbour, and precise definitions of the terms relate sizes of independent sets and clique cover numbers.
- For certain classes of well-covered graphs, $d_0(G) = \Gamma(G) + 1$, where a graph is well-covered if every maximal independent set has the same cardinality.

Open Problem Presentations

Henning Fernau (Universität Trier, Germany) Various extensions and generalizations of reconfiguration were proposed, as detailed below.

Building on ideas on generalizing the notion of adjacency, Fernau proposed approaching problems by determining what definition of adjacency would result in a connected reconfiguration graph. It was noted that for many problems there is a trivial solution, such as in the case of vertex cover, when all intermediate solutions are vertex covers of any size (a vertex cover V_1 can be reconfigured to a vertex cover V_2 by adding all vertices in V_2 and then deleting all vertices in V_1).

Another proposal was to consider the relationship between reconfiguration and reoptimization, where the goal is to determine a solution for an instance to a problem given a "similar" instance and its solution. Combining and generalizing the concepts could lead to the notion of a sequence of steps that changes both the instance and the solution.

It was further observed that although many reconfiguration problems turn out to be computationally difficult, it may be beneficial to focus on results in areas where hardness is a positive attribute. One example could be taken from computational social choice, considering how easy it is for an election to be rigged.

Other possible topics include the reconfiguration of problems in computational geometry as well as that of string editing.

Tatsuhiko Hatanaka (Tokohu University, Japan): Optimizing a Colouring via a Reconfiguration Sequence In this presentation, a new variant of reconfiguration was introduced. In the optimization variant, the goal is to find the optimal solution over all solutions reachable from a given solution. In particular, for the problem of colouring reconfiguration, an optimal solution is one that minimizes the number of colours used.

Preliminary results include a proof that the problem is NP-hard when the number of colours is at least five, and a polynomial-time algorithm for numbers of colours no greater than three.

Jan van den Heuvel (London School of Economics and Political Science, UK) The focus of this presentation was the k-colouring reconfiguration problem, considering both the reachability problem and the connectivity problem. The reachability problem can be solved in polynomial time when the number of colours, k, is at most three, and is PSPACE-complete otherwise (as a consequence of [8]).

For the connectivity problem, the answer is always no when $k = 2$ and is co-NP-complete when $k = 3$. The complexity remains open for $k \geq 4$. There followed a discussion of the co-NP-completeness result, and in particular the folding operation, where a vertex has two nonadjacent neighbours that can be folded together. It can be shown that a bipartite instance G is a no-instance of the connectivity problem for $k = 3$ if and only if G can be folded to a cycle on six vertices. One might wish to try to generalize this to $k = 4$ by showing that G can be folded to a cube; however, this is not the case. Further work is required on this problem, perhaps using another technique, and also perhaps by considering techniques for nonbipartite graphs.

Nicolas Bousquet (Institut Polytechnique de Grenoble, France): Graph Recolouring on Sparse Classes of Graphs Based on a conjecture by Cereceda in 2007, one possible approach to characterizing the reconfiguration graph for k-colouring is by considering the degeneracy of the graph and using that to determine the diameter of the reconfiguration graph. In particular, the conjecture states that the reconfiguration graph is of quadratic diameter when the number of colours is two greater than the degeneracy of the input. The conjecture has been shown to be true for k-regular graphs, trees, chordal graphs, bounded treewidth graphs, and distance-hereditary graphs.

So far, all the results have used nothing more than a tree decomposition of the input graph, raising the question as to whether other techniques might be applied.

Another question concerns the relationship between mixing time and the diameter of the reconfiguration graph, and in particular a characterization of classes of graphs for which the diameter of the reconfiguration graph is linear. Results are known for trees and k-degenerate graphs.

Kunihiro Wasa (National Institute of Informatics, Japan)

This talk focused on the reconfiguration of optimal amidakujis. An amidakuji, also known as a ladder lottery, is a way of generating a permutation by creating a series of vertical and horizontal bars. The number of vertical bars is the number of items to be permuted. The placement of horizontal bars joining adjacent vertical bars dictates the permutation, where from the top of each vertical bar, a path is traced following all horizontal bars encountered. An amidakuji is optimal if it uses the minimum number of bars to achieve the particular permutation.

The talk presented an algorithm that can be used to determine the shortest reconfiguration sequence between optimal amidakujis, and presented open problems on various variants.

Ryuhei Uehara (Japan Advanced Institute of Science and Technology (JAIST), Japan)

Starting with a general overview of the complexity of various games and puzzles, it was observed that most one-player games are NP-complete whereas most two-player games are PSPACE-complete or EXP-complete [7]. As an exception to this general rule, many sliding block puzzles, in which it is possible to return to the same state, are PSPACE-complete.

Uehara is the director of the JAIST Gallery, which houses NOB's Puzzle Collection, a collection of about 10,000 puzzles from around the world. Although he did not bring the entire collection to share with us, he did share his latest purchase, the Qubigon, which is a generalization of the well-studied slide-block 15-puzzle. The observation that the puzzle makes use of 18 of the 20 possible locations at once led to questions about the relationship between the number of tokens and the number of locations more generally. (This was later addressed in a presentation by van den Heuvel.)

An additional open problem that was discussed related to token sliding for independent sets.

Scientific Progress Made

Matching Shortest Transformation

Many techniques have been developed in the last few years in order to design polynomial time algorithms to determine the existence of a transformation between two configurations (reachability). On the other hand, very few of them have been proposed to compute shortest transformations between two configurations. One of the main reasons is that the shortest transformation problem seems to be much harder than the reachability problem even if only little evidence has been found so far in that direction (i.e. we need a very simple reconfiguration problem in order to obtain a polynomial-time algorithm for the shortest transformation problem). One of our motivations was to find problems on which the shortest transformation problem is hard while the reachability problem is very simple. Another motivation consisted in developing new tools to prove that the shortest transformation problem is hard.

During the workshop, Nicolas Bousquet, Tatsuhiko Hatanaka, Takehiro Ito and Moritz Muehlenthaler found preliminary results on the hardness of shortest transformation for the matching problem on graphs of maximum degree 4. In particular it implies that the shortest transformation is hard for the reconfiguration of independent sets on the token jumping and token sliding variants on line graphs. It answers a question of Ryuhei Uehara on the complexity of the TJ problem on claw-free graphs raised during the workshop. The proof technique might be generalized in order to show that there is no constant factor approximation algorithm for the Matching Shortest Transformation problem.

Equitable Colouring Reconfiguration

A *colouring* of a graph is an assignment of colours to its vertices so that no two adjacent vertices have the same colour, whereas an *equitable colouring* of a graph is a colouring where the difference between the number of vertices coloured by any two colours is at most one. Equitable colourings have many real-world applications in scheduling, load-balancing, and timetabling, and has been the subject of much research interest. The question of whether one colouring can be transformed to another colouring by changing the colour of only one vertex at each step has also been studied in the field of reconfiguration problems and has been proved to be PSPACE-complete.

Tatsuhiko Hatanaka, Haruka Mizuta, and Krishna Vaidyanathan considered the reconfiguration of equitable colourings, showing that the problem is PSPACE-complete, even for planar graphs using only four colours.

String Editing Reconfiguration

Discussions among Henning Fernau, Ruth Haas, Matthew Johnson, Naomi Nishimura, and Karen Seyffarth focused on the issues surrounding solution spaces in which each solution consists of a sequence (in this case, a sequence of edit operations). Preliminary observations included the fact that a reconfiguration sequence is itself a solution in the form of a sequence, allowing for the discussion of the metaproblem of reconfiguration of reconfiguration.

Additional observations found connections to work on permutations, including Kunihiro Wasa's open problem presentation on amidakujis.

Homomorphism Reconfiguration

Vijay Subramanya and Ben Moore considered the homomorphism reconfiguration problem for graphs on at most four vertices. They showed that H-reconfiguration is in P when H is C_4 or the diamond graph.

Furthermore, they conjectured that if H_1- and H_2-reconfiguration are in P, then H-reconfiguration is also in P, where H is obtained by joining H_1 and H_2 at a vertex. They proved the conjecture when H_1 and H_2 are both equal to C_3. This result extends to the case where H is a series of C_3's.

NCL and Matroids

Two open problems presented by Akira Suzuki about restricted variants of nondeterministic constraint logic can be rephrased in terms of connectivity of common independent sets of two partition matroids. Since NCL is a tool which has been developed for proving hardness of puzzles, this is quite an unexpected connection.

Outcome of the Meeting

The objectives of the workshop were all met, as elaborated below; further outcomes are listed in Section 2.

Providing an Opportunity for Joint Discussion by Researchers in Reconfiguration from All over the World

The workshop was very successful in bringing together researchers from different countries and different research groups. Unfortunately, funding and visa issues resulted in last-minute cancellations by researchers from India and Lebanon.

In the interests of maintaining our world-wide inclusiveness, in our business meeting we decided to try to alternate the continents on which future workshops will be held, with the 2019 workshop to be held in France.

Identifying Future Research Directions

The many open problem sessions and follow-up discussions generated many future research directions. As detailed in Section 2, progress has already been made in several of these directions.

Deepening the Area by Establishing a Set of Common Methods and Algorithmic Techniques

At the business meeting we discussed possible approaches to collecting and displaying known results. The prototype of a website was shown and discussed.

Broadening the Area by Making Connections to Related Areas and Problems

Several of the presentations introduced the audience to new applications of reconfiguration, such as the application of colouring to the Potts model (Section 2). In other talks, unexpected connections were discovered, such as between ladder lotteries, string editing, and permutations (Sections 2 and 2) and between NCL and matroids (Section 2).

Additional Outcomes

In addition to the planned outcomes, to further the research community we are planning to hold a minisymposium on reconfiguration at the 6^{th} biennial Canadian Discrete and Algorithmic Mathematics Conference in 2017, and to have a special issue on reconfiguration in the journal Algorithms.

Participants

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Chapter 3

Brain Dynamics and Statistics: Simulation versus Data (17w5036)

February 27- March 3, 2017

Organizer(s): Susanne Ditlevsen (University of Copenhagen), Priscilla Greenwood (University of British Colombia), Andre Longtin (University of Ottawa)

Motivation

The human brain is arguably the most complex biological structure. Making sense of its inner workings poses an enormous challenge. Different components of its activity evolve over periods ranging from one thousandth of a second to years, and over distances of one millionth of a meter to the size of the skull. Experiments perturb and probe activity over these scales of time and space, yielding reams of data. Our cohort of scientists is leading the effort to organize these data into a coherent picture using state-of-the-art mathematical modelling techniques. This picture encompasses deterministic and random components. This is a consequence of the fact that part of the brain's activity seems to evolve in a predictable structured manner, while other parts are unpredictable, even when responding to the same stimulus or initiating the same action.

The goal of this workshop was to develop methods to characterize these components, how they interact with each other, and how this interaction changes when stimuli impinge on the brain or the brain acts upon its environment. The techniques and advances that were discussed are at the forefront of two fields that seldom come together, namely dynamics and statistics, which respectively address the predictable and unpredictable aspects of the brain's behaviour. The rising and established experts brought together by this workshop aim to gain a deeper understanding of the brain's operational principles in the presence of variability, an understanding grounded in quantifiable "statistical" certainty. The group is also driven by potential novel directions in mathematics that the intersection of concepts from dynamics and statistics will offer.

Overview of the topic

The primary goal of the workshop was to inspire collaboration among researchers in theoretical and experimental neuroscience, by discussing statistical methods, stochastic modeling tools, and experimental possibilities.

The data come from intra- and extracellular recordings of single neurons, revealing both spiking and subthreshold patterns, local field potential data, spatio-temporal measurements which show us spatial patterns with the extra dimension of time, EEG and fMRI data. Specific questions are: What are the important modes of information transmission in the brain? Does it occur via spiking patterns, or time averages of patterns? Are various rhythms also important? Surprisingly little is known. We know about some simple aspects of the operation of primary visual cortex, such as how single cells code for edges or movement at different angles; but the representation of the many features that make up a stimulus across different spatio-temporal scales of activity is an unsolved puzzle.

In order to make progress we needed to bring together people who gather neural data and understand lab work, people who have an extensive but usually specialized current knowledge about the brain, people who work in the mathematics of dynamical systems, particularly stochastic dynamical systems, and people who have facility in the statistics of stochastic processes. The common requirement is that these people are devoted to understanding the dynamics of the brain.

Despite a considerable recent body of literature on statistical inference and stochastic modeling in neuroscience, there remain substantial unsolved problems and challenges, some of which we addressed during the workshop. Examples are: understanding the role of stochasticity in the brain, developing efficient Monte Carlo methods for inference, understanding the relationship between system behavior (i.e. bifurcations and their stochastic counterparts) and statistical properties of estimates based on data from these systems, and reconstructing the structure of network models from partially observed data.

The workshop discussed the way data are processed and understood in the biological sciences, from a dynamical stochastic approach. Neuroscience is experiencing an imbalance between prowess in data collection and deficiency in data interpretation. The improved models for data processing are expected to be of significant theoretical interest from biophysical, mathematical and statistical viewpoints. Our work contributes to all parts of neuroscience, from single cell analysis to the understanding of neuronal network dynamics, and potentially beyond.

Main subjects of the workshop

- The role of stochastic dynamics in the Brain
  - Single neuron models
  - Mean field models
  - Network models
  - Hypoelliptic models and the statistical challenges
- Information transmission, encoding and decoding
- Applications of neural modeling to brain physiology
  - Control of Oscillatory Brain Dynamics
  - EEG data during anaesthesia in humans
  - Artifical neural network, biological instantiation of processing
  - The weakly electrical fish
  - Red-eared turtle

Presentation Highlights

We next give an overview of the presentations within the overall headings above, together with abstracts.

The role of stochastic dynamics in the Brain

Peter Thomas: Noise in the Brain: Statistical and Dynamical Perspectives There is growing interest in applying statistical estimation methods to dynamical systems arising in neuroscience. The discipline of statistics provides an intellectual framework for quantifying and managing uncertainty. For statistical methods to apply, one must consider a system with some variability. Depending on where one locates the source of variability, different methods suggest themselves. The talk will discuss some challenges and opportunities in linking statistical and mathematical perspectives in theoretical neuroscience, including the problem of quantifying ”phase resetting” in stochastic oscillators, the problem of identifying the most significant sources of noise in a finite state Markov model, the problem of inferring control mechanisms in brain-body motor control systems, and the problem of parameter identification in noisy conductance-based models.

Adeline Samson: Hypoelliptic stochastic FitzHugh-Nagumo neuronal model: mixing, up-crossing and estimation of the spike rate Joint work with J.R. Leon (Universidad Central de Venezuela). In this presentation, we will introduce the hypoelliptic stochastic neuronal model, as diffusion approximation of intra-cellular or extra-cellular models. Then we will specifically focus on the FitzHugh-Nagumo model, and detail some of its properties: stationary distribution, definition of spikes, estimation of spiking rate.

Eva Löcherbach: Memory and hypoellipticity in neuronal models I will discuss the effect of memory in two classical models of neuroscience, first the stochastic Hodgkin-Huxley model, considered in a series of papers by R. Höpfner, M. Thieullen and myself, and second, a model of interacting Hawkes processes in high dimension which has been studied recently by S. Ditlevsen and myself. I will explain why memory leads to hypoellipticity and then discuss the probabilistic and statistical consequences of this fact.

Antoni Guillamon: At the crossroad between invariant manifolds and the role of noise I will drift along several topics in which we are currently involved. They all contain some treatment of recent tools from dynamical systems, mainly related to invariant manifolds, intertwined with the role of stochasticity. We aim at sharing our research interests to boost discussion and eventual collaborations. The first example will be on bistable perception where we are exploring up to which extent noise is necessary to explain data obtained from psycophysical experiments, as opposed to the information provided assuming quasi-periodic forcing. Second, we will focus on phase response curves (PRC) for oscillating neurons, a well-known tool to study the effectiveness of information transmission between cells; we have extended the notion of PRC to that of phase response field/function (PRF), also valid away from pure periodic behaviour. Stochastic PRCs have been already studied but PRFs constitute a more natural setting when neurons are subject to noisy inputs; we have neither explored the impact of noise on PRFs nor the extension of the concept to stochastic processes yet and, moreover, we also seek for experimental tests of our theoretical findings. Therefore, we think it can be interesting to address these open questions in this audience. If time allows, we will comment on some other issues (estimation of conductances and short-term synaptic depression) that combine concepts of dynamical systems with the presence of noise.

Zachary Kilpatrick: Maintaining spatial working memory across time in stochastic bump attractor models We discuss various network mechanisms capable of making spatial working memory more robust to noise perturbation and error. The canonical example we begin with arises from classic oculomotor delayed response tasks whereby a subject must maintain the memory of a location around a circle over the period of a few seconds. Asymptotic methods are used to reduce the dynamics of a bump attractor to a stochastic differential equation whose dynamics are governed by a potential that reflects spatial heterogeneity in the network connectivity. Heterogeneity can serve to reduce the degradation of memory overtime, ultimately increasing the transfer of information forward in time. We also show that connectivity between multiple layers of a working memory can further serve to stabilize memory, especially if they possess propagation delays. We conclude by discussing recent work, where we are modeling the phenomenon whereby a previous trials response attracts the current trials response, sometimes called repetition bias.

Lawrence Ward: Pattern formation via stochastic neural field equations The formation of pattern in biological systems may be modeled by a set of reaction-diffusion equations. A form of diffusion-type coupling term biologically significant in neuroscience is a difference of Gaussian functions used as a space-convolution kernel. Here we study the simplest reaction-diffusion system with this type of coupling. Instead of the deterministic form of the model, we are interested in a stochastic neural field equation, a space-time stochastic differential-integral equation. We explore, quantitatively, how the parameters of our model that measure the shape of the coupling kernel, coupling strength, and aspects of the spatially-smoothed space-time noise, control the pattern in the resulting evolving random field. We find that a spatial pattern that is damped in time in a deterministic system may be sustained and amplified by stochasticity, most strikingly at an optimal space-time noise level.

Laura Sacerdote: Integrate and Fire like models with stable distribution for the Interspike Intervals In 1964, Gernstein and Mandelbrot [1] proposed the Integrate and Fire model to account for the observed stable behavior of the Interspike Interval distribution. Their study of histograms of ISIs revealed the stable property and they suggested modeling the membrane potential through a Wiener process in order to get the inverse Gaussian as first passage time distribution, i.e. a stable distribution. Holden (1975) [2] observed that stable distributions determine a simple transmission pathway. Later many variants of the original model appeared with the aim to improve its realism but meanwhile researches forgot the initial clue for the model. The Leaky Integrate and Fire model that has not stable FPT distribution gives an example. The same holds for many other variants of this model. More recently Persi et al. (2004) [3] studying synchronization patterns, proposed a time non homogeneous integrate and fire model accounting for heavy tail distributions. The existence of heavy tails, typical of stable distributions is well recognized in the literature (see for example Tsubo et al [4] , Gal and Morom [5] and references cited therin) Signals from different neurons are summed up during the elaboration. Different ISIs distributions would determine an incredible variety of firing distributions as the information progresses in the network. However, it seems unrealistic to admit ISIs that cannot reproduce the same distribution when summed. This suggest the development of Integrate and Fire models using stable distributions. Furthermore, the stable ISIs paradigm gives rise to a more robust transmission algorithm since a possible lack of detection of some spike from the surrounding neurons does not change the nature of the final distribution. Here we rethink to the problem, taking advantage of the mathematical progresses on Lévy processes [6]. Hence, we propose to start the model formulation from the main property, i.e. the stable nature of the ISIs distribution. We follow the Integrate and Fire paradigm but we model the membrane potential through a randomized random walk whose jumps are separated by intertimes with stable distribution (or in the domain of attraction of such distribution). Observing that the supremum of the modeled membrane potential results to be the inverse of a stable subordinator allows to determine the Laplace transform of the ISIs distribution. This is a preliminary contribution since we limit ourselves to some aspects of the modelling proposal ignoring any attempt to fit data. We are conscious that these are preliminary results and some further mathematical study will be necessary to obtain models fitting data. In particular we expect an important role of tempered stable distributions [7]. [1] Gerstein G.L., Mandelbrot B. (1964) Random walk models for the activity of a single neuron. Biophys. J. 4: 41-68. [2] Holden, A.V. (1975) A Note on Convolution and Stable Distributions in the Nervous System. Biol. Cybern. 20: 171-173. [3] Persi E., Horn D., Volman V., Segev R. and Ben-Jacob E. (2004) Modeling of Synchronized Bursting Events: the importance of Inhomogeneity. Neural Computation 16: 2577-2595. [4] Tsubo Y., Isomura Y. and Fukai T. (2012) Power-Law Inter-Spike Interval Distributions infer a Conditional Maximization of Entropy in Cortical Neurons PLOS Computational Biology 8, 4: e1002461. [5] Gal A., and Marom S. (2013) Entrainment of the Intrinsic Dynamics of Single Isola Neurons by Natural-Like Input The Journal of Neurosciences 33(18), 7912-7918. [6] Kyprianou, A. (2014) Fluctuations of LÃ©vy Processes with Applications. Springer Verlag, Berlin/Heidelberg. [7] Rosinski, J. (2007) Tempering stable processes. Stochastic Processes and their Applications. 117, 6: 677-707

Jonathan Touboul: Noise in large-scale neuronal networks, brain rhythms and neural avalanches It is now folklore that intracellular membrane potential recordings of neurons are highly noisy, owing to a variety of random microscopic processes contributing to maintenance. At macroscopic scales, reliable, fast and accurate responses to stimuli emerge, and are experimentally described through various quantities. I will focus in particular on mathematical models of large neuronal networks can account for the type of experimental data observed in synchronized oscillations (sources of brain rhythms) or neural avalanches. These two quantities are relevant in that rhythms reportedly support important functions such as memory and attention, while distributions of avalanches were reported to reveal that the brain operates at criticality where it maximizes its information processing capacities. I will show that a simple theory of large-scale dynamics allows understanding under a common framework both phenomena. In particular, I will show the relatively paradoxical phenomenon that noise can contribute to synchronization of large neural assemblies, and that the experimentally reported heavy-tailed distributions of avalanche durations and sizes may be in fact related to Boltzmanns molecular chaos that naturally emerges in limits of large interacting networks with noise. These works were developed with Alain Destexhe and Bard Ermentrout.

Janet Best: Variability and regularity in neurotransmitter systems In this talk I will discuss a couple of recent themes in my work: (1) Population models of deterministic neural systems enable one to capture the biological variability in individuals. These models can explain how the same circuits can operate differently in different individuals or can change in time in individuals because of dynamic changes in gene expression. They can be used to discover the characteristics of subpopulations in which drugs are efficacious or deleterious and are therefore useful for clinical trials; (2) In volume transmission neurons don't engage in one on one transmission but project changes in biochemistry over long distances in the brain. Thus, the nuclei containing the cell bodies (like the dorsal raphe nucleus for serotonin) are basically acting as endocrine organs. Recent work with Sean Lawley shows why volume transmission works, that is why concentrations of neuromodulator in the extracellular space are very even despite the fact that terminals and varicosities are distributed unevenly and firing may be random.

Shigeru Shinomoto: Emergence of cascades in the linear and nonlinear Hawkes processes The self-exciting systems as represented by neural networks are known to exhibit catastrophic chain reaction if the internal interaction exceeds an epidemic threshold. Recently, we have shown that the same systems may exhibit nonstationary fluctuations already in the subthreshold regime [1,2]; cascades of events or spikes may emerge in the Hawkes process even in the absence of external forcing. In a practical situation, however, systems are subject to time-varying environment in addition to internal interaction between elements. Here we attempt to estimate the degree in which the nonstationary fluctuations occurring in the system are induced by external forcing or internal interaction. [1] T. Onaga and S. Shinomoto, Bursting transition in a linear self-exciting point process. Physical Review E (2014) 89:042817. [2] T. Onaga and S. Shinomoto, Emergence of event cascades in inhomogeneous networks. Scientific Reports (2016) 6:33321.

Romain Veltz: Quasi-Synchronisation in a stochastic spiking neural network I will discuss a recent mean field (from E. Löcherbach and collaborators, R. Robert and J.Touboul) of a stochastic spiking neural network from a dynamical systems perspective. More precisely, I will present some recent results concerning the quasi-synchronisation of the neurons as function of the different parameters of the network (gap junction conductances, synaptic strength...).

Information transmission, encoding and decoding

Massimiliano Tamborrino: Novel manifestation of the noise-aided signal enhancement We discuss various type of noise-aided signal enhancements. Since not all stimulus levels can be decoded with the same accuracy, it is of paramount interest to determine which stimulus intensities can be discriminated most precisely. It is well known that the presence of noise corrupts signal transmission in linear systems. Nevertheless, noise may have a positive effect on signal processing in nonlinear systems, as confirmed by the stochastic resonance phenomenon. Stochastic resonance is typically observed in systems with a threshold in presence of a weak signal. However, the subthreshold regime is not a necessary condition when considering more than one neuron, since, for example, a suprathreshold signal may be also enhanced by noise in a network of threshold devices. Other phenomena where noise enhances the signal are for example coherence resonance and firing-rate resonance. We present a study of the decoding accuracy of the stimulus level based on either the first-spike latency coding or the rate coding (from the exact spike counting distribution) in a neuronal model as simple as the perfect integrate-and-fire model. We report counter-intuitive results, representing a novel manifestation of the noise-aided signal enhancement which differs fundamentally from the usual kinds reported on.

Benjamin Lindner: Spontaneous activity and information transmission in neural populations In my talk I will first review features of the spontaneous activity of nerve cells in neural populations and in recurrent networks, ranging from effects of cellular properties (e.g. noise and leak currents) and slow external noise (up/down transitions in a driving population) to the slow fluctuations that can build up due to the recurrent connectivity. I will then discuss how these features affect information transmission and stimulus detection, for instance, enable or suppress information filtering, benefit overall population coding, or lead to the detection of a short single cell stimulation in a large recurrent network.

Richard Naud: Burst ensemble multiplexing: connecting dendritic spikes with cortical inhibition Two distinct types of inputs impinge on different spatial compartments of pyramidal neurons of the neocortex. A popular view holds that the input impinging on the distal dendrites modulates the gain of the somatic input encoding. This gain modulation is thought to participate in top-down processes such as attention, sensory predictions and reward expectation. Here we use computational and theoretical analyses to determine how the two input streams are represented simultaneously in a neural ensemble. We find that dendritic calcium spikes in the distal dendrites allows multiplexing of the distal and somatic input streams by modifying the proportion of burst and singlet events. Two ensemble-average quantities encode the distal and somatic streams independently: the event rate and the burst probability, respectively. Simulations based on a two-compartment model reveal that this novel neural code can more than double the rate of information transfer over a large frequency bandwidth. To corroborate these findings, we determined analytically the parameters regulating mutual information in a point process model of bursting. Secondly, we find that an inhibitory microcircuitry combining short-term facilitation and short-term depression can decode the distal and somatic streams independently. These results suggest a novel functional role of both active dendrites and the stereotypical patterns with which inhibitory cell types interconnect in the neocortex. Burst ensemble multiplexing, we suggest, is a general code used by the neural system to flexibly combine two distinct streams of information.

Applications of neural modeling to brain physiology

Jeremie Lefebvre: State-Dependent Control of Oscillatory Brain Dynamics Numerous studies have shown that periodic electrical stimulation can be used not only to interfere with the activity of isolated neurons, but also to engage population-scale synchrony and collective rhythms. These findings have raised the fascinating prospect of manipulating emergent brain oscillations in a controlled manner, engaging neural circuits at a functional level to boost information processing, manipulate cognition and treat neurobiological disorders (so called oscillopathies). Capitalizing on this, it has been shown that brain stimulation can be tuned to alter perception and task performance. Rhythmic brain stimulation forms the basis of a control paradigm in which one can manipulate the intrinsic oscillatory properties of cortical networks via a plurality of input-driven mechanisms such as resonance, entrainment and non-linear acceleration. But the brain is not a passive receiver: outcomes of brain stimulation, either intracranial or non-invasive, are highly sensitive to ongoing brain dynamics, interfering and combining with internal fluctuations in non-trivial ways. Exogenous control on brain dynamics has indeed been shown to be gated by neural excitability, where effects of brain stimulation are both state-dependent and highly sensitive to stimulation parameters. To understand this phenomenon, we here used computational approach to study the role of ongoing state on the entrainment of cortical neurons. We examined whether state-dependent changes in thalamo-cortical dynamics could implement a gain control mechanism regulating cortical susceptibility to stimulation. We found that the resulting increase in irregular fluctuations during task states enables a greater susceptibility of cortical neurons to entrainment, and that this phenomenon can be explained by a passage through a bifurcation combined to stochastic resonance. We also investigated the relationship between the stimulation parameters, such as amplitude and frequency, on entrainment regimes for different levels of sensory input. Taken together, our results provide new insights about the state-dependent interaction between rhythmic stimulation and cortical activity, accelerating the development of new paradigms to interrogate neural circuits and restore cognitive functions based on the selective manipulation of brain rhythms.

Rune W Berg: Neuronal population activity involved in motor patterns of the spinal cord: spiking regimes and skewed involvement Motor patterns such as chewing, breathing, walking and scratching are primarily produced by neuronal circuits within the brainstem or spinal cord. These activities are produced by concerted neuronal activity, but little is known about the degree of participation of the individual neurons. Here, we use multi-channel recording (256 channels) in turtles performing scratch motor pattern to investigate the distribution of spike rates across neurons. We found that the shape of the distribution is skewed and can be described as log-normal-like, i.e. normally shaped on logarithmic frequency-axis. Such distributions have been observed in other parts of the nervous system and been suggested to implicate a fluctuation driven regime (Roxin et al J. Neurosci. 2011). This is due to an expansive nonlinearity of the neuronal input-output function when the membrane potential is lurking in sub-threshold region. We further test this hypothesis by quantifying the irregularity of spiking across time and across the population as well as via intra-cellular recordings. We find that the population moves between supra- and sub-threshold regimes, but the largest fraction of neurons spent most time in the sub-threshold, i.e. fluctuation driven regime. Read more about this work here: Peter C Petersen, Rune W Berg "Lognormal firing rate distribution reveals prominent fluctuation driven regime in spinal motor networks" eLife 2016;5:e18805

Axel Hutt: Model and prediction of anaesthetic-induced EEG The monitoring of patients under general anaesthesia is an essential part during surgery. To interpret correctly measured brain activity, such as electroencephalogram (EEG), it is essential to first understand the possible origin of EEG and hence classify the physiological state of the patient correctly. Moreover, it would be advantageous to even predict the development of the brain activity to anticipate severe changes of the physiological state. The talk presents a novel model explaining major EEG features under light anaesthesia, such as the spectral smile effect, by a denoising of brain dynamics. In a second part, the talk shows how to predict EEG under anaesthesia by applying a data assimilation technique.

Mark McDonnell: What can we learn from deep-learning? Models and validation of neurobiological learning inspired by modern deep artificial neural networks In the field of machine learning, deep-learning has become spectacularly successful very rapidly, and now frequently achieves better-than-human performance on difficult pattern recognition tasks. It seems that the decades-old theoretical potential of artificial neural networks (ANNs) is finally being realized. For computer vision problems, convolutional ANNs are used, and are often characterized as biologically inspired. This is due to the hierarchy of layers of nonlinear processing units and pooling stages, and learnt spatial filters resembling simple and complex cells. However, this resemblance is superficial. An open challenge for computational neuroscience is to identify whether the spectacular success of deep-learning can offer insights for realistic models of neurobiological learning that are constrained by known anatomy and physiology. I will discuss this challenge and argue that we need to validate proposed neurobiological learning rules using challenging real data sets like those used in deep-learning, and ensure their learning capability is comparable to that of deep ANNs. To illustrate this approach, in this talk I will show mathematically how a standard cost-function used for supervised training of ANNs can be decomposed into an unsupervised decorrelation stage and a supervised Hebbian-like stage. With this insight, I argue that this form of learning is feasible as a neurobiological learning mechanism in recurrently-connected layer 2/3 and layer 4 cortical neurons. I will further show that the model can learn to very effectively classify patterns (e.g. images of handwritten digits from the MNIST benchmark); error rates are comparable to state of the art deep-learning algorithms, i.e. less than 1.

Jonathan D. Victor: How high-order image statistics shape cortical visual processing Several decades of work have suggested that Barlow's principle of efficient coding is a powerful framework for understanding retinal design principles. Whether a similar notion extends to cortical visual processing is less clear, as there is no "bottleneck" comparable to the optic nerve, and much redundancy has already been removed. Here, we present convergent psychophysical and physiological evidence that regularities of high-order image statistics are indeed exploited by central visual processing, and at a surprising level of detail. The starting point is a study of natural image statistics (Tkacic et al., 2010), in which we showed that high-order correlations in certain specific spatial configurations are informative, while high-order correlations in other spatial configurations are not: they can be accurately guessed from lower-order ones. We then construct artificial images (visual textures) composed either of informative or uninformative correlations. We find that informative high-order correlations are visually salient, while the uninformative correlations are nearly imperceptible. Physiological studies in macaque visual cortex identify the locus of the underlying computations. First, neuronal responses in macaque V1 and V2 mirror the psychophysical findings, in that many neurons respond differentially to the informative statistics, while few respond to the uninformative ones. Moreover, the differential responses largely arise in the supragranular layers, indicating that the computations are the result of intracortical processing. We then consider low- and high-order local image statistics together, and apply a dimension-reduction (binarization) to cast them into a 10-dimensional space. We determine the perceptual isodiscrimination surfaces within this space. These are well-approximated by ellipsoids, and the principal axes of the ellipsoids correspond to the distribution of the local statistics in natural images. Interestingly, this correspondence differs in specific ways from the predictions of a model that implements efficient coding in an unrestricted manner. These deviations provide insights into the strategies that underlie the representation of image statistics.

Volker Hofmann: Population coding in electric sensing: origin and function of noise correlations In many cases, great knowledge regarding single neuron activity during the encoding of sensory signals or the generation of behavioral outputs has been achieved. On another scale, however, we still lack detailed information of the mechanisms of concerted neuronal activity, i.e. population coding. This is crucial to understand the neuronal code and remains a central problem in neuroscience. Extrapolating the knowledge of single unit activity to the scale of a neuronal population is often complicated by the fact that the activities of neurons are typically correlated rather than being independent. Such correlations, which can arise in terms of the average responses to different stimuli as well as in terms of trial to trial variability, were shown to substantially impact the efficacy of population codes. To investigate the sources and function of correlations we use the weakly electric fish Apteronotus leptorhynchus as a model system, due to the wealth of physiological and anatomical knowledge that is available with regard to electrosensory processing. These fish sense electric fields with an array of electroreceptors that project to three parallel segments of the medullary electrosensory lateral line lobe (ELL). Previous studies established, that the size and the organization of receptive fields differs substantially between pyramidal neurons in the different segments, which should consequently result in very different amounts of correlations in each segment. In contrast to this, our experimental recordings revealed very similar levels of correlation magnitudes. To explain this surprising result, we investigated the differential receptive field interactions using a modeling approach. Considering the antagonistic center-surround organisation of receptive fields, we were able to show that very different receptive field organization can give rise to very similar amounts of correlations. After establishing the presence of noise correlations in the ELL, we assessed their potential impact and function for the encoding of electrosensory stimuli. Our preliminary results suggest that ELL noise correlations encode stimuli independent of classical measures of neuronal activity (i.e. firing rate). The stimulus dependent changes in correlation levels are potentially modulated via the recurrent ELL connectivity which will be a focus of our future investigations to unravel the mechanisms mediating this independent additional channel of information transmission in the brain.

Posters

Each poster presenter had 5 minutes to introduce their poster to the full workshop. During our 45 minutes breaks, the posters were discussed in smaller groups. Having the posters up for the full meeting gave ample time to discuss and study the posters.

- Catalina Vich: Different strategies to estimate synaptic conductances
- Timothy Whalen: Pallidostriatal Projections Promote Beta Oscillations in a Biophysical Model of the Parkinsonian Basal Ganglia
- Jacob Østergaard: Capturing spike variability in noisy Izhikevich neurons using point process GLMs
- Peter Rowat: Stochastic network thinking applied to firing patterns of stellate neurons
- Wilhelm Braun: Spike-triggered neuronal adaptation as an iterated first-passage time problem

- Mareile Grosse Ruse: Modeling with Stochastic Differential Equations and Mixed Effects
- Kang Li: Mathematical neural models for visual attention
- Pietro Quaglio: SPADE: Spike Pattern Detection and Evaluation in Massively Parallel Spike Trains
- Alexandre René: Dimensionality reduction of stochastic differential equations with distributed delays

Impact Reports from Participants

Participants were asked to write a few lines telling about what they learned. Below are individual reports, also giving hints at outstanding problems in the field, and new directions for future research.

Laura Sacerdote, Torino: The workshop was a great occasion to meet very interesting people in a wonderful environment. We had contributions from theoricians and experimentalists with a challenging alternance that allowed each one to learn from the others. I already knew some of the scientists attending to the workshop but the discussions after each talk, as well as the long coffee breaks, gave me the opportunity to know new people and to propose new scientific links for my research. The presence of a very mixed group during my talk, gave me hints for interesting improvements of my work. The size of the workshop was perfect, in a small group it becomes easy to start to talk with the others, learning from posters and talks but also from many private discussions.

Benjamin Lindner, Berlin: The schedule was much better than that of many other workshops I have attended: long talks with sufficient time for questions during and after the talks, long breaks for even more informal discussions. The relaxed schedule, the number and kind of participants, the way the sessions were chaired - all of this was encouraging for a creative atmosphere during this workshop that I enjoyed a lot.

I met old friends and colleagues (Susanne Ditlevsen, Axel Hutt, Jeremie Lefebvre, Andre Longtin, Mark McDonnell, Richard Naud, Laura Sacerdote, and Peter Thomas) and talked with them about old times and new research projects. I was happy to make new acquaintances (Wilhelm Braun, Cindy Greenwood, Volker Hofmann, Zachary Kilpatrick, Eva Löcherbach, Massimiliano Tamborrino, Jonathan Touboul, Romain Veltz, Jonathan Victor, Lawrence Ward), to learn about exciting new research directions, and to get a bit of feedback on my own research efforts too.

I was especially inspired by the mixture of presentations on fundamental math problems related to stochastic processes in neuroscience (stochastic dynamics of excitable systems) and talks that were devoted to more data-driven and biology-centered topics - I think this was an excellent mixture that met many of my research interests. I know from many conversations that I was not the only one who felt like that.

Thank you again for organizing this marvelous workshop.

Mariele Groβe Ruse, Copenhagen, Denmark: What I appreciated a lot was the atmosphere among our group and that I all the time felt very welcome and included. No matter the "academic status", all participants were open, happy to talk/discuss, happy to provide inputs/feedbacks/ideas and also to chat about topics beyond research. I got very good inputs regarding applicability of my model to different kinds of input, which is useful for my PhD project. I talked to people with various background, which I always perceive as very interesting. Definitely, I learned a lot, although there was as much I did not understand, due to me having not really a background in neuroscience.

Mark McDonnell, Australia: I enjoyed discussions with a lot of slightly more junior researchers who were familiar with my work on stochastic resonance from the time of my PhD and just after. This was an enjoyable experience to see the new directions these researchers are following in that field, and have the opportunity to discuss with them. I also enjoyed many discussions with people about their experiences with machine learning and how it might benefit their future research.

A potential collaboration is in progress with Jonathan Touboul, whom I met at the workshop for the first time.

And of course the workshop was an opportunity to continue work on existing collaborations with Cindy Greenwood and Lawrence Ward.

Zachary Kilpatrick, Boulder Colorado: Thanks so much for organizing this workshop! It was wonderful. My current work and future plans were impacted in several different ways from my experience at the BIRS workshop. One of the first conversations I had at the conference was over dinner with Andre Longtin, and he shared his experience in balancing research, teaching, and service with me, providing some useful ideas for how to use my time wisely in my career. Peter Thomas provided some excellent feedback after my talk on different information theoretic measures and error quantification I could use to better think about the way the brain might optimize working memory systems. Furthermore, he was gracious enough to give me extensive details on his recent work on asymptotic phase for stochastic systems, taking me through some special cases. I learned a lot! Other conversations with Jonathan Touboul, Benjamin Lindner, and Axel Hutt gave me excellent ideas for future projects. Wilhelm Braun took the time to ask me some of my own advice about how to proceed at his stage (as an early postdoc) in view of getting an academic job in the future. Furthermore, I had an excellent conversation with Susanne Ditlevsen about planning current and future meetings for the International Conference on Mathematical Neuroscience.

Peter Thomas, Case Western University: Benjamin Lindner and I, who had been working together previously, were able to complete and finally submit a paper to the Journal of Statistical Physics while we were at Banff together, and began work on our next paper. I also learned a great deal from the workshop talks. Adeline Samson and Eva Löcherbach's talks on Hypoelliptic stochastic systems gave me ideas for how to show that a certain kind of nerve cell "resets" (in a stochastic sense) each time it fires, which is something my biological collaborator has conjectured repeatedly but which we did not know how to formulate. Antoni Guillamon's talk give me new ideas about how to represent the phase and amplitude of stochastic oscillators, a long standing interest of mine. Rune Berg's talk gave me new ideas for a project with a different collaborator involving mathematical modeling of motor control systems. I benefitted from side conversations with many participants, including the three organizers (Greenwood, Ditlevsen, and Longtin). I particularly appreciated the chance to discuss data assimilation techniques with an expert in this area, Axel Hutt. And snowshoeing up the back side of Mount Sulphur in subzero temperatures with Zach and Wilhelm was a real treat.

Jonathan Victor, Cornell: First, Romain Veltz: following my presentation, we had a substantive discussion about new approaches to understanding cortical dynamics, which may provide a way to account for some of the puzzling features of the data I had discussed. (The basic idea here is that patches of visual cortex don't represent just an orientation (as is traditionally thought), but rather, three parameters – an orientation, a dispersion, and a scale – effectively a 2x2 symmetric matrix. The implications are that cortical dynamics don't live on a torus, but rather, on a stack of Poincare disks.) Second, Volker Hofmann: with whom I had a series of discussions on active sensation, and the parallels and contrasts between the electric sense in weakly electric fish, and vision.

But most importantly, I really want to thank you for inviting me to this very stimulating event.

Timothy Whalen, Carnegie Mellon University: As a younger PhD student, it was a great learning experience to meet with experts in neural data analysis from across the experimental-mathematical spectrum. Several of the talks and discussions in the poster hall inspired ideas on how to address difficult, quantitative questions in my data and better relate these to my simulations. The Banff Center proved to be a fantastic venue for this purpose, streamlining everything so that we attendees could focus on the content of the workshop. The fellow researchers I've connected with and the new ideas I've learned from the workshop will not soon be forgotten!

Lawrence Ward, UBC, Vancouver: I was very happy to meet and converse with Axel Hutt and Jonathan Touboul, having previously known of their work in this field. I of course learned many new things about brain dynamics from the talks, but also from my conversations with their authors. Again, the work of Hutt, Touboul, Lindner, Kilpatrick, Veltz and Thomas was very revealing, and in some cases new to me (esp. Hutt on effects of anesthesia, Kilpatrick on memory and bumps). All of this will stimulate my future work on stochastic neuron models as well as on brain dynamics.

Axel Hutt, INRIA, France: The workshop gathered most of the top-level researchers working in the research field. This allowed to learn directly the primary current hot topics in the field and who works on what. The rather informal meeting made it possible to chat with researchers at a beer on general and specific science what was elucidating in several cases. I would say the workshop was quite valuable scientifically and definitely worth making the long trip from Europe.

Kang Li, Copenhagen: It was a great experience for me, in terms of both academic studies and mountain viewing. I brought a poster and there were some who came by and discussed with me. Among them were Eva, Adeline, Laura, and some others but I can't remember their names. I presented my work and future ideas and received interesting and useful feedbacks. The posters and talks from participants are brilliant and inspiring, from parameter estimations of point & diffusion processes, to the synchronization and coherence of spike trains, and to the emerging application of deep neural networks, etc. I was trained as a statistician and have been focusing on model inference, but there are obviously many different areas beyond mine which I really learned a lot from.

Eva Löcherbach, Paris: Personally, I found it very interesting to meet Peter Thomas; in particular his question how to distinguish between variability that is intrinsically generated versus the one that is extrinsically generated is a problem that I would like to study in models such as Hawkes processes in high dimension, subject to mean-field interactions within a spatially structured system of neurons which are arranged in populations. I also would like to study phase response curves in such models - and this is certainly inspired by some of the talks presented at the conference. The talk of Adeline Samson was also very interesting for me, in particular her question how to decide whether there is presence or not of channel noise for example (can we write a real statistical test for this?). Other talks that inspired me a lot were those of Zachary Kilpatrick, of Shigeru Shinomoto and certainly also Benjamin Lindner. Finally, let me say that the posters that were presented were very interesting and that I really appreciated the way those posters were shortly presented during the conference.

Adeline Samson, Grenoble: This conference was very productive for me, starting a new project with Massimiliano Tamborrino around some good beers in the cafe!

Toni Guillamon, Barcelona: The facilities provided by BIRS and the Banff Center are excellent; staying there is so easy and confortable that you can keep your mind occupied with the scientific problems and, at the same time, you forget you are working. The information provided along the whole process is very clear and all is prepared to arrive at Banff and work efficiently. The lodging quality is excellent, as well as the food in the restaurant, and the leisure activities. Among them, I will remember our walk on the Tunnel Mountain, the excursion to the Sulphure mountain, and the chats during the meals and around a glass of wine (sorry for the alcoholic reference) at the "bistrot".

I find also worth to mention the job made by the organizers of the workshop, mainly in selecting the lectures and the attendants. They were able to put together a group of excellent persons, with common interests but non-coincident backgrounds, and with a sincere desire to understand each other's work. More personally speaking, I come from dynamical systems theory and I've been working mainly on deterministic treatment of brain dynamics with an increasing interest in statistics and stochastic processes. In this workshop, it has been very rewarding to see how my research could link to the interests of other participants, most of them experts on aspects that I do not know but I do need! At the end, I came back home with a long list of stimulating suggestions as well as a group of colleagues which whom I can eventually interact in the future. I wish I could say the same of every meeting I'm attending!

In virtue of my sincere words in the above paragraph, thanks again for organizing it.

Volker Hofmann, McGill University: 1. I met researchers that work in my immediate field (weakly electric fish) and approach common question with employing different methods (i.e. by theoretical means). I am happy to have met Andre Longin and Benjamin Lindner at this work shop that allowed me to discuss a few things regarding recent and coming publications in the field. In addition to that I have had conversations with a number of researchers that share interests in certain more general research questions and use different animal models and/or methods. Also seeing how these people combine methodological expertise which I share, with further methods and techniques was of particular interest and will most probably help me to develop my own profile further down the road. Particularly my conversations with Rune Berg and Jonathan Victor were interesting in this respect.

2. To me, who is doing his very first steps in computational modeling this workshop was a jump start into the field and gave me a good overview of the aspects that are ”cutting edge problems” in neuronal modeling and ”mathematical neuroscience”. While I will probably not be working on any of these problems soon, I think this broad overview is helpful for me to see where the field of computational modeling is at the moment and where it is going in the near future. I actually learned about some problems that I didn't know were problems.

3. The very positive and accommodating atmosphere at this workshop has positively reinforced my own work. I have become some good feedback after the presentation of my talk and this feedback has led to ideas that I am testing at the moment in the revision of one of my submitted publications.

Massimiliano Tamborrino, Linz: I have been honoured of being invited as a speaker to the workshop ”Brain Dynamics and Statistics: Simulation versus Data”. Without any doubt I can state that it has been the most inspiring and successful workshop I have ever attended. First of all, it gave me the opportunity of presenting myself and my research to a group of eminent professors and researchers, half of them I never met before. The organizers made an excellent job selecting the workshop participants and I believe that was one of the reason of the success of the workshop. All participants share an interest towards neuroscience, coming from different background though, e.g. statistics, mathematics, physics, experimental neuroscience, etc. More importantly, each participant was willing to listen, share and discuss ideas and open problems, making the workshop the perfect spot for starting new collaborations and being inspired by new topics. I wish there could be more of these workshops.

Outcome of the Meeting

This workshop brought together researchers from several different disciplines to work on the problem of understanding stochastic brain dynamics and statistics. This involved statisticians, probabilists, physicists, mathematicians and experimentalists. The experimental neuroscientists attending the workshop have been selected, in part, because of their interest in neuronal modeling and understanding stochastic effects in the brain. Their input was essential in our discussion.

The workshop has enabled us to cristalize and update the current directions and main goals in this area. We have forged links among people working in various directions, all being important for the advancement of the field. We have obtained a more focussed view of the current state of the field, and become aware of directions and activities of the various groups, the latest results in diverse areas, and where to find expertise within subtopics.

Our field has had a slow development. It is a rather complicated topic, having many modeling and physiological aspects, as well as statistical challenges. Treatments of this subject have generally been focussed on one or another of these aspects. In this meeting we have had a group of people with different emphasis of research. Several people join biological research with modeling, and have presented creative methods to address the statistical challenges of this particular field.

The workshop integrated graduate students and postdocs. All of the participants presented their work during the meeting, either with a talk or a short presentation and subsequent discussion of their poster in smaller groups. It is important to note that these young people were excellent in their ability to communicate their ideas and research to the diverse audience attending the workshop.

Informal feedback from workshop participants has been very positive with many new ideas emerging and the participants have all learned a great deal. All participants heartily thank BIRS and its staff for providing truly exceptional facilities and organization to support the meeting.

Participants

Berg, Rune W (University of Copenhagen)
Best, Janet (Ohio State University)
Braun, Wilhelm (University of Ottawa)
Ditlevsen, Susanne (University of Copenhagen)
Greenwood, Priscilla (University of British Colombia)
Grosse Ruse, Mareile (University of Copenhagen)
Guillamon, Antoni (Universitat Politchnica de Calatunya)
Hofmann, Volker (McGill University)

Hutt, Axel (German Weather Service)
Kilpatrick, Zachary (University of Colorado)
Lefebvre, Jeremie (Krembil Research Institute)
Li, Kang (University of Copenhagen)
Lindner, Benjamin (Humboldt-Universitat zu Berlin)
Locherbach, Eva (Universite de Cergy-Pontoise)
Longtin, Andre (University of Ottawa)
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Østergaard, Jacob (University of Copenhagen)
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Ren, Alexandre (Forschungszentrum caesar)
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Sacerdote, Laura (University of Torino)
Samson, Adeline (Universite Grenoble Alpes)
Shinomoto, Shigeru (Kyoto University)
Tamborrino, Massimiliano (Johannes Kepler University Linz)
Thomas, Peter (Case Western Reserve University)
Touboul, Jonathan (College de France & Inria)
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Vich, Catalina (Universitat de les Illes Balears)
Victor, Jonathan D. (Weill Cornell Medical College)
Ward, Lawrence (University of British Columbia)
Whalen, Timothy (Carnegie Mellon University)

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Chapter 4

Optimization and Inference for Physical Flows on Networks (17w5165)

March 5 - 10, 2017

Organizer(s): Michael Chertkov (Los Alamos National Laboratory), Sidhant Misra (Los Alamos National Laboratory), Marc Vuffray (Los Alamos National Laboratory), Anatloy Zlotnik (Los Alamos National Laboratory)

Overview of the Field

Mathematical models that describe the flow of fluids, movement of particles, or transfer of information over a network of channels appear in a wide range of fields of theoretical and scientific study, as well as important engineering applications. Particular interest is devoted to models for which the network structure is represented using graphs where flows on edges and conditions at nodes are governed by sets of physical laws, which may be steady or dynamic, and deterministic or stochastic. Important examples of such physics-governed network flow systems include the electric power grid, natural gas transmission systems, and traffic flows of air or land vehicles. While these large-scale engineered systems have been the subject of extensive study, recent developments in applied mathematics and computation promise a breakthrough in our ability to analyze them and develop new methods for efficient and reliable operations through optimization and control.

Numerous problems of practical interest in applications of flow networks can be mathematically posed as optimization and control. and inference and learning problems. For example, the Optimal Power Flow (OPF) problem in electric transmission networks is an optimization problem that seeks to minimize the cost of power generation among distributed sources subject to the physical constraints of transmission networks. Alternatively, given a continuous stream of noisy/uncertain phase measurements throughout the power grid, we may infer congestion in the network. The relation between these problems may be used to actively control reactive power generation at different locations or change the network topology. As in the case of the power grid, the infrastructure networks of interest may be very large, and may consist of sub-networks or components characterized by qualitatively different physics and/or dynamics on multiple spatial and temporal scales. Therefore, the scaling properties for the mathematics and computational algorithms used to address these problems require careful investigation. Moreover, these problems tend to have intrinsic features such as non-linearities and non-convexities that are dictated by the physical laws governing flows through the network. Such properties make the design of scalable algorithms for these problems a challenge.

The theoretical sides of the relevant mathematical fields, which include optimization, graph theory, statistical inference, dynamical systems, partial differential equations (PDEs), among others, have each given rise to sophisticated computational techniques. However, problems of optimization and inference for physical flow networks require methods from each of these disciplines to be further developed in coordination with one another. For instance, a very accurate and efficient method for simulation of flow dynamics governed by PDEs may be unsuitable for use within an optimization algorithm. Instead, appropriate models that are tractable for optimization, yet accurately represent the physics involved, may be used to develop optimal control algorithms with significantly improved scalability and stability. Similarly, inference algorithms for stochastic state estimation on networks, and the related marginal probabilities, need to be designed in a way that they provide accurate input for stochastic optimization problems. The objective of this workshop was to bridge these mathematical and methodological gaps in the context of large-scale networks. by bringing together an interdisciplinary community of researchers working in the fields of network science, optimization, dynamical systems, optimal control, machine learning, and statistical physics to provide an environment of interaction at the interfaces of these fields. The purpose was to exchange and combine new and emerging ideas with particular focus on inference, learning and optimization for physical networks characterized by dynamics and uncertainty.

Recent Developments and Open Problems

Recently, significant progress progress has been made in mathematical techniques for Optimization, Optimal Control, and algorithms for learning and inference in Graphical models. At the same time need for new methods in the aforementioned topics have exploded in many application areas. The growing use of renewable energy for power generation and need for improved economic and reliable operation of the power grid requires methods for non-linear non-convex optimization and optimization under uncertainty in large networks. Due to the increase in the use of gas generators for balancing out fluctuations in the electric power grid, the gas transmission systems are witnessing an unprecedented level of fluctuation and stochasticity, creating a need for new and efficient methods for optimal and robust optimal control that can be applied to the non-linear PDE governed gas network physics. The advent of large data, need for state estimation in infrastructure networks require fast and sample optimal learning algorithms for graphical models, as well as new methods for inference.

While several methods and algorithms have been developed recently for these application areas, there are still several open problems and unmet requirements that can be potentially resolved by interaction between the applied mathematics and algorithms community and the engineering community. Several convex relaxations and approximation methods such as the Lasserre's hierarchy have been successfully used to solve the so-called Optimal Power Flow Problem, but further specialization is required to make the algorithms scalable. Optimization under uncertainty has been extensively studied recently for the linear DC approximation of the power flow equations, but methods for uncertainty quantification and optimization under uncertainty for the full non-linear AC power flow physics remain unexplored.

Traditional high fidelity simulation methods perform poorly to solve optimal control problems in network flows governed by non-linear partial differential equations. Recently it has been shown that a lower fidelity representation of the non-linear network dynamics that is optimization friendly can solve the same problem much more efficiently. The extention of these methods to accommodate uncertainty, i.e., to solve a robust optimal control problem remains open. In addition, there is still much to be explored in the field of PDE constrained network optimal control problems.

Graphical Models have received a lot of attention due to their ability to naturally model network problems by capturing the network structure and the corresponding dependency between variables succinctly via the underlying graph. Recent work has demonstrated successfully the use of Belief Propagation on tree networks to solve the Optimal Power Flow problem. However, the case of loopy networks, and general methods to handle continuous Graphical Models remains open. In the field of learning structure/parameters of a Graphical Model from data, numerous methods based on regularized M-estimators that exploit convex optimization have been proposed. However, with huge advances in mixed integer programming and the availability of powerful commercial solvers allow the use of learning algorithms based on non-convex optimization, which is an active area of research with several open ends.

This workshop aimed at generating new ideas to solve the above-mentioned open problems by bringing together a multi-disciplinary community from both the applied mathematics and the engineering applications side. The next section summarizes the various presentations in the three broad topics mentioned above.

Presentation Highlights

The proposed schedule involved surveys of mathematical fields and motivating applications so that all participants learned the fundamental assumptions and perspectives of each field. In addition, leaders in each area have delivered focused technical presentations of recent ideas, techniques, and research results that could be adopted by participants from other fields and disciplines. Significant focus was placed on inference, learning and optimization problems involving physical processes that occur over network systems. Some of the problems that we considered, such as optimization of electric power systems, are themselves mature fields under active investigation, while others are in an exploratory stage and have not yet been addressed by the mathematically oriented research community. All of the problems have important common features. They are (a) stated on large graphs/networks; (b) constrained by physics-related constraints that are expressed through algebraic equations, ordinary or partial differential equations, or a mix; (c) require re-formulation to facilitate optimization; (d) depend on practical algorithmic solutions such as approximations or relaxations with analytic or numerical guarantees. The presentations can be categorized into three broad categories: (i) Optimization methods in Power Systems, (ii) Optimal Control in Power Systems and Gas Networks, and (iii) Graphical Models - inference, learning and optimization.

Optimization methods in Power Systems

Miles Lubin and Line Roald: Chance Constraints for Improving Security of AC Optimal Power Flow In the two linked talks the speakers presented a new method to account for uncertainty in the AC optimal power flow problem. Using a linearized model around an operating point, the presenters showed how to solve a convex optimization problem that simulates real-time proportional response throughout the system and uses chance constraints to enforce physical limits. For the chance constraints enforcing line flow, an approximation based on two-sided chance constraints was developed. The speakers have also discussed experimental part where they reported that solution to this convex problem can be corrected back to AC feasibility and is significantly more robust than the solution to the deterministic AC-OPF problem [19].

Andy Sun: Optimal Transmission Switching with DC and AC Power Flows: New Formulations and Strong Relaxations In this talk, the problem of optimal transmission switching (OTS) in a power grid to reduce operating cost and improving reliability was discussed. The author explained study of OTS problems under both DC and AC power flow models. For the DC-OTS problem, a cycle-based reformulation and characterize the convex hull of the cycle-induced relaxation was proposed; for AC-OTS, an MISOCP relaxation and strengthen it with SDP and McCormick-based disjunctions were proposed. It was shown that the proposed reformulations and algorithms efficiently solved IEEE and NESTA instances and lead to significant cost benefits with provably tight bounds [8].

Cédric Josz: Application of complex polynomial optimization to optimal power flow Multivariate polynomial optimization where variables and data are complex numbers is a non-deterministic polynomial-time hard problem that arises in various applications such as electric power systems, signal processing, imaging science, automatic control, and quantum mechanics. Complex numbers are typically used to model oscillatory phenomena which are omnipresent in physical systems. A complex moment/sum-of-squares hierarchy of semidefinite programs to find global solutions with reduced computational burden compared with the Lasserre hierarchy for real polynomial optimization was proposed in this talk. The authors applied the approach to large-scale sections of the European HV electricity transmission grid. Thanks to an algorithm for exploiting sparsity, instances with several thousand variables and constraints can be solved to global optimality. In discussions after the presentation the author has mentioned, that other approaches for solving the deterministic AC OPF were presented at the workshop. These include SOCP and SDP relaxations proposed by Andy Sun, as well as the graphical models approach by the team at Los Alamos National Laboratory. In addition, more complicated problem were presented, such a chance-constrained AC OPF. In the following discussions it was also noted that link between the moment approach and graphical models need to be elucidated in the future. Synergies between both approaches could lead to advances in power systems optimization and to future collaborations between researchers in different areas of applied mathematics (statistics, information theory on one hand, and numerical optimization on the other hand) [7].

Dvijotham Krishnamurthy: Robust feasibility analysis and robust optimization for infrastructure networks The presenter addressed the problem of insuring network security under uncertainties, caused by renewable energy sources, and non-traditional energy demands. He presented a fast and reliable tool for constructing inner approximations of steady state voltage stability regions in multidimensional injection space such that every point in our constructed region is guaranteed to be solvable. The method is supported by numerical simulations that demonstrate that this approach outperforms all existing inner approximation methods in most cases. The stability regions were also shown to cover substantial fractions of the true voltage stability region. This technique has important applications, such as fast screening for viable injection changes, constructing an effective solvability index and rigorously certified loadability limits. [14].

Dongchan Lee: Reachability analysis of transient stability with linear programming relaxation Transient security assessment of power system remains highly challenging task due to the nonlinear and complex nature of the system. The current practices in the industry are mostly based on time-domain simulations, which are computationally expensive especially under uncertainties. In order to overcome the limitations of the existing techniques, the author proposed a reachability analysis approach. He first described a generator model with a set of algebraic equation with the Euler's method. Then the author considered evolution of the system states in a polytope template, which can be solved with linear programming relaxation at every time step. This approach allows to consider potentially any high order generator model. The effect of uncertainty was discussed which, as the workshop showed, is a growing theme in the community. The author has looked at this problem from the prospectives of transient stability and has reported development of algorithms to efficiently assess the security of the power grid. In the result of the follow up discussions the presenter has stated that the workshop was a great opportunity to talk to other researchers and think about future directions. Thanks to the workshop the presenter got the opportunity to meet researchers from Los Alamos National Lab, whom he is planning to collaborate with in the summer of 2017 [10].

Jean Bernard Lasserre: A moment SOS approach for some chance constraints issues on semi-algebraic sets In the talk the author has described a general framework to handle chance-constraints. It consists of posing the problem as an infinite-dimensional LP in an appropriate space of measures. Then when data are polynomials and semi-algebraic sets one may invoke the moment-SOS approach. For instance it allows to approximate the feasible set associated with a chance constraint by the superlevel set x: p(x) >=0 of some polynomial of degree "d" whose coefficients are optimal solutions of an SDP. The larger "d" the better is the approximation. The presenter has mentioned in the discussion period that he was particularly interested by the various papers related to the OPF. The presenter have discussed with Sidhant Misra extensively about how to handle chance constraints via the moment-SOS approach. This could be further developed and extended during the internship of the presenter PhD student Tillman at LANL in the summer of 2017 [9].

Daniel Molzahn: Error Bounds on Power Flow Linearizations: A Convex Relaxation Approach The power flow equations model the relationship between the voltages phasors and power flows in an electric power system. The nonlinearity of the power flow equations results in algorithmic and theoretical challenges, including non-convex feasible spaces for optimization problems constrained by these equations. Accordingly, many practical approaches for solving power system optimization and control problems employ linearizations of the power flow equations. By leveraging developments in convex relaxation techniques, this presentation describes recent progress regarding a method for bounding the worst-case errors resulting from power flow linearizations. Specifically, with a focus on the DC power flow approximation, this presentation characterized the worst-case error in the line flows over a specified range of operational conditions. In terms of links to other presentations and attendees, the presenter noticed in the follow up discussions that the audience was exactly the right group of people for my work. For instance, the approach the presenter used to develop worst-case error bounds on power flow linearizations applies the complex hierarchy of moment/sum-of-squares relaxations presented by Cedric Josz. This hierarchy is also closely related to a hierarchy of relaxations proposed by Jean Bernard Lasserre. The inner approximations proposed by Dj (Krishnamurthy Dvijotham) are also closely related in that the convex relaxations develop outer approximations that complement Dj's inner approximations of the feasible spaces of power system optimization problems.The presenter has also discussed an application of the worst-case error bounds to the topology identification algorithm presented by Deepjyoti Deka. Regarding the development of new work and extending existing ideas, the presenter specifically had discussions with Cedric Josz and Jean Bernard Lasserre on new applications of the hierarchies of convex relaxations, with Dj, Florian Dorfler, Savario Bolognani, and Deepjyoti Deka on extensions and applications of the worst-case error bounds I presented, and with Sidhant Misra and Line Roald on issues related to chance constraints in power system optimization problems with AC power flow models. The presenter has emphasized that he had enjoyed discussions with a variety of other researchers on their work [4].

Control in Power Systems and Gas Networks

Florian Dörfler: Control of Low-Inertia Power Systems: Naive & Foundational Approaches A major transition in the operation of electric power grids is the replacement of bulk generation based on synchronous machines by distributed generation based on low inertia power electronics sources. The accompanying "loss of rotational inertia" and fluctuations by renewable sources jeopardize the system stability, as testified by the ever-growing number of frequency incidents. As a remedy, numerous studies demonstrate how virtual inertia can be emulated through various devices, but few of them address the question of "where" to place this inertia. It is however strongly believed that the placement of virtual inertia hugely impacts system efficiency, as demonstrated by recent case studies. A major transition in the operation of electric power grids is the replacement of synchronous machines by distributed generation connected via power electronic converters. The accompanying "loss of rotational inertia" and the fluctuations by renewable sources jeopardize the system stability. The problem of inertia allocation has been hinted at before, where the grid is modeled by the linearized swing equations, and eigenvalue damping ratios as well as transient overshoots (estimated from the system modes) are chosen as optimization criterion for placing virtual inertia and damping. The resulting problem is highly non-convex, but a sequence of approximations led to some insightful results. The presented work focused on network coherency as an alternative performance metric, that is, the amplification of stochastic or impulsive disturbances to a quadratic performance index measured by the H2 norm. As performance index, a classic coherency criterion was chosen that penalizes angular differences and absolute frequencies, which has recently been popularized for consensus and synchronization studies as well as in power system analysis and control. A foundational analysis was presented for model reduction for low-inertia power systems, and a coherency metric was proposed. A system-theoretic methodology for optimal inertia allocation was presented, which included theorems on performance bounds and optimal resource allocation given the assumption of uniform disturbance-damping. Finally, a tractable computational method for the general case of optimal inertia allocation was presented [15].

Saverio Bolognani: Networked Feedback Control for a Smart Power Distribution Grid Emerging challenges in distribution grids were discussed, including distributed microgenerators, high spatio-temporal electric mobility, the constraints of grid congestion, and unsustainable grid reinforcement. Tractable methodologies for virtual grid reinforcement using feedback control was then presented, based on distributed "model-free" control and centralized chance-constrained decision-making. A fundamental control system model for power distribution grids was stated with underdetermined sensing using meters for voltage, line currents, and transformer loading, and underactuated control using tap changers, reactive power compensators, and active power management. The collection of hard, soft, and chance- constraints used in the optimal power flow (OPF) problem were reviewed, and disturbance attenuation for distribution grid dynamics was developed using a linearization approach on the tangent space to the solution manifold [2].

Michael Herty: Modeling, Simulation and Optimization of Gas Networks There has recently been an intense discussion on physical phenomena on graphs and in particular in the context of gas pipeline network dynamics. An overview of the physics and engineering modeling typically used in the academic literature was given. A new model for representing gas flow dynamics in pipeline networks by asymptotic analysis was presented. The model is derived from the isothermal Euler equations using phenomenological approximations. The derivation of the model was presented as well as numerical results to illustrate order of accuracy, validity, and properties of the approximation. A comparison of the new model with existing models from the mathematical and engineering literature was given, in particular with respect to solution of the full-physics Euler equations. Concepts for nodal control of gas networks using the developed model were discussed [6].

Anatoly Zlotnik: Discretization for optimal control of physical flows on networks A fundamental application for optimal control of physical flows on networks is the operation of large-scale natural gas transmission pipelines. A control system model was outlined for the distributed dynamics of compressible gas flow through large-scale pipeline networks with time-varying injections, withdrawals, and control actions of compressors and regulators. The gas dynamics PDE equations over the pipelines, together with boundary conditions at junctions, were shown to be reducible using lumped elements to a sparse nonlinear ODE system with compact expression in graph theoretic notation. This system was developed as a consistent discretization of the PDE equations for gas flow that can be used to represent the dynamic constraints for optimal control problems for pipeline systems with known time-varying withdrawals and injections and gas pressure limits throughout the network. A problem formulation for intra-day economic optimal control of large-scale pipelines was presented, and an example of tractable, validated solution was discussed [23].

Anders Rantzer: H-infinity optimal control on networks Classical control theory does not scale well for large systems like traffic networks, power networks and chemical reaction networks. However, in this lecture we will present a class of networked control problems for which scalable distributed controllers can be proved to achieve the same performance as the best centralized ones. The control objective is stated in terms of frequency weighted H-infinity norms. This makes it possible to combine disturbance rejection at low frequencies with robustness to high frequency measurement noise and model errors. An optimal controller is given in the form of a multi-variable PI (proportional integrating) controller, which is distributed in the sense that control action along a given network edge is entirely determined by states at nodes connected by that edge. Fundamental bounds on the achievable performance are given in terms of the algebraic connectivity of the network graph [16].

Mihailo Jovanovic: Low-complexity modeling of partially available second-order statistics: theory and an efficient matrix completion algorithm State statistics of linear systems satisfy certain structural constraints that arise from the underlying dynamics and the directionality of input disturbances. The problem of completing partially known state statistics was described. The aim of this work is to develop tools that can be used in the context of control-oriented modeling of large-scale dynamical systems. For the type of applications of interest, the dynamical interaction between state variables is known while the directionality and dynamics of input excitation is often uncertain. Thus, the goal of the mathematical problem that we formulate is to identify the dynamics and directionality of input excitation in order to explain and complete observed sample statistics. More specifically, we seek to explain correlation data with the least number of possible input disturbance channels. We formulate this inverse problem as rank minimization, and for its solution, we employ a convex relaxation based on the nuclear norm. The resulting optimization problem is cast as a semidefinite program and can be solved using general-purpose solvers. For problem sizes that these solvers cannot handle, we develop a customized alternating minimization algorithm (AMA). We interpret AMA as a proximal gradient for the dual problem and prove sub-linear convergence for the algorithm with fixed step-size. We conclude with an example that illustrates the utility of our modeling and optimization framework and draw contrast between AMA and the commonly used alternating direction method of multipliers (ADMM) algorithm [22].

Dennice Gayme: Evaluating performance in linear oscillator networks: from vehicle platoons to power grids Two measures of synchronization performance for networks of coupled linear oscillators were defined and characterized. The first is the aggregate steady state variance of the system due to a disturbance at a single node. The second is the steady state variance of the phase difference between a given pair of nodes due to distributed disturbances. Both metrics are quantified in terms of the H2-norm of an appropriately defined linear system. A systematic framework was presented that allows computation of these metrics for networks over graphs with arbitrary structure. The framework was used to derive relationships between both measures of synchronization performance and the effective resistance between particular nodes in the graphs underlying the oscillator networks. Interpretations were given of the second performance metric as both a local and global measure of network performance. Finally, the theory was applied to two applications; the evaluation of the transient real power losses in power grids and the characterization of coherence for a certain class of vehicle platoons with relative and absolute velocity feedback [5].

Graphical Models

Nicholas Ruozzi: Continuous Graphical Models Computing the mode or MAP assignment in a probabilistic graphical models is generally intractable. As a result, for discrete graphical models, the MAP problem is often approximated using linear programming relaxations. When much of the research has focused on characterizing when these relaxations are tight, only a few results are known for their continuous analog. The presenter focused on the latter problem showing that one can use graph covers to provide necessary and sufficient conditions for continuous MAP relaxations to be tight. In particular this characterization give simple proofs that the relaxation is tight for log-concave decomposable and log supermodular decomposable models. The presenter concludes by exploring the relationship between these two seemingly distinct classes of functions and provides specific conditions under which the MAP relaxation can and cannot be tight [18].

The presenter asked other participants for feedback and discussions about other classes of continuous graphical models where such questions could be ask and answered. In particular if it could be relevant and possible to find the mode of the probability distribution with a message-passing technique beyond Gaussian graphical models.

Patrick Rebeschini: Min-sum and network flows This talk was about message-passing algorithms for solving systems of linear equations in the Laplacian matrices of graphs and to compute electric flows. These two problems are fundamental primitives that arise in several domains such as computer science, electrical engineering, operations research, and machine learning. Formers algorithms that have been proposed are typically centralized and involve multiple graph theoretic constructions or sampling mechanisms that make them difficult to implement and analyze. On the other hand, message-passing routines are distributed, simple, and easy to implement. The presenter establishes a framework to analyze message-passing algorithms to solve voltage and flow problems. The presenter shows that the convergence of the algorithms is controlled by the total variation distance between the distributions of non-backtracking random walks that start from neighbor nodes. More broadly, his analysis of message-passing introduces new insights to address generic optimization problems with constraints [17].

This talk connected the community of graphical modeling, optimization and power systems by combining techniques of the earlier to solve problems for the latter in an efficient and decentralized fashion. In follows up discussions both theoretical aspects and applicability to concrete problems were discussed.

Sungsoo Ahn: Optimizing Gauge transformation for inference in graphical model Computing partition function is the most important inference task arising in applications of Graphical Models (GM). Since it is computationally intractable, approximate algorithms are used to tackle the problem. In this talk, the presenter introduced the technique, coined gauge transformation, modifying GM factors such that the partition function stay the same (invariant), and propose two optimization formulations which generalize the Bethe Free Energy, Belief Propagation approach. Then the optimizations are solved efficiently by Alternating Direction Method of Multipliers (ADMM) algorithms. The first algorithm provides deterministic lower bounds of the partition function. The algorithm is exact for GMs over a single loop with a special structure, even though the popular Belief Propagation algorithm performs badly in this case. The second algorithm is of a randomized, Monte Carlo, type. It lowers sample variance, which can be further reduced with the help of annealed/sequential/adaptive importance sampling. The experiments show that the newly proposed Gauge-ADMM algorithms outperform other known algorithms for the approximate inference task [1].

The talk shares some common interest to that of Nicholas Ruozzi, ”Continuous Graphical Models”. In the follow up discussions it was suggested to consider the gauge transformation approach of discrete GMs to the framework to continuous graphical models and analyze its connection to existing algorithms for continuous GMs.

Marc Vuffray: Optimal Learning of Sparse Graphical Models Graphical Models represent multivariate probability distributions for which direct dependencies between random variables are captured by a network. Graphical Models are widely used for uncertainty management, inference and model reductions. In many applications the network of a Graphical Models is expected to be sparse i.e. the number of edges is of the same order than the number of nodes. As Graphical Models are often not known a priori or cannot always be deduced from first principles, it is of importance to learn Graphical Models from data. In this talk the presenter has focused on efficient methods for learning Graphical Models over discrete random variables and Gaussian random variables when data take the form of several independent observation of random variables. Several low-complexity learning-algorithms that achieve or quasi-achieve the information theoretical lower-bound on data-requirement were proposed [21, 13, 12].

The talk was well received in particular by non-experts in the field of machine learning. The comments and discussions undertaken after the presentation focused on being aware of the problem of data acquisition and data processing in a network. In particular, how reconstruction imprecision based on finite data can have an impact in the control of these networks and what tools can help to overcome this problem. The methods and setting in this talk connected the fields of Gaussian graphical models and learning or solving inverse problems.

Andrey Lokhov: Reconstructing the power grid dynamic model from sparse measurements The presenter discusses the problem of parameter reconstruction in the power grid dynamics from the observations coming from the phasor measurement units (PMUs). The presenter assumes that the dynamics is described by a system of coupled linearized second-order differential swing equations. The fluctuations on individual nodes are assumed to be independent and Gaussian. Given the data coming from the sparsely located PMU sensors, the goal is to learn the inertia and the damping coefficients of the generators using the statistics of the ambient fluctuations. Adopting the description of the process as a multivariate stochastic Ornstein-Uhlenbeck process, the presenter discusses the strengths and the limitations of several approaches to this problem, based on the method of covariance matrices, as well as the maximum likelihood and least-squares estimators.

The framework presented in this talk interested specialists in control for developing & using better real time actuation of the system. The presenter engaged in further discussions for possible collaborations on this interdisciplinary integration with Pr. Jovanovic & Pr. Gayme.

Deepjoti Deka: Topology Learning in Power Grids from Ambient Dynamics Estimation of the operational topology of the power grid is necessary for optimal market settlement and reliable dynamic operation of the grid. The presenter introduces a novel framework for topology estimation for general power grids (loopy or radial) using time-series measurements of nodal voltage phase angles that arise from the swing dynamics. The learning method utilizes multivariate Wiener filtering to unravel the interaction between fluctuations in voltage angles at different nodes and identifies operational edges by considering the phase response of the elements of the multivariate Wiener filter. The performance of this learning framework has been demonstrated through simulations on standard IEEE synthetic test cases [20].

A talk similar in flavor to that of Andrey Lokhov but focused on a concrete application in power systems. The presenter engages in discussions afterwards with A. Lokhov and with the same participant in control theory of power networks.

Yury Maximov: Statistical Learning with Proxies for Power Flow Feasibility In power systems and in particular in energy transmission systems it is of importance to determine if a state, i.e. a set of power productions and consumptions, can be realized in a safe and secure fashion. The presenter shown an approach that uses sampling and machine learning to develop a classifier for secure and insecure states. This statistical machine learning can provide probabilistic guarantees for the quality of the classifier, i.e., it guarantees that we do not treat an unsolvable state as a solvable one. This algorithm approximate the solvability region, which enables in particular to provide polyhedral approximations of the solvability region, as well as more sophisticated convex approximations.

This talk was of interested and well received by scientists from the power system community present at the conference with many implementations related questions asked after the talk. This approach for finding the feasibility set is an original and novel approach which combines knowledge from machine learning and power systems.

Other Topics

Vijay Subramanian: Mean field Games: Incentive and Lottery Mechanisms Motivated by systems with a large number of strategic players, such as in Internet marketplaces, the speaker explored the use of incentive and lottery mechanisms in mean-field games.

First, he considered real-time streaming of video to co-located wireless devices where cooperation among the devices would lead to greater system efficiency. Based on ideas drawn from truth-telling auctions, a mechanism that achieves this cooperation via appropriate transfers (monetary payments or rebates) in a setting with a large number of devices, and with peer arrivals and departures was designed. Furthermore, the complexity of calculating the best responses under this regime is low. This allowed to implement the proposed system on an Android testbed, and illustrate its efficient performance using real world experiments. The tools developed here easily generalize to other resource allocation problems.

Second, the general problem of resource sharing in societal networks, consisting of interconnected communication, transportation, energy and other networks important to the functioning of society was considered. Participants in such network need to take decisions daily, both on the quantity of resources to use as well as the periods of usage. With this in mind, the presenter discussed the problem of incentivizing users to behave in such a way that society as a whole benefits, specifically by rewarding users with lottery tickets based on good behavior, and periodically conducting a lottery to translate these tickets into real rewards. The user decision problem was posed as a mean field game (MFG), and the incentives question as one of trying to select a good mean field equilibrium (MFE). The existence of such an MFE under different settings was shown, and it was also illustrated how to choose an attractive equilibrium using as an example demand-response in energy networks [11].

The two topics discussed are based on the joint work of the speaker with Srinivas Shakkottai (TAMU) and Jian Li (UMass Amherst), with the first also with Rajarshi Bhattacharyya (TAMU) and Suman Paul (TAMU), and the second also with Le Xie (TAMU), Bainan Xia (TAMU), Xinbo Geng (TAMU), Hao Ming (TAMU).

In the following discussions the presenter stated his intent to look into Belief Propagation with mixed integer and real variables as well as the LP-BP versions of it.

Gunnar Flötteröd (KTH) and Carolina Osorio (MIT): Stochastic network link transmission model Stochastic network link-transmission model

In this presentation based on a joint work with Carolina Osorio (MIT) – the presenter has described the link transmission model which has recently gained popularity as an efficient yet exact model of kinematic waves on road networks. The presenter formulated a stochastic instance of this model, where the number of vehicles anywhere in the network is a distributed quantity. The model approximates network-wide stochastic dependencies. Multiple questions related to connections between the model setting and other physical network flow problems discussed during the workshop was raised. The subject requires further work and investigation.

Philippe Jacquod: Multistability and topologically protected loop flows in meshed planar networks The presenter shown that the number of stable fixed points of locally coupled Kuramoto models depends on cycles in meshed networks. In particular one should expect that the number of fixed points increases for meshed networks with more cycles and with longer cycles. The presenter shows an upper-bound for the number of stable fixed point in planar networks which support this intuition and has identified network topologies carrying stable fixed points with angle differences larger than $\Pi/2$. Compared to earlier approaches this bounds is lower and hence much closer to the true number of stable fixed points. [3].

Outcome of the Meeting

The workshop helped to highlight and appreciate new connections between the mathematical areas of dynamical systems, network science, control theory, optimization, and applied probability. Bringing together the theory and applications community helped emphasize the important modeling aspects in each application, problems of importance. Developing efficient algorithms for optimization under uncertainty in networks invariably needs additional structures, assumptions and simplifications, and domain specific knowledge from the applications community helped identify what aspects of the system modeling are mandatory, and in what parts simplification is acceptable. Presentations from the theory/methods community brought to attention potential new directions to solve challenging problems in the application areas of infrastructure networks.

The schedule involved an entire day for discussions. Several new problems, solution methods, and new collaborations were conceived during these discussions. Some examples of new ideas include using Lasserre's hierarchy to solve stochastic optimization problems in non-linear networks, direct approximations to non-linear network physics around an operating point that allow for efficient optimization and uncertainty quantification.

The workshop has received very positive informal feedback, and we anticipate the new ideas and collaborations to lead to new and exciting advancements in the field. All of this was made possible by the BIRS through its excellent organization, facilities and support staff.

Participants

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Chapter 5

New trends in arithmetic and geometry of algebraic surfaces (17w5146)

March 12 - 17, 2017

Organizer(s): Noam D. Elkies (Harvard University), Keiji Oguiso (University of Tokyo), Matthias Schütt (Universität Hannover)

Overview

The interplay of arithmetic and geometry has had a major impact on the recent development of algebraic geometry and number theory. Over the years, algebraic curves have been a driving force, culminating with ground-breaking results such as Faltings' finiteness theorem for rational points on algebraic curves of general type, and the proof of Fermat's Last Theorem by Wiles and Taylor. With curves much better understood, it seems natural to turn to higher-dimensional varieties, starting with surfaces.

Algebraic surfaces have been at the heart of algebraic geometry ever since the Italian school shaped the subject at the beginning of the 20th century. There is little doubt that algebraic surfaces support rich arithmetic structures; precise results on their arithmetic properties, however, are still widely conjectural, despite spectacular recent achievements such as the proof of the Tate conjecture for K3 surfaces in many important settings due to Madapusi Pera, Maulik, and Charles.

The astounding facets of the rich interplay of arithmetic and geometry, which have featured in many recent developments, were a key motivation to propose a workshop at BIRS. While K3 surfaces were a large focus of the workshop, important roles were also reserved for Enriques surfaces and certain aspects of general type surfaces.

Recent Developments and Open Problems

Algebraic surfaces are two-dimensional varieties (usually considered to be smooth and projective) over some algebraically closed field. Over the complex numbers, we can identify them with suitable manifolds, to which we can then apply important principles such as GAGA following Serre's classical work. To avoid confusion (arising from the fact that varieties over $\mathbf{C}$ of complex dimension d are real manifolds of dimension $2d$), we emphasize that Riemann <u>surfaces</u> correspond to algebraic <u>curves</u>, whose properties and structures we take to be fairly well understood (even though in the arithmetic fine print one may argue about this).

Thus turning to algebraic surfaces, we note immediately that they offer much more space for us to manoeuvre. Notably, an algebraic surface admits as subvarieties not only points but also curves, which in fact govern much of the structure of the surfaces. Another crucial difference is fibrations with fibers and base both of dimension one. Fibrations have played a central role in the study and classification of algebraic surfaces, as they come with several critical advantages. To name just two, they provide an important insight into the surface's structure — dividing curves into horizontal and vertical ones, for instance; on the other hand, they allow us to carry over concepts from curve theory, most canonically through the scheme-theoretic concept of the generic fiber.

We next detail some of the most recent developments and describe open problems in the area of algebraic and arithmetic of surfaces, especially those relevant to the present workshop.

Rational points

Given an algebraic variety X defined over some field K, one fundamental problem is to determine the set $X(K)$ of K-rational points on X. Of course, this problem is most interesting (albeit less geometric) when K is not algebraically closed, especially when K is a number field. Already very basic questions turn out to be often hard to answer, starting with whether $X(K)$ is finite or infinite, which can be made more complicated (and interesting) by removing certain algebraic subvarieties, such as those dominated by rational and abelian varieties. For the workshop, this circle of problems was mostly a very important motivation lurking in the background, but a quick discussion will still set the scene for most of what is to come.

It may be instructive to recall the now classical case of curves, where the picture is clear-cut. Let C, then, be a smooth projective curve defined over some number field K, and let g be the genus of C.

If $g = 0$, then either C has a K-rational point or $C(K) = \emptyset$. In the first case, it quickly follows that C is rational (i.e. isomorphic to $\mathbf{P}^1$ over K), and in particular $C(K)$ is infinite. In the latter case, the same properties hold true over any extension of K where C acquires a rational point.

At the other end of the scale, i.e. if C is of general type (which means $g > 1$), then it was conjectured by Mordell in the 1920's that $C(K)$ would always be finite. After celebrated early contributions, for instance by Siegel, Mordell's conjecture was finally proved by Faltings in the landmark paper [Fa83].

This leaves the case of genus one. If $C(K) \neq \emptyset$, then C is an elliptic curve; this means that $C(K)$ has the structure of an abelian group, which (for any number field K, but also under mild conditions over function fields) is finitely generated. More precisely, $C(K)$ can contain any abelian finite group of length at most two as torsion subgroup (although the precise groups that occur are rather limited by the given field K; for instance, over $\mathbf{Q}$ the torsion subgroups run through all cyclic groups until $\mathbf{Z}/10\mathbf{Z}$ together with $\mathbf{Z}/12\mathbf{Z}$, and the non-cyclic groups $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2n\mathbf{Z}$ for $n = 1, 2, 3, 4$ by Mazur's famous theorem). As for the rank of $C(K)$, this could also a priori be any nonnegative intger r, and in fact it is an open problem whether given K, the rank of $C(K)$ for all elliptic curves C over K is bounded. For instance, the the current record over $\mathbf{Q}$ stands at 28, due to Elkies [Elk07] (whose construction uses elliptic fibrations on K3 surfaces!). The rank can be made arbitrarily large by passing to finite extensions K'/K.

Note, however, that these considerations have taken us far beyond the original question, as we are already investigating the precise structure of $C(K)$ as opposed to its cardinality. Indeed, for the cardinality, the answer is simple: $C(K)$ could be either finite or infinite, but on increasing the base field K by a finite extension K', we can first ensure that C is elliptic (over K') and then endow it with a rational point of infinite order so that $C(K')$ becomes infinite, and indeed dense in the associated Riemann surface. This property is also called potential density, and it is of great relevance for us because it is geometric: it does not depend anymore on the field K or the chosen model of C, but only on the isomorphism class of C over $\mathbf{C}$ (given that C may be defined over some number field).

A similar picture persists for algebraic surfaces — as far as we can see. Here the genus is replaced by the Kodaira dimension. Surfaces of Kodaira dimension $\kappa = -\infty$ are well understood as far as the cardinality of the set of rational points, though one can then ask for more subtle properties such as density — or even ask to remove a countable union of rational subvarieties. This leads to Manin's conjectures for Fano varieties, which have seen substantial progress in recent years (in any dimension, in fact). In the opposite direction, for surfaces of general type (i.e. $\kappa = 2$), Lang's conjecture predicts that rational points accumulate on a Zariski-closed proper subset, but this problem still seems hopelessly out of reach. We thus turn to the intermediate case of $\kappa = 0$. Here abelian surfaces can be treated just like elliptic curves, but it is the case of K3 surfaces where new phenomena arise. Notably, potential density is guaranteed by special structures on the surface. Not too surprisingly, an infinite automorphism group suffices for this end; much less trivial is that the same applies to genus one fibrations by a result of Bogomolov and Tschinkel [BT00] (and carries over to Enriques surfaces). This should serve as an indication that one ought to take a closer look at the structure of the curves on a surface (and at fibrations).

Curves on surfaces

The curves on an algebraic surface S generate a huge group of formal linear combinations, the divisor group Div(S). As with curves, we can retrieve essential information about the surface S after dividing out by suitable equivalence relations; presently these could be linear equivalence, algebraic equivalence or numerical equivalence. All quotients share the same discrete part — up to torsion, which in fact is eliminated by numerical equivalence so that the resulting group $\mathrm{Num}(S)$ turns out to be an integral lattice with the natural intersection pairing. Its rank $\rho(S)$ is called the Picard number of S and is one of the most fundamental invariants of an algebraic surface. We emphasize that the Picard number is in general not preserved by deformations, but in some special instances, as for abelian or K3 surfaces (or irreducible holomorphic symplectic manifolds), there is a nice compatibility with moduli theory.

Despite its importance, the problem of computing the Picard number of a given algebraic surface S remains notoriously difficult. However, if S is defined over some number field K, there is an arithmetic approach, which can serve as a prototypical example for this kind of question. Fix some prime p of K where S has good reduction (denoted by S_p). Then there is an embedding

$$\mathrm{Num}(S) \hookrightarrow \mathrm{Num}(S_p),$$

primitive on the level of lattices. This yields the bound

$$\rho(S) \leq \rho(S_p). \tag{5.0.1}$$

It might seem that we have merely reduced one intractable problem to another. But crucially $\mathrm{Num}(S_p)$ lends itself, at least in principle, to explicit computations via the ℓ-adic cycle class map

$$\mathrm{Num}(S_p) \hookrightarrow \mathrm{H}^2_{ét}(\bar{S}_p, \mathbf{Q}_\ell(1)).$$

What's more, the embedding is Galois equivariant, so the image lies in the invariant part under Frobenius. In fact, a famous conjecture of Tate states that this should be an equality after tensoring with $\mathbf{Q}_\ell$:

Conjecture (Tate). In the above setting, one always has

$$\mathrm{Num}(S_p) \otimes \mathbf{Q}_\ell = \mathrm{H}^2_{ét}(\bar{S}_p, \mathbf{Q}_\ell(1))^{\mathrm{Frob}}.$$

This conjecture, one of the most important and influential in the field, has seen great progress in the last 5 years: it has been proved competely for K3 surfaces. The pioneering work of Artin and Swinnerton-Dyer [ASD73] on homogeneous spaces proved the Tate conjecture for elliptic K3 surfaces; Nygaard [Nyg83a] proved it for ordinary K3 surfaces in any characteristic; and the remaining cases were recently settled by work of Maulik [Mau14], Charles [Cha13], Madapusi Pera [Mad15] (for odd characteristic), and Kim–Madapusi Pera [KM15] (for characteristic 2).

This paves the way for several applications and new directions. First, along the above lines, one can try to compute the action of Frobenius on cohomology to obtain upper bounds as in (5.0.1). For parity reasons, these bounds alone on $\rho(S)$ are never sharp if $\rho(S)$ is odd, but this subtlety can be overcome by varying p and examining the lattice structures more closely (see work of van Luijk [vLui07] and Kloosterman [Klo07]). It is another result of Charles [Cha14] that all this information together will be sufficient to derive a sharp upper bound on $\rho(S)$ in any case (at least in theory, and depending on the Hodge conjecture for the self-product $S \times S$). We note that for all of this, we do not need the full action of Frobenius on $\mathrm{H}^2_{ét}(\bar{S}_p, \mathbf{Q}_\ell(1))$, only its eigenvalues with multiplicity. These can be computed, for instance, by extensive point counts (and applying the Lefschetz fixed point formula), or through p-adic cohomological methods and p-adic approximation.

We highlight two further applications of the above ideas. The first concerns very special curves on K3 surfaces, namely rational ones. The problem has seen remarkable progress recently, starting from work of Bogomolov-Hassett-Tschinkel [BHT11] and then greatly extended by Li and Liedtke in [LL12]. The main novel technique is to use Picard jumps upon reduction from S to S_p in (5.0.1), forced for instance by parity, to detect infinitely many rational curves, first on S_p, but then, using arguments going back to Bogomolov and Mumford, also on the original surface S. In particular, this applies to any K3 surface S over $\bar{\mathbf{Q}}$ with $\rho(S)$ odd or at least 5. The second application draws the connection back to rational points. Wondering whether K3 surfaces over number fields satisfy potential density, the key case to consider seems to be Picard number 1. In order to check this, one may combine the above methods with elementary deformation theory to exhibit explicit K3 surfaces with Picard number one, say over $\mathbf{Q}$, and use them as test cases.

Good reduction and Honda-Tate

The problem of good reduction lies at the heart of arithmetic geometry, and even more so do integral models, say over integer rings in number fields. Prototypica examples are often provided by moduli spaces; especially for Shimura varieties integrality questions are one of the key issues for a better understanding, both of the varieties and the objects parametrized.

The classical examples are provided by abelian varieties, for which we have a uniform treatment thanks to the existence of Néron models. Indeed, Serre and Tate showed in [ST68] that an abelian variety over a number field has good reduction if and only its ℓ-adic cohomology is unramified (for some ℓ or for all ℓ, and even H^1 suffices). Later Fontaine proved that there cannot be any (positive-dimensional) abelian schemes over $\mathbf{Z}$ [Fo85]. However, already in dimension one, i.e. for elliptic curves, it is not clear yet precisely which quadratic fields support everywhere integral models of some elliptic curve.

A similar picture persists for K3 surfaces (and higher-dimensional Calabi–Yau varieties) where no models over $\mathbf{Z}$ may exist as proven independently by Abrashkin [Ab89] and Fontaine [Fo93]. Surprisingly, there really seem to be no known explicit integral models of K3 surfaces (unless one considers [Ma15] as explicit); but on the theoretical side there has been substantial progress recently, although part of it depends on the arithmetic analogue of parts of the Minimal Model Program. More precisely, assuming the existence of so-called Kulikov models (a strong form of semi-stable reduction which under certain conditions follows from work of Maulik [Mau14]), Liedtke and Matsumoto prove potential good reduction for K3 surfaces over a discrete valuation ring, i.e. unramified Galois representations imply good reduction over some unramified finite extension (which generally cannot be circumvented, see [LM17]). Very recently, there has also been a crystalline version in [CLL17].

In a similar, but somewhat converse direction, one may also ask whether given a candidate zeta function (satisfying all the standard conditions over some finite field imposed by the Weil conjectures, compatibilities etc.), there is a variety within a given class (say abelian or K3) over the fixed finite field with this exact zeta function. Classically this is known for abelian varieties as the theorem of Honda–Tate (cf. [Hon68]). Taelman proved it recently for K3 surfaces in [Tae16], again assuming a strong form of semi-stable reduction and possibly after an extension of the finite field (which, for instance, makes the Galois action on $\mathrm{Num}(S)$ trivial). We point out that it is not known whether the extension can be avoided in general, or under certain conditions; but experiments conducted by Kedlaya and Sutherland provide rich test data for quartic surfaces over $\mathbf{F}_2$ [KS16].

Automorphisms of Enriques surfaces

Enriques surfaces probably form the most mysterious surfaces of Kodaira dimension zero, because they are governed by their universal covers, namely by K3 surface, and thus by lattice theory, but yet they sometimes show a completely different behaviour. For instance, a very general complex K3 surface has finite automorphism group (or even trivial or of order two), but a general Enriques surface has infinite automorphism group by [2]. In fact, Enriques surfaces with finite automorphism group are very rare; over $\mathbf{C}$, they have been classified completely into seven types by Kondō [Kon86]. In (small) positive characteristics, different situations may persist; recently, the odd characteristic case (and part of characteristic two, see below) was solved completely by Martin [Ma17]. In characteristic two, however, the moduli space of Enriques surfaces decomposes into two 10-dimensional components, corresponding to singular (or ordinary) Enriques surfaces on the one hand and classical Enriques surfaces on the other, as was shown by Liedtke in [Li15]. The singular Enriques surfaces still admit a smooth universal K3 cover; in practice this means that they lend themselves to a treatment very similar to all other characteristics, which was exploited with great success in [KK15] and [Ma17]. The classical Enriques surfaces, however, behave rather differently: much of their geometry seems to be closer to the complex counterpart (for instance genus one fibrations still have two ramified fibers), but the universal covers are no longer smooth K3 surfaces, but only K3-like. The Enriques surfaces at the intersection of the singular and the classical component, the so-called supersingular Enriques surfaces, might be considered even more mysterious.

In a different direction, one may ask for finite subgroups of the (usually infinite) automorphism group. For complex K3 surfaces, this problem leads to now classical work of Mukai [Muk88] which classifies finite groups acting symplectically (i.e. leaving the regular 2-form invariant) in terms of the Mathieu group M_{23}. Recently Mukai and Ohashi started to apply this approach to complex Enriques surfaces and M_{12} in [MO15].

Other aspects

Of course, there are many other open problems on algebraic surfaces which we cannot mention in great detail here. They range from very classical problems, such as the geography of surfaces of general type, through modern topics such as the ubiquitous moduli theory (especially for K3 surfaces, Enriques surfaces, but also for irreducible holomorphic symplectic manifolds) to the most recent advances in connection with dynamics, with emphasis on rational and K3 surfaces.

Presentation Highlights

The workshop featured a great number of very interesting talks. There were numerous experienced leaders as speakers, but we made an effort to schedule a good portion of talks by junior participants, one even before receiving his Master's degree. Throughout the audience was very appreciative and took an active role in the talks, with intensive discussions regularly continuing during the breaks. Below we lay out some of the presentation highlights.

Anthony Varilly-Alvarado (Rice University): A conjecture on Brauer groups of K3 surfaces

Torsion points govern much of the theory of elliptic curves (as indicated in 5). Varilly-Alvarado reported on recent attempts to find a good replacement for torsion points on K3 surfaces, ideally lending itself to uniform bounds such as Merel's 1996 results for elliptic curves [Mer96]. Recent findings suggest that the Brauer groups might be a good candidate yielding analogous statements for K3 surfaces. Growing evidence was lately provided by works of several researchers, including Cadoret–Charles, Orr–Skorobogatov [OS17], and the speaker with collaborators.

Lenny Taelman (University of Amsterdam): Equivariant Witt groups and zeta functions

Taelman presented work that should have a high impact on the Honda–Tate problem for K3 surfaces (cf. 5). The talk itself (which has in the meantime resulted in a joint paper with Bayer-Fluckiger [BFT17]) was almost purely concerned with quadratic forms, but it shed a completely new light on them by working out necessary and sufficient criteria for the following problem: Given a discrete valuation ring R with field of fractions K, consider a symmetric bilinear space V over K. Assume that some group G acts on V by isometries. Does V contain a unimodular lattice stabilized by G?

Taelman's result also provides restrictions on the possible characteristic polynomials of Frobenius on the middle cohomology of a smooth projective variety of even dimension over a finite field, thus generalizing a theorem of Elsenhans and Jahnel [EJ15] (in a rather conceptual way).

Kazuhiro Ito (Kyoto University): On the construction of K3 surfaces over finite fields with given L-function

Ito improved on Taelman's result on Honda-Tate for K3 surfaces [Tae16] (cf. 5) by giving a construction of K3 surfaces over finite fields with given L-function, independent of the previously crucial good reduction criterion [It16]. The proof combines Taelman's approach with showing the existence of an elliptic fibration with a section, or of an ample line bundle of low degree. It still involves a potential finite extension of the base field, and a mild condition on the characteristic (either $p \geq 7$, or $p = 5$ plus one of three extra conditions, e.g. $\rho \geq 4$).

Yuya Matsumoto (Nagoya University): Degeneration of K3 surfaces and automorphisms

Matsumoto's talk provided a good reduction criterion for K3 surfaces (as in 5) that works not only for isolated examples but in full families [Ma16]. Again assuming the existence of the so-called Kulikov models, his methods require only the existence of an automorphism (of finite or infinite order) that acts on the regular 2-form by a primitive root of unity of order $m = 5$ or $m \geq 7$. To this end, the author studies the weight filtration of the ℓ-adic cohomology groups. Then the compatibility with the induced action of the automorphism group lets him exclude certain types of degeneration.

Curtis McMullen (Harvard University): Algebraic integers and surfaces dynamics

McMullen discussed reverse engineering at the interface of algebraic geometry and dynamical systems. In detail, he explained how to create dynamical systems on varieties starting with an algebraic integer (a Salem number, to be precise, which will serve as eigenvalue of some automorphism on cohomology). Explicit dynamical systems of minimal entropy on K3 surfaces, rational surfaces and complex tori were described in detail.

Simon Brandhorst (Leibniz Universität Hannover): On the dynamical spectrum of projective K3 surfaces

Brandhorst focused on the above mentioned problem of realizing a Salem number $\lambda \in \mathbf{C}$ as entropy of an automorphism of a complex projective K3 surface. He gave a affirmative answer, except that his methods only cover some power of λ. Afterwards, the techniques were extended to Enriques surfaces and 2-dimensional tori, both projective and non-projective [Bra17].

François Charles (Université Paris-Sud): Arithmetic ampleness and an arithmetic Bertini theorem

Charles discussed properties of certain ample line bundles in arithmetic geometry, aiming for analogues of well-known geometric results in the arithmetic setting [Cha17]. The key case consisted in an ample hermitian line bundle $\mathcal{L}$ on a projective arithmetic variety $\mathcal{X}$ of dimension at least 2 where the proportion of the effective sections of $\mathcal{L}^{\otimes n}$ defining irreducible divisors on $\mathcal{X}$ was shown to tend to 1 as n tends to ∞. Applications included restriction theorems and an arithmetic analogue of the Bertini irreducibility theorem, improving on the results from [CP16] (while at the same time building on them).

Asher Auel (Yale University): Decomposition of the diagonal and phantom categories on surfaces

Auel surveyed recent developments on the problems of rationality and stable rationality, combining perspectives from unramified cohomology and zero-cycles as well as derived categories and semiorthogonal decompositions. In particular, he reported on joint work with Bernardara (cf. [AB17]), concerning phantom subcategories of the derived category of an algebraic surface. He showed how these can be viewed as a stronger measure of rationality than the existence of a decomposition of the diagonal.

Jörg Jahnel (University of Siegen): On the distribution of the Picard ranks of the reductions of a K3 surface

Jahnel reported on joint work with Costa and Elsenhans [CEJ16] on the distribution of the Picard ranks of a K3 surface under the reduction map (5.0.1) as the prime p varies. To this end, he considered the determinant of the Galois representation on the transcendental cycles. After a Tate twist, this gives either a trivial or a quadratic character. Whenever this character evaluates to -1, he showed that the Picard number jumps. What's more, the character can be related to the discriminant of the K3 surface. This paves the way to a sufficient criterion for the existence of infinitely many rational curves on K3 surfaces not covered by the cases laid out in 5.

Toshiyuki Katsura (Hosei University): Classification of Enriques surfaces with finite automorphism groups in characteristic 2

Katsura's talk concerned the problem of finite automorphism groups of those Enriques surfaces in characteristic two whose universal cover is not a (smooth) K3 surface; that is, classical or supersingular Enriques surfaces – exactly those which were not covered in 5. In joint work with Kondō and Martin [KKM17], they use genus one fibrations (much like Kondō did in [Kon86], but now also including quasi-elliptic fibrations) and the classification of conductrices from [ESB04] to establish the full classification of the configurations of smooth rational curves (which happen to form a finite set) on Enriques surfaces with finite automorphism group. In particular, they provide examples for each configuration, some supporting different finite groups of automorphisms. This leaves open only the determination of the underlying moduli spaces.

Outcome of the Meeting

The workshop brought together experts from all over the world working on algebraic surfaces, ranging from leaders in the field to rising stars and newcomers. It featured exciting talks about the latest developments, which were extremely well received and led to lively discussions. The workshop strengthened existing interactions and at the same time initiated several new collaborations and directions.

Participants

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Chapter 6

Generated Jacobian equations: from Geometric Optics to Economics (17w5078)

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Overview of the Field

The field of generated Jacobian equations (GJE) is a nascent field, motivated on one hand by questions in geometric optics and partial differential equations, and on the other by structures that arise naturally in economics. It may be cast as an outgrowth of the theory of optimal transport (OT), which has seen a lot of development and progress over the last couple of decades.

As may be gleaned from the fields mentioned above, the span of areas involving the analysis of generated Jacobian equations is vast. This ubiquity of GJE and OT reflects the fact that they are concerned with the study of the following common situation: one has mappings (alternatively "changes of variables", or "matchings") with a prescribed action on volume and which satisfy an optimality condition. Such situations include measure preserving mappings that minimize a cost functional, optimal matchings between two or more populations, couplings of random variables that minimize covariance, ray tracing maps from one reflecting surface to another, to name a few.

As we are dealing with mappings with a prescribed action on given measures, we are led naturally to the notion of a **prescribed Jacobian equation**. This is an equation where the unknown is a map T (sometimes, but not always a diffeomorphism) between two spaces X and Y (frequently Riemannian manifolds or subsets of Euclidean space), and the distortion of the volume by this map is **prescribed**. This leads to equations of the form

$$\det DT(x) = \psi(x, T(x)).$$

In this generality, this is a nonlinear system of PDEs, however, we are interested in a more special situation. If T is a gradient map, that is if $T(x) = \nabla u(x)$ for some scalar valued potential function u, the equation turns into the following second order scalar PDE, known as the **Monge-Ampère** equation:

$$\det D^2 u(x) = \psi(x, \nabla u(x)).$$

The most studied class of solutions to this equation is the class of convex u where ψ is nonnegative, in which case the equation is degenerate elliptic. In the study of more general prescribed Jacobian equations, there is an analogue of "mapping arising from a convex potential" through the introduction of what is known as a **generating function**

G. This refers to a real valued function,

$$G : (x, y, z) \in X \times Y \times \mathbb{R} \mapsto \mathbb{R},$$

which determines a **duality structure** between the spaces X and Y. This generalizes the notion of duality between a vector space and its dual, which corresponds to the choice of

$$G(x, y, z) = -\langle x, y\rangle + z,$$

(here $\langle \cdot, \cdot\rangle$ is the duality pairing between a vector and covector). The theory of GJE is concerned with prescribed Jacobian equations that arise from a generating function. Concretely, they involve an unknown scalar function $u : \Omega \mapsto \mathbb{R}$ solving a nonlinear PDE of the form

$$\det D(T_u(x)) = \psi(x, u(x), \nabla u(x)), \;\; T_u(x) = T(x, u(x), Du(x)),$$

where T_u is a generalization of the gradient map from the classical Monge-Ampère case above, which is determined by solving the following system of equations in T and Z:

$$D_x G(x, T(x, u, \bar{p}), Z(x, u, \bar{p})) = \bar{p}, \; G(x, T(x, u, \bar{p}), Z(x, u, \bar{p})) = u.$$

As with the classical Monge-Ampère equation, it is advantageous to restrict attention to the class of $\underline{G\text{-convex}}$ solutions u, that is, those functions that can be expressed in terms of some dual function $v : Y \mapsto \mathbb{R}$ and G via the generalized Legendre transform defined by

$$u(x) = \sup_{y\in Y} G(x, y, v(y)). \tag{6.0.1}$$

Then, the map $T_u(x)$ (under additional conditions) is characterized by

$$T_u(x) = y, \quad \text{where } y = \operatorname*{argmax}_{y\in Y}\; G(x, y, v(y)). \tag{6.0.2}$$

The classical Monge-Ampère equation corresponds to the generating function $G(x, y, z) = -\langle x, y\rangle + z$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, in which case u is a convex function and $T_u(x) = \nabla u(x)$.

Applications of Generated Jacobian Equations

Let us elaborate further on the ways GJE arise in areas such as geometric optics, differential geometry, mathematical physics, and economics. We could start by pointing out that the wide applicability of GJE may already be inferred from the fact that GJE "contains" Optimal Transport as a "special case", however, this does not seem like a fair or illustrative approach. A more interesting measure of the merits for GJE is given by the many examples of problems in GJE are not treatable by the optimal transport framework.

Let us start with an important class of examples of GJE, namely the "near field" reflector problems in geometric optics, which fall outside the scope of OT[1]. In fact, at a previous BIRS workshop (Workshop # 12w5118), Trudinger posited the concept of GJE and proposed it as a framework that covers both optimal transport as well as near field reflector problems. The report for this workshop [] remarks that "a natural question is how to extend the results known up to now for optimal transport maps, to this more general setting, and it seems to give an interesting and challenging line of research".

One particular example is the point source near field reflector problem. In this problem there is a set of directions $X \subset \mathbb{S}^2$ through which light emanates from a point source at the origin O, and there is a region $Y \subset \Sigma$ in a smooth surface Σ. The goal is to illuminate Y by constructing a mirror to catch and reflect the light emitted by the source. Since the problem will be underdetermined otherwise, there are also two absolutely continuous mass distributions with densities $f(x)$ and $g(y)$ on X and Y respectively, with equal masses, f representing the initial distribution of light through X and g the desired output pattern on Y.

[1]"far field" problems, which are a limiting case of the near field regime, can be understood via OT

Then make the choice

$$G(x,y,z) = \frac{z^{-2} - \frac{1}{4}|y|^2}{z^{-1} - \frac{1}{2}(x,y)}. \tag{6.0.3}$$

Here, the radial graph of $G(\cdot, y, z)$ over $\mathbb{S}^2$ is an ellipsoid of revolution whose foci are at O and y, and whose eccentricity is determined from z. Then, a G-concave function u (defined as in (6.0.1), but with an inf instead of a sup) represents a reflective surface given by envelope of ellipsoids of revolution with one foci at O and the other at points in Y. Assuming the reflective surface obeys Snell's law, a ray emanating from $\mathcal{O}$ in the direction x will bounce off the reflective surface and eventually hit Σ at a point $T_u(x) \in Y$, where $T_u(x)$ is as in (6.0.2) but with an argmin instead of argmax. Then in order to construct the desired mirror, one should take the radial graph of a G-convex solution $u : X \mapsto \mathbb{R}$ of the GJE

$$\det DT_u(x) = f(x)/g(DT_u(x)). \tag{6.0.4}$$

By considering other source and target domains X and Y, and different G corresponding to other basic reflective surfaces (e.g. paraboloids or cartesian ovals) one may treat many different near field geometric optic problems, such as parallel beam sources, or refraction problems.

In another direction, note all of the geometric optic problems above involve determining a surface whose normal vector behaves in such a way so that the ray tracing map sends one measure into another. This situation is very similar to that of prescribing the Gauss curvature of a convex surface (the Minkowski problem), which leads us to differential and convex geometry. A number of problems in convex analysis and convex geometry, generalizing the Minkowski problem, all lead naturally to Monge-Ampère type equations and generating functions. The Minkowski problem, which involves finding a closed convex surface with prescribed Gauss curvature, may, be posed as the analysis of a function of the form

$$\rho(x) = \inf_y G(x, y, z(y)), \quad G(x,y,z) := \frac{z}{x \cdot y}, \quad x, y \in \mathbb{S}^2,$$

for some strictly positive function $z(y)$ (and $G(x,y,z)$ being defined only for $x, y \in \mathbb{S}^2$ with $x \cdot y > 0$). There are also other related problems of great interest in the study of convex bodies, the L^p- and log-Minkowski problems. It is reasonable to expect that an analogue of this problem when $\mathbb{R}^3$ is replaced by a general Riemannian manifold will also lead to a GJE.

Alternatively, from the viewpoint of dynamics an important notion –currently limited to OT, with no parallel at the more general level of GJE– is that of **displacement convexity** and **Wasserstein barycenters**. Recall that if (X, d) is a metric space, OT imbues the space $\mathcal{P}_2(X)$ (probability measures on X with finite second moment) with a metric $d_{\mathrm{MK},2}$, defined via the Kantorovich problem

$$d_{\mathrm{MK},2}(\gamma_0, \gamma_1)^2 := \inf_{\pi \in \Pi(\gamma_0,\gamma_1)} \int_{X \times X} d(x,y)^2 \, d\pi(x,y),$$

where $\Pi(\gamma_0, \gamma_1)$ is the space of probability measures in $X \times X$ whose left and right marginals are γ_0 and γ_1, respectively. Then $d_{\mathrm{MK},2}$ makes $\mathcal{P}_2(X)$ a Polish and/or geodesic space as long as (X, d) is itself a Polish and/or geodesic space respectively, and thus in the latter case one can consider geodesics via this metric between two probability measures on X.

Displacement convexity is a property of functionals E on $\mathcal{P}_2(X)$, requiring the convexity of E along geodesics in this $d_{\mathrm{MK},2}$ metric. This notion of convexity for functionals is particularly useful in analyzing gradient flows for functionals on $\mathcal{P}_2(X)$ (such as relative entropy, as a famous example) and in particular for understanding the rate of convergence to equilibrium.

Lastly, let us discuss the emergence of generated Jacobian equations in economics, concretely, in relation to both matching problems and principal/agent problems. In either case, the set up involves compact metric spaces X and Y, and (in the greatest generality) a generating function $\phi : X \times Y \times \mathbb{R} \mapsto \mathbb{R}$, which is assumed to be continuous, strictly decreasing in its third argument, and such that $\phi(x, y, \mathbb{R}) = \mathbb{R}$ for all x, y. Moreover, ψ represents the inverse generating function. In this context, ϕ and ψ will represent the utility functions of certain parties.

In the matching context, $\phi(x, y, v)$ represents the utility that an agent x obtains when matched with agent y who is receiving a utility of v; the quantity $\psi(x, y, u)$ then corresponds to the utility agent y obtains when matched with agent x, who is receiving a utility of u. By contrast, in the principal-agent context $\phi(x, y, v)$ corresponds to the utility earned by an agent of type x who chooses decision y and makes a transfer of v to the principal, and $\psi(x, y, u)$ is the transfer which yields a utility of u to an agent of type x who chooses decision y. One is given distributions for agents x and agents y in the former case, and a utility function for the principal and a distribution for the agents in the latter one, and the goal is to find a stable matching: an admissible matching between parties where no pair has incentive to change their assigned pairing. Utility functions ϕ are called quasilinear if they are of the form $\phi(x, y, z) = b(x, y) + z$, for some $b(x, y)$, and previous work has been done connecting OT to this case of quasilinear utility. However, it can be more natural to assume a non-quasilinear utility, which now becomes part of the realm of general GJEs.

Recent Developments and Open Problems

It seems that generated Jacobian equations and the analysis of G-convex functions first appeared in the literature in two independent, vastly different contexts. The first is in the PDE literature motivated by near-field geometric optics problems, and the second in the economics literature in relation to principal-agent and matching problems in the non-quasilinear setting.

On the PDE side, in [34] Trudinger introduced GJE as a convenient framework covering a broad type of problems –motivated by geometric optics problems that are not covered by OT. Up until this point, far-field optics problems had been recognized as being covered by the OT framework (for example, [35]). The work of Trudinger in [34], besides introducing the GJE framework, studies existence of weak solutions subject to the second boundary value condition (the natural boundary condition when one prescribes the action of a volume deforming transformation), and provides (C^2 and higher) regularity results for smooth, positive densities under natural structural conditions on the generating function and domains. One of the conditions is a natural analogue of the Ma-Trudinger-Wang condition which plays a central role in regularity of OT ([28]). Since then, Guillen and Kitagawa have developed regularity theory ($C^{1,\alpha}$) for weak solutions of GJE with possibly discontinuous densities, while Jhaveri has obtained partial regularity results under minimal assumptions (see also discussion of presentations below). In terms of pure geometric optics, Gutierrez and Tournier analyzed the regularity for the near field parallel reflector/refractor problem [19] and also studied problems for media with different refractive indices [18]. Gutierrez and Sabra also studied models for scattering on free form lenses [17] and aspherical lenses [16]. There is also work of Karakhanyan and Wang [20] regarding the near field point source problem, which obtains regularity of reflectors for smooth densities under natural structural assumptions, but furthermore, provides an example of singular behavior of a nature not exhibited in OT. This brief and incomplete lists illustrates just some of the recent activity from the PDE side concerning GJEs, and the many geometric optic problems that GJE encompasses.

Independently, from the economics community, Nöldeke and Samuelson [32] initiated a systematic study of generating functions and G-convex analysis in the context of the "implementation duality" in matching and principal-agent problems. Previous works [5] and [12] analyzed the quasilinear case, which falls within the realm of OT, but the work of Nöldeke and Samuelson deals specifically with utility functions which are not quasilinear. Another interesting development is in relation to the second welfare theorem, which in the quasilinear case essentially indicates that a matching in equilibrium coincides with the optimality conditions arising in OT. In the non-quasilinear case, non-transferable utilities may make it impossible to characterize equilibrium matchings as optimal, which has led Galichon to propose "equilibrium" as a replacement for "optimal" in matching problems. This in turn leads to a question that arises naturally and has connections with geometric optics as well: what is the analogue for GJEs of Kantorovich's relaxation of OT? This is an unresolved question, in fact, it is not even clear what a corresponding notion of a transport plan should be within GJE. This issue not withstanding, work by Liu [26] shows there is a variational characterization behind certain reflector antenna problems, and shows this question can form a unifying thread between economics and geometric optics within GJE.

Finally, we mention the great deal of activity relating to numerical schemes for GJEs. There is work in the semi-discrete setting by Merigot, Meyron, and Thibert [31] as well as by Kitagawa, Merigot, and Thibert [25] for optimal transport. Monotone approximation schemes for viscosity solutions of the Monge-Ampère equation have been obtained recently by Froese and Oberman [13], which are notable for their handling of the second boundary value problem. Developing respective numerical schemes for various geometric optic problems is a worthwhile research direction that is still in its early stages.

Presentation Highlights

This workshop brought together mathematicians (notably, experts in differential geometry, PDE, convex analysis), engineers and applied mathematicians (numerical analysis, optimization, mathematical physics), and economists (stable matching theory, principal/agent problems). Our chief goals were to compare each field's methods, research directions, and conjectures, and to lay the ground for future interdisciplinary work.

General Theory of GJEs

Nestor Guillen began with an overview of generated Jacobian equations, presenting the basic setup and terminology as first introduced by Trudinger in [34]. This included a brief overview of the applications to geometric optics and economics, a discussion of his recent work in regularity of weak solutions with Kitagawa [15], and key differences with the classical Monge-Ampère and optimal transport theory (as shown by Karakhanyan and Wang [20]).

Yash Jhaveri discussed a partial regularity result for GJE, showing that for generating functions that satisfy structural conditions and continuous, bounded data, there are two closed sets of measure zero, outside of which the associated transport mapping is a $C^{0,\alpha}$ homeomorphism. One significance of this result is that it does not require an MTW like condition that is usually necessary for full regularity results. The result first appeared in the context of the real Monge-Ampère equation by Figalli and Kim [11] and OT by De Philippis and Figalli [10].

Brendan Pass presented recent work with McCann on OT problems between spaces of unequal dimension. They have shown that when the dimension of the source domain is larger than that of the target domain, the problem can be described by a non-local analogue of the Monge-Ampère equation that arises for transport between spaces of equal dimensions. Such problems are of great interest in economic matching problems, as the spaces to be matched (buyers and sellers, employers and employees, etc.) naturally depend on different numbers of parameters. In another recent work with Chiappori and McCann [6], Pass has exhibited some conditions under which one can obtain uniqueness and regularity of transport maps when the target space is one dimensional.

Numerical Analysis

Boris Thibert presented two numerical schemes for the semi-discrete optimal transport problem based on damped Newton methods. The first is applicable to transportation of an absolutely continuous measure to a discrete measure subject to a cost function satisfying the MTW condition, under only mild connectivity assumptions on the source of the source measure (joint with Kitagawa and Mérigot [25]). This scheme relies on the geometric implications of convexity coming from the MTW condition, which was first shown by Loeper ([27]). The second considers the case where the source measure is supported on a simplex soup and the target is discrete (joint with Mérigot and Meyron [31]). In this case the convex structure of the MTW case is lost, hence the scheme requires the points in the support of the discrete target satisfy a natural non-degeneracy type condition. Both results include rigorous proofs of convergence rates.

Farhan Abedin discussed an iterative method for computing solutions to GJE also in the semi-discrete case. The method is in the vein of Caffarelli, Kochengin, and Oliker's in the far-field reflector problem [3] and Kitagawa's for strongly MTW costs [24], but Abedin's method is novel in that it does not require any MTW like conditions on the generating function to obtain termination after a finite number of steps (which is new even in the OT case) and thus applies to an extremely wide variety of generating functions.

Brittany Froese introduced a general framework to construct monotone approximation schemes for Monge-Ampère type equations given a point cloud satisfying certain structural conditions. With this framework and a careful resolution of boundary geometry, given a degenerate elliptic equation with a comparison principle, Froese is able to construct approximation schemes which converge to viscosity solutions of the PDE [14]. Numerical results were also presented for examples including the prescribed Gauss curvature problem with Dirichlet boundary data, the quadratic optimal transport problem, and problems arising from seismic imaging and beam shaping.

GJE and OT in economics

Larry Samuelson's talk dealt with generalized notions of convexity and how they are a means to understand the "implementability" of assignment maps in natural problems in economics, again in instances where the underlying utilities may not be quasilinear. A natural concept is that of a profile $v : Y \mapsto \mathbb{R}$ implementing a pair (u, y), where $u : X \mapsto \mathbb{R}$ (another profile) and $y : X \mapsto X$ (an assignment), which means that

$$u(x) = \max_{y \in Y} \phi(x, y, v(y)), \quad y(x) \in \text{argmin}_{y \in Y} \phi(x, y, v(y)).$$

In the matching model, u and v are profiles of utilities for the buyers and sellers. In the principal-agent model, u is a rent function for the agent, giving a utility $u(x)$ for agent x, and v is a tariff function, giving the tariff $v(y)$ at which any agent can execute contract y. One result presented by Samuelson was that the set of pairwise, stable, full outcomes satisfying a given initial condition (y_1, v_1) is nonempty and closed in the right topology. Additionally, he mentioned that implementation maps are examples of a (antitone) **Galois connection**: a pair of order reversing mappings between posets of a set.

Alfred Galichon introduced the framework of equilibrium transportation, which encompasses problems arising naturally in economics which often fall outside the scope of OT. As mentioned in the previous section, the second welfare theorem gives the equivalence of stability and optimality in matching problems involving quasilinear utilities, but this may not be the case in the non-quasilinear case. However, Galichon posits that it does make sense to talk about equilibrium in matchings in the non-quasilinear case, and perhaps this is the correct notion to pursue rather than optimality. Galichon illustrated this situation by introducing a variation on the matching problem involving taxation. In this setting, workers x are paired with firms y and receive a wage $w(x, y)$ while the worker obtains a utility $\alpha(x, y)$ (this may represent aspects different from pay, such as job satisfaction). The firm obtains a utility $\gamma(x, y)$, hence their total profit is $\gamma(x, y) - w(x, y)$. However, there is a tax levied on the worker's pay, thus the actual benefit a worker gains is $\alpha(x, y) + N(w(x, y))$ for some function N. The goals is then to match a group of workers with firms in a stable manner, in the case of no taxation ($N(w) = w$) we recover OT and stability is equivalent to optimality. In the case of taxation, Galichon introduces the notion of feasible sets defined by

$$\mathcal{F}_{xy} = \{(u, v) \in \mathbb{R}^2 : u - \alpha(x, y) \leq N(\gamma(x, y) - v)\},$$

and a triple (P, u, v) where $P \in \mathcal{P}(X \times Y)$, $u, v : X, Y \to \mathbb{R}$ is said to be an equilibrium when

$$(u(x), v(y)) \notin \mathcal{F}^{\circ}_{xy} \text{ and } (x, y) \in \text{spt}(\pi) \Rightarrow (u(x), v(y)) \in \mathcal{F}_{xy}. \tag{6.0.5}$$

Finding (π, u, v) such that (6.0.5) holds is known as a Nonlinear Complimentary Problem (NCP). In the continuum case, if the matching between x and y is given by a smooth map $x \to y(x)$, then $y(x)$ is the G-gradient map of a G-convex function u which (using the mass balance) solves a corresponding GJE. The map $y(x)$ and scalar $u(x)$ are examples of an implementation map, as discussed by Nöldeke and Samuelson [32].

Shuangjian Zhang discussed a principal-agent problem modeling a pricing problem for a monopolist involving non-quasilinear utility functions, and presented necessary and sufficient conditions for the principal's problem to be a linear program (joint with McCann, [30]). Here the principal-agent problem is where a principal and agents attempt to maximize their utilities which arise through the execution of certain contracts, but often with some amount of information asymmetry. In the context of Zhang and McCann's work the principal is a monopolist manufacturer and agents are potential buyers, the different contracts to be executed are the purchase of different models of whatever product the principal manufactures. The monopolist then has a (possibly non-quasilinear) profit function $\pi(x, y, v)$, which is the profit generated by agent x buying the product y at price v. Then the monopolist knows the distribution $d\mu$ of different types of buyers in the market and the utility that a given buyer type derives from purchasing a given model of product, and seeks to set the price $v(y)$ for each model y in a manner which maximizes their profit, i.e. they wish to maximize the functional

$$\int_X \pi(x, y(x), v(y(x)))\, d\mu(x),$$

where here $y(x)$ is the choice of model that a buyer x will select based on their own utility and a given pricing scheme. Zhang has shown that under a convexity condition on the agents utility function along certain curves, the principal's problem is a linear program. This convexity condition follows from a strengthening of the analogue of MTW condition for GJEs, which corresponds to the condition of non-negative cross curvature used in previous work of Figalli, Kim, and McCann ([12]) for the case of quasilinear utilities.

Guillaume Carlier presented a model for equilibrium prices with constraints stemming from work with Ekeland and Galichon. In the model, one has a set Z representing feasible goods, and which is assumed to be a compact metric space. The goal is to determine a price system that clears the market, that is, a price function $p : Z \mapsto \mathbb{R}$ which leads to equilibrium.

One is given a set of producers $y \in Y$ (assumed also a compact metric space), a cost function $c : Y \times Z \mapsto \mathbb{R}$ (what it costs to y to produce z), and a distribution of producers corresponding to a probability distribution ν over Y. Then, given $p(z)$, it's c transform is $p^c(y) = \max_z\{c(y, z) - p(z)\}$ and it represents what the maximum cost to producer y when selling good z in this price system. Consumers are represented by a compact metric space X with a probability distribution $\mu \in \mathcal{P}(X)$.

So far, this is standard, the novelty arises in that consumers are modeled to have a budget: each consumer x has a revenue $\phi(x)$, $\phi \in C(X)$, as well as a utility function given by $U : X \times Z \mapsto \mathbb{R}$, such that $U \in C(X \times Z)$. Then, given a price function $p(z)$, we define

$$U_p(x) = \max\{U(x, z) : z \text{ s.t. } p(z) \leq \phi(x)\}$$

Then, the problem is to determine a price controls p and matching plans for consumers/goods $\gamma \in \mathcal{P}(X \times Z)$ and producers/goods $\sigma \in \mathcal{P}(Y \times Z)$ satisfying the following. The prize function is such that $p \in C(Z, \mathbb{R})$ and $\min_Z p(z) = \min_X \phi$, while $\pi_x\#\gamma = \mu$, $\pi_y\#\sigma = \nu$, and $\pi_z\#\gamma = \pi_z\#\sigma$. Then, the equilibrium corresponds to these objects satisfying

$$p^c(y) + p(z) = c(y, z)\ \sigma\text{-a.e.}, \quad \text{while } p(z) \leq \phi(x) \text{ and } U(x, z) = U_\rho(x)\ \gamma\text{-a.e.}$$

This a case of nontransferable utility and OT methods do not apply. An existence result is obtained for equilibria in the above sense.

Beatrice Acciaio provided an introduction to the theory of causal transport and some of its applications, based on joint work with Backhoff and Zalashko [1]. In probabilistic language, classical OT corresponds to minimizing the expectation of the cost function:

$$\inf\{\mathbb{E}_\pi[c(x, y)]\ : P \in \Pi(\mu, \nu)\}$$

where as usual, $\Pi(\mu, \nu)$ denotes the space of probability measures over $X \times Y$ with marginals μ and ν. Acciaio introduced the variant of causal transport. Here one takes filtrations $\mathcal{F}^X$ and $\mathcal{F}^Y$ over X and Y respectively, then $\pi \in \Pi(\mu, \nu)$ is said to be a causal plan if for all t and $D \in \mathcal{F}^Y_t$ the map $x \mapsto P^x(D)$ is measurable w.r.t. to $\mathcal{F}^X_t$. Then, the causal transport problem is the minimization problem

$$\inf\{\mathbb{E}_\pi[c(x, y)]\ : P \in \Pi(\mu, \nu) \text{ such that } P \text{ is causal}\},$$

corresponding to a constrained version of the classical OT problem where time matters, hence the involvement of causality. Attainability for the causal transport problem, as well as the expected duality relations are satisfied. Acciaio also discussed an application of causal transport, to the following question of importance in stochastic integration: given a space of events Ω imbued with two filtrations $\mathcal{F}$ and $\mathcal{G}$, and a probability measure $\mathbb{P}$, then given X a semimartingale in $(\Omega, \mathcal{F}, \mathbb{P})$ when can we say that X is also a semimartingale in $(\Omega, \mathcal{G}, \mathbb{P})$? With an appropriate choice of cost function, causal transport provides an optimal transport characterization for those processes B which are a Brownian motion with respect to a given filtration and remain a semimartingale with respect to an enlarged filtration.

Dynamics

Bernhard Schmitzer's presentation was concerned with the problem of "unbalanced" optimal transport and various approaches. This problem, considered in joint work with Benedikt Wirth [33], is relevant to situations where the two measures under consideration mays not have the same total mass –e.g. a partial transport problem or an inference problem where one needs to estimate the similarity between two distributions that may have different total mass. Such problems can arise for example in image interpolation between images that are not normalized (such as medical images). Schmitzer's talk considered both static as well as dynamic formulations (much in the spirit of work of Benamou and Brenier). Such considerations are important as frequently unbalanced transport is initially defined via a dynamic formulation, such as in the case of the Wasserstein-Fisher-Rao metric, and a major question is if there is a corresponding static formulation. Schmitzer and Wirth's work deals with an interpolation involving the OT metric derived from the distance cost, and not from the distance squared as in the case of Wasserstein-Fisher-Rao.

Codina Cotar presented recent developments in the connections between multimarginal optimal transport and density functional theory. Density functional theory or DFT, is a simplified version of quantum mechanics in which an N-particle system (represented by a probability distribution on $\mathbb{R}^{3N}$) is described by the behavior of its one-particle marginals on $\mathbb{R}^3$, and models electron interactions of molecules as described by the Schrödinger equation.

In recent joint work with Friesecke and Klüppelberg [7], Cotar has investigated an approximate DFT ground state energy functional where the electron-electron interaction in the Hohenberg-Kohn functional is replaced with one coming from optimal transport. This transport is a multimarginal transport problem with a Coulomb cost function, of the form $c(x_1, \ldots, x_N) := \sum_{i \neq j} |x_i - x_j|^{-1}$. In the above joint work, Cotar has shown that this optimal transport problem has a unique solution, the minimal energy in the DFT approximation is always a lower bound for the true quantum mechanical ground state energy, and in the two particle case this optimal transport energy gives the semiclassical limit of the Hohenberg-Kohn functional. Finally, it was announced that there is an ongoing joint work with Petrache in which they are analyzing higher order corrections to this semiclassical limit in the general N-particle case.

Katy Craig presented recent results concerning existence, uniqueness, and stability of solutions to PDE involving both degenerate diffusion and aggregation. Specifically, the focus was on equations of the form

$$\partial_t \rho = div((\nabla K * \rho)\rho) + \Delta(\rho^m)$$

which arise in many contexts such as biological chemotaxis, swarming, and modeling granular media. Such equations are known to correspond, at least formally, to gradient flows of certain energies under the Wasserstein metric. Questions such as existence and uniqueness are known for cases where the kernel K is λ-convex for some $\lambda \in \mathbb{R}$, but there are many interesting examples (such as those raised above) that do not fall in this framework. In [8], Craig has shown existence, uniqueness, and quantitative stability estimates (double-exponential bounds in time) for gradient flows of an energy functional that is ω-convex, for some modulus of convexity ω satisfying the Osgood criterion. In further joint work with I. Kim and Y. Yao [9], Craig uses this framework to analyze the Keller-Segel equation with a hard height constraint (which can be viewed as a limiting case of the above diffusion-aggregation equation as $m \to \infty$).

Micah Warren discussed some new ideas related to a mean curvature flow of the pseudo-Riemannian structure given by optimal transport. In 2010, Kim and McCann [21] introduced a pseudo-Riemannian metric defined using the cost function in an optimal transport problem, which in particular captures the MTW condition as positivity of certain sectional curvatures under this metric. In an interesting development, Kim, McCann, and Warren [22] showed that minimizers in the optimal transport problem give rise to codimension n, volume maximizing submanifolds under the pseudo-Riemannian metric of Kim and McCann (after a conformal change). Warren suggests then that a natural line of investigation is to consider the gradient flow of the volume functional, which leads to the pseudo-Riemannian mean curvature flow. It is noted that this could be a very promising direction, as generally mean curvature flows tend to be better behaved in the pseudo-Riemannian framework compared to the positive definite case, as in for example, a result of Li and Salavessa regarding the second fundamental form of space-like manifolds under mean curvature flow in pseudo-Riemannian spaces.

Geometric optics

Cristian Gutiérrez presented recent results on free-form lens design. Taking the point source refractor problem as an example, traditional methods encase the light source in some medium, then cut a ball out centered at the light source to create a lens. However, in practice such a model results in very thick, large lenses, which are heavy and difficult to manufacture. Thus a new goal is to construct a lens sandwiched between two optically active surfaces to focus a light source into some desired energy distribution, which can potentially be made thinner. Gutiérrez discussed recent joint work with Sabra [16, 17] in which they show that given one optically active surface, they can solve a system of PDE in order to find a corresponding second surface which will combine with the first to create such a desired lens, in the far-field regime. They give conditions under which the representation of this second surface is physically realizable, along with estimates that can be given for the thickness of the resulting lens. It is noted than an important question that remains is to find an appropriate first surface so that the second surface may be relatively simple to manufacture.

Ahmad Sabra presented results on an alternative framework to construct optical surfaces in the near-field regime when the usual convex model may face physical obstructions. Using the near-field parallel, point source reflector problem as an example, Sabra discussed how the usual model of constructing a reflecting surface as intersections of ellipsoids may encounter obstructions such as self-blocking of reflected rays by the surface itself. In recent work with Gutiérrez, he proposes to instead take unions of ellipsoids which will result in reflectors that are no longer convex. Sabra is able to prove existence of such reflectors when the target energy distribution is discrete, and then pass to a limit in order to recreate absolutely continuous densities. It should be noted that regularity for this model is still unexplored and falls outside of the usual elliptic framework, in fact it would give rise to a GJE of mixed hyperbolic - elliptic type.

Geometry

Aram Karakhanyan talked about generalizations of Blashke's rolling ball theorem. The original theorem deals with the following geometric situation (which explains the name of the theorem), we are given M and M', two smooth and strongly convex surfaces whose second fundamental forms II_M and $\mathrm{II}_{M'}$ are such that $\mathrm{II}_M(x) \geq \mathrm{II}_{M'}(x')$ whenever $x \in M$ and $x \in M'$ are such that the outer unit normal vectors satisfy $n_M(x) = n_{M'}(x')$. Blashke's theorem says that if M and M' are internally tangent at one point then M is contained in the convex region bounded by M'. In the OT community, this statement is reminiscent of the local-to-global property for costs satisfying the MTW condition.

Karakhanyan presented a generalization of this result where instead, one considers M being strongly c-convex and M' a c-hyperplane with respect to a cost function c satisfying certain natural structural assumptions. He also demonstrated one important application of this result, to parallel reflector and refractor problems.

Young-Heon Kim presented a new attempt at defining a canonical notion of the barycenter of a measure on a metric measure space, via a process of Wasserstein regularization. In joint work with Pass [23], if $\mu \in \mathcal{P}(M)$ for some metric measure space (M, d, m), and $\varepsilon > 0$, then they consider the problem of minimizing the functional

$$\nu \mapsto \int_X \int_X d^2(x,y) d\mu(x) d\nu(y) + \varepsilon W_2^2(\nu, m)$$

over $\nu \in \mathcal{P}(M)$, where here W_2 is the 2-Wasserstein distance between probability measures on M. For each $\varepsilon > 0$, there exists a unique minimizer μ_ε, and moreover the μ_ε converge weakly to some limiting measure $B(\mu)$ as $\varepsilon \searrow 0$, which is supported in the set $b(\mu)$, the set of barycentric points of μ that minimize the quantity $\int_M d^2(x, \cdot)\mu(dx)$. Moreover, $B(\mu)$ can uniquely be characterized as the minimizer of $W_2(\cdot, \mu)$ over probability measures supported on $b(\mu)$. Additionally, Kim and Pass have analyzed some of the dynamics that the mapping $\mu \mapsto B(\mu)$ induces over the space $\mathcal{P}(M)$ of probability measures, including fixed points and periodic orbits.

Yi Wang presented recent developments on a nonlinear Sobolev trace inequality involving the k-Hessian energy in place of the usual Dirichlet energy. It is a classical result that if u is a function on Ω with $u = f$ on $\partial\Omega$ and u_f is the harmonic extension of f to Ω, then

$$\int -u\Delta u \, dx + \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u \partial_\nu u \, \sigma(dx) \geq \frac{1}{|\partial\Omega|} u_f \partial_\nu(u_f) \, \sigma(dx)$$

where ν is the outward unit normal and σ the surface measure on $\partial\Omega$. One question that arises is if the first term above (which corresponds to the 1-Hessian energy) can be replaced by the k-Hessian energy for $k > 1$, and in this case what the appropriate choice of boundary operator to replace ∂_ν may be. In joint work with Case [4], Wang has shown that such a boundary operator exists, along with a corresponding "polarized" version of the inequality, yielding a sharp Sobolev trace inequality for k-admissible functions.

Robin Neumayer discussed a one parameter family of variational problems on the upper half space in $\mathbb{R}^n$, where certain values of the parameter correspond to optimizers of the classical L^p Sobolev inequality with zero boundary condition, and the L^p Sobolev trace inequality. In joint work with Maggi [29], in the case of $n \geq 2$ and $p > 1$, Neumayer succeeded in characterizing minimizers for all positive values of the parameter, along with a detailed analysis of the properties of the optimal value as a function of the parameter. Additionally when $p = 2$ and $n \geq 3$, the variational problem can be interpreted as minimizing certain curvature quantities over a class of conformal changes of the Euclidean metric, and the aforementioned characterizations of minimizers prove that in for various ranges of the parameter one recovers exactly the spherical, flat, and hyperbolic geometries.

Deane Yang gave an overview of the current state of the L^p Brunn-Minkowski theory. The L^p Minkowski problem is, given a Borel measure μ on $\mathbb{S}^{n-1}$, to find a convex body K such that the L^p surface area measure of the body is μ. Here, the L_p-surface area of a convex body K with outer normal ν_K is a Borel measured over $\mathbb{S}^{n-1}$ given by

$$S_p(K,\omega) := \int_{x\in {\nu_K}^{-1}(\omega)} (x\cdot \nu_K(x))^{1-p}\, d\mathcal{H}^{n-1}(x), \ \ \forall\, \omega \in \mathcal{B}(\mathbb{S}^{n-1}).$$

The case $p = 1$ corresponds to the classical Minkowski problem of prescribed Gauss curvature, and for $p \geq 1$, the problem can be attacked with a variational approach. Yang presented recent results on a particularly interesting case which is when $p \to 0$, which is the logarithmic Minkowski problem of prescribing the cone volume measure of a body. In joint work with Böröczky, Lutwak, and Zhang [2], Yang has shown that under something called the subspace concentration inequality, the log Minkowski problem has a solution. Finally, Yang closed with a question, asking if the case corresponding to $p = -n$ can be recast as an optimal transport or generated Jacobian equation, which could provide a new tool for this difficult and important case.

Reception of the workshop and participant's feedback

Given the reception of the workshop, it is safe to say that the goal of bringing several different communities of researchers together was achieved successfully. In particular, several new collaborations were initiated at the conference. Katy Craig reports that motivated by the discussions during the conference, she has started a project with Micah Warren where they will be exploring connections between the Jordan-Kinderlehrer-Otto scheme from gradient flows and game theory. Galichon and Carlier indicated that as a result of the workshop they have started on a new project involving a hedonic model with tight budget constraints. Abedin expressed that as a result of the workshop he is considering embarking on a project involving regularity for GJE equations under a strong MTW type condition.

There was also a very successful open problem session which was originally planned to last for one hour, but ended up going for twice that. Problems were proposed by Alfred Galichon, Jun Kitagawa, Aram Karakhanyan, Ahmad Sabra, and Alessio Figalli.

The participants' feedback was extremely positive and enthusiastic, with many highlighting the interdisciplinary nature of the workshop, here we present quotes from four different participants:

I learned about the fascinating parallel between problems in reflector design in geometric optics and problems on taxation in matching markets.

The connection between generated Jacobian equations and economics seems very intriguing and hopefully will spark some ideas for future research.

My work is focused on economics, and before this conference, I was unaware of both the vast theoretical literature in economics as well as applications to optics and other areas. The conference was an invaluable point of entry into these areas, accomplishing in a few days what would have taken me months or years to do on my own. I will make good use of what I've learned in my continuing research.

The introductory lecture was very useful, helping me put many of the subsequent talks into context. The open problem session was also great.

Participants

Abedin, Farhan (Temple University)
Acciaio, Beatrice (London School of Economics)
Bowles, Malcolm (University of British Columbia)
Carlier, Guillaume (Université Paris Dauphine)
Chen, Eric (Princeton University)
Cotar, Codina (University College London)
Craig, Katy (University of California Santa Barbara)
Figalli, Alessio (ETH Zurich)
Galichon, Alfred (New York University)
Guillen, Nestor (University of Massachusetts Amherst)
Gutierrez, Cristian (Temple University)
Hamfeldt, Brittany (New Jersey Institute of Technology)
Jhaveri, Yash (ETH Zurich)
Karakhanyan, Aram (University of Edinburgh)
Kim, Young-Heon (University of British Columbia)
Kitagawa, Jun (Michigan State University)
Mawi, Henok (Howard University)
McCann, Robert (University of Toronto)
Mérigot, Quentin (Université Paris-Dauphine / CNRS)
Neumayer, Robin (University of Texas)
Pass, Brendan (University of Alberta)
Sabra, Ahmad (University of Warsaw)
Saldanha Salvador, Tiago Miguel (Mcgill University)
Samuelson, Larry (Yale University)
Schmitzer, Bernhard (TU Munich)
Thibert, Boris (LJK Universite Grenoble)
Wang, Yi (Johns Hopkins University)
Warren, Micah (University of Oregon)
Yang, Deane (NYU Tandon School of Engineering)
Zhang, Shuangjian (University of Toronto)

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Chapter 7

Quantum Field Framework for Structured Light Interactions (17w5079)

April 23 - 28, 2017

Organizer(s): David L. Andrews (University of East Anglia, UK), Robert Boyd (University of Ottawa, Canada), Konstantin Y. Bliokh (RIKEN, Japan), Mark R. Dennis (University of Bristol, UK), Alexander Lvovsky (University of Calgary, Canada), Duncan O'Dell (McMaster University, Canada)

Overview

It is widely recognised that structured light, whose theoretical foundations were first laid twenty-five years ago, serves as a near-perfect test case for studying the quantum-classical boundary in physics. The pace of advances in both theory and experiment, since the original concept was developed, has led to the research community identifying a wide range of issues demanding urgent attention, to resolve conflicts between representations, with the aim of securing a consistent, agreed framework to describe the photonic interactions of such light. Specific problems had already been identified in connection with canonical and non-canonical forms of operator, coherence and quantum uncertainty, and the quantized forms of both linear and angular optical momentum.

Such is the backdrop to the initial 2015 proposal for this workshop which, focused on the theory and mathematics of the subject, from the outset received enthusiastic international support from all quarters. Although unconnected, the importance of this topical area was substantially underscored by the publication, only three months before the workshop took place, of a major 'roadmap' review of the significance and practical potential for the whole field of structured light. [1] Six of the participants at this workshop featured in the long list of authors.

It was rewarding to find that only a handful of the experts who had been informally approached were ultimately unable to accept formal invitation; shortly before the workshop took place, almost every one of the places available was filled. Most participants were theorists, but a handful of especially knowledgeable experimentalists were also included, to sharpen the focus on issues of key practical significance. A few individuals had to pull out just before the conference began, all of them for family or health-related reasons. In the event, thirty-five individuals were able to participate, representing ten nations: Australia, Canada, Germany, Japan, Kazakhstan, Netherlands, Poland, UK, USA and Singapore.

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The workshop was structured to provide an opportunity, without obligation, for each member to deliver a short lecture (some participants needed this explicit opportunity in order to secure independent funding to travel to the meeting). In the event, twenty-four talks were given, as listed in Section 2 below, and several other papers were also circulated during the meeting. The aim was to allow some preliminary points of discussion to be recognized, prior to full discussion sessions later in the week.

By mid-week the workshop had identified eight different topics, as shown in Section 3, which a show of hands indicated were considered the most worth revisiting in more detailed discussions. Together, these addressed a sizeable subset of the issues that had been identified as possible topics, at the planning stage. Not surprisingly, many participants were interested in all of them. Nonetheless, for expeditious discussion in groups of a practicable size, three groups were formed to pursue, in parallel sessions, loosely cognate sets of topic. Each group was assigned two members to report back to the reassembled full meeting the essential content and upshot of their discussions, as detailed in Section 4. A summing up session for all attendees was held on the last day of the workshop.

Immediately following the meeting, the Contact Organizer solicited brief feedback from all participants. Along with numerous messages of congratulations and thanks for the meeting, some very significant and promising new lines of investigation were identified, many of them involving a prospective collaboration between workshop members – and in many cases interactions between individuals who had not worked or even met before. They represent a substantial legacy, one that can be expected to lead to tangible fruit in the form of new work and publications in due course.

Lecture schedule

1. Iwo Bialynicki-Birula. *Do electromagnetic waves with fixed orbital angular momentum exist?* Examining orbital angular momentum and spin angular momentum shows that a relativistic treatment introduces a coupling between them. Neither of them satisfies a conservation relation; only the total angular momentum is a conserved quantity.

2. Elliot Leader: *The elementary particle angular momentum controversy: lessons from laser optics.* Informing participants in the field of laser optics about the recent angular momentum controversy in the particle physics community.

3. Ivan Fernandez-Corbaton: *An algebraic approach to light-matter interactions.* Expounding an S-matrix based methodology for the study and engineering of light-matter interactions, exhibits the ease with which symmetries and conservation laws enter the formalism, with an algebraic character that is suitable for computer implementation.

4. Garth Jones: *Spherical descriptions of photon fields compared with plane wave descriptions.* Typically when describing photonic events, the photon fields are cast as plane wave vector fields. In an alternative approach the fields are described in terms of spherical waves, expressible in the mathematical language of vector spherical harmonics. This approach is complementary to that of plane waves, and is more amenable to modelling specific types of phenomena, such as isotropic processes and condensed phase energy transfer.

5. Wolfgang Löffler: *Perfect darkness in exact solutions of Maxwell's equations.* Possibilities exist to identify nontrivial lines of dark vortices in exact solutions, leading to an examination of the importance of relativistically invariant 'null' solutions. One issue with such solutions is their very broad plane-wave frequency spectrum; there are a number of strategies to narrow this down, that could lead to experimental demonstration.

6. Alexander Lvovsky: *Beating the Rayleigh limit*.

7. Miguel Alonso: *Structured Gaussian beams: ray and wave pictures*. The analogy between structured Gaussian beams and other systems can be described by two-dimensional harmonic oscillators. A ray-based (semiclassical) treatment based on a Poincaré sphere reveals the underlying geometry including beam shape and geometric (Berry) and Gouy phases, and new families of structured Gaussian beams emerge.

8. Masud Mansuripur: *Electromagnetic energy, force, torque, linear momentum, and angular momentum*. Maxwell's equations, manipulated in different ways, allow both familiar and unfamiliar treatments of electric and magnetic dipole interactions with the electromagnetic field. Different definitions of the Poynting vector and the stress tensor lead to different expressions for energy, force, torque, linear momentum, and AM. Each approach

to the classical theory is self-consistent, as well as consistent with conservation laws; each has advantages and disadvantages. The traditional definition of the Poynting vector in conjunction with the Einstein–Laub stress tensor provides the most reliable and physically sensible approach to the interactions of light and matter.

9. Duncan O'Dell: *The Abraham-Minkowski controversy: an ultracold atom perspective*. Whilst the momentum of light in vacuum is well understood, the complexity of real media means that when light propagates inside a dielectric it is hard to separate the momentum of the electromagnetic field from that of the constituent atoms. A dilute gas of ultracold atoms constitutes a particularly simple dielectric where experiments and theory can be understood from first principles. In particular, the well-known experiment by Campbell et al. [Phys. Rev. Lett. 94, 170403 (2005)] is usually considered to verify the Minkowski result. But this is wrong: both Abraham and Minkowski give the same result for their particular set-up. Also, a geometric phase is associated with the Abraham-Minkowski problem – an optical analogue of the He-McKellar-Wilkens phase of an electric dipole moving in a magnetic field. This phase can be measured using an atom interferometer.

10. Halina Rubinsztein-Dunlop: *Quantum aspects of structured light interactions, and ultracold atomic ensembles*.

11. Mohamed Babiker: *The current status and future of structured matter vortex waves.* A critical review of current work focuses on the practical limitations, i.e. the feasibility of experimental work in various types of matter wave, questioning what new physics is expected in each category, and what new applications may be envisaged.

12. Konstantin Bliokh: *Edge modes, degeneracies, and topological numbers in non-Hermitian systems.* Fundamental problems arise at the interface between two hot areas of modern physics: (a) non-Hermitian quantum mechanics and (b) topological states in wave systems. Non-Hermitian systems can have nontrivial topological properties and new type of chiral edge modes, determined by so-called 'exceptional points' (i.e. degeneracies in the bulk spectrum) and two types of topological numbers.

13. Kobus Kuipers: *Using local field topology for controlling quantum transitions.* Two topics: firstly, how to measure various vectorial components of both the electric and magnetic fields of light guided in nanophotonic structures, where phase- and polarization singularities are ubiquitous near the nanostructures. Secondly, it emerges that C-points and the field topology around them could be used to deterministically direct the emission of quantum transitions involving a change in orbital angular momentum.

14. Robert Boyd: *The promise of orbital angular momentum for quantum communications.* Hanbury Brown and Twiss interferometry with twisted light.

15. Robert Fickler: *Experimental limitations for large orbital angular momentum quantum numbers.* Several approaches generate photons with large OAM quantum numbers. One technique involving spin-to-orbit coupling uses tailored q-plates to generate photons with up to 200 quanta of the reduced Planck's constant, $\hbar$. Off-the-shelf spatial light modulators are limited to generating photons with up to 300 quanta. A third, recently established spiral phase mirror technique makes outputs with quantum numbers of 10000 possible. The mirrors are cut surface into such that the depth modulation corresponds to the azimuthal phase modulation, on optical reflection.

16. Ebrahim Karimi: *High-dimensional intra-city quantum cryptography, and quantum hacking with structured photons.*

17. David Andrews: *Representations of structured light and their quanta.* Questions arise over the extent to which common figurative representations of structured light and photons are defensible, and the possibility of their conveying misinformation. A look at quantum uncertainty in the light of a reformulated expression for optical mode orthogonality suggests implications for the number-phase quantum uncertainty in vortex light.

18. Enrique Galvez: *High-dimensional spaces of polarization and spatial mode of photons.* There are modes of light that give rise to spatially-variable polarization; this is a topic that touches of the fundamentals of light, and especially on the information that can be encoded into it.

19. Peter Banzer: *Transverse angular momentum of light*.

20. Martin Lavery: *Atmospheric turbulence and the propagation of structured modes.*

21. Alexander Khanikaev: *Photonic topological insulators: from theory to experimental realization*.

22. Daniel Leykam: *Topological phases and topological photonics*. Connections are established between topological photonic systems and structured light. Firstly it can be shown how Bloch band degeneracies such as Dirac points display analogies with optical vortices. Secondly, examination of the microscopic structure of optical fields in topological photonic systems, such as the presence of polarization singularities or optical vortex rings, may be used to diagnose topological phases

23. Akbar Salam: *Quantisation of the electromagnetic field in magnetodielectric media*. Considering the formulation of electrodynamics in a magnetoelectric medium, from both the classical and quantum mechanical points of view, raises questions regarding the potential utility of employing a macroscopic theory of QED to study structured light-matter interactions.

24. Nader Engheta: *Zero-index materials: some wave physics and quantum optical features*.

Headline topics

The following listing of topics were decided upon as especially worth focusing upon in detailed discussion:

(i) the separability of quantum spin and orbital angular momentum, and the role laser optics might play in determining their limits of measurability;

(ii) degenerate down-conversion, and the implications of quantum uncertainty on the spatial origin of correlated photon pairs;

(iii) defensible forms of the quantum number - phase uncertainty relation, as regards structured modes of light;

(iv) the relativistic origin of spin-orbit coupling, and its implications for electron vortices;

(v) dispersive effects in the Abraham and Minkowski formulations of linear and angular momentum, for photons travelling through a dispersive medium;

(vi) potential limitations of a fundamental quantum nature on the information content of an individual photon;

(vii) the role of local field symmetry and topology in structured light and other chiroptical interactions with matter;

(viii) cold atoms, elementary particles and their connections with quantised orbital angular momentum.

Session discussions

Three discussion groups were subsequently convened to address these issues, with each participant free to attend and participate in whichever proved to interest them most. Two reporters were assigned from within each group:

(A). Spin and orbital angular momentum separability [Konstantin Bliokh (Japan) and Ivan Fernandez-Corbaton (Spain)] (Room 202);

(B). Structured matter waves and quantised angular momentum [Halina Rubinsztein-Dunlop (Australia) and Duncan O'Dell (Canada)] (Room 107);

(C). Quantum uncertainty in structured light and down-conversion [Martin Lavery (United Kingdom) and Jörg Götte (China)] (Lecture Theatre).

The following summarizes the outcome of these discussions, as presented to the reconvened full workshop.

In *Group A*, where there were extensive discussions about the separability of orbital and spin angular momentum, it became clear that there are two distinct frameworks, whose individual usefulness depended to some extent on the realm of application.

In Framework 1, helicity is an independent physical quantity related to dual symmetry. The integral values of spin and orbital angular momentum (AM) can be separated in the most generic EM fields in free space. However, local spin and orbital AM densities can be separated only in monochromatic optical fields. To obtain these quantities one can use either (i) canonical operators – which do not satisfy the transversality constraint, but obey canonical SO(3) commutation relations – or (ii) modified operators compatible with transversality, but obeying modified commutation relations. Modified operators with unusual commutation relations are natural in the presence of gauge fields (e.g. the momentum operator of a charged particle in a magnetic field). In the case of light, this is the Berry-connection field in momentum space. Importantly, the densities of the spin AM and momentum in a monochromatic field are directly measurable via radiation torque and force on a point dipole particle. They also satisfy separate local conservation laws (continuity equations).

In Framework 2, only transverse operators are allowed. The operators that are used are connected with symmetry transformations, and this is exploited in applications to experiments. The typical explanations of experimental results that are made using spin and orbital AM have a symmetry explanation in this framework. Sometimes, there are two different symmetry explanations for what would correspond to spin to orbital AM transfer in Framework

1. Total angular momentum is the only operator considered to be connected with physical rotations. Helicity plays an important role in this framework.

In *Group B* there were three sessions focused on structured matter waves: 1) Cold atoms and optical angular momentum; 2) Electron, neutron, and other matter waves; 3) The Abraham-Minkowski controversy.

1) The term 'cold atoms' refers to atoms cooled to less than one millionth of one degree above absolute zero, sometimes even 1000 times colder. At these temperatures the de Broglie wavelength becomes long enough to extend between the atoms in a trapped gas so that the gas becomes quantum degenerate. If the atoms are bosons they will Bose condense into a single quantum state forming a macroscopic quantum matter wave with a high degree of coherence. Bose-Einstein condensates (BECs) with orbital angular momentum (OAM) were first made by stirring a condensate with a laser beam, forming quantized vortices in the superfluid. The meeting discussed a more sophisticated method developed in a series of experiments by Bill Phillip's group at NIST [M. F. Andersen et al, "Quantized Rotation of Atoms from Photons with Orbital Angular Momentum", Phys. Rev. Lett. 97, 170406 (2006); C. Ryu, "Observation of Persistent Flow of a Bose-Einstein Condensate in a Toroidal Trap", Phys. Rev. Lett. 99, 260401 (2007)] where a Laguerre–Gaussian laser beam was interfered with a counter-propagating Gaussian beam to form a corkscrew-like interference pattern. The atoms diffract off this intensity pattern (via a stimulated Raman transition) and in so doing worth of angular momentum is transferred from the light to the external motion of each atom. This method can be used to introduce a singly-quantized vortex into a BEC, and also to rotate a BEC in a ring trap. Related experiments have also been performed in Bigelow's group [K. C. Wright, L. S. Leslie, and N. P. Bigelow, "Optical control of the internal and external angular momentum of a Bose-Einstein condensate", Phys. Rev. A 77, 041601(R) (2008)].

An interesting basic question that has been tackled by some of the participants is whether optical OAM influences transitions between internal states in atoms. The selection rules for a dipole transition are $\Delta L = \pm 1$, $\Delta m = 0, \pm 1$. In traditional experiments where the laser carries no OAM the change in angular momentum of the atom is supplied by the 'spin angular momentum' (SAM) of the light: circularly polarized light carries $\pm\hbar$ of SAM depending on its handedness. The question is then whether OAM can play a similar role in transitions as SAM. A theory paper in 2002 [M. Babiker, C. R. Bennett, D. L. Andrews, and L. C. Davila Romero, "Orbital Angular Momentum Exchange in the Interaction of Twisted Light with Molecules", Phys. Rev. Lett. 89, 143601] suggested that dipole transitions would be unaffected by OAM and the lowest multipole transition that could be affected by OAM is the electric quadrupole transition. A very recent experiment has confirmed the first part of this prediction [F. Giammanco et al., "Influence of the photon orbital angular momentum on electric dipole transitions: negative experimental evidence", Opt. Lett. 42, 219 (2017)].

Open questions in this area include whether there are efficient methods to generate OAM in beams of atoms (as opposed to the case where the atoms are held in a trap) and whether having cold atoms with OAM would facilitate precision measurements. In contrast to light, atoms are massive and thus sensitive to gravity, and typically they have magnetic moments and are also electrically polarizable, and thus sensitive to material properties. Moreover, in contrast to electrons they are neutral and 1800 times more massive.

2) Electron beams are a very active area of investigation, not least because of their technological applications such as in electron microscopes. Like light and atoms, beams of these particles can be structured, including Airy and vortex beams [J. Harris et al., "Structured quantum waves", Nature Phys. 11, 629 (2015); H. Larocque and E. Karim, "A New Twist on Relativistic Electron Vortices", Physics Viewpoint Physics 10, 2 (2017)]. A significant motivation for studying such structured beams is the possibility of developing probes for magnetic structure at the nanoscale, benefitting from the fact that a vortex is a sub-wavelength structure.

Airy beams are wave-packets with the remarkable property that they appear as though they are accelerating, even though they are propagating in free space [N. Voloch et al., "Generation of Electron Airy Beams", Nature 494, 331 (2013)]. These beams also 'self-heal' if an obstacle is placed in their path. Vortex beams of electrons were first created in the lab in 2010 [J. Verbeeck, H. Tian, and P. Schattschneider, "Production and application of electron vortex beams", Nature 467, 301 (2010)]. The theoretical analysis of electron beams is involved and depends upon whether one is in the non-relativistic regime (described by the Schrödinger-Pauli equation) or the relativistic regime (described by the Dirac equation). In non-relativistic electron beams OAM and SAM are independently conserved [K. Y. Bliokh, Y. P. Bliokh, S. Savel'ev, and F. Nori, "Semiclassical Dynamics of Electron Wave Packet States with Phase Vortices," Phys. Rev. Lett. 99, 190404 (2007)]. However, in the relativistic regime these two angular momenta become coupled [K.Y. Bliokh, M. R. Dennis, and F. Nori, "Relativistic electron vortex beams:

Angular momentum and spin-orbit interaction" Phys. Rev. Lett. 107, 174802 (2011)]. There has recently been some controversy as to whether a true vortex can exist in an electron beam. In particular, in a theoretical paper by one of the participants [I. Bialynicki-Birula and Z. Bialynicka-Birula, "Relativistic Electron Wave Packets Carrying Angular Momentum," Phys. Rev. Lett. 118, 114801 (2017)] it was shown that the SAM and OAM contributions to the vorticity exactly cancel each other out. A non-zero vorticity is seen as a crucial feature of a vortex. Nevertheless, the probability density does coil around the beam in a manner highly reminiscent of a vortex; practical applications are unlikely to be affected by this cancellation.

A fundamental issue with free electrons is that they are hard to polarize; as first realized by Pauli and also Bohr. The standard Stern-Gerlach method of sorting particles according to their magnetic moment by using an inhomogeneous magnetic field fails for electrons, as the Lorentz force on such light particles leads to transverse forces which completely overwhelm the force on the magnetic moment. However, recent work by some of the participants [E. Karimi, L. Marrucci, V. Grillo, E. Santamato, "Spin-to-orbital angular momentum conversion and spin-polarization filtering in electron beams", Phys. Rev. Lett. 108, 044801 (2012); E. Karimi, V, Grillo, R. W. Boyd, and E. Santamato, "Generation of a spin-polarized electron beam by multipole magnetic fields", Ultramicroscopy 138, 22 (2014)] has shown that so-called 'q-filters', i.e. inhomogeneous magnetic fields with a quadrupole distribution, can give rise to a spin-orbit interaction that couples the OAM and the SAM of the electron beam. If a beam with the correct OAM is input to the system then a spin-polarized beam is produced.

Questions for future investigation include the further development of efficient methods for producing spin-polarized electrons, and also extending the idea of matter-wave vortex beams to neutrons [C.W. Clarke et al, "Controlling neutron orbital angular momentum", Nature 525, 504 (2015)]. Concerning this last point, it was pointed out during the meeting that the neutron interferometer used to detect the vortex beam was essentially a white light (Michelson) style interferometer, due to the very low degree of transverse coherence in the neutron beam, and that this meant that the results were not very clear cut. The important problem of vortex beams of neutrons therefore remains ripe for further work. It was also noted that there are intriguing suggestions that self-accelerating beams can be used to prolong the lives of unstable particles [I. Kaminer et al., "Self-accelerating Dirac particles and prolonging the lifetime of relativistic fermions", Nature Physics 11, 261 (2015)].

3) The Abraham-Minkowski controversy refers to two different expressions for the momentum density of light inside a dielectric medium derived separately by Abraham (1909) and Minkowski (1908). Although these are classical effects it is often convenient to express them in terms of the photon momentum: $p_A = p_0/n$ and $p_M = p_0 n$, where $p_0 = \hbar k$ is the free space momentum of a single photon and n is the refractive index of the dielectric. This topic was first discussed during and following talks 8 and 9 during the earlier part of the workshop.

The momentum density of light can be extracted from the stress-energy tensor, but it turns out that there is a considerable degree of arbitrariness in writing down stress-energy tensors. Examples include those due to Abraham, Minkowski and also Einstein and Laub [M. Mansuripur, Electromagnetic Force and Momentum, in *Roadmap on Structured Light*, edited by H. Rubinsztein-Dunlop, Journal of Optics 19, 013001, 8 (2017)]. This has led some authors to prefer to focus on the actual forces involved by calculating the Lorentz force on each current (bound and unbound) in the material due to the electromagnetic field [J. P. Gordon, "Radiation forces and momenta in dielectric media", Phys. Rev. A 8, 14 (1973)].

Experiment is the ultimate arbiter in science, and over the years there have been a large number of experimental investigations that have variously agreed with either Abraham or Minkowski or have come up with their own expressions. The essential difficulty lies in separating the total momentum (which all agree must be conserved) into that due to the optical field and that due to the material. In other words, the polarization excitation induced by the optical field that propagates inside a medium is part atom, part electromagnetic field. Also, the result one finds depends upon precisely what measurements are made. One school of thought [championed by Loudon and collaborators: C. Baxter, M. Babiker, and R. Loudon, "Canonical approach to photon pressure", Phys. Rev. A 47, 1278 (1993)] is that the Abraham expression corresponds to kinetic momentum of the light and the Minkowski expression to a canonical momentum. For example, if an experiment could measure the recoil velocity of a transparent block of dielectric as a pulse of light enters it, then using the concept of the uniform motion of the centre-of-mass of energy [a principle emphasized by Einstein: A. Einstein, "The principle of conservation of motion of the center of gravity and the inertia of energy", Ann. Phys. 20, 627 (1906)] leads to the expectation that the momentum of the light in the block takes the Abraham value [S. M. Barnett and R. Loudon, "The enigma of optical momentum in a medium", Phil. Trans. R. Soc. A 368, 927 (2010)]. By contrast, in experiments involving diffraction it can be argued that it is more likely that the canonical (Minkowski) momentum sill be found because the wavelength, and hence the wave vector, is associated with canonical momentum.

During the discussions, participant Masud Mansuripur gave a tutorial on the classic Balazs thought experiment where a pulse of light enters a dielectric block [N. L. Balazs, "The energy-momentum tensor of the electromagnetic field inside matter", Phys. Rev. 91, 408 (1953)] and using only a knowledge of the Fresnel reflection coefficients obtained the result that the momentum density is the average of the Abraham and Minkowski results, $p = p_0(n + n^{-1})/2$ (in contrast to the Abraham momentum density result found using the principle of uniform motion of the centre-of-mass energy). Mansuripur's result is also obtained if one instead calculates the Lorentz force on the bound currents and charges in the material [M. Mansuripur, "Radiation pressure and the linear momentum of the electromagnetic field", Opt. Express 12, 5375 (2004)]. Mansuripur went on to discuss the case of the recoil of a mirror immersed in a fluid, investigated in the classic experiment by Jones [R. V. Jones and J. C. S. Richards, "The pressure of radiation in a reflecting medium", Proc. Roy. Soc. A 221, 480 (1954)] whose results support the Minkowski result, the idea being that the mirror's recoil is a direct measure of the momentum of light in a medium. One issue that Mansuripur highlighted, yet to be tested experimentally, is the effect of varying the Fresnel reflection coefficients of the mirror. This controls the interference fringes that form in the medium between the incident light and that reflected from the mirror. These intensity fringes lead to forces on the molecules in the fluid as momentum transfers between light and medium.

A number of suggestions for future investigations were made: firstly, to repeat the Jones experiment as discussed above, with varying mirror reflectivity. Secondly, it was suggested to repeat Ashkin's surface bulge experiment in which the reaction of a fluid surface to an incident beam of light is measured (the surface can either bulge outward or be pressed inward) [A. Ashkin and J.M. Dziedzic, "Radiation pressure on a free liquid surfac", Phys. Rev. Lett. 30, 139 (1973)] as the interpretation of this experiment is controversial. Thirdly, there has been a suggestion [E. A. Hinds and S. M. Barnett, "Momentum Exchange between Light and a Single Atom: Abraham or Minkowski?", Phys. Rev. Lett. 102, 050403 (2009)] for measuring the recoil of a single atom due to a laser pulse – which could potentially be attempted with cold atomic gases, provided a clever way is found to circumvent the problems associated with the random recoil due to spontaneous emission.

In *Group C* discussions centred on: 1) Degenerate down conversion; 2) Number-phase uncertainty; 3) Information content of photons, and its limits.

1) Discussion was launched by considering the consequence of a recent publication [K. A. Forbes, J. S. Ford and D. L. Andrews "Nonlocalized generation of correlated photon pairs in degenerate down-conversion", Phys. Rev. Lett. 118, 133602 (2017)]. This suggests that each photon pair produced by spontaneous parametric down conversion (SPDC) can originate from spatially distinct locations. Interest centred on whether this would affect the information content of the down-converted biphoton, and how any such change of information might manifest itself. It was agreed that considering only plane waves as excitation during SPDC is a limiting case, which led to the question of whether a superposition of wave-vectors is necessary to explain and measure this effect.

2) Number-phase quantum uncertainty remains a vexed issue, and despite the special significance of phase in connection with structured light it seems to have received little attention in the literature. It is well known that there is no simple or completely defensible form for an optical phase operator, partly due to the cyclical nature of its parameter space. In the course of the discussions on number-phase uncertainty, participant Iwo Bialynicki-Birula gave a brief tutorial on a negentropic representation, cast in terms of limits on azimuthal measurability. This leads to questions about the phase of a single photon, which it was agreed should be rephrased with regard to some form of reference. Phase is only relevant relative to another photon, or material oscillation, begging the question of whether there is an operator which could realise this and how it could best be implemented.

3) This group then discussed that the extent of accessible information of a photon, recognising that it is subject to boundary condition of the measurement process, so that in a sense every measurement of information relies on a restricted Hilbert space. This in turn should the affect the fundamental uncertainty relation and leads to the question whether there is a correction required to account for this. Michael Mazilu pointed to his work on optical eigenmodes as offering a possible way forward [M. Mazilu et al., "Optical eigenmodes; exploiting the quadratic nature of the energy flux and of scattering interactions", Opt. Express 19, 933 (2011)].

Other, informal group discussions focused on the C points occurring in photonic crystal waveguide modes, as observed in experiments by Kopus – a relatively unexplored area of research at the intersection between structured light and topological phases of light and matter. This group identified pressing questions that should be addressed in the near future:

1) What is the relation between the C points observed experimentally in real space modal profiles, and momentum space C points occurring in models of photonic topological insulators?

2) Present designs for photonic topological insulators are basis on silica-rod photonic crystals. For experimental probes of near-field properties, it would be better to employ a design utilising air-hole photonic crystals. Accordingly, there is a need to investigate whether such a design supports topological edge modes.

3) Once an appropriate design is identified, it would be interesting to study the near field structure of the topological edge modes, and compare against the occurrence of C points in previously-studied photonic crystal waveguides.

Outcomes and future plans

In the closing session, in addition to many voices of satisfaction with the meeting there were calls for a follow-up workshop, perhaps in two or three years' time. In the meanwhile there will be a follow-up to another suggestion, that proceedings of work initiated by this workshop might be combined with content from this summer's now-biennial International Conference on Optical Angular Momentum [Capri, 18-22 September 2017] in a special issue of some suitable journal. The organisers of that conference have now been approached, to explore this possibility.

Numerous new collaborations were instigated directly through the workshop; those that have been reported thus far involve groupings of individuals who together represent more than a third of the workshop participants, specifically including Alonso, Andrews, Bliokh, Fernandez-Corbaton, Galvez, Jones, Karimi, Khanikaev, Leader, Leykam, Loeffler, Mazilu, O'Dell and Salam.

Press release

At this early point in the twenty-first century it is already apparent that a paradigm shift is taking place in numerous areas of technology, as the new science of photonics increasingly outperforms and displaces twentieth century electronics. Prominent examples are high throughput nanoscale connectivity in IT systems, secure and enhanced telecommunications, and quantum computing. Some of the latest advances, aiming to exploit the distinctive quantum properties of light, convey information in the form of highly unconventional structured beams of light. This potentially transformative field is revolutionising many areas of optical physics, yet it has raised numerous questions in urgent need of address at the level of fundamental theory. This workshop brought top experts together in the aim of resolving these issues, laying new mathematical foundations to secure the ground for further progress.

Acknowledgement

Thanks to all other organisers and participants: above all, and on behalf of all, hearty thanks to BIRS for making this workshop possible, intellectually profitable, and thoroughly enjoyable.

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Chapter 8

Phase Transitions Models (17w5110)

April 30 - May 5, 2017

Organizer(s): Yaniv Almog (Louisiana State University), Petru Mironescu (University Claude Bernard Lyon 1)

Overview

The term phase transition (or phase change) is most commonly used to describe transitions between different states of matter (solid/liquid/gas, superconducting/normal, ferromagnetic/paramagnetic, isotropic/nematic, etc.). Second-order phase transition models have been in the focus of mathematical research for a number of decades. As opposed to first-order models, where the transition occurs along a sharp interface (e.g. the surface of a droplet), second-order ones, developed under the impulsion of Lev Landau, phenomenologically include a smoothing (gradient like) term leading to diffuse interfaces and transition phenomena. Mathematically, one is lead to consider in the models a small parameter (typically, $\varepsilon > 0$) interpreted as the characteristic length of the interface, and to replace the discrete order parameter u (with e.g. $u = 0$ or $u = 1$ in the case of a two-phase fluid) by a continuous one. This order parameter could be either a scalar, as in the discrete case, or a complex u (with $|u|^2$ interpreted as a density) or a probabilistic function of the orientation of molecules (the Q-tensor below).

Mathematical issues within these models are related to the study of the well-posedness of the boundary conditions, the behavior of minimizers or critical points of the appropriate energy functional, approximate description of transition layers, description of metastable states, asymptotics as $\varepsilon \to 0$, justification of first-order transition models as limits of the second-order one. More specialized directions concern the sound justification of physical observation as expulsion of magnetic flux (Meissner effect), existence of critical fields, existence of lattices of vortices in type II superconductors (Abrikosov lattices), etc.

A very partial list of second-order phase transition models include

1. The van der Waals-Cahn-Hilliard theory of phase transitions in a fluid. In this theory, the total energy of the fluid filling the container $\Omega \subset \mathbb{R}^n$ is assumed to be given by $\varepsilon^2|\nabla\rho|^2 + W(\rho)$, where ε is a small parameter and $W : \mathbb{R} \to [0, \infty)$ is a two-well potential, vanishing at a and b. A typical example is $W(\rho) = \rho^2(1-\rho)^2$, with $a = 0$ and $b = 1$. When $\varepsilon = 0$, minimizers take only the values a and b. In the limit $\varepsilon \to 0$, minimizers or critical points of bounded energy of the rescaled energy

$$F_\varepsilon(\rho) = \int_\Omega \left(\frac{1}{\varepsilon}|\nabla\rho|^2 + \varepsilon W(\rho)\right),$$

are expected to get closer and closer to a and b. In the typical example above, the limit should be the characteristic function of a subset $A \subset \Omega$. The shape of A depends on the constraint on ρ: mass constraint

$\int_\Omega \rho = m$, contact energy with the walls of the container, for example of the form $\varepsilon\sigma(\rho)$, etc. In each case, minimizers approximate a two-phase configuration satisfying a variational principle related to the equilibrium configuration of liquid drops.

Variants of the above functional include vector-valued unknowns $\rho : \Omega \to \mathbb{R}^d$ and adapted potentials $W : \mathbb{R}^d \to [0, \infty)$.

2. The Ginzburg-Landau model. This model and its variant are relevant for the theory of superconductivity and Bose-Einstein condensates.

   Its simplest version is given by the energy functional

$$E_\varepsilon(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 \right) , \tag{8.0.1}$$

   where $u \in H^1(\Omega, \mathbb{C})$, $\Omega \subset \mathbb{R}^n$, and $\varepsilon > 0$ is a (small) parameter. The physically relevant quantity here is $|u|^2$, which plays the role of a density. The unknown function u can be subject to various boundary conditions (Dirichlet, semi-stiff, etc.).

   There are numerous generalizations and modifications for this functional. One can, for instance, discuss the above model on manifolds, to consider $u \in H^1(\Omega, \mathbb{R}^n)$, etc. One can also consider time-dependent gradient flow (with either heat, Schrödinger or wave operator for the time derivatives).

   This simplified energy is obtained by neglecting the effect of magnetic field in the Ginzburg-Landau model of superconductivity. In two dimensions, the full energy functional takes the form

$$G_\varepsilon(u) = \int_\Omega \left(|\nabla_A u|^2 + |\nabla A - h_{ex}|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right) . \tag{8.0.2}$$

   In the above $A \in H^1(\Omega, \mathbb{R}^2)$ denotes the magnetic vector potential (∇A is the magnetic field), h_{ex} is the applied magnetic field and $u \in H^1(\Omega, \mathbb{C})$ denotes the superconductivity order parameter. The minimization of G_ε can be carried out without any prescribed boundary conditions, but one can also prescribe boundary conditions other than the natural ones, on some subset of the boundary.

   Another variant is obtained by adding a traping potential, leading to the Bose-Einstein model for condensates. The effect of electric current can also be added to the Ginzburg-Landau model of superconductivity, in which case the problem is not variational anymore (it cannot be formulated in terms of an energy functional that needs to be minimized).

   Other variants include spin models for ferromagnetic materials. Mathematically, this involves two unknown wave functions u_1, u_2, weakly coupled through the potential terms.

3. Oseen-Frank model of nematic and smectic liquid crystals. Liquid crystals are an intermediate state between liquids and solid crystals. Their main mathematical feature is the orientation of the molecules. In the simplest model, one associates to each molecule in the container $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ an orientation $n = n(x) \in \mathbb{RP}^2$, and the energy to minimize is $\int_\Omega |\nabla n|^2$ (equal elastic constants). One may further simplify by replacing the physically realistic target $\mathbb{RP}^2$ by $\mathbb{S}^2$.

   A first sophistication of this model consists of introducing a more general dependence of the energy on n. The general form of the Oseen-Frank energy density for a nematic liquid crystal in the absence of external electromagnetic fields is given by

$$\begin{aligned} \mathcal{F}_N(n, \nabla n) = K_1|\nabla \cdot n|^2 + K_2|n \cdot (\nabla n) + \tau|^2 + \\ + K_3|n(\nabla n)|^2 + (K_2 + K_4)(\mathrm{tr}(\nabla n)^2 - |\nabla \cdot n|^2) . \end{aligned} \tag{8.0.3}$$

   In the above, K_i ($i = 1, 2, 3, 4$) are dimensionless elastic coefficients which depend on the material, and τ is a real number representing the chiral pitch in some liquid crystals.

Smectic crystals are liquid crystals disposed in twisted or tilted layers. A typical Oseen-Frank model for such crystals is

$$J(\psi, n) = \int_\Omega \left(|(i\nabla + qn)\psi|^2 + \mu \left(-|\psi|^2 + \frac{1}{2}|\psi|^4 \right) + \mathcal{F}_N(n, \nabla n) \right) . \quad (8.0.4)$$

The number q is the characteristic smectic layer number: $2\pi/q$ is the characteristic layer thickness. The constant μ must be positive. The additional complex order parameter ψ has the following interpretation: $|\psi|$ characterizes the strength of the density modulation, while the (suitable normalized) phase of ψ measures the smectic layer displacement relative to that of perfect 1D crystalline order of layer periodicity $2\pi/q$.

The minimization of (8.0.2) is usually carried over all $(\psi, n) \in H^1(\Omega, \mathbb{C})H^1(\Omega, \mathbb{S}^2)$.

4. The Landau-de Gennes phase transition models. Oseen-Frank models cannot explain lines of disclination observed in 3D liquid crystals. The Landau-de Gennes phenomenological model describes liquid crystals not via the director n, but uses instead a 33 symmetric matrix $Q(x)$, interpreted as the (suitably normalized) covariance matrix of the probability distribution of $n(x) \in \mathbb{RP}^2$.

   In this case, the total energy takes the form

$$E_\varepsilon(Q) = \int_\Omega \left(f_e(\nabla Q) + f_{LdG}(Q) \right) ,$$

   where the elastic energy is given by

$$\begin{aligned} f_e(\nabla Q) &:= \frac{L_1}{2}|\nabla Q|^2 + \frac{L_2}{2} Q_{ij,j}Q_{ik,k} + \frac{L_3}{2} Q_{ik,j}Q_{ij,k} \\ &= \sum_{j=1}^{3} \left\{ \frac{L_1}{2}|\nabla Q_j|^2 + \frac{L_2}{2} (\mathrm{div} Q_j)^2 + \frac{L_3}{2} \nabla Q_j \cdot \nabla Q_j^T \right\}, \end{aligned} \quad (8.0.5a)$$

   while the bulk Landau-de Gennes energy density is

$$f_{LdG}(Q) := a \,\mathrm{tr}\left(Q^2\right) + \frac{2b}{3}\mathrm{tr}\left(Q^3\right) + \frac{c}{2}\left(\mathrm{tr}\left(Q^2\right)\right)^2 . \quad (8.0.5b)$$

   Temperature and material depending constants a, b,c determine the critical points of the bulk energy, whose nature governs the (meta)stability of the isotropic and nematic phases and the behavior of the liquid crystal. In type II liquid crystals, the molecules will tend to be aligned most of the time, leading to a Q-tensor of the approximate form $Q(x) = n(x) \otimes n(x) - (1/3)I_3$. In small regions (diffuse points or lines) Q will differ much from this form, one of the options (but not the only one) consisting of being close to $Q = 0$ (as in isotropic liquids).

5. Micromagnetics. In certain asymptotic regimes, the magnetization of a material contained in a thin film in $\mathbb{R}^3$ is described (at the mesoscopic level) by a 3D vector field $m : \Omega \subset \mathbb{R}^2$ taking values into $\mathbb{S}^2$. In dimensional reductions procedures, one is then led to the study of the asymptotic behavior of functionals of the form

$$E_\varepsilon(m) = \varepsilon \int_\Omega |\nabla m|^2 + f(\varepsilon) \int_\Omega \left| |\nabla|^{-1/2} (\nabla \cdot m) \right|^2 ,$$

   where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$; the given f accounts of the characteristics of the asymptotic regime.

6. The Bloch-Torrey model of diffusion-weighted magnetic resonance imaging (DW-MRI). For a magnetization M and an external magnetic field B the time-dependent equations

$$\frac{dM}{dt} = MB + h^2 \Delta M ,$$

   where h is a parameter. In two dimensions the steady-state problem reduces to a Schrödinger equation with purely imaginary potential.

Recent Developments

The problems that have been addressed in the workshop were spanned over a wide area of interest. Some of the problems that were considered are listed below.

1. Linear stability of trivial solutions. In particular, some attention was devoted to the spectral theory of non selfadjoint operators, which appear, for instance, when one considers the effect of electric current into the Ginzburg-Landau model, or when considering the Bloch-Torrey model.

2. Tunneling effects for either the Robin-Laplacian (or p-Laplacian) or the linearized Ginzbug-Landau model.

3. Weakly nonlinear analysis of either the Ginzburg-Landau model or for the smectic A liquid crystals model. These problems are characterized by boundary layer solution, decaying exponentially fast away from the boundary.

4. Analysis of singularities for the Ginzburg-Landau model. The issues considered include the analysis of local minimizers in the sense of De Giorgi in dimension 3, the interaction between local (singularities) and global analysis (genus) in the asymptotic analysis of minimizers.

5. Existence and linear stability of radially symmetric solutions of the Landau-de Gennes model.

6. New models of liquid crystals and their analysis. In this direction, topics include models for the free energy accounting of the shape the molecules and the role of the penalization terms in the free energy in preventing the escape of the Q-tensor from the physically realistic region. A related topic is the one of discrete models to nematic configuration, and the justification of the derivation of the continuous model as limit of the discrete one.

7. The transitions from isotropic to uniaxial, from uniaxial to biaxial nematic solutions for the Landau-de Gennes model, and the transition from smectic A to smectic C states for the Chen-Lubensky model. In particular, the results presented include a rigorous justification of the existence of Saturn rings and/or biaxial transitions around spherical droplets in nematic liquids.

8. Some general issues of analysis related to phase transition models. In particular, description of an appropriate functional space for the analysis of semi-stiff boundary problems, homotopy classes of Sobolev maps with prescribed singularities, lifting problems for Sobolev maps, justification of 3D to 2D dimensional reductions via asymptotic analysis of variational inequalities in large cylinders, and the rigorous analysis of variational problems under prescribed mass and energy constraints. In a different direction, applications of the ideas developed in relation with phase transition theories to the relaxation of total length minimization Plateau type problems were discussed. Attention was also focused on the use of Allen-Cahn type regularizations in the modeling of the motility of eukaryotic cells on substrates.

Presentation Highlights

Non selfadjoint operators

Bernard Helffer considered the complex Airy operator $-d^2/dx^2 + ix$ defined on the domain

$$D = \{(u_+, u_-) \in H^2(\mathbb{R}_+)H^2(\mathbb{R}_-) \,|\, u'_+(0) = u'_-(0) = \kappa[u_+(0) - u_-(0)]\} \,.$$

The transmission boundary condition satisfied at $x = 0$ is in frequent use when studying the above-mentioned Bloch-Torrey equations. Among other things, he presented a proof of completeness of the eigenspace, and of the simplicity of the eigenvalues. Some additional results were presented for the Robin realization of the same operator acting on $\mathbb{R}_+$ (i.e., $u'(0) = \kappa u(0)$) (cf. [29, GH]).

Yaniv Almog considered the Schrödinger operator with a purely imaginary potential $\mathcal{A}_h = -h^2\Delta + iV$ in the semiclassical limit $h \to 0$. Assuming a smooth domain and a smooth potential with no critical points ($\nabla V \neq 0$ in $\bar{\Omega}$) he obtained the left margin of the spectrum of $\mathcal{A}_h$. A lower bound of $\inf \Re\sigma(\mathcal{A}_h)$ have previously been obtained in [6, 30] for the Dirichlet realization of $\mathcal{A}_h$, an upper bound, under some significant assumptions on V, has been obtained in [8] for the Dirichlet realization. Here both upper bounds and lower bounds have been obtained in much greater generality for a variety of boundary conditions, including the above-mentioned transmission boundary condition.

Petr Siegl presented a study of the eigenfunctions of operators of the type

$$-d^2/dx^2 + |x|^\beta + iV(x) \quad ; \quad |V| \lesssim |x|^\gamma \text{ as } |x| \to \infty$$

acting on the real line, where $\beta \geq 2$ and $\gamma \geq 0$. It has been established that the system of eigenfunctions is complete. Nevertheless, very often it does not form a basis. The main result for this specific operator is that whenever $\gamma < \beta/2 - 1$ the eigenfunctions form a Riesz basis. Some negative results for the case $\gamma > \beta/2 - 1$ have been demonstrated as well (cf. [39, 43]).

Surface superconductivity

Soeren Fournais considered the 3D Ginzburg-Landau energy functional (8.0.2) in the limit $\varepsilon \to 0$. It is well known that when the applied magnetic field satisfies $\varepsilon^{-2} < h_{ex} < \varepsilon^{-1}/\Theta_0$ ($\Theta_0 \approx 0.59$ is known as de Gennes value) the minimizer decays exponentially fast away from the boundary. The energy density function on the surface (the local energy of the minimizer per unit surface area) has been studied. A proof was presented to the fact that this energy density is monotone non-decreasing with the angle between the applied magnetic field and the local tangent plane.

Ayman Kachmar presented a study of the smectic A energy (8.0.3)–(8.0.4) in the limit where the large elastic constants force the director field to be in the chiral nematics state, i.e.,

$$n^* = (\cos(\tau x_3), \sin(\tau x_3), 0)\,,$$

where the x_3 axis direction is arbitrary. In previous works [5, 31] a reduced functional where $n = n^*$ (and hence $\mathcal{F}_N = 0$) was considered. Kachmar demonstrated that the results obtained for the reduced function can be reproduced for the full functional (8.0.3)–(8.0.4), in the limit of large elastic constants (cf. [41]).

Michele Correggi began by reviewing some of the results he has obtained with Rougerie on the behavior of the minimizer of (8.0.2) near the boundary, in the surface superconductivity regime. It has been conjectured by Pan [45] that for any critical point of (8.0.2) the solution becomes uniformly distributed along the boundary as $\varepsilon \to 0$. Coreggi and Rougerie proved this conjecture for the global minimizer (cf. [24]). He then discussed the effects of curvature and corners on the energy [22, 25].

Tunneling and Weyl asymptotics

Nicolas Raymond considered the Robin Laplacian $\mathcal{L}_h = -h^2\Delta$ defined on

$$D = \{u \in H^2(\Omega) \,|\, h^{1/2}\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = u\}$$

in the semiclassical limit $h \to 0$. The leading behavior (as well as higher order terms) of the negative eigenvalues has been obtained. Assuming two points of maximal curvature on $\partial\Omega$ for a 2D symmetric domain, it is demonstrated that the gap between the first two eigenvalues can be approximated by the gap between the first pair of eigenvalues of a simple boundary operator. The presentation also contained some insights of the L^p spectral theory of magnetic Laplacians. In particular, a localization result, similar in nature to existing L^2 results, has been obtained (cf. [40, 26]).

Hynek Kovařík considered the p-Laplace operator with Robin boundary conditions on Euclidean domains with sufficiently regular boundary. In particular, the asymptotic behavior of the first eigenvalue, given by

$$\Lambda(\Omega,p,\alpha) := \inf_{\substack{u\in W^{1,p}(\Omega)\\ u\not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla u|^p \mathrm{d}x - \alpha\int_{\partial\Omega} |u|^p \mathrm{d}\sigma}{\displaystyle\int_\Omega |u|^p \mathrm{d}x},$$

when the strength, α, of the (negative) boundary term grows to infinity, depends on the geometry of the domain. Localization of the corresponding minimizers, near the point of maximal mean curvature, has been obtained as well [38].

Virginie Bonnaillie-Noël devoted her presentation to the semiclassical analysis of the Neumann realization of the magnetic Laplacian $(-ih\nabla + A)^2$ (where $\nabla A = \hat{\imath}_3$) on a smooth planar domain. As in the presentation of Raymond a symmetric domain is considered, with precisely two points on the boundary where the maximal curvature is attained (an ellipse for example). By obtaining WKB expansions of the eigenfunctions a formal derivation of the gap between the first two eigenvalues is achieved (cf. [15]).

Liquid crystals

Peter Pallfy-Muhoray reported a study where the effects of particle shape on the behavior of soft condensed matter systems were considered. A simple orientational density functional form of the Helmholtz free energy including both long-range attractive and short-range repulsive interactions is provided. It is applied to an example, taking into account nematic order due to both temperature–dependent attractive (Maier-Saupe) and concentration dependent repulsive (Onsager) interactions. The shape dependence of the attractive interactions originates in the polarizability, while the shape dependence of the repulsive interactions arises through the excluded volume. The varying phase behavior due to the relative contributions of these two effects has also been considered (cf. [46]).

Xiaoyu Zheng continued the presentation of Pallfy-Muhoray, and provided more details as to how the Onsager model can be modified to account for hard core repulsion, including the effect of shape-dependent excluded volume. A detailed derivation of the new model from the Helmholtz free energy was presented. Unlike the Onsager model where the pressure remains finite for arbitrarily high densities, the new model predicts divergence of the pressure at a critical density.

Dmitry Golovaty presented a Gamma-convergence result of the Landau-de Gennes (LdG) model (8.0.5) for a nematic liquid crystalline film in the limit of vanishing thickness. The film is assumed to be attached to a fixed surface, where an anchoring energy of the form

$$f_s(Q,\nu) = \alpha\left[(Q\nu\cdot\nu) - \beta\right]^2 + \gamma\left|(I_3 - \nu\otimes\nu)\, Q\nu\right|^2 ,$$

is assumed, where ν is a unit normal vector, and $\alpha, \gamma > 0$, $\beta \in \mathbb{R}$ are given parameters. In the limit of vanishing thickness, Γ-convergence to a limiting surface energy is proved. Then, the limiting problem was discussed in several parameter regimes (cf. [27]).

Dan Phillips considered a surface stabilized ferroelectric liquid crystal cell which, after being cooled from the smectic-A to the smectic-C phase forms V-shaped (chevron like) defects. The defects create an energy barrier that prevent (for the Γ-limit) switching between equilibrium patterns. To obtain such a transition, a gradient flow for a Chen-Lubensky energy is examined. The flow allows the order parameter to vanish, and consequently it is possible to overcome the energy barrier (cf. [44]).

Patricia Bauman presented a study of minimizers, on a bounded domain, of the Landau-de Gennes functional (8.0.5) where instead of (8.0.5a) use is made of the Maier-Saupe energy

$$f_{ms}(Q) = \begin{cases} -K|Q|^2 + \inf\limits_{\rho\in A_Q} \int_{\mathbb{S}^2} \rho(\mathbf{p})\ln(\rho(\mathbf{p}))d\mathbf{p}\,, & \text{if } Q\in\mathcal{M} \\ \infty\,, & \text{otherwise} \end{cases},$$

where

$$
\begin{aligned}
A_Q := \{\rho \in L^1(\mathbb{S}^2;\mathbb{R}) : \rho \geq 0, s\int_{\mathbb{S}^2} \rho(\mathbf{p})d\mathbf{p} = 1,\\
Q = \int_{\mathbb{S}^2}\left(\mathbf{p}\otimes\mathbf{p} - \frac{1}{3}I_3\right)\rho(\mathbf{p})d\mathbf{p}\},
\end{aligned}
\tag{8.0.6}
$$

and

$$
\mathcal{M} = \{Q \in \mathbb{R}^{33} \,|\, Q^t = Q\,;\ \mathrm{Tr}Q = 0\,;\ \sigma(Q) \in (-1/3, 2/3)\,\}\,.
$$

Note that $\mathcal{M}$ is the set of physically acceptable states, in view of the relation betwen Q and ρ given in (8.0.6). Using the fact that the energy density is singular outside the physically realistic range, it is proved that minimizers are regular and, in several model problems, take on values strictly within the physical range (cf. [9]).

Lia Bronsard provided an instructive illustration of the advantage of the Landau-de Gennes model over the more rigid Oseen-Frank or Ericksen theories. She considered a spherical colloidal particle immersed in a liquid crystal, satisfying homeotropic weak anchoring at the surface of the colloid (with strength parameter W) and approaching a uniform uniaxial state at infinity. The analysis is performed in the case of the Landau-de Gennes model with equal elastic constants, i.e., in (8.0.5) one takes $L_1 = L_2 = L_3 := L$.

For small balls (i.e., $r \ll L^{1/2}$) and relatively strong anchoring, she proved the existence of a limiting quadrupolar configuration, with a "Saturn ring" defect, corresponding to an exchange of eigenvalues of the Q-tensor. Relatively strong anchoring means that r is of the order of L/W; the limiting shape depends on the limiting ratio $\dfrac{rL}{W}$. An interesting feature of this result is the explicit form of the quadrupolar configurations. The Saturn ring defect appears as a discontinuity in the principal eigenvector of $Q(x)$, the Q-tensor passing through a uniaxial state as eigenvalue branches cross via an "eigenvalue exchange" mechanism. In the large particle limit, she was able to analyze the strong anchoring (Dirichlet) condition under the extra assumption that the minimizer is axially symmetric. In this setting, she obtained that the limiting map when $r \gg L^{1/2}$ is the unique uniaxial axially symmetric minimizer of the Oseen-Frank energy which is compatible with the behavior at infinity of the liquid crystal (cf. [4]). Some of the striking features of this analysis: the Oseen-Frank model does not allow line defects as in the Saturn ring, and the numerical simulations based on the Ericksen model do not provide the correct radius of the ring.

Arghir Zărnescu gave a general overview of the research project of the group he forms with Radu Ignat, Luc Nguyen and Valeriy Slastikov, and of some of their main achievements. His presentation served as an introduction to the one of Luc Nguyen. In particular, he put in perspective the Landau-de Gennes and the Ginzburg-Landau models, the former one being a higher dimensional, more complex version than the latter. It turns out that new interesting features appear, connected to new phenomena which are specific to this type of models. A typical example is the uniqueness and the stability of radial solutions. While it is well understood for decades for the Ginzburg-Landau model, its resolution for the Landau-de Gennes model required new ideas and techniques which are of independent interest, in particular new separation of variables and uniqueness techniques. His presentation was based on [33, 34, 35, 36].

Luc Nguyen detailed one of the new features presented in the previous lecture. In Ginzburg-Landau theory, it is known that the hedgehog $f(r)e^{i\theta}$ is energy minimizing. In the Landau-de Gennes setting, the hedgehog $f(r)\dfrac{x}{|x|}$ is minimizing in certain regimes, but not in all of them. Another way to present this is that even if the energy functional and the boundary data are invariant under the orthogonal group, the minimizer need not have this property. A striking result presented is that even for two dimensional liquid crystals, uniqueness is a non-trivial matter. More specifically, if the boundary datum on $\partial\mathbb{D}$ has no topological obstruction, the minimizers are "unique and rotationally symmetric" (cf. [36]). As an application, he derived the existence of multiple non-minimizing rotationally symmetric critical points.

Giacomo Canevari presented recent progress related to nematic shells. These are rigid colloidal particles with a typical dimension in the micrometer scale coated with a thin film of nematic liquid crystal whose molecular orientation is subjected to a tangential anchoring. The recent interest in their study is related to the possibility of using them as building blocks of meso-atoms with a controllable valence. Mathematically, they are described as compact boundaryless two dimensional surfaces M embedded into $\mathbb{R}^3$, together with a unit-norm tangent vector field $n : M \to TM$. Such a smooth n need not exist, and even in H^1 n does not exist in a surface of non zero Euler characteristic, by the weak form of the Poincaré-Hopf's theorem (see also Robert Jerrard's presentation). He presented a discrete-to-continuous approach of nematic shells. It consists of discretizing the surface using triangulations, and to define the tangent vectors only on the vertices of the triangulation, avoiding in this way all topological obstructions. A discrete energy is defined, mimicking the standard Dirichlet energy. The main result presented is the Γ-convergence of the discrete energy as $\varepsilon \to 0$. The second order expansion involves, at the minimal energy level, the Euler characteristic of the surface and a renormalized energy in the spirit of [13] (cf. [19]). A striking fact about the renormalized energy is that it keeps trace of the discrete structures used to discretize the surface M around each singularity.

Jinhae Park discussed the nature of minimizers for isotropic-nematic 1D interface problem in case of equal elastic constants in (8.0.5), i.e., $L_1 = L_2 = L_3 := L$. In this setting, it is interesting to allow non positive values of the parameter L. The matter of discussion is the stability and/or minimality of 1D interfaces, i.e., of entire critical points of (8.0.5) depending only on x_3 and satisfying appropriate anchoring conditions at $x_3 = \pm\infty$. This issue is investigated under the assumption $b^2 = 27ac$, meaning that the bulk energy of the normal and the nematic phase are equal. When $L = 0$, it is proved that the minimizer must be a kink similar to the one for the Modica-Mortola functional. Assuming $Q(-\infty) = 0$, the anchoring condition at ∞ is proved to play a crucial role when $L \neq 0$: homeotropic, planar and tilt anchoring lead to different analysis. For example in the case of homeotropic anchoring it is proved that the uniaxial equilibrium is stable when $L \leq 0$, but unstable when $L > 0$ (cf. [47]).

Other aspects of the Ginzburg-Landau theories

Nicolas Rougerie considered a mean-field approximation of multi-particle Hamiltonian, appropriate for "almost bosonic" 2D anyons. The order parameter associated with these anyons minimizes the energy functional

$$\mathcal{E}_\beta^{\mathrm{af}}[u] := \int_{\mathbb{R}^2} \left\{ \left| \left(-i\nabla + \beta \mathbf{A}[|u|^2] \right) u \right|^2 + V(\mathbf{r})|u|^2 \right\} d\mathbf{r}, \tag{8.0.7}$$

where V is a trapping potential and

$$\mathbf{A}[\lambda] := \nabla_\perp \log |\mathbf{r}| * \lambda\,.$$

After providing some physical background, the energy and the density (u) asymptotics in the limit $\beta \to \infty$ were presented [23].

Kirill Samokhin presented the challenges of the study of non centrosymmetric superconductors. One of their noticeable features is the possibility of unusual nonuniform superconducting states even without any external field. In such situations, the strong spin-orbit coupling of electrons with the crystal lattice makes it necessary to describe superconductivity in terms of one or more nondegenerate bands characterized by helicity. He also explained how the origin of non uniform superconducting states can be traced in the first-order gradient terms in the Ginzburg-Landau free energy (cf [48]). He gave some insight concerning the topological classification of the superconducting states using the integer-valued Maurer-Cartan invariants and the Bogoliubov-Wilson loops, with focus, for a two-dimensional setting, on the corresponding wave function topology.

Étienne Sandier presented a first progress towards the description of local minimizers of the simplified Ginzburg-Landau functional (8.0.1) in 3D. An entire solution of (8.0.1) is locally minimizing in the sense of De Giorgi if $E_\varepsilon(u + \phi, B_R) \geq E_\varepsilon(u, B_R)$ for every $R > 0$ and $\phi \in C_c^\infty(B_R; \mathbb{C})$. Such solutions are classified in 2D (they are either constant, or, up to an isometry, the unique radial solution of degree 1 at infinity, u_{rad}). Their classification in dimension ≥ 3 is widely open. He presented the following result: an entire solution whose average energy is sufficiently small, in the sense that

$$\liminf_{R\to\infty} \frac{E(u, B_R)}{R \ln R} < 2\pi,$$

must be constant [49]. Here, 2π is precisely the density of the 2D solution u_{rad} considered as a function of three variables. This may open the way for classifying the local minimizers in 3D as constants or, up to isometries, the 2D solution u_{rad}.

Robert Jerrard focused on the analysis of the functional

$$G_\varepsilon(u) = \int_S \left(\frac{1}{2}|Du|_g^2 + \frac{1}{4\varepsilon^2} F(|u|_g^2) \right),$$

with $u : S \to TS$ a tangent vector field, and S a (two dimensional) surface. He explained a Γ-convergence result at the second order of G_ε as $\varepsilon \to 0$. In particular, he showed that energy minimizers u_ε converge outside a finite number of singularities to a canonical vector field u^* of unit length, whose number of singularities is $|2 - 2\mathbf{g}|$, $\mathbf{g}$ being the genus of the (compact) surface S (cf. [32]). In addition, he obtained a second order expansion of the energy $G_\varepsilon(u_\varepsilon)$ involving a renormalized energy in the spirit of the one devised by Bethuel, Brezis and Hélein for the Dirichlet problem in the plane (cf. [13]). The remarkable aspect of this analysis is that the role of the Dirichlet condition is played by the topology of S, more specifically by its Euler characteristic.

Radu Ignat presented a uniqueness (up to symmetries) asymptotic result for minimizers of the Ginzburg-Landau E_ε functional for $\mathbb{R}^3$ (and not complex) valued unknown functions u_ε. His starting point is the following well-known symmetry result for $\mathbb{S}^2$-valued harmonic maps: the Dirichlet energy $\int_{\mathbb{D}} |\nabla n|^2$, with $\mathbb{D}$ the unit disc and $n : \mathbb{D} \to \mathbb{S}^2$ subject to the boundary condition $n(e^{i\theta}) = (e^{ki\theta}, 0)$ has exactly two solutions, one such that $n_3 > 0$, the other one such that $n_3 < 0$. These solutions are of the form $(f(r)e^{ki\theta}, \pm g(r))$, with $r := |x|$. One expects this uniqueness (up to the $\pm$ sign) and special form of the solutions to propagate to minimizers of E_ε on $\mathbb{D}$, at least for small ε. This is indeed the case. He presented elements of the proof, with emphasis on the crucial steps and ideas in [37].

Miscelaneous topics

Leonid Berlyand presented a study of two types of models describing the motility of eukaryotic cells on substrates. The first one, a phase-field model, consists of the Allen-Cahn equation for the scalar phase field function coupled with a vectorial parabolic equation for the orientation of the acting filament network. In the sharp interface limit the equation of the motion of the cell boundary has been derived. The existence of two distinct regimes of the physical parameters is then established and the existence of traveling waves in one of them is proved. The second model type is a non-linear free boundary problem for a Keller-Segel type system of PDEs in 2D with area preservation and curvature entering the boundary conditions. A family of radially symmetric standing wave solutions (corresponding to a resting cell) are obtained. Then, with the aid of topological tools, traveling wave solutions (describing steady motion) with non-circular shape are shown to bifurcate from the standing wave. The bifurcation analysis explains, how varying a single (physical) parameter allows the cell to switch from rest to motion (cf. [12, 10]).

Gershon Wolansky considered a non-local Liouville equation

$$\pm\Delta\psi = \frac{e^{-\lambda\psi}}{\|e^{-\lambda\psi}\|_1} \text{ in } \Omega \subset \mathbb{R}^n, \quad \psi|_{\partial\Omega} = 0,$$

where $\lambda > 0$. The physical motivation behind this equation is a variational problem for entropy maximization under prescribed mass and energy. He presented an unconditional existence and uniqueness proof in case of electrostatic (repulsive) self-interaction, and conditional existence and uniqueness in dimension two in the case of gravitational (attractive) self-interaction.

Stan Alama considered a a nonlocal isoperimetric functional with a confinement term, derived as the sharp interface limit of a variational model for self-assembly of diblock copolymers under confinement by nanoparticle inclusion. This functional appears as the sharp interface limit of a model of diblock copolymer/nanoparticle blend where a large number static nanoparticles serve as a confinement term, penalizing the energy outside of a fixed region. In a periodic setting, this energy takes the form

$$\mathcal{E}_{\gamma,\sigma,\mu}(u) := \int_{\mathbb{T}^3} |Du| + \gamma \|u-m\|^2_{H^{-1}(\mathbb{T}^3)} + \sigma \int_{\mathbb{T}^3} (u-1)^2.$$

Here, μ is a measure with density on $\mathbb{T}^3$ and the BV unknown density function u is subject to the mass constraint $\displaystyle\int_{\mathbb{T}^3} = m$.

The following asymptotic regime naturally appears in connection with the sharp interface limit of the Ohta-Kawasaki functional, modeling the self-assembly of diblock copolymers: $m \to 0, \gamma, \sigma \to \infty$.

In this regime and after a suitable rescaling, a two terms expansion for the energy without confinement term was presented, involving a small parameter η (representing, roughly speaking, the radius of the small particles). The first term captures the location of droplets in the limit $\eta \to 0$ (through a sum of Dirac masses); the second one, a Coulomb type interaction between droplets. Much more delicate is the analysis of the case with confinement. This is obtained in presence of confinement densities of μ which are spatially variable and attain a nondegenerate maximum. Under such an assumption, it is possible to obtain a two-terms asymptotic expansion exhibiting a separation of length scales due to competition between the nonlocal repulsive and confining attractive effects in the energy. Two different asymptotic behaviours may occur, according to the total volume of the droplets: either the minority phase splits into several droplets at an intermediate scale, or the minimizers form a single droplet converging to the maximum of the confinement density (cf [1]).

Michel Chipot presented new aspects of the justification of dimensional reduction for infinitely large cylinders, theory he developed over the last decade [20, 21]. More specifically, he established the existence of minimal solutions (local minimizers in the sense of De Giorgi) to some variational elliptic inequalities. These solutions are obtained by appropriate truncations of the unbounded domain. A typical situation is the one of an infinite (straight) cylinder, but the techniques allow considering more general cylinder like domains, in particular bounded by appropriate graphs. Uniqueness of the minimal solutions is obtained via adapted comparison principles.

Peter Sternberg introduced a non-standard isoperimetric problem in the plane associated with a metric having a degenerate conformal factor at two points. More specifically, he considers the minimization problem

$$\inf E(\gamma) \text{ with } \gamma : [0,1] \to \mathbb{R}^2,\ E(\gamma) := \int_0^1 F(\gamma)|\gamma'|.$$

The competitors must satisfy $\gamma(0) = \mathbf{p}_-$, $\gamma(1) = \mathbf{p}_+$ and the "enclosed area" condition $\displaystyle\int_\gamma \omega = const.$, with $\omega := -p_2 dp_1$.

The conformal factor $F : \mathbb{R}^2 \to [0,\infty)$ vanishes exactly at $\mathbf{p}_\pm$.

Existence of minimizers is related to the existence of traveling wave solutions to a Hamiltonian system associated with the energy functional

$$H(u) := \int \left(\frac{1}{2}|\nabla u|^2 + W(u)\right), \text{ with } W(u) := F^2(u).$$

Minimizer for the one-well problem exist for quadratic potentials F^2, and need not exist for more degenerate ones. He explained how to obtain existence for the two-wells problem assuming separate existence for the one-well ones at $\mathbf{p}_\pm$ (cf. [3]). This result has implications on the existence of traveling waves. It is also possible to derive a limitation for the maximal propagation speed for these traveling waves through an explicit upper bound depending on the conformal factor F.

Giandomenico Orlandi discussed a relaxation of the Steiner tree problem, consisting of finding a minimal (for various metrics) connected graph containing given points. He proposed a variational approximation of this problem, as well as of the Gilbert-Steiner irrigation problem for the Euclidean distance in the plane. The relaxation involves Modica-Mortola or Ginzburg-Landau type functionals. More generally, he considered the Plateau problem in 2D with coefficients into a normed group. He presented elements of the proof of the fact that these approximations do indeed Γ converge to the original problems (cf. [14]), and thus offer an alternative approach for handling such questions known to be NP hard.

Itai Shafrir presented a natural notion of class within Sobolev spaces of sphere-valued maps, with focus on the spaces $W^{1,1}(\Omega;\mathbb{S}^1)$ and $W^{1,2}(\Omega;\mathbb{S}^2)$. More specifically, two maps are in the same class if they have "the same singularities" (the same distributional Jacobian). The latter functional setting is relevant for liquid crystals. He introduced the metric and Hausdorff distance between two classes, and gave their precise values (cf. [18, 17, 16]). The former distance can be interpreted as the minimal energy required to move the singularities from one place to the other, and its exact value involves the Wasserstein distance between the Jacobians of maps.

Petru Mironescu presented a functional analytic setting adapted to the study of the semi-stiff problems. A typical relevant space in this case is $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$, with $1 < p < \infty$. Such a space is critical for lifting and weak convergence matters: bubbling phenomena appear, and there is no phase control or preservation of the degree in the weak limit. He described the behavior of weakly convergent sequences, demonstrating that non compactness can appear only via the conformal group of the unit disc (profile decomposition cf. [42]) and gave applications of the profile decomposition to the existence of critical points for semi-stiff problems [11].

Future Perspectives

The meeting revealed the emergence of new techniques and intermediate problems that in the near future should facilitate the ground for approaching several problems that seemed out of reach before:

1. Study of stability, uniqueness and other qualitative properties of radial solutions of vector-valued radially symmetric functionals.

2. Classification of locally minimizing entire solutions of the Ginzburg-Landau functional in dimension ≥ 3.

3. Complete analysis of Landau-de Gennes models with penalization terms confining the Q-tensor within a physically realistic region.

4. Sharp analysis of the biaxial escape phenomenon in nematics.

5. Rigorous asymptotic analysis of many order parameters Ginzburg-Landau type functionals – beyond the p-waves models.

6. Linear stability analysis of hydrodynamic flow in a tube of arbitrary cross section.

7. Rigorous analysis of tunneling effects for the Neumann magnetic Laplacian and other operators.

8. Determination of the preferred orientation of the helical axis in nearly cholesteric smectic A liquid crystals.

9. Analyzing the behaviour of the superconductivity order the boundary layer in a three-dimensional setting, in the surface superconductivity parameter regime.

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Chapter 9

Algebraic Combinatorixx 2 (17w5012)

May 14 - 19, 2017

Organizer(s): Julie Beier (Earlham College), Patricia Muldoon Brown (Armstrong State), Rosa Orellana (Dartmouth), Stephanie van Willigenburg (University of British Columbia)

Overview

Algebraic combinatorics is a large branch of mathematics with strong ties to many areas including representation theory, computing, knot theory, mathematical physics, symmetric functions and invariant theory. This workshop focused on the three areas below, inspired by the previous Algebraic Combinatorixx workshop and current trends.

- Algebraic combinatorics and representation theory: The representation theory of Lie algebras, quantum groups, Hecke algebras / double affine Hecke algebras, diagram algebras, symmetric groups and more utilize algebraic combinatorics and often take advantage of combinatorial objects such as crystals, Littlemann paths, tableaux, quivers, lattice paths, knots, and alternating sign matrices to discover new results.
- Algebraic combinatorics and geometry: This area includes the understanding of posets, lattices, simplicial complexes, CW-complexes, polytopes, and groups or arrangements also associated with these such as reflection groups, braid groups, hyperplane arrangements and descent algebras. This subarea of algebraic combinatorics connects to tropical geometry, Schubert calculus, Grassmannians and techniques in geometric representation theory.
- Algebraic combinatorics and combinatorial functions: In this category we are referring to symmetric functions such as Hall-Littlewood polynomials, Schur functions / k-Schur functions, and Weyl group characters as well as quasisymmetric functions such as those arising from random walks, P-partitions, 0-Hecke algebras and Hopf algebras. In addition, the Kazhdan-Lusztig polynomials fit into this category.

These three areas are clearly interconnected and all are of current interest in research. For example, the search for an algorithmic description of the coefficients that arise in the inner product of two Schur functions, known as Kronecker coefficients, has formed a very active research area over the past twenty years. Most recently this is due to deep connections with quantum information theory and the central role it plays within geometric complexity theory, which is an approach that seeks to settle the celebrated P versus NP problem – one of the several 1,000,000 Millennium Prize Problems set by the Clay Mathematics Institute. Meanwhile, the search for a combinatorial formula for the product of two Schubert polynomials is another longstanding open problem that has fuelled much research and development in the area due to its connections to algebraic geometry and the resolution of Hilbert's 15th Problem.

In addition to these mathematical goals, the workshop also sought to foster the community of women in algebraic combinatorics, cutting across the false research versus teaching institution divide. Thus, in addition to presentations, work was done during the week in teams; each team focusing on a different problem.

Presentation Highlights

The first day contained eight talks by the team leaders. In these, they presented background and problems that the teams would work on during the week. They are discussed in the following section. On the remaining days we heard a total of 15 excellent talks on current research. It is impossible to do justice to all of these talks but some highlights are mentioned below.

Hélène Barcelo spoke about discrete homotopy theory for graphs, known work of Babson, Barcelo, Kramer, de Longueville, Laubenbacher, Severs, Weaver and White, and argued that this theory is a good analogy to classical homotopy theory (cf. [3]). She went on to present a new idea developed by Barcelo, Capraro and White for a discrete homology on graphs and argued that it gives expected results such as Hurewicz Theorem and a discrete version of the Eilenbrerg-Steenrod axioms [4]. Moreover, she presents a conjecture that the path homologies of Grigor'yan, Muranov, and Yau are isomorphic to these discrete homologies for undirected graphs (cf. [10]).

Yui Cai presented work that was joint with Margaret Readdy and published this month on q-stirling numbers [8]. First, she argues that the q-Stirling numbers of the second kind can be understood using weights on allowable restricted growth words. Next we are presented with a new poset called a Stirling poset of the second kind. Cai was then able to give a poset explanation and homological interpretation to this work. Finally, we saw that using rook placements similar techniques could be used to produce the q-stirling numbers of the first kind.

The relationship between claw-contractible-free graphs and the e-positivity of a graph's chromatic symmetric function has been discussed since Stanley's 1995 paper [16]. Samantha Dahlberg presented results by Dahlberg, Foley and van Willigenburg, just recently posted on the archive, that settle this matter [9]. In this talk, we finally find the answer by learning of infinite families of graphs that are not claw-contractible and do not have chromatic symmetric functions that are e-positive. Moreover, we also saw one such family that is claw free but not e-positive.

Scientific Progress Made

During the week, we were split into eight teams working on eight different projects. Approximately eight hours were specifically designated for work in teams, and a number of teams met outside of the other workshop activities to work farther. In this section we briefly describe each project and summarize the progress made so far.

Chromatic Symmetric Functions

Team Leader: Angèle Hamel
Team Members: Julie Beier, Samantha Dahlberg, Maria Gillespie, Stephanie van Willigenburg

Consider all of the proper colorings of a graph G. With each coloring, K, associate a monomial $x_{K(v_1)}x_{K(v_2)}\ldots$. The chromatic polynomial is defined to be the sum of these monomials, and is known to be symmetric. Stanley [and Gasharov] conjecture that all claw free graphs have chromatic polynomials that are Schur positive [17].

We know from the work of Schilling that if one can create a crystal structure on the proper colorings of a graph G then the chromatic polynomial will indeed be Schur positive. We started with a special simple case of claw free graphs that are already known to be Schur positive, paths on n vertices. We spent the week working on a definition of Kashiwara operators f_i, e_i and of ϕ_i, ε_i for path colorings. Subsequently, we started to show that these satisfy the Stembridge axioms, which would give us a crystal structure. At the end of our time, we are still working on refining our rules so that they will generalize to colorings of non-path graphs, and proving our rules for the path case.

Quasisymmetric Analogue of Macdonald Polynomials

Team Leader: Sarah Mason
Team Members: Cristina Ballantine, Zajj Daugherty, Angela Hicks, Elizabeth Niese

This project was initiated to work toward finding a quasisymmetric analogue to the Macdonald polynomials. It became quickly evident that the naive approach of combining suitable nonsymmetric Macdonald polynomials would not lead to the desired outcome. Since the initial definition of the Macdonald polynomials is based on a modification of the Hall inner product on the power sum symmetric functions, the team decided to change approaches and work toward an appropriate modification of the inner product on QSym and NSym. In order to do this, a quasisymmetric power sum basis is needed, so this is the task the group spent time on and made significant progess.

Calculation 1: An explicit calculation of the Cauchy kernel:

$$\sum_\beta H_\beta^N(X)M_\beta^Q(Y) = \prod_j \left(1 - \left(\sum_{i\geq 1} x_i\right) y_j\right)^{-1}$$

Result 1: A quasisymmetric power sum (type 1) P_α^Q such that $\langle P_\alpha^Q, P_\beta^N \rangle = \delta_{\alpha\beta} z_\alpha$ where $z_\alpha = z_{\widetilde{\alpha}}$.

Result 2: A monomial expansion for the quasisymmetric power sum (type 1):

$$P_\alpha^Q = \sum_{\alpha\leq\beta} c_\beta^\alpha M_\beta^Q$$

Conjecture 1: For $\lambda \vdash n$,

$$p_\lambda = \sum_{\widetilde{\alpha}=\lambda} P_\alpha^Q.$$

Algebraic Voting Theory & Representations of $S_m[S_n]$

Team Leader: Kathryn Nyman
Team Members: Hélène Barcelo, Megan Bernstein, Sarah Bockting-Conrad, Erin McNicholas, Shira Viel

We consider the problem of selecting a committee consisting of one member chosen from m candidates in each of n departments. Voters rank the possible committees and a positions scoring method (such as the Borda count) can then be used to select a winning committee. We can view the profile and results spaces of voting information as $\mathbb{Q}S_m[S_n]$ modules, where the wreath product acts on the committees by permuting candidates within a department and permuting the departments. Since the positional voting procedure is a $\mathbb{Q}S_m[S_n]$ module homomorphism between the spaces, Schur's Lemma provides information on the voting information lost in the election process as well as the voting information that determines the outcome.

Our first goal is to decompose the results space R into simple $\mathbb{Q}S_m[S_n]$-submodules. Let $p(m)$ denote the number of partitions of m, and enumerate the set of partitions as $(\gamma_1, \ldots, \gamma_{p(m)})$. These partitions index the irreducible representations of S_m. Note that m of these partitions are hooks; we refer to the others as non-hook. In addition, we define a flat partition of k to be the partition consisting of a single part of size k.

We conjecture R has the following decomposition into simple $\mathbb{Q}S_m[S_n]$-modules:

$$R = \oplus_{\binom{a_1+a_2+\ldots+a_{p(m)}=n}{a_i=0\ \forall \text{ non-hook } \gamma_i}} S^{(a_1,\ldots,a_{p_m})} \tag{9.0.1}$$

where $S^{(a_1,\ldots,a_{p(m)})}$ denotes the simple $\mathbb{Q}S_m[S_n]$-module indexed by the multi-partition $(\alpha_1, \ldots, \alpha_{p(m)})$ of n where each α_i is the flat partition of $|\alpha_i| = a_i$. A proof of Matt Davis shows that Equation (9.0.1) holds in the case when $m = 2$; we are working on generalizing his proof. Towards that end, our primary result is a computation of the character of V_n, the representation of $S_m[S_n]$ corresponding to R.

Let $g \in S_n[S_m]$, viewed as a permutation of $\{1_1, 1_2, \ldots, 1_m, \ldots, n_1, n_2, \ldots, n_m\}$. (Think of i_k as candidate k from department i). Writing g as a product of disjoint cycles, for each $i \in [n]$ define the <u>cycle family</u> $F_i(g)$ to be the set of cycles in g containing i_k for some $k \in [m]$. By definition of the action of the wreath product, π determines the permutation of the departments. Thus, if i_k and j_l appear together in some cycle of g, π sends department i to department j and for all $s \in [m]$, i_s is sent to j_t for some $t \in [m]$. We define $m(F_i(g)) = \#\{j \in [n] : F_j(g) = F_i(g)\}$, i.e. the number of departments appearing in cycles containing candidates from department i. If $F_i(g) = F_j(g)$ we will say j is a <u>member</u> of $F_i(g)$. Thus, $m(F_i(g))$ is the number of members in $F_i(g)$. Finally, we let $\mathcal{F}(g) = \{F_i(g) : i \in [n]\}$.

Theorem 9.0.1 *For each $g \in S_n[S_m]$,*

$$\chi_{V_n}(g) = \prod_{F \in \mathcal{F}(g)} \#\{ \text{ cycles } c \text{ in } g : \text{the length of } c \text{ is } m(F)\}. \tag{9.0.2}$$

cd-Index of Eulearian Posets

Team Leader: Margaret Readdy
Team Members: Yue Cai, Anastasia Chavez, Gizem Karaali, Heather Russell

The Readdy group is currently looking at two questions: how to compute the cd-index of important families of Eulerian posets and to discover combinatorial interpretations of the coefficients of the cd-index invariant. A lot of time was spent getting the group to speed on the geometry and coalgebraic structure of flag enumeration for polytopes. We are in the midst of understanding the cd-index of the family of cyclic polytopes and finding a more streamlined way to compute the cd-index.

The cyclic polytopes are a family of polytopes central to the Upper Bound Theorem; that is, for fixed dimension n and vertices f_0 the cyclic polytope maximizes the face vector. Further demonstrating this family's importance, for the cd-index of polytopes, more generally, Eulerian posets, Billera and Ehrenborg showed the cyclic polytope maximizes each coefficient of the cd-index [5]. In essence, this is an Upper Bound Theorem for the cd-index. The Readdy group also began discussing a combinatorial interpretation for the cd-index of the permutahedron. This polytope will very likely have a more tractable answer as we can use its inherent group structure.

On the Kronecker quasi-polynomials

Team Leader: Mercedes Rosas
Team Members: Marni Mishna, Sheila Sundaram

A central result in the representation theory of the symmetric group says that the set of partitions λ of n label the irreducible representations S^λ of $\mathbb{S}_n$.

In this setting, a major open problem is to understand the decomposition into irreducibles for the tensor product of representations. The Kronecker coefficients $g^\lambda_{\mu,\nu}$ are the coefficients governing this decomposition,

$$S^\mu \otimes S^\nu \cong \bigoplus_{\mu,\nu} g^\lambda_{\mu,\nu} S^\lambda$$

The coefficients $g^\lambda_{\mu,\nu}$ are intriguing and poorly understood, and determining a satisfactory formula for them is one of the major open questions in algebraic combinatorics.

Our project focuses on a related function. Let $Q^\lambda_{\mu,\nu}$ be the stretching function defined by

$$Q^\lambda_{\mu,\nu}(n) = g^{n\lambda}_{n\mu,n\nu}$$

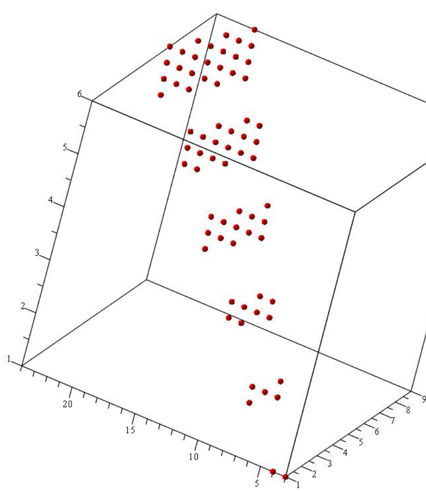
It is known that $Q^{\lambda}_{\mu,\nu}(n)$ is a quasi–polynomial in n, but little else is known about this function [11]. Recently, Balloni and Vergnes have developed algorithms to compute these polynomials when the length of the partitions is small [1, 2].

On the other hand, quasi–polynomials appear naturally in the study of dilations of rational polytopes. In this situation, Ehrhart quasi-polynomials count the number of integer points inside dilations of polytopes. Of course, not all quasi–polynomials correspond to such a counting function.

It is known that $Q^{\lambda}_{\mu,\nu}(n)$ is not, in general, the counting function for the number of integral points inside the dilations of any polytope, [11, 6].

In our research team we explored the *reduced* Kronecker coefficients, or equivalently, the $Q^{\lambda}_{\mu,\nu}(n)$ when μ, ν and λ are stable. Here, we refer to a stabilization phenomenon first observed by Murnaghan, [13]. These triples are important because they include all Littlewood–Richardson coefficients [12], they are themselves Kronecker coefficients, and the value of the usual Kronecker coefficients can be recovered from them, [7]. Finally, there is a wide consensus that the reduced Kronecker coefficients should be easier to understand than the general Kronecker coefficients.

In our work at Banff, we characterized the Kronecker polytope corresponding to those triples μ, ν and λ with $\ell(\mu), \ell(\nu) \leq 2$ and $\ell(\lambda) \leq 4$, i.e, $(2, 2, 4)$, and programmed our results into MAPLE. This is probably the situation where the Kronecker coefficients are best known; it belongs to the case of two-rowed shapes, one of a handful of cases for which explicit formulas exist [15]. In the following picture there is an example of the resulting polytope and some of its dilations.



We plan to continue working on this project. Currently we are looking at the following case $(3, 3, 9)$. We are optimistic that getting a better understanding of this polynomial in this more unknown setting will lead to results in the general situation. In particular, we are trying to determine whether the resulting quasi-polynomial is the counting function for the dilations of a polytope, with the eventual goal of fully describing it.

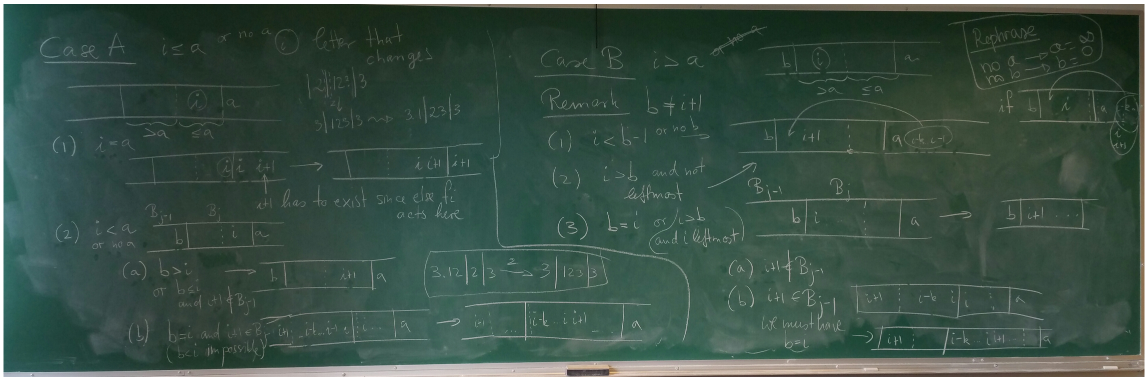
Minimaj Crystal

Team Leader: Anne Schilling
Team Members: Georgia Benkart, Laura Colmenarejo, Pamela Harris, Rosa Orellana, Greta Panova, Martha Yip, Meesue Yoo

Suppose f is a symmetric function defined on a combinatorial object C, and f has a positive integer expansion in terms of Schur functions. Our team explored the question of whether this implies C has a crystal structure. As a test case for this investigation, we focused on the particular case that C is the set of ordered multiset partitions of

$\{1, 2, \ldots, n\}$ into $k + 1$ nonempty blocks. The motivation to consider this particular example was a recent paper by Rhoades, “Ordered Set Partition Statistics and the Delta Conjecture” [14].

We have defined operators on C that we conjecture to be crystal operators and that preserve the minimaj statistic (see the picture below). We have worked out many examples that indicate that the conjecture is correct. We intend to prove that they satisfy Stembridge’s axioms, which will imply that C has a crystal structure. We have a representation theoretic interpretation of C that should serve as a guide.



Reduced Word Bounds

Team Leader: Bridget Tenner
Team Members: Susanna Fishel, Elizabeth Milićević, Rebecca Patrias

Let $R(w)$ denote the set of reduced words for a fixed permutation w in the symmetric group S_n. Identifying all reduced words which are related by a sequence of commutation relations, one obtains the well-studied set of commutation classes for w, defined by $C(w) = R(w)/[ij \sim ji]$ where $|i - j| > 1$. Similarly, one defines the set of braid classes for w to be $B(w) = R(w)/\left[i(i+1)i \sim (i+1)i(i+1)\right]$, which have received comparatively less attention in the literature. Our goal for the week at BIRS was to study the braid classes of a fixed permutation, and we decided to focus on an enumerative problem in particular.

Problem: What is the number $|B(w)|$ of braid classes in the set of all reduced words $R(w)$?

Our strategy for approaching this problem was to simultaneously study the number of commutation and braid classes. Since $R(w)$ is a finite set, we can organize the elements of $R(w)$ into a table which lists the elements in each distinct commutation class $C_1, C_2, \ldots, C_k$ across in rows, and the elements in each distinct braid class $B_1, B_2, \ldots, B_m$ in columns. If we only record nonempty classes, there must be at least one element of $R(w)$ in each row and column. It is straightforward to prove that each entry in this table contains at most one element of $R(w)$. Combining this observation with the fact that any two reduced words for w are related by a sequence of commutation relations and/or braid relations, we obtain sharp upper and lower bounds on the number of reduced words in terms of the number of braid and commutation classes.

Proposition: For any $w \in S_n$, we have $|C(w)| + |B(w)| - 1 \leq |R(w)| \leq |C(w)||B(w)|$.

Having proved this proposition early in the week, our goal for the remainder of the workshop was to determine precisely which permutations realize these upper and lower bounds. Since both $|R(w)|$ and $|C(w)|$ have been fairly well-studied, achieving this goal would then yield closed formulas for $|B(w)|$ for such families of permutations.

Theorem 9.0.2 (Fishel-Milićević-Patrias-Tenner) *There is a classification of the elements $w \in S_n$ which achieve the above upper and lower bounds on $|R(w)|$:*

1. $|R(w)| = |C(w)||B(w)|$ *if and only if* $|C(w)| = 1$ *or* $|B(w)| = 1$.

2. $|R(w)| = |C(w)| + |B(w)| - 1$ *if and only if w lies in one of a few special families.*

Regarding those elements which achieve the upper bound, the permutations which have a single commutation class are called fully commutative and can be characterized in terms of pattern avoidance. Similarly, it is possible to completely describe those permutations which have a single braid class. For the lower bound, in fact we prove something stronger by listing all permutations such that $|R(w)| = |C(w)| + |B(w)| - e$, where $e \in \{0, 1\}$. For each such permutation, there exists an ordering on the rows and columns in the table discussed above such that the resulting pattern is "serpentine", and this visualization provided a useful platform upon which to build our analysis. We will continue by writing up these results and submitting them for publication.

Ehrart Theory of Alcoved and Tropical Polytopes

Team Leader: Josephine Yu
Team Members: Emily Barnard, Carolina Benedetti, Patricia Brown

Our project focuses on characterizing the Ehrhart polynomials of alcoved and tropical polytopes. To this end we started our analysis focusing on 2-dimensional alcoved polytopes. It is known that the leading coefficient of the Ehrhart polynomial of a lattice polytope P is the (Euclidean) volume of the polytope and that the second leading coefficient is half of the boundary volume (normalized so that the standard simplex in a lattice has volume 1). For example, the standard 3-dimensional simplex has Ehrhart polynomial $x^3 + 3x^2 + 3x + 1$. We obtained the following lemma whose proof we omit.

Lemma: Let A be the normalized area (twice Euclidean area) and P be the (lattice) perimeter of an alcoved polygon. Then the pair (A, P) satisfies the following relations.

1. If the alcoved polygon has no interior lattice points, then $A + 2 = P$.
2. If the alcoved polygon contains an interior lattice point and is not three times a smallest simplex, then $A + 8 \geq 2P$ (Scott's inequality).
3. $6A \leq P^2$, and equality is achieved exactly for regular hexagons.

We know that not every pair (A, P) satisfying the conditions above can be achieved by an alcoved polygon. In view of this, we are making steps towards the following problems:

1. Characterize the pairs (A, P) that come from alcoved polygons. The next section may be useful.
2. Study higher dimensions.
3. Study h^* vector, which coincides with (in reverse order) the h-vector of the alcoved triangulation (or any unimodular triangulation) of the polytope.
4. Understand families of alcoved polytopes whose Ehrhart and/or h^* polynomial are palindromic and unimodal.
5. Some alcoved polytopes arise from matroids. Using the known Hopf algebras structures on matroids, can we define similar operations on (families of) alcoved polytopes?

Some of the results above can be extended in the setting of tropical polytopes as follows.

Lemma: For lattice tropical polygons, the area A and perimeter P satisfy $6A \leq P^2$, where the perimeter P of a pendant line segment is defined as twice its lattice length. The Ehrhart coefficients have the same interpretation.

We can decompose an alcoved polygon into ribbons by slicing it with parallel hyperplanes (lines). An alcoved polygon is thus determined by the numbers of triangles in the layers. A sequence of positive integers come from an alcoved polygon this way if and only if

1. it is unimodal,

2. only the peak can be even, and

3. only the peak can repeat.

Since there are three different hyperplane directions, different sequences may give the same alcoved polygon. We plan to characterize ”canonical” sequences giving rise to alcoved polytopes.

Each alcoved polytope has an “alcoved triangulation” obtained by slicing with all hyperplanes of the form $x_i - x_j = c$ where $c \in \mathbb{Z}$. The dual graph of an alcoved triangulation is a subgraph of the tiling of $\mathbb{R}^n/(1, 1, \ldots, 1)$ by regular permutohedra. Recall that all the two dimensional faces of a permutohedron (in any dimension) is either a square or a hexagon.

Conjecture: A subgraph of the permutohedral tiling is dual to the alcoved triangulation of an alcoved polytope if and only if

1. it is connected and induced,

2. if three vertices of a square are in the subgraph, then so is the fourth, and

3. if four vertices of a hexagon are in the subgraph, then so are the other two.

This gives us a way to study higher dimensional alcoved polytopes using graphs only.

Open problems.

1. Make connections to reduced words in affine Weyl groups.

2. Generalize to Weyl groups of other types.

Outcomes of the Meeting

As detailed above, all teams made significant progress on their problems. According to our team reports, they all have concrete plans to continue to work together to finish at least the problem they started at the meeting. Several teams are planning to submit papers within the next year and some have already scheduled times to meet in person again. Moreover, many people commented that they learned new mathematics or who to talk to about their work. This is fantastic and was an important goal.

MSRI generously agreed to provide \$25,000 of funding for participants from this workshop to meet as teams this summer for continued work. We began accepting applications on June 1st and have selected four teams that will be meeting at MSRI during July and August of 2017. We are appreciative of the MSRI’s support and believe their support illustrates the significance of the research completed during this workshop.

We also would like to acknowledge that the Association for Women in Mathematics, AWM, provided funding for several of the US based participants to attend this workshop. This was absolutely vital to us creating a community with such diversity, and we appreciate their support. The financial support came from both their established travel funding program and from the new Advance grant program, NSF-HRD 1500481 - AWM ADVANCE grant.

In addition to the mathematical outcomes, it is clear from the evaluations that the community was enhanced as almost every respondent referred to the value of the networking, mentoring and conversations that occurred during the week. We had participants from numerous countries, people at many different points in their career, and women who work at a wide range of institutions. Respondents overwhelmingly valued the team structure and felt the conference was well organized. Every respondent said that they believed this was a valuable use of their time and several commented that they strongly hope there will be another Algebraic Combinatorixx in the future.

The comments from the week and in the evaluations were incredibly positive, and we share the participants’ gratitude to BIRS for providing such a fantastic space to work and amazing staff to help. Below is a small sample of some of the comments about the significance of the week for them:

- “I feel like I learned more algebraic combinatorics in 4 days than in years before. I learned so much. I also made personal and professional connections that I will maintain going forward”

- "It has helped to re-energize me. ... I learned a lot of cool math, and my group made serious progress on our project and has plans to continue."
- (Listed as a strength of the workshop) "Bringing together an extraordinary group of mathematicians - [the] talks were so impressive and informative"
- "I think my group will write a paper and continue to collaborate."
- "My team made great progress and I am confident that we will be able to write a nice paper."
- "The supportive community here was exceptional."
- "The discussions at meals, in groups, in the evening discussions were all amazingly friendly and helpful."
- "I most appreciated the way the organizers and more experienced attendees fostered a positive, supportive environment for collaboration and mentoring. I received so much valuable advice and mathematical insight."
- "I really appreciated that there were different kinds of mathematicians here - those from liberal arts colleges [and] those from R1 schools. That mix was helpful; I benefited from many perspectives and many different ways of doing math."
- "This networking is alone worth the trip."
- "I just really enjoyed the experience overall and felt energized professionally and personally."
- "I think this has been my favorite conference/workshop!"
- "I would do this every summer if I could."

Participants

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Chapter 10

Recent Advances in Discrete and Analytic Aspects of Convexity (17w5074)

May 21 - 26, 2017

Organizer(s): Karoly Bezdek (University of Calgary), Alexander Koldobsky (University of Missouri), Dmitry Ryabogin (Kent State University), Vladyslav Yaskin (University of Alberta), Artem Zvavitch (Kent State University)

Overview of the Field

Convexity is a very old topic which can be traced back at very least to Archimedes. It shares ideas and methods from many fields of Mathematics, including Differential Geometry, Discrete Geometry, Functional Analysis, Harmonic Analysis, Geometric Tomography, Combinatorics, Probability, and it has numerous applications. The aim of the meeting was to discuss most recent developments in the area and interrelate new discrete and analytic methods.

Presentation Highlights

The topics of the workshop included convex geometry, discrete geometry, theory of valuations, geometric inequalities, probability, and geometric functional analysis.

Grigoris Paouris presented his work "Affine isoperimetric inequalities on flag manifolds" joint with S. Dann and P. Pivovarov. Let K be a compact set in $\mathbb{R}^n$ (and convex, depending on the context) and k be an integer $1 \leq k \leq n-1$. The following are known as affine quermassintegrals and dual affine quermassintegrals correspondingly:

$$\Phi_k(K) = \left(\int_{G(n,k)} |K|F|^{-n} \right)^{-1/(nk)} dF,$$

$$\Psi_k(K) = \left(\int_{G(n,k)} |K \cap F|^{n} \right)^{1/(nk)} dF,$$

where $G(n,k)$ is the Grassmanian of the k-dimensional subspaces in $\mathbb{R}^n$.

The results of Furstenberg-Tzkoni and Grinberg show that Ψ_k is invariant under linear volume-preserving transformations, and Φ_k is invariant under affine volume-preserving transformations. In this talk affine and dual affine quermassintegrals were extended from the Grassmanian to the flag manifolds. The corresponding invariant properties were also established.

Busemann-Strauss and Grinberg established the following extremizing inequality for Ψ_k. Let K be of volume 1 and D_n be the Euclidean ball of volume 1, then $\Psi_k(K) \leq \Psi_k(D_n)$. Lutwak asked whether the following inequality is true: $\Phi_k(K) \geq \Phi_k(D_n)$. Paouris and Pivovarov earlier proved that the latter holds with an absolute multiplicative constant. In this talk the authors extended these results to the setting of flag manifolds. Some functional forms of these constructions were also discussed.

Peter Pivovarov spoke about his joint work "On a quantitative reversal of Alexandrov's inequality" with G. Paouris and P. Valettas. Alexandrov's inequality implies that for a convex body K we have the following:

$$\left(\frac{V_n(K)}{V_n(B)}\right)^{1/n} \leq \left(\frac{V_{n-1}(K)}{V_{n-1}(B)}\right)^{1/(n-1)} \leq \cdots \leq \frac{V_1(K)}{V_1(B)},$$

where B is the unit Euclidean ball and V_1,..., V_n are the intrinsic volumes.

Milman's random version of Dvoretzky's theorem shows that a large initial segment of this sequence is essentially constant, up to a critical parameter called the Dvoretzky number. The authors showed that this near-constant behavior actually extends further, up to a different parameter associated with K. Namely, set

$$\beta_* = \beta_*(K) = \frac{Var(h_K(g))}{(\mathbb{E}h_K(g))^2},$$

where h_K is the support function of K and g is a standard Gaussian random vector in $\mathbb{R}^n$. It is shown that there exists a constant $c > 0$ such that if K is a symmetric convex body in $\mathbb{R}^n$ and $1 \leq k \leq c/\beta_*(K)$, then

$$\frac{V_1(K)}{V_1(B)} \leq \left(1 + c\sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)}\right)\left(\frac{V_k(K)}{V_k(B)}\right)^{1/k}.$$

This yields a new quantitative reverse inequality that sits between the approximate reverse Urysohn inequality, due to Figiel–Tomczak-Jaegermann and Pisier, and the sharp reverse Urysohn inequality for zonoids, due to Hug–Schneider.

Liran Rotem gave a talk "Powers of convex bodies" based on a joint work with V. Milman. The main question discussed in the talk is the following: given a convex body K in $\mathbb{R}^n$ and a number $\alpha \in \mathbb{R}$, is there a natural definition for the power K^α. If $\alpha = -1$, then the natural definition of K^{-1} is K^o, the polar of K. This is due to the fact that polarity satisfies properties analogous to those of the function $x \mapsto 1/x$. The authors suggested a construction of K^α for $0 < \alpha < 1$ (again the motivation was that this operation should be similar to the function $x \mapsto x^\alpha$). This is done by first defining this operation for ellipsoids, in a natural way, and then passing to general bodies by using ellipsoidal envelopes.

In her talk "Recent results on approximation of convex bodies by polytopes", Elisabeth Werner discussed two results. The first one, joint with J. Grote, generalizes a theorem by Ludwig, Schuett and Werner on approximation of a convex body K in the symmetric difference metric by an arbitrarily placed polytope with a fixed number of vertices. Namely, let K be a convex body in $\mathbb{R}^n$, $n \geq 2$, that is C^2_+. Let $f : \partial K \to \mathbb{R}_+$ be a continuous strictly positive function with

$$\int_{\partial K} f(x) d\mu_{\partial K}(x) = 1,$$

where $\mu_{\partial K}$ is the surface area measure on ∂K. Then for sufficiently large N there exists a polytope P_f in $\mathbb{R}^n$ having N vertices such that

$$\mathrm{vol}_n(K \Delta P_f) \leq a N^{-2/(n-1)} \int_{\partial K} \frac{\kappa_K(x)^{1/(n-1)}}{f(x)^{2/(n-1)}} d\mu_{\partial K}(x),$$

where $a > 0$ is an absolute constant, and κ_K is the Gaussian curvature.

The second recent result, joint with S. Hoehner and C. Schuett, gives a lower bound, in the surface deviation, on the approximation of the Euclidean ball by an arbitrary positioned polytope with a fixed number of k-dimensional faces.

Dan Florentin spoke about "New Prékopa Leindler Type Inequalities and Geometric Inf-Convolution of Functions". The talk is based on a joint work with S. Artstein and A. Segal. Consider the class $Cvx_0(\mathbb{R}^n)$ of non-negative convex functions on $\mathbb{R}^n$ vanishing at the origin (these are called geometric convex functions). The authors define a geometric inf-involution $\phi \boxdot_\lambda \psi$ by first providing a geometric interpretation and then showing the precise formula:

$$(\phi \boxdot_\lambda \psi)(z) = \inf_{0<t<1} \inf_{z=(1-t)x+ty} \max\left\{\frac{1-t}{1-\lambda}\phi(x), \frac{t}{\lambda}\psi(y)\right\}.$$

Then they prove that for $\lambda \in (0,1)$ and $\phi, \psi \in Cvx_0(\mathbb{R}^n)$ one has

$$\int e^{-\phi \boxdot_\lambda \psi} \geq \left(\frac{1-\lambda}{\int e^{-\phi}} + \frac{\lambda}{\int e^{-\psi}}\right)^{-1}.$$

They also discuss other forms of this inequality and explain why this is an analogue of the Prékopa-Leindler inequality.

Bo'az Klartag presented his work on "Convex geometry and waist inequalities". The spherical waist inequality states that any continuous function f from the unit sphere S^n to $\mathbb{R}^l$ has a large fiber, i.e., the $(n-l)$-dimensional volume of some fiber $f^{-1}(t)$ is at least as large as that of S^{n-l}.

Here the author proves the following result. Let K be a convex body in $\mathbb{R}^n$ and $1 \leq 1 \leq n$. Then for any continuous function $f : K \to \mathbb{R}^l$,

$$\sup_{t\in\mathbb{R}^l} \mathrm{vol}_{n-l}(f^{-1}(t)) \cdot \sup_{E\in AG(n,l)} \mathrm{vol}_l(K \cap E) \geq \mathrm{vol}_n(K).$$

Furthermore, if $K \subset \mathbb{R}^n$ is a convex body of volume 1, then there exists a volume-preserving linear map T_K such that $\tilde{K} = T_K(K)$ has the following property. Let $1 \leq l \leq n$ and $f : \tilde{K} \to \mathbb{R}^l$ be a continuous map. Then there exists $t \in \mathbb{R}^l$ with

$$\mathrm{vol}_{n-l}(f^{-1}(t)) \geq c^{n-l},$$

where $c > 0$ is a universal constant.

The theme of waist inequalities is continued in the talk of Arseniy Akopyan "Waists of balls in different spaces", based on a joint work with R. Karasev and A. Hubard. The speaker starts with the following curious question. Can one map a shape of width 1 into a shape inside the strip of width 0.99 in such a way that the distances do not decrease? The answer is "No", as follows from the following result of the authors. For any convex body $K \in \mathbb{R}^n$ and a continuous map $f : K \to \mathbb{R}^{n-1}$ there exists a fiber $f^{-1}(y)$ of 1-Hausdorff measure at least the width of K.

Further the speaker presents a number of waist inequalities for balls in spaces of constant curvature, tori, parallelepipeds, projective spaces and other metric spaces.

Christos Saroglou spoke about his joint work with S. Myroshnychenko and D. Ryabogin on "Star bodies with completely symmetric sections". The starting point of this research was the question: Let K be a convex body in R^n, $n \geq 3$. If all orthogonal projections of K are 1-symmetric, is it true that K must be a Euclidean ball? (A body is said to be 1-symmetric if its symmetry group contains the symmetry group of a cube of the same dimension).

The authors gave a positive answer to this problem, as well as other closely related questions. For, example the corresponding problem for sections also has a positive answer. In fact, a more general statement is true. If f is an even function on the sphere whose restriction to every equator is isotropic, then f is a constant.

Jaegil Kim presented a joint work with S. Dann and V. Yaskin titled "Busemann's intersection inequality in hyperbolic and spherical spaces". A version of Busemann's intersection inequality says that ellipsoids in $\mathbb{R}^n$ are the only maximizers of the quantity

$$\int_{S^{n-1}} |K \cap \xi^\perp|^n d\xi \tag{10.0.1}$$

in the class of star bodies of a fixed volume.

The authors study this question in the hyperbolic space $\mathbb{H}^n$ and the sphere $\mathbb{S}^n$. It is shown that in $\mathbb{H}^n$ the centered balls are the unique maximizers of (10.0.1) in the class of star bodies of a fixed volume. However, on the sphere the situation is completely different. It is surprising that in $\mathbb{S}^n$ with $n \geq 3$ the centered balls are neither maximizers nor minimizers of (10.0.1), even in the class of origin-symmetric convex bodies.

Ning Zhang spoke about his work "On bodies with congruent sections by cones or non-central planes". Let K and L be convex bodies in $\mathbb{R}^3$ such that the projections $K|H$ and $L|H$ are congruent for every subspace H. Does this imply that K is a translate of $\pm L$. This question is open. Ryabogin solved a version of this problem when congruency is replaced by rotation. In the language of functions his result can be stated as follows. Let f and g be continuous functions on S^2 such that for every 2-dimensional subspace H there is a rotation ϕ_H in H such that $f(\theta) = g(\phi_H(\theta))$ for all $\theta \in S^2 \cap H$. Then $f(\theta) = g(\theta)$ or $f(\theta) = g(-\theta)$ for all $\theta \in S^2$.

The speaker considered a similar question for small circles on the sphere, instead of great circles. Namely, consider a fixed $t \in (0, 1)$ and let f and g be continuous functions on S^2 such that for every 2-dimensional affine subspace H that is distance t from the origin there is a rotation ϕ_H in H such that $f(\theta) = g(\phi_H(\theta))$ for all $\theta \in S^2 \cap H$. Is it true that $f(\theta) = g(\theta)$ for all $\theta \in S^2$. In this talk it is shown that the answer to this question is affirmative if f and g are of class $C^2(S^2)$.

Petros Valettas gave a talk about "A Gaussian small deviation inequality", a joint work with G. Paouris. The Gaussian concentration phenomenon states that for any L-Lipschitz map $f : \ mathbbR^n \to \mathbb{R}$ one has

$$\mathbb{P}\left(|f(Z) - M| > t\right) \leq \exp\left(-\frac{1}{2}t^2/L^2\right),$$

for all $t > 0$, where Z is an n-dimensional standard Gaussian random vector and M is the median for $f(Z)$.

The authors are interested in refining (a one-sided version of) this inequality by replacing L with the variance $\mathrm{Var}f(Z)$. It is known that $\mathrm{Var}f(Z) \leq L^2$. However, there are many Lipschitz maps for which $\mathrm{Var}f(Z) \ll L^2$.

Their main result reads as follows. For any convex map $f \in L_2(\gamma_n)$ one has

$$\mathbb{P}\left(f(Z) - M < -t\right) \leq \exp\frac{1}{2}\left(-\frac{\pi}{1024}t^2/\mathrm{Var}f(Z)\right),$$

for all $t > 0$.

Monika Ludwig presented her joint work with L. Silverstein "Valuations on lattice polytopes". The study of valuations on convex bodies is a classical area. There are well-known classifications of such valuations. In this talk the authors study valuations on the space $\mathcal{P}(\mathbb{Z}^n)$ of lattices polytopes (these are convex hulls of finitely many points from $\mathbb{Z}^n$). A natural question is to classify $SL_n(\mathbb{Z})$ and translation invariant valuations on $\mathcal{P}(\mathbb{Z}^n)$. Some work in this direction was done by Betke and Kneser in the case of real-valued valuations. Here the authors prove the following result about vector-valued valuations. $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ is an $\mathrm{SL}_n(\mathbb{Z})$ equivariant, translation covariant valuation if and only if there exist c_1,..., $c_{n+1} \in \mathbb{R}$ such that

$$Z = c_1 l_1 + \cdots + c_{n+1} l_{n+1}.$$

Here, l_i's are defined by the formula

$$l(kP) = \sum_{i=1}^{n+1} l_i(P)k^i,$$

where P is a lattice polytope, k is a positive integer, and

$$l(P) = \sum_{x \in P \cap \mathbb{Z}^n} x$$

is the discrete moment vector of P.

The next natural step is to look at what happens for tensor-valued valuations. They find a classification of $\mathrm{SL}_n(\mathbb{Z})$ equivariant, translation covariant valuations with values in the space of symmetric tensors of rank at most 8. The case of tensors of rank 9 and higher appears more complicated.

Franz Schuster talked about "Even $SO(n)$ Equivariant Minkowski Valuations – An Update". The main theme is finding Hadwiger-type theorems for Minkowski valuations. Recall that a map Φ from the set $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$ to itself is called a Minkowski valuation if

$$\Phi(K \cap L) + \Phi(K \cup L) = \Phi(K) + \Phi(L),$$

whenever K, L, and $K \cup L$ are in $\mathcal{K}^n$.

The first problem is to describe the set $\mathrm{MVal}^{SO(n)}$ of continuous Minkowski valuations, which are translation invariant and $SO(n)$ equivariant.

The second problem is to find a classification/description of $\mathrm{MVal}_i^{SO(n)}$. Here, $\mathrm{MVal}_i^{SO(n)} = \{\Phi \in \mathrm{MVal}^{SO(n)} : \Phi(\lambda K) = \lambda^i \Phi(K)\}$.

The speaker discussed recent progress and insights on these problems. In particular, Wannerer and Schuster obtained the following result. If $\Phi_i \in \mathrm{MVal}_i^{SO(n),\infty}$ (∞ refers to smooth valuations), $1 \le i \le n-1$, then there exists a unique function $g_{\Phi_i} \in C^\infty([-1,1])$ such that

$$h(\Phi_i K, u) = \int_{S^{n-1}} g_{\Phi_i}(\langle u, v\rangle) dS_i(K, v), \qquad u \in S^{n-1},$$

where dS_i is the i-th surface area measure.

Of course, the natural goal is to remove smoothness in the above theorem. Recently, Dorrek proved the following. If $\Phi_i \in \mathrm{MVal}_i^{SO(n)}$, then there exists a unique function $g_{\Phi_i} \in L^1([-1,1])$ such that

$$h(\Phi_i K, u) = \int_{S^{n-1}} g_{\Phi_i}(\langle u, v\rangle) dS_i(K, v), \qquad u \in S^{n-1}.$$

The description of such functions g_{Φ_i} is still an open problem.

Wolfgang Weil spoke about "Integral representations of mixed volumes", a joint work with D. Hug and J. Rataj. Recall that mixed volumes arise as coefficients in the following expansion:

$$\mathrm{vol}(t_1 K_1 + \cdots + t_m K_m) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} t_{i_1} \cdots t_{i_n} V(K_{i_1}, ..., K_{i_n}),$$

where K_1,...,K_m are convex bodies and t_1,...,t_m are positive numbers.

It would be important to represent mixed volumes by integrals of local quantities of K_1,...,K_m. In the case when there are only two bodies such a formula was obtained earlier by the same authors. There is a function $f_{j,n-j}$ such that for all K, M (in suitable general position)

$$V(K[j], M[n-j]) = \int_{F(n,n-j+1)} \int_{F(n,j+1)} f_{j,n-j}(u_1, L_1, u_2, L_2) \psi_j(K, d(u_1, L_1)) \psi_{n-j}(M, d(u_2, L_2)),$$

where $\psi_j(K, \cdot)$ and $\psi_{n-j}(M, \cdot)$ are flag measures of K and M and the integration is over the manifolds of flags $F(n, n-j+1)$ and $F(n, j+1)$ respectively, which are defined as follows:

$$F(n, j+1) = \{(u, L) : L \in G(n, j+1), u \in S^{n-1} \cap L\},$$

$$\psi_j(K, \cdot) = \int_{G(n,j+1)} 1((u, L) \in \cdot) S_j'(K|L, du) dL.$$

Recently the authors extended this result to the case of more than two bodies. The formula is similar to the one above.

Martin Henk gave a talk about "The even dual Minkowski problem", based on joint works with K. Böröczky and H. Pollehn. The Minkowski-Christoffel problem asks for characterizations of area measures $S_i(K, \cdot)$ of a convex body K among all finite Borel measures on the sphere. When $1 < i < n-1$ the problem is still open.

Lutwak initiated the dual Brunn-Minkowski theorem. Instead of the Minkowski addition as in the classical setting, here one uses the radial addition. In this theory there is a local dual Steiner formula, which allows to define ith dual curvature measures $\tilde{C}_i(K, \cdot)$. Moreover, there are explicit formulas for these measures that allow an extension from an integer i to a real number q. The dual Minkowski problem asks for necessary and sufficient conditions for a given finite Borel measure on the sphere to be the qth dual curvature measure of some convex body K. The authors found the following necessary condition. Let K be an origin-symmetric convex body, $q \in (1, n)$, and L be a proper subspace. Then

$$\tilde{C}_i(K, S^{n-1} \cap L) < \min\left\{1, \frac{\dim L}{q}\right\} \tilde{C}_i(K, S^{n-1}).$$

Later Zhao, Böröczky, Lutwak, Yang, Zhang showed that this condition is also sufficient.

What happens for other values of q? Zhao found a necessary and sufficient condition when $q < 0$. Henk and Pollehn gave a necessary condition for $q \geq n+1$. Let K be an origin-symmetric convex body in $\mathbb{R}^n$. Then for every proper subspace $L \subset \mathbb{R}^n$,

$$\tilde{C}_i(K, S^{n-1} \cap L) < \frac{q - n + \mathrm{dim}L}{q} \tilde{C}_i(K, S^{n-1}),$$

and the bound is best possible.

An open question is whether this condition is also sufficient.

Hermann König presented his work "Submultiplicative operators in C^k-spaces", joint with D. Faifman and V. Milman. Multiplicative operators have been studied by many authors and the corresponding characterization were obtained. In this talk the goal is to look at the stability properties. Can we go from multiplicative operators to submultiplicative?

The first main result is the following. Let $I \subset \mathbb{R}^n$ be an open set and $k \in \mathbb{N}_0$. Consider a map $T : C^k(I) \to C^k(I)$ that is bijective and submultiplicative, meaning that for all $f, g \in C^k(I)$ one has

$$T(f \cdot g) \leq T(f) \cdot T(g).$$

T is also assumed to be pointwise continuous and satisfying the property $f \geq 0$ if and only if $Tf \geq 0$. Then there exist functions p, $A \in C(I)$, satisfying $p > 0$, $A \geq 1$, and a C^k-diffeomorphism u so that

$$\begin{aligned} &\text{for } k = 0, \quad Tf(u(x)) = \begin{cases} f(x)^{p(x)}, & f(x) \geq 0, \\ -A(x)|f(x)|^{p(x)}, & f(x) < 0, \end{cases} \\ &\text{for } k > 0, \quad Tf(u(x)) = f(x). \end{aligned}$$

The second main result, which is joint with V. Milman, reads as follows. Let $\phi : \mathbb{R} \to \mathbb{R}$ be submultiplicative, i.e.

$$\phi(xy) \leq \phi(x)\phi(y), \qquad x, y \in \mathbb{R}.$$

ϕ is also assumed to be measurable, continuous at 0, 1, and satisfying $\phi(-1) < 0 < \phi(1)$. Then there exist numbers $p > 0$, $A \geq 1$ so that

$$\phi(x) = \begin{cases} x^p, & x \geq 0, \\ -A|x|^p, & x < 0. \end{cases}$$

Konstantin Tikhomirov spoke about "Superconcentration, and randomized Dvoretzky's theorem for spaces with 1-unconditional bases". For an origin-symmetric convex body B in $\mathbb{R}^n$ and a linear operator $U : \mathbb{R}^n \to \mathbb{R}^n$ define

$$\ell(B, U) = \left(\mathbb{E}\|U(G)\|_B^2\right)^{1/2},$$

where G is the standard Gaussian vector in $\mathbb{R}^n$. The body B is said to be in ℓ-position if $\ell(B, \mathrm{Id}_n) = 1$ and

$$1 = \det \mathrm{Id}_n = \sup\left\{|\det U| : U \in \mathbb{R}^{n \times n}, \ell(B, U) \leq 1\right\}.$$

The main result is the following. Let B be an origin-symmetric convex body in $\mathbb{R}^n$ in the ℓ-position, and such that the space $(\mathbb{R}^n, \|\cdot\|_B)$ has a 1-unconditional basis. Further, let $\varepsilon \in (0, 1/2]$ and $k \leq c\varepsilon \log n / \log \frac{1}{\varepsilon}$. Then for a random k-dimensional subspace $E \subset \mathbb{R}^n$ uniformly distributed according to the Haar measure, one has

$$\mathbb{P}\{B \cap E \text{ is } (1+\varepsilon)\text{-spherical}\} \geq 1 - 2n^{-c\varepsilon},$$

where $c > 0$ is a universal constant.

This shows that the "worst-case" dependence on epsilon in the randomized Dvoretzky theorem in the ell-position is significantly better than in John's position.

Alexander Litvak presented his joint work with K. Tikhomirov titled "Order statistics of vectors with dependent coordinates". In 2000 Mallat and Zeitouni posed the following question. Let $X = (X_1, ..., X_n)$ be an

n-dimensional random Gaussian vector with independent centered coordinates (possibly with different variances). Let T be an orthogonal transformation of $\mathbb{R}^n$ and $Y = T(X)$. Is it true that for every $k \leq n$ one has

$$\mathbb{E}\sum_{j=1}^{k} j\text{-}\min_{i\leq n} X_i^2 \leq \mathbb{E}\sum_{j=1}^{k} j\text{-}\min_{i\leq n} Y_i^2?$$

Here, for a sequence of numbers a_1,..., a_n and $j \leq n$, $j\text{-}\min_{1\leq i\leq n} a_i$ denotes the j-th smallest element of the sequence.

In their work the authors solve this problem in the affirmative (up to an absolute multiplicative constant). That is, there is an absolute constant $C > 0$ such that

$$\mathbb{E}\sum_{j=1}^{k} j\text{-}\min_{i\leq n} X_i^2 \leq C\mathbb{E}\sum_{j=1}^{k} j\text{-}\min_{i\leq n} Y_i^2.$$

Matthew Stephen presented his joint work with N. Zhang "Grünbaum's inequality for projections". Let K be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. Let $\xi \in S^{n-1}$ and denote $\xi^+ = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq 0\}$. Grünbaum's inequality says that

$$\frac{\mathrm{vol}_n(K \cap \xi^+)}{\mathrm{vol}_n(K)} \geq \left(\frac{n}{n+1}\right)^n,$$

with equality at the cone.

Another closely related inequality is due to Minkowski and Radon. If K is a convex body with centroid at the origin and h_K is its support function, then

$$\frac{h_K(\xi)}{h_K(\xi) + h_K(-\xi)} \geq \frac{1}{n+1},$$

with equality at the cone.

Stephen and Zhang realized that there should be a connection between these two inequalities. Moreover, they found a general inequality that has these two as particular cases. They proved the following. Fix $1 \leq k \leq n$ and take $E \in G(n, k)$, $\xi \in S^{n-1} \cap E$. If K is a convex body with centroid at the origin, then

$$\frac{\mathrm{vol}_k((K|E) \cap \xi^+)}{\mathrm{vol}_k(K|E)} \geq \left(\frac{k}{n+1}\right)^k.$$

The equality condition is also characterized.

Susanna Dann talked about "Flag area measures", a joint work with J. Abardia and A. Bernig. Let V be an Euclidean vector space of dimension n and let $\mathcal{K}(V)$ be the space of non-empty compact convex subsets in V. A valuation on V is a map $\mu : \mathcal{K}(V) \to \mathbb{R}$ satisfying

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L),$$

whenever $K, L, K \cup L \in \mathcal{K}(V)$. We say that μ is continuous if it is so with respect to the topology on $\mathcal{K}(V)$ induced by the Hausdorff metric. A flag area measure on V is a continuous translation-invariant valuation with values in the space of signed measures on the flag manifold consisting of a unit vector v and a $(p+1)$-dimensional linear subspace containing v where $0 \leq p \leq n-1$.

Using local parallel sets Hinderer constructed examples of $\mathrm{SO}(n)$-covariant flag area measures. There is an explicit formula for his flag area measures evaluated on polytopes involving the squared cosine of the angle between two subspaces. The authors construct a more general space of $\mathrm{SO}(n)$-covariant flag area measures via integration of appropriate differential forms. They also compute the dimension of this space, discuss their properties and provide explicit formulas on polytopes, which are similar to the formulas for Hinderer's examples, however with an arbitrary elementary symmetric polynomial in the squared cosines of the principal angles between two subspaces. Hinderer's flag area measures correspond to special cases where the elementary symmetric polynomial is just the product. Moreover, they construct an explicit basis for this space, which gives a classification result in the spirit of Hadwiger's theorem.

Boaz Slomka gave a talk "On convex bodies generated by Borel measures", a joint work with H. Huang. Their goal is to introduce a natural way of generating convex bodies from Borel measures. They suggest the following construction. Given a Borel measure μ on $\mathbb{R}^n$, define

$$M(\mu) = \bigcup_{\substack{0\le f\le 1\\ \int_{\mathbb{R}^n} f d\mu=1}} \left\{ \int_{\mathbb{R}^n} y f(y) d\mu(y) \right\},$$

where the union is taken over all measurable function $f : \mathbb{R}^n \to [0,1]$ with $\int_{\mathbb{R}^n} f d\mu = 1$. The set $M(\mu)$ is called the metronoid generated by μ. In particular, if μ is a finite combination of δ-measures, then $M(\mu)$ is a polytope. The latter suggests that some classical quantities related to approximation of convex bodies by polytopes can be extended to the class of metronoids.

For example, one such quantity is

$$d_R(K) = \inf \left\{ N \in \mathbb{N} : \exists P = \operatorname{conv}(x_1, \dots, x_N) \subset \mathbb{R}^n, \frac{1}{R} P \subset K \subset P \right\},$$

which naturally extends in the case of metronoids as follows:

$$d_R^*(K) = \inf \left\{ \mu(\mathbb{R}^n) : \frac{1}{R} M(\mu) \subset K \subset M(\mu) \right\}.$$

Similarly, the vertex index introduced by Bezdek and Litvak:

$$\operatorname{vein}(K) = \inf \left\{ \sum_{i=1}^{N} \|x_i\|_K : K \subset P = \operatorname{conv}(x_1, \dots, x_N) \right\}$$

in the setting of metronoids can be modified as follows:

$$\operatorname{vein}^*(K) = \inf \left\{ \int_{\mathbb{R}^n} \|x\|_K d\mu(x) : K \subset M(\mu) \right\}.$$

Next, the authors establish some bounds for the quantities they introduced. For example, they show that for an origin-symmetric convex body K in $\mathbb{R}^n$ one has

$$c\sqrt{n} \le c \operatorname{vein}^*(B_2^n) \le \operatorname{vein}^*(K) \le C_1 \operatorname{vein}^*(B_1^n) \le C_2 n.$$

Matt Alexander presented his work done with M. Fradelizi and A. Zvavitch "Polytopes of Maximal Volume Product". Let K be an origin-symmetric convex body in $\mathbb{R}^n$, and let

$$K^o := \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1 \ \forall y \in K\}$$

be its polar body. Denote by

$$P(K) = \operatorname{vol}_n(K)\operatorname{vol}_n(K^o)$$

the volume product of the body K.

In the non-symmetric case the volume product is defined as follows. The polar body of a convex body K in $\mathbb{R}^n$ with respect to a point z is

$$K^z := \{x \in \mathbb{R}^n : \langle x - z, y - z\rangle \le 1 \ \forall y \in K\}.$$

For a convex body K the Santaló point is the unique point $s(K)$ such that

$$\operatorname{vol}(K^{s(K)}) = \min_{z \in \operatorname{int} K} \operatorname{vol}(K^z).$$

Define the volume product of K to be

$$P(K) = \mathrm{vol}_n(K)\mathrm{vol}_n(K^{s(K)}).$$

The maximum of the volume product in the class of convex bodies is known: the maximizers are centered ellipsoids. However, it is interesting to find the maximum in some specific classes of convex bodies.

The authors examine the volume product for classes of restricted polytopes. Let $\mathcal{P}_m^n$ be the set of all polytopes in $\mathbb{R}^n$ with non-empty interior having at most m vertices. Denote

$$M_n^m = \sup_{K \in \mathcal{P}_m^n} P(K).$$

They prove that this supremum is achieved at some polytope with exactly m vertices and the sequence M_m^n is strictly increasing in m.

A polytope is called simplicial if every facet is a simplex. The authors show that if K is of maximal volume product among polytopes with at most m vertices, then K is a simplicial polytope.

They also investigate some particular classes $\mathcal{P}_m^n$. In particular, if K is an origin-symmetric convex body in $\mathcal{P}_8^3$, then the maximal volume product of such bodies is given by the double cone on a regular hexagonal base. They also find the maximizers of the volume product for convex hulls of $n+2$ points in $\mathbb{R}^n$.

Gideon Schechtman presented his joint work with A.Naor on lower bounds for the distortion of bi-Lipschitz embeddings of metric spaces. A metric space (X, d_X) is said to admit a bi-Lipschitz embedding into a metric space (Y, d_Y) if there exist $s \in (0,\infty), D \in [1,\infty)$ and a mapping $f : X \to Y$ such that

$$sd_X(x,y) \le d_Y(f(x), f(y)) \le Dsd_X(x,y), \qquad \forall x, y \in X.$$

When this happens it is said that (X, d_X) embeds into (Y, d_Y) with distortion at most D. The authors have previously done some work on bounding from below the distortion of embedding certain metric spaces into L_p.

In this talk the speaker considered embedding certain grids in Schatten p-classes S_p into L_p. In particular, let $M_n[m]$ be the grid of all $n\times n$ matrices, whose entries have values in the set $[m] = \{-m, -(m-1), ..., m-1, m\}$, equipped with the S_1 norm. The authors prove that for any n and m large enough with respect to n, the distortion of embedding $M_n[m]$ into a Banach space X is at least of order $n^{1/2}/\alpha(X)$. Here $\alpha(X)$ is the smallest constant K satisfying the so-called linear upper α inequality

$$\mathbb{E}_{\varepsilon_{ij}=\pm 1} \| \sum_{i,j=1}^{n} \varepsilon_{ij} x_{ij} \| \le K \mathbb{E}_{\varepsilon_i, \delta_j = \pm 1} \| \sum_{i,j=1}^{n} \varepsilon_i \delta_j x_{ij} \|$$

for all $x_{ij} \in X$. Much of the talk was devoted to α inequalities in different settings.

Carsten Schuett spoke about his joint work with O.Giladi, J.Prochno, N.Tomczak-Jaegermann and E.Werner on the geometry of triple tensor products of ℓ_p^n-spaces. Let X be an n-dimensional normed space with the unit ball B_X . The volume ratio of X is defined by

$$vr(X) = \inf_{E \subset B_X} \left(\frac{vol_n(B_X)}{vol_n(E)} \right)^{1/n},$$

where the infimum is taken over all ellipsoids contained in B_X, and vol_n is volume in $\mathbb{R}^n$. In their earlier work, Schuett and Tomczak-Jaegermann established the exact behavior of the volume ratio of tensor products of the spaces ℓ_p^n and ℓ_q^n. In the talk, Schuett explained the extension of this result to triple tensor products of the spaces ℓ_p^n, ℓ_q^n and ℓ_r^n for all choices of $p, q, r \in [1,\infty]$. The authors established the exact behavior of the volume ratio of these triple tensor products.

Károly Bezdek has given a talk under the title “From dual bodies to the Kneser-Poulsen conjecture”. Let $\mathbb{M}^d$ denote the d-dimensional Euclidean, hyperbolic, or spherical space. The r-dual set of a given set in $\mathbb{M}^d$ is the intersection of balls of radii r centered at the points of the given set. As a Blaschke–Santaló-type inequality for r-duality it was shown in the talk that for any set of given volume in $\mathbb{M}^d$ the volume of the r-dual set becomes maximal if the set is a ball. As an application also the following was proved. The Kneser–Poulsen Conjecture states that if the centers of a family of N congruent balls in Euclidean d-space is contracted, then the volume of the intersection does not decrease. A uniform contraction is a contraction where all the pairwise distances in the first set of centers are larger than all the pairwise distances in the second set of centers. Finally, the talk presented an outline of the proof of the Kneser–Poulsen conjecture for uniform contractions (with N sufficiently large) in $\mathbb{M}^d$.

Márton Naszódi spoke about his joint work with Károly Bezdek under the title "The Kneser–Poulsen conjecture for special contractions". Recall that the Kneser–Poulsen Conjecture in general states that if the centers of a family of N unit balls in $\mathbb{E}^d$ is contracted, then the volume of the union (resp., intersection) does not increase (resp., decrease). They consider two types of special contractions. First, a uniform contraction is a contraction where all the pairwise distances in the first set of centers are larger than all the pairwise distances in the second set of centers. The authors obtain that a uniform contraction of the centers does not decrease the volume of the intersection of the balls, provided that $N \geq (1+\sqrt{2})^d$. Their result extends to intrinsic volumes. They prove a similar result concerning the volume of the union. Second, a strong contraction is a contraction in each coordinate. They show that the conjecture holds for strong contractions. In fact, the result extends to arbitrary unconditional bodies in the place of balls.

Igors Gorbovickis spoke about "The central set and its application to the Kneser-Poulsen conjecture". He has given new results about central sets of subsets of a Riemannian manifold and applied those results to prove new special cases of the Kneser-Poulsen conjecture in the two-dimensional sphere and the hyperbolic plane.

Robert Connelly talked about "The isostatic conjecture", which is a joint work of him with Evan Solomonides and Maria Yampolskaya. They show that a jammed packing of disks with generic radii, in a generic container, is such that the minimal number of contacts occurs and there is only one dimension of equilibrium stresses. They also point out some connections to packings with different radii and results in the theory of circle packings whose graph forms a triangulation of a given topological surface.

Oleg Musin has given a talk under the title "Representing graphs by sphere packings". His talk surveyed recent major advances on the following three topics: Euclidean and spherical graph representations as two–distance sets; Euclidean and spherical graph representations as contact graphs of congruent sphere packings; generalizations of Steiner's porism and Soddy's hexlet.

János Pach talked about "Disjointness Graphs", which is a joint work of him with Gábor Tardos and Géza Tóth. The disjointness graph $G = G(S)$ of a set of segments S in $\mathbb{R}^d$, $d \geq 2$, is a graph whose vertex set is S and two vertices are connected by an edge if and only if the corresponding segments are disjoint. They prove that the chromatic number of G satisfies $\kappa(G) \leq (\omega(G))^4 + (\omega(G))^3$, where $\omega(G)$ denotes the clique number of G. It follows , that S has $\Omega(n^{\frac{1}{5}})$ pairwise intersecting or pairwise disjoint elements. Stronger bounds are established for lines in space, instead of segments. They show that computing $\omega(G)$ and $\kappa(G)$ for disjointness graphs of lines in space are NP–hard tasks. However, they can design efficient algorithms to compute proper colorings of G in which the number of colors satisfies the above upper bounds. One cannot expect similar results for sets of continuous arcs, instead of segments, even in the plane. They construct families of arcs whose disjointness graphs are triangle-free ($\omega(G) = 2$), but whose chromatic numbers are arbitrarily large.

Outcome of the Meeting

The meeting was very successful, we were lucky to bring together mathematicians from many countries and many research areas, such as convex geometry, discrete geometry, probability, functional analysis. Besides the leading scientists, we also had 6 graduate students and 12 postdocs or recent PhDs participating in the workshop. Female participation was above 21%. The friendly atmosphere created during the workshop helped many participants not only to identify the promising ways to attack the old problems but also to get acquainted with many open new ones.

Participants

Akopyan, Arseniy (IST Austria)
Alexander, Matt (Kent State University)

Alfonseca-Cubero, Maria de los Angeles (North Dakota State University)
Artstein, Shiri (Tel-Aviv University)
Bezdek, Karoly (University of Calgary)
Connelly, Robert (Cornell University)
Dann, Susanna (University of Bogota)
Florentin, Dan (Kent State University)
Gorbovickis, Igors (Uppsala University)
Henk, Martin (Technische Universität Berlin)
Khan, Muhammad (University of Calgary)
Kim, Jaegil (University of Alberta)
Klartag, Boaz (Weizmann institute)
Koenig, Hermann (Universitaet Kiel)
Koldobsky, Alexander (University of Missouri)
Litvak, Alexander (University of Alberta)
Livshyts, Galyna (Georgia Institute of Technology)
Ludwig, Monika (Technische Universität Wien)
Musin, Oleg (University of Texas Rio Grande Valley)
Myroshnychenko, Sergii (University of Alberta)
Naszodi, Marton (Alfred Renyi Institute of Mathematics)
Oliwa, Michael (University of Calgary)
Pach, Janos (Ecole Polytechnique Federale de Lausanne)
Paouris, Grigoris (Texas A&M University)
Pivovarov, Peter (University of Missouri)
Rotem, Liran (Technion)
Ryabogin, Dmitry (Kent State University)
Saroglou, Christos (Kent State University)
Schechtman, Gideon (Weizmann Institute of Science)
Schuster, Franz (Vienna University of Technology)
Schütt, Carsten (Christian-Albrechts-Universitaet)
Slomka, Boaz (University of Michigan)
Stancu, Alina (Concordia University)
Stephen, Matthew (University of Alberta)
Tikhomirov, Konstantin (Princeton University)
Tomczak-Jaegermann, Nicole (University of Alberta)
Valettas, Petros (Universtiy of Missouri)
Vritisou, Beatrice (University of Alberta)
Weil, Wolfgang (Karlsruhe Institute of Technology)
Werner, Elisabeth (Case Western Reserve University)
Yaskin, Vladyslav (University of Alberta)
Zhang, Ning (Huazhong University of Science and Technology)

Chapter 11

Arithmetic Aspects of Explicit Moduli Problems (17w5065)

May 28 - June 2, 2017

Organizer(s): Nils Bruin (Simon Fraser University), Kiran Kedlaya (University of California San Diego), Samir Siksek (University of Warwick), John Voight (Dartmouth College)

Overview of the Field

A central theme of modern number theory is understanding the absolute Galois group of the rational numbers $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and more generally $G_K = \mathrm{Gal}(\overline{K}/K)$ where K is a number field. The principal approach to this has been to study the action of $G_{\mathbb{Q}}$ on objects arising in geometry, especially the p-torsion of elliptic curves. Let E be an elliptic curve defined over $\mathbb{Q}$. This can be given by an equation of the form

$$E \;:\; Y^2 = X^3 + AX + B, \qquad (A, B \in \mathbb{Z}, \quad 4A^3 + 27B^2 \neq 0).$$

The points of E form a group. The action of $G_{\mathbb{Q}}$ on the p-torsion subgroup $E[p]$ gives rise to a mod p representation $\overline{\rho}_{E,p} \;:\; G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_p)$. Such representations are the subject of two of the most important conjectures in number theory. Both are due to Fields Medalist and Abel Prize winner Jean-Pierre Serre: Serre's uniformity conjecture (1968) and Serre's modularity conjecture (1986). Serre's modularity conjecture was recently proved by Khare and Wintenberger (2009). Another closely related conjecture is the modularity conjecture for elliptic curves over $\mathbb{Q}$, proved by Wiles and Taylor (1995) for semistable elliptic curves over $\mathbb{Q}$, and by Breuil, Conrad, Diamond and Taylor (2001) for all elliptic curves over $\mathbb{Q}$. In proving modularity for semistable elliptic curves, Wiles proved Fermat's Last Theorem, a question that had vexed mathematicians for 350 years.

A modular curve classifies elliptic curves whose torsion points enjoy certain Galois properties. Understanding the points of modular curves over number fields is key to many great theorems in number theory. Let us mention a few:

(I) Heegner's resolution of Gauss's class number 1 problem required the determination of rational points on various modular curves.

(II) Mazur proved that if E is an elliptic curve defined over $\mathbb{Q}$ and $p > 163$ then E does not have a rational p-isogeny (an equivalent formulation is that $\overline{\rho}_{E,p}$ is irreducible for $p > 163$). The proof involved the determination of rational points on the family of modular curves $X_0(p)$. Mazur's theorem is one of the three great pillars on which the proof of Fermat's Last Theorem rests; the other two are Ribet's level-lowering theorem, and Wiles' modularity of semistable elliptic curves.

(III) Building on earlier work by Mazur and Kamienny, Merel proved the uniform boundedness conjecture: for any $d \geq 1$, there is a B_d such that if E is an elliptic curve over a number field K of degree d and $p > B_d$ a prime, then E does not have K-rational p-torsion. Merel's theorem involves the study of rational points on symmetric powers of $X_0(p)$ and $X_1(p)$.

(IV) The obstruction to modularity lifting for elliptic curves is represented by rational points on modular curves. The proof of the full modularity theorem for elliptic curves by Breuil, Conrad, Diamond and Taylor required the determination of rational points on several modular curves; this computation was carried out by Elkies.

(IV) Serre's uniformity conjecture that asserts that if $p > 37$ is prime and $E/\mathbb{Q}$ an elliptic curve without complex multiplication then $\overline{\rho}_{E,p}$ is surjective. This conjecture reduces to the determination of rational points on three families of modular curves $X_0(p)$, $X_{\mathrm{s}}^+(p)$, $X_{\mathrm{ns}}^+(p)$.

Beyond modular curves there are equally intriguing but harder moduli problems for curves of higher genus, abelian varieties, abelian varieties with level structure, abelian varieties with certain endomorphism rings, etc.

Recent Developments

In recent years there have been many spectacular breakthroughs (both theoretical and algorithmic). These have formed a strong motivation for the workshop. Among them we mention the following:

(I) The proof by Bilu, Parent and Rebolledo [2], [3] of the split Cartan case of Serre's uniformity conjecture: they have determined the rational points on the modular curves $X_{\mathrm{s}}^+(p)$ for $p = 11$ and $p \geq 17$.

(II) The recent proof of modularity of elliptic curves over real quadratic fields by Le Hung, Freitas and Siksek [8]. This required the determination of quadratic points on several complicated modular curves.

(III) Systematic tables of modular curves of small genus due to Zywina and Sutherland.

(IV) Equations for Hilbert modular surfaces for all thirty fundamental discriminants D of level $1 < D < 100$ due to Elkies and Kumar [7].

(V) Work of Derickx, Kamienny, Stein and Stoll [6] who determined the possible prime orders of torsion points on elliptic curves over number fields of degrees 4, 5, 6.

(VI) A database of genus 2 curves due to Booker, Sijsling, Sutherland, Voight and Yasaki [4].

(VII) Work of Bruin and Nasserden [5] elucidating the arithmetic of the Burkhardt quartic which is the moduli space for principally polarized abelian surfaces with full level 3 structure.

Open Problems

An open problems session formed one of the highlights of the workshop. The participants were encouraged to suggest good open problems as a means of stimulating further progress in the field.

David Zureick–Brown

Compute $X_H(\mathbb{Q})$ from the following list of curves.

```
P2<x,y,z> := ProjectiveSpace(Rationals(),2);

// level 3^n curves
X33:= Curve(P2, -x^3*y + x^2*y^2 - x*y^3 + 3*x*z^3 + 3*y*z^3);
X43:= Curve(P2, x^3*z - 6*x^2*z^2 + 3*x*y^3 + 3*x*z^3 + z^4);
```

```
// level 5^n curves
R<x> := PolynomialRing(Rationals());
S<a,b,c,d> := PolynomialRing(Rationals(),4);

h := x^3 + x + 1;
f := 6*x^6 + 5*x^5 + 12*x^4 + 12*x^3 + 6*x^2 + 12*x - 4;
X11 := HyperellipticCurve([f,h]);

h2 := x^3 + x + 1;
f2 := x^6 - 13*x^4 - 38*x^3 + 6*x^2 + 22*x + 6;
X15 := HyperellipticCurve([f2,h2]);

f1 := a^2 + 51*a*b + 648*b^2 - 900*a*c - 22086*b*c + 211572*c^2 - 25650*a*d
      - 629856*b*d + 11499732*c*d + 156402576*d^2;
f2 := a*b^2 + 24*b^3 - 438*a*b*c - 10818*b^2*c - 11232*a*c^2 -  186732*b*c^2
      - 243648*c^3 - 12996*a*b*d - 320382*b^2*d -  285444*a*c*d - 2161728*b*c*d
      - 104818536*c^2*d +  992412*a*d^2 + 90530136*b*d^2 - 5156170344*c*d^2
      - 67660478712*d^3;
X16 := Curve(ProjectiveSpace(Rationals(),3),[f1,f2]);
```

David Zureick–Brown

In Theorem 1.4 of Várilly-Alvarado–Viray

https://sites.math.washington.edu/~bviray/papers/VAV_UniformBoundRank19K3.pdf

and degree $r'' = 2$ (so over quadratic fields), apply results of Bruin–Najman

https://arxiv.org/pdf/1406.0655.pdf

so with finitely many exceptions, an elliptic curve over a quadratic extension with a cyclic n-isogeny is a $\mathbb{Q}$-curve.

Eric Katz

A question related to the Chabauty method: define iterated p-adic integrals in a down-to-earth way without using Frobenius. Suppose C over $\mathbb{Q}_p$ has good reduction. Classically, a p-adic integral comes about via

$$C(\mathbb{C}_p) \hookrightarrow J(\mathbb{C}_p) \xrightarrow{\text{Log}} \text{Lie}J(\mathbb{C}_p);$$

so for iterated integrals, we need to replace J by a unipotent analogue.

René Schoof

Let X be a nice curve over $\mathbb{Q}$ of genus $g \geq 1$ given by equations in projective space $\mathbb{P}^n$ equipped with a height function h. Let $P_0 \in X(\mathbb{Q})$, and use P_0 to embed $X(\mathbb{Q}) \hookrightarrow J(\mathbb{Q})$ by $P \mapsto [P - P_0]$. One has the canonical height $\widehat{h}$ on $J(\mathbb{Q})$. Are there bounds for $h(P)$ in terms of $\widehat{h}([P - P_0])$? If $g = 1$, there are bounds in Silverman. (We would use this to say that points in a box on $J(\mathbb{Q})$ determine points in a box on $X(\mathbb{Q})$.)

Kiran Kedlaya

By an old result of Mumford, the closure of the moduli space of principally polarized abelian fourfolds with trivial geometric endomorphism algebra but the Mumford-Tate group is nontrivial ($\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$) is nonempty and a countable union of components of dimension 1.

- Give an explicit model for one or more components.
- Give explicit points, especially on the Torelli locus.
- For points on the Torelli locus, what fields of definition are possible? (Is it possible to show or rule out the existence of an example over $\mathbb{Q}$?)

Jeroen Sijsling

As in Problem 4, let X be a nice curve over $\mathbb{Q}$ of genus $g \geq 1$ given by equations in $\mathbb{P}^n$. Embed $X(\mathbb{Q}) \hookrightarrow J(\mathbb{Q})$ by $P \mapsto [P - P_0]$ for $P_0 \in X(\mathbb{Q})$.

Now let $M : H^0(X, \omega_X) \to H^0(X, \omega_X)$ be a matrix representing a candidate endomorphism α of J. To check if α is an endomorphism, we compute

$$\alpha([P - P_0]) = \sum_{i=1}^{g} [Q_i - P_0]$$

and make the corresponding graph $Y \subset X \times X$, the closure of the points (P, Q_i) so obtained.

- The projection onto the first component is degree g. What is the degree of the projection onto the second projection?
- Which monomials are needed to define $Y \subseteq \mathbb{P}^n \times \mathbb{P}^n$, i.e., those monomials in some set of generators for the ideal of vanishing of Y?
- What can one say about the sizes of the coefficients in the equations defining Y?

Maarten Derickx

Derickx–Kamienny–Mazur

`http://www.math.harvard.edu/~mazur/papers/For.Momose20.pdf`

prove that every point on $X_1(17)$ defined over a quartic field comes from a rational function of degree 4 on $X_1(17)$; moreover, up to $(\mathbb{Z}/17\mathbb{Z})^*/\{\pm 1\}$, there are three such functions, with Galois group once S_4 and twice D_4. Note there exists an elliptic curve E over a number field K with $\mathrm{Gal}(K/\mathbb{Q}) \simeq C_4$ cyclic which has a direct explanation.

Find the rational points on those curves that classify when the Galois group of these points is smaller: for the normal closure $X \to X_1(17) \to \mathbb{P}^1$ and a subgroup $H \leq \mathrm{Gal}(X/\mathbb{P}^1)$, we find modular curves $X/H \to \mathbb{P}^1$ and there are six left.

For more detail, see the file

`http://www.birs.ca/workshops/2017/17w5065/files/X_1(17)_D4_S4.txt`

Jennifer Johnson–Leung

Let F be a Siegel paramodular form of level N with Fourier–Jacobi expansion

$$F(\tau, \tau', z) = \sum_k f_k(\tau, z) q^k.$$

Let χ be a quadratic character of conductor p, and consider the twist

$$F(\tau, \tau', z; \chi) = \sum_k \chi(k) f_k(\tau, z) q^k;$$

the twist is no longer a Siegel paramodular form, but rather, it is stable under the stable paramodular group $K_s(p^n) = K(p^n) \cap K(p^{n-1})$ where $p^n \parallel N$ and $K(m)$ is the paramodular group of level m. The representation theory of the group $K_s(p^n)$ is very nice, worked out by Ralf Schmidt, with newspaces of dimension 1 when they are supposed to be—and there are Hecke operators.

Is there a geometric object associated to $F(\tau, \tau', z; \chi)$? And is there some class of abelian surfaces for which the Galois representations coincide?

Bjorn Poonen

Let $p > 2$ be a prime, let $k = \mathbb{F}_p(t)$ and $X : y^p = tx^p + x$. Compute $X(k)$. Is there a nice way to do it?

This curve is smooth and has the structure of an additive group. But over a base extension, the genus goes down, and by work of Voloch the set of points is finite, so the answer is a finite abelian group. (For $p = 2$, the curve is a conic birational to $\mathbb{P}^1$.)

Several people suggested an argument to prove that $(0, 0)$ is the only solution. In particular, Bas Edixhoven used a parametrization of the curve over $\mathbb{F}_p(u)$ with $u^p = t$, and then imposed the conditions that dx/du and dy/du be zero to ensure that x and y are in $\mathbb{F}_p(t)$ instead of just $\mathbb{F}_p(u)$.

Drew Sutherland

Given a smooth plane quartic X over $\mathbb{Q}$ compute $\mathrm{Jac}(X)(\mathbb{Q})_{\mathrm{tors}}$ efficiently. This would be useful for the database of genus 3 curves going into the LMFDB.

For hyperelliptic of genus 3, in principle it has been worked out. Work modulo many primes to get an upper bound and look for rational points to match. Perhaps Chaubauty's method works (make Manin–Mumford effective)? Perhaps a Hensel lifting method works?

(It may also be interesting to work out the geometrically hyperelliptic but non-hyperelliptic curves.)

Elisa Lorenzo Garcia

What modular curves $X(\Gamma)$ have a smooth plane model? (In particular, all genus three non-hyperelliptic modular curves.) Then $g = (d-1)(d-2)/2$ for a degree d, and we need a g^2_d-linear system on X. Such a curve has gonality $\sqrt{g}$, so using an effective bound on the gonality this should reduce the problem to a finite list?

David Zureick–Brown

Is there a surface S which is not the quotient of the product of two curves, with a nontrivial Albanese variety, such that one can apply Chabauty's method?

Armand Brumer

We leave it to the reader to generalize this in the obvious manner. It is motivated by making sure that we might someday be able to find all abelian surfaces over $\mathbb{Q}$ of given conductor.

Let S be a finite set of primes, $\mathcal{A}(S)$ be the finite set of abelian surfaces good outside S, and $\mathcal{J}(S)$ the set of Jacobians in $\mathcal{A}(S)$. Introduce an invariant $d(S)$ and a set $T(S)$ as follows. For each isogeny class in $\mathcal{A}(S)$, take the minimum degree of any polarization and then let $d(S)$ be the maximum over the isogeny classes in $\mathcal{A}(S)$. Let $T(S)$ be a minimal set of places such that each isogeny class in $\mathcal{J}(S)$ contains a Jacobian $\mathrm{Jac}(C)$ such that C is good outside $T(S)$.

What can be said about $d(S)$ and $T(S)$. Is $d(S)$ bounded as S grows?

Even 30 years after Faltings, the only case understood is $S = \emptyset$! Even for $S = \{2\}$ neither $d(S)$ nor $T(S)$ are known. The work of Merriman–Smart only find the curves good outside 2, but there are many other examples beyond this list.

The problem is slightly easier if one restricts to semistable abelian varieties: for a few sets S, one may find all semistable surfaces good outside S, up to isogeny, thanks to Schoof or Brumer–Kramer.

Samuele Anni

Let $E/\mathbb{Q} : y^2 + y = x^3 - x$ (LMFDB label 37.a1). For every prime ℓ we have that $\mathrm{Gal}(\mathbb{Q}(E[\ell])/\mathbb{Q}) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$. This gives a realization of $\mathrm{GL}_2(\mathbb{F}_\ell)$ as Galois group over $\mathbb{Q}$ for all primes ℓ using "one object". Is there an analogous construction, i.e. simultaneous realization of $\mathrm{GL}_2(\mathbb{F}_\ell)$ for all ℓ as Galois group using the "same object", over any number field different from $\mathbb{Q}$?

John Voight

Computations with paramodular forms and L-functions suggest that there is an abelian surface A over $\mathbb{Q}$ of conductor 550 whose first few Euler factors (computed by David Farmer and Sally Koutsoliotas, the first few by Cris Poor and David Yuen) are as follows:

$$\begin{aligned}
L_2(T) &= (1+T)(1+2T^2) \\
L_3(T) &= 1 - T^2 + 9T^4 \\
L_5(T) &= 1 + 3T + 5T^2 \\
L_7(T) &= 1 + 4T^2 + 49T^4 \\
L_{11}(T) &= (1+T)(1-3T+11T^2) \\
L_{13}(T) &= 1 - 8T^2 + 169T^4
\end{aligned}$$

Show that such a surface exists! Because $L_3(T)$ is irreducible, if A exists then A is simple over $\mathbb{Q}$. The abelian surface A may or may not have a principal polarization over $\mathbb{Q}$. We expect that $A[2]$ is an extension of $E_1[2]$ by $E_2[2]$, where E_1 and E_2 are elliptic curves of conductors 11 and 50 respectively. The first few Dirichlet coefficients of the L-function are:

$$\begin{aligned}
\{&1, -1, 0, -1, -3, 0, 0, 1, 1, 3, 2, 0, 0, 0, 0, 3, -3, -1, 1, 3, 0, -2, -3, 0, 4, 0, 0, 0, 0, 0, -5, -3, 0, 3, 0, -1, \\
&3, -1, 0, -3, -3, 0, 12, -2, -3, 3, 6, 0, -4, -4, 0, 0, -6, 0, -6, 0, 0, 0, 3, 0, -14, 5, 0, -5, 0, 0, 0, \\
&3, 0, 0, 3, 1, -3, -3, 0, -1, 0, 0, 10, -9, -8, 3, -3, 0, 9, -12, 0, 2, 0, 3, 0, 3, 0, -6, -3, 0, 9, 4, 2, -4, 12, \\
&0, 6, 0, 0, 6, 21, 0, 4, 6, 0, 0, 0, 0, 9, 0, 0, -3, 0, 0, -4, 14, 0, 5, 3, 0, -18, 5, 0, 0, 0, 0, 0, 0, 0, -3, -6, 0, 1, \\
&0, 0, -3, 0, 3, 0, 3, 0, -3, -6, 0, -8, 1, -3, 0, 15, 0, -21, -10, 0, 9, 0, 8, 9, 3, 0, 3, 6, 0, 8, -9, \\
&1, -12, -18, 0, 0, 6, 0, 0, 12, 3, 13, 0, 0, -3, -9, 0, -6, -6, 0, 3, -3, 0, -15, -9, 0, 4, 12, -2, -32, 4, \\
&0, -12, 0, 0, 9, -6, -3, 0, 2, 0, 1\}.
\end{aligned}$$

Drew Sutherland

Let $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\ell)$ be an odd irreducible mod-ℓ Galois representation associated to a classical modular form f, and let p be a prime not dividing the level of f. Is there a way to determine the conjugacy class of $\rho_f(\mathrm{Frob}_p)$ directly from f (given by its q-expansion, say)?

When the eigenvalues of $\rho_f(\mathrm{Frob}_p)$ are distinct, this is clear, but if $\rho_f(\mathrm{Frob}_p)$ has trace 2 and determinant 1, for example, is it possible to distinguish the conjugacy classes of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ from the identity without computing separately the torsion of an associated abelian variety?

John Voight

Is there an efficient (or at least practical) algorithm that, given a genus 2 curve X over $\mathbb{Q}$, computes the isogeny graph of abelian surfaces isogenous to $\mathrm{Jac}(X)$ as principally polarized abelian varieties over $\mathbb{Q}$, and the minimal degree of isogenies between them—like for elliptic curves?

If one allows isogenies that do not respect the principal polarization (so we allow polarizations of arbitrary degree), is the corresponding set finite?

Presentation Highlights

Balakrishnan and Müller: Rational Points on $X_{\text{ns}}^{+}(13)$

At the workshop Jennifer Balakrishnan and Jan-Steffen Müller created much excitement by announcing the determination of the rational points on the modular curve $X_{\text{ns}}^{+}(13)$, as part of joint work with Netan Dogra, Jan Tuitman and Jan Vonk. This has been a famous open problem for many years. More significantly, it demonstrates the ideas pioneered initially by Minhyong Kim, which were extensively studied at BIRS workshop 07w5063, February 4-9, 2007, have substantial applications to modular curves.

Let $X/\mathbb{Q}$ be a curve of genus $g \geq 2$ with Jacobian J and let ℓ be a prime of good reduction. Using Selmer varieties, Kim defines a decreasing sequence

$$X(\mathbb{Q}_\ell) \supseteq X(\mathbb{Q}_\ell)_1 \supseteq X(\mathbb{Q}_\ell)_2 \supseteq \cdots$$

all containing $X(\mathbb{Q})$. Thanks to the work of Coleman, the 'Chabauty set' $X(\mathbb{Q}_\ell)_1$ is known to be finite provided the 'Chabauty condition' $\text{rank}J(\mathbb{Q}) < g$ holds. In this case one has a practical strategy that often succeeds in computing the set of rational points $X(\mathbb{Q})$. Alas, for the family of modular curves $X_{\text{ns}}^{+}(p)$ with $p \geq 13$ it is known (assuming BSD) that $\text{rank}J(\mathbb{Q})$ is at least the genus, making the methods of Mazur, Kamienny and Merel (as well as Coleman–Chabauty) inapplicable.

Balakrishnan and Dogra [4] have recently shown that the 'quadratic Chabauty set' $X(\mathbb{Q}_\ell)_2$ is finite provided

$$\text{rank}J(\mathbb{Q}) < g + \text{rankNS}(J) - 1,$$

where $\text{NS}(J)$ is the Néron-Severi group of $J/\mathbb{Q}$. It is known for modular curves X of genus $g \geq 3$ that $\text{rankNS}(J) \geq 2$, and thus quadratic Chabauty is strictly more powerful than classical Chabauty in the modular context. The joint work alluded to above turns quadratic Chabauty into a practical computational tool that can be used to attack explicit examples, and the application to $X_{\text{ns}}^{+}(13)$ is expected to the first of many breakthroughs with this method.

Zureick-Brown: Mazur's Problem B

Mazur's Problem B (also known as Mazur's vertical uniformity problem) asks for the determination of possible images of the representations $\rho_{E,p^\infty} : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{Z}_p)$ for elliptic curves E over the rationals and all primes p. For $p > 37$ it is easy to give an answer conditional on Serre's uniformity conjecture. Recently this question has been resolved completely by Rouse and Zureick-Brown [9] for $p = 2$. Zureick-Brown's talk gave a detailed overview of the proof which involves the computation of models of modular curves X_H and rational points on these modular curves for around 700 arithmetically minimal subgroups H of $\text{GL}_2(\mathbb{Z}_2)$.

Andrew Sutherland: Modular curves of prime-power level with infinitely many rational points

For each open subgroup G of $\text{GL}_2(\hat{\mathbb{Z}})$ containing $-I$ and having full determinant there is a a modular curve X_G defined over $\mathbb{Q}$ whose non-cuspidal points parametrize elliptic curves $E/\mathbb{Q}$ such that the image of $\rho_E : G_{\mathbb{Q}} \to \text{GL}_2(\hat{\mathbb{Z}})$ is contained in G. When the index of G is sufficiently large, the curve X_G has genus ≥ 2 and so by Faltings has finitely many rational points. This raise the interesting question of for which G does the modular curve X_G have infinitely many rational points. This talk gave an overview of recent work by Sutherland and Zywina in which they give a full answer to this question where G has prime-power level. They find (up to conjugacy) 248 such groups where $X_G(\mathbb{Q})$ is infinite, with 220 being curves of genus 0, and 28 being elliptic curves with positive rank. This is indeed a step towards Mazur's vertical uniformity conjecture, as for each prime p it gives an explicit classification of possible p-adic images with the possible exception of finitely many j-invariants.

Pierre Parent: Rational points of Modular Curves–An Arakelovian Point of View

Let p be a prime and let J_e denote Merel's winding quotient of $J_0(p)$; this is the maximal quotient that has analytic rank 0. One knows thanks to deep work of Kolyvagin, Logachev and Kato that $J_e(\mathbb{Q})$ is finite. Write $J \sim J_e \times J_e^{\perp}$. Let P be a degree d point on $X_0(p)$ (that is a point defined over a number field of degree d). Let $\tilde{P}$ denote the corresponding rational point on the d-th symmetric power $X_0(p)^{(d)}$. The image of this on $J_0(p)$ belongs to the intersection of the two cycles $(J_e^{\perp} + \text{torsion})$ and $X_0(p)^{(d)}$. Parent's talk explained that knowing the heights and degrees of the two cycles, allows via an arithmetic Bezout theorem, to give an upper bound for the height of the intersection. Of course the smaller the dimension of $J_e^{\perp}$ (and hence equivalently the larger the dimesion of J_e), the better control we have on the intersection. A theorem of Iwaniec and Sarnak gives

$$\frac{1}{4} + o(p) \leq \frac{\dim(J_e)}{\dim(J_0)} \leq \frac{1}{2} + o(p).$$

A conjecture of Brumer asserts

$$\frac{\dim(J_e)}{\dim(J_0)} = \frac{1}{2} + o(p).$$

Parent sketched a proof of the following theorem: under Brumer's conjecture, the j-height of quadratic points on $X_0(p^2)$ is $O(p^5 \log p)$. Remarkably the bound is independent of the quadratic field!

Scientific Progress Made

The workshop schedule was designed to give participants plenty of time for collaboration and discussions. We have asked the participants to report on the progress to existing projects made and also on any new projects initiated during the workshop.

- Samuele Anni and Elisa Lorenzo Garcia: we are designing an algorithm to compute endomorphism rings of threefolds in positive characteristic. Using this algorithm, we are also studying endomorphisms in characteristic zero through liftings, giving a completely algebraic alternative to the known algorithms.
- Samuele Anni and Ekin Özman. We want to study local points on fibred products of X0(p) for different primes p. This is connected to local-global questions studied in Ekin and my thesis from different points of view.
- Samuele Anni and Samir Siksek. A new paper on modularity of elliptic curves over totally real subfields of cyclotomic fields is in preparation.
- Andrew Sutherland, Jeroen Sijsling and John Voight. We have worked on our Genus 2 automorphy paper. This is still a work in progress, but we moved the ball forward.
- David Zureick-Brown received lots of advice and help from other participants towards completing Mazur's programme B. The progress made includes:
  - Determination of rational points on several X_H.
  - Andrew Sutherland was able to compute traces of Frobenius at the first 1000 primes for some of the large genus subgroups H. This allowed the recognition that a few pairs of X_H, X_K, with H, K not conjugate, were accidentally isomorphic (and then it was proved). Sutherland will optimize his code to allow computation of zeta functions (and hence, whether J_H is simple, etc) for several of the H for which there are currently no nice equations.
- Maarten Derickx and David Zureick-Brown. We were able to find the rational points on 4 of Maarten's 6 curves, and in one of the remaining 2 cases we were able to rule out most of the standard techniques from working.

- Eric Katz and David Zureick-Brown. We made fair progress on our Buium project (that was the subject of Katz's talk), and some progress on another project (about "Total Jet Spaces").

- Sara Arias-de-Reyna, Elisa Lorenzo Garcia and Christophe Ritzenthaler discussed some aspects of Jacobians of genus 3 curves. This is expected to lead to improvements on the Arias-de-Reyna's work, presented at the workshop, on the realisation of $\mathrm{GSp}_6(\mathbb{F}_\ell)$ as a Galois group of a tamely ramified extension.

- Rachel Pries and Ekin Özman were able to complete their project on p-ranks of trielliptic curves.

- Mark van Hoeij and David Zureick-Brown started an new collaboration. In Zureick-Brown's talk he displayed several curves having high degree plane models. Van Hoeij computed plane models of much lower degrees, which will help Zureick-Brown study the arithmetic of these curves.

- Francesc Fite, Elisa Lorenzo Garcia and Andrew Sutherland. We have continued our work on our paper Sato-Tate groups of twists of Fermat and Klein quartics. This is a project we have been working on for a long time, but we finally cracked the last stumbling block during the week and should be able to wrap up the paper shortly.

- Francesc Fite: In my talk I explained the theorem that ensures that if the square of an elliptic curve with CM admits a rational model up to $\overline{\mathbb{Q}}$-isogeny, then the quadratic imaginary field of the CM has either class number 1, class number 2, or class group $C_2 \times C_2$. While one easily shows that all quadratic imaginary fields with class numbers 1 and 2 arise, the question on whether quadratic imaginary fields with class group $C_2 \times C_2$ actually occur was open before the workshop. The afternoon after my talk John Voight showed to me an example having class group $C_2 \times C_2$. We expect to start a collaboration in which we determine exactly which quadratic imaginary fields with class group $C_2 \times C_2$ can arise.

- Nils Bruin, Armand Brumer, Chritophe Ritzenthaler and Jaap Top: we had a discussion which may lead to a new way to compute Serre's obstruction for abelian threefolds. We have not proved anything yet but the strategy seems coherent and effective. We consider a non-hyperelliptic genus 4 curve C in $\mathbb{P}^3$ as the intersection of a (unique) quadric Q and cubic over a field K. We ask that the discriminant of Q is a square in K and $Jac(C)$ to have a rational non-zero two-torsion point. We can then construct an unramified double cover $D \to C$ and its Prym is a principally polarized abelian threefold (A, a). We conjecture that if (A, a) is geometrically undecomposable, then (A, a) is the Jacobian of a genus 3 curve over K if and only if the Galois closure of $D \to C \to \mathbb{P}^1$ (the last map coming from any of the rational rulings on Q) is defined over K.

- Christophe Ritzenthaler: I received interesting feedback after my talk from Brumer, Viray, Elkies and Voight. During the problem session, I think we also proved that except for $X(7)$, none of the other $X(n)$ (with $n > 6$) can have a plane model because their automorphism groups are not automorphism groups of plane curves (by results from Harui).

- Ekin Özman and Samir Siksek: we are now many steps closer to completing our project of determining the quadratic points on $X_0(p)$ of genera 3, 4 and 5.

Outcome of the Meeting

As the above feedback amply demonstrates, this was a great meeting at which much progress has been made towards fundamental questions in the arithmetic of moduli problems. Many of our participants wrote to tell us how useful the workshop has been to them. We conclude with a few quotes from our participants highlighting the success of the meeting.

Jeroen Sijsling had this to say:

> "The venue and facilities of the workshop were top-notch. The planning encouraged the researchers involved to discuss as much as possible, an invitation that was certainly taken up. Especially useful

was the open problems session on Tuesday, where some participants, myself included, asked some open questions of theirs to the audience. My question got resolved quite rapidly by other participants. Also in this way the workshop contributed to advancing its field of research."

Jennifer Johnson-Leung said:

"The BIRS workshop 17w5065 had a strong positive impact on my research program, I had several valuable interactions with my colleagues. In particular, I learned of certain surfaces that Brumer has recently constructed that he believes to be paramodular. However, he expects the representation to have a non-trivial central character. This is not possible in the paramodular theory for essential reasons. He hopes that I and my collaborators will be able to reconcile this. I also had useful conversations with Balakrishnan, Sutherland and Voight about classes of known examples of paramodular surfaces. I found that I had a basic error in my understanding which I was able to correct. I also had the opportunity to meet several colleagues in person for the first time. This makes it much easier for me to write to them with specific questions or ideas. I was able to attend this conference only because of the generous family accommodations. I travel very little due to my husband's disability, and this workshop provided me with the opportunity to interact with collaborators and colleagues very close to my research area. I also learned of useful results and techniques from the lectures."

Andrew Sutherland said:

"This workshop was an extremely productive one from my perspective. The participants included many leading experts in the field, and I was able to make forward progress on two existing projects with collaborators who were also in attendance, as well as obtaining an entirely new result. I can pinpoint the exact moment when the new insight occurred: it was on the trail up Tunnel Mountain while taking a quick hike I took during the lunch break before the afternoon session. The combination of the theoretical beauty of the mathematical content of the talks and the natural beauty of the environment around BIRS was wonderfully exhilirating."

Finally we quote Armand Brumer, one of our most distinguished participants:

"It was a great pleasure having a chance to participate in the workshop. Neither the beauty of the surroundings nor the great amenities could make a dent on the stimulating talks and mathematical conversations!"

Participants

Achter, Jeff (Colorado State University)
Anni, Samuele (Heidelberg University)
Arias-de-Reyna, Sara (University of Sevilla)
Balakrishnan, Jennifer (Boston University)
Bruin, Nils (Simon Fraser University)
Bruin, Peter (Universiteit Leiden)
Brumer, Armand (Fordham University)
Cesnavicius, Kestutis (University of California, Berkeley)
Derickx, Maarten (Universiteit Leiden)
Edixhoven, Bas (Mathematical Institute, Universiteit Leiden)
Elkies, Noam D. (Harvard University)
Fite, Francesc (Essen)
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Ho, Wei (University of Michigan)

Johnson-Leung, Jennifer (University of Idaho)
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Sijsling, Jeroen (Universität Ulm)
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Stoll, Michael (Universität Bayreuth)
Sutherland, Andrew (Massachusetts Institute of Technology)
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Voight, John (Dartmouth College)
Zureick-Brown, David (Emory University)
Zywina, David (Cornell University)

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Chapter 12

Mathematical Analysis of Biological Interaction Networks (17w5099)

June 4 - 9, 2017

Organizer(s): David Anderson (University of Wisconsin-Madison), Matthew Johnston (San Jose State University), Casian Pantea (West Virginia University), Lea Popovic (Concordia University), Anne Shiu (Texas A&M University)

Overview of the Field

Recent breakthroughs in experimental and high-throughput data analysis techniques have provided an unprecedented window into the complexity of the biochemical interactions in living cells. Fundamental metabolic and signaling pathways corresponding to basic cellular functions such as osmolarity, chemotaxis, the cell cycle, and apoptosis are now known to consist of dozens of metabolites interacting via hundreds of interdependent chemical reactions. New experimental techniques such as fluorescent proteins, meanwhile, have allowed real-time observations of such complex dynamical behavior as periodicity, hysteresis, and stochastic fluctuations.

Mathematics has emerged as a pivotal player in grappling with the complexity of these biochemical systems, and is a cornerstone of current systems biology research. Two common mathematical modeling frameworks for networks of biochemical interactions include systems of ordinary differential equations (ODEs) and stochastic continuous-time Markov chains (CTMCs). The classical mass-action form for the interaction rates reduces the ODE formulation to a polynomial dynamical system, which has led to wide-spread interest from researchers in algebraic geometry. When properly posed, these models have the ability to capture important properties of the physical system without the expense and time-consumption of laboratory experimentation and data collection. These models can also allow the generation of hypotheses which can then be tested in the laboratory.

Significant mathematical research has been conducted on relating structural properties of the underlying network of chemical interactions to admissible dynamical behaviors and properties. These approaches have attempted to identify motifs which contribute to biologically important behaviors such as biochemical switching, oscillatory behavior, concentration robustness, and persistence (i.e. non-exhaustion) of metabolites. Classical network-based approaches have included flux balance analysis, stoichiometric network analysis, chemical reaction network theory, and monotone systems theory. Remarkably, many results have been derived which guaranteed qualitative behavior which is robust to the system's parameters and initial conditions.

Research on network-based approaches to biochemical reaction models is now conducted by prominent researchers in a wide breadth of complementary mathematical disciplines, including differential equations and control theory, stochastic processes, algebraic geometry, optimization and computation, and analysis. While work in these areas is often distinct in flavor, it has been the community's experience that the most successful approaches come from sharing ideas across lines of expertise. This workshop focused on establishing communication among the different disciplines and building new collaborations.

Workshop Structure

The workshop contained a mix of prepared talks and free time for collaboration and discussion. Since the workshop participants had areas of expertise spanning many distinct mathematical disciplines, the invited talks were chosen to reflect a diversity of mathematical approaches. Eight speakers were selected to give 40 minute talks on Monday and Tuesday which served as overviews of the notation and common approaches taken in their respective disciplines. On Monday morning, David F. Anderson and Gheorghe Craciun (both UW-Madison) introduced the notation relevant to two common dynamical modeling frameworks for biochemical interaction networks: deterministic ODE models and stochastic CTMC models. On Monday afternoon, Anne Shiu (Texas A&M) and Alicia Dickenstein (University of Buenos Aires) outlined algebraic approaches for the study of chemical interaction networks. Early Tuesday morning, Anne Condon (UBC) and Erik Winfree (Caltech) explained how reaction networks are used as computational tools, both in theory and in the laboratory. Later on Tuesday morning, Yiannis Kaznessis (University of Minnesota) and Eduardo Sontag (Rutgers University) discussed some engineering applications of reaction network theory. The remaining talks were 25 minutes in length and focused predominantly on current research. Speakers were also encouraged to present open problems for discussion. In total, 26 of the 42 participants gave talks, with many of the other participants leading breakout sessions.

The schedule was left deliberately more open in the later part of the week, so that as the week progressed it would allow for the open problems that were raised to have more time to be discussed in self-organized sessions. In total, two hours of breakout session / free time was offered during Monday (late afternoon), two and a half hours during Tuesday (early afternoon), all of Wednesday afternoon, two and a half hours on Thursday (early afternoon), and complete free time on Friday. Prior to the daily breakout sessions, participants were invited to propose open problems / discussion areas on a white board in the TCPL foyer. At the beginning of the breakout session time, the participants self-organized into groups and session rooms were assigned according to the size of the interested groups. Several of these groups continued discussing over multiple days, this included a group discussing extinction events in discrete reaction networks and one discussing the possibility of lifting oscillations from small networks to large ones.

Presentation Highlights

David Anderson (University of Wisconsin-Madison): Stochastic models of CRNs. An overview.

The workshop opened with a survey talk on the field of reaction networks, highlighting one of the core questions in the field: what are the connections between dynamical behavior and network structure in CRNs? This presentation set the stage for many subsequent talks by reviewing CRN terminology (reaction networks, stoichiometric space, linkage classes, weak reversibility, complex-balancing, deficiency, etc.) and CRN mass-action models, both deterministic and stochastic. In particular, a number of ways to represent stochastic CRNs were surveyed, including Gillespie algorithms, Kurtz's random time change representation, and the chemical master equation. An overview of current research directions in stochastic CRNs followed. One of the important open questions is determining when well-behaved deterministic dynamics of a CRN (for example, uniqueness of positive equilibria, stability of equilibria, etc) guarantees that the stochastic dynamics is also well-behaved (for example, positive recurrent). This is known to be true for deficiency zero, weakly reversible networks, which are known to be positive recurrent [21]. The search for other classes of networks with this property is another area of current interest; it is conjectured that weak reversibility is a sufficient condition for positive recurrence. The list of open questions also included finding conditions for stochastic CRNs to undergo an extinction event, and studying when combining well-behaved stochastic CRNs results in a well-behaved CRN. These questions have reoccured in later talks, and have also been the topic of some of the breakout sessions.

Gheorghe Craciun (University of Wisconsin-Madison): On possible future directions for the mathematical analysis of reaction networks.

Prof. Craciun's talk comprised a historical overview of the developments leading to the current state-of-the-art in deterministic CRNs (including the proof of the global attractor Conjecture). Many of the ideas in the field (for example detail and complex balancing) go back to Boltzmann, and attempts at proving the stability of detailed (complex) balanced equilibria go back to work of Shear and Higgins in the 1960's and before the celebrated work of Horn and Jackson. A first possible research direction identified in this talk is applying the machinery of deterministic CRNs to the Boltzmann equation. There is an interesting connection between the convergence to equilibrium in complex-balanced CRNs and Boltzmann's H-theorem (the distribution of molecule velocities in an ideal gas converges to the Maxwell distribution). This connection has been studied with the simplifying assumption of spacial homogeneity and when finite number of velocities are possible [22], and an open question was proposed on extending this approach to an infinite number of velocities. A theory of detail balanced and complex-balanced infinite reaction networks would be a key ingredient in this direction. Another area of exploration is the study of piecewise constant differential inclusions, a far-reaching generalization of complex-balanced systems. These objects have a very rich algebraic and combinatorial structure, are connected to toric dynamical systems, and are a key ingredient in the recent proof of the global attractor conjecture [5]. An interesting open question proposed in the talk is whether piecewise linear differential inclusions or some related general object can give rise to a Lyapunov function the way toric dynamical systems do.

Daniele Cappeletti (University of Wisconsin-Madison): Stochastically Modeled Reaction Networks with Absolute Concentration Robustness.

It is known [20] that certain structural conditions of a deterministic mass-action CRN imply absolute concentration robustness (i.e. some of the species have the same same concentration at all positive steady states). Recent work [21] has however also shown that under the same conditions on the network the stochastic CRN undergoes an extinction event with probability 1. This talk discussed a new result resolving this discrepancy between the deterministic and stochastic models. A key observation is that while in general when rescaled stochastic trajectories converge to a deterministic trajectory, in complex systems with differences of orders of magnitude the hypothesis needed for such general rescaling and convergence is violated, so the result does not apply. The talk presented the idea to divide the dynamics into a slow and a fast subsystem, and hypothesize that the fast subsystem has a product form stationary distribution. Under certain conditions, one can then show that up to any fixed time the averages of the ACR species counts tend to their ACR equilibria. It is an open question how fast this convergence is, and whether the convergence rate depends on the initial state. Finally, a conjecture proposing that all ACR systems have extinction events was shown false by way of counterexample.

Badal Joshi (California State University San Marco): Graphically balanced equilibria and stationary measures of reaction networks.

Graph-related symmetries of a reaction network give rise to certain special equilibria (such as complex balanced equilibria) in deterministic models. Correspondingly, in stochastic models these symmetries give rise to certain special stationary measures. Some new balance measures were discussed in the stochastic setting: the reaction balanced measure, the complex balanced measure, the reaction vector balanced measure and the cycle balanced measure. The reaction vector balancing is the same notion as detailed balancing in Markov chain theory. Relations between these balance measures in the stochastic setting were discussed. For example, reaction balance implies complex balance, reaction vector balance, and cycle balance. The talk discussed the idea of decomposing both deterministic and stochastic systems into so-called "factor systems" and establishing the correspondence between factors of a complex balanced deterministic system and those of the corresponding stochastic system. It is known by work of Dickenstein and Perez Millan that complex balance and cycle balance implies reaction balance. In the stochastic setting complex balance and reaction vector balance implies cycle balance and reaction balance, but this is not true in the deterministic setting.

Alicia Dickenstein (University of Buenos Aires): Algebraic tools in the study of reaction systems.

This overview talk focused on questions of multistationarity in mass-action CRN models. Large classes of systems (especially enzymatic mechanisms) give rise to steady state varieties having particularly nice parametrizations (in work of Gunawardena, Thompson, Feliu, Wiuf, Dickenstein, Perez Millan, etc.). These parametrizations rewrite the steady state algebraic equations using a minimal number of variables. For example, while the n site phosphorylation network involves $3n + 3$ variables, its steady state variety can be parametrized using only three variables. This reduction proves very useful in answering questions of existence and number of positive steady states, and finding regimes of parameters (reaction rate constants, total masses) for which multistationarity is possible. A recent paper of Conradi et al. uses degree theory and the parametrization of the steady state variety to devise explicit conditions on parameters that guarantee the existence of multiple equilibria. On the other hand, purely algebraic techniques (not relying on the Jacobian) are applicable for classes of enzymatic systems. The example of enzymatic cascades was presented: algebraic methods show that if the enzymes are different, only one positive steady state is possible; however, multistationarity is possible if the enzymes coincide.

Anne Shiu (Texas A& M): Algebraic methods for analyzing bistability and oscillations in reaction systems.

This overview talk reviewed the current state of the art in the dynamics of phosphorylation networks (where the phosphorylation mechanism is either processive, distributive, or mixed), as well as other important biological pathways, including the ERK network. The talk focused on the questions of multistationarity and oscillation, although results on stability of equilibria were also briefly mentioned. These networks exhibit an variety of combinations of multistationarity/oscillation behaviors. For example, processive networks are globally stable (Conradi and Shiu 2015); distributive networks are multistationary; mixed networks admit oscillations; the ERK network admits multistationarity. The proofs of these facts rely on parametrizations of the steady state varieties, and use of the Routh-Hurwitz conditions, or of Yang's Hopf-bifurcation criterion. It is an open question if the double phosphorylation mechanism can admit oscillations. One possible approach to attacking this question is lifting the periodic orbits of the corresponding mixed phosphorylation network (which are known to exist), using techniques from the recent work of M. Banaji. This became the topic of one of the breakout sessions.

Anne Condon (University of British Columbia): An introduction to molecular programming with stochastic CRNs.

CRNs can be viewed as a programming language. Prof. Condon introduced this idea by way of examples: the sum of two integers $n_1 + n_2$ can be computed by the network $X_1 \to Y, X_2 \to Y$ as the count of Y when all reactions stopped firing and the initial molecule counts for X_1 and X_2 are n_1 and n_2, respectively. Similarly, the network $X_1+X_2 \to Y$ computes $\min(n_1, n_2)$, and network $X_1 \to T, X_2+Y \to 0$ computes n_1-n_2. Thinking of them as programs, it is desirable that networks are output stable (i.e. output does not change in any reachable state from the output state) and that they stably compute the output (i.e. output is always reachable from all possible inputs). A small discussion on conditions implying these properties followed, along with considerations on the speed of the computations. These have to do the propensity of the reactions in the stochastic model; it can be showed that computing the sum is $O(\log n)$, whereas computing the minimum is $o(n)$. CRNs can also be used to compute logical predicates, like "$n_1 > n_2$": a corresponding network is $L \to Y, X_2 + Y \to N, X_1 + N \to Y$, where the output is Y if $n_1 > n_2$ and N otherwise. It can be shown that union and intersections of predicates can be computed using CRNs. However, many of these predicates are semi-linear and are slow computing (they require linear time). In these cases once can construct faster programs (networks) that converge to the correct output with high probability.

Erik Winfree (Caltech): Computing with CRNs.

Prof Winfree started his talk by discussing the similarities between programming molecules to perform a certain task in synthetic biology and programming electrical systems, but also pointing out that they are also vastly different. Our gap in the component-wise understanding of biological systems (smallest living thing is composed of thousands of interacting components) requires a distinct approach to bioengineering. The dynamics of CRNs are extremely varied, encompassing stable systems, oscillations, and chaos. He then proceeded to describe the implementation of the kinetic control of DNA strand displacement, and how it can be used to create desired CRN. He presented toehold-mediated strand displacement, where an incumbent strand is initially hybridized to a complementary ÔdisplacementÕ domain of a target strand by base pairing, and an invader strand is complementary both to the displacement domain of the target and to an adjacent ÔtoeholdÕ domain, allowing it to displace the incumbent to form a more stable duplex. The displacement is initiated by binding of the invader to the toehold, this is then followed by a branch migration process in which base pairs between target and incumbent break and are replaced by base pairs with the invader, and the overall reaction rate for this displacement is strongly dependent on the toehold binding strength. Distinct species get distinct domains, and reactions in the system are then based on displacement. Experimental displacillator results were presented: the dynamics is slow but works reasonably well (no sustained oscillations, though). Probabilistic inference was discussed, as were chemical Boltzmann machines, and the complexity of robust CRN distributions.

Yiannis Kaznessis (University of Minnesota): Closure scheme for chemical master equations.

Probability distributions (in stochastic CRNs) can be described by moments, which encode the complete information on the distribution. Moments obey differential equations that govern the dynamic and steady state behavior of stochastic reaction networks. However, moment equations are generally not closed (derivatives of moments depend on higher order moments). The zero-information closure scheme for the moment equation starts with the premise that while higher order probability moments may not be numerically negligible, they contain little information to reconstruct the master probability. Higher order moments are then related to lower order ones by maximizing the information entropy $-\sum P(x)\ln(P(x)$ of the network; this is done by solving a system of nonlinear algebraic equations coming from the corresponding Lagrange multipliers. Once the moments are computed, they are used to reconstruct the probability distribution, and the stability around steady states can be assessed by computing the eigenvalues of the moment Jacobian. The Schlogl model was used to illustrate the power of the approach, but also its limitations: for example, bimodality is lost when the system size is increased.

Eduardo Sontag (Rutgers): Dynamic response phenotypes in systems biology: Scale-invariance and monotone I/O systems.

This overview talk focused on output responses to external inputs. Some adaptive systems (ranging from bacterial chemotaxis pathways to signal transduction mechanisms) have an additional feature: scale invariance, i.e. the property that output depends not on the absolute change of the input, but rather on the scaling factor by which the input changes. A theoretical framework characterizing scale invariance was discussed, and typical structures for scale invariant mechanisms were presented (for example, incoherent feed-forward loops). The theory correctly predicts that the E. coli output behavior is unchanged under scaling of its ligand input signal, which has been verified experimentally. Chemosensing mechanisms, as well as cancer immunotherapy mechanisms have also been observed experimentally to exhibit scale invariance, with implications to model selection and validation.

Robert Brijder (Hasselt University): Sufficient Conditions for the Eventual Dying of Reactions in Discrete Chemical Reaction Networks.

In this talk the speaker presented conditions on CRNs for some of the reactions to stop firing in the long term, followed by a corollary that gives a sufficient condition for an extinction event to occur. The condition is based on the structure of the network alone (phrased in terms of Petri nets), and it produces a statement about the reachability of certain states, being therefore independent of the stochastic rates one may assign. A subconservative CRN of stoichiometric matrix Γ is defined by the existence of a a positive vector c such that $c\Gamma \leq 0$. A CRN is enlarged by adding edges from componentwise "larger" to "smaller" complexes, and in such a way that the reactions added do not end at complexes of a certain set $\mathcal{Y}$ (the enlarged network is called dom-CRN Y-admissible). It turns out that if the set $\mathcal{Y}$ of complexes has a certain structural property, if the CRN is subconservative, then any $\mathcal{Y}$-admissible dom-CRN is guaranteed an extinction event. Finding sufficient conditions for extinction in discrete CRNs that enlarge the class presented here is an open problem, as is the study of computational decidability of extinction questions. These issues have been the focus of one of the breakout sessions.

Robert Johnson (Caltech): Formal Verification of Chemical Reaction Network Equivalence: A Bisimulation Approach

This talk defined two CRNs to be equivalent if the have the same simulated behavior. This notion of equivalence is relevant in molecular programming, where the simplest mechanism for performing a task are often desirable. The setup is that of classical CRNs endowed with mass-action kinetics of rate constants set to one. The equivalence definition relies on syntactic Markovian bisimulation, which is also central to devising algorithms that check CRN equivalence. This area of research presents a series of open questions, for example can bisimulation prove some feature that a correct implementation must have, or can it give bounds bounds on the size of a correct implementation?

Nicolette Meshkat (Santa Clara University): Using algebraic matroids and avoiding differential algebra in identifiability, observability, and indistinguishability.

The questions of structural identifiability (i.e. uniquely recovering parameters), structural observability (uniquely recovering trajectories), structural indistinguishability (uniquely recovering models) in CRN input/output systems can be attacked using algebraic tools. The example of a linear 2-compartment model was used to exemplify the techniques, but these are applicable to general systems. If one of the species is used as the output (and thus observed without error for all time, and for any input), can one then uniquely identify all the parameters? It turns out that the answer is "no" even in this simple case. An algorithm for identifying rational combinations of parameters is based on the input/output equations and using differential algebra. This information can be then used to extract individual parameters that are identifiable using Groebner basis and elimination ideals; however, choosing a monomial order that leads to the desired result is in general hard to do. In new work, the speaker used elements of matroid theory to devise methods of choosing the correct order that allows certain parameters to be uniquely identified. Moreover, similar techniques are applicable to questions of indistinguishability.

Atsushi Mochizuki (RIKEN, Japan): Observation and Control of Complex Nonlinear Systems Based on Network Structures.

Understanding the dynamics of CRNs is a difficult task even when measurements on all species are possible. For example, measuring responses to changes in total amounts of enzymes is very subtle, as some changes are counter-intuitive, while some metabolites do not change at all. This talk presents a theory based in part on sensitivity analysis that links qualitative responses of a CRNs to its structure alone. In particular, it is shown that a particular set of nodes in the CRNs the feedback vertex set determines the CRN dynamics. In other words, this implies that steady states, periodic oscillations, quasi-periodic oscillations and other long-term behaviors, can be identified by measurements of a subset of molecules in the network, and this subset is determined by the network structure alone. Moreover, change in the dynamics (for example switching from one stable steady state to another) can be achieved by controlling only the feedback vertex set. A concrete regulatory network of 90 genes was analyzed using these techniques, and the theory predicted five genes that can be used to control the overall dynamics, in agreement with experimental observations.

Ankit Gupta (ETH Zurich): Numerical estimation of the stationary solution of the chemical master equation.

This talk focused on numerical estimation of the stationary solution for the chemical master equation. Solving the (generally infinite) system of ODE corresponding to the chemical master equation is a daunting task, as is, in general, finding the stationary solution of the chemical master equation. If the state space is infinite, the latter is equivalent to solving an infinite dimensional linear system, which can't be done directly. Instead, a truncation of the state space is considered (e.g. by truncating at certain a order of magnitude, by cutting off absorbing states, and by sending reactions to absorbing space to a designated state in the truncated space). It is shown that the stationary distribution corresponding to the truncated system approximates the stationary distribution of the original system, if the infinite state-space is irreducible, and if the reaction dynamics is exponential ergodic (which can be checked by constructing a Foster-Lyapunov function). The algorithm is shown to be effective even for very large state spaces ($\sim 10^9$ states), and on par with SSA performance for concrete protein pathways examples.

Matthew Johnston (San Jose State University): Network Translation and Absolute Concentration Robustness.

A species is said to have absolute concentration robustness (ACR) if its attains the same value at all positive steady states. This suggests evolved structures in biochemical reaction networks which are capable of maintaining narrow ranges for certain reactants. This talk presented new results expanding the class of networks known to have ACR (e.g., from work of Feinberg and Shinar), and conditions that ensure that the ACR values can be determined. The technique uses tools from generalized CRNs and translated CRNs, and show that (1) if translation is deficiency zero and weakly reversible, all complexes on common linkage class are robust; (2) if translation is deficiency one, then all pairs of nonterminal complexes are robust; and (3) the ACR value can be determined if deficiency is zero and under a few additional conditions. These results are illustrated by way of examples, including the futile cycle and the EnvZ-OmpR signaling pathway.

Heinz Koeppl (Technische Universität Darmstadt): Biochemical networks in random environments: modeling and inference.

Many cellular processes on the single-cell level show significant cell-to-cell variability even when the cells are genetically identical and share the same growth conditions. An additional source of variation in intracellular systems may be needed to account for this, as the noise from standard stochastic CRN models is insufficient to explain the magnitude of these differences. This talk focused on adding a source of variability from the random environment in which the cellular process is embedded. A generative joint model was presented in which the reactions had additional static random variables factoring multiplicatively into the kinetic rate. The main question of interest was whether one could represent this enhanced model with a simpler model which behaves as if embedded in a random environment. Various approaches, based on the chemical master equation, the Fokker-Planck equation, and the Liouville equation were discussed. In an example of a first order decay reaction filtering theory was used to make a successful simplification, in which conditional information on the additional static random variable factors was obtained from observations of the species counts over time.

Mercedes Perez Millan (University of Buenos Aires): Checking multistationarity in MESSI systems.

Many enzyme mechanisms fall under the class of MESSI (Modifications of type Enzyme-Substrate or Swap with Intermediates) systems. Here swaps includes transfer reactions, e.g. $X_p + Y \rightleftharpoons X_pY \rightarrow X + Y_p$. This talk presented new results on the study of steady states of MESSI systems; in particular, the focus was on toric MESSI systems, i.e. systems whose steady state variety can be described by binomial equations. A theorem characterizing multistationarity (and based on elimination of intermediates) for these systems is presented. It is shown that there exists a unique positive steady state if and only if there is a certain sign compatibility between minors of two matrices (one of which is the stoichiometric matrix, and the other comes from exponents of binomials in the steady state variety). A number of common examples can be studies using this results: for example, it follows that enzymatic cascades with common phosphotase for both layers is multistationary. The sign conditions can be rephrased in terms of certain subspaces intersecting certain orthants. The talk concluded with a discussion of algorithms checking these sign conditions and their connection with finding circuits in oriented matroids.

Lea Popovic (Concordia University): Rare event calculations for chemical reaction systems.

Theory of large deviations for random processes and rate calculations for frequencies of rare events was applied to stochastic models of CRNs. In order to define "rare" events one first needs to establish a concentration of measure property for the sequence of variables / sequence of processes based on some scaling parameter N (typical examples include the sample average of i.i.d variables and the empirical measure of i.i.d variables or of sequential outcomes of a Markov chain). In the chemical reaction network context the standard scaling would be to consider large amounts of all species and fast rates of all reactions. Rare events can then play a key role in switching between stable states in CRNs whose mean dynamics exhibits bistability. This talk showed how to determine the time scale of switching based on calculations from large deviation theory, and also compared the calculations for the rate of switching based on a scaled Markov chain model for the CRN to the calculations for this rate based on a small diffusion model for the same CRN. A framework for large deviation theory and results for multi-scale CRNs was also presented.

Greg Rempala (The Ohio State University): Law of large numbers for the SIR process on a random graph.

Motivated by the recent Ebola outbreak, and informed by epidemiological data coming from this outbreak, the speaker developed an SIR model evolving on a multilayer random graph with given degree distribution, where edges in different layers correspond to potentially infectious contacts of different types. The evolution of the graph follows the assumption that infectious individuals drop each of their contacts according to an exponential distribution, and that the dropped contacts, which could account for behavioral changes due to disease such as isolation or decreased mobility, cannot be recovered. This setup allows the derivation of a law of large numbers for a large graph that results in a system of ODEs which drive the evolution of various quantities of interest, such as the proportions of infected and susceptible vertices, as the number of nodes tends to infinity. The result is shown to be in concordance with data collected during the recent Ebola outbreak in Africa.

Alan Rendall (Johannes Gutenberg-Universitï¿œt Mainz): Multiple steady states in models for the Calvin cycle of photosynthesis.

A number of models have been proposed in the literature for the Calvin cycle, leading to a number of statements (many of which are based on numerical simulations) on the number and stability of steady states (see Prof. Rendall's recent paper "A Calvin Bestiary"). However, many of these results are not very careful, and sometime overlook physical requirements on the models (e.g. that the steady states must be positive). The focus of the talk is the rigorous analysis of a simple model with five unknowns: it is proved that there are parameters for which there are two positive steady states, one of them stable, and a special case where there is a continuum of steady states. The analysis is made possible by identifying a weak fold bifurcation with 1D center manifold. More detailed mechanisms were also discussed. Using elementary flux modes, the Pettersson model (with fifteen unknowns and expressed as a system of differential-algebraic equations) is shown to have two positive equilibria for certain choices of parameters. The Poolman model is also discussed, and it can give rise to three positive steady states.

Carsten Conradi (HTW Berlin): Multistationarity in biochemical reaction networks.

There is an extensive literature on necessary conditions for multistationarity for mass-action CRNs. Most of those results depend on the structure of the network alone, are based on injectivity of the vector field, and go back to the work of Craciun and Feinberg. Roughly speaking, if the Jacobian of the system doesn't change sign on the positive orthant for all values of rate constants, multistationarity is ruled out. On the other hand, finding sufficient conditions for multistationarity, and parameter regimes that allow it, is still, by and large, an open question. This talk presented new results in this direction (Conradi, Mincheva, Feliu, Wiuf). Using degree theory, one shows that if the vector field is dissipative, does not admit boundary steady states, its Brouwer degree restricted to compatibility classes has a predetermined sign, and if the Jacobian does not change sign, then multistationarity is not possible. On the other hand (and perhaps most importantly), if the determinant changes sign and a certain parametrization of the steady state variety is possible, then the system admits at least two positive steady states.

Stefan Mueller (University of Vienna): Sign conditions in chemical reaction network theory.

Many results on existence and uniqueness of equilibria in mass-action, or generalized mass-action systems are expressed in terms of sign conditions on certain subspaces. This talk is an overview of these results. In the classical theory of complex-balanced mass-action networks, there is a very useful parametrization of the steady states. This extends to generalized mass action (where the monomial rates of reactions are not necessarily linked to reactant complexes). Two notions of deficiency can be defined in this setting, and the kinetic deficiency is intimately linked to complex-balancing: zero kinetic deficiency implies existence of complex balanced equilibria for all rate constants, while kinetic deficiency equal to $\delta > 0$ implies the existence of δ conditions on the rate constants for complex balancing. The existence and uniqueness of steady states for generalized mass-action systems corresponds to surjectivity and injectivity of generalized polynomial maps, which can be translated in terms of sign conditions on vectors of the stoichiometric subspace and its orthogonal complement. The talk concluded with a complete characterization of long-term dynamics of generalized Lotka systems: depending on the kinetic monomial, the system exhibits various behaviors, from center oscillations to global stability.

Maya Mincheva (Northern Illinois University): Interplay between diffusion and delay.

Diffusion and delay play a common role in biological pattern formation, and have mathematical similarities as well: they both destabilize a stable equilibrium of a CRN system of differential equations. The question considered here is whether, when both delay and diffusion are present, the destabilizing can be achieved by diffusion or by the delay alone. It is known since Turing that diffusion alone can destabilize an equilibrium if the diffusion constants are not equal; but does that happen if the delay cannot destabilize the system equilibrium? The analysis presented in this talk shows that the answer is no: If the nonsingular matrix A is strongly stable with respect to delay (i.e. linearly stable for any choice of delays), then $A - D$ is strongly stable with respect to delay for any diagonal diffusion matrix D. (Mincheva, Hinow, 2016). The analysis makes use of a matrix-theoretic characterization of strong delay stability due to Hopfbauer, and involves calculations on the (quasi-linear) Jacobian of the delay system.

Casian Pantea (West Virginia University): Inheritance of multistationarity in CRNs: lifting nondegenerate (stable) steady states). This talk presented a series of modifications on a CRN that preserves its capacity for multistability/multistationarity. Some of these modifications have already appeared in the literature (in work of Craciun, Feinberg, and Joshi, Shiu); however, the talk discusses a new common framework for all these results using the implicit function theorem on local coordinates on stoichiometric classes. The notion of "reduced Jacobian", i.e. the Jacobian of the vector field restricted to stoichiometric classes) is a central object of the analysis. Examples of modifications that preserve multistability are (1) adding dependent reactions (e.g. reverse reactions); (2) adding inflow and outflow; (3) adding new species and inflows and outflows of that species; (4) adding new reversible reactions (with a nondegenerancy condition: full column rank of new species/reaction submatrix is required); (5) adding intermediate complexes involving new species (with a certain rank nondegeneracy condition); (6) adding enzymatic mechanism, i.e. replacing $y \to y'$ by $E + y \leftrightharpoons I \to E + y'$.

Jinsu Kim (University of Wisconsin-Madison): Lyapunov functions and Tiers for ergodicity and mixing times of stochastic reaction networks.

In this talk it was shown how the standard Lyapunov function for ODE models of mass-action CRN can be used to verify ergodicity of the stochastic CRN model. More precisely, it was shown that under certain conditions the Lyapunov-Foster criteria are satisfied. The approach is based on splitting monomials in tiers (following Anderson's proof of the global attractor conjecture for the one linkage class case). A sufficient condition based on the network alone is presented: if (1) the network is double full (i.e. it contains complexes $2A$ for all species A) and (2) every double complex has a path to a single or empty complex, then the Markov chain is positive recurrent. The talk also included a discussion on mixing time (time until distance between probability profile and stationary distribution is epsilon small). It was shown that the conditions above give rise to a "super Lyapunov function" (in which case the expected time to extinction has a uniform upper bound regardless of initial condition, and convergence to the stationary distribution is in fact exponential). It is conjectured that detailed balance also implies exponential convergence to the stationary distribution.

James Brunner (University of Wisconsin-Madison): Robust permanence of deterministic reaction network models.

In this talk the notion of tropically endotactic differential inclusions is presented. This is a far-reaching generalization of toric differential inclusions, and it emphasizes the rich combinatorial structure of vector fields corresponding to, for example, weakly-reversible mass-action networks. Tropically endotactic networks are defined starting with a certain polyhedral fan, whose exponentiation cuts out regions in the positive orthant on each of which the differential inclusion is defined by a constant cone. These regions correspond to areas of dominance of specific source monomials, which drive the dynamics in the direction of small cones around their respective reactions. A theorem is presented for 2D tropically endotactic differential inclusions, which are shown to be permanent – a continuum of trapping regions (much like level sets of a Lyapunov function) is defined, which drive the trajectory towards a compact set in the interior of the positive orthant. As an example, it is shown that if the Lotka-Volterra network is perturbed by pushing reaction vectors inwards, then the resulting system gives rise to a tropically endotactic differential inclusion, and the system is therefore persistent.

Germán Enciso (University of California-Irvine): Gave a tutorial on the stochastic models of reaction networks on Wednesday evening.

Scientific Progress Made

Breakout sessions summaries

Tuesday June 6

Cycle balance I (reported by G. Craciun). This group focused on determining whether cycle balance could replace complex balance in the construction of a Lyapunov function, along the lines of the Horn and Jackson theorem. The group established that, as stated, the question can be answered in the negative by way of counterexample: they constructed a cycle balanced network with three positive equilibria, which therefore can not have a global Lyapunov function.

Constructing Lyapunov functions for stochastic CRNs (reported by J. Kim). The standard Lyapunov function from the ODE CRN model does not readily work in the stochastic case. This group started considering alternative constructions, including piecewise linear Lyapunov functions. For bimolecular networks, the group also considered defining Lyapunov functions as solutions of a certain PDE, using the classical Zubov's method.

Stochastic CRNs with extinction events I (reported by R. Brijder). The group considered the question of whether a given CRN has an extinction event, and what conditions on the network characterize this property. Issues of decidability and complexity of this question were also considered.

Oscillations in ODE phosphorylation models I (reported by A. Shiu). This group considered the existence of oscillations in the double phosphorylation system. One approach discussed here was "lifting" the oscillation from the mixed mechanism phosphorylation (shown to exist in a recent paper by Suwanmajo and Krishnan). Using a recent result of Banaji, a new model of mixed mechanism network was shown to exhibit oscillation.

General mass-action I (reported by M. Johnston). The group discussed results from the paper of Mueller and Regensburger, focussing on how existence of steady states and complex-balanced steady states relate to zero network deficiency and zero kinetic deficiency. Connections with toric dynamical systems were also discussed.

Thursday June 8

Combining networks in stochastic CRN models (reported by D. Anderson). The group considered the question of combining two "well-behaved" (for example, positive recurrent) stochastic CRNs: under what conditions is the resulting network also well-behaved? The group proved a series of preliminary results for small cases.

Stochastic CRNs with extinction events II (reported by R. Johnson). The use of Petri nets was considered in relation to the question: what are the states starting at which all reactions are guaranteed to fire for all times with nonzero probability? Complexity of this and related questions have been discussed, as well as decidability issues: the group identified four quantifiers relevant to these questions.

General mass-action and cyclic balance II (reported by D. Siegel). Relevant results due to Volpert were discussed, in particular the invariance of the positive orthant without monotonicity hypotheses, the connection between balance equalities and convergence to equilibrium, and the role of the Petri net cycles.

Oscillations in ODE phosphorylation models II (reported by A. Shiu). The group performed numerical searches of parameters for Hopf bifurcations in the double phosphorylation system, and also discussed the gaps in a recent paper claiming to have found such parameters. Moreover, candidates for minimal models of oscillations were considered.

Identifiability in stochastic CRNs models (reported by G. Rempala). The group fixed the gap in a long-standing proof about using MLE methods to consistently estimate parameters in stochastic models of CRNs, when incomplete observations of trajectories are observed.

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Chapter 13

Nilpotent Fundamental Groups (17w5112)

June 18 - 23, 2017

Organizer(s): Ján Mináč (Western University), Florian Pop (University of Pennsylvania), Adam Topaz (Oxford), Kirsten Wickelgren (Georgia Institute of Technology)

Overview of the Field

One of the main guiding principles in modern Galois theory is the deep interaction between arithmetic and geometry. This principle is evident in Grothendieck's étale fundamental group of a geometrically connected variety, which is an extension of the absolute Galois group of the base field by the geometric fundamental group. Such groups have been studied via their nilpotent/unipotent completions for many decades, simply because such completions are more amenable to cohomological calculations. This point of view has seen a recent surge of activity, primarily revolving around new results in anabelian geometry, as well as new structural results about absolute Galois groups arising from the norm-residue isomorphism.

Recent Developments

Anabelian Geometry

Nilpotent and solvable quotients of fundamental/Galois groups play a central role in recent results in anabelian geometry. In relationship with the section conjecture, nilpotent quotients of geometric fundamental groups provide a plethora of obstructions for the existence of sections, and therefore also to the existence of rational points. As mentioned above, the main benefit of working with such obstructions is that they can be studied using cohomological methods. More precisely, these obstructions manifest themselves as Massey products in cohomology; such results have been pervasive in the work of Wickelgren. However, it is still unknown whether these obstructions are complete. For instance, there are known counter-examples due to Hoshi to the geometrically pro-p variant of the section conjecture, as well as counter-examples due to Wickelgren to the geometrically 2-step nilpotent section conjecture. But it is still unknown which minimal variants of the section conjecture are plausible.

Nilpotent quotients of Galois groups also play a key role in the birational setting. For instance, Pop proved a refinement of the birational p-adic section conjecture using mod-p meta-abelian quotients of absolute Galois groups. This strengthens Koenigsmann's original proof of the birational section conjecture which used absolute Galois groups. It's important to mention, however, that many of Pop's methods actually work with the smaller "abelian-by-central" (i.e. 2-step nilpotent) quotients, but it is still open whether or not the mod-p abelian-by-central variant of the birational p-adic section conjecture holds true.

Abelian-by-central quotients of Galois groups play a central role in Bogomolov's program in birational anabelian geometry, whose goal is to reconstruct higher-dimensional geometric function fields using the pro-ℓ abelian-by-central quotient of their absolute Galois group. This program was formulated into a precise functorial conjecture by Pop, and this formulation is now commonly known as the Bogomolov-Pop conjecture. While the general statement of the conjecture is still wide open, it has been proven in several important cases by Bogomolov-Tschinkel, Pop and Silberstein. If successful, Bogomolov's program would go far beyond Grothendieck's original anabelian philosophy for two primary reasons. First, it considers purely geometric objects whereas Grothendieck's anabelian philosophy revolved around a strong interaction of arithmetic and geometry. And second, and it considers "almost-abelian" Galois groups, whereas Grothendieck's anabelian philosophy revolved around Galois groups which were highly non-abelian.

Some of the key steps in Bogomolov's Program are by now well-developed, but there are still many open questions. For instance, the "local theory" detects decomposition/inertia groups of so-called "quasi-divisorial valuations" using pro-ℓ abelian-by-central Galois groups; this theory was developed over the last several years by Bogomolov-Tschinkel, Pop, and Topaz. It is still a major open problem to give similar recipes for divisorial valuations. The "global theory" is the second major component of Bogomolov's Program, and Pop has some general results in this direction. In the global theory, it is still a major open problem to construct so-called "rational quotients" using group-theoretical methods. Finally, there is some recent work due to Topaz which suggests the plausibility of a mod-ℓ abelian-by-central variant of Bogomolov's Program, although there currently no known complete case of this mod-ℓ variant.

Lastly, we discuss the question of Ihara from the 1980's about finding a (non-tautological) combinatorial description of absolute Galois groups. Being motivated by Ihara's work on braid groups, this combinatorial description should arise from the Galois action of absolute Galois groups on geometric fundamental groups. This question was formulated into a precise conjecture by Oda-Matsumoto, based on motivic evidence, and this question/conjecture is now commonly referred to as the "I/OM." The original I/OM using the full geometric fundamental groups of algebraic varieties was proven by Pop (unpublished) in the 90's. Nevertheless, using anabelian techniques mentioned above, Pop recently proved a strengthening of this result which instead considers the Galois action on pro-ℓ abelian-by-central quotients of geometric fundamental groups. It is still a major open question whether similar results hold for the geometric fundamental group of certain small categories of varieties, such as the Teichmüller modular tower.

Galois Groups and the Norm-Residue Isomorphism

In another direction, one has the subject of determining explicit properties about the structure of large Galois groups of fields, specifically absolute Galois groups and maximal pro-ℓ Galois groups. Most such results rely on the recent proof of the norm-residue isomorphism theorem due to Voevodsky-Rost et al. This highly-celebrated theorem imposes strict restrictions on the Galois cohomology ring, and determining the precise implications of these restrictions on the structure of absolute Galois groups is an active research area. In the same vein, there is also the open question of making some aspects of the norm-residue isomorphism explicit by tying it to explicit constructions of certain pro-ℓ Galois extensions.

In the last few years, there has been a surge of interest in this area, specifically concentrated around the vanishing of higher Massey products in Galois cohomology. This is particularly highlighted in the works of Hopkins-Wickelgren, Mináč-Tân and Efrat-Matzri, which show that certain higher Massey products vanish over essentially all fields. Still, at this point it remains open whether all higher Massey products vanish when they are defined. In any case, such results have important group-theoretical consequences which are summarized in the so-called Kernel-Unipotent Conjecture. Some partial results towards this conjecture were recently obtained by Efrat-Mináč and Mináč-Tân, but the general conjecture remains open.

A related topic is about formality and 1-formality in Galois cohomology. Massey products are a basic obstruction to the formality and 1-formailty of the differential graded algebra (DGA) underlying Galois cohomology. Positselski has recently provided examples where absolute Galois groups are not formal. This implies that there is more information in the DGA than is available from the Norm Residue theorem. The work discussed above gives information about this DGA, which could potentially lead to a refinement of the Norm Residue theorem itself. There are also interesting questions worth exploring in connection with the cyclotomic character and formality of Galois cohomology in certain cases, and some striking connections, due to Positselski, between Koszulity and Bogomolov's conjecture about the freeness of the commutator subgroup of p-Sylow subgroups of absolute Galois groups.

A second facet in this subject comes from the Elementary Type Conjecture, which aims to determine the structure of finitely-generated maximal pro-ℓ Galois groups of fields. These results are closely related to the "local-theory" in anabelian geometry which was mentioned above. In the last several years, it has become apparent that abelian-by-central quotients of absolute Galois groups completely control much of the local theory, in "tame" situations. On the other hand, the very recent work of Koenigsmann-Strommen suggests that similar results might also hold in "wild" situations. A complete picture of the local theory in both the tame and wild situations could potentially lead to the resolution of the Elementary-Type conjecture.

Unipotent Fundamental Groups and Hyperplane Arrangements

The pro-nilpotent completion of the geometric fundamental group has a natural algebraic analogue in the pro-unipotent fundamental group. The pro-unipotent fundamental group has been an object of great interest ever since the work of Deligne, Sullivan, Morgan and Griffiths, who studied the de Rham fundamental groups of compact Kahler manifolds using Hodge theory. More generally, Deligne's theory of mixed hodge structures and the weight filtration has been extensively used to study unipotent fundamental groups of possibly non-proper varieties. By now, unipotent fundamental groups are fairly ubiquitous, and they are particularly pervasive throughout the work of Hain-Matsumoto, Deligne-Goncharov, as well as many others.

Such ideas have also taken center stage in the theory of hyperplane arrangements, as appears in the work of Cohen, Dimca, Denham, Papadima, Suciu, Wang and others. In this context, the connection between the (unipotent) fundamental group of the complement of a hyperplane arrangement and the combinatorial structure (matroid) of the hyperplane arrangement is of particular interest, and determining the precise connection is a major research area. A very similar question comes up in higher-dimensional (birational) anabelian geometry, where one must understand the precise connection between the geometry of a divisor and the group-theoretical structure of inertia elements in the fundamental group of its complement. In connection with section (2), formality, 1-formality, graded-formality, etc. for fundamental groups of complements of a hyperplane arrangements are again questions which are of great interest in the theory, where such questions can be studied using the weight filtration in de-Rham and/or ℓ-adic cohomology.

The theory of unipotent fundamental groups and hyperplane arrangements can therefore be seen as a bridge between subjects (1) and (2) above. Moreover, there is a clear overlap between the three subjects, and some similarity between the techniques they use. But there are also some significant differences between their methods, as they strive for different goals. Therefore, we believe that increased interaction between these three subjects could lead to new collaborations and new results, by rethinking current research areas from new points of view. In this respect, the primary goal of this workshop was to facilitate such new interaction between the three subjects mentioned above.

Open Problems

During two open problem sessions on Thursday and Friday, workshop participants presented and discussed open problems in the field. Other open problems were discussed during talks and breaks. Here is a list of these open problems.

Thursday

1. (K. Wickelgren, presenting an open problem posed by Micheal J. Hopkins and K. Wickelgren). Let k be a field. The nth Milnor K-theory group $K_n^M(k)$ is defined by tensoring the underlying multiplicative group k^* of k with itself n-times, and quotienting out by the subgroup generated by elements $a_1 \otimes a_2 \otimes \ldots \otimes a_n$ where $a_i + a_j = 1$ for some $i \neq j$, i.e.,

$$K_n^M(k) = (k^*)^{\otimes n} / \langle a_1 \otimes a_2 \otimes \ldots \otimes a_n : a_i + a_j = 1 \rangle. \tag{13.0.1}$$

Milnor K-theory thus combines the multiplication on k with the addition in some interesting manner. View the presentation (13.0.1) as a field arithmetic description.

Let Gal_k denote the absolute Galois group of k. The Milnor/Bloch-Kato Conjecture proved by Voevodsky and Rost provides the following description of the cohomology ring $H^*(\mathrm{Gal}_k)$ of absolute Galois groups. Let N be prime to the chracteristic of k. Applying $H^*(\mathrm{Gal}_k, -)$ to the Kummer exact sequence

$$1 \to \mu_N \to \mathcal{G}_m \overset{z\mapsto z^N}{\to} \mathcal{G}_m \to 1$$

produces the sequence

$$k^* \overset{z\mapsto z^N}{\to} k^* \to H^1(\mathrm{Gal}_k, \mu_N) \to 0,$$

and therefore the Kummer isomorphism $k^*/(k^*)^N \cong H^1(\mathrm{Gal}_k, \mu_N) = H^1(\mathrm{Gal}_k, \mathbb{Z}/N(1))$. The cup product then produces a map $(k^*)^{\otimes n} \to H^n(\mathrm{Gal}_k, \mathbb{Z}/N(n))$. The Steinberg relation implies that this map factors through a map from mod N Milnor K-theory

$$K_n^M(k)/N \cong H^n(\mathrm{Gal}_k, \mathbb{Z}/N(n)),$$

and the Milnor/Bloch-Kato Conjecture is that this map is an isomorphism. For example, we have a ring isomorphism

$$H^*(\mathrm{Gal}_k, \mathbb{Z}/2) \cong \oplus_n (k^*)^{\otimes n}/\langle x \otimes (1-x), 2\rangle,$$

(the Milnor conjecture). The Milnor/Bloch-Kato Conjecture is also called the Norm-Residue isomorphism theorem.

There is a differential graded algebra DGA of cochains $C^*(\mathrm{Gal}_k, \mathbb{Z}/2)$ computing the cohomology with $\mathbb{Z}/2$-coefficients, and the coefficient group can be generalized as well, but with $\mathbb{Z}/2$-coefficients, the complication from the twists is absent. One could alternatively assume that k contains roots of unity, or perhaps use all twists. Let $C^*(\mathrm{Gal}_k)$ denote a DGA arising in this way, but not specifying the coefficients.

Question: Is there a field arithmetic description of the DGA of cochains $C^*(\mathrm{Gal}_k)$ refining the Milnor/Bloch-Kato description of Galois cohomology?

In [13], it was asked if $C^*(\mathrm{Gal}_k)$ is formal, i.e., quasi-isomorphic to $H^*(\mathrm{Gal}_k)$. If it were, then the field arithmetic description given in Milnor/Bloch-Kato would be the desired refinement. Interestingly, it is not, as shown by Positselski [30]. So there is information contained in $C^*(\mathrm{Gal}_k)$ which is lost on passage to cohomology. Is it possible to give a generator-relation description of $C^*(\mathrm{Gal}_k)$ in terms of the arithmetic of k?

In his paper [30, page 226] Positselski mentions that he is not aware of any counterexample to formality of $C^*(\mathrm{Gal}_k)$ if all roots of unity of powers l^n are in the field where $\mathbb{Z}/l$ are considered coefficients in Galois cohomology. Also it is interesting to notice that Example 6.3 can be considered as analogue of an example in [MS]. This was discussed with J. Mináč and A. Topaz during the conference, and there is still a possibility that that Galois cohomology and cyclotomic character can contain information obtained from $C^*(\mathrm{Gal}_k)$. In fact there maybe possibly "enhanced notion of formality" taking into account also some Bockstein maps in Galois cohomology which could possibly explain Positselski's examples and could still lead to a deep formality-like property of $C^*(\mathrm{Gal}_k)$.

2. (F. Bogomolov) Let K be a function field over an algebraically closed field k of characteristic $\neq \ell$. Let G be a Sylow-ℓ subgroup of the absolute Galois group of K. It turns out that the isomorphism class of G (as a pro-ℓ group) only depends on $\mathrm{trdeg}(K|k)$ and k. Let $G^{(2)}$ denote the commutator subgroup of the profinite group G.

   Conjecture: $G^{(2)}$ is a free pro-ℓ group.

   This conjecture provides a strategy to give an alternative proof of the Bloch-Kato conjecture. Indeed, consider the spectral sequence associated to $G \twoheadrightarrow G^{\mathrm{ab}}$:

$$\mathrm{H}^i(G^{\mathrm{ab}}, \mathrm{H}^j(G^{(2)}, \mathbb{Z}_\ell)) \Rightarrow \mathrm{H}^{i+j}(G, \mathbb{Z}_\ell)$$

The conjecture implies that the $E_2^{i,j}$ terms of this spectral sequence all vanish for $j \geq 2$. The main point is that elements of $E_2^{i,0} = \mathrm{H}^i(G^{\mathrm{ab}}, \mathbb{Z}_\ell)$ can all be represented as sums of symbols.

The cohomology of G^{ab} behaves similarly to the cohomology of an abelian variety. It might be the case that G^{ab} can be seen as a fundamental group of some sort of "universal Albanese variety" associated to k and $\mathrm{trdeg}(K|k)$. For a more detailed discussion, refer to [1].

3. (D. Litt) Let X be a smooth geometrically-connected curve over a finite field k of characteristic $\neq \ell$. After extending the base field, we may assume that the Frobenius acts trivially on $\mathrm{H}^1(\bar{X}, \mathbb{F}_\ell)$. Consider the action of Frobenius (Frob) on the $\mathbb{F}_\ell$ group-algebra

$$\mathbb{F}_\ell[[\pi_1(\bar{X})]].$$

Question: What are the sizes of the Jordan blocks of this action? More precisely, define:

$$r(n) = \min\{m \ : \ (\mathrm{Frob} - I)^m \mid \mathbb{F}_\ell[[\pi_1]]/\mathcal{I}^n = 0\}$$

(here $\mathcal{I}$ denotes the augmentation ideal.) Here are some interesting questions one might ask:

(a) Can $r(n)$ be bounded non-trivially?

(b) How does $r(n)$ change on passing to finite étale covers of X?

Satisfactory answers to these questions would lead to an alternate proof of a conjecture of de Jong (which has been proven by Gaitsgory). **Idea:** The above construction should give a notion of weights which should behave well when dealing with $\mathbb{F}_\ell$ coefficients.

4. (M. Florence) Let U be an open subscheme of $\operatorname{Spec}\mathbb{Z}$ and let $X \to U$ be a smooth projective (arithmetic) curve. For each closed point $x \in U$, consider the fibre X_x above x, as well as the number of rational points $X_x(k(x))$.

Question: Is there an algorithmic way to describe $\#(X_x(k(x)))$, as x varies, using the polynomials defining X?
Comment: Since we expect equidistribution of Frobenius elements, the number of points should behave randomly.

5. (D. Harbater) Let X be a smooth affine hyperbolic curve over an algebraically closed field k of characteristic $p > 0$.

Question: Does $\pi_1^{\text{ét}}(X)$ determine X up-to isomorphism?

If X is the affine line over the algebraic closure of a finite field, then the answer is yes (this is a theorem of Tamagawa). It was pointed out by Hoshi during the discussions, that the answer is also yes for a once-punctured elliptic curve.

In general, it is known that $\pi_1^{\text{ét}}(X)$ (in the above context) determines the genus of X, the number of punctures, the cardinality and the characteristic of the base field. (This is also due to Tamagawa.)

Remark (Bogomolov): In the cases that we know, it suffices to use certain small quotients of π_1.

6. (M. Florence) Let G be a profinite group endowed with a character $\chi : G \to \mathbb{Z}_p^\times$. Let $\mathbb{Z}/p^n(1)$ denote the G-module whose underlying abelian group is $\mathbb{Z}/p^n$, and such that G acts on $\mathbb{Z}/p^n(1)$ via χ. Assume that (G, χ) satisfies a formal version of Hilbert 90. In other words, for all open subgroups U of G, the canonical map

$$\mathrm{H}^1(U, \mathbb{Z}/p^n(1)) \to \mathrm{H}^1(U, \mathbb{Z}/p(1))$$

is surjective. Some questions that one could ask about such (G, χ):

(a) Which groups satisfy this property? (Examples include free profinite groups, Demuškin groups.)

(b) Do such groups always arise as Galois groups? I.e., does there exist a field F of characteristic $\neq p$, and an extension E of F which is p-closed, such that $G = \mathrm{Gal}(E|F)$ and χ is the cyclotomic character?

Related topics were discussed in C. Quadrelli's talk. See [7, 16, 21] for related papers.

Friday

1. (A. Topaz) Let g be a positive integer, and consider the (discrete) group:

$$\Sigma_g := \langle a_1, b_1, \ldots, a_g, b_g \ : \ \prod_i [a_i, b_i] = 1 \rangle.$$

Let $\widehat{\Sigma}_g$ denote the pro-ℓ completion of Σ_g. Does there exist a field F of characteristic $\neq \ell$ which contains μ_{ℓ^∞}, such that $\widehat{\Sigma}_g$ is isomorphic to $\mathrm{Gal}(F(\ell)|F)$? Does such a field have any "distinguished" valuations? **Note:** This is a special case of question (6)(b) above.

2. (J. Mináč, continuing with (1)) More generally, do all pro-p Demuškin groups arise as maximal pro-p Galois groups of fields (of characteristic $\neq p$)? See [15, 16, 25, 26].

3. (J. Mináč) For a field F of characteristic $\neq 2$, let $W(F)$ denote the Witt ring of quadratic forms of F. Does there exists a field F such that $W(F)$ is isomorphic to $W(\mathbb{Q}_2)$, and such that 2 is a square in F?

4. (J. Mináč) Let F be a field of characteristic $\neq p$ which contains μ_p, and let $\mathrm{Gal}_F^{(n)}$ denote the p-Zassenhaus filtration on Gal_F. Can one describe the cohomology rings $\mathrm{H}^*(\mathrm{Gal}_F \,/\, \mathrm{Gal}_F^{(n)}, \mathbb{Z}/p)$ in terms of generators and relations? (Note, the answer is "yes" for $n = 2$ and $n = \infty$).

5. (A. Topaz) Let X be a smooth variety over $\mathbb{C}$, and let U be a (Zariski) open subvariety of X. Does there exist a smaller (Zariski) open subvariety V of U such that V is (rationally) 1-formal?

6. (F. Pop, J. Mináč) Does there exist a proof of the Bloch-Kato conjecture which is purely formal in nature? In other words, what group-theoretical properties of the absolute Galois group of a field (endowed with a character mimicking the cyclotomic character) ensure that the Bloch-Kato conjecture holds true?

7. (P. Guillot, J. Mináč, A. Topaz, N. D. Tân) Let F be a field of characteristic $\neq \ell$ such that $\mu_\ell \subset F$, and let $x_1, \ldots, x_n \in \mathrm{H}^1(F, \mathbb{Z}/\ell)$ be given. Assume that the n-fold Massey product $\langle x_1, \ldots, x_n \rangle$ is non-empty. Does $\langle x_1, \ldots, x_n \rangle$ contain 0? This is the so-called n-Massey vanishing conjecture in Galois cohomology, which was formulated by Mináč-Tân [24, 23], and which was strongly influenced by the previous work of Hopkins-Wickelgren [13]. For recent results towards this conjecture and beyond, refer to [11, 12, 19]. One can also consider a weaker version of the above question, where one additionally requires that all the cup-products $x_i \cup x_j$ vanish; this weaker version seems easier, but it is still far from trivial.

Presentation Highlights

The presentations in the workshop consisted of five survey talks (60 minutes) and 14 research talks (45-50 minutes). The survey talks were certainly among the highlights, while the shorter research talks presented the state of the art on the various topics covered in the workshop. All of the talks in the workshop were extremely interesting and added to the success of the workshop. Several speakers provided supplemental materials, all of which has been made available on the workshop web-page. Finally, almost all of the talks were recorded and can be viewed on the workshop web-page.

Survey Talks

The survey talks were some of the major highlights of the workshop. They helped considerably to introduce some of the main topics and results of the workshop, and they were successful in stimulating many further discussions throughout the week. Here is a brief summary of each of these survey talks.

A. Topaz gave a survey of almost-abelian anabelian geometry. As described in this talk, the goal of almost abelian anabelian geometry is to recover arithmetic and geometric information from two-step nilpotent Galois-theoretical data. The talk focused primarily on two topics: Bogomolov's Programme in birational anabelian geometry, and the Ihara/Oda-Matsumoto conjecture. Aspects from both the pro-ℓ and mod-ℓ variants of the theory were discussed. Some relevant references include [3, 4, 28, 29, 27, 33].

T. Szamuely gave a survey of fundamental groups and their variants. This was an introduction to several variants of fundamental groups that arise in algebraic and arithmetic geometry. In addition to the usual topological fundamental group and Grothendieck's étale fundamental group, this survey included a description of Nori's fundamental group scheme, and Deligne's pro-algebraic fundamental group. Wherever possible, this talk highlighted both the Tannakian point of view, and the point of view using torsors. The speaker graciously provided typeset notes from his talk, which are available on the workshop webpage.

J. Stix gave a survey of the section conjecture. This is a major open conjecture, which he explained and then discussed examples and analogues, including a proof for abelian varieties over finite fields, and birational and minimalistic analogues. The speaker has written an excellent book on the subject [31].

K. Wickelgren gave a survey of Massey products in Galois cohomology, discussing vanishing results, notably the major advance to 4-fold Massey products in [12], as well as applications to the structure of Galois groups and automatic realization results. The first problem listed in the Open Problems section was also discussed.

A. Suciu gave a survey of formality notions for spaces and groups, and the slides from his talk are available on the workshop webpage. His talk provides many open problems: namely, he gave results on cdga's over fields of characteristic 0 and explained to the audience that many of these results may hold (and may not hold) in finite characteristic, and suggested the possibility of understanding the analogous structure in the case of finite characteristic. He discussed resonance varieties, characteristic varieties, and the Tangent cone Theorem as an alternative to Massey products for disproving the formality of a space.

Research Talks

The first day of the workshop had two research talks which focused on birational anabelian geometry and the Galois action on fundamental groups. M. Lüdtke spoke about his recent results [18] concerning anabelian geometry for one-dimensional function fields over algebraically closed fields. D. Litt spoke about his recent results [17] concerning the Galois action on ℓ-adic unipotent completions of geometric fundamental groups.

One primary focus of the second day was the section conjecture and its variants. I. Dan-Cohen spoke about his recent preprint [9], joint with T. Schlank, about rational motivic homotopy theory and its connections with Kim's unipotent variant of the section conjecture. A. Betts spoke about his recent preprint [5] which studies local hights on abelian varieties from a motivic/anabelian point of view; a handout from this lecture is available on the workshop web-page. F. Bogomolov spoke about a rational, almost-abelian variant of the section conjecture for higher-dimensional function fields over algebraically closed fields, which appears in his joint work with M. Rovinsky and Y. Tschinkel [2].

There were two additional talks on the second day whose topic was unrelated to the section conjecture. E. Bayer-Fluckiger spoke about cohomological invariants of G-Galois algebras, and recent results establishing the Hasse principle for their self-dual normal bases; the slides from this talk are available on the workshop web-page. D. Neftin spoke about local approximation and specialization, emphasizing recent new results in the context of the Grunwald problem.

The third day of the workshop focused on recent developments in the context of Massey products in Galois cohomology. E. Matzri spoke about his recent work [19] concerning the vanishing of triple Massey products with higher weight in Galois cohomology. P. Guillot spoke about his recent joint work with J. Mináč, A. Topaz, N. D. Tân and O. Wittenberg [11, 12], concerning vanishing of quadruple Massey products in the Galois cohomology of number fields.

The morning session on the fourth day of the workshop saw two talks related to Galois cohomology. A. Schultz spoke about the parametrizing space of bi-cyclic elementary p-extensions and its Galois-module structure, in analogy with Kummer theory. S. Chebolu spoke about his joint work with J. Carlson and J. Mináč [6], concerning the finite-generation problem in Tate cohomology of finite groups.

The afternoon session on the fourth day saw two additional talks with different focuses. Y. Hoshi spoke about his recent joint work with S. Mochizuki and A. Minamide [14], which gives a new simple group-theoretical characterization of the Grothendieck-Teichmüller group. D. Harbater spoke about the Galois theory of arithmetic function fields, especially his work on local-to-global principles using patching techniques, which is part of his long-term ongoing project joint with J. Hartmann, D. Krashen, J.-L. Colliot-Thélène, Parimala and Suresh.

The last day of the workshop saw a single research talk by C. Quadrelli, who spoke about his ongoing joint work with I. Efrat, J. Mináč and T. Weigel, which studies the group-theoretical structure of absolute Galois groups, endowed with their cyclotomic character, from a cohomological point of view.

Informal Discussions

In addition to the survey and research talks, many participants of the workshop also engaged in several informal discussions and explanations of recent research. Here are three which we are aware of. T. Feng explained his recent paper [8] using cohomology operations to study a question of Tate. M. Florence explained some ideas behind his recent joint work with C. De Clercq [10], which aims to give a new simple and purely algebraic proof of the norm-residue isomorphism theorem. A. Topaz explained his recent work [32] concerning reconstruction of higher dimensional function fields from their generic cohomology endowed with their infinite-dimensional mixed Hodge structure.

Scientific Progress Made

J. Mináč and A. Topaz discussed ideas related to the differential graded algebra of étale cochains on $\operatorname{Spec} k$. R. Davis and K. Wickelgren made progress on a project on the 2-nilpotent mod p quotient of the étale fundamental group, which is joint with R. Pries. F. Bogomolov, F. Pop, A. Suciu and A. Topaz had some discussions around the open problem concerning formality (c.f. §3.2 (5)) and its connection with structure of cohomology and anabelian geometry. P. Guillot, J. Mináč, A. Topaz and O. Wittenberg had some discussions concerning Massey products in Galois cohomology, which could lead to further progress in their work. Further discussions between the team above, E. Matzri, and D. Neftin also took place, which could possibly lead to new advances towards the n-Massey vanishing conjecture.

These are just a few of the research discussions that we are aware of, but certainly many more took place. Especially important were several informal discussions between young researchers, including graduate students and postdocs, and more senior researchers, which will lead to new research exchanges. We expect that many of these discussions will eventually lead to new collaborations and new results, which combine ideas and tool from different areas.

Outcome of the Meeting

The primary goal of this workshop was to increase interaction between the three subjects mentioned in the overview: anabelian geometry; the group theory of pro-ℓ and absolute Galois groups via the norm-residue isomorphism; and (unipotent) pro-algebraic fundamental groups. From this point of view, the workshop was extremely successful, and this is backed up by comments we received from several participants. We hope to organize a similar workshop again in the near future.

Participants

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Betts, Alexander (University of Oxford)
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Dan-Cohen, Ishai (Ben Gurion University of the Negev)
Davis, Rachel (University of Wisconsin-Madison)
Feng, Tony (MIT)
Florence, Mathieu (Université Paris 6)
Frankel, Brett (Northwestern University)
Gille, Stefan (University of Alberta)
Guillot, Pierre (Universite de Strasbourg)
Haase, Bastian (Emory University)
Hall, Chris (Western University)
Harbater, David (University of Pennsylvania)
Hoshi, Yuichiro (Kyoto University)
Litt, Daniel (Columbia University)
Lüdtke, Martin (Universität Frankfurt)
Matzri, Eliyahu (Bar-Ilan University)
Merling, Mona (Johns Hopkins University)
Mináč, Ján (Western University)
Neftin, Danny (Technion)
Obus, Andrew (University of Virginia)
Palaisti, Marina (The University of Western Ontario)
Pop, Florian (University of Pennsylvania)
Quadrelli, Claudio (University of Milan-Bicocca)
Schultz, Andrew (Wellesley College)
Silberstein, Aaron (University of Chicago)
Srinivasan, Padmavathi (Georgia Institute of Technology)
Stix, Jakob (Goethe-Universität Frankfurt)
Suciu, Alexandru (Northeastern University)
Szamuely, Tamás (Rényi Institute, Hungarian Academy of Sciences)
Topaz, Adam (University of Oxford)
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Chapter 14

Diophantine Approximation and Algebraic Curves (17w5045)

July 2 - 7, 2017

Organizer(s): Michael Bennett (University of British Columbia), Aaron Levin (Michigan State University), Jeff Thunder (Northern Illinois University)

Introduction

The first topic of the workshop, Diophantine approximation, has at its core the study of rational numbers which closely approximate a given real number. This topic has an ancient history, going back at least to the first rational approximations for π. The adjective Diophantine comes from the third century Hellenistic mathematician Diophantus, who wrote an influential text solving various equations in integers and rational numbers (and whose name is now also attached to such "Diophantine equations"). The subject rose to prominence in the last century beginning with the seminal work of Thue, and it has been the source of a large number of deep and influential results, including far-reaching work of Siegel, Roth, Baker, Schmidt, and Faltings, among others. It remains a highly active area at the forefront of number theory and mathematics.

The second topic of the workshop, algebraic curves, has long been implicit and in the background of Diophantine problems. In this direction, the fundamental role of algebraic curves was cemented in the influential 1922 conjecture of Mordell that an algebraic curve of genus at least two possesses only finitely many rational points. The interplay between geometry and arithmetic has increased rapidly since that time, and the use of increasingly advanced tools from algebraic and arithmetic geometry has led to the solution of many outstanding and previously inaccessible problems, including the resolution of Mordell's conjecture (by Faltings).

The workshop centered on the interplay between Diophantine approximation and algebraic curves, with interconnections to a diverse array of topics in algebra, geometry, analysis, and logic, among others.

Overview of the Field

A foundational topic in Diophantine approximation is the study of rational numbers which closely approximate a given real number. Naïvely, since the rational numbers are dense in the real numbers, one may approximate a real number arbitrarily well by a rational number. Diophantine approximation studies the closeness of this approximation in terms of the "size" of the rational number (e.g., the denominator). Specifically, if α is a real number, one can study the rational solutions $\frac{p}{q}$ to an inequality of the form

$$\left|\alpha - \frac{p}{q}\right| < \frac{c(\alpha)}{q^\delta}, \tag{14.0.1}$$

where $c(\alpha)$ and δ are positive real numbers.

If $\delta = 2$, then an elementary result of Dirichlet shows that for any irrational real number α, the set of rational solutions $\frac{p}{q}$ to (14.0.1) is infinite when $c(\alpha) = 1$ (or even $c(\alpha) = \frac{1}{\sqrt{5}}$, due to Hurwitz). The theory of continued fractions provides a method for explicitly calculating such good approximations.

In a complementary direction, Liouville (1844) showed that if α is an algebraic number of degree $d \geq 2$, then there exists an explicit positive constant $c(\alpha)$ such that (14.0.1) has no solutions for $\delta \geq d$. Liouville's result is already sufficient to show that numbers such as $\sum_{i=1}^{\infty} 10^{-i!}$ are transcendental, giving a first link between Diophantine approximation and transcendence problems. On the other hand, Liouville's inequality is typically too weak to obtain interesting consequences for Diophantine equations.

A breakthrough occurred in 1909 when Thue [53] improved the exponent in Liouville's result to $\delta > \frac{1}{2}d+1$. As an application, Thue proved the finiteness of integer solutions $x, y \in \mathbb{Z}$ to equations (now called Thue equations) of the form $F(x, y) = a$, where $a \in \mathbb{Z}$ and $F \in \mathbb{Z}[x, y]$ is an irreducible binary form of degree $d \geq 3$.

Siegel [50], in 1921, expanded on Thue's method to prove finiteness for exponents $\delta > 2\sqrt{d}$. The improvement to an exponent δ of order $o(d)$ proved essential when, in 1929, Siegel [52] proved his famous theorem on integral points on curves: if $C \subset \mathbb{A}^n$ is an affine curve defined over a number field k with positive geometric genus or possessing at least 3 (geometric) points at infinity, then the set of points of C with coordinates in the ring of integers of k, $C(\mathcal{O}_k)$, is finite. More generally, in this case finiteness holds for any set of S-integral points $C(\mathcal{O}_{k,S})$, where $\mathcal{O}_{k,S}$ (S finite) is a ring of S-integers of k. Siegel's theorem on integral points provides a basic link between the two topics of the workshop, and more generally, represents an early instance of the deep and fundamental interplay between geometry and arithmetic. In the case of rational affine curves, Siegel's theorem is equivalent to the finiteness of solutions to the so-called S-unit equation

$$au + bv = c, \quad u, v \in \mathcal{O}_{k,S}^*, \tag{14.0.2}$$

where $a, b, c \in k^*$ are nonzero constants. This equation had been studied earlier by Siegel (when $\mathcal{O}_{k,S}^* = \mathcal{O}_k^*$) and in the above generality by Mahler. In fact, one can study the S-unit equation (and Siegel's theorem) in far more general contexts (e.g., when u, v lie in a finitely generated subgroup of $\mathbb{C}^*$).

Siegel's result was improved to $\delta > \sqrt{2d}$ by Gelfand and Dyson (1947) (independently), and finally, the optimal result $\delta > 2$ was obtained in 1955 by Roth [43]: If α is an algebraic number, $\varepsilon > 0$, and $C > 0$, then there are only finitely many rational numbers $\frac{p}{q} \in \mathbb{Q}$ satisfying

$$\left|\alpha - \frac{p}{q}\right| < \frac{C}{q^{2+\varepsilon}}. \tag{14.0.3}$$

Beginning with work of Mahler, Roth's theorem was subsequently generalized by Ridout and Lang [33, 42] to an arbitrary fixed number field k (in place of $\mathbb{Q}$) and to allow for finite sets of absolute values (including non-archimedean ones).

In contrast to Liouville's result, Roth's theorem is ineffective. That is, given α an algebraic number, the method of proof does not yield a way, in general, to compute the finitely many solutions to (14.0.3). In fact, this defect is already present in all of the results on the inequality (14.0.1) beginning with Thue (consequently, Siegel's theorem on integral points is also ineffective). This problem of effectivity remains a major open problem. On the other hand, it is possible to derive an upper bound on the number of solutions to Roth's inequality (14.0.3) and related problems. For instance, beginning with work of Evertse [25], many authors have studied the problem of producing bounds for the number of solutions to the S-unit equation (14.0.2) that, remarkably, depend only on the cardinality $|S|$. For example, Beukers and Schlickewei [7] have proven the bound $2^{16|S|}$.

In the 1960's, in a series of papers, Alan Baker [5] revolutionized the subject by producing effective lower bounds for linear combinations of logarithms of algebraic numbers. Baker's results found numerous applications, including effective bounds for solutions to the unit equation (14.0.2) (from which many other Diophantine equations may be effectively solved). Analogous lower bounds for linear forms in p-adic logarithms were obtained by van der Poorten [40], Yu [59], and others.

In a different direction, Schmidt [48] proved a deep higher-dimensional generalization of Roth's theorem to the setting of hyperplanes in projective space. Schmidt's Subspace Theorem, as it is known, has found numerous applications to Diophantine problems (more precisely, the version typically applied includes subsequent improvements due to Schlickewei [45], analogous to Ridout-Lang's generalization of Roth's theorem). Among the many applications is the fundamental result of Evertse [24] and van der Poorten and Schlickewei [41] on the n-term unit equation: all but finitely many solutions of the unit equation

$$u_1 + u_2 + \ldots + u_n = 1, \quad u_1, \ldots, u_n \in \mathcal{O}_{k,S}^*, \tag{14.0.4}$$

satisfy an equation of the form $\sum_{i \in I} u_i = 0$, where I is a nonempty subset of $\{1, \ldots, n\}$.

In 1983, another leap forward occurred when Faltings [27] proved Mordell's conjecture dating from 1922: for any number field k and any smooth projective curve C of genus at least two, the set of rational points $C(k)$ is finite. Thus, the qualitative behavior of rational points on algebraic curves is completely determined by the crudest geometric invariant of a curve, the genus. Although Faltings' original proof avoided Diophantine approximation, Vojta [55] subsequently found a new and influential proof of the Mordell conjecture based in Diophantine approximation. Building on this work, Faltings [28] proved a conjecture of Lang classifying the subvarieties of an abelian variety which contain a Zariski dense set of rational points, and another conjecture of Lang generalizing Siegel's theorem to affine open subsets of an abelian variety. Even more generally, Vojta [56, 57] extended Faltings' results to subvarieties of semi-abelian varieties.

Generalizing Mordell's conjecture, Bombieri and Lang conjectured that if X is a variety of general type, over a number field k, then the set of rational points $X(k)$ is not Zariski dense in X. In fact, the conjecture can be extended to include integral points and varieties of log general type. In this form, the Bombieri-Lang conjecture unifies all of the aforementioned qualitative results on integral and rational points. In particular, it contains as special cases the results of Siegel and Faltings for curves, and the results of Faltings and Vojta for integral and rational points on subvarieties of semi-abelian varieties.

Returning to Diophantine approximation, and ending the overview, we mention Vojta's conjectures [54], which posit a precise inequality for algebraic points on a variety, at once quantifying the Bombieri-Lang conjecture and generalizing the Diophantine approximation inequalities of Roth and Schmidt. Vojta's conjectures came as the result of the surprising discovery of analogies between Diophantine approximation and Nevanlinna theory, the quantitative theory that grew out of Picard's classical theorem in complex analysis on entire functions omitting two values. This has led to the development of strong ties and analogies between arithmetic and complex geometry, in addition to the more classical analogies between number fields and function fields.

Recent Developments and Presentation Highlights

In this section we discuss select recent developments, with a view towards the presentations given at the workshop, and give a description of some of the presentation highlights.

Schmidt's Subspace Theorem and Its Generalizations

Schmidt's Subspace Theorem remains one of the most powerful Diophantine approximation results available, and constitutes the central tool in a diverse and increasingly large number of applications (e.g., see Bilu's survey article [8]). Generalizing the Subspace Theorem to other contexts began with work of Faltings and Wusthölz [29], and more recently, Corvaja and Zannier [22] and Evertse and Ferretti [26] have proven generalizations to higher-degree hypersurfaces in projective space and to more general projective varieties. Closely related techniques have been used to study integral points on varieties, beginning with a new proof of Siegel's theorem by Corvaja and Zannier [21], and continuing in their work [23] and work of Levin [34], Corvaja, Levin, and Zannier [19], and Autissier [2].

Continuing this line of research, **Paul Vojta** (University of California, Berkeley) spoke on joint work with Min Ru on birational Nevanlinna constants. In earlier work of Ru, a new invariant, the Nevanlinna constant, was introduced to clarify and unify the Diophantine approximation method (and analogues in Nevanlinna theory) developed in the previous work. Vojta spoke on variants of this constant, and corresponding generalizations of the

Subspace Theorem, which go further by incorporating the filtrations introduced in Autissier's work, in addition to other novel ideas.

Unit equations and other related equations

The unit equation (14.0.2) (and more generally (14.0.4)) is ubiquitous in number theory, and many other famous Diophantine equations (e.g., Thue-Mahler equations, Mordell equations, hyperelliptic equations) can be reduced to it. Such an observation goes back to at least the 1926 work of Siegel [51] (written under the pseudonym X), who used the unit equation to study integral points on affine hyperelliptic curves. The groundbreaking work of Baker led to effective height bounds for solutions to the unit equation (and a host of other Diophantine equations) and the possibility of practical general algorithms. We can (loosely) organize a number of the presentations around these core themes.

Height bounds and algorithmic aspects

The classical approach to producing effective height bounds for solutions to the unit equation (and related Diophantine equations) comes from the theory of linear forms in logarithms (including p-adic and elliptic versions). Since the bounds produced by this method are typically too large to admit a naïve exhaustive search, finding practical algorithms for the resolution of such Diophantine equations has been intensively studied, with key developments going back to Baker and Davenport [4] and the introduction of the LLL-algorithm by de Weger [58].

Building on observations of Frey [30] and the proof of the Taniyama-Shimura conjecture [14], recently "modular" approaches to the unit equation, and related equations, have been developed (over the rational numbers). Work in this direction includes results of Murty and Pasten [37], Bennett and Billerey [5], and **Benjamin Matschke** (University of Bordeaux), in joint work with Rafael von Känel, who spoke on solving S-unit, Mordell, Thue, Thue-Mahler, and generalized Ramanujan-Nagell equations via the Taniyama-Shimura conjecture. There are close relations between all of these classical Diophantine equations, and Matschke focused on the Mordell equation $y^2 = x^3 + a$, where $a \in \mathbb{Z}$ is fixed and x and y lie in a ring of S-integers $\mathbb{Z}_S = \mathbb{Z}[1/N_S]$, where S is a given set of rational primes and $N = \prod_{p \in S} p$. Several improvements to the theory and practical resolution of the Mordell equation were given, including improved height bounds for the solutions (using an approach based on the Taniyama-Shimura conjecture, as opposed to linear forms in logarithms) and improvements to the known algorithms for solving the Mordell equation, including novel sieving techniques. Applications to elliptic curve databases were discussed, including the explicit computation of all elliptic curves (up to isomorphism) with good reduction outside a set of of sufficiently small primes.

Beth Malmskog (Villanova University) discussed implementing a solver for S-unit equations in Sage and applications to algebraic curves (joint work with Alejandra Alvarado, Angelos Koutsianas, Christopher Rasmussen, Christelle Vincent, and Mckenzie West). In contrast to Matschke's talk, a key focus of the algorithm was handling S-unit equations over arbitrary number fields. As an application, all 63 $\mathbb{Q}$-isomorphism classes of Picard curves with good reduction away from 3 were computed (by a result of Börner-Bouw-Wewers, all Picard curves over $\mathbb{Q}$ have bad reduction at 3).

Adela Gherga (University of British Columbia) spoke on implementing algorithms to compute elliptic curves over $\mathbb{Q}$. Specifically, she discussed difficulties with implementing the algorithms of de Weger and Tzanakis, particularly with respect to performing p-adic computations. This is an area where many current computer algebra packages are rather lacking.

Closely related inequalities and equations

Jan-Hendrik Evertse (Universiteit Leiden) discussed joint work with Yann Bugeaud and Kálmán Györy on S-parts of values of binary forms and decomposable forms. If $S = \{p_1, \ldots, p_n\}$ is a finite set of primes, $a \in \mathbb{Z}$, $a \neq 0$, and we have a factorization $a = p_1^{i_1} \cdots p_n^{i_n} a'$ with $p_i \nmid a'$, $1 \leq i \leq n$, then the S-part of a is defined to be $[a]_S = p_1^{i_1} \cdots p_n^{i_n}$. For a decomposable form $F \in \mathbb{Z}[X_1, \ldots, X_m]$, let

$$\alpha(F) = \inf\{\alpha \mid \forall S, \exists c > 0 \text{ such that } [F(\mathbf{x})]_S \leq c|F(\mathbf{x})|^\alpha \text{ for all } \mathbf{x} \in \mathbb{Z}^m, \gcd(x_1, \ldots, x_m) = 1, F(\mathbf{x}) \neq 0\}.$$

If F is a binary form ($m = 2$) of degree $n \geq 3$ and nonzero discriminant, then it is proven that $\alpha(F) = \frac{2}{n}$. More generally, they prove results for decomposable forms under certain conditions, and give quantitative results counting the number of solutions to related inequalities.

Yann Bugeaud (University of Strasbourg) discussed the binary representation of smooth numbers. He proved a series of results all with the general theme that a large integer cannot simultaneously have only very small prime divisors and very few nonzero binary digits. The given new results are effective, and rely on estimates for linear forms in complex and p-adic logarithms of algebraic numbers.

Shabnam Akhtari (University of Oregon) discussed joint work with Jeffrey Vaaler on finding height inequalities for units in the ring of S-integers of a number field. More specifically, if $h(\cdot)$ denotes the (absolute) Weil height, then Akhtari discussed the problem of finding estimates for the quantity

$$\min\{h(\beta\gamma) \mid \gamma \in \Gamma\},$$

where Γ is a given subset of a group of S-units $\mathcal{O}_S^*$ of k, and $\beta \neq 0$ is a nonzero S-integer. Aside from intrinsic interest, such height inequalities have applications to Diophantine equations, and in particular, the relevance to solving norm form equations was discussed.

Unit equations and arithmetic dynamics

The application of unit equations to arithmetic dynamics, and in particular as a tool to study periodic and preperiodic points of a rational function, goes back to at least Narkiewicz [38], who used unit equations to study rational periodic points of monic polynomials. In the more general setting of rational functions, recent works of Canci [15], Canci and Paladino [16], and Canci and Vishkautsan [3] have used unit equations as an essential tool to study rational preperiodic points and give partial results towards the Morton-Silverman conjecture in arithmetic dynamics.

Extending this line of results, **Sebastian Troncoso** (Michigan State University) spoke on rational preperiodic points and hypersurfaces in projective space. In the one-dimensional case, if $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is an endomorphism of degree $d \geq 2$ over a number field k, and ϕ has good reduction outside of a set of places S of k (including archimedean places), then Troncoso shows that the number $|\mathrm{PrePer}(\phi, k)|$ of k-rational preperiodic points satisfies

$$|\mathrm{PrePer}(\phi, k)| \leq 5 \cdot 2^{16|S|d^3} + 3.$$

A sharper bound is obtained for strictly preperiodic points. In higher dimensions, quantitative results are obtained for the number of rational preperiodic hypersurfaces satisfying certain conditions. Key ingredients in the proofs are bounds for the number of solutions to S-unit equations and Thue-Mahler equations.

Runge's method

An old method of Runge [44], dating from 1887, proves the effective finiteness of the set of integral points on certain affine curves. In its most basic form, Runge proved that if $f \in \mathbb{Q}[x, y]$ is an absolutely irreducible polynomial and the leading term of f factors nontrivially over $\mathbb{Q}$ into nonconstant coprime polynomials, then the set of solutions to

$$f(x, y) = 0, \quad x, y \in \mathbb{Z},$$

is finite and can be effectively computed. More generally, Runge's method can be used to prove effective finiteness of S-integral points on an affine curve C when $|S|$ is small compared to the number of rational components of C at infinity. Bombieri [12] proved a uniform version of Runge's theorem, allowing the number field K and set of places S to vary.

More recent developments include a generalization of Runge's method to higher dimensions by Levin [35], a series of papers by Bilu and Parent [9, 10, 11] applying Runge's method to modular curves and solving certain cases of a well-known conjecture of Serre, an application in arithmetic dynamics to integral points in orbits by Corvaja, Sookdeo, Tucker, and Zannier [20], and an application to the Nagell-Ljunggren equation by Bennett and Levin [6].

Further developing these themes, **Samuel Le Fourn** (ENS Lyon) spoke on Runge's method and its application to integral points on Siegel modular varieties. Expanding on Levin's higher-dimensional version of Runge's method, Le Fourn proves a "tubular" version of Runge's method which permits wider applicability to higher-dimensional problems on integral points. As an application, Le Fourn proves an explicit finiteness result for integral points on the Siegel modular variety $A_2(2)$.

Unlikely intersections

If X and Y are subvarieties in a space of dimension $n > \dim X + \dim Y$, then one usually expects that the intersection $X \cap Y$ will be empty (the intersection is "unlikely"). In fact, if X is fixed and Y runs through an infinite family of suitable subvarieties with $\dim X + \dim Y < n$, then one still expects that X has small intersection with the entire family unless X is of a certain special type related to the family. One may view conjectures of Lang, Manin-Mumford, Zilber, André-Oort, Bombieri-Masser-Zannier, and others in this general setting. The topic has been developing rapidly, with notable contributions including work of Bombieri, Masser, and Zannier [13], Maurin [36], Habegger [31, 32], and Pila [39].

Within this framework, **Laura Capuano** (Oxford University) spoke on joint work with Barroero on unlikely intersections in families of abelian varieties. In particular, the case where X is a curve in a family of abelian varieties was studied, and applications were given to the study of families of polynomial Pell equations.

Perfectoid spaces and p-adic hyperbolicity

Vojta's dictionary [54] gives a precise way to "translate" between statements in Diophantine approximation and statements in (complex) Nevanlinna theory. This analogy between the two subjects has proved very fruitful and influential, and has led to substantial progress in both subjects. Since at least the 1980's, a theory of p-adic (or non-Archimedean) Nevanlinna theory has been developed, and in particular, various notions of p-adic hyperbolicity were introduced by Cherry [18]. Similar to the classical case, a dictionary between p-adic Nevanlinna theory and Diophantine approximation over the rational numbers was studied by An, Levin, and Wang in [1].

The recent theory of perfectoid spaces, due to Scholze [49], has been quickly recognized as a major advance, and has already found many deep applications in diverse areas. In this context, **Ariyan Javanpeykar** (University of Mainz) discussed the notion of p-adic hyperbolicity and its relation to other common notions of hyperbolicity. Using Scholze's theory of perfectoid spaces, he presented a proof of a strong p-adic hyperbolicity statement for moduli spaces of polarized abelian varieties.

Transcendence Theory

Michael Coons (University of Newcastle) spoke on joint work with Yohei Tachiya on problems in the transcendence theory of functions of one complex variable. The talk centered on proving generalizations and extensions of a result of Duffin and Schaeffer. A primary result yielded the transcendence, over the field of meromorphic functions, of complex functions with a finite (positive) bounded radius of convergence satisfying certain radial unboundedness conditions. As an application, one obtains that the field $\mathbb{C}(z)(e^z, \sum_{n\geq 0} z^{2^n})$ has transcendence degree two over $\mathbb{C}(z)$, answering a function-theoretic analogue of a question of Nishioka on the algebraic independence of e and the number $\sum_{n\geq 0} 2^{-2^n}$.

Noriko Hirata-Kohno (Nihon University) discussed the independence of values of the polylogarithm function and a generalized polylogarithm function with periodic coefficients. The method of proof relies on the construction of new Padé approximations built on earlier (unpublished) work of Nishimoto.

Further Presentation Highlights

Richard Guy (University of Calgary) spoke on some relations between number theory and the Euclidean geometry of triangles. He discussed some classical geometric configurations of lines and circles associated to triangles, including configurations arising from the Simson line theorem, and raised some open problems about the structure of a related graph.

Lajos Hajdu (University of Debrecen) discussed the problem of finding well approximating lattices for a finite set of points (joint work with András Hajdu and Robert Tijdeman). After giving some motivation for the problem coming from medical image processing, Hajdu defined an appropriate measure for the approximation of a lattice to a finite set of points. Under this notion, results on finding well-approximating lattices were given in all dimensions, with more complete results in the one-dimensional case. The techniques used include the LLL algorithm and techniques from simultaneous Diophantine approximation.

Fabien Pazuki (University of Copenhagen) spoke on inequalities for the j-invariants of isogenous elliptic curves. Specifically, Pazuki showed that if h denotes the absolute logarithmic Weil height, E_1 and E_2 are elliptic curves over $\overline{\mathbb{Q}}$ with j-invariants j_1 and j_2, respectively, and $\phi : E_1 \to E_2$ is an isogeny, then the following two inequalities hold:

$$\begin{aligned} |h(j_1) - h(j_2)| &\leq 9.204 + 12 \log \deg \phi, \\ h(j_1) - h(j_2) &\leq 10.68 + 6 \log \deg \phi + 6 \log(1 + h(j_1)). \end{aligned}$$

Some applications of the inequalities were discussed, including the problem of bounding the coefficients of modular polynomials.

Cameron Stewart (University of Waterloo) spoke on joint work with Stanley Xiao on the representation of integers by binary forms. Let F be a binary form with integer coefficients, non-zero discriminant, and degree $d \geq 2$. Let $R_F(Z)$ denote the number of integers of absolute value at most Z which are represented by F. Computing $R_F(Z)$ in the case of binary quadratic forms ($d = 2$) is classical, and goes back to work of Fermat, Lagrange, Legendre, and Gauss. The cubic case ($d = 3$) was studied by Hooley, and many authors have studied various higher degree classes of binary forms. For $d \geq 3$, Stewart and Xiao are able to give a precise asymptotic result, showing that there is a positive number C_F such that $R_F(Z)$ is asymptotic to $C_F Z^{2/d}$. The techniques used include Heath-Brown's p-adic determinant method, as refined by Salberger.

Amos Turchet (University of Washington) discussed joint work with Kenneth Ascher on the uniformity of integral points on curves and surfaces. A well-known result of Caporaso-Mazur-Harris shows that, assuming the Bombieri-Lang-Vojta conjecture, there should exist a uniform bound $B(k, g)$ for the number of k-rational points on a smooth projective curve C, where $B(k, g)$ depends only on the number field k and the genus g of C. Higher-dimensional uniformity results on rational points in the same vein were proven by Hassett, Abramovich, and Abramovich-Voloch. Turchet discussed partial results towards a program for proving an integral points analogue of Caporaso-Mazur-Harris's result, extending results of Abramovich for integral points on elliptic curves. Geometrically, this involves extending the previous uniformity results from varieties of general type to varieties of log general type.

Participants

Akhtari, Shabnam (University of Oregon)
Bauer, Mark (University of Calgary)
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Bremner, Andrew (Arizona State University)
Bugeaud, Yann (Université de Strasbourg)
Capuano, Laura (Oxford University)
Coons, Michael (University of Newcastle)
Evertse, Jan-Hendrik (Universiteit Leiden)
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Guy, Richard (The University of Calgary)
Hajdu, Lajos (university of Debrecen)
HIRATA-Kohno, Noriko (Nihon University)
Ingram, Patrick (York University)
Javanpeykar, Ariyan (University of Mainz)
Kundu, Debanjana (University of Toronto)
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Morrow, Jackson (Emory University)
Mosunov, Anton (University of Waterloo)
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Sharma, Divyum (University of Waterloo)
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Troncoso, Sebastian (Michigan State University)
Tucker, Tom (University of Rochester)
Turchet, Amos (University of Washington)
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Chapter 15

Women in Control: New Trends in Infinite Dimensions (17w5123)

July 16 - 21, 2017

Organizer(s): Luz de Teresa (Universidad Nacional Autónoma de México), Irena Lasiecka (University of Memphis), Kirsten Morris (University of Waterloo)

Overview

Control theory as a discipline has received considerable attention over the years. There are important applications to such diverse fields as engineering and medical sciences and challenging mathematical questions have made it relevant from both the abstract and applications point of view. The first developments in mathematical theory were for ordinary differential equation models. The need for more accurate descriptions and the constitutive relations in physical models such as structures and fluids give rise to partial differential equations (PDE's). The analysis of these models is strongly tied to several areas of mathematics, both pure and applied. Besides PDE's, geometric analysis, functional analysis, harmonic analysis, differential geometry, combinatorics and graph theory, optimization, numerical analysis and computational mathematics are all relevant. The last 20 years or so have witnessed great interaction and synergism where long-standing questions have found unexpected answers but these results have opened up new avenues of research. Because the state of systems with PDE models is a time-varying function that evolves on an infinite-dimensional state space, such systems are often referred to as infinite-dimensional systems.

With the impetus of technological problems demonstrating a need for models based on a continuum rather than a discrete description of phenomena, control theory has evolved into infinite-dimensions. This means the use of models described as the objects in infinite-dimensional Hilbert/Banach spaces with concrete representations stemming from PDE's. It was the goal of this workshop to review the major trends in this area and to introduce a wealth of problems and techniques which could provide a springboard to further research. This conference was a perfect opportunity to put together people working in the field of infinite-dimensional systems from diverse backgrounds and with different expertise.

Control theory in the last 20 years or so has undergone tremendous evolution. From vibrant developments in linear finite-dimensional control, with almost complete understanding of fundamental concepts such as stabilization, controllability, optimal control in this context. A wealth of applications pushed the borders into PDE's modeling by asking similar questions for infinite-dimensional systems. The unifying language then became that of semigroups which provide a common description of dynamics in both finite and infinite-dimensions. Not surprisingly, the answers are not the same, and in depth study of analysis, geometry, topology has paved the way to new control theoretical results within the realm of infinite-dimensions. These results were often formulated first in an abstract form in the language of functional analysis. However new conditions which transpired had to be verified by concrete representations described by PDE's or other infinite-dimensional functional equations. Thus the theory progresses in parallel-at an abstract level and also on a more specific PDE level, where verification of the sought-after properties became an art in itself. In infinite-dimensional systems research the focus is on development of an abstract and axiomatized framework. The talks presented at the workshop represented both trends: PDE's and infinite-dimensional system theory.The interaction between these two approaches and the resulting benefits as well as for collaborations were heavily emphasized. Applications to biological, fluid dynamics and mechanical models give a very tangible payoff in the applied world with solutions obtained via numerical methods; extension of algorithms to this class is non-trivial. Plenary talks were meant to represent diverse areas of infinite-dimensional control theory where new developments have emerged and are projected to flourish into the future. It was hoped that these talks would be accessible to non-specialists and providing a basis for the material in a number of shorter contributed talks.

Control theory, like other STEM disciplines, has a marked gender gap. Gender studies confirm that having scientists and engineers with diverse backgrounds, interests, and cultures assures better scientific and technological results. Improving female representation in the control field is not just an equity issue, but also an opportunity to improve the quality of work in the branch by having an impact on highly dedicated female researchers with varied expertise ranging from very applied work being done in engineering, to very theoretical mathematical work. All branches of control are inter-disciplinary. Inter-disciplinary work can add to the difficulties faced by minority researchers since it is easy to criticize contributions and grant proposals as not being "mainstream". Another difficulty, faced by female researchers in all STEM fields, is in finding collaborations since men tend to collaborate with other men.

The meeting had several inter-connected objectives: to provide support for young women interested in this field, introduce young female graduate students to potential future advisors and collaborators. We hope the workshop helped retention and will increase the participation of women in research activities in control of PDE's in the long-term. Another objective was to deepen worldwide networks of collaboration by inviting women from Latin America.

State of the Field

Developments in PDE optimal control theory started with considerations of parabolic partial differential equations - see for instance, the celebrated work of J.L. Lions [2]) . Later developments brought up front hyperbolic systems -first in one space dimension, starting with the pioneering papers of D. Russell [4] and W. Littman [3]. Already in one space dimension, it became clear that the nature of control mechanisms and capabilities is vastly different from that in finitedimensional spaces. This has provided an drive in abstract infinite-dimensional theory to seek counter-examples and also positive results within more restrictive frameworks. One can get a basic understanding of the area by realizing that properties such as stabilization are relatively simple for parabolic systems (this is essentially finite-dimensional theory due to spectral analysis), while (exact) controllability becomes almost impossible. In hyperbolic dynamics, controllability and more generally inverse problem are practical and solvable. On the other hand, stabilization is much more difficult for hyperbolic systems and requires special feedback that is intrinsically infinite-dimensional. While controllability is an *open loop* control problem, stabilization is realized by a *closed loop or feedback* control acting on an infinite time horizon $T = \infty$. Optimal control could be either open loop or closed loop and either finite or infinite horizon. As we shall see later, there is strong synergism between these concepts - solution of one problem feeds into another problem.

Consider an abstract model that provides a unified treatment for most of infinite-dimensional control problems. Let $T > 0$ be finite or infinite and let H and U be two separable Hilbert spaces with scalar product and associated norm respectively denoted by $(\cdot,\cdot)_H$, $(\cdot,\cdot)_U$, $\|\cdot\|_H$ and $\|\cdot\|_U$. Let us consider the abstract control system:

$$\begin{cases} y(t) = \ Ay(t) + Bu(t) \quad \text{in } \ (0,T)\,, \\ y(0) = y_0 \in H. \end{cases} \tag{15.0.1}$$

In this system $y_0 \in H$ is the initial datum and $u \in L^2(0,T;U)$ is the control, is exerted on the system through the operator B. Assume that A and B are unbounded operators respectively defined on $D(A) \subset H$ and $D(B) \subset U$. Let us also assume that system (15.0.1) is well-posed ; that is, for any $(y_0, u) \in H \times L^2(0,T;U)$ there exists a unique weak solution $y \in C^0([0,T];H)$ to problem (15.0.1) which depends continuously on the data. Denote by $y(t; y_0, u) \in H$ the solution to system (15.0.1) at time $t \in [0,T]$ corresponding to $(y_0, u) \in H \times L^2(0,T;U)$. Controllability is an open loop control problem where one seeks a control $u \in L_2(0,T;U)$ in order to "hit" a given target. Hitting may be *exact* or *approximate*. These concepts are quantitized by the following definitions.

1. System (15.0.1) is exactly controllable at time T if, for all $(y_0, y_1) \in H \times H$, there exists $u \in L^2(0,T;U)$ such that the solution y of (15.0.1) satisfies
$$y(T; y_0, u) = y_1.$$
2. System (15.0.1) is controllable to trajectories at time T if, for every $(y_0, \widehat{y}_0) \in H \times H$ and $\widehat{u} \in L^2(0,T;U)$, there exists $u \in L^2(0,T;U)$ such that the corresponding weak solution to (15.0.1) satisfies
$$y(T; y_0, u) = y(T; \widehat{y}_0, \widehat{u}).$$
3. System (15.0.1) is null controllable or exactly controllable to zero at time T if, for every $y_0 \in H$ there exists $u \in L^2(0,T;U)$ such that the corresponding weak solution to (15.0.1) satisfies
$$y(T; y_0, u) = 0.$$
4. System (15.0.1) is approximately controllable at time T if, for every $(y_0, y_1) \in H \times H$, and every $\varepsilon > 0$, there exists $u \in L^2(0,T;U)$ such that the corresponding weak solution to (15.0.1) satisfies
$$\|y(T; y_0, u) - y_1\|_H < \varepsilon.$$

Stabilization can be formulated as seeking a feedback control $u = F(y)$ where F is an operator from $U \to H$ [bounded or unbounded, linear or nonlinear] with the property that when inserted into the dynamics, it leads to stable dynamics. Stability may be asymptotic or uniform. In a simplest case when F is bounded, asymptotic stability amounts to the property that the semigroup $e^{(A+BF)}$ is asymptotically stable, that is,

$$e^{(A+BF)t} y_0 \to 0, \text{ as } t \to \infty \forall y_0 \in H \tag{15.0.2}$$

Uniform stability requires that stability property be uniform with respect to the underlined topology. This is to say

$$||e^{(A+BF)t}||_{t\to\infty} \to 0 \tag{15.0.3}$$

Several developments and new trends in various areas were discussed during the workshop.

Controllability and Stabilization of Parabolic Systems

The theory of infinite-dimensional systems often bifurcates into parabolic or hyperbolic frameworks. This is due to axiomatized assumptions imposed where parabolicity exhibits very special properties such as smoothing and infinite speed of propagation while hyperbolicity is marked by finite speed of propagation and intrinsic lack of smoothing by the dynamics. These properties have a fundamental effect on abstract control theory. Treatments of both type of dynamics often require different methods. These differences were also emphasized at the workshop where parabolicity [or more generally analyticity of the underlying semigroups] was exhibited by systems of heat transfer, Navier Stokes equations, viscoelastic systems, equations of thermoelasticity. In the case of parabolic dynamics expected properties are stabilization and null-controllability [or controllability to trajectories]. Talks by Assia Benabdallah, Catherine Lebiedzik, Suzanne Lenhart, Jing Zhang provide good representation of the area. However, the results presented by Assia Benabdallah showed that parabolic coupled equations can present hyperbolic behavior related to controllability; more specifically, minimal time of controllability and regional dependence of the control region.

Controllability and Stabilization of Hyperbolic Systems

Controllability and stabilization of hyperbolic systems is different than for parabolic systems. Here the situation is opposite: exact controllability is a natural property to expect due to time reversibility and propagation property. which does not change the regularity. However, it takes time for the signal to travel. Thus finite speed of propagation occurs and a typical results on localized/boundary controllability takes place after some time. Stabilization, instead, is more demanding due to the fact that instability in hyperbolic dynamics is typically infinite-dimensional. Typically the unstable part of the spectrum lies on the imaginary axis. In parabolic dynamics the unstable subspace is finite-dimensional. The workshop had many talks devoted to this issue. Valeria Cavalcanti presented some results on stabilization of wave equation via the transmission of viscoelasticity. Daniela Sforza considered systems of equation with memory terms. Paola Loreti presented an approach where controllability of a large class of hyperbolic systems can be resolved by methods of Fourier's Analysis and exponential functions.

Optimal control

In optimal control the goal is to find a control to minimize a given performance index, $J(u, y_0)$, subject to dynamics (15.0.1). Thus one seeks an optimal control $u^0 \in L_2(0,T;U)$ such that

$$J(u^0, y(u^0)) = \inf_{u \in L_2(0,T;U)} J(u, y(u)). \tag{15.0.4}$$

Optimal control has strong synergy with controllability and stabilization. For instance, consider $T = \infty$. Seeking an optimal control requires that the functional cost be finite for some control. This is guaranteed by the so called *finite cost condition*, and this may be deuced from controllability. Once the optimal control is found, for linear systems it can be synthesized in a feedback form. There exists an operator P so that the optimal control

$$u^0(t) = -B^* P y(t).$$

In addition, subject to coercivity of the functional cost, the corresponding optimal system is exponentially stable.

Classical optimal control involves minimization of a chosen functional (representing the cost and the target) subject to dynamics depending on control function and possibly constraints on both state and control. This constitutes the core of calculus of variations and optimal control theory. This class of control problems has been studied first within the context of ODE's, then evolved in the direction of PDE's, mostly abstract systems with bounded control operators. Unbounded controls such as point or boundary control have been considered more recently starting with parabolic systems, due to their nice regularity properties. Again, motivated by important applications in mechanics and medicine optimal control problems, both boundary and point controls have been considered, and some theory has been obtained for hyperbolic dynamics. We have new phenomena coming into picture, such as finite speed of propagation or unexpected hidden regularity. These properties have a heavy influence on the overall theory. New trends that are emerging in the field include optimal and constrained control of nonlinear systems and optimal control of hyperbolic- like models with boundary controls.

There were several talks on this topic, For instance, Suzanne Lenhart and Wendi Ding on optimal control of parabolic PDE's with application to biology and Francesca Bucci on optimal control including optimal feedback control of the third order system with boundary control. This latter reduces to a second-order system with a finite speed of propagation. All these talks, in addition to presenting mathematical contribution to the field, were strongly routed in life science applications. Lenhart's and Ding in biology while Bucci's presentation dealt with a problem of controlling radiation and high frequency ultrasounds. Constanza Sanchez presented optimal control problem applicable to a Schrödinger equation, an equatio thatn in some sense combines parabolic and hyperbolic effects - it displays infinite speed of propagation but lack of smoothing on bounded domains.

Abstract Infinite-Dimensional Systems

Infinite-dimensional system theory is an area where the goal is to axiomatize certain properties in control theory within the context of infinite-dimensional theory with standard matrices replaced by operators. The aim is to verify which properties from finite-dimensional control theory persists and which are no longer true and thus need to be replaced by an appropriate framework. The importance of the field lies in its generality. Of course, such abstract theory must be supported by examples that provide a verification and justification for the content. Thus, not surprisingly, there is a strong synergism between infinite-dimensional system theory and PDE control. This workshop was a good example of existing collaborations and the need for future research exchanges. Talks by Birgit Jacob and Jacquelien Scherpen provided overviews of some recent developments in these areas.

Control of Hybrid Systems and Systems with Transmission

An important area of current theoretical and practical interest is combining several different types of dynamics (as dictated by modeling -e.g. fluid–structure interaction) where control acquires a unique role in transferring desirable properties from one part of dynamics onto another. Propagation of stability, controllability and regularity [the latter in optimal control problems] through an interface becomes a centerpiece of the underlying analysis.

Canonical examples of such systems are fluid/ flow structure interactions, structural acoustic interactions and composite structures. Applications are abundant in various other areas of sciences such as medicine, biology, engineering, aeroelasticity etc. The main characteristic feature of such systems is the interaction of two different PDE dynamics via some interface (a manifold of co-dimension one) or via transmission conditions separating two different media. From the mathematical point of view control of such systems present formidable challenges. Any progress in this area must be preceded by good understanding of control theoretic aspects of a single PDE equation -both parabolic and hyperbolic.. Only then one can venture into trying to control the interaction by taking advantage of the desirable properties each PDE brings to the table. For instance, dissipativity and natural stabilizing mechanism in parabolic flows (Navier Stokes equation) enables controlling vibrations/oscillation in mechanical structures (bridges, vessels, arterias). In order to exhibit such phenomena one must assure appropriate geometric conditions. Several talks at the workshop [Valeria Cavalcanti, Lorena Bociu, Irina Ryzhkova, Weiwei Hu, Katie Szulc, Jing Zhang] were devoted to this topic which then was discussed in smaller groups of interest with plans for further collaboration. The topic, itself, is very rich and interdisciplinary. It involves modeling, PDE analysis, fluid mechanics and experiment/numerics. Speakers at the workshop were able to provide different perspectives and aspects of the problem, so that the cumulative experience is greater than the sum of ingredients. Formulation of relevant mathematical problem requires strong back up of modeling and of applied relevance. Thus it was natural to engage in collaborative work between USA, Brazil and France.

Control of models Arising in Biology and Medicine

This is a field where theoretical methods of optimal control find their way into truly applied problems. Due to the array of PDE models describing biological phenomena, a maximum principle in optimal control as well as feedback stabilization and synthesis provide for critical tools in improving therapies and treatments. Typical theoretical results are local (as relevant in applications), due to a nonlinear nature of the models. Suzanne Lenhart's talk on optimal control provided an overview of math biology applications, Francesca's Bucci lecture dealt with medical applications in inverse problems-ultrasound technology. Bianca Calsavara's presentation described local controllability results in genetic networks.

Nonlinear control

All these problems can be also be studied within the nonlinear framework, integral equations, delayed equations, Banach spaces etc. The applications give a vast scenario of models in the infinite-dimensional framework. Non-linear models with nonlinear controls can be written

$$\dot{y} = A(y) + B(u)$$

where $A(y)$ and $B(u)$ are nonlinear operators may also be considered. With feedback control this becomes $\dot{y} = A(y) + BF(y)$ -where the ultimate equation may represent a well studied dynamical system. Clearly, many possible scenarios are under consideration, leading to an array of results with strong applied ramifications. Many talks at the workshop addressed these nonlinear aspects of control theory.

Every nonlinear problem needs its own precise statement and development. Some of the techniques presented in the workshop include fixed point arguments (from Browder to Kakutani fixed point theorems), Coron's "return method" [1] or the use of the Inverse Function Theorem.

Panel Discussions

We had several panel discussions that participants found useful. Our area is very inter-disciplinary which can be exciting but also an issue for women, particularly in hiring and promotion, granting agencies etc. The fact that research may not fit into a mainstream area can be used as an excuse for turning down a proposal or promotion. The efficacy of some kind of mentoring was clear. Of course this can be of great help for any young researcher but in particular for women since gender imbalance and society pressures and attitudes discourage women in a number of aspects of the academic life.

Open research problems

Identification of research problems that are interesting (either for applications or theory) but not already been studied by another group, is critical to success as an academic. This, and effective time management, are two skills that need to be developed by young academics in order to obtain tenure and promotion. New faculty need to distance themselves from their advisor to develop their own program and must write papers independently in order to obtain tenure. The following points were also made during the discussion.

- Mathematical biology has a lot of job and grant opportunities and there are lots of interesting unsolved problems.
- For inter-disciplinary work, such as biology, working with an expert in the non-math area is very helpful.
- Slow/fast dynamics arise in many dynamics and methods are needed for them.
- Coupled PDE's, particularly hyperbolic/parabolic equations, have many interesting mathematical issues with consequences for applications (a number of talks at the workshop were concerned with such. systems)
- Systems with memory are receiving interest.
- Fractional Laplacians and other fractional derivatives have interesting theoretical issues and connections with applications, such as thermo-elastic materials, but some people won't touch them.
- Interdisciplinary work can be a good source of interesting problems, but there is a danger of falling out of the mainstream work in your department or the granting agencies
- Illustrations of differences with finite-dimensional systems useful.
- Keep in touch with numerical and experimental people. They can be a source of insight and new problems even if you are only interested in theory.
- Try to find problems that will develop collaborators in your department.
- Go to workshops in your field and related fields.
- Think of your strengths/uniqueness and play to those.
- Choose problems that excite you!

Grant Writing

Grants are important for academics in several respects. On a practical level, they provide funds for travel, collaboration and graduate students. They are also used by institutions as a criterioni for tenure and promotion, partly because they are regarded as an indication of the respect that a researcher's work holds and partly because grant overhead is important to university budgets.

It was clear in the discussion that several younger faculty had been discouraged by unsuccessful grant applications. However several respected and well-funded senior participants shared that often applications are turned down. The success rate for most programs is quite low, and applications for interesting research are turned down. Sometimes it is just bad luck. The important thing is use any reviews as useful information for revising and rewriting and to persist; either with a revised proposal to the same program or else to a different program. A number of other comments and advice were made:

- Look at all sources of funding, be creative in thinking of programs to which to submit applications.
- Think of who will be reading the application. Is it government personnel ? Experts in the field? This should affect how the research is presented.
- Useful to have someone read and advise on the proposal. Many universities have personnel to help.
- Start with applications for smaller grants.
- Proposal writing is time-consuming.
- Find out and address the priorities of the grant program.
- If you're trying to involve an industrial partner, "talk their language" and think about their priorities.
- Use common language for inter-disciplinary work.
- Collaborators can be helpful, particularly for inter-disciplinary work.
- Read reviews carefully and use to make changes for new proposals. Often you can make changes and resubmit to the same program. Or, the research may be more suitable for another program.
- Even unsuccessful proposals can help in focusing ideas and identification of open problems.
- Networking is important; go to conferences and meet people.

Highlights

As outlined above, there is a synergistic diagram

$$exact\ controllability \ \leftrightarrow\ solvability\ of\ optimal\ control \leftrightarrow exponential\ stabilizability$$

and also important applications of stabilizability and optimal control. There was a balance in the topics: not everything was theoretical nor everything was applied. A wide variety of different aspects of control theory of PDE's was covered. The workshop gave an opportunity to explore links and future collaborations. Although this was not planned, a number of talks featured work on different aspects of coupled systems. This was marked by a number of participants as a possibly fruitful area of research and also as a good topic for a conference invited session.

This was a very good meeting from the scientific point of view. All participants expressed enthusiasm for the high technical content and atmosphere of the workshop.The research presented was of the usual calibre of major international meetings. This provided encouragement to the younger women about the quality of women's contributions to our field.

The participants obtained interesting new research ideas and bibliography. Some of the participants have already invited to give seminars at other participants' universities and we all have new names of excellent researchers in the area of control of PDE's that can be invited to meetings, workshops and seminars. The experience was great in the sense that we put in touch different groups working on this interesting subject all around the world, with different techniques, backgrounds and area of application. Talks included very abstract setting to very applied one, from biology to engineering, from physics to abstract analysis.

Typical comments were:

> "I want to thank you for inviting me to the workshop women in control, it's really the best conference I have attended till now. Thank you."

> "First of all, thank you again for organizing the workshop! It was a great learning experience, and I found it much more beneficial than the usual conferences I attend. The schedule was great, and allowed participants to interact, collaborate, discuss potential common projects, and give/receive advice regarding professional development. I found all the small group research talks and panels very helpful. Also, I really enjoyed the atmosphere of the workshop. I wish we had more workshops like this."

> "In general, I think that the Workshop "Women in Control: New Trends in Infinite Dimensions" was a really useful and reach experience in many senses. One of the interesting things I think I obtained from the Workshop is that I could make contact with people with a lot of knowledge in control theory and learn and see other applications and techniques of different areas of control problems. I keep in touch with them and I am opened to establish common work with some of them."

Meeting Outcome

The high quality of the talks, despite the restriction to women working in a small field, was very encouraging to all participants. There was discussion at the workshop about organization of sessions at international conferences in 2018, such as MTNS and IFIP, that will involve participants as speakers.

The format of the meeting facilitated discussion in small groups around open problems which helped participants develop their networks and think about research problems. Networks in five sub-areas were create to establish new collaboration, to mentor young women with their research proposals, to share specific material and papers, propose new research, and so on. This list will also help in organization of sessions at conferences. The list of sub-areas, along with acronyms, is

- CIDS: Control of infinite Dimensional Systems
- OCO: Optimal control and Optimization
- CMBM:Control of mathematical-biological Models
- SLTB: Stabilization and long time behaviour
- NM: Numerical Methods

Table 15.1 lists the members of each group.

We have also created a closed group in Facebook and a "basecamp" webtool for our group to share files and emails. Only invited members can receive the updates and to see the files.

Some representatives from Asia were missing. Although possible participants in China were identified, lack of support for their travel expenses did not allow them to attend. Also the control of PDE's community in France is large but not well represented. We believe that after the present positive experience a future meeting will be even more successful, particularly with financial support. We would like to encourage more participation from Asia and France, as well as women working on numerical issues. The financial and administrative support of BIRS that enabled this meeting to happen was very much appreciated. A future meeting, maybe in China, is already being discussed.

As described above the topics and lectures presented are characterized by great synergy and gradual buildup of knowledge in the area which is progressive with a strong slant to modern applications and technologies. For

Person	Country	Networkings
Apraiz, Jone	Spain	CIDS/OCO
Belkhatir, Zehor	Saudi Arabia	CMBM/NM
Benabdallah, Assia	France	CIDS/CMBM
Bociu, Lorena	U.S.A	OCO/CMBM
Bucci, Francesca	Italy	OCO
Calsavara, Bianca	Brazil	CIDS/CMBM/OCO
Cavalcanti, Valéria Neves Domingos	Brazil	SLTB
de Teresa, Luz	Mexico	CIDS/OCO
Ding, Wandi	U.S.A	CMBM
Doubova, Anna	Spain	CIDS/CMBM/NM
Fu, Xiaoyu	China	SLTB
Gomes, Susana N.	U.K.	SLTB/NM/OCO
Hu, Weiwei	USA	SLTB/NM/OCO
Jacob, Birgit	Germany	CIDS/SLTB
Jamieson, Jessie	USA	NM
Laasri, Hafida	Gernamy	CIDS/SLTB
Lasiecka, Irena	USA	SLTB/CIDS
Lebiedzik, Catherine	USA	SLTB
Lenhart, Suzanne	USA	CMBM
Loreti, Paola	Italy	CMBM
Morris, Kirsten	Canada	SLTB/OCO/NM
Rivas, Ivonne	Colombia	CIDS/SLTB
Ryzhkova-Gerasymova, Iryna	Ukraine	SLTB
Sánchez de la Vega, Constanza	Argentina	CIDS/OCO
Scherpen, Jacquelien	Netherlands	NM/SLTB
Sforza, Daniela	Italy	CMBM
Szulc, Katarzyna	Poland	NM
Tang, Shuxia	Canada	SLTB/OCO
Tegling, Emma	Sweden	OCO
Zhang, Jing	USA	SLTB/OCO

Table 15.1: Networkings

example: the results presented in topics 1-4 display great synergy. Developments achieved at the abstract level, when combined with parabolic and hyperbolic discoveries culminate in the study of *systems* where various types of dynamics interact. This led to strong interaction among participants who find proper mentoring in the areas complementary [but important] to their expertise. Similarly topics 5-7 pave the way to a treatment of applied problems that can be resolved with a help of computational mathematics.

Both senior and junior researchers found the workshop useful for advancing themselves and the field. The senior ones by being exposed to disciplines a bit outside their expertise and the junior ones by being introduced gradually into the arcanes and developments in each area. On site contacts, formalized by creation of the Working Groups [see above] are expected to provide an effective mentoring forum with the goal of advancing the field and careers of women researchers.

Participants

Apraiz, Jone (University of the Basque Country)
Belkhatir, Zehor (King Abdullah University of Science and Technology (KAUST))
Benabdallah, Assia (Aix Marseille University)
Bociu, Lorena (NC State University)
Bucci, Francesca (Universita' degli Studi di Firenze)
Calsavara, Bianca (University of Campinas)
Cavalcanti, Valéria Neves Domingos (University of Maringà Brazil)
de Teresa, Luz (Universidad Nacional Autonoma de Mexico)
Ding, Wandi (Middle Tennessee State University)
Doubova, Anna (Universidad de Sevilla Spain)
Fu, Xiaoyu (Sichuan University China)
Gomes, Susana N. (Imperial College London)
Hu, Weiwei (Oklahoma State University)
Jacob, Birgit (University of Wuppertal)
Jamieson, Jessie (University of Nebraska-Lincoln)
Laasri, Hafida (Fern Universität Hagen)
Lasiecka, Irena (University of Memphis)
Lebiedzik, Catherine (Wayne State University USA)
Lenhart, Suzanne (University of Tennessee)
Loreti, Paola (University of Rome)
Morris, Kirsten (University of Waterloo)
Rivas, Ivonne (Universidad del Valle Colombia)
Ryzhkova-Gerasymova, Iryna (Kharkiv National University)
Sánchez de la Vega, Constanza (Universidad de Buenos Aires Argentina)
Scherpen, Jacqueline (University of Groningen Netherlands)
Sforza, Daniela (Sapienza Università di Roma)
Szulc, Katarzyna (Polish Academy of Sciences)
Tang, Shuxia (University of Waterloo)
Tegling, Emma (KTH Royal Institute of Technology)
Zhang, Jing (Virginia State University)

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Chapter 16

Latest Advances in the Theory and Applications of Design and Analysis of Experiments (17w5007)

August 6 - 11, 2017

Organizer(s): Holger Dette (Ruhr-Universität Bochum), Douglas Wiens (University of Alberta), Weng Kee Wong (University of California, Los Angeles)

Design and analysis of experiments is an area in statistical methodology, which has applications in many research fields where experiments are performed to develop new theory and/or improve the quality of processes and services. For example, experiments are continually carried out to investigate and improve manufacturing processes, understand climate changes, provide better utilization of large-scale primary care services, evaluate environmental damages, and develop more effective drugs for patients, to name a few. Optimal design techniques are also used in education testing, cryptology or fMRI analysis of brain data. Even small improvements in methodology can have huge effects in applications. Because of rapidly rising costs in experimentation, researchers are increasingly interested in using efficient designs that save costs without sacrificing the quality of ensuing statistical inferences. In toxicology, for example, in addition to saving labour, time and material cost, researchers are increasingly under pressure to use the fewest possible animals in laboratory research.

There have been many exciting new developments in this area in the last two decades. In particular, researchers have made advances in both theory and applications of optimal designs for mixed effects models, longitudinal models and nonlinear models, including more efficient designs in health care studies, computer experiments, and experiments for processes with time or space dynamics. The main aims of the workshop were to develop new theory for optimal design for new and more challenging problems, apply novel applications of optimal designs to new problems and keep us up to date of the state-of-the-art optimization techniques used in engineering and computer science. Some focus areas of the workshop are described below.

New Theory and Construction Methods for Nonlinear Models

The theory of optimal designs for non-linear models is very important because they have many applications in the medical and pharmaceutical sciences, biometry and in various other industrial experiments. However, most design work has focused on models with independent and normally distributed errors, fixed effects, and assuming the outcome is either continuous or binary. The workshop provided a forum to stimulate development of advanced methods for constructing optimal designs in nonlinear models with mixed effects, longitudinal models, and models with non-normally correlated errors or models with outcomes that may comprise of continuous and discrete responses. Progress was also made in the construction of optimal designs with correlated observations, which has particular mathematical challenges, due to the non-convexity of the optimization problems. Other attendees had already made, and continue to make, important contributions to design in health care and clinical trials. These include strategies for designing studies where the main outcomes are categorized (treatment success, partial success or failure), optimal randomization allocation plans for patients in a multi-arm clinical trial, and optimal design strategies for testing in enzyme-kinetic studies.

Two different approaches of experimental design for the so called Fractional Polynomials (FP) family of models were described, independently, by Professors Steven Gilmour and Jesus López-Fidalgo. This family has been shown to be much more flexible than polynomials for fitting continuous outcomes in the biological and health sciences. In particular, being linear models they can be used instead of a variety of non-linear models that add much complexity to the usual inferences. Despite their increasing popularity, design issues for FP models had never been addressed. D- and I-optimal experimental designs were computed for prediction using FP models. Detailed proofs of the closed-formed design were given when possible. Otherwise numerical computations were made. Their properties were shown and a catalogue of design points useful for FP models was provided in illustrative graphics. As applications, linear mixed effects models for longitudinal studies were considered. An example using gene expression data was considered by Professor López-Fidalgo, comparing the designs used in practice. Taking into account the final choice of one of the potential models, an interesting problem is finding designs for effective model discrimination for FP models. They are exploring KL-optimality and he showed some of their advances.

On the other hand Professor Gilmour noted that with the availability of routines for fitting nonlinear models in statistical packages, FP models are increasingly being used. However as in all experiments the design should be chosen such that the model parameters are estimated as efficiently as possible. The design choice for such models involves the usual difficulties of nonlinear models design. He presented Bayesian optimal exact designs for several fractional factorial models. The optimum designs were compared to various standard designs in response surface problems. Some unusual problems in the choices of prior and optimization method were noted. As an introduction to his presentation Professor López-Fidalgo made some comments about the present and future of experimental design. In particular he mentioned the paper "Manifesto for reproducible science" published recently in Nature, where the authors, coming from different sciences, ask for a rigorous research methodology and give key measures to optimize the scientific process. He pointed out the word 'design' appears 25 times in 7 pages in all sections of the paper.

Experimental design applications for discriminating between models have been hampered by the assumption to know beforehand which model is the true or more adequate one, which is counter to the very aim of the experiment. Previous approaches to alleviate this requirement were either symmetrizations of asymmetric techniques such as compound T-optimality, or Bayesian, minimax and sequential approaches. In their joint talk Professors Radoslav Harman and Werner Müller presented a novel, genuinely symmetric criterion based on a linearised distance between mean-value surfaces and the newly introduced notion of nominal confidence sets . The computational efficiency of the proposed approach was shown and a Monte-Carlo evaluation of its discrimination performance on the basis of the likelihood-ratio was provided. Additionally these authors demonstrated the applicability of the new method for a pair of competing models in enzyme kinetics.

Professor Anatoly Zhigljavsky (Dette et al. 2016, 2017) reviewed a series of works related to optimal designs with correlated errors and noted the potential importance of implementing designs with correlated errors in practical applications. His talk highlighted new and important theoretical investigations toward that direction. He presented a complete solution of this challenging optimal design problem for a broad class of regression models and covariance kernels. As a by-product he derived explicit expressions for the BLUE in the continuous time model and analytic expressions for the optimal designs in a wide class of regression models. Professor Anatoly Zhigljavsky also reviewed the celebrated Sacks-Ylvisaker approach and discussed several approximate methods of construction of optimal estimators.

Yu Shi from UCLA Biostatistics Department described her group's recent research into utilizing advanced optimization and integration techniques to find Bayesian optimal designs for random effects models, which potentially has profound applications in HIV studies, PK/PD studies and many other studies in biomedicine. She also talked about using these techniques in developing biomarkers in oncology studies. This work is interesting both methodologically and in practice, and the group puts great efforts into developing user-friendly software that is promising to benefit scientist and practitioners at large. Her message reinforced Dr. Kim's use of novel algorithms used by engineers and computer scientists to solve hard design problems. This algorithm is a member of the class of nature-inspired metaheuristic algorithms which has been shown to be very flexible, assumptions free and easy to implement and use. They are powerful in that they have been reported in computer science and engineering arenas to be able to solve optimization problems with hundreds of variables in a relatively very short period of time. In short, they offer exciting opportunities and great promise when current numerical methods in statistics do not work well. When designing a study for a complicated nonlinear problem or a high dimensional design problem, these algorithms can also help determine the structure of the optimal design quickly and hence help the researcher makes an informed guess on the correct number of points required, their locations and the number of replicates at each of the design points. Subsequent applications of an equivalence theorem can then produce a formula for the optimal design, which can be elusive. Chen el at. (2017) provide examples.

Professor Mong-Na Lo Huang (Huang, et al., 2017) proposed optimal design methodology for group testing experiments, where the goal is to estimate the prevalence of a trait using a test with uncertain sensitivity and specificity. She and her co-authors determined approximate optimal designs for simultaneously estimating the prevalence, sensitivity, and specificity The optimal designs require three different group sizes with equal frequencies. On the other and, if estimating prevalence as accurately as possible is the only focus, the optimal strategy is to have three group sizes with unequal frequencies. Based on a Chlamydia study in the United States, she compared the competing designs and provided Additionally she discussed extensions on budget-constrained optimal group testing designs, where both subjects and tests incur costs, and assays have uncertain sensitivity and specificity that may be linked to the group sizes.

Dr. Stephanie Biedermann (Lee, et al., 2017) described her newly developed methodology for designing experiments when outcome values are not missing at random. This is the first explorative investigation into this topic, with some promising results so far, and thus opens up a new avenue of research on the interface of design of experiments and incomplete data problems, in particular the notoriously challenging situation when missing data are not missing at random. In the ensuing discussion, an interesting modification to the optimality criterion was proposed, which may lead to an increased benefit of the approach in the planning of scientific experiments. Professor Rosemary Bailey described recent work obtaining good block designs for 36 treatments in blocks of size 64, thus filling in a gap in the famous square lattice designs introduced by Frank Yates 80 years ago. Professor Jeff Wu also referred to the ongoing importance of Yates's work in comments on another talk.

Prof Henry Wynn introduced a new optimal design criterion (Kuriki and Wynn, 2017), which can be used to determine an experimental design that minimise the width of simultaneous confidence bands for nonlinear regression models, which are constructed by evaluating the volume of a tube about a curve that is defined as a trajectory of a regression basis vector. Professor France Mentre investigated optimal designs for nonlinear mixed effect models (NLMEM), which are used in longitudinal studies. She reviewed methods evaluating the expected Fisher information based on Monte-Carlo Hamiltonian Monte-Carlo (MC/HMC), which require a priori knowledge on models and parameters, and extended these MC/HMC-based methods to account for uncertainty in parameters and in models.

Professor Min Yang described his recent research into data reduction of big data. Extraordinary amounts of data are being produced in many branches of science. Proven statistical methods are no longer applicable with extraordinary large data sets due to computational limitations. A critical step in Big Data analysis is data reduction. Professor Yang introduced a novel approach called Information-Based Optimal Subdata Selection (IBOSS) for data reduction. The proposed research will open an entirely new avenue for research. It will make a significant contribution by developing methods that extract information from big data more efficiently. The impact will be felt in many areas of science. Existing sampling-based methods face the limitation that the amount of information that they extract is constrained by the subsample size, and does often not increase with the full data size. The proposed IBOSS strategy alleviates this problem. Preliminary results show that the IBOSS strategy retains much more information than the sampling-based methods. During the discussion that followed, the comparison between divide-and-conquer method, and new and unexpected applications of this work to high-dimensional data, such as genomics data were brought forth.

Robustness of Design

Suppose that an investigator anticipates planning a study that will result in a number of observations on a random variable Y, whose probability distribution – often merely through its expected value – depends on a vector x of covariates *that can be set by the investigator* – hence the design. After the data are gathered the relationship between Y and x is to be assessed. This will generally involve both estimation and prediction, and is often done in the context of a particular model *of which the experimenter might have only partial knowledge and in which he might have little faith* – hence the robustness requirement. 'Robustness' has numerous meanings in statistics. The notion appears to have been introduced by Box (Box, 1953), and was given a firm mathematical basis by Huber (Huber, 1964, 1981), for whom it generally – but certainly not exclusively – meant the relative insensitivity of a statistical procedure to departures from the assumed Gaussian error distribution. In design the usual performance measures depend on the error distribution only through the first two moments, and beyond this the distributional shape is not so relevant. One does however have in mind a particular model to be fitted once the data are gathered. In 'classical' optimal design theory, one believes explicitly that the model one fits is the correct one, and measures the quality of a design through a 'loss function' such as the determinant, or trace, or maximum eigenvalue of the covariance matrix, corresponding to the well known D-, A- and E-optimality criteria. In 'model robust' design theory one instead anticipates that the model that will be fitted by the experimenter is not necessarily the true one – a simple example to bear in mind is that of fitting a straight line regression when the true response function is possibly not exactly linear in the covariate – and so the loss function highlights some more general feature such as the mean squared error (*mse*). This will of course depend on the true, rather than fitted, model and so one seeks a design minimizing some scalar quantity summarizing the increased loss – perhaps the maximum, or average, of the *mse* over the predicted values – as the true model varies. A common framework is that an experimenter plans to fit a particular model to his data, while realizing that any model is at best an approximation. He seeks protection, at the design stage, from increased loss incurred by model mis-specification. Canada has become a noted region of concentration for research in Robustness of Design, through the work of (contact organizer) D. Wiens (Wiens, 2015), with his colleagues and former students, several of whom attended and gave presentations.

Professor Julie Zhou (U. Victoria) presented her recent research on optimal regression designs under the second-order least squares estimator (SLSE). Since the SLSE is more efficient than the least squares estimator when the error distribution in regression model is asymmetric, new optimality criteria under the SLSE have good design properties. An attendee from outside the design community commented on the usefulness of Julie's work to his work, involving the modelling of survival data.

The workshop provided an excellent opportunity for sharing and discussing research ideas. After Professor Mong-Na Lo Huang gave her presentation on optimal group testing designs, Professors Zhou and Huang started to explore possible joint research projects on algorithms for computing optimal group testing designs and robust designs.

In a thought-provoking and elegantly presented address Professor Tim Waite (Manchester U.) proposed usage of minimax efficient random designs based on a frequentist decision-theoretic experimental design approach, with application to model-robust design for prediction. These seem to address in an efficient manner a long open problem – the inadmissibility of a class of design presented by Wiens (Wiens, 1992).

Professor Dave Woods (University of Southampton, UK) presented recent results and ongoing research on sequential design for Bayesian optimisation of physical systems with application to automatic experimentation in the chemical and pharmaceutical sciences. This work, in collaboration with chemists and chemical engineers, aims to develop novel design of experimental methods and apply them in conjunction with automated equipment to increase lab efficiency and reduce costs. A particularly important element of the work is methodology to automatically detect and down-weight potentially outlying observations. Useful feedback from the talk included links to other optimisation methods for maximising expected improvement in computer experiments.

Xiaojian Xu (Brock U.) discussed the effects of maximum likelihood estimation for a generalized linear mixed model (GLMM) when possible departures may appear from its assumed form. The commonly occurring departures involved in a GLMM may include imprecision in the assumed linear predictor, misspecified random effect distribution, or possibly both. She outlined the construction of D-optimal sequential designs which are robust under consideration of these types of departures. Since the computational work involved in GLMMs can be very intensive, an approximate approach was also proposed. Some comparisons were given through simulations.

Computer Experiments

The traditional idea of an experiment involves observation of a system of interest under controlled conditions, with the intent of learning something about that system. The system of interest varies by discipline: engineers and physicists may be interested in systems involving physical material, biologists may focus on living organisms (or collections or components of them), while social scientists may be interested in experiments involving the behaviour of human beings. In contrast, the system of interest in a computer experiment is often a computer model, usually a mathematical description of a real system of interest, represented as a computer program. The computer representation is usually necessary due to the complexity of the model. Experimental goals are often similar to those in traditional experiments. While the computer model must, in principle, be fully known, it is generally so complex that a useful understanding of its behavior requires the empirical approach of an experiment (Morris and Moore, 2015).

Computer experiments are increasingly used in many important scientific studies, in part because they can account for the sophisticated correlation structures often found in complex studies that generate very large data sets quickly and continuously, such as in climate changes. For such complicated and expensive physical experiments, a well-designed computer experiment using, say, Latin square designs and sequential designs can reduce the experimental costs significantly. Canada has become somewhat of a hotbed for such work; evidence of this was provided by the attendance of D. Bingham and B. Tang from Simon Fraser University, William Welch from University of British Columbia, Devon Lin from Queen's University and other world leaders in this area, such as Jeff Wu from the Georgia Institute of Technology and Peter Chien from the U. Wisconsin-Madison, together with some of their current and former students.

Professor Will Welch described his recent research into emulation of computer experiments. His talk motivated some to think of new ways to link Gaussian process models with additive correlation functions to general functional additive models. This could lead to a new research direction in emulation. Will pointed out that Gaussian processes (GPs) are widely used for analysis of the input-output relationship(s) of a deterministic computer experiment. While there are many tweaks of the basic model, they turn out to be fairly unimportant for prediction accuracy. In particular, complex input-output functions remain difficult to model with useful accuracy whatever method is used within the usual GP universe of models.

Professor Bingham introduced new types of space-filling designs motivated by several real applications. The proposed designs are unconventional and novel. This is a product of the cross-fertilization between experimental design, combinatorics, and applied mathematics. It shows a new aspect of space-filling designs. The approach was demonstrated on several examples, including the cosmology study of non-linear power spectrum simulation models that motivated the work.

Devon brought 'big data' into focus, via 'fat data' with high dimensional inputs and/or outputs. She proposed a computationally efficient modelling approach to build emulators for large-scale dynamic computer experiments. This approach sequentially finds a set of local design points based on a new criterion specifically designed for emulating dynamic computer simulators. Singular value decomposition based Gaussian process models are built with the sequentially chosen local data. To update the models efficiently, an empirical Bayesian approach was introduced. When a target observation is available, estimating the inputs of the computer simulator that produce the matching response as close as possible is known as inverse problem. She proposed a new criterion-based estimation method to address the inverse problem of dynamic computer experiments.

Professor Jeff Wu described his recent and potentially groundbreaking research into rocket injection design through spatial-temporal modelling; issues on uncertainty quantification were also discussed. He detailed a recent study to illustrate how physics and data are used jointly to learn about the "truth" of the physical world. In the quest for advanced propulsion systems, a new design methodology is needed which combines engineering physics, computer simulations and statistical modelling. There are two key challenges: the simulation of high-fidelity spatial-temporal flows (using the Navier-Stokes equations) is computationally expensive, and the analysis and modelling of this data requires physical insights and statistical tools. In this method, a surrogate model is first presented for efficient flow prediction in swirl injectors with varying geometries, devices commonly used in many engineering applications. The novelty lies in incorporating properties of the fluid flow as simplifying model assumptions, which allows for quick emulation in practical turnaround times, and also reveals interesting flow physics which can guide further investigations. Next, a flame transfer function framework is used for modelling unsteady heat release in a rocket injector. Such a model is useful not only for analyzing the stability of an injector design, but also identifies key physics which contribute to combustion instability. During the discussion that followed, Professor Wu specifically mentioned the importance of computer experiments as well as sequential designs for future developments, especially in industrial applications. This has provided a very good direction for future study on design of experiments.

Professor Peter Chien's talk consisted of three topics on the design and analysis of computer experiments with complex data. The first topic dealt with computer codes with gradients. The gradient-enhanced Gaussian process emulator is widely used to analyze all outputs from a computer model with gradient information. The gradient-enhanced Gaussian process emulator has more numeric problems than in many multivariate cases because of the dependence of the model output and each gradient output. He had derived, and presented, a statistical theory to understand why this problem happens and proposed a solution using a data selection approach. The second topic concerned computer models with invariance properties, which appear in materials science, physics and biology. He proposed a new statistical framework for building emulators to preserve invariance. The framework uses a weighted complete graph to represent the geometry and introduces a new class of function, called the relabeling symmetric functions, associated with the graph. The effectiveness of the method was illustrated by several examples from materials science. The third topic presented a new class of statistical design inspired by the Samurai Sudoku puzzle. These designs have overlapping components and are useful for cross-validating data or models from multiple sources.

Advanced Numerical Methods for Searching Optimal Designs

Analytical optimal designs are typically available only for relatively simple models or models with a very special structure, where mathematics can be cleverly applied to find formulas for the optimal designs and to study their properties. With large data becoming increasingly more available and improved modelling techniques, more realistic but more complicated models are employed to draw accurate inference from scientific studies; for example, in fMRI studies on how the brain functions and reacts to different stimuli or how water resources can be better conserved and efficiently utilized by studying the whole water system involving hundreds of variables in hydrology. A focus in the workshop was to develop more efficient numerical methods for finding optimal designs for high dimensional problems. To this end, the workshop promoted cross-disciplinary research by having engineers and computer scientists at the workshop as well. Eminent engineering experts in optimization, as well as current graduate students and recent graduates from statistics programmes, demonstrated how to apply state-of-the-art algorithms, such as nature-inspired metaheuristic algorithms to solve complex design problems.

Professor Luc Pronzato discussed the problem of obtaining good experimental designs for Gaussian process modelling in computer experiments. To determine designs, which have satisfactory space-filling properties he proposed minimax-distance (sub-)optimal designs, which minimise the maximum distance between a point of the region of interest and its closest design point, and thus have attractive properties in this context. As their construction is difficult, even in moderate dimension Professor Pronzato considered several discretisation methods of the experimental region, such as the determination of Chebyshev-centroidal Voronoi tessellations obtained from fixed-point iterations of Lloyds' method, and the construction of any-time (nested) suboptimal solutions by greedy algorithms applied to submodular surrogates of the minimax-distance criterion.

Dr. Seongho Kim, an Associate Professor of Biostatistics at Wayne State University/Karmanos Cancer Institute, described his pioneering developments in pharmacokinetics and phase II clinical trial designs using a stochastic global optimization algorithm, particle swarm optimization (PSO). During his presentation, he brought up three crucial but challenging issues in parameter estimation, global optima, speed of convergence, and statistical identifiability, and demonstrated that PSO is able to deal with those challenges efficiently, leading to the possible opening of a new avenue of research.

Some members of the optimization community attended and gave interesting talks on this important aspect of design construction. In particular, Professor Lieven Vandenberghe from Department of Engineering Departmenat at UCLA described connections between experimental designs and semidefinite programming, and provided up-to-date computing algorithms, useful for finding optimal designs efficiently. Semidefinite programming (SDP) has important applications in optimisation problems that involve moment cones or, by duality, cones of nonnegative polynomials. Examples can be found in statistics, signal processing, control, and non-convex polynomial optimisation. Professor Guillaume Sagnol used semidefinite programming (SDP) and second-order cone programming (SOCP) to compute distributionally-robust (DRO) optimal designs. The distributionally robust approach can be seen as a robust counterpart of the Bayesian approach, where one optimises against the worst-case of all priors belonging to a family of probability distributions. Both these speakers talked about techniques to solving optimization problems not commonly used in the subfield of optimal designs and so attendees are now likely more equipped with additional tools to tackle tough optimization design problems, including high dimensional optimizationa design problems with many covariates.

New Researchers

A morning was devoted to presentations from graduate students and recent graduates.

Dr. Rui Hu – a freshly minted Ph.D. – presented her method of constructing robust sampling designs for the estimation of threshold probabilities in spatial studies. Her method, at the design stage, addressed the problem of possible model misspecification when estimating threshold probabilities which are critical in many disciplines such as ecology, epidemiology, etc. This work is a successful application of the theory of robust optimal design and sparked a lot of interest and discussion among participants.

The work presented by Dr. Kirsten Schorning (Feller, et al., 2017) discussed a new field of optimal design theory, namely the construction of optimal designs for several models, sharing some common parameter. Her results have many applications in pharmaceutical investigations where good designs for dose response curves with common parameters are required (for example the placebo effect in daily or twice daily treatment). In particular she determined optimal designs for a recently preformed Phase II clinical trial, where the goal goal was a comparison of two dose response curves corresponding to different treatment groups

A third talk was delivered by Dr. Maryna Prus (Prus and Schwabe, 2016), who investigated optimal design problems for random coefficient regression (RCR) models, which are frequently used in many statistical applications; especially in biosciences and medical research. In the case of multiple group RCR models individuals in different groups get different kinds of treatment and Dr. Maryna Prus determined the optimal group sizes for the prediction of the individual parameters in multi group RCR models with a common population mean for all individuals across all groups.

One of the young researchers scheduled to present at the meeting is Ahipasaoglu Selin from Singapore. She is an Assistant Professor in the Engineering Systems & Design Pillar at Singapore University of Technology and Design since April 2012. She received her doctorate at Cornell University from Prof. Michael Todd in 2009. Afterwards, she worked as a researcher at the Operations Research and Financial Engineering Department in Princeton University and the Management Science Group at London School of Economics. She is interested in Convex Optimization, Sparse Learning, Combinatorial Auctions, and Global Optimization. She is new to optimal design of experiments and was looking very forward to attend and meet top researchers in the field. However, being a Turkish citizen, her visa application was surprisingly denied. The organizers arranged to her present her talk via video and it was very well received. Her talk followed up on Drs. Zhou and Vandenburghe's talks on using semi-definite programming to solve difficult optimization design problems in statistics. She presented a new perspective that many in the audience had not seen before.

Special Event

The meeting featured an evening group presentation, with much audience participation, on the Future of Design of Experiements, with presenters Joachim Kunert, Dennis Lin, John Stufken and Boxin Tang, with Rainer Schwabe as moderator and discussant. Joachim emphasized the continuing role of the concepts of blinding, randomization and permutation tests. Dennis shared some of his personal views on the evolution of DOE since R. A. Fisher, and went on to talk about some of his recent work on Design for Order-of-Addition. John asked general questions about our relevance as a group, with suggestions for meeting current challenges. His comments regarding the potential of design of experiments in Big Data inspired comments on the use of such methods in methods – design and analysis – for the IT industry. Boxin took a look at some possible future developments in computer experiments, factorial designs and beyond, through the lenses of optimality, robustness and fusion. Probably more discussion was generated in this session than in any others. Some key topics raised were the need to interact with other disciplines, including computer science, and the need to be involved in big data and data science applications to help ensure scientific rigour through the application of design principles. Comments of Professor Lin regarding our collective and sometimes discouraging methods of statistical nomenclature led to the possible opening of a new avenue of research – 'heartfelt learning' – to exploit methodological links with AI.

Another point that was emphasised in this discussion was the ongoing need for the foundation ideas of experimental design, such as randomisation, replication, blinding and blocking. These are sometimes overlooked in new areas of research. All found the panel on the future of design of experiments extremely insightful. New ideas discussed in the panel, such as subsampling of Big Data, are highly relevant to current research projects and can open new angles to solve those problems.

Some take home messages from the workshop are

- All agreed that design is an important component of many scientific studies and these are well reported and reaffirmed in regulatory and scientific committee reports,
- Researchers in optimal design should work more collaboratively to realize a common goal - greater impact of the subfield of optimal experimental designs in scientific discovery,
- Discussions on how to work strategically to realize the common goal were discussed but there was no clear consensus,
- Some thought that it is increasingly important to do more interdisciplinary work and learn from related disciplines, such as engineering and computer science, in how they solve optimization problems.
- All agreed that the workshop we had at Banff was intellectually stimulating, very helpful in exchanging research ideas and in promoting the subfield of optimal design.

Participants

Bailey, Rosemary (University of St Andrews)
Biedermann, Stefanie (University of Southampton)
Chien, Peter (University of Wisconsin-Madison)
Dette, Holger (Ruhr-Universität Bochum)
Gao, Lucy (University of Washington)
Gilmour, Steven (Kings College London)
Harman, Radoslav (Comenius University)
Hu, Rui (MacEwan University)
Huang, Mong-Na Lo (National Sun Yat-sen University)
Kim, Seongho (Wayne State University)
Kim, Skim (University of Alberta)
Kunert, Joachim (TU Dortmund)
Lin, Devon (Queen's University)
Lin, Dennis (PennState)
Lopez-Fidalgo, Jesus (University of Navarre)
Lu, Xuewen (University of Calgary)
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Müller, Werner (Johannes Kepler Universität Linz)
Plagemann, Angela (Canadian Statistical Sciences Institute)
Pronzato, Luc (CNRS/Université de Nice–Sophia Antipolis)
Prus, Maryna (Otto von Guericke Universität Magdeburg)
Sagnol, Guillaume (Technical University of Berlin)
Schorning, Kirsten (Ruhr-Universität Bochum)

Schwabe, Rainer (Otto von Guericke Universität Magdeburg)
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Stufken, John (Arizona State University)
Tang, Boxin (Simon Fraser University)
Vandenberghe, Lieven (UCLA)
Waite, Timothy (University of Manchester)
Welch, Will (University of British Columbia)
Wiens, Douglas (University of Alberta)
Wong, Weng Kee (University of California, Los Angeles)
Woods, Dave (University of Southampton)
Wu, Jeff (Georgia Institute of Technology)
Wynn, Henry (London School of Economics & Political Science)
Xu, Xiaojian (Brock University)
Yang, Min (University of Illinois at Chicago)
Zhang, Zizhao (UCLA)
Zhigljavsky, Anatoly (Cardiff University)
Zhou, Julie (University of Victoria)

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Chapter 17

Women in Numbers 4 (17w5083)

August 13 - 18, 2017

Organizer(s): Chantal David (Concordia University), Michelle Manes (University of Hawaii at Manoa), Jennifer Balakrishnan (Boston University), Bianca Viray (University of Washington)

Conference at BIRS

Rationale and Goals

The first Women in Numbers workshop was held at BIRS in 2008, with the explicit goals of increasing the participation of women in number theory research and highlighting the contributions of women who were already doing high-quality research in the field. In their original proposal, the organizers cited the lack of representation of women in major institutions and at major international conferences.

Since that first workshop, 84 different women have participated in three WIN conferences at BIRS, with more than 60 other participants at the Women in Numbers – Europe conferences held at CIRM in Luminy and at the Lorentz Center. The number of women doing research in number theory is steadily growing. Though it is difficult to know exactly how much of this increase is due (directly or indirectly) to WIN, the visibility of the WIN conferences — due both to the community that has developed and the high quality of research output from these workshops — has helped to raise awareness in the broader community of the important contributions made by these researchers.

Another major indicator of the success of the WIN conferences is the publication output: three proceedings volumes containing a total of 36 papers (including some survey papers), as well as more than 10 journal articles published elsewhere. This far surpasses the output of a typical research workshop or graduate focused workshop such as the Arizona Winter School.

Less tangible but equally important outcomes include the lasting bonds formed between beginning researchers and their mentors, and the invigorating effect of the conferences on this mathematical community. WIN created a new model of working research conference designed to build supportive networks for women. Inspired by the success of WIN conferences, women in other fields have organized similar conferences at math institutes over the past few years, each focused on building collaboration groups consisting of senior and junior women in a given area. These include: Algebraic Combinatorixx and Women in Topology (WIT) at BIRS; Women in Shape (WiSh) at IPAM; and two Women in Applied Math conferences at IMA, Dynamical Systems with Applications to Biology and Medicine (WhAM!) and Numerical PDEs and Scientific Computing (WhAM2!). Each of these conferences has resulted in new, high-quality mathematics research as well as lasting collaborations among attendees.

These are positive changes, but our work is not done. Visibility of women at international number theory workshops and conferences, though increasing, continues to be low, with percentages of speakers who are women rarely exceeding 20%. Women are still underrepresented in major research universities, especially at the most prestigious institutions. Women receive disproportionately less grant money from the NSF than their peers. It is imperative that we continue to build on the success of the WIN and other research collaboration conferences for women, creating supportive networks for women at the early stage of their careers. The WIN4 workshop was the next step in this continuing story.

The specific goals of the workshop were:

- **to generate research in significant topics in number theory;**
- **to broaden the research programs of women and gender minorities working in number theory, especially pre-tenure;**
- **to train graduate students and postdocs in number theory, by providing experience with collaborative research and the publication process;**
- **to strengthen and extend a research network of potential collaborators in number theory and related fields;**
- **to enable faculty at small colleges to participate actively in research activities including mentoring graduate students and postdocs; and**
- **to highlight research activities of women in number theory.**

We would like to thank the following organizations for their support of this workshop: the Association for Women in Mathematics, BIRS, the Clay Mathematics Institute, Microsoft Research, the National Science Foundation, the Number Theory Foundation, and the Pacific Institute for the Mathematical Sciences.

Participants and Format

The focus of the workshop was on supporting new research collaborations within small groups. Before the workshop, each participant was assigned to a working group according to her research interests. Each group had two leaders chosen for their skill in both research and communication. These leaders designed projects and provided background reading and references for their groups. At the conference, there were a few talks, but most of the time was dedicated to working groups. Each group also submitted a short written progress report on their project. These reports, along with the project title and the names of the group members, are included in Section 17.

There were a total of forty-two participants (with sixteen of those project leaders[1]), all women or gender minorities. Of those, we had:

- **16 in tenured or tenure-track faculty positions,**
- **12 early-career faculty or postdocs,**
- **13 graduate students, and**
- **1 industry mathematician.**

Significant effort was made to enlarge the WIN community as much as possible: we had 4 project leaders who had never attended a WIN before and 24 out of 26 group members were first-time WIN-ers.

[1] 3 additional project leaders participated remotely.

Schedule

Sunday

16:00 – 17:30	**Check-in begins**
18:00 – 19:30	**Dinner**
20:00 – 22:00	**Informal Gathering**

Monday

07:00 – 08:45	**Breakfast**
08:45 – 09:00	**Introduction and Welcome by BIRS Station Manager**
09:00 – 10:00	**Introduction and Welcome by Organizers**
10:00 – 10:30	**Coffee Break**
10:30 – 12:00	**Group Work**
12:00 – 13:30	**Lunch**
13:30 – 14:30	**Group Work**
14:30 – 15:30	**Coffee Break**
15:30 – 16:30	**Group Work**
16:30 – 17:50	**Project Introductions**
18:00 – 19:30	**Dinner**

Tuesday

07:00 – 09:00	**Breakfast**
09:00 – 10:00	**Group Work**
10:00 – 11:00	**Coffee Break**
11:00 – 12:00	**Group Work**
12:00 – 13:30	**Lunch**
13:30 – 14:30	**Group Work**
14:30 – 15:30	**Coffee Break**
15:30 – 16:30	**Group Work**
16:30 – 17:50	**Project Introductions**
18:00 – 19:30	**Dinner**
20:30 – 21:30	**Panel Discussion**

Wednesday

07:00 – 09:00	**Breakfast**
09:00 – 10:00	**Group Work**
10:00 – 11:00	**Coffee Break**
11:00 – 12:00	**Group Work**
12:00 – 13:30	**Lunch**
13:30 – 17:30	**Free Afternoon**
18:00 – 19:30	**Dinner**
20:30 – 21:30	**Panel Discussion**

Thursday

07:00 – 09:00	**Breakfast**
09:00 – 10:00	**Group Work**
10:00 – 11:00	**Coffee Break**
11:00 – 12:00	**Group Work**
12:00 – 13:30	**Lunch**
13:30 – 14:30	**Group Work**
14:30 – 15:30	**Coffee Break**
15:30 – 16:30	**Group Work**
16:30 – 16:35	**Group Photo**
16:35 – 17:55	**Project Introductions**
18:00 – 19:30	**Dinner**

Friday

07:00 – 09:00	**Breakfast**
09:00 – 10:10	**Project Presentations**
10:15 – 10:45	**Coffee Break**
10:50 – 11:20	**Project Presentations**
11:30 – 12:00	**Checkout by Noon**
12:00 – 13:30	**Lunch**

Outcomes

The obvious outcomes of WIN4 are the research advances made by each of the nine working groups. As with the previous WIN workshops, these results will appear in peer-reviewed journals over the next few years. In addition, two of the organizers (Jennifer Balakrishnan and Michelle Manes) together with Amanda Folsom and Matilde Lalín will edit a Proceedings volume. We have recently signed a contract with Springer, and the Proceedings will appear in their AWM Series. We anticipate that each group will submit at least one paper — either a survey paper or a paper with new results — for peer review. We have announced a deadline of

March 31, 2018 for submissions, with the Proceedings appearing in 2019.

In our minds, more important than the specific mathematical results that emerge from the conference are the collaborations and mentoring that happens there. In particular, we held two after-dinner mentoring panels in the lounge at BIRS. The first was primarily for early-career mathematicians, focused on job applications, interviews, publishing, and grant proposals. The second was focused on long-range career planning, relevant to both early- and mid-career participants.

Only time will prove the success of the WIN4 workshop, but the excitement and energy there were palpable. Besides plans for the Proceedings volume, plans are already in place for a third WINE (Women in Number-Europe) conference and potential organizers for a WIN5 workshop are being contacted. We had a number of WIN4 participants volunteer to help keep the WIN network active, whether it be through organizing special sessions at AMS or AWM meetings, leading projects at future WIN/WINE workshops, editing future WIN proceedings volumes, or even organizing a future WIN or WINE workshop. We take the enthusiasm of these volunteers as additional evidence of the benefits of the mentoring and collaboration of that occurs at WIN workshops.

Project Reports

Horizontal distribution questions for elliptic curves over $\mathbb{Q}$

Participants: Chantal David, Ayla Gafni, Amita Malik, Lillian Pierce, Neha Prabhu, Caroline Turnage-Butterbaugh

Let E denote an elliptic curve over $\mathbb{Q}$. There are many "horizontal distribution" questions about elliptic curves in the literature, that is, questions about the properties of the reduced curves modulo primes p on average over the primes. Most of them remain open questions, such as the Lang-Trotter conjecture, or the Koblitz conjecture, while some of were recently solved, as the Sato-Tate conjecture. The following conjecture for "extremal primes" was considered recently by [25] and [26] (where James and Pollack prove the conjecture for CM elliptic curves in the second paper). For a non-CM elliptic curve, it is conjectured that

$$\#\{p \leq x : a_p(E) = [2\sqrt{p}]\} \sim \frac{16}{3\pi} \frac{x^{1/4}}{\log x}. \tag{17.0.1}$$

This conjecture is completely open, and there are not even non-trivial upper bounds for the number of such primes.

Similar to Birch's take on the Sato-Tate conjecture [5], one can look at the "vertical distribution" to get evidence for conjecture (17.0.1). This means to consider the number of curves over $\mathbb{F}_p$ such that $a_p(E) = [2\sqrt{p}]$ on average over the primes. In a recent paper, David, Koukoulopoulos and Smith [13] presented a general framework to address such vertical distribution, which makes it possible to attack the vertical distribution for extremal primes.

Our first goal is then to use the axiomatic framework of [13] to prove the "vertical distribution"

$$\sum_{p \leq x} \frac{\#\{E/\mathbb{F}_p : a_p(E) = [2\sqrt{p}]\}'}{p} \sim \frac{16}{3\pi} \frac{x^{1/4}}{\log x},$$

where the dash in the numerator indicates that we are counting the number of curves E up to isogeny.

A key ingredient that is needed to apply the results of [13] is to know that $[2\sqrt{p}]$ and p are independently well distributed modulo ℓ, i.e., a result of the type

$$\#\{p \leq x : [2\sqrt{p}] \equiv a \bmod \ell,\, p \equiv b \bmod \ell\} \sim \frac{\pi(x)}{\ell(\ell-1)}, \tag{17.0.2}$$

with a good error term with sufficient uniformity in ℓ.

This leads to the distribution of the fractional part $\{\sqrt{p}\}$. Indeed, it is easy to see that

$$[2\sqrt{p}] = k\ell + a \iff \left\{\frac{2\sqrt{p}}{\ell}\right\} \in \left(\frac{a}{\ell}, \frac{a+1}{\ell}\right).$$

The distribution of $\{\sqrt{p}\}$ has been studied by Balog [4].

The first goal of our project was then to generalize Balog's result to include more general intervals for $\{\sqrt{p}\}$, to include a congruence condition on the primes p, and with sufficient uniformity in ℓ. We started to work on that specific problem during the week at BIRS, and have made very good progress: the approach of Balog generalizes to this case, with more work to control the various parameters involved. This should lead to a vertical distribution for champion primes, and we are currently writing the details of the argument.

After looking at the vertical distribution, we also want to address the problem of finding a non-trivial upper bound for (17.0.1), and this will lead to mixed distribution between the fractional parts $\{\sqrt{p}\}$ and primes splitting in the Galois extensions obtained by adding the ℓ-torsion point of E to $\mathbb{Q}$, i.e. a mix between the results of Balog and the Chebotarev Density Theorem. This is analogous to (17.0.2), which can be seen as the independence between Balog's result on the equidistribution of fractional parts and the equidistribution of primes in arithmetic progressions (which is the Chebotarev Density Theorem for the cyclotomic extensions).

Apollonian circle packings

Participants: Holley Friedlander, Elena Fuchs, Piper Harron, Catherine Hsu, Damaris Schindler, Katherine Stange

Apollonian circle packings are fractal sets in the plane which are obtained by repeatedly adding circles into an initial constellation of three mutually tangent circles. Starting from such a collection of three mutually tangent circles C_1, C_2, C_3 one adds all circles that are tangent to all three of those. By a theorem of Apollonius there are exactly two such circles (viewing a line as a circle with infinite radius). Given those five circles one can continue the process of adding circles that are tangent to three of the circles that are already in the picture. We repeat this process ad infinitum and let $\mathcal{P}$ be the union of all the circles obtained in this way.

Apollonian circle packings are very interesting from a number theoretic point of view. By a theorem of Descartes, four mutually tangent circles C_1, C_2, C_3, C_4 with curvatures a, b, c, d fulfill the following quadratic equation

$$(a+b+c+d)^2 - 2(a^2+b^2+c^2+d^2) = 0.$$

From this equation one can deduce that if four mutually tangent circles C_1, C_2, C_3, C_4 in the starting configuration have integer curvatures, then all curvatures in the packing $\mathcal{P}$, as constructed above, are integers. This gives rise to a number of questions on arithmetic properties of the set of curvatures that appear in such a packing.

It is known that for a fixed primitive integral packing $\mathcal{P}$ not all integers $m \in \mathbf{N}$ can occur as the curvature of a circle in the packing. As Fuchs showed in [21], for every fixed packing $\mathcal{P}$ there is a non-trivial subset of residue classes modulo 24, say $\Sigma \subset \mathbf{Z}/24\mathbf{Z}$, to which the curvatures are restricted, and this is the only congruence obstruction. The local-global conjecture for Apollonian packings states that every sufficiently large integer m with $m \bmod 24 \in \Sigma$, also occurs as a curvature in $\mathcal{P}$. Building on work of Sarnak [35] and Bourgain and Fuchs [6], Bourgain and Kontorovich showed in [7] that the local-global conjecture holds for almost all positive integers $m \leq X$, with an exceptional set of size at most $O(X^{1-\varepsilon})$ for some positive ε.

The goal of our project is to understand the local-global conjecture for certain subcomponents of a primitive integral packing $\mathcal{P}$. We start by constructing a "tree" $T_{\mathcal{P},r}$ (it isn't quite a tree) of tangent circles inside a primitive integral Apollonian packing $\mathcal{P}$. Fix a positive integer r and two tangent circles C_a, C_c in the packing of curvatures a and c, such that a and c are both r-almost primes (their existence follows from [35]). We begin drawing $T_{\mathcal{P},r}$ by drawing the (tangent) circles C_a, C_c of curvature a and c. Now, add into $T_{\mathcal{P},r}$ all those circles in $\mathcal{P}$ that are tangent to either C_a or C_c. Of those, let $W_{r,1}$ be the set of the circles in

$T_{\mathcal{P},r}$ so far that have r-almost prime curvature, and add into $T_{\mathcal{P},r}$ all circles tangent to a circle in $W_{r,1}$. Of these new circles, let $W_{r,2}$ be the set of the circles that have r-almost prime curvature, and add into $T_{\mathcal{P},r}$ all circles tangent to a circle in $W_{r,2}$. Continue this process indefinitely (this process does indeed continue indefinitely as shown in [35]) to construct the tree $T_{\mathcal{P},r}$. Note that if $r = 1$, when we delete from $T_{\mathcal{P},r}(\mathbf{v})$ all of the circles whose curvatures are not prime we will get precisely a full prime component as described at the end of [35].

Our question is then, does there exist an r so that the resulting curvatures of circles in $T_{\mathcal{P},r}$ must satisfy a local-global principle?

During the workshop we worked on numerical experiments to test such a conjecture and we discussed ways to define Cayley-like graphs (which play an important role in [7]) for r-almost prime trees. Moreover, we worked on lower bounds on the number of curvatures less than X appearing in an r-almost prime component. Following the methods in Bourgain and Fuchs [6], we expect that we can show that the number $N(X)$ of curvatures of size at most X that appear in the r-almost prime tree $T_{\mathcal{P},r}$ is at least

$$N(X) \gg \frac{X}{(\log\log X)^{1/2}}.$$

We hope to improve this to a positive density result in the future.

Arithmetic dynamics and Galois representations

Participants: Jamie Juul, Holly Krieger, Nicole Looper, Michelle Manes, Bianca Thompson, Laura Walton

Let K be a number field and $f \in K(z)$ a rational function of degree $d \geq 2$. Let T_f be the infinite preimage tree rooted at 0, with absolute Galois group G_f acting as a subgroup of the automorphisms of T_f. There is a growing body of literature on these arboreal Galois representations. In particular, Joes [28] has a specific conjecture for quadratic functions:

Conjecture1 1 *Let K be a number field, and $f(z) \in K(z)$ have degree 2. Then the index $[\mathbf{Aut}(T_f) : G_f]$ is finite unless one of the following holds:*

1. *f is post-critically finite*
2. *f commutes with a non-trivial Möbius transformation that fixes 0*
3. *the critical points c_1, c_2 of f satisfy $f^r(c_1) = f^r(c_2)$ for some $r \geq 2$*
4. *0 is a periodic point of f*

Cases of this conjecture have been dealt with by Jones [28], Jones-Manes [29], and others. It is known that if any of these conditions holds, then the arboreal Galois representation cannot be finite index. Our plan was to investigate this conjecture with an eye towards a very difficult question:

Question1 1 *$\mathcal{F} = \{f_t\}$ be a one-parameter family of quadratic rational maps, defined over $\overline{\mathbb{Q}}$, and a_t a marked point defined over $\overline{\mathbb{Q}}$. Let T_t be the infinite preimage tree rooted at a_t and G_t the absolute Galois group, as a subgroup of* $\mathbf{Aut}(T_t)$. *When is*

$$S_{\mathcal{F}} := \{t : [\mathrm{Aut}(T_t) : G_t] = \infty\}$$

a set of bounded height?

It is known by Pink [33] that the set $S_{\mathcal{F}}$ will <u>not</u> have bounded height if any of the following conditions are met:

- if the family is a curve of quadratic rational maps defined by a specific critical orbit relation — specifically, there is some iterate $r > 0$ such that the critical points c_1, c_2 satisfy $f^r(c_1) = f^r(c_2)$ and $a_t \equiv 0$,

- **if the family is a curve for which a_t is periodic, or**
- **if f commutes with a non-trivial Möbius transformation.**

However, there is in all those cases a reasonable alternative to the full automorphism group (for example in the latter case, the centralizer of Aut(f) in Aut(T_f)) in which the Galois group has finite index, and the question above can be modified accordingly. The set of post-crticially finite maps in the space of quadratic rational maps is known to have bounded height by work of Benedetto-Ingram-Jones-Levy. Hence the question can be viewed as a weaker version of the conjecture; however, the question could be refined to search for a subgroup for which the index is finite (as in the case of the centralizer for f with nontrivial automorphisms).

During our week at BIRS, we investigated a particular family of quadratic rational functions. We proved some results related to height bounds on the parameter outside of which we had an attracting fixed point that would control much of the dynamics, and for several particular parameter choices we were able to prove that the image of Galois is the full automorphism group of the tree. We also proved that assuming Vojta's Conjecture, a similar set of conditions to those in Conjecture 1 characterizes the set of cubic polynomials $f \in K[x]$ such that $[\mathbf{Aut}(T) : G_K(f)] < \infty$, where K is a number field. To do so, we make use of a result from [24]. Specifically, we have the following.

Theorem 17.0.1 ***Let K be a number field, and let $f \in K[x]$ have degree $d \geq 2$, where d is prime. Suppose $f^n(x)$ has at most r irreducible factors over K as $n \to \infty$, so that***

$$f^{N+n}(x) = f_{N,1}(f^n(x))f_{N,2}(f^n(x)) \cdots f_{N,r}(f^n(x))$$

is the prime factorization of $f^{n+N}(x)$ in $K[x]$ for any sufficiently large N. Suppose that there is an M such that the following holds: for all $n \geq M$, and for all $1 \leq j \leq r$, there is a multiplicity one critical point γ_1 of f and a prime $\mathfrak{p}_n$ of K such that:

- $v_{\mathfrak{p}_n}(f_{N,j}(f^n(\gamma_1))) = 1$
- $v_{\mathfrak{p}_n}(f_{N,j}(f^n(\gamma_t))) = 0$ ***for all*** $t \neq 1$
- $v_{\mathfrak{p}_n}(f^m(\gamma_t)) = 0$ ***for all critical points γ_t, and all*** $m < N + n$
- $v_{\mathfrak{p}_n}(d) = 0$.

Then $[\mathbf{Aut}(T) : G_K(f)] < \infty$.

Chabauty-Coleman experiments on genus three hyperelliptic curves

Participants: Jennifer Balakrishnan, Francesca Bianchi, Victoria Cantoral-Farfán, Mirela Çiperiani, Anastassia Etropolski

Let C be a hyperelliptic curve of genus $g = 3$ defined over $\mathbb{Q}$. An affine model for C can be written in the form $y^2 = f(x)$, where $f(x)$ is a polynomial with rational coefficients and $7 \leq \deg f(x) \leq 8$. As a special case of Faltings' theorem, we know that $C(\mathbb{Q})$, the set of rational points on C, is finite; moreover, by the Mordell-Weil theorem, the set of rational points of the Jacobian J of C forms a finitely generated abelian group, i.e., $J(\mathbb{Q}) \simeq \mathbb{Z}^r \oplus T$.

When $r < g$, the Chabauty-Coleman method [12] can be used to help compute $C(\mathbb{Q})$. The method produces a finite set of p-adic points on C containing $C(\mathbb{Q})$, via the construction of p-adic integrals of annihilating regular 1-forms. In the case when $r = 1$ (and $g = 3$), the method produces two independent regular 1-forms α, β whose (p-adic) Coleman integrals can be used to compute $C(\mathbb{Q})$. It is of interest to see what other points on C the integrals of α and β cut out.

Indeed, under these hypotheses (hyperelliptic curves with $g = 3$ and $r = 1$), our goal is to produce algorithms that would allow us to use the Chabauty-Coleman method to compute the rational points of

any such curve. One fully worked out example of this technique appears in the Ph.D. thesis of Wetherell [39] where he does various computations on the Jacobian of a genus 3 hyperelliptic curve to carry out Chabauty-Coleman; our goal is to automate this method, while replacing the computations on the Jacobian with computations directly on the curve.

We now also fix a prime p of good reduction and a naive height bound $B \in \mathbb{N}$. We first determine the set $S_B \subseteq C(\mathbb{Q})$ of points of naive height bounded by B. We proceed if S_B is non-empty. Suppose, for ease of exposition, that we have $\deg f(x) = 7$. Fix the embedding $C \hookrightarrow J$ that sends P to the class $[(P) - (\infty)]$ of the divisor $(P) - (\infty)$. If $P \in C(\mathbb{Q})$, then $Q := P - \infty$ gives us a rational point in the Jacobian of C, and if Q is non-torsion, by computing the three Coleman integrals $\int_\infty^P \omega_0$, $\int_\infty^P \omega_1$, $\int_\infty^P \omega_2$, where $\{\omega_i = \frac{x^i dx}{2y}\}$ forms a basis of $H^0(C, \Omega^1)$, we may determine α and β.

We then consider the indefinite Coleman integrals of α and β as p-adic power series over all residue disks of C. The zero locus of these integrals gives a set $C(\mathbb{Q}_p)_1 \subseteq C(\mathbb{Q}_p)$ such that $C(\mathbb{Q}) \subseteq C(\mathbb{Q}_p)_1$. We want to determine, in practice, how much larger $C(\mathbb{Q}_p)_1$ is than $C(\mathbb{Q})$, and further, if we can describe $C(\mathbb{Q}_p)_1$ in terms of $C(\mathbb{Q})$ and other arithmetically relevant p-adic points on C.

During our week at BIRS, we implemented the algorithm described above to determine $C(\mathbb{Q}_p)_1$ and studied the data produced by a short list of curves pulled from the genus 3 database of hyperelliptic curves soon to be in the L-functions and Modular Forms DataBase (LMFDB) [37]. We plan on running these algorithms on further curves in the LMFDB and analyzing the resulting data.

Computational aspects of supersingular elliptic curves

Participants: Efrat Bank, Catalina Camacho, Kirsten Eisenträger, Jennifer Park

Project Goal: Our goal during the week at BIRS was to solve the following problem:

Problem: Given a supersingular elliptic curve, compute its endomorphism ring.

Setup and context: One of the methods for computing the endomorphism ring is to study the ℓ-isogeny graph of supersingular elliptic curves defined over a field of characteristic p, where p and ℓ are distinct primes. Kohel's thesis [30] outlines an algorithm to try to compute this endomorphism ring, but it is not completely explicit in the sense that some parts of the arguments are only sketched. Thus, we hope to make this algorithm more concrete, and prove the existence of some extra features in the ℓ-isogeny graph that could eventually aid in the efficiency of the algorithm to computing the endomorphism rings.

The setup that we will consider is as follows: For distinct primes p and ℓ, we can define the ℓ-isogeny graph $G(p, \ell)$ as follows:

1. **The vertex set V is the set of isomorphism classes of supersingular elliptic curves over the finite field $\mathbf{F}_{p^2}$. (This gives us, up to isomorphism, all supersingular elliptic curves in characteristic p because every supersingular elliptic curve has a model defined over $\mathbf{F}_{p^2}$.) We can think of the vertices labeled by the j-invariants of the supersingular elliptic curves. The number of vertices is roughly $p/12$ and depends on the congruence class of p modulo 12 (see [36]).**

2. **The edge set can be described as follows: given a supersingular j-invariant j, write down an elliptic curve E in Weierstrass form with $j(E) = j$. This can be done explicitly.**

   Choose a subgroup H of E of order ℓ, and let E' be the elliptic curve $E' = E/H$. Now connect the vertices $j = j(E)$ and $j' = j(E')$ by an edge, which represents the ℓ-isogeny with torsion H. Since the ℓ-torsion on the elliptic curve E is isomorphic to $\mathbf{Z}/\ell\mathbf{Z} \times \mathbf{Z}/\ell\mathbf{Z}$, there are $\ell + 1$ subgroups of order ℓ. Hence, with this construction, we get an $\ell + 1$-regular directed graph, and the edges leaving a vertex $j(E)$ are labelled by subgroups of E of order ℓ. The graph is a multigraph, which means that two vertices can be connected by more than one edge, and allows self-loops.

In this setup, a loop beginning and ending at a vertex j can be interpreted as an element of the endomorphism ring of the supersingular elliptic curve E with $j(E) = j$.

By Deuring's theorem, each endomorphism ring corresponds to a maximal order in the unique quaternion algebra ramified at p and ∞. When one can identify the maximal order corresponding to the endomorphism ring of a particular elliptic curve, we say that the endomorphism ring has been computed.

By identifying the elements of the quaternion algebra corresponding to these loops, one has a chance of finding the generators of the maximal order, according to Kohel's thesis [30, Chapter 7].

Parts of the project that were finished during the week at BIRS: During the week, we were able to:

1. Find a short list of at most eight elements of the quaternion algebra that could correspond to a given loop;

2. For certain primes ($p = 31, 101, 103$), identify the maximal orders attached to all of the supersingular elliptic curves over $\mathbf{F}_p$ by using the above short list;

3. Identify some criteria for when two given loops generate an order of rank < 4;

4. Identify some criteria for when two given loops generate a non-maximal order;

5. Conjecturally identify some criteria for when two given loops generate an order of rank 4.

Quantum modular forms and singular combinatorial series

Participants: Amanda Folsom, Min-Joo Jang, Susie Kimport, Holly Swisher

Mock modular forms are complex-valued functions which share similar transformation properties to ordinary modular forms with respect to the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper-half complex plane $\mathbb{H}$, but gain their modular transformation properties on $\mathbb{H}$ at the expense of losing their holomorphic properties there. The theory of mock modular forms, and the more overarching theory of harmonic Maass forms, has largely developed during this century [9]; however, their origins trace back to both the original Maass forms and Ramanujan's mock theta functions from the early-to-mid 1900s. The development of the theory of harmonic Maass and mock modular forms has been of great importance within the theory of modular forms, itself a major area of research in Number Theory related to the Langlands Program, the BSD Conjecture, and Fermat's Last Theorem, for example. Applications of harmonic Maass and mock modular forms have also emerged in the diverse areas of Mathematical Physics, Representation Theory, Topology, and more.

Quantum modular forms were defined by Zagier in 2010 [40]; they are similar to mock modular forms in that they feign modularity in some way, with the notable exception that their domain is not $\mathbb{H}$, but rather $\mathbb{Q}$. More precisely, a <u>weight k quantum modular form</u> ($k \in \frac{1}{2}\mathbb{Z}$) is a complex-valued function f on $\mathbb{P}^1(\mathbb{Q}) \setminus S$, for some appropriate S, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, the functions $h_\gamma(x) = h_{f,\gamma}(x) := f(x) - \varepsilon^{-1}(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$ satisfy a 'suitable' property of continuity or analyticity in $\mathbb{R}$ (such as real analyticity). Here, the $\varepsilon(\gamma) \in \mathbb{C}$ satisfy $|\varepsilon(\gamma)| = 1$.

Questions of interest to many (see for example [9, 18, 19, 22, 23, 27, 40]) have been to understand spaces of quantum modular forms, determine explicit examples of and sources of quantum modular forms, and to understand the relationship, if any, between quantum modular and mock modular forms. Rational cusps are a natural boundary to the fundamental domain of a mock modular form, so on one hand, the latter problem is a natural one to study. On the other hand, a relationship is not immediate as the domains and analytic properties of mock and quantum modular forms are a priori different.

In this project, we address the above questions in our study of a $(k+1)$-variable combinatorial generating function $R_k(w_1, w_2; \ldots, w_k; q)$, $k \geq 1$, for k-marked Durfee symbols introduced by Andrews [3]. Historically, combinatorial functions are known to occasionally be a source of automorphic functions; likewise, such automorphic properties can sometimes be used to prove combinatorial theorems. Andrews' celebrated $(k+1)$-variable function R_k is a vast generalization of the one-variable combinatorial generating function for integer partitions, an ordinary modular form whose automorphic properties were famously studied and developed along with the Circle Method by Hardy and Ramanujan. The automorphic properties of R_k when viewed as a function of $\tau \in \mathbb{H}$ with $q = e^{2\pi i \tau}$ and fixed w_j have slowly been uncovered in a series of papers [8, 10, 17], the last of which was authored by participants Folsom and Kimport. R_k indeed exhibits "mock" behavior, though the exact type of transformations it exhibits on $\mathbb{H}$ depend on k, and also the w_j. Our project seeks to understand the quantum properties of R_k, if any, on $\mathbb{Q}$.

A number of complications arose in our study. First, it is unclear that R_k is even defined when viewed as a function on $\mathbb{Q}$. Second, the automorphic properties of R_k on $\mathbb{H}$ depend on k and the w_j. Third, showing errors to modularity extend to analytic functions on $\mathbb{R}$ is non-trivial. Barring these obstacles, our strategy is to ultimately make use of the mock-automorphic transformation properties possessed by R_k on $\mathbb{H}$, and to use analytic continuation to obtain quantum transformation properties on $\mathbb{Q}$.

To this end, our first result establishes a quantum set of rationals on which R_k is defined; this involves working with a $(k+1)$-fold q-hypergeometric series related to R_k by work of Andrews [3]. We then work with general automorphic transformation properties on $\mathbb{H}$ established by Folsom and Kimport in [17] to produce explicit errors to modularity exhibited by R_k. The errors which emerge are in terms of the non-holomorphic function

$$S(z;\tau) := \sum_{n\in\frac{1}{2}+\mathbb{Z}} \left(\operatorname{sgn}(n) - E\left(\left(n+\tfrac{\operatorname{Im}(z)}{\operatorname{Im}(\tau)}\right)\sqrt{2\operatorname{Im}(\tau)}\right)(-1)^{n-\frac{1}{2}}\, e^{-2\pi i n z} q^{-\frac{n^2}{2}}\right),$$

where $E(z) := 2\int_0^z e^{-\pi t^2}\,dt$. In certain settings, our errors to modularity in fact involve limiting versions of S, further complicating their understanding. We next transform our errors to modularity from functions involving the non-holomorphic S to period integrals $\int_{a/b}^{i\infty} g(\rho)(-i(\tau+\rho))^{-\frac{1}{2}}\,d\rho$, where the g are ordinary modular forms, and $\frac{a}{b}\in\mathbb{Q}$. We are also able to turn the limiting versions of S which appear into derivatives of period integrals after working with properties of S as proved in [41], among other things. We then employ delicate analytic arguments to deduce that the errors to modularity indeed extend to analytic functions in $\mathbb{R}$, and use analytic continuation to establish that the R_k is a quantum modular form. As corollaries to our results, we also obtain – non-trivially – exact formulas for period integrals of modular forms and their derivatives as evaluations of simple finite q-hypergeometric multi-sums when parameters are specialized to roots of unity.

Newton polygons of cyclic covers of the projective line

Participants: Wanlin Li, Elena Mantovan, Rachel Pries, Yunqing Tang

A fundamental problem in arithmetic geometry is understanding which abelian varieties arise as Jacobians of (smooth) curves. This question is equivalent to studying (the interior of) the Torelli locus in Siegel varieties.

In positive characteristic p, there are an abundance of discrete invariants associated with abelian varieties, e.g., the p-rank, the Newton polygon, and the Ekedahl–Oort type. These invariants give information about the Frobenius morphism and the number of points of the abelian variety defined over finite fields. It is a natural question to ask which of these invariants are realized by Jacobians. As each type of discrete invariant yields a stratification of the reduction modulo p of the Siegel variety, this problem is equivalent to understanding which strata intersect the Torelli locus, and its interior.

Ultimately, one would like to understand the geometry of the induced stratifications of the Torelli locus (e.g., the connected components of each stratum and their closure), in the same way that the geometry of the corresponding stratifications of Siegel varieties is understood. For example, in [16], Faber and van der Geer prove that for any genus g and prime p, the p-rank strata are non-empty and have the appropriate codimension. See also [1] and [2].

In the case of the Newton polygon, more precisely the Newton polygon of the characteristic polynomial of Frobenius, much less is known beyond genus 3. Most interestingly, in 2005, Oort observed that for genus $g \geq 9$, a dimension count suggests that it is unlikely for all Newton polygons to occur for Jacobians.

This project focuses on understanding the Newton polygons of cyclic covers of the project line. In [15], for any family T of cyclic covers of the projective line, Deligne and Mostow construct a PEL-type Shimura variety containing T. In [31], Moonen shows that there are precisely twenty families of cyclic covers of the

projective line which give rise to special subvarieties of Siegel varieties, and each of these twenty special families agrees with the associated Shimura variety constructed in [15]. The Newton polygon and Ekedahl–Oort stratifications of PEL-type Shimura varieties are well understood by the work of Viehmann and Wedhorn [38]. Combining these results thus enables us to compile the list of all Newton polygons and Ekedahl–Oort types of Jacobians in each of the twenty special families in [31]. (Some of the necessary computations were carried out using MAGMA.)

Given this list, we proceed to investigate which invariants arise for smooth curves in the families. For the Newton polygon corresponding to the open stratum (i.e., the μ-ordinary polygon) the answer is always affirmative. Beyond that it is less clear. In the cases when the closed stratum (which corresponds to the Newton polygon called basic) has codimension 1, a count of the number of connected components of the boundary implies that the answer is affirmative for sufficiently large primes. These two results yield several new examples of Newton polygons of Jacobians of smooth curves for low genera for infinitely many p.

Finally, to attack the same questions for arbitrarily large genera, we develop a new induction argument, similar in flavor to the one initially used in [16] for p-ranks and its most recent refinement for Newton polygons in [34]. The value of the induction argument is that it yields information about the geometry of the Newton polygon strata for higher genus, starting from geometric information for the strata in low genus. This yields new occurrences for Newton polygons of smooth cyclic covers of the projective line varying in non-special families. Unfortunately, one hypothesis is in general not stable under the inductive argument, and so far our result only yields finitely many new Newton polygons. As we have not yet exhausted all possible settings, there is still hope to construct new examples for arbitrary large genera. Either way, we expect to establish the non-emptiness of many/most/all Newton polygon strata for genera $g \leq 8$ under certain congruence conditions on the prime p.

Ramanujan graphs in Cryptography

Participants: Anamaria Costache, Brooke Feigon, Kristin Lauter, Maike Massierer, Anna Puskás

Our WIN4 group studied the security of a new proposal for Post-Quantum Cryptography (PQC) from both a number theoretic and cryptographic perspective. National Institute of Standards and Technology (NIST) will be running an international competition over the next few years to select a new system for PQC. One of the possible candidates is based on the hardness of finding isogenies between supersingular elliptic curves. This hard problem was first proposed by Charles-Goren-Lauter in 2006 ([11]) as the basis for a new cryptographic hash function construction. A Pizer graph is the isogeny graph of supersingular elliptic curves, $SS(\ell, p)$, which can be interpreted in terms of Brandt matrices representing Hecke operators acting on spaces of modular forms. The idea behind the cryptographic applications is to use the hardness of finding paths in these Ramanujan graphs (or inverting random walks) as a way to construct a one-way function. The hard problem is then to find paths in the graph, given the starting and ending point. In the same paper, Charles-Goren-Lauter also proposed a PQC system based on another family of Ramanujan graphs, those of Lubotzky-Phillips-Sarnak (LPS). A 2008 paper by Petit-Lauter-Quisquater ([32]) breaks the hash function based on LPS graphs, whereas recent work has built on the hardness of finding isogenies between supersingular elliptic curves. In particular, in [14] De Feo-Jao-Plût proposed a cryptographic system based on supersingular isogeny Diffie-Hellman as well as a set of five hard problems.

Our group had two goals related to the crypto-systems on these Ramanujan graphs. On the one side we wanted to study the weaknesses and reductions in the problems proposed by De Feo-Jao-Plût. On the other side we wanted to study the relation between the Pizer and LPS graphs by viewing both from a number theoretic perspective.

For the first goal, we started to understand the relationships between the 5 problems posed by De Feo-Jao-Plût and to relate them to the hardness of the Supersingular Isogeny Graph problem which is the foundation for the CGL hash function.

For the second goal, we started explicating the relation between LPS graphs and Pizer graphs. The vertex set of both of these graphs can be viewed as a set of double cosets obtained from the adelic points of the multiplicative group of a particular quaternion algebra, B, acted on by an order of the algebra.

The particular algebra and the order one must choose depend on the graphs. For LPS graphs we must fix $B = B_{2,\infty}$, the quaternion algebra over $\mathbb{Q}$ that is ramified at 2 and ∞, and choose a non-maximal order that will depend on a prime p. Whereas for the Pizer graphs we choose $B = B_{p,\infty}$, the quaternion algebra over $\mathbb{Q}$ that is ramified at p and infinity, and take the maximal order in B. Thus we were able to see that the Pizer and LPS considered in [11] are not isomorphic. However, there are generalizations of both of these graphs that we believe do come from the same choices of quaternion algebras and orders. We are continuing to explore how explicit we can make this isomorphism and whether or not this can provide any insight into why the hash-function for LPS graphs has been broken, but the one for Pizer graphs has not been.

Torsion structures on elliptic curves

Participants: Abbey Bourdon, Özlem Ejder, Yuan Liu, Frances Odumodu, Bianca Viray

A result of Frey, which relies on Faltings's theorem about rational points on subvarieties of abelian varieties, states that a curve $C_{/\mathbb{Q}}$ has infinitely many points of degree at most d only if $2d$ is at least the $\mathbb{Q}$-gonality of C [20]. Moreover, the proof shows that if C has infinitely many points of degree at most d and $d < \text{gon}_{\mathbb{Q}}(C)$, then the image of Sym$^d(C)$ in Jac(C) contains a translate of a positive rank subabelian variety. One can think of Frey's result as saying that there are infinitely many points of degree at most d only when there is an infinite family parametrizing them (e.g., a $\mathbb{P}^1$ or a positive rank abelian variety).

Degree d points that are not parametrized by an infinite family are less understood. Such points are called sporadic; precisely, a closed point x on a curve C is sporadic if C has only finitely many closed points of degree at most $\deg(x)$. In our project, we study sporadic points on the modular curves $\{X_1(N)\}_{N\geq 1}$, whose non-cuspidal points correspond to isomorphism classes of pairs of elliptic curves with a marked point of order N. In particular, we seek to better understand the properties of elliptic curves that can give rise to sporadic points on $X_1(N)$.

Let $E_{/\mathbb{Q}}$ be an elliptic curve and let ℓ be a prime. If there is a sporadic point $x \in X_1(\ell)$ with $j(x) = j(E)$, then E must achieve a rational point of order ℓ in unusually low degree. This in turn implies that the image of the mod ℓ Galois representation must be unusually small. Thus one may anticipate that elliptic curves with complex multiplication (CM) are good candidates for producing sporadic points, and indeed any CM elliptic curve will give rise to infinitely many sporadic points in $\bigcup_\ell X_1(\ell)$. In contrast, if $E_{/\mathbb{Q}}$ is a non-CM elliptic curve, Serre's Uniformity Conjecture states that the image of the mod ℓ Galois representation associated E is surjective for all primes $\ell > C$, where C is a constant that does not depend on E (a standard guess is that $C = 37$). If true, this would imply there are only finitely many non-CM non-cuspidal sporadic points on $\cup_\ell X_1(\ell)$ that correspond to elliptic curves E with $j(E) \in \mathbb{Q}$. This observation inspires the following question:

Question 17.0.2 ***Does there exist an absolute constant C such that if $N > C$, there are no non-CM non-cuspidal sporadic points on $X_1(N)$ corresponding to elliptic curves with rational j-invariant?***

More generally, one may ask:

Question 17.0.3 ***Fix a positive integer d. Does there exist a constant $C = C(d)$ such that if $N > C$, there are no non-CM non-cuspidal sporadic points x on $X_1(N)$ with $[\mathbf{k}(j(x)) : \mathbb{Q}] \leq d$, where $j\colon X_1(N) \to X(1)$ denotes the natural map to $X(1) = \mathbb{P}^1_j$?***

During our week at BIRS, our group succeeding in proving the following partial affirmative answer to this question. We write Spor(N) for the closed subset of $X_1(N)$ consisting of all sporadic points on $X_1(N)$.

Theorem 17.0.4 ***Fix a number field k and assume Serre's Uniformity Conjecture for k. Then there exists a positive integer $A = A(k)$ such that***

$$j\left(\bigcup_{N\in\mathbb{N}} \mathbf{Spor}(N)\right) \cap \mathbb{P}^1_j(k) \subset j\left(\bigcup_{N\mid A} \mathbf{Spor}(N)\right).$$

In particular, the set of k-rational j-invariants of sporadic points is finite.

Moreover if the constant in Serre's Uniformity conjecture can be taken to depend only on the degree d of k for all number fields k of degree d, then the same is true for A. In particular, then there are only finitely many j-invariants corresponding to sporadic points x with $[\mathbf{k}(j(x)) : \mathbb{Q}]$ bounded.

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Chapter 18

Future targets in the classification program for amenable C*-algebras (17w5127)

September 3 - 8, 2017

Organizer(s): George Elliott (University of Toronto), Zhuang Niu (University of Wyoming), Aaron Tikuisis (University of Ottawa)

Overview of the Field

C*-algebra theory is a branch of mathematics with connections to other areas such as topology, symbolic and measurable dynamics, harmonic analysis, number theory, differential geometry, set theory, and geometric group theory. A C*-algebra is an algebra of bounded operators on Hilbert space, that is closed under taking adjoints and in the operator norm topology. Operator algebras – C*-algebras and von Neumann algebras – arose in the mathematical rigorization of quantum mechanics in the 1920s and '30s.

Constructions of C*-algebras from other mathematical objects, such as groups or dynamical systems, appeared shortly after the original physics-inspired inception of C*-algebras. These constructions encode interesting information about the input objects: e.g., for groups the C*-algebra captures unitary representation theory; for dynamical systems it concerns the orbit-equivalence relation. Many other constructions have since arisen, producing C*-algebras associated to such objects as directed graphs, foliations, coarse metric spaces, integral domains, higher-rank graphs, and C*-dynamical systems. Understanding the structure of such C*-algebras, and how this structure reflects the input, has been a driving force behind C*-algebra research.

The classification program is a major research direction in C*-algebra theory, with the aim of showing that C*-algebras which agree on K-theoretic (and thereby computable) invariants are isomorphic. K-theory for C*-algebras is a generalization of topological K-theory; it is also necessary to include the simplex of tracial states and its pairing with the K_0-group in the K-theoretic invariants used for classification. The classification program systematically tackles the question of the structure of C*-algebras, since a given abstract algebra will then be isomorphic to a model with known structure – constructed to have the same invariant.

Classification of C*-algebras has its origins in the work of Dixmier and Glimm in the '60s; in the early '90s, Elliott brought it to prominence as a problem in its own right, and it has subsequently seen tremendous development. Much of the recent focus has been on simple, separable, amenable, unital, infinite dimensional C*-algebras; the nonunital case has also seen very recent developments. The following are the major turning points in the theory, which have set the foundations for exciting recent developments:

- **The classification, by Bratteli and Elliott, of approximately finite dimensional (AF) C*-algebras [2, 6].**
- **The development by Brown–Douglas–Fillmore and Kasparov of KK-theory, a bivariant version of K-theory [3, 18].**
- **Elliott's conjecture of the classifiability of simple separable amenable unital C*-algebras [7].**
- **The Kirchberg–Phillips classification of purely infinite simple separable amenable unital C*-algebras which satisfy the Universal Coefficient Theorem (UCT) [29].**
- **The unexpected discovery of simple separable amenable unital C*-algebras which cannot be distinguished by traditional K-theoretic invariants, with obstructions related to high topological dimension [45, 46, 47, 36, 42, 38].**
- **The Toms–Winter conjecture, postulating, in response to the last point, a robust characterization of low topological dimension for simple separable amenable unital C*-algebras (see more in Recent Developments and Open Problems.) [44, 10]**
- **Tracial approximation, pioneered by Lin based on results of Elliott, Gong, and L. Li (see [23, 8]), and on Popa's local quantization ([31]).**

Recent Developments

The recent developments in C*-algebra theory may be viewed in the context of three streams:

***Classification*. Classification, underpinned by the concept of tracial approximation, culminated recently in the following:**

Theorem 18.0.1 (Elliott, Gong, Lin, Niu) ***[9] Let A, B be simple unital infinite dimensional separable amenable C*-algebras with finite nuclear dimension and for which all traces are quasidiagonal, which satisfy the Universal Coefficient Theorem for KK-theory, and which have a non-zero trace. Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.***

In the above theorem, the Elliott invariant, $\mathrm{Ell}(A)$ of a C*-algebra A is

$$\mathrm{Ell}(A) := (K_0(A), K_0(A)_+, [1_A]_{K_0(A)}, K_1(A), T(A), \rho_A : K_0(A) \times T(A) \to \mathbb{R}).$$

The hypotheses of finite nuclear dimension and the Universal Coefficient Theorem are discussed below. The case with no non-zero trace was dealt with by Kirchberg and Phillips (see [20, 29]). (In both cases, the result followed a significant amount of earlier work, such as [35, 24, 25, 50, 51].)

In fact, this result came about in two stages. In the first stage, Gong, Lin, and Niu proved classification of $\mathcal{Z}$-stable C*-algebras which satisfy certain tracial approximation hypothesis. Then, using a key result of Winter, it was proven that the hypotheses of the above theorem are sufficient to establish the tracial approximation hypothesis.

These ideas have been pushed further, yielding certain (but not yet complete) classification results in the nonunital setting. For instance, Elliott, Gong, Lin, and Niu have classified simple separable C*-algebras

with finite nuclear dimension which are KK-contractible (UCT with zero K-groups) – the Elliott invariant in this (non-unital) case reduces to the cone of lower semicontinuous traces, together with the subset of tracial states

***Structure*. The Toms–Winter conjecture is an idea central to the structure theory of C*-algebras, which complements the classification results mentioned above. It states the following:**

Conjecture 18.0.2 (The Toms–Winter conjecture) ***[44, 10] Let A be a simple separable unital infinite dimensional amenable C*-algebra. The following conditions are equivalent.***
(i) A ***has finite nuclear dimension:*** $\dim_{\mathrm{nuc}}(A) < \infty$.
(ii) A ***is $\mathcal{Z}$-stable:*** $A \cong A \otimes \mathcal{Z}$.
(iii) A ***has strict comparison of positive elements.***

The three eclectic properties in this conjecture have varied pertinence in different settings, or constructions. Powerful structural consequences ensue for C*-algebras known to have all three of these properties.

A C*-algebra is said to have nuclear dimension at most n if it has the completely positive approximation property (the identity map approximately factorizes via uniformly bounded completely positive maps through finite dimensional algebras), in a particularly controlled manner which is inspired by Lebesgue's notion of covering dimension. This property was introduced by Winter and Zacharias, and is based on an earlier property called decomposition rank due to Kirchberg and Winter [22].

The property of $\mathcal{Z}$-stability says that the C*-algebra A is isomorphic to its tensor product with a certain special C*-algebra, namely the Jiang–Su algebra $\mathcal{Z}$. This algebra $\mathcal{Z}$ has the important property of being a strongly self-absorbing C*-algebra, meaning that $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$, and the isomorphism can be chosen to be approximately unitarily equivalent to the first-factor embedding $\mathcal{Z} \to \mathcal{Z} \otimes 1 \subset \mathcal{Z} \otimes \mathcal{Z}$, and it is not isomorphic to $\mathbb{C}$ [43]. Every other strongly self-absorbing C*-algebra is $\mathcal{Z}$-stable [48], and the Jiang–Su algebra is in fact characterized by this property. The Jiang–Su algebra may be built indirectly using UHF-algebras; this enables certain properties of UHF algebras to be generalized to the algebra $\mathcal{Z}$, a prominent example being [50].

Strict comparison of positive elements is a property for a C*-algebra that factors through a particular invariant, the Cuntz semigroup. The Cuntz semigroup is an ordered abelian semigroup which captures certain cohomological information about the C*-algebra. Strict comparison is equivalent to almost unperforation of the Cuntz semigroup, a property which captures a sense of good (or expected) behaviour in this invariant.

The conjecture is closely linked to the classification of C*-algebras. It arose in response to the counterexamples to Elliott's classification conjecture. The hypothesis of finite nuclear dimension appears in Theorem 18.0.1, and is known to be a necessary hypothesis. In fact, $\mathcal{Z}$-stability is also used crucially in parts of the proof of Theorem 18.0.1, whence the known implication (i) $\Rightarrow$ (ii) of the Toms–Winter conjecture is essential to this theorem.

In fact, this conjecture is very close to being established, owing to results that were announced at this workshop. (i) $\Rightarrow$ (ii) is due to Winter [49] and (ii) $\Rightarrow$ (iii) to Rørdam [37]. After being established in various special cases (including [26, 39, 1]), it was announced at this workshop that the implication (ii) $\Rightarrow$ (i) has been proven in full generality, by Castillejos, Evington, Tikuisis, White, and Winter. While the validity of (iii) $\Rightarrow$ (ii) has yet to be determined (there are partial results, such as [21]), a significant step forward was presented at the workshop by Thiel. He showed that C*-algebras with stable rank one and strict comparison of positive elements automatically have almost divisible Cuntz semigroups; when one additionally assumes locally finite nuclear dimension, $\mathcal{Z}$-stability follows by [49].

Quasidiagonality, of C*-algebras and of traces, is another key hypothesis in Theorem 18.0.1. In 2015, it was proven by Tikuisis, White, and Winter that for amenable C*-algebras which satisfy the Universal Coefficient Theorem, all faithful traces are automatically quasidiagonal [41]. It remained mysterious what

role the Universal Coefficient Theorem really played in this result: was it just a technical hypothesis, or was it truly necessary? Indeed, many questions around the Universal Coefficient Theorem remain mysterious (more on this below).

Putting together the exciting developments mentioned above yields the following result of a fairly definitive nature.

Theorem 18.0.3 *Let A, B be a simple unital infinite dimensional separable amenable C*-algebras with the following properties:*

(i) either $\mathcal{Z}$-stability, or stable rank one plus locally finite nuclear dimension, and

(ii) satisfying the Universal Coefficient Theorem.

Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

A separable C*-algebra A satisfies the Universal Coefficient Theorem if a canonical sequence,

$$0 \to \mathrm{Ext}_1^{\mathbb{Z}}(K_*(A), K_*(B)) \to KK(A,B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0,$$

is exact, for every separable C*-algebra A. Many C*-algebras are known to satisfy the Universal Coefficient Theorem; one of the key results in this direction is due to Tu, which says that C*-algebras constructed from amenable groupoids automatically satisfy the Universal Coefficient Theorem. However, the following is a major open question.

Question 18.0.4 *Does every separable amenable C*-algebra satisfy the Universal Coefficient Theorem?*

Dadarlat gave an excellent talk at the workshop discussing some of the subtleties of this problem.

Group actions and crossed products. As the classification of simple unital separable amenable C*-algebras reaches maturity, there is increasing interest in looking at group actions on C*-algebras and at crossed products, where the classification results may be put to use.

Given a group G and an action α of the group (by *-automorphisms) on a C*-algebra A, the crossed product, denoted $A \rtimes_\alpha G$, is a new C*-algebra which contains A and encodes the action. An important special case is when A is a commutative C*-algebra, necessarily of the form $C_0(X, \mathbb{C})$; in this case, the action corresponds to an action of G on X by homeomorphisms.

There have been a variety of approaches to showing classifiability of crossed products; however, while the standard classification hypotheses (separability, unitality, simplicity, amenability) are either automatic or well-understood for crossed products, we still have a lot to learn about how the hypotheses of $\mathcal{Z}$-stability/finite nuclear dimension and of the Universal Coefficient Theorem behave.

Some highlights among what is already known:

- If G is a finitely generated group with polynomial growth (equivalently, a virtually nilpotent group [14, 27]) and X is a finite dimensional locally compact Hausdorff space, then for any action of G on X, the crossed product $C_0(X) \rtimes G$ has finite nuclear dimension. (A result of Hirshberg and Wu, presented in the workshop.)
- For any countable amenable group G, the generic action α of G on the Cantor set has a $\mathcal{Z}$-stable crossed product [4]. This crossed product is also known to satisfy the other hypotheses of Theorem 18.0.1, so it is classifiable.

- **If G is an inductive limit of residually finite groups and A is a classifiable C*-algebra (i.e., satisfying the hypotheses of Theorem 18.0.1) for which the trace space $T(A)$ is a Bauer simplex, and α is an action of G on A which is pointwise strongly outer, and which acts trivially (or at least periodically) on $T(A)$, then the crossed product $A \rtimes G$ has finite nuclear dimension (a result of Gardella–Hirshberg and Gardella–Phillips–Wang, presented in the workshop). If the crossed product also satisfies the Universal Coefficient Theorem, then it is classifiable.**

- **Kerr recently defined a notion of "almost finite" actions, and proved that the crossed products of such actions are $\mathcal{Z}$-stable.**

Another natural problem to turn to is the classification of actions of groups on C*-algebras. A group action consists of a homomorphism from the given group to the group of *-automorphisms of the C*-algebra; in case the group is equipped with a topology, one asks that this homomorphism be continuous, using the point-norm topology on the set of automorphisms. There are several notions of equivalence which one may consider; although this is not immediately apparent, cocycle conjugacy turns out to be one of the most natural. However, the classification problem is believed to be intractable for general actions, and generally it is natural to restrict to actions with the Rokhlin property.

This problem is at an early stage, and there is much to do. A few highlights:

- **Actions with the Rokhlin property of finite groups on a classifiable Kirchberg algebra or on a classifiable C*-algebra with tracial rank zero have been classified by Izumi [15, 16].**

- **Actions of the circle $\mathbb{T}$ on classifiable Kirchberg algebras with the continuous Rokhlin property have been classified by Gardella.**

- **Actions of the real numbers, $\mathbb{R}$, also called flows, on classifiable Kirchberg algebras with the Rokhlin property have been classified by Szabó [40], an exciting new result that was presented at the workshop.**

Open Problems

All participants contributed to the following list of open problems. Our thanks to Hannes Thiel for recording the list.

(i) **The UCT-problem: Does every nuclear C*-algebra satisfy the universal coefficient theorem (UCT)?**

(ii) **How bad can crossed products be? When is $C(X) \rtimes G$ classifiable?**

(iii) **Let D be a strongly self-absorbing (s.s.a.) C*-algebra with $D \not\cong \mathcal{O}_2$. Is there a unital embedding $D \hookrightarrow \mathcal{O}_\infty \otimes \mathcal{Q}$, where $\mathcal{Q}$ denotes the universal UHF-algebra? (This can be considered as an infinite version of the quasidiagonality problem [52].)**

(iv) **Develop applications of classification. Establish Giordano–Putnam–Skau type (orbit equivalence) theorems ([13]) for spaces that are not Cantor spaces.**

(v) **More examples of actions of $\mathbb{Z}^d$ on Cantor sets ([11, 12]).**

(vi) **Sell classification. Simplify the proof and make it accessible to other areas of mathematics.**

(vii) **Is conjugacy of shifts of finite type (SFT) decidable? Can this be rephrased in terms of C*-algebras?**

(viii) **Do simple C*-algebras with real rank zero necessarily have strict comparison of positive elements? Are they $\mathcal{Z}$-stable?**

(ix) **Range results for Cuntz semigroups.**

(x) Which rigid C*-tensor categories embed into the representation theory of $\mathcal{Z}$?

(xi) Let D be a s.s.a. C*-algebra, let G be a torsion-free amenable group. Does there exist a unique, strongly outer action of G on D?

(xii) Develop a theory of subfactors for inclusions of classifiable C*-algebras ([17, 30]).

(xiii) Let G be an étale, amenable groupoid with $G^{(0)}$ a Cantor set such that C$^*(G)$ is simple. Is C$^*(G)$ $\mathcal{Z}$-stable?

(xiv) Let A be a simple, classifiable C*-algebra, and let G be a (finite) group of automorphisms of the Elliott invariant $\mathrm{Ell}(A)$. Does G lift to an action on A? Does the map $\mathrm{Aut}(A) \to \mathrm{Aut}(\mathrm{Ell}(A))$ split?

(xv) Let A be a simple, classifiable C*-algebra, let $a \in A$, and let $\pi\colon A \to B(H)$ be an irreducible representation. Does H have an $\pi(a)$-invariant subspace?

(xvi) Is every simple, classifiable C*-algebra a groupoid C*-algebra ([5, 34])? Is every simple (exact) C*-algebra a groupoid C*-algebra?

(xvii) Develop analogies of Popa's rigidity theory (for actions of property (T) groups on s.s.a. C*-algebras) ([32, 28, 33]). Is there a theory of intertwining by bimodules for C*-algebras?

(xviii) Classify non-simple TAF algebras.

(xix) Is every real rank zero ASH algebra with slow dimension growth TAF?

(xx) Develop a better understanding of completely positive approximations. In particular, when is an inductive limit, in the category of operator systems, of finite-dimensional C*-algebras, order-isomorphic to a C*-algebra?

(xxi) What are the possible Cowling–Haagerup invariants for separable, simple, exact C*-algebras?

Scientific Progress Made

The workshop brought together experts in various aspects of C*-algebra theory, and created an environment which fostered collaboration and the generation of new ideas. Here are a few examples of specific progress made during the meeting.

- **Francesc Perera, Leonel Robert, and Hannes Thiel made advances in understanding the Cuntz semigroup, and specifically addressing the question of when infima exist in this ordered C*-algebra invariant.**

- **Søren Eilers, Jamie Gabe, and Takeshi Katsura made a decisive breakthrough on the problem of determining when an extension of two simple graph C*-algebras is again a graph C*-algebra.**

- **Gábor Szabó benefitted from discussions with experts in C*-algebra classification, leading to an insight that will likely lead to new results.**

- **José Carrión, Chris Schafhauser, Aaron Tikuisis, and Stuart White initiated a new collaborative project on aspects of C*-algebra classification.**

Outcome of the Meeting

There was a lot of sharing of expertise and learning at this meeting. This led to new collaborations, and inspired serious discussions of future directions for researchers in the classification of C^*-algebras, and more generally, directions for the field of C^*-algebras itself. It is expected that there will be a number of future meetings which follow up on these directions, including Noncommutative Geometry and Operator Algebras 2018: C^*-algebras and Dynamics (Münster).

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Chapter 19

Lattice walks at the Interface of Algebra, Analysis and Combinatorics (17w5090)

September 17 - 22, 2017

Organizer(s): Marni Mishna (Simon Fraser University), Mireille Bousquet-Mélou (Université de Bordeaux/ Centre National de la Recherche Scientifique), Michael Singer (North Carolina State University), Stephen Melczer (University of Waterloo & ENS Lyon)

Overview of the field

Lattice Path Models

Lattice paths are a classic object of mathematics, with applications in a wide range of areas including combinatorics, theoretical computer science and queuing theory. In the past ten years, several new approaches have emerged to determine formulas for exact enumeration. In particular, several systematic methods have arisen from very disparate areas of mathematics. The study of lattice walks restricted to the first quadrant is surprisingly illuminating; it reveals connections between abstract combinatorial structures and classes of analytic functions.

Here we define a lattice path model to be a combinatorial class of walks on the integer lattice, typically starting at the origin, and remaining in some cone. It is prescribed by a finite set of vectors $\mathcal{S}$, the allowable steps one can take in the lattice. Much attention has been paid to walks with small steps, where $\mathcal{S} \subseteq \{-1, 0, 1\}^2$, constrained to the quarter plane, $\mathbb{Z}^2_{\geq 0} = \{(x, y) \in \mathbb{Z}^2 : x \geq 0, y \geq 0\}$. We frequently abbreviate the small steps using compass directions or vector diagrams. A walk of length n which starts at (x_0, y_0) is a sequence of steps $(s_1, \dots s_n)$ such that $s_i \in \mathcal{S}$ and all points $\{(x_0, y_0) + \sum_{i=1}^{\ell} s_i : 0 \leq \ell \leq n\}$ remain in the designated cone. Figure 19.1 is a walk from the quarter plane model with step set $\mathcal{S} = \{(0, 1), (0, -1), (1, 0), (-1, 0)\}$.

The basic enumerative problem is to determine formulas (exact or asymptotic) for the number of walks of length n in a given model. Early studies [24, 21] determined formulas for particular models, generally using adhoc methods. Fayolle, Iagnogorski and Malyshev completed a comprehensive study of stationary distribution of small step models in the quarter plane. They built a functional equation, and provided an analytic toolkit to resolve cases. The work inspired Bousquet-Mélou, who adapted their methods for

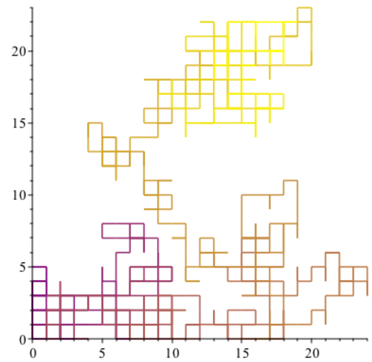


Figure 19.1: A walk of length 1000 from the simple quarter plane model with step set $\mathcal{S} = \{(-1,0),(1,0),(0,-1),(0,1)\}$

exact enumeration. She was able to directly derive the results of Kreweras and Gessel in a straightforward manner. This approach was adapted to many models by Bousquet-Mélou and Mishna [12], who provided a structured analysis of the generating functions for the small step models restricted to the first quadrant.

Specifically, they determined that of the 2^8-1 possible small step models, there were precisely 79 that were combinatorially distinct and nontrivial. They provided generating function expressions for 22 of them, and made conjectures on the nature of the generating functions of the rest. The conjectures use a related group defined for each step set (inspired by the work of Fayolle et al.) This presented a tidy, and clearly intriguing, classification challenge.

A wide variety of techniques have been developed in the interim, and this meeting sought to bring researchers together to discuss commonalities, feasible extensions, and future directions. The approaches discussed at this meeting included bijective methods, probability, computer algebra, complex analysis, algebraic geometry, and differential Galois theory. There are many references for these methods [12, 4, 19, 7, 27, 16]. In this document we outline some of these topics as they apply to lattice path enumeration, and discuss the implications in generating function classification.

The generating function and its kernel

We revisit the idea of more general cones towards the end of the document. For now, consider the quarter plane and a fixed given step set $\mathcal{S}$. Let $e_{(k,\ell)\to(k',\ell')}(n)$ denote the number of walks in the cone starting at (k,ℓ), ending at (k',ℓ'), taking n steps, each from $\mathcal{S}$. The complete generating function for these numbers is the following series, viewed to be an element of $\mathbb{Q}[x,y][\![t]\!]$:

$$\sum_{k,\ell,n\geq 0} e_{(0,0)\to(k,\ell)}(n)\, x^k\, y^\ell t^n.$$

Formally, weights on the steps add no significant difficulty to the set up. Each step is assigned a (real, positive) weight, and the weight of a walk is the product of weights of its component steps. Remark, we recover the unweighted model by setting the weight of each step to be 1 (and hence the weight of each walk is 1). Alternatively, if the sum of the weights is 1, we describe a probability model. More precisely, let $(d_{i,j})_{(i,j)\in\{0,\pm1\}^2}$ be a family of elements of $\mathbb{Q}\cap[0,1]$ such that $\sum_{i,j} d_{i,j}=1$. The weight $d_{i,j}$ can be viewed as the probability that a walk take a step in the direction (i,j), with $d_{0,0}$ the probability that it does not move.

For any $(i,j)\in\mathbb{Z}^2_{\geq 0}$ and any $n\in\mathbb{Z}_{\geq 0}$, we let $q_{i,j}(n)$ be the sum of the weights of all walks of length n

starting at the origin, and ending at the point (i, j). Our main focus here is the trivariate generating series

$$Q(x, y; t) := \sum_{i,j} q_{i,j}(n) x^i y^j t^n. \tag{19.0.1}$$

It is sufficient to consider the case $\sum_{i,j} d_{i,j} = 1$, by suitably scaling the t variable (although many techniques are custom made for the unweighted version). Then, for any $n \in \mathbb{Z}_{\geq 0}$, $|q_{i,j}(n)| \leq \sum_{i,j} |q_{i,j}(n)| \leq (\sum_{i,j} |d_{i,j}|)^n = 1$. We deduce that $Q(x, y; t)$ converges for all $(x, y; t) \in \mathbb{C}^3$ such that $|x| < 1$, $|y| < 1$ and $|t| \leq 1$.

The inventory of the step set is a Laurent polynomial which we might write as a function of either variable x or y:

$$\begin{aligned} S(x, y) &= \textstyle\sum_{(i,j)\in\{0,\pm1\}^2} d_{i,j} x^i y^j \\ &= A_{-1}(x)\frac{1}{y} + A_0(x) + A_1(x)y \\ &= B_{-1}(y)\frac{1}{x} + B_0(y) + B_1(y)x. \end{aligned} \tag{19.0.2}$$

Here $A_i(x) \in x^{-1}\mathbb{Q}[x]$, and $B_i(y) \in y^{-1}\mathbb{Q}[y]$. The kernel of the walk is defined by

$$K(x, y, t) := xy(1 - tS(x, y)). \tag{19.0.3}$$

A walk is either a walk of length 0, or a walk followed by a valid step. This straightforward decomposition has a natural translation into generating functions, into what we call the kernel equation. A property specific to models of walks in the quarter plane is that the generating function for walks that end on a boundary can be obtained from an evaluation of $Q(x, y; t)$. For example, the family of walks that begin and end at the origin have generating function $\sum_{n\geq0} q_{0,0}(n)t^n = Q(0, 0; t)$. The walks that end on the y-axis have generating function $\sum_{n,k\geq0} q_{k,0}(n)x^k t^n = Q(x, 0; t)$. This is a particularly convenient observation.

Following [18, Section 1], and translating the aforementioned combinatorial recurrence, we may prove that the generating series $Q(x, y, t)$ satisfies the following functional equation:

$$K(x, y, t)Q(x, y; t) = xy - R(x, t) - S(y, t) + td_{-1,-1}Q(0, 0, t) \tag{19.0.4}$$

where

$$R(x, t) := K(x, 0, t)Q(x, 0; t), \quad \text{and} \quad S(y, t) := K(0, y, t)Q(0, y; t). \tag{19.0.5}$$

Many enumeration strategies start with Equation (19.0.4), and then differ in their subsequent manipulations. This report summarizes only some of them. Central to many of these methods is the property of small step walks is that $K(x, y; t)$ is quadratic. Considering larger steps is considerably more complicated. Nonetheless, stepsets with negative steps of length 2 have recently been thoroughly considered [10].

Classification of generating functions

Lattice walks provide a unified context to examine combinatorial origins of different function classes. The combinatorial complexity of a given model is correlated to the analytic complexity of its generating function. It remains open to completely formalize many of the known connections.

Rational **A generating function is rational if it is the quotient of two polynomials. Generating functions of regular languages are rational. The coefficients of rational series, and their asymptotics are extremely well understood, see [20] for a pedagogical treatment. For example, the the quarter plane model with step set $\{N, NE, E\}$ is unconstrained, and consequently it is easy to show the generating function is rational: $Q_{N,NE,E}(x, y; t) = \frac{1}{1-t(x+xy+y)}$.**

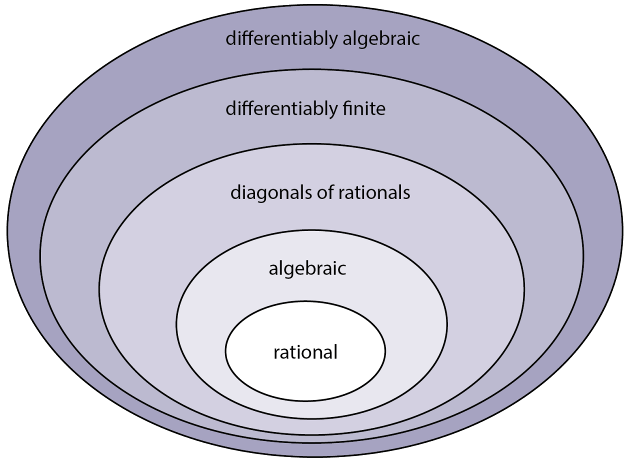


Figure 19.2: Different classes of generating functions

Algebraic **An univariate generating function $f(t)$ is algebraic if there is a polynomial $P(u, v) \in \mathbb{C}[u, v]$ such that $P(f(t), t) = 0$. The polynomial P provides a finite encoding for $f(t)$, and can be used in computations and closure properties. The asymptotics are also relatively well understood. A multivariate function of m varables is algebraic if it solves a polynomial equation in $m + 1$ variables. Rational functions are algebraic.**

From the combinatorial perspective, combinatorial classes such as trees and maps possess the archetypal algebraic structure, and have algebraic generating functions.

In the realm of lattice models, unidimensional walks have algebraic generating functions, and many two-dimensional models can be reduced to this case. Expressions for the generating functions, and the coefficient asymptotics are straightforward to compute.

Transcendental D-finite **A univariate function is differentiably finite, commonly called D-finite or holonomic, if it satisfies a linear differential equation with polynomial coefficients. In 1980 Stanley highlighted the interest to combinatorialists by pointing out the very combinatorial collection of closure properties: including differentiation, algebraic substitution and Hadamard product. It remains open to find a combinatorial mechanism that adequately captures this class of functions. It strictly contains algebraic functions.**

The following result (compiled from work of Kath, André, and Garoufalidis) is useful to show that a function is not D-finite.

Theorem 19.0.1 ***If a series $\sum a_n t^n$ in $\mathbb{Z}[t]$ is D-finite, with radius of convergence in $(0, \infty)$, then its singular points are regular with rational exponents. Consequently, the asymptotic expansion has the form of a finite sum***

$$a_n \sim \sum \lambda^{-n} n^{\alpha} \log^k(n) f_{\lambda,\alpha,k}(1/n)$$

where α is rational, λ is algebraic, k is a nonnegative integer, and f is a polynomial over $\mathbb{Q}[x]$.

For example, Bostan, Raschel and Salvy proved that several models of excursions have non-D-finite generating functions because the sub-exponential growth is irrational [7].

A multivariate function is D-finite with respect to a set of variables if it is D-finite with respect to each of the variables.

Diagonals **The diagonal of a formal power series $\sum_{i_1,\ldots,i_d\geq 0} f_{i_1,\ldots,i_d} z_1^{i_1}\cdots z_d^{i_d}$ is the series $(\Delta F)(t) = \sum_{n\geq 0} f_{n,\ldots,n} t^n$.**

The diagonal of a multivariate function is defined for a particular series expansion of that function around the origin.

Furstenburg showed in 1967 that algebraic functions can be expressed as a diagonal of a bivariate rational function by providing the rational based on the polynomial that the function satisfies. Lipshitz showed that the diagonal of a D-finite multivariate series was D-finite. There do exist D-finite functions which could not be a diagonal of a rational, for example e^x, but perhaps if the conditions are right, there might be an equivalence. The following conjecture has been considered for a while:

Question 19.0.2 (Christol, 1990) ***Can any D-finite series with integer coefficients and a positive radius of convergence can be expressed as a diagonal of a rational series?***

Differentiably algebraic **A function is differentiably algebraic, or D-algebraic, if there exists an algebraic relationship between $f(x)$ and its derivatives. More precisely, there exists a polynomial $P \in \mathbb{C}[x]$ so that**

$$P(f(x), f'(x), \ldots, f^{(k)}(x)) = 0.$$

This class clearly contains the D-finite functions, but is otherwise less well understood.

Beyond this **Finally, we say that the remaining functions are hyper transcendental.**

Classifying Lattice paths

So where do the generating functions of the 79 small step quarter plane models fall? Attempts to answer this question have spawned a very active area of research. The answers provide insight on the subtle differences of the function classes. Figure 19.3 is a compilation of results from numerous articles.

Slightly less is known about the univariate counting generating functions. Since these classes are closed under algebraic substitutions, which includes evaluations, we can deduce some containments, but we cannot conclude that many of the step sets with non-D-finite complete generating function have a non-D-finite counting generating function.

Question 19.0.3 ***Describe combinatorial criteria on 2D lattice models which determine the nature of the generating functions. Which criteria hold in problems of higher dimension?***

Question 19.0.4 ***Prove the remaining unclassified small step univariate counting generating functions non-D-finite.***

Computer Algebra

Part of the appeal of lattice path models is the ease with which one can experiment using computer algebra tools. It is relatively straightfoward to develop a series expansion from the basic recurrence, and then form hypotheses and conjectures about the generating functions.

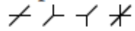
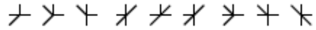
Class	Count	Step Sets
Algebraic	4	
Transcendental D-finite	19	
Non-D-finite D-algebraic	9	
Hyper transcendental	42	
D-transcendental	5	

Figure 19.3: The nature of the complete generating function $Q(x, y; t)$ for the 79 small step quarter plane lattice path models

Guess 'n' prove

Software to guess the algebraic or differential equation satisfied by a series has been used enumerative combinatorics for nearly 30 years. The main implementions work by generating a high order truncation of the (possibly multivariate) generating function, followed by a guessing stage which tries to fit this truncation into algebraic or differential equations of various orders and degrees using Padé approximants. In Maple, this is implemented in the gfun [30] package.

The equations that are found remain conjectural, until they are proved. In this field, the most famous example, perhaps, is known as "Gessel's walk". In 2000 Ira Gessel, via personal communication, suggested that the generating function for the quarter plane walks with step set $\{SW, W, NE, E\}$ should be D-finite. This was based on the observation that the counting sequence for excursions, that is, the walks that start and end at the origin, appeared to fit very nice formula. This lead to a very vigourous discussion, as the other models with D-finite generating functions were easily guessed to be, and this model resisted the approach.

The D-finiteness of the excursion generating function was published in 2009 [23]. The proof used a cutting edge holonomic systems approach, and creative telescoping to confirm Gessel's conjecture. Still, it remained open to determine the nature of the complete generating function.

Around the same time, Bostan and Kauers [4] used the aforementioned approach based on Padé-Hermite approximants to systematically guess algebraic and differential equations satisfied by the 79 non-isomorphic two dimensional models in the quarter plane. In addition to detailing several algebraic and arithmetic conditions which help one believe that a guessed differential equation truely annihilates the multivariate generating function, the authors produced an influential table of guessed asymptotics for the 23 models they predicted to have D-finite generating function.

It surprised many when Bostan and Kauers [5] proved not only the D-finiteness, but the algebraicity of the multivariate Gessel generating function $Q(x, y; t)$! Their approach rigorously certified a guessed minimal polynomial of $Q(x, y; t)$ using algebraic elimination theory. The result was a computational tour de force – the algebraic equation that $Q(x, y; t)$ satisfies requires several MB to store. Researchers were challenged to demonstrate the algebraicity using "human" techniques. The community delivered, and such results appeared in the following few years. Indeed, resolving this problem is a recurrent theme. For more details, see the approachable survey of Bostan and Raschel [6].

Computer algebra methods have proved indispensible for providing bulk results. Notably, Bostan, Raschel and Salvy [7] iterated through to show that the excursions in 56 of the models did not have a D-finite generating function because the asymptotic expressions were incompatible. Specifically, the subexponential growth had an irrational exponent.

Recently, Bostan et al. [11] proved all guessed annihilating algebraic and differential equations for the 23 D-finite models using creative telescoping on diagonal expressions, and wrote lattice path generating functions as explicit integrals of hypergeometric functions. These expressions, in turn, allowed the authors

to classify all transcendental D-finite generating functions, and determine which combinatorially interesting specializations of the transcendental generating functions are algebraic.

In this workshop, the topic was introduced by a thorough mini-course given by Manuel Kauers. Several talks addressed computational aspects, including how to prove the transcendance of a D-finite generating function from an annihilating differential equation.

This is but a small snapshot of the computer algebra methods in this field. A far more complete and detailed summary is the habilitation thesis of Bostan [8].

Question 19.0.5 ***The computational methods of Bostan et al. [9] show that the number of excursions on the step set*** $\{(-1,0),(-1,0),(-1,1),(-1,-1),(1,0),(1,1)\}$ ***(note the double*** $(-1,0)$ ***step) satisfy***

$$a_{2n} = \frac{6(6n+1)!(2n+1)!}{(3n)!(4n+3)!(n+1)!}$$

(there are no excursions of odd length). Find a 'human' proof of this result.

Question 19.0.6 ***The four quadrant models with algebraic multivariate generating functions have zero orbit-sum, meaning the kernel method does not allow one to represent these generating functions as diagonals of rational functions. As algebraic functions, they can be written as diagonals of bivariate rational functions through known constructions, but these representations are unweildy and hard to analyze compared to the nice 'combinatorial' expressions arising from the kernel method. Do there exist simple 'combinatorial' rational functions whose diagonals give the generating functions of the 4 models with algebraic generating function?***

Enhanced data manipulation

The strategies for guessing apprimations of asymptotic formulas from initial series data are becoming increasingly sophisticated. If one assumes that the generating function $f(z) = \sum a_n z^n$ **has a power law singularity of the form**

$$f(x) \sim C\left(1 - \frac{x}{x_0}\right)^{\alpha},$$

the series can be analyzed to produce high precision numerical estimates for the unknown quantities. In this workshop, Tony Guttman talked about how to use differential approximants to predict subsequent terms from some initial data. Every differential approximant naturally reproduces exactly all coefficients used in its derivation. Being a D-finite differential equation, it implies the value of all subsequent coefficients. These subsequent coefficients will usually be approximate.

These techniques have been used by the Melbourne school for a variety of combinatorial problems, and recent work by Guttman and Elvey Price on the enumeration of Planar Eulerian Orientations [17] was presented at this workshop.

The Kernel Method

Kernel equations here denote functional equations of a particular form that arise from combinatorial recursions. One separate the terms into two classes: on the left hand side is an expression with the complete generating function, known in some circles as the <u>bulk</u>. One the right hand side, there are functions which involve the boundary or more generally, evaluations of the complete generating function. The kernel determines the bulk behaviour. The two main ways to proceed are to manipulate the equation to either eliminate the boundary conditions, or to eliminate the bulk.

Andrew Rechnitzer provided a short course on variants of the kernel method for lattice walks, with a view to the study of polymers. He identified the main steps:

- identify a catalytic or auxillary variable to track a useful parameter;
- use a combinatorial recursion to establish a functional equation for the complete generating function;
- rearrange the terms and write the equation in kernel form;
- evaluate the auxiliary variables to either eliminate the kernel, or to eliminate the boundary terms;
- extract the complete generating function.

Remark the two principle variants: the first one uses substitutes solutions to the kernel into the main equation, and manipulates the boundary terms ("eliminate the bulk"); the second uses invariants of the kernel to eliminate the boundary terms ("eliminate the boundary").

Eliminating the bulk can find explicit generating expressions. In polymer modelling problems, it is used to understand the behaviour for different evaluations of the auxiliary parameter. A typical example is contacts with a boundary, well explained by the talk of Aleks Owczarek on three interacting friendly walks [31]. This is also central to several analytic approaches which we describe below.

Eliminating the boundary using invariants of the kernel has been very useful for lattice path enumeration. The finiteness of a subgroup of the kernel invariants is implicated in the classification of the complete generating function. Bousquet-Mélou and Mishna described how to use such invariants to express the complete generating function as a sub-series of a Laurent series expansion of a computable rational function.

Analytic combinatorics of several variables

Complex analysis methods to determine asymptoic enumeration formulas for combinatorial classes have become standard [20]. Initially the majority of the result came from the singularity analysis of of univariate generating functions. The multivariate problem is substantially harder, but steady progress over the past two decades has lead to some systematic strategies. Relevant here has been the work developed by Pemantle, Wilson, Baryshnikov, and coauthors under the name of Analytic Combinatorics in Several Variables, abreviated (ACSV) [28].

Melczer and Mishna [26] noted the applicability of these methods to lattice enumeration. They restricted to models with significant symmetry, but were able to generalize the strategy to higher dimensions. Melczer and Wilson performed a case by case analysis [27] to determine asymptotic formulas for most of the small step 2D models with a finite group. Courtiel et al. [15] used ACSV to determine asymptotic formulas for the weighted Gouyou-Beauchamps model.

Differential Galois Theory

The strategy to eliminate the "bulk" leads to important theoretical results. The key is to understand that the kernel is eliminated on an elliptic curve. Charlotte Hardouin gave an excellent course on how to apply methods in differential Galois theory to prove the hypertranscendance of lattice model generating functions. Similar strategies have been used to show the transcendance of the Gamma function.

We first fix the value of t, admittedly a nontraditional strategy for generating function combinatorics. Specifically, let us fix $0 < t < 1$ with $t \notin \overline{\mathbb{Q}}$.

We start from Equation (19.0.4)

$$K(x,y,t)Q(x,y;t) = xy - R(x,t) - S(y,t) + td_{-1,-1}Q(0,0,t). \tag{19.0.6}$$

Consider the algebraic curve $\overline{E_t}$, which is defined as the zero set in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ of the following homogeneous polynomial

$$\overline{K}(x_0,x_1,y_0,y_1,t) = x_0x_1y_0y_1 - t\sum_{i,j=0}^{2} d_{i-1,j-1}x_0^i x_1^{2-i} y_0^j y_1^{2-j} = x_1^2y_1^2K(\frac{x_0}{x_1},\frac{y_0}{y_1},t). \tag{19.0.7}$$

This curve is an an elliptic curve. This is the situation studied in [12, 25], and for the Galoisian point of view, see [16].

Thanks to uniformization, we can identify E_t with $\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Specifically, consider the map

$$\begin{array}{ccc} \mathbb{C} & \to & \overline{E}_t \\ \omega & \mapsto & (\mathfrak{q}_1(\omega), \mathfrak{q}_2(\omega)), \end{array}$$

where $\mathfrak{q}_1, \mathfrak{q}_2$ are rational functions of $\mathfrak{p}$ and its derivative $d\mathfrak{p}/d\omega$, $\mathfrak{p}$ the Weirestrass function associated with the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ (cf. [25, Section 3.2]). Therefore one can lift the functions $R(x,t)$ and $S(y,t)$ to functions $r_x(\omega) = R(\mathfrak{q}_1(\omega),t)$ and $r_y(\omega) = S(\mathfrak{q}_2(\omega),t)$.

One can deduce from [25, Theorems 3 and 4], that the functions $r_x(\omega)$ and $r_y(\omega)$ can be continued meromorphically as univalent functions on the universal cover $\mathbb{C}$. Furthermore, for any $\omega \in \mathbb{C}$, we have

$$\tau(r_x(\omega)) - r_x(\omega) = b_1, \text{ where } b_1 = \iota_1(\mathfrak{q}_2(\omega))(\tau(\mathfrak{q}_1(\omega)) - \mathfrak{q}_1(\omega)) \tag{19.0.8}$$
$$\tau(r_y(\omega)) - r_y(\omega) = b_2, \text{ where } b_2 = \mathfrak{q}_1(\omega)(\iota_1(\mathfrak{q}_2(\omega)) - \mathfrak{q}_2(\omega)) \tag{19.0.9}$$
$$r_x(\omega+\omega_1) = r_x(\omega) \tag{19.0.10}$$
$$r_y(\omega+\omega_1) = r_y(\omega). \tag{19.0.11}$$

where τ is the automorphism of the field of meromorphic functions sending $f(\omega)$ onto $f(\omega+\omega_3)$ and ω_3 is explicitely given in [25, Section 3.2].

One can show that $r_x(\omega)$ is differentially algebraic with respect to $\frac{d}{d\omega}$ over C_E, the field of elliptic functions with respect to $\overline{E_t}$ if and only if $Q(x,0,t)$ is differentially algebraic with respect to $\frac{d}{dx}$ over $\mathbb{C}(x)$.

The order of the differential operator

In [16], it is shown that if $r_x(\omega)$ is differentially algebraic, then there is a second order linear differential operator L with constant coefficients such that $L(r_x) - g$ is an ω_3 periodic function, that is, it is left invariant by τ.

For the 51 unweighted walks associated to genus 1 curve and infinite group of the walk, only 9 walks give rise to a differentially algebraic generating function (see [16]). For these walks, it seems that the results of [3] show that there exists $g \in C_E$ such that $r_x(\omega) - g(\omega)$ is ω_3-periodic.

This line of research has great potential, and there remain many natural questions.

Question 19.0.7 ***For weighted walks associated to genus 1 curve and infinite group of the walk, is it always true that when $r_x(\omega)$ is differentially algebraic then there exists $g \in C_E$ such that $r_x(\omega) - g(\omega)$ is ω_3-periodic?***

Question 19.0.8 ***What can be dervied from this approach about the differential dependence in t for the 51 unweighted walks associated to genus 1 curve and infinite group of the walk?***

Finite group of the walk

Let us consider a walk associated with an elliptic curve $E_t = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ (the curve defined by the kernel equation) and let us assume that this walk has finite group. This is equivalent to the fact that an integer multiple of ω_3 above belongs to $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. One can ask the following questions:

Question 19.0.9 ***The prolongation of $r_x(\omega)$ into a meromorphic function over $\mathbb{C}$ as well as the functional equation (19.0.8) obtained in [25] for the 51 unweighted walks associated to genus 1 curve and infinite group of the walk, depend heavily on the fact that the group of the walk is infinite. Can one still obtain analogous results in the finite group case?***

Question 19.0.10 ***For weighted walks associated to genus 1 curve and finite group of the walk, is there a general closed form for the generating series?***

Random walk processes and the critical exponent

Random walk processes are a classic topic of probability. Processes conditioned never to leave the cone are connected to Harmonic functions and Random Matrix theory. The persistance probabilities of random walks are linked to statistical physics and population models.

Many results on the expected first exit time from a cone can be converted into asymptotic enumeration results. The course of Kilian Raschel provided many very clear connections. If the number $q(n)$ of walks of length n grows asymptotically like $\kappa\rho^{-n}n^{-\alpha}$ as n tends to infinity, then we call α the critical exponent. Probability results give straightforward access to information on the critical exponents. The classical exponents for 1 dimensional models are 0, 1/2 and 3/2 which depend on the drift. The simple walk in two dimensions is a far richer. This is modelled by the simple $\{N, E, S, W\}$ walks in an arbitrary angular sector θ. Varopoulos, and Denisov and Wachtel give that the exponent for the total number of walks is $\pi/2\theta$ and for excursions is $\pi/\theta + 1$. These results imply that the generating functions are not D-finite whenever θ is not a rational multiple of π.

The non simple models can be similarly analyzed, once the model is suitably transformed. This plan of attack was used with great success by Bostan, Raschel and Salvy who demonstrated via a case analysis that for all small step models, if the group of the step set was infinite, the excursion generating function is not D-finite [7].

Question 19.0.11 ***Is there a direct connection between the infiniteness of the group and the irrationality of critical exponent? If so, does this correspondence hold in higher dimnesions?***

In fact, the results of Denisov and Wachtel are for arbitrary dimension, so in theory a similar approach should work in higher dimension. The first model under consideration is the model known as the 3D Kreweras model, with step set $\{(1,1,1),(-1,0,0),(0,-1,0),(0,0,-1)\}$ restricted to the orthant $\mathbb{Z}^3_{\geq 0}$. Several groups [1, 22] have tried to estimate the critical exponent with a goal of establishing its irrationality.

Question 19.0.12 ***Is the generating function for the 3D Kreweras model D-finite?***

Extensions of the original problem

Non convex regions

Non convex regions require a modified analysis. The 3/4 plane is the next candidate for a systematic study. Bousquet-Mélou has launched this for the simple steps [13], and has demonstrated that, while more technical, many of the same basic strategies can apply.

The most recent advances on this problem were presented by Trotignon during the meeting. Raschel and Trotignon have expressed the generating function for simple walks in the 3/4 plane as the solution to a boundary value problem [29].

Winding numbers

The majority of enumerative works on lattice walks to date use a last step recursive decomposition: A walk is either empty, or it is a walk plus a step. Timothy Budd described a very promising alternative decomposition [14]. He considers models with step sets that have a high amount of symmetry, and cones whose boundaries on the angles that are integer multiples of $\pi/4$. He considers a new decomposition for walks as a sequence of excursions. The work offers explicit formulas for 2D walks in a cone which avoid a boundary except at the start, and at the end. He considers the generating function where length and winding angle of the walk are tracked. The winding angle is the angle formed by the first and last step. This set up is sufficient to determine an explicit formula for the usual Gessel walks in the first quadrant. He gives some clear criteria for algebraicity, and the D-finiteness of the models that he considers is easy to establish.

This work raises a number of questions, and his intermediary objects may be useful to determine the combinatorics behind the classification of the generating functions.

Question 19.0.13 ***Can his argument deduce the generating function for simple walks in the 3/4 plane tracking winding number? How does it compare with the work of Raschel and Trotignon?***

New Collaborations

Many participants have reported new collaborations as a result of this workshop, as per the intended goal. Several participants have launched collaborations to extend, and address many of the questions that were listed here. Several participants noted the quality of the open questions presented, and the applicability of techniques across domains.

Of particular note is an entirely new seminar series that was created at the historic Institut Henri Poincaré (IHP): The Groupe de travail autour des marches dans le quart de plan[1] is organized by Lucia Di Vizio and Alin Bostan.

Michel Drmota and Andrew Rechnitzer have begun discussions on analytic properties of solutions of systems of functional equations.

Thomas Dreyfus reported several new collaborations, including new work on the algebraic independence of Mahler functions with Jason Bell, Boris Adamscewski and Michael Singer; and a new collaboration with Carlos Arreche on the elliptic hypergeometric functions.

Mark Wilson noted a boost to several projects in progress with Stephen Melczer.

[1] https://divizio.joomla.com/seminaires-et-gdt/gdt-marches-dans-le-quart-de-plan

Miklos Bona started a new collaboration with Jay Pantone at Dartmouth college on the log-convexity of certain permutation classes. They have used lattice paths to settle one of the special cases. Some of these classes are counted by generating functions that are not algebraic, but differentiably finite, and Miklos Bona is trying to use what he learned from the talk of Bruno Salvy about such functions.

Jason Bell and Marni Mishna considered some of the questions on the cogrowth problem raised by Igor Pak. Their work resulted in a preprint [2].

Michael Singer and Charlotte Hardouin have made progress towards Open Problem 6.

Participants

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Chapter 20

Symmetries of Surfaces, Maps and Dessins (17w5162)

September 24 - 29, 2017

Organizer(s): Marston Conder (University of Auckland, New Zealand), Nigel Boston (University of Wisconsin at Madison, USA), Gabino González-Diez (Universidad Autónoma de Madrid, Spain), Gareth Jones (University of Southampton, UK), Thomas Tucker (Colgate University, USA)

Introduction

The main objective of this workshop was to bring together a number of leading and emerging researchers in the area of study of actions of discrete groups on Riemann and Klein surfaces and algebraic curves, and related topics such as symmetric embeddings of graphs and dessins d'enfants on surfaces. It followed up on a meeting held at the CIEM in Spain in 2010.

The last two decades have seen a burgeoning of activity in these fields, with various strands coming together, exploiting the linkages established by Belyi and Grothendieck, and some increasingly useful techniques from combinatorial and computational group theory. In particular, computational experiments and searches have produced a wealth of examples (either for small genera or infinite families of a particular type), and these serve as a useful test-bed for conjectures and potential new approaches.

Workshop format

The workshop began with introductory lectures, one on each of the eight themes, presented by experts in the area. These addressed recent developments and described important open problems, and possible approaches to answering them. Then the participants formed groups to work separately on the first four themes from late morning on the Tuesday to lunchtime on the Wednesday, and rearranged into new groups to work on the other four themes from the Wednesday afternoon to the Friday morning.

Also during the week an opportunity was given for open problems to be presented, and ten short talks were presented by workshop participants on topics related to the workshop themes, on the Monday, Tuesday and Wednesday afternoons. A conference excursion was made to Lake Louise on the Thursday afternoon, and enjoyed by all who took part in that.

Themes

Introductory talks on the main themes of the workshop were given on the first two days, as follows:

- Theme 1: Regular and edge-transitive maps,
  introduced by Marston Conder and Tom Tucker
- Theme 2: Dessins d'enfants, Belyi's theorem and Galois action,
  introduced by Gabino González-Diez and Ernesto Girondo
- Theme 3: Defining equations for Riemann surfaces,
  introduced by Allen Broughton and Aaron Wootton
- Theme 4: Polytopes, hypertopes, maps and maniplexes,
  introduced by Dimitri Leemans
- Theme 5: Cayley maps and skew morphisms,
  introduced by Robert Jajcay
- Theme 6: n-gonal automorphisms of Riemann surfaces,
  introduced by Mariela Carvacho
- Theme 7: External symmetries of regular maps,
  introduced by Jozef Širáň
- Theme 8: Pseudo-real surfaces, and large automorphism groups of surfaces,
  introduced by Javier Cirre and Grzegorz Gromadzki.

Working groups

Below we report on the activities of the eight working groups that were formed during the workshop. The first four convened early in the week, with two of them merging together, and the last four convened in the second half of the workshop.

Regular and Edge-Transitive Maps

[Workshop leaders: Marston Conder, Jozef Širáň, and Tom Tucker]

By a 'map' we mean a 2-cell embedding of a connected graph or multigraph on a closed surface (which may be orientable or non-orientable). Such a map M is edge-transitive if its automorphism group $\mathrm{Aut}(M)$ acts transitively on edges. Graver and Watkins [5] introduced a classification of such maps into 14 types, according to the local effects of certain generators of the automorphism group. Then Širáň, Tucker and Watkins [10] showed how to construct orientable maps of each type from certain presentations satisfying an 'index two subgroup and a 'forbidden automorphism' condition, and used this construction to give finite orientable maps of each type with $\mathrm{Aut}(M) \cong S_n$ for certain n.

Recently Jones [7] extended this work to surfaces with boundary, and among other things showed that all but a small number of simple groups occur as the automorphism group of various edge-transitive maps of all 14 types (with all of these maps being chiral or non-orientable). Gareth Jones was a co-organiser of this workshop, but sadly was unable to attend, and so we made available to participants a copy of a talk that he gave on edge-transitive maps at a BIRS meeting in Oaxaca in August 2017.

There have been only a few papers on edge-transitive maps published since 2001. Many questions have been suggested by some of the above people and others for further study: classifying all such maps for a given group or family of groups, or on a given surface, or for a given underlying graph or family of graphs; determining the relative frequency of each type; and seeking generalisations, say to polytopes.

At this workshop, the working group spent time reviewing the 14 types of edge-transitive maps, including important details about (a) the automorphism group $G = \mathrm{Aut}(M)$ with marked generators as a quotient of some universal group U for the appropriate type, (b) the image G^+ of the orientation-preserving subgroup U^+ of U and its meaning for orientability of the map, (c) the nature of forbidden automorphisms permuting the marked generators and their inverses, and (d) the roles of duality and Petrie duality.

The working group then focused on one of the open problems described in the opening presentation on the first day, namely Classifying the edge-transitive maps for which the subgroup G^+ is abelian. This problem turned out to be a great deal more complicated than the presenters had thought, but was highly illuminating. The discussion concentrated mostly on a small number of the 14 types, and eventually led to the discovery that there are no ET maps with abelian G^+ for six of the 14 types, and there do exist such maps for six of the other eight types. Two of the 14 types remained unresolved.

This problem provided an excellent introduction to the study of edge-transitive maps, and clarified a number of issues for the main presenters — indeed so much so that a publication on the findings is likely (once the investigations are completed).

Belyi Theory

[Workshop leaders: Allen Broughton, Gabino González-Diez and Aaron Wootton]

This theme was combined with 'Defining Equations for Surfaces' — see the next subsection.

Defining Equations for Surfaces

[Workshop leaders: Allen Broughton, Gabino González-Diez and Aaron Wootton]

(a) ***Defining a surface.***

In both of these themes, each surface S of interest typically has either a suitably 'large' group of automorphisms G, with quotient surface S/G having genus 0, or a 'nicely structured' branched covering $\pi : S \to \mathbb{P}^1$ (n-gonal morphism). Such surfaces may be constructed in two different ways.

(b) ***Defining equations.***

Let $f_1, \dots, f_{N-1}$ be homogeneous polynomials in $X \in \mathbb{P}^N$ of varying degrees. In general, the algebraic variety

$$V(f_1, \dots, f_{N-1}) = \Big\{X \in \mathbb{P}^N : f_1(X) = \cdots = f_{N-1}(X) = 0\Big\} \tag{20.0.1}$$

will be an irreducible, complex algebraic curve, and its normalisation will be a smooth surface. The desirable case is a plane curve where $N = 2$.

(c) ***Group actions and monodromy groups.***

A conformal G-action on S is described by the set of points $\{Q_1, \dots, Q_t\} \subset \mathbb{P}^1$ over which $\pi : S \to S/G = \mathbb{P}^1$ is ramified, and a t-tuple $(c_1, \dots, c_t)$ of elements of G, called a generating vector, satisfying

$$G = \langle c_1, \dots, c_t\rangle \text{ and } c_1c_2 \cdots c_t = 1. \tag{20.0.2}$$

The t-tuple $(c_1, \dots, c_t)$ is determined by the monodromy of $\pi : S \to S/G$.

More generally, given any branched cover $\pi : S \to \mathbb{P}^1$ of degree n, ramified over $\{Q_1, \dots, Q_t\}$, a transitive monodromy group $M_\pi \leq \Sigma_n$ is determined by a monodromy system $(p_1, \dots, p_t)$:

$$M_\pi = \langle p_1, \dots, p_t\rangle \text{ with } p_1p_2 \cdots p_t = 1. \tag{20.0.3}$$

The cycle structures of the permutations p_i are determined by the cycle structures of the exceptional fibres $\pi^{-1}(Q_i)$. Conversely, given points $\{Q_1, \dots, Q_t\}$ and systems as in (20.0.2) or (20.0.3), a branched covering surface S may be defined. This is a typical starting point for defining S, and we want to determine a defining equation for S as in (20.0.1).

(d) ***Quasi-platonic and Belyi surfaces.***

Currently there is intense interest in quasi-platonic surfaces, Belyi surfaces, and their dessins d'enfant. Quasi-platonic surfaces form the G-action case with $t = 3$, and Belyi surfaces include all monodromy groups with $t = 3$. Both types of

surfaces have equations defined over number fields, and the absolute Galois group acts on both the surfaces and their dessins.

(e) *Questions.*

At the workshop, the following questions were posed:

- **Given a group action of G or monodromy group M_π with generating systems as in (20.0.2) and (20.0.3), is there an algorithm to find defining equations as in (20.0.1)? The cyclic case is easy and well known. In the important Belyi surface case, Monien [9] and Voight et al [12] have developed computationally intensive methods for determining equations.**
- **In the quasi-platonic case, can the G-action be used to speed up computation in the methods of Monien and Voight? Are there estimates for the running time of these algorithms?**
- **Can a set of defining equations as (20.0.1) be found so that the G-action is induced by a linear action of G on the ambient space $\mathbb{P}^N$? See [11].**
- **What are the defining equations of a curve S with cyclic group of automorphisms G such that S/G has genus 1? This generalises the cyclic n-gonal case.**

(f) *Discussion, outcomes, and further work.*

The easy case of cyclic n-gonal surfaces was presented and discussed. The easy examples of explicit equations of higher genus Belyi surfaces have almost all been determined and exploited. Further work on Galois action on dessins needs a good library of examples.

Hartmut Monien and John Voight gave extended presentations of their work (as mentioned above) on constructing a defining equation for any monodromy triple given in (20.0.3). The monodromy triple is used to compute a fundamental region for a Fuchsian group Π such that $\mathbb{H}/\Pi \cong S$. One then works with modular forms for Π, using either numerical linear algebra or numerical PDEs.

The working group enthusiastically supported the future construction of a widely available data base of equations for those surfaces of low genus or those defined by monodromy systems (generating vectors) of interesting groups.

Maniplexes and Incidence Geometries

[Workshop leader: Dimitri Leemans]

This working group aimed to gain a better understanding of the link between maniplexes and incidence geometries, and to develop a kind of 'dictionary' that would permit people working in these fields to understand each other's research.

Steve Wilson introduced maniplexes as a means of constructing maps, either by drawing a structure on a surface, or by assembling polygons (just as a 4-cube can be constructed by gluing 3-cubes along faces).

An $(n+1)$-<u>maniplex</u> is a set Ω of <u>flags</u>, together with an ordered set $R = [r_0, \ldots, r_n]$ of sets of pairs of flags, where each r_i may be seen as an involution permuting the flags. Then for $0 \leq i < n$, an i-face is a connected component of $R_i = \cup_{j \neq i} r_j$, and the 0-faces are called the vertices, the 1-faces called the edges, and the n-faces called the facets. In particular, each facet is itself a maniplex.

The flag-graph of any polytope is a maniplex, but there are many other kinds of examples. A 1-maniplex is just a set Ω of cardinality 2 (and is the flag graph of the graph K_2, while a 2-maniplex is a polygon, and a 3-maniplex may be viewed as a transitive action of a non-degenerate string group.

The working group focused on understanding maniplexes as chamber systems of incidence geometries obtained from polytopes. The group also worked on ways of removing the 'string' condition from the definition of maniplexes, and developing a notion of hyperplexes **which could generalise hypermaps. This work is still in progress, and is likely to enable people working on maniplexes to exploit research on incidence geometries, and vice-versa.**

Cayley Maps and Skew Morphisms

[Workshop leader: Robert Jajcay]

A skew-morphism of a group A is a permutation φ of A preserving the identity and a function $\pi : A \to \mathbb{Z}_{|\varphi|}$, called the power function associated with φ, satisfying the property $\varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b)$ for all $a, b \in A$. Skew-morphisms were originally introduced for the study of regular Cayley maps, but eventually became significant in the study of complementary cyclic group extensions.

An orientable map $\mathcal{M}$ is called orientably-regular if for every pair of arcs of $\mathcal{M}$ there exists an orientation-preserving automorphism of $\mathcal{M}$ that takes the first arc to the second. A Cayley map $CM(A, X, p)$ is an embedding of a connected Cayley graph $C(A, X)$, that has the same local orientation p at each vertex. All left multiplications within the Cayley group A induce automorphisms of the Cayley map, and many of the well-known families of orientably-regular maps turn out to be Cayley maps. A Cayley map $CM(A, X, p)$ is regular if and only if there exists a skew-morphism φ of A with the property that $\varphi(x) = p(x)$ for all $x \in X$. Thus regular Cayley maps on A correspond to orbits of skew-morphisms of A that generate A and are closed under inverses.

In the context of group extensions, if A is a group and φ is a skew-morphism of A, then the skew-product of A and $\langle\varphi\rangle$ is defined as $G = (A \times \langle\varphi\rangle, *)$ under the operation $(a, \varphi^i) * (b, \varphi^j) = (a\varphi^i(b), \varphi^{\sigma(i,j)})$, for a suitably defined function $\sigma(i, j)$. This product is a group, with a complementary factorisation $G = AY$. Conversely, if G is any finite group with a complementary factorisation $G = AY$ where Y is cyclic, then the quotient $\overline{G} = G/\mathrm{Core}_G(Y)$ is a skew product group associated with the skew morphism φ. Hence the classification of skew-morphisms of a finite group A allows for classification of all regular Cayley maps for A, as well as all complementary extensions of a group G in the form $G = AY$ with Y cyclic.

Much effort has been devoted recently to classifying skew-morphisms of various classes of groups. Classifications have been achieved for finite abelian groups of odd prime-power order, elementary abelian 2-groups, and groups whose order is a product of two primes. (Various product results have also been found.) It was announced at the workshop that the classification of skew-morphisms of finite dihedral groups has been completed. The classification for all abelian finite groups now seems within reach. On the other side of the spectrum, the classification of skew-morphisms of finite simple groups was also announced.

Also discussed at the workshop was a generalised definition of skew-morphisms, related to the concept of a generalised Cayley map (which is a map admitting a group of automorphisms that acts regularly on vertices). A suitable definition has been found, and basic properties have been determined. The concept appears to be related to the theory of 2-extensions of groups.

n-gonal Surfaces

[Workshop leaders: Mariela Carvacho and Aaron Wootton]

An n-gonal morphism of a Riemann surface S is map $\phi : S \to \mathbb{P}^1$ of degree n, and an n-gonal surface is one that exhibits such a ϕ. When ϕ is regular, we say that S is regular with group of deck transformations $\mathrm{deck}(\phi) = G$. Literature on n-gonal morphisms is scant, most being limited to regular morphisms with $G = \langle\sigma\rangle$ cyclic, and the so-called cyclic n-gonal surfaces) Most of the discussion at and after this workshop focused on generalising known theory of cyclic n-gonal surfaces to other regular n-gonal surfaces.

(a) ***The cyclic n-gonal case.***

A complete picture is known for cyclic n-gonal surfaces and their automorphism groups when n is a prime number. This was made possible through a consequence of the Castelnuovo–Severi theorem, which states that if the genus g of a cyclic p-gonal surface S satisfies $g > (p-1)^2$, then $G = \langle\sigma\rangle$ is normal in the full automorphism group of S. When n is not prime, additional assumptions on the automorphism group, or on the branching data of the quotient map $\phi : S \to S/\langle\sigma\rangle = \mathbb{P}^1$ yield a similar inequality. In particular, a similar complete description appears to be tractable.

(b) ***Strong branching.***

Much of the above theory depends on the concept of 'strong branching' and the normality of G. A covering $f : S_1 \to S_2$ of Riemann surfaces of degree n is said to be *strongly branched* if $g_1 > n^2 g_2 + (n-1)^2$, where g_i is the genus of S_i

(for $i = 1, 2$). Accola proved that when f is a strongly branched regular map with $\mathrm{deck}(f) = G$ and G is simple, then $G \lhd \mathrm{Aut}(S_1)$.

(c) *Questions.*

At the workshop, the following questions were posed:

- **What conditions can be imposed on G to ensure that $G \lhd \mathrm{Aut}(S)$, and in particular, what weakening of 'strongly branched' can be used?**
- **Given a regular n-gonal surface S, how can $\mathrm{Aut}(S)$ be computed for non-normal G?**

(d) *Discussion, outcomes and further work.*

- **It was proved that if the core of G in $\mathrm{Aut}(S)$ is trivial, then $G \lhd \mathrm{Aut}(S)$ when ϕ is strongly branched.**
- **It was agreed that a list should be built of examples of pairs $G < \mathrm{Aut}(S)$ where S/G has genus 0, to help provide a conjectural picture.**
- **Kay Magaard explained the work of Guralnick and Thompson on monodromy groups of rational maps. It was envisioned that this could be used as a tool in answering the second question in (c) above.**

External Symmetries of Regular Maps

[Workshop leader: Jozef Širáň]

A map M on a surface is called regular if its automorphism group $\mathrm{Aut}(M)$ acts transitively on incident vertex-edge-face triples. Every such map has constant valency m and constant face-size ℓ, and this pair of numbers defines the type of M. Also every such M may be identified with a presentation for its automorphism group G in the form $G = \langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^\ell = (bc)^m = (ca)^2 = \ldots = 1\,\rangle$, as a quotient of the full $(\ell, m, 2)$-triangle group, and denoted by the symbol $M(G; a, b, c)$.

This working group explored duality, Petrie-duality and combinations of these, as well as exponents for self-invariance under the Coxeter 'hole' operators. Together, these concepts are sometimes colloquially called external symmetries of regular maps, and a regular map of type (ℓ, m) is said to be kaleidoscopic if it admits all exponents in the multiplicative group of units mod m.

A regular map $M(G; a, b, c)$ is self-dual if its group G admits an automorphism that swaps a with c while fixing b, and self-Petrie-dual if G admits an automorphism that swaps a with ac while fixing b and c, and kaleidoscopic if for every unit j mod m the group G has an automorphism that fixes a and c while taking b to $(bc)^j c$. Then further, M is said to have trinity symmetry (or to be completely self-dual, or 'self-everything') if it is both self-dual and self-Petrie-dual, and to be super-symmetric if it is kaleidoscopic and has trinity symmetry as well.

In his introductory lecture, Jozef Širáň cited four research-driving questions:

(a) **Does there exist a completely self-dual regular map of valency n for every odd $n \geq 5$?**
(b) **Does there exist a kaleidoscopic completely self-dual regular map of valency n for every odd $n \geq 5$?**
(c) **What kind of structure has the external symmetry group of a kaleidoscopic completely self-dual regular map?**
(d) **Is it true that for every $m \geq 3$ and every subgroup U of $C_m^* \times C_2$, there exists a non-orientable regular map of valency m with exponent group U?**

Affirmative answers to the first two questions for all even valencies are already known, and the analogue of (d) in the orientable case was shown to be true by Conder and Širáň in [3]. Also for valency 8 it was also shown by Conder, Kwon and Širáň [4] that the order of the external symmetry group can be larger than any pre-assigned positive integer.

The working group focused on Question (a). Because explicit generating triples of involutions can be found in the case where $G = PSL(2, q)$ or $G = PGL(2, q)$, it was suggested that suitable candidates for completely self-dual regular maps of arbitrary valency $m \geq 5$ should be sought in these families of groups. Marston Conder checked and confirmed the feasibility of such an approach for odd valencies 5 to 17 (with the help of the MAGMA computer system [2]), and he and Steve Wilson and Jozef Širáň made further suggestions about how this problem might be approached algebraically. The group also had a brief discussion about a possible approach to Question (b), using parallel products (which were developed several years ago by Steve Wilson and recently re-considered by Gareth Jones).

Some of the participants in this working group will take these investigations further.

Pseudo-real Riemann Surfaces, and Large Group Actions on Surfaces

[Workshop leaders: Javier Cirre and Grzegorz Gromadzki]

A compact Riemann surface X is said to be pseudo-real if it admits anti-conformal automorphisms but no such automorphisms of order 2. Any such surface lies in the so-called real moduli of the moduli space of compact Riemann surfaces, but cannot be defined by real polynomials. There is no pseudo-real surface of genus 0 or $1,$ but it has been known since 2010 that there exist pseudo-real surfaces of genus g for every $g \geq 2.$

An important aspect of current research on pseudo-real surfaces has to do with the largest order $M(g)$ of the full automorphism group of a pseudo-real surface of genus $g \geq 2.$ It is known that $M(g) \leq 12(g-1),$ and that this upper bound is attained for infinitely many values of $g,$ but before this workshop, relatively little was known about lower bounds for $M(g)$. Accordingly, several questions regarding potential lower bounds for $M(g)$ were considered during the workshop. It was proved that $M(g) \geq 2g$ for all even $g,$ and that $M(g) \geq 4(g-1)$ for all odd $g.$ This left the question of sharpness of these lower bounds.

In some work that took place after the workshop, a proof of sharpness of the bound $M(g) \geq 4(g-1)$ for odd g was obtained for a large (and likely infinite) set S of odd values of g. Sadly a similar theorem for even genera seems well out of reach at this stage.

In the case of conformal actions on all Riemann surfaces, the values of the corresponding parameter $M(g)$ are known to lie between $8(g+1)$ and $84(g-1)$, by the celebrated theorems of Accola–Maclachlan [1, 8] and Hurwitz [6] respectively. The precise values of $M(g)$ are known for $2 \leq g \leq 300$ and for some particular series of values for g, but despite a lot of work on the topic, this function is still not well understood.

Accordingly, instead of looking at individual values of $M(g)$, it makes sense to study the asymptotic properties of the function M, via the set $\mathcal{A}^d$ of accumulation points of the set $\mathcal{A}$ of values of the ratio $M(g)/g$. It was known before the workshop that for the second derived set $\mathcal{A}^{(2)} = (\mathcal{A}^d)^d$, we have $12 \in \mathcal{A}^{(2)} \subseteq \{8, 12\}$, and during the workshop we succeeded in essentially reducing the problem of deciding whether or not $8 \in \mathcal{A}^{(2)}$ to deciding whether or not certain very particular Belyi actions exist.

Articles on both of the above topics are planned to be written.

Short talks

The following 20-minute talks on topics related to the workshop themes were given by workshop participants, on the Monday, Tuesday and Wednesday afternoons:

- **Jen Paulhus: A database of group actions**
- **Alina Vdovina: Buildings and generalisations of dessins**
- **Dimitri Leemans: Almost simple groups and polytopes**
- **Alexander Zvonkin: Diophantine invariants of dessins d'enfants**
- **Becca Winarski: Homomorphisms between mapping class groups of surfaces**
- **Roman Nedela: Skew morphisms of cyclic groups and complete regular dessins**
- **Charles Camacho: Counting quasiplatonic cyclic group actions of order n**
- **Shaofei Du: Nilpotent primer hypermaps with hypervertices of prime valency**
- **Milagros Izquierdo: Dessins d'enfants and a curve of Wiman**
- **Dimitri Leemans: Almost simple groups and polytopes**
- **David Torres: Teichmüller curves and Hilbert modular surfaces.**

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Chapter 21

p-adic cohomology and arithmetic applications (17w5118)

October 1 - 6, 2017

Organizer(s): Ambrus Pal (Imperial College London), Tomoyuki Abe (Kavli Institute for the Physics and Mathematics of the Universe), Kiran Kedlaya (University of California, San Diego), Christopher Lazda (Universita di Padova)

Overview

The subject of p-adic cohomology, a Weil cohomology theory for algebraic varieties defined over fields of positive characteristic p, has a long and distinguished history. It began with Dwork's proof in [Dwo60] of one of the most important results in arithmetic geometry, the rationality of zeta functions of algebraic varieties defined over finite fields, originally conjectured by Weil. This proof in fact predated that given by Artin and Grothendiek, and, very surprisingly at the the time, used p-adic analytic methods. Soon the cohomology theory that was hiding behind these methods was identified by Monsky and Washnitzer in [MW68]. At the same time, Grothendieck and Berthelot introduced crystalline cohomology [Ber74,Gro68], motivated by the desire to capture p-torsion phenomena in the cohomology of varieties in characteristic p. These two approaches were later unified, at least rationally, via Berthelot's theory of rigid cohomology [Ber97b]. After a period of intensive work, due to the efforts of Kedlaya, Caro, and others, the theory reached a new level of sophistication with the proof of Grothendieck's six functors formalism, replicating those properties of ℓ-adic cohomology which occupied Grothendieck and his school for many years, and considered as a central achievement in arithmetic geometry.

While a major driving factor in the subject is often expected analogies with other cohomology theories, such as ℓ-adic étale cohomology or algebraic de Rham cohomology, in fact p-adic cohomology generally contains much more information, and displays a wealth of extra subtlety that both complicates and enriches the subject. Indeed, as Ogus has long argued, crystalline cohomology is often much more closely analogous to Hodge theory for complex varieties, for example in how one can use p-adic invariants to control certain deformation problems in mixed characteristic or equicharacteristic p.

This added subtlety makes p-adic cohomology a more challenging and intricate subject of study than the ℓ-adic theory, and is partially the reason why progress in the field has often been slower and more painstaking, usually lagging behind results obtained for other Weil cohomology theories. When the p-adic world does catch up, however, the results can be far more significant that their ℓ-adic counterparts. A good example of this is provided by the (full) Weil conjectures: although the ℓ-adic proof preceded the purely p-adic proof by over 30 years, it is only by using p-adic cohomology that we now have algorithms that can actually compute the zeta functions of varieties [Ked04, Lau06].

Recent years have seen somewhat of a maturation of the subject; many basic foundational results are now settled, and interest has started to turn towards applications of these results to long standing arithmetic questions and conjectures, some of which were the original motivation for the development of p-adic cohomology. The most important of these are, in no particular order:

- various manifestations of p-adic Hodge theory;
- the p-adic Langlands correspondence both for number fields and function fields;
- applications of the theory of arithmetic $\mathcal{D}$-modules to p-adic representation theory;
- the p-adic properties of special values of L-functions, including refined conjectures of the Birch–Swinnerton-Dyer and Beilinson–Bloch–Kato type;
- Iwasawa theory in positive characteristic.

Another important new research direction is to explore some more refined aspects of the theory such as the question of integrality and p-torsion phenomena. New avenues are starting to be opened up in extending the scope of p-adic cohomology beyond the classical situation of perfect ground fields. There has also been a surge of interest in the more subtle cousin of p-adic cohomology theory, namely p-adic homotopy theory, which attempts to apply p-adic methods to study non-abelian phenomena on varieties in characteristic p.

These two directions of research are deeply interwoven. For example a major motivation for the study of integrality and p-torsion in p-adic cohomology is to develop tools for attacking refined Birch–Swinnerton-Dyer-, Beilinson–Bloch–Kato- and Iwasawa-type conjectures. The desire to develop p-adic homotopy theory is partially driven by problems encountered in the p-adic Langlands conjecture, and to have the usual homotopical tools available for the study of p-adic algebraic cycles, such as oriented cobordism theory. Part of the motivation for exploring overconvergent cohomology over non-perfect ground fields is to have an analogue of p-adic Hodge theory over Laurent series fields, especially for families of coefficients, and to develop tools for p-adic cycle and period maps, similar to those appearing in the regulators of Gros [Gro90] and the integration theory of Besser [Bes00].

There are two more areas of mathematics which are closely connected to the main topic of the conference: p-adic or non-archimedean analytic geometry, via the study of p-adic differential equations, and function field arithmetic, especially its local and cohomological aspects (see for example [?, ?]). Non-archimedean analytic geometry went through something of a revolution in the last few years, with the methodical introduction of a new concept of points, due to Berkovich and Huber. This strongly extended the scope of its foundational results, but at the same time streamlined the proof of many classical theorems. In the setting of p-adic differential equations these methods also found applications, and via the work of Baldassarri, Poineau–Pulita and Kedlaya [Bal10, PP13, Ked16] now we have beautiful sufficient conditions for the finiteness of cohomology for p-adic local systems on curves.

All these various strands of research were represented at the workshop, which drew together a wide range of researchers all with a broad interest in p-adic cohomology. There were a large number of excellent talks given by a diverse selection of speakers, all generating lively further discussions among participants. A lot of the work discussed at the workshop concerned exciting progress towards major open problems in the area, as well as interesting new perspectives that will provide fertile ground for the continuing development of the subject.

Main Themes

As we have already explained, all these exciting new trends emerging in the field are of course deeply interwoven, and the aim of the workshop was to encourage new progress in these areas by promoting both predictable and unpredictable synergies between them. Here we describe in more details the specific topics that were covered during the workshop, and some of the interactions that were generated.

Foundations and theory over non-perfect fields

Traditionally, p-adic cohomology theories have been expressed for varieties over perfect ground fields of characteristic p. While much of the theory still works over non-perfect fields, arithmetic considerations (in particular the general phenomenon of semistable reduction, as well as analogies with the ℓ-adic theory) lead one to expect certain refinements

of existing p-adic cohomologies (such as rigid cohomology) when working over such non-perfect fields. There have been several recent attempts to work out what such a more general theory might look like - for example in [LP16] the case of Laurent series fields over perfect fields was considered, and in [LS17] B. Le Stum has given a completely general definition of "overconvergent cohomology" for locally Noetherian schemes.

During the workshop, a major advance in these foundational matters was announced in the talk given by Richard Crew on Rings of arithmetic differential operators on tubes. He described an extension of Berthelot's theory of arithmetic differential operators to a class of morphisms of adic formal schemes that are not necessarily of finite type, or even adic, by introducing a new finiteness condition, that of being 'universally Noetherian'. After an overview of the necessary commutative algebraic foundations, he went onto to explain how one can use this theory to give generalisations of the theory of convergent and overconvergent isocrystals.

There are also other approaches to generalising the theory of arithmetic $\mathcal{D}^\dagger$-modules, as studied for example by Caro–Vauclair [CV15]. These are based on a slightly different finiteness condition - being 'p-smooth' rather than 'universally Noetherian'. These two conditions are somewhat orthogonal, and provide good frameworks in which to study Grothendieck's 6 operations in different situations. One particularly interesting question would be to see whether or not there is a natural way to unify these two approaches.

The Langlands program and links with representation theory

Berthelot's theory of arithmetic $\mathcal{D}^\dagger$-modules was created to try to give a good analogue of the theory of algebraic $\mathcal{D}$-modules in mixed characteristic or characteristic p situations, and thanks to recent work of Caro, many of the foundations are now essentially in place.

In the classical case, over the complex numbers, or more generally fields of characteristic zero, one especially important area of application is to the study of representation theory, the most famous example of this being the Beilinson–Bernstein correspondence. It is similarly hoped that the theory of arithmetic $\mathcal{D}^\dagger$-modules will prove to be a powerful tool in the study of representation theory, in particular that of p-adic Lie groups, and thus in the p-adic Langlands program. A similar such application of $\mathcal{D}$-module theory on rigid analytic spaces over p-adic fields has been already found by Ardakov–Wadsley [AW13], who used their theory to answer some representation theoretical problems which arose in the local Langlands program.

There is also closely related work of Huyghe, Patel, Schmidt and Strauch on localisation theorems in the setting of arithmetic $\mathcal{D}^\dagger$-modules [HPSS15], in which they show that there is an equivalence of categories between the category of locally analytic admissible representations of some split reductive group over a finite extension of $\mathbb{Q}_p$, and the category of coadmissible arithmetic $\mathcal{D}^\dagger$-modules over the rigid analytic space attached to the flag variety of the group. The talk by Matthias Strauch entitled Arithmetic structures in sheaves of differential operators on formal schemes and $\mathcal{D}^\dagger$-affinity reported on this exciting piece of work. First he defined certain integral structures, depending on a congruence level, for the sheaves of differential operators on a formal scheme which is a blow-up of a formal scheme which itself is formally smooth over a complete discrete valuation ring of mixed characteristic. When one takes the projective limit over all blow-ups, one obtains the sheaf of differential operators on the associated rigid space, introduced independently by K. Ardakov and S. Wadsley. In the second part he explained what it means for a formal model of a flag variety to be $\mathcal{D}^\dagger$-affine (this concept is analogous to that of Beilinson–Bernstein and Brylinski–Kashiwara in the algebraic context), and he then explained an example illustrating how to use these results together with methods from rigid cohomology to analyze locally analytic representations of p-adic groups.

Similarly, the 6 operations formalism has been used by Abe [Abe13] to prove a p-adic Langlands correspondence in the function field setting, and thus verify Deligne's "petits camarades cristallins" conjecture on the existence of p-adic companions to compatible systems of ℓ-adic Galois representations (at least over curves). This has been the starting point of a lot of new research which we will talk about in more detail in the next section.

Higgs bundles, Simpson correspondence, Simpson's conjecture and p-adic companions

Since the foundational work of Grothendieck, Deligne, and Tate, it has been clear that ℓ-adic local systems (or more generally, complexes of ℓ-adic constructible sheaves) should be the ℓ-adic incarnation of a family of motives in characteristic p. The situation for p-adic coefficient objects was only more recently clarified, through foundational work of P. Berthelot and R. Crew and, more recently, breakthroughs due to D. Caro and especially T. Abe.

As we already mentioned above, Abe used his robust p-adic cohomology theory to resolve part of Deligne's Companions Conjecture from Weil II for curves: he constructs, for every sufficiently irreducible lisse ℓ-adic sheaf on a smooth curve $X/\mathbb{F}_q$, a compatible overconvergent F-isocrystal. However the p-adic Companions Conjecture for higher dimensional bases remains wildly open. Morally, this is because p-adic coefficent objects contain dramatically more information than their ℓ-adic cousins; they look and act much more like complex variations of Hodge structures. Nevertheless, in the talk by Kiran Kedlaya entitled Update on the companion problem, the current status of the problem was surveyed. He first described the problem in detail, and the previous work of Drinfeld, Deligne, and Abe–Esnault, then he talked about his new result, which establishes a weak form of the conjecture.

In Simpson's foundational work on non-abelian Hodge theory, he conjectures that rigid local systems on smooth, complex, quasi-projective varieties are of geometric origin. This problem was discussed in the talk given by Hélène Esnault entitled Rigid systems and integrality, where she reported on her remarkable joint work with M. Gröchenig. She described how they proved that the monodromy of a cohomologically rigid integrable connection (E, ∇) on a smooth complex projective variety X is integral, a statement sometimes known as Simpson's integrality conjecture. In order to achieve this, they first proved that the mod p reduction of a rigid integrable connection (E, ∇) has the structure of an isocrystal with Frobenius structure. She also explained how they proved that rigid integrable connections with vanishing p-curvatures are unitary, and how they used the latter result to prove new cases of Grothendieck's p-curvature conjecture. The result is both interesting for its philosophical implications and the link it creates between p-adic cohomology and Simpson's original conjecture.

A fundamental tool used by Esnault–Gröchenig is a certain p-adic analogue of Simpson's correspondence, relating local systems with Higgs bundles. The talk by Bernard Le Stum on A quantum Simpson correspondence was a report on a result of this type, but for 'quantum' local systems, i.e. non-commutative deformations through a formal parameter q. He first explained how the Simpson correspondence establishes an equivalence between a category of modules equipped with a connexion and a category of Higgs bundles. Then he talked about his joint work with Michel Gros and Adolfo Quirós, where they describe such an equivalence in the case of modules equipped with a q-connexion when q is a p-th root of unity. This is modelled on the characteristic p case and relies on the notion of quantum divided powers that he also discussed.

In contrast with what happens over the complex number field, for normal varieties over a finite field, recent work of Deligne, Drinfeld, Abe–Esnault, and Kedlaya establishes that for fixed ℓ, there are only finitely many ℓ-adic coefficient objects with finite determinant, bounded degree, and 'bounded ramification'. This suggests that they should exhibit some form of rigidity, and that some form of Simpson's conjecture should be true for them. Indeed, in the talk by Subrahmanya Krishnamoorthy entitled Rank two F-isocrystals and abelian varieties the speaker reported on his joint work-in-progress with Ambrus Pál on a Simpson-like conjecture for F-isocrystals. This conjecture states that for smooth varieties X over finite fields, any overconvergent F-isocrystal which 'looks' as though it should appear in the cohomology of a family of abelian varieties over X does in fact come from such a family. The methods are similar to the work above: it starts with the curve case, which can be proved via a refined form of the Langlands correspondence, then uses tools from geometry to extend the abelian variety from a sub-curve to the whole variety, using Serre–Tate type deformation theory and algebraization results.

Finally let us mention the talk by Paul Ziegler entitled Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration, which was a report on a very surprising application of p-adic methods to the study of Higgs bundles. The speaker talked about a recent proof, joint with M. Gröchenig and D. Wyss, of a conjecture of Hausel and Thaddeus which predicts the equality of suitably defined Hodge numbers of moduli spaces of Higgs bundles with SL_n- and PGL_n-structure. The proof, inspired by an argument of Batyrev, proceeds by comparing the number of points of these moduli spaces over finite fields via p-adic integration. This then provides information about Hodge numbers through the Weil conjectures and p-adic Hodge theory. Particularly exciting is the prospect of using these methods to shed new light on aspects of the Langlands program.

The de Rham–Witt complex, Iwasawa theory and aspects of integral p-adic cohomology

One of the original motivations of Grothendieck and Berthelot for inventing crystalline cohomology, as a p-adic companion to the family of ℓ-adic cohomologies produced by the étale theory, was to explain p-torsion phenomena. While integral crystalline cohomology achieves this for smooth and proper varieties, the extension to a 'good' cohomology theory for arbitrary varieties, which reached its zenith in the proof of the 6 operations formalism by Caro, has been achieved only for rational coefficients, i.e. after tensoring with $\mathbb{Q}$. This therefore still leaves open the question of what

an integral p-adic theory should look like for open or singular varieties, which has been the subject of much recent work in the field, in particular the study of the overconvergent de Rham–Witt complex by Davis, Langer and Zink [DLZ11]. This now seems to provide a good candidate for smooth (but possibly open) varieties, although a priori it is not clear how it compares with other candidates such as integral Monsky–Washnitzer cohomology, or even if the latter is well-defined.

Such a comparison theorem between these two approaches was the subject of the talk given by Veronika Ertl, entitled Integral Monsky-Washnitzer and overconvergent de Rham-Witt cohomology. In it she reported on joint work with Johannes Sprang in which they first of all show that Monsky–Washnitzer cohomology is well defined on the integral level, and secondly prove a comparison isomorphism between this and the overconvergent de Rham–Witt cohomology. This extends previous work of Davis and Zurieck-Brown by removing all restrictions on the cohomological degree. Interesting further questions in this direction would be to calculate the p-torsion of these groups in new and interesting cases, and to give a geometric interpretation of this torsion.

The study of integral properties of p-adic cohomology is very closely related to that of the p-adic properties of L-functions in characteristic p. Most of the work recently has been done on 1-dimensional families of abelian varieties, for example [KT03], [Pal10] and [TV15] which look at the refined Birch–Swinnerton-Dyer conjecture, the integrality of p-adic L-functions and the equivariant Tamagawa number conjecture, respectively. What is common in these works is the crucial use of integral p-adic cohomology theories predating the construction in [DLZ11], typically log crystalline cohomology. Therefore they are forced either to reduce the general case to the semi-stable one, or worse, restrict to the situation when the abelian scheme is semi-stable and the considered Galois covers of the base are tame. This demonstrates the limitations of these methods, but with sufficient progress on the finiteness properties of the the overconvergent de Rham–Witt complex we expect that this area would start to develop very rapidly.

Another talk with connections to integral aspects of p-adic cohomology and Iwasawa theory was by Ambrus Pál on the subject of Formal deformations of crystals and arithmetic applications. The speaker first described two problems in the arithmetic of elliptic curves over function fields, the analogue of a classical conjecture of Stevens in the function field setting, originally asked by Mazur, and his own conjecture on the integrality properties of Hecke eigenforms attached to elliptic curves. He then spoke about his work in progress on these conjectures, using tools from p-adic cohomology. The first ingredient is a pro-representability theorem of what could be called arithmetic deformations of crystals and Dieudonné crystals. The second ingredient is the application of the Taylor–Wiles method in this setting, which shows that the first conjecture implies the second.

One exciting recent trend in integral p-adic cohomology, and in particular integral p-adic Hodge theory, has seen the application of methods and techniques from higher category theory and higher algebra to the study of crystalline cohomology and K-theory. The talk given by Lars Hesselholt on Higher algebra and arithmetic provided an entertaining introduction to some of these ideas, and how they can be used to circumvent problems in arithmetic geometry, in particular in characteristic p, caused by the introduction of denominators.

As explained in the talk, these often arise because of the fact that the natural numbers record only the result of counting, and not the process of counting. The higher algebra of Joyal, Lurie and others replaces the initial ring of algebra $\mathbb{Z}$ by a more fundamental object, the sphere spectrum $\mathbb{S}$, which also retains information about the counting process. The 'brave new algebra' long advocated by Waldhausen uses $\mathbb{S}$ as the basis for arithmetic, and doing so can often lead to the elimination of 'unwanted' denominators.

Notable manifestations of this vision include the Bökstedt–Hsiang–Madsen topological cyclic homology, which receives a denominator-free Chern character, and the related Bhatt–Morrow–Scholze integral p-adic Hodge theory, which makes it possible to exploit torsion cohomology classes in arithmetic geometry. In the talk, the speaker explained the construction of a certain 'higher' analogue of de Rham cohomology, and gave some calculations in p-adic and characteristic p settings. In the former case, one recovers quite naturally certain constructions that appear in integral p-adic Hodge theory. In the latter, one can use this theory to give a cohomological interpretation of the Hasse-Weil zeta function of smooth and proper varieties over finite fields by regularized determinants, as envisioned by Deninger.

The recent revolution in the higher category theory and higher algebra has the potential to become a powerful tool in all sorts of areas of mathematics. This talk gave some excellent concrete examples of this potential in the particular fields of algebraic K-theory and p-adic Hodge theory.

F-isocrystals and p-adic representations, and homotopy theory

Most current research on the homotopy-theoretical aspects of p-adic cohomology concentrates on two particular strands: the theory of the p-adic, or crystalline, fundamental group via various Tannakian categories of isocrystals, and applications of the latter to Diophantine problems. The talk by Atsushi Shiho entitled On de Jong conjecture concerned the first aspect, covering a fascinating conjecture due to de Jong predicting that any isocrystal on a geometrically simply connected smooth projective variety over a perfect field of characteristic $p > 0$ should be constant. The speaker reported on his joint work in progress with Hélène Esnault where they proved several results related to this conjecture, including some special cases and a reduction to the case of convergent isocrystals.

On the other hand the talk by Ishai Dan-Cohen entitled Rational motivic path spaces concerned the second aspect. A central ingredient in Kim's work on integral points of hyperbolic curves is the "unipotent Kummer map" which goes from integral points to certain torsors for the pro-unipotent completion of the fundamental group, and which, roughly speaking, sends an integral point to the torsor of homotopy classes of paths connecting it to a fixed base-point. The speaker reported on recent joint work of his with Tomer Schlank, which provides liftings of Kim's construction to more refined rational homotopy types, and thus gives rise to factorizations of Kim's original period map. He explained the construction of a certain space Ω of rational motivic loops, and use this to provide a factorization of the unipotent Kummer map which can be summarized schematically as

$$\text{points} \to \text{rational motivic points} \to \Omega\text{-torsors} \to \pi_1\text{-torsors.}$$

For affine curves, Ω-torsors are the same thing as π_1-torsors, essentially because these spaces are $K(\pi, 1)$'s, however, their approach has the potential to give more refined information for higher dimensional varieties.

A key link between the p-adic, and the usual étale fundamental groups is given by the Katz correspondence, or the study of unit root F-isocrystals, and the associated p-adic Galois representations. The talk by Joe Kramer-Miller on Slope filtrations of F-isocrystals, log decay, and genus stability for towers of curves reported on work which shed new light on this classical, but very important connection. The speaker introduced a notion of F-isocrystals with logarithmic decay, gave a conjectural description of how this should relate to the slope filtrations, and sketched a proof of this conjecture when the unit-root subcrystal has rank one. This then leads to a new proof of a recent theorem of Drinfeld–Kedlaya, as well as a generalized version of Wan's conjecture on genus stability for towers of curves coming from geometry.

Other topics

Finally we report on those activities of the workshop which cannot be easily classified into one of the categories mentioned above, but nevertheless demonstrate the richness of the subject and its many connections to other areas of algebraic geometry and number theory.

The talk by Gebhard Böckle entitled Compatible systems of Galois representations of global function fields reported on joint work of the speaker with W. Gajda and S. Petersen on the important open image conjecture. If K is a finitely generated infinite field, then it was shown recently by Cadoret, Hui and Tamagawa that for almost all ℓ, the cohomology group $H^i_{\text{ét}}(X_{K^s}, \mathbb{F}_\ell)$ is semisimple as a representation of the geometric fundamental group $G_K^{\text{geo}} = \text{Gal}(K^s/K\mathbb{F}_p^s)$. This has an important consequences for the image of G_K^{geo} under its action on the adèlic module $H^i_{\text{ét}}(X_{K^s}, \mathbb{A}_\mathbb{Q})$. The speaker described how to extend these results to the case where $\mathbb{F}_\ell$ is replaced by the mod-ℓ reduction of a compatible system of Galois representations, using automorphic methods.

The talk given by Masha Vlasenko on Atkin and Swinnerton-Dyer congruences for toric hypersurfaces was about a new result on a classical application of p-adic cohomology to arithmetic. It concerned certain kinds of explicit crystals called Dwork modules, which in the 1990's were used by V. Batyrev to describe the mixed Hodge structure on the middle cohomology of affine hypersurfaces in algebraic tori. In the talk the speaker reported on her work in progress done jointly with Frits Beukers, in which they they used these crystals to show several p-adic congruences for the coefficients of powers of a Laurent polynomial. This then leads to congruences involving the L-function of toric exponential sums, and yields new p-adic unit-root formulas.

One of the early indications of the power and utility of p-adic methods was the Serre–Tate theorem on the deformation theory of abelian varieties, showing that this is in fact controlled by the deformation theory of its p-divisible group, or equivalently its first crystalline cohomology. This basic idea of using p-adic invariants to control deformation problems has proved to be extremely fruitful, and the talk given by Ananth Shankar provided an excellent example of this. Entitled Serre-Tate theory for Shimura varieties of Hodge type, it was a report on work of the speaker and his coauthor Rong Zhou in which they studied the formal neighbourhood of a point in μ-ordinary locus of an integral model of a Hodge type Shimura variety. They proved the existence of a structure analogous to the Serre–Tate structure on the deformation space of an ordinary abelian variety, in particular providing an analogue of 'canonical lifts' in these situations.

Finally, the talk by Edgar Costa entitled Computing zeta functions of nondegenerate toric hypersurfaces was a very useful update on the ongoing revolution on the point counting problem via p-adic methods. The speaker reported on an ongoing joint project with Kiran Kedlaya and David Harvey on the computation of zeta functions of nondegenerate toric hypersurfaces over finite fields, using p-adic cohomology. By exploiting a new way of computing sparse approximations to overconvergent cohomology classes, they were able to drastically improve the feasibility of computing zeta functions of certain smooth varieties, in particular for much larger values of p than were previously accessible.

Outcomes

The subject of p-adic cohomology is often characterised by a plethora of different approaches to the subject, each of which has its own particular perspective and scope of application. The workshop successfully drew together people working on all aspects of the theory, and the various topics generated many discussions, kick-starting new research and collaborations across the whole breadth of the subject.

For example, the talk given by Veronika Ertl inspired much speculation on the expected properties and possible scope of any integral theory of p-adic cohomology. Richard Crew in particular suggested that in fact a fully robust theory cannot exist for finite coefficients, and sketched during discussions the reasons why.

Another topic that was constantly in the air during the workshop concerned problems surrounding the Companion Conjecture, the main remaining case now being how to produce p-adic companions on higher dimensional varieties. This is a topic that several of the participants at the workshop have been working on recently, and one in which we can hope to see some more progress in the near future.

Let us also mention several interesting discussions that took place concerning the p-adic homotopical properties of varieties in characteristic p, and to what extent these mirror or deviate from 'classical' topological behaviour. As well as basic foundational matters, such as deformation invariance, or the existence of moduli spaces for p-adic local systems, there were also interesting problems posed on the behaviour of particular kinds of isocrystals, often inspired by Hodge theory and Simpson-like conjectures. This reflects a general shift in focus in the subject, and is likely to become an important area of future research. Some of these debates have been followed up since the conference, and have formed the germ of potential new collaborations.

Participants

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Chapter 22

New Perspectives in Representation Theory of Finite Groups (17w5003)

October 15 - 20, 2017

Organizer(s): Gunter Malle (TU Kaiserslautern), Gabriel Navarro (University of Valencia), Britta Späth (Bergische Universität Wuppertal), Pham Huu Tiep (Rutgers University)

Overview of the Field

Group Theory is essentially the theory of symmetry for mathematical and physical systems, with major impact in diverse areas of mathematics. The Representation Theory of Finite Groups is a central area of Group Theory, with many fascinating and deep open problems, and significant recent successes. In 1963 R. Brauer [B] formulated a list of deep conjectures about ordinary and modular representations of finite groups. These have led to many new concepts and methods, but basically all of his main conjectures are still unsolved to the present day. A new wealth of difficult problems, relating global and local properties of finite groups, was opened up in the seventies and eighties (of the 20th century) by subsequent conjectures of J. McKay [Mc], J. Alperin [Al], E. Dade [D1, D2], M. Broué [Br], and others, all remain open up to date.

The classification of finite simple groups raised the hope that one should be able to reduce some of the aforementioned conjectures to statements about simple groups, and subsequently establish these statements by exploring deep knowledge about simple groups provided by the Deligne–Lusztig theory and other recent fundamental results in representation theory. In fact, this reduction to simple groups was accomplished by several of the organizers in [IMN] for the McKay conjecture, in [NT1] for the Alperin weight conjecture and in [S1], [S2] and [S3] for the blocks versions. This hope has recently materialized since the proof of the McKay conjecture for the prime 2 by two of our organizers: G. Malle and B. Späth [MS]. The road to prove the McKay conjecture for odd primes is now paved.

Recent Developments and Open Problems

There were two main themes in our meeting, with several remarkable highlights, which are discussed in more detail below. One of the directions that became even more apparent is the advent of powerful geometric methods to solve problems in finite group representation theory. This was witnessed for example in the impressive talks by Raphaël Rouquier and by Cedric Bonnafé. One of the most important problems in Representation Theory is to find Brauer's

decomposition matrices of symmetric groups and finite groups of Lie type. R. Rouquier announced an astonishing conjecture, joint work with O. Dudas, that will have a lasting impact on our field. The other theme has been remarkable progress on the so called McKay–Galois conjecture, proposed in 1994 by one of the organizers ([N]). Other themes are related to another open problem: how the absolute Galois group act on the irreducible characters of the simple groups.

Decomposition Numbers

Decomposition numbers are the data relating the modular representations of a finite group G over a field of positive characteristic to the generally better known complex characters Irr(G). The two representation theories (modular and over the complex numbers) are very different in nature. Complex characters were introduced by Frobenius at the end of the 19th century and from the start corresponding sets Irr(G) were quickly determined for many of the finite simple groups known at the time, as for example for the symmetric and alternating groups. Modular representation theory was only introduced much later by Brauer in the 1940's and very simple questions about the representations of the symmetric groups are still unanswered — and intensely investigated, including in the present meeting.

A related question is the one of basic sets, that are subsets of Irr(G) somehow standing for the much more mysterious irreducible modules over fields of positive characteristic.

Decomposition numbers and basic sets are thus a vital information to tackle conjectures about modular representations since their versions over the complex numbers are generally much simpler. Any progress in the knowledge of decomposition numbers for finite simple groups, i.e., now essentially groups of Lie type, is of foremost importance.

It was a big success of this meeting to arrange the presence of Raphaël Rouquier (UCLA) one of the main contributors to the field of representation theory as a whole, and a very fruitful presence in our meeting as testified by numerous participants (see testimonials below). The meeting was the place where it was announced what is probably a breakthrough in the knowledge on decomposition numbers. In joint work with Olivier Dudas (Paris), Rouquier gives a conjectural description of the decomposition matrix of finite unitary groups with respect to the so-called non linear (non-defining) primes. This is the main case left open by the first generation of work on the subject (Dipper–Gruber–Hiss, 1985–1997) treating essentially linear primes [Di], [GH]. This has been confirmed for dimensions up to n by comparison with the known results. Though still a conjecture this description has the advantage of being very precise and concrete. The matrix should correspond to an explicit operator ∇ on the algebra of symmetric functions defined in terms of Macdonald polynomials.

Rouquier adds even more perspective to the subject by relating this algebraic setup to algebraic geometry. Indeed he provides a categorification of the operator ∇ in the category of coherent sheaves on Hilbert schemes on the complex plane. This interpretation should also open the door to (conjectural) generalizations to other types of classical groups for non linear primes.

The McKay Conjecture and refinements

If G is a finite group, we denote by Irr(G) the set of irreducible complex characters of G and by Irr$_{p'}(G)$ its subset consisting of characters having degree not divisible by p. The McKay conjecture [Mc] from 1972 is at the center of representation theory. If p is a prime and G is a finite group, then it asserts that

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|\,,$$

where P is a Sylow p-subgroup of G and $\mathbf{N}_G(P)$ its normalizer. That is to say, a certain fundamental information on G is encoded in some local subgroup of G, namely the Sylow normalizer $\mathbf{N}_G(P)$.

This conjecture somehow gave a new direction to representation theory of finite groups. Many results and conjectures, even recently, would not have been possible without it. We have already mentioned the conjectures by Alperin–McKay, Alperin, Dade and Broué. In fact major contributions to the representation theory of finite groups of Lie type such as [BMM] originate in the same kind of ideas.

Since our previous meeting in 2014, there has been an absolute highlight in this subject: Malle and Späth (two of our organizers) proved the McKay conjecture for $p = 2$ (see [MS]), the case that was originally conjectured. This came after the reduction theorem by I. M. Isaacs, G. Malle and G. Navarro in [IMN], proving that the the path that started in

2007 is a successful one. Of course, now the big question is what happens if p is odd? In our meeting we had a fruitful session on a discussion on what remains to do next, and on several possible ideas to tackle this long-standing problem.

In 2004, Navarro proposed a deep refinement of the McKay conjecture [N] that takes into account fields of values over the p-adics. This conjecture has consequences in determining local structure of the group from global structure and vice versa. In our meeting, A. Schaeffer Fry announced a proof, by her and in part also in collaboration with J. Taylor, of the self-normalizing 2-Sylow conjecture, which is one of the most remarkable consequences of this McKay-Galois conjecture. This proof comes after 14 years since the conjecture was announced (and ten years after the proof [NTT] of the self-normalizing Sylow conjecture for odd primes). Recent work of Schaeffer Fry and Taylor also led to a proof of another conjecture of Navarro, Tiep, and Vallejo [NTV] concerning groups whose Sylow 2-normalizers contain a single irreducible 2-Brauer character (which was recently reduced to simple groups by Navarro and Vallejo). O. Brunat announced a proof of the McKay–Galois conjecture for symmetric groups and odd primes (the case $p = 2$ was announced in our 2014 meeting) and Lucas Ruhstorfer announced a proof of the conjecture for groups of Lie type in their defining characteristic.

In recent years, there has been considerable interest in fields of values of characters, and a main open problem in this area is to determine how the absolute Galois group acts on the irreducible complex characters of a group. D. Rossi announced a proof of one direction of the 3-rational conjecture (by two of the organizers, Navarro and Tiep) that asserts that the finite groups with 3 rational conjugacy classes are the finite groups with 3 rational characters. For the other direction, he was also able to get a very tight control over a minimal nonsolvable counterexample (if any).

J. Tent announced his $\mathbb{Q}_p$-conjecture, that would extend work of J. Thompson, W. Feit and R. Gow. Almost since the beginning of character theory, the connections between fields and characters have been studied. An object of interest is the so called rational groups, or groups all of whose characters have values in the field of the rationals. While the non-abelian composition factors of rational groups were classified by W. Feit and G. Seitz long time ago, it is a conjecture of J. Thompson that the abelian composition factors are cyclic groups of prime order p, for $p < 7$. The conjecture is still open. Tent conjectures that if the field of values of the characters are contained in the field $\mathbb{Q}_p$, then the abelian composition factors of G are also under absolute control. Perhaps, further relations hold for arbitrary fields.

Future directions

Recent progress on all these fundamental conjectures raises the hope that complete proofs of some of them may be possible in the not too distant future.

It has become even clearer in the past few years that any further significant progress can only be achieved once we can resolve a number of basic questions on representations of finite groups of Lie type. With many experts in the Deligne–Lusztig theory present at the meeting, we had a discussion session on Tuesday afternoon to draw a roadmap towards possible solutions of some of these principal obstacles.

Presentation Highlights

We already highlighted the spectacular announcement by Raphaël Rouquier of his and Dudas' conjecture on the decomposition numbers of the finite unitary groups, as well as the impressive talk of Cedric Bonnafé presenting his results, joint with P. Shan and R. Maksimau, that established a conjecture concerning equivariant cohomology and fixed points of Calogero–Moser spaces in the smooth case.

Jesper Grodal presented in Banff a novel approach to endo-trivial modules. These representations of finite groups over fields of positive characteristic p were introduced by Dade in his study of the Glauberman correspondence. While modules of a given finite group G over a field k of characteristic p have been shown to be in general beyond classification, the subset of endo-trivial has been shown to be endowed with the structure of a finitely generated abelian group $T_k(G)$ for the tensor product. This also relates with the Picard group of the stable category for the corresponding group algebra kG. Results by Alperin and Bouc–Carlson–Thévenaz (2001–2006, see [CT1]) have given a complete structure theorem in the case where G is a p-group.

In the general case of a finite group with Sylow p-subgroup S, one classically defines a restriction morphism

$$T_k(G) \to T_k(S)$$

whose kernel $T_k(G, S)$ contains in fact all the relevant information to deduce $T_k(G)$ from the case of p-groups thanks to work of Carlson–Thévenaz–Mazza–Nakano giving the image and proving the splitness of the above map. Grodal presented spectacular theorems about $T_k(G, S)$, identifying it with the first cohomology group of the orbit category $\mathcal{O}$ on non-trivial p-subgroups with values in the units $k^\times$, viewed as a constant coefficient system. He uses homotopical techniques to give a number of formulas for $T_k(G, S)$ in terms of one-dimensional representations of normalizers and centralizers. This allows him to prove a conjecture by Carlson–Thevenaz (see [CT2]). He also provides strong restrictions on when such representations of dimension greater than one can occur, in terms of the p-subgroup complex and p-fusion systems. He recovers and extends in a spectacularly simple fashion several computational results in the literature. The computational potential of his methods is illustrated by calculating $T_k(G, S)$ and therefore $T_k(G)$ in other sample new cases, e.g., for the Monster at all primes.

N. Rizo introduced a generalization of Brauer p-blocks with respect to an invariant character of a normal subgroup, which might give a different perspective and help with the reduction in some of the block theory conjectures, including the $k(B)$-conjecture. These blocks, which are related to twisted group algebras, seem to unify several statements, like Brauer's Height Zero conjecture and the Gluck–Wolf–Navarro–Tiep theorem [NT2].

B. Sambale reported on the efforts to find a counterexample to the defect group Morita equivalent block problem: what is the exact relationship between the defect groups of two Morita equivalent blocks? It has been claimed, and proved in important cases, that these groups might be isomorphic. The experience in group algebra isomorphism problems tell us, though, that there might exist counterexamples but that these might be too big to find, at least with computers.

P. Fong revisited his celebrated paper with Jon Alperin [AF] giving verifications of Alperin's weight conjecture for symmetric and general linear groups. This paved the way to subsequent verifications of the conjecture for other simple groups but also to the Dade conjecture. The paper has a strong group theoretical side that has been seminal to studies of distinguished classes of nilpotent subgroups of simple groups. But the talk by Fong exemplified rather the program of finding deep connections between combinatorics arising from Alperin's weight conjecture for the general linear group with the work of Lascoux–Leclerc–Thibon. Indeed a quite classical operation on partitions with a given r-core was re-interpreted by them in terms of the action of quantum groups on Fock space, thus making a combinatorial bijection more natural. This is of great relevance given how central this combinatorics or their relatives are to representations of finite groups.

A. Paolini lectured on representations of maximal unipotent subgroups U of finite groups G of Lie type. He discussed some history and methods about the problem of parametrizing the set Irr(U). He went on to mention some joint results with Goodwin, Le and Magaard about the parametrization of Irr(U) when G is of small rank, including the exceptional types F_4 and E_6.

H. Nguyen reported on several recent results, obtained by him and his collaborators, that utilize the average degree over a subset of complex irreducible characters of a finite group and generalize various classical theorems in Character Theory of Finite Groups, including theorems of Thompson and Ito–Michler.

The concept of regular embeddings, first formulated by G. Lusztig, is nowadays a standard reduction technique to deal with finite connected reductive groups with disconnected centre. J. Taylor lectured on smooth regular embeddings, a strengthening of this concept which includes the usual embedding of the special linear group into the general linear group. He presented a construction of smooth regular embeddings for any algebraic group, which is done using root data, and then used smooth regular embeddings to provide new proofs of unpublished reduction techniques due to Asai.

Another recent highlight was explained in the talk of Z. Halasi. He reported on the final solution of a 20 year old conjecture by L. Pyber on base sizes for permutation groups. Base sizes for permutation groups have outside applications to computer science. This has been a remarkable tour de force in which many mathematicians have been involved, that required an incredible amount of delicate techniques.

Scientific Progress Made and Outcome of the Meeting

General Comments

The meeting featured 20 talks, given by well-known experts in the area as well as many younger participants (including postdocs and two Ph. D. students).

Aside from the officially scheduled talks, ample time was allocated to informal discussions. As mentioned above, a discussion session took place on Tuesday afternoon to draw a roadmap towards possible solutions of some of these principal obstacles on the way to complete proofs of some of the fundamental local-global conjectures in representation theory.

Collaboration Started or Continued During the Meeting

We received an enthusiastic response to our request for feedback from the participants of our meeting which we think illustrates very clearly the many cooperations started or continued during the workshop and how the talks given fostered new ideas and developments.

R. Boltje: I was impressed by the high quality of the talks and enjoyed them very much. Benjamin Sambale's talk for instance was very closely related to my own research and gave me some new perspectives. Besides that I would like to mention three more specific developments.

— With Burkhard Külshammer we proved that p-permutation equivalences between two block algebras A of a group G and B of a group H preserve a certain family of invariants that is associated to the two blocks. It was already known that a p-permutation equivalence between A and B determines an isomorphism φ between defect groups of A and B and that this isomorphism is also an isomorphism between the fusion systems of A and B. To each centric A-Brauer pair (P, e), Külshammer and Puig associated an extension of the group $N_G(P, e)/C_G(P)$ by $Z(P)$. This construction is related to the construction of a centric linking system for the fusion system of A. During the conference, Külshammer and myself proved that if (P, e) is a centric A-Brauer pair and (Q, f) is the corresponding centric B-Brauer pair (via φ), then φ also induces an isomorphism between the extension groups associated to (P, e) and to (Q, f).

— With Susanne Danz we started the project to determine the group of perfect self-isometries of a block algebra A with cyclic defect groups. This is still in the beginning stages.

— Jesper Grodal made some very interesting remarks related to my own talk, saying that the results with my coauthors Kessar and Linckelmann that describe the Picard groups of a block algebra B use terms that can be derived from homotopy invariants of the centric linking system of the fusion system of B. We had a very long discussion about these connections that gave me enough incentive to start learning the necessary material from algebraic topology to understand these connections. Although nothing concrete came out of it so far, I'm very excited and hopeful that these connections will give me a much better insight into the topic of my own research.

C. Bonnafé: Raphaël Rouquier and me worked on two different subjects. First, we tried to correct an error discovered in our common paper with Jean-Francois Dat on Jordan decomposition of representations of finite reductive groups. This work is going on and will hopefully lead to a positive conclusion. Also, after Rouquier's talk on Hilbert scheme and interesting properties of the canonical line bundle, this suggested to us to look at other blowing-up of varieties of the form $(V \times V^*)/W$: we realized that, in some small examples, such blowing-up satisfies analogous properties (one can recover Lusztig's fundamental a- and A-invariant geometrically). We intend to work in this direction in the near future.

S. Danz: I would like to thank the organisers for organizing this inspiring workshop and for giving me the opportunity to take part in it. As for concrete mathematical discussions and collaborations, I would like to mention the following:

— Thanks to the free afternoon on Wednesday I was able to continue a collaboration with Robert Boltje that we had started this summer. Together with Jürgen Müller we are studying perfect isometry groups of blocks with cyclic defect groups, building on work done by Pornrat Ruengrot (student of Charles Eaton) in his PhD thesis. During the week at BIRS we, in particular, discussed some ideas how to use Boltje's recent work with Markus Linckelmann and Radha Kessar, which he presented in his talk, to make progress on describing these isometry groups.

— After my talk I had some interesting discussions with Benjamin Sambale concerning certain simple QG-modules that are related to 2-transitive permutation modules, in order to gain new information on integral forms of these modules.

— In his talk on the AWC Paul Fong presented his bijection between the isomorphism classes of simple modules of the symmetric group S_n in a p-block B and the conjugacy classes of B-weights of S_n. In doing so, every simple B-module is, in particular, assigned a certain conjugacy class of p-subgroups of S_n. The natural question (also asked by

Gabriel Navarro) is whether these groups are related to the vertices of the respective simple modules. It turned out that the group occurring in a B-weight is not the vertex of the corresponding simple B-module, in general. But there might still be a connection, and we are in touch to discuss this.

C. Eaton: I think the balance between lectures and free time was about right.

With regards to outcomes: As a direct result of the workshop I'll be visiting Gerhard Hiss to work on the problem of Donovan's conjecture for defect groups $C_3 \times C_3$. Whilst in Canada I also discussed this problem with Gunter Malle and Shigeo Koshitani. I discussed the problem of the structure of the Picard group for a block with Robert Boltje, which amongst other things highlighted the importance of parts of his current work with Kessar and Linckelmann. Thanks again for the invitation to the workshop - it was very useful.

P. Fong: The conference at BIRS two weeks ago was a very stimulating and enjoyable one — the particular mix of interests and people worked out well. I found the "New Perspectives" in the title of the conference covered not only new mathematical insights, but also the promising young researchers in the representation theory of finite groups who were there.

As for my particular interests: The natural bijections for the Alperin Weight Conjecture in symmetric groups and general linear groups in my talk gave rise to some interesting questions. First, what is the relation of the vertex of an irreducible modular representation of S_n and the radical subgroup of the weight associated with it? Susanne Danz and Burkhard Külshammer have computed such vertices. We looked at the case $p = 2$, but no obvious relation seems to hold. The matter may be different for odd p and this is being examined now. Another question is whether the natural bijection for symmetric groups passes nicely to alternating groups. This may be a tricky matter - one only has to recall the talk of Olivier Brunat and the inherent difficulties in passing from S_n to A_n. But it deserves study. Also of relevance is whether the natural bijection for GL(n, q) behaves well with respect to automorphisms. Britta Späth and I had a chance to talk about this — the bijection should behave well and I need to write this up. The question is also relevant to a result of Jiping Zhang and he and I had the opportunity to talk about this. A natural bijection should also exist for unitary groups and some of the ground work has been done by Jianbei An.

Banff also gave me the opportunity to talk with Michel Broué about resuming our collaboration (together with Bhama Srinivasan) on the Dade conjecture for unipotent blocks in classical reductive groups. The discussion workshop that Britta Späth ran on the inductive local-global conjectures provided an incentive to write-up this project next year during the period our stays at MSRI overlap.

One perspective on the representation theory of finite groups I find particularly intriguing is the application of the representation theory of algebras, be they affine Lie algebras, affine Hecke algebras, Ariki-Koike or any other algebras. That such theories can help settle special problems in finite groups is a fascinating phenomenon. The talk given by Raphaël Rouquier illustrates this well, linking as it did a special case of the decomposition matrix of a unitary group with an algebra over a coefficient domain $Q(q, t)$ in two transcendental variables. One may hope for more since the representation theory of algebras is still developing.

E. Giannelli: The week in Banff has been extremely productive.

In my talk I presented a conjecture concerning the number of linear constituents appearing in the restriction of an irreducible character of a finite group to a Sylow p-subgroup (this is part of a joint work with Gabriel Navarro). During my stay in Banff I discussed this problem with many researchers that gave me valuable ideas to prove that statement. In particular Michel Broué and Joan Tent suggested different but very promising strategies of proof. This might lead to future collaborations on the topic.

Moreover, I received considerable feedback and suggestions that led me to sensibly improve my analysis of the decomposition into irreducible constituents of the permutation character obtained by inducing the trivial character of a Sylow p-subgroup of S_n to the full symmetric group. The paper containing this study is a joint work with Stacey Law and I have been working on it during my stay in Banff. This will be submitted for publication in the next few weeks. The comments of Gerhard Hiss definitely contributed to improving the above mentioned work.

J. Grodal: A belated thank you for a great meeting. I really enjoyed it.

It was hugely inspiring to hear the many different approaches to the "standard" conjectures in modular representation theory. Even for the talks which were further away from me, it was interesting to hear what sort of problems people were working with.

In terms of interaction, I spent a good deal of time talking to Raphaël Rouquier about the conjectural role of loop

groups in modular representation theory, and with David Craven (and Rouquier) about conjectural ways to approach Broué's conjecture using subgroup complexes and homotopy theory. We've kept up the dreaming after the conference, but it is still very speculative.

Concretely, I also benefited from Robert Boltje's talk, which seemed to suggest relations to things I've been thinking about earlier. I hope to continue the discussions with him, as well as his collaborators Kessar and Linckelmann at MSRI...

It was also nice to be able to quiz the assembled experts about points in the literature, large and small.

S. Koshitani: It is all the time a great pleasure to stay and research Mathematics in BIRS. I have had wonderful experience this time as well.

Actually I was able to discuss the joint work with Caroline Lassueur quite a lot, and therefore our research has had a big improvement. We are researching Puig's Finiteness Conjecture (PFC), that is a more precise conjecture than Donovan's conjecture and that can be stated if we replace "a Morita equivalence" by "a splendid Morita (= Puig) equivalence". The co-worker and me both were suggested by Michel Broué to change (to uniform) the name to a splendid Morita equivalence instead of Puig's equivalence. This was also very helpful.

I could discuss Morita equivalent blocks when our blocks have elementary abelian defect groups of order 9, with Gerhard Hiss and Charles Eaton. This is related to Donovan's Conjecture and is really one of the global-local conjectures in the representation theory of finite groups.

I have learned quite a lot from the many interesting and nice talks held during the week. Finally I really would like to thank the BIRS for giving me such a wonderful opportunity for our research, and also for the four organizers. Hopefully I would like to come back again here in near future.

C. Lassueur: The BIRS workshop was very helpful to my current research. In particular a current project with S. Koshitani could be finished and discussions with R. Boltje and C. Eaton brought us some new ideas for future developments of this project. Furthermore, the talk by R. Boltje was particularly relevant to my own research. It let me understand that a result I have with J. Thévenaz on the lifting of endo-p-permutation modules can be extended to modules with fusion-stable endo-permutation sources with a possible application to the lifting of endo-permutation Morita equivalences considered by Boltje–Kessar–Linckelmann through alternative arguments.

A. Maroti: Gabriel Navarro and I began to work on a conjecture of his which concerns the possibility of inducing an irreducible character of a nilpotent subgroup of a group to an irreducible character of the group. More precisely, he conjectures that if α is a complex irreducible character of a nilpotent subgroup of a finite group G such that α^G is irreducible, then the generalized Fitting subgroup of G is nilpotent. We think that we managed to prove a partial result in this direction. Namely, that if G is a group having the property that it has an irreducible character which can be induced from an irreducible character of a nilpotent subgroup, then the Fitting subgroup of G is non-trivial or G is almost simple. There is good evidence that perhaps the second part of the conclusion of this latter claim may be omitted.

At the conference I was happy to see Hung Nguyen for the first time after almost ten years. We had lots of conversations about mathematics. One of these was with Hung and Zoltan Halasi. We had partial results (joint work also with James Cossey) on Gluck's conjecture. This conjecture states that if G is a finite solvable group, $b(G)$ is the largest degree of a complex irreducible character of G, and $F(G)$ is the Fitting subgroup of G, then $|G : F(G)| \leq b(G)^2$. We had long conversation about how to prove this conjecture. One important special case is when a group has the form HV where V is an irreducible and faithful H-module and V can only be viewed over a field of size 2, 3, or 4. The difficulty in this case is that there may not exist two vectors v and w in V such that $C_H(v) \cap C_H(w) = 1$. This would have been convenient for this latter identity implies that there is a vector u in V with $|C_H(u)|$ at most $|H|^{1/2}$, and so there is a large orbit of H on V. We tried many ways to get around this difficulty but so far it seems that we will aim to find a (single) vector in Irr(V) with a large H-orbit, at least in as many cases as it is possible, giving us an irreducible character of $G = HV$ having large degree.

G. Navarro: I have now new joint projects with B. Späth and C. Vallejo on the reduction of the McKay-Galois conjecture to finite groups, and several new projects with Pham Huu Tiep, Gunter Malle, Benjamin Sambale and Attila Maroti. Overall, I was impressed by the Rouquier talk. It is quite impressive to witness the announcement of a conjecture of this depth and of this impact. BIRS has become one of the best places to do mathematics and interact with our colleagues in the perfect environment. I want to thank BIRS for making our visit the most enjoyable and fruitful one.

H. Nguyen: This was my first time participating in a workshop in BIRS and I had enjoyed every moment of it. I had a chance to discuss some topics and problems that I am particularly interested in with my collaborators. Maroti, Zoltan Halasi and I have a hours-long conversation on a possible completed proof of an old conjecture of Gluck on bounding the index of the Fitting subgroup in terms of the largest character degree. I also discussed with Vallejo about the relation between the average of p'-degrees of characters in the principal block and the p-nilpotency of finite groups. During the workshop, Tiep and I were able to finish an ongoing project on an improvement of the Ito–Michler theorem using the notion of average degree. I also had a chance to learn various new perspectives and trends in group representation theory from other participants. I am especially interested in a very recent conjecture proposed by Giannelli and Navarro on the number of distinct linear characters in the restriction of an irreducible character of degree divisible by p to Sylow p-subgroups.

A. Schaeffer Fry: The BIRS workshop was extremely helpful to my current research. The talks were closely related to my own research and provided me with important updates and new insights to the area.

During the workshop, I made significant progress on a project with J. Taylor, which resulted in finishing and submitting a paper. We also began discussing our next projects.

I also had very fruitful conversations with the organizers and other participants of the workshop. For example, G. Malle suggested a new direction for my research, which led to one of the new projects with Taylor. I spoke with A. Turull about an ongoing project and agreed to keep in touch regarding the problem. I also discussed a potential project with J. Brough, which we have since began work on.

B. Späth: The conference was very useful and inspiring. I could meet and discuss the progress of various inductive conditions of the so-called global-local conjectures. It was very interesting to meet Paul Fong and Jiping Zhang. With Paul Fong I discussed his recent refinements of his deep but older results on the Alperin weight conjecture for symmetric and finite general linear groups. Also Jiping Zhang presented several results of his in this direction. The discussions clarified the current difficulties with the inductive conditions. Also it was useful to get informed on recent results of Jianbei An and Gerhard Hiss. Meeting these experts and discussing their on-going research project gave me the feeling that a new approach to the inductive conditions will be needed. I started thinking about that I further discussed in this context with Cedric Bonnafé the action on unipotent characters that might simplify the proof of a recent result joint with Marc Cabanes.

Several talks showed the interest in the Galois refinement of the McKay conjecture. This encouraged me to start a project with Gabriel Navarro and Carolina Vallejo in this direction.

The workshop was made very comfortable thanks to the BIRS center providing a very convenient family suite and babysitter time for my family, Marc and our baby son Maurice. The breakthrough I may remember most is the fact that Maurice started to walk for good that very week in the corridors of Corbett Hall!

J. Taylor: This present meeting at Banff was one of the most useful meetings I have attended in my career thus far. After my talk on root data I had a lengthy and interesting discussion with Frank Lübeck about how one could implement these theoretical ideas computationally. I hope this will result in something useful. My collaborator, Mandi Schaeffer Fry, and I managed to complete our current project on principal 2-blocks, which appeared on the arXiv shortly after the meeting. In discussions with Gunter Malle and Pham Tiep we came up with some new ideas on our current joint project, which will hopefully be successful. Alex Turull shared with me his work on the Glauberman correspondence and pointed me towards an open problem on Green Functions for finite reductive groups, which I am still thinking about. Finally, I was really encouraged by the number of younger participants at the meeting. I think this is a great testament to the depth and fortitude of finite groups as an area of study.

P. Tiep: I have had fruitful discussions with G. Malle and J. Taylor on our ongoing joint project. We have also started new joint projects with G. Navarro and with H. N. Nguyen.

A. Turull: I was very impressed by the organization, and how well the facilities were administrated. The scientific aspects of the conferences were likewise excellent. The topics of the conference had a focus on new perspectives in the representation theory of finite groups. It was interesting for me to see and hear what other experts on this are thinking currently. There is currently progress on the many important conjectures in representation theory, and it was very useful for me to hear the details of their perspectives. I was particularly interested to hear some work on the Navarro Conjecture for particular groups. I think this may lead to the proof of the stronger version of this conjecture that I proposed for those same groups. I was also interested to discuss some new research avenues with some of the younger participants. It was an excellent and productive experience all around.

C. Vallejo: Thanks for making possible such a great week at BIRS. It has been an especially enjoyable workshop for me. Despite not having given a talk, I have found it very fruitful. I have talked with Nguyen about the possibility of extending some of the results he presented in his talk by considering the average of degrees of characters in the principal block and not in the whole group. Also, I have had the opportunity to talk with Britta Späth about Navarro's conjecture. She has explained to me a new way, that she recently discovered, of relating central isomorphisms of character triples with Galois action. This might lead to a reduction theorem for Navarro's conjecture in a near future.

J. Zhang: Banff workshop was just excellent.

I talked with many participants and discussed with Fong, Broué, Koshitani, etc. on parametrizations of characters of groups of Lie type related to the Alperin–McKay conjecture. In my Banff talk I mentioned our verification of the Inductive Blockwise Alperin weight condition for PSL$(3, q)$. However there exists a small inaccuracy in a result on radical p-subgroups we cited, however this is ok for $G =$PSL$(3, q)$. The paper based on my Banff talk is now accepted by J. Alg. Encouraged by the Banff workshop we will continue to verify further Inductive Conditions. During the Banff workshop I also discussed with Külshammer, Nguyen Hung, etc. about further research collaborations. Külshammer will visit us for a month next year, and Nguyen Hung and I will possibly organize a research program on character degree problems in 2018–2019.

All the workshop's participants agreed that Banff lived up to its promises of a quiet, inspiring and very comfortable place to make science. We are all very grateful to the BIRS for providing such excellent facility for discussing and doing mathematics, and hope to return some time in the future.

Participants

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Chapter 23

Automorphic Forms, Mock Modular Forms and String Theory (17w5097)

October 29 - November 3, 2017

Organizer(s): Terry Gannon (University of Alberta), David Ginzburg (Tel Aviv University), Axel Kleinschmidt (Max Planck Institute for Gravitational Physics), Stephen D. Miller (Rutgers University), Daniel Persson (Chalmers University of Technology), Boris Pioline (LPTHE)

Overview

Automorphic forms in string theory

Over the last years there have been the beginnings of cross-interactions and cross-fertilizations between the theory of automorphic representations and string theory. The well-understood connections are mainly restricted to cases that in string theory go under the name of BPS-protected couplings with at least 8 residual supersymmetries. Such couplings arise for example in the lowest orders in a low-energy expansion of certain string theory scattering amplitudes. These can be represented by terms in an effective low-energy action that is of the schematic form

$$S_{\text{eff}} = \int_M d\mu \left(R + \mathcal{E}_{(0,0)} R^4 + \mathcal{E}_{(1,0)} \nabla^4 R^4 + \mathcal{E}_{(0,1)} \nabla^6 R^4 + \dots \right) \tag{23.0.1}$$

where R, R^4, $\nabla^4 R^4$ and $\nabla^6 R^4$ denote certain invariants built from curvature tensors (or their covariant derivatives) on space-time M. The expansion of the effective action is ordered by the number of derivatives (any curvature tensor has two derivatives) and since derivatives translate into momentum p after Fourier transform, and hence energy by Einstein's relation $E = p^2$, this is an expansion with later terms becoming more important when more energy is involved in an interaction.

The coefficient functions $\mathcal{E}_{(p,q)}$ appearing in the effective action (23.0.1) are constrained to be automorphic functions as functions of the so-called string theory moduli. For type II string theory on toroidal backgrounds the moduli live on Riemannian symmetric spaces based on split real groups and the automorphy groups are the corresponding Chevalley groups. Supersymmetry (a.k.a. BPS-protection) requires the coefficient functions $\mathcal{E}_{(p,q)}$ to satisfy differential equations and these differential equations can be translated into ideals of the universal envelopping algebra. This is the link to automorphic representations that are also characterised by being annihilated by ideals in the universal envelopping algebra.

This link was reviewed in the first introductory talk of the conference by Michael Green and further elaborated on in the talk by Guillaume Bossard. For example, the first coefficient function $\mathcal{E}_{(0,0)}$ is associated with the minimal representation while the function $\mathcal{E}_{(1,0)}$ is associated with the so-called next-to-minimal representation.

The cross-fertilization now arises from the fact that (i) having identified the right automorphic representations allows the use of representation theory to deduce constraints on the associated physics that is typically revealed by considering the Fourier expansion of the automorphic function. Moreover, (ii) the analysis of the effective action (23.0.1) also gives important impetus for mathematical research since the ideal that is determined by the coefficient function $\mathcal{E}_{(0,1)}$ of the $\nabla^6 R^4$ term is not of any type normally considered in the theory of automorphic representations. In mathematical terms, its maximal orbit in the wave-front set is not a single nilpotent orbit. This is a phenomenon that is not expected to arise in the theory of automorphic functions and can be traced back to $\mathcal{E}_{(0,1)}$ not being finite under the action of the center of the universal envelopping algebra. In physical terms, this new feature is due to the fact the BPS-protection of this term is reduced to only four supersymmetries for which many new features arise (like Kontsevich–Soibelman wall-crossing).

Both aspects (i) and (ii) have been discussed in talks at the workshop and also in many lively discussions outside the lecture hall. In particular, the generalisation of the standard definition of automorphic representations that is suggested by (ii) was the subject of numerous discussions. Approaches discussed for understanding the associated automorphic forms were

- Spectral methods
- Poincaré series
- Explicit constrained lattice sum constructions

The automorphic forms appearing (23.0.1) are associated with space-time dualities and, in the instances described above, are related to non-perturbative effects in the string coupling. In a different vein, one can also study string theory in perturbation theory. This is the theory of computing certain integrals over the moduli spaces of (punctured) Riemann surfaces that represent the world-sheet of the string. The integrands of these integrals are determined by conformal field theory correlation functions and become complicated quickly as the genus and the number of punctures of the Riemann surface increases. Again, one is often restricted to a low-energy expansion.

One case of particular interest is when the world-sheet of the string is torus, i.e. of genus one. The moduli space of all such tori is $SL(2,\mathbb{Z})\backslash\mathcal{H}$ where $\mathcal{H}$ is the Poincaré upper half-plane describing the ratio of the two periods of the torus. The integrand must be a doubly periodic function on the torus, in other words an automorphic function of $SL(2,\mathbb{Z})$. As mentioned above, this automorphic function is determined by a field theory correlator that is in general not known in closed form. Performing, however, a low-energy expansion one can (often) represent the integrand a given order in this expansion in terms of a graph on the torus world-sheet where the punctures are connected by lines. The associated integrands have been dubbed modular graph functions and have recently a lot of attention as discussed in the talks of Michael Green and Eric D'Hoker.

Modular graph functions on the torus are interesting since they contain interesting elliptic multiple zeta values when one considers certain degeneration limits of the torus. (Elliptic) multiple zeta values represent an active current research field in number theory, in particular their single-valued projection at the one-loop level. The modular graph functions that contain them satisfy interesting systems of differential equations that can sometimes be used to extract relations between modular graph functions and hence between the multiple zeta values they contain. A full understanding of this is an open problem and was one of the topics discussed at the conference.

Automorphic representations

Mathematically, the functions $\mathcal{E}_{(0,0)}$ and $\mathcal{E}_{(1,0)}$, associated with R^4- and $\nabla^4 R^4$-couplings, are attached to certain special automorphic representations which have unusually small functional dimension. Such small automorphic representations are usually representations of metaplectic groups. These small representations proved important in number theory as well as physics, where for example integral representations of L-functions, from Shimura's 1975 construction of the symmetric square L-function through constructions such as recent work of Cai-Friedberg-Ginzburg-Kaplan. The process of unifying linear algebraic groups and their covers leads to helpful new insights, often tending in the direction of quantum groups. An example is the representation of Fourier coefficients of Eisenstein series as "Tokuyama formulas" that are sums over crystal bases. Very recently work of Brubaker, Buciumas, Bump and Friedberg and others has revealed further unexpected relations between quantum groups and automorphic forms on metaplectic groups. In particular, the Fourier coefficients of metaplectic Eisenstein series are multiple Dirichlet series whose local factors can be interpreted as partition functions of supersymmetric solvable lattice models. Functional equations of such Fourier coefficients, which were studied by Kazhdan and Patterson and by Chinta and Gunnells can be interpreted as R-matrices of quantum groups. These ideas have equal origins in number theory and physics.

Approach of the workshop

Developments in recent years have made it clear that automorphic representations provide a crucial ingredient in our understanding of non-perturbative aspects of string amplitudes. With the advent of Umbral moonshine we have also seen new cross-fertilisations between string theory, conformal field theory, finite groups and mock modular forms. These developments provided ample motivation for having a new channel of communication between researchers in both communities, and the workshop succeeded in doing precisely this.

One of the main goals of the workshop was to provide an opportunity for interaction between mathematicians and physicists working on different aspects of automorphic forms, mock modular forms and string theory. The talks were selected in such a way as to give maximum exposition to current research topics from these fields and the lively discussions during the talks and in the breaks gave us the impression that our concept worked. The feedback we received from individual participants confirmed this impression and we think that the goal of creating new research ideas and projects at this exciting interface between mathematics and physics was certainly met.

The workshop brought together many researchers interested in similar questions and the talks and discussions at the workshop revealed many common questions that need to be researched further. There are some new collaborations that were initiated at the workshop as well as some ongoing collaborations that were propelled further as a consequence of the workshop. We felt the workshop was very successful and the academic concept "works" such that we plan to continue similar events in the future. A follow-up program will take place at the Simons Center during the spring of 2019 and we are aware of number of mutual visits between participants to further advance on research questions identified during the workshop.

The workshop was organized into 4 overall themes, and each theme was introduced during a 1-hour lecture and followed by a selection of shorter research seminars spread throughout the week.

Themes and talk descriptions

Automorphic forms in string theory

The keynote speakers were Michael B. Green and Eric D'Hoker, who each gave a one hour lecture. The first lecture reviewed two main instances of where automorphic forms and automorphic functions occur in string theory, namely the low-energy expansion of graviton amplitudes in a U-duality invariant framework with automorphic forms on exceptional duality groups and, secondly, the integrands of string scattering amplitudes at a fixed loop order. This latter subject has mainly been developed at one-loop order corresponding to toroidal world-sheets and automorphy group $SL(2,\mathbb{Z})$. This was reviewed in the talk by D'Hoker and he also explained recent work on the extension two loops and $Sp(4,\mathbb{Z})$ and even higher loops, work that was considerably extended at the BIRS workshop (see outcome section).

Guillaume Bossard discussed a new approach to the automorphic couplings appearing in the low-energy expansion. This approach is based on considering so-called exceptional field theory amplitudes at loop order where certain classes of supersymmetric excitations (BPS states) circulate in the loop. The resulting sum over such states leads to automorphic forms expressed through constrained lattice sums, different from the usual coset sum description of Langlands. This approach also provides expressions for automorphic functions that violate the usual $\mathcal{Z}$-finiteness condition. Discussions between physicists and mathematicians following this talk led to the proof of a conjecture of the rewriting of orbit sums.

Jeff Harvey presented his recent work with G. Moore on subtleties in the way Weyl reflections act in toroidal compactifications. The hitherto unnoted feature is that the combination of the usual geometric action of the reflection group with an action on the states leads to a cover (eight-fold in cases) of the standard Coxeter group.

Automorphic representations

The keynote speaker was Gordan Savin who gave an overview of the field of small automorphic representations. Global uniqueness of the minimal representation was recently shown by Kobayashi and Savin. He also explained the ubiquity of Weissman's construction of the Fourier-Jacobi functor for the minimal representation. As already mentioned the importance of the minimal representation in string theory has become increasingly clear in recent years, thereby creating a surge of interest in the field also from physics.

Marie-France Vignéras reported on here recent results on the existence of supercuspidal p-adic representations for G_2, D_n and E_n, a project that was initiated as a direct result of her preparation for the workshop.

Dihua Jiang presented recent progress on the theory of endoscopic classification. He reviewed his conjecture with Zhang on the large cuspidal spectrum, called the Global Large Cuspidal Packet Conjecture (GLCP-conjecture), and its ramifications.

Birgit Speh discussed her work on symmetry breaking of infinite-dimensional representations which deals with real representations of a group G and their restriction to a subgroup $H \subset G$, in particular problems with estimating multiplicities of representations of H in the case of infinite-dimensional representations. The main focus was for the case $O(n, 1) \subset O(n + 1, 1)$ and unitary representations.

Henry Kim reported on his joint work with Yamauchi on Ikeda-type lifts. Using the Ikeda-type lift they construct a higher level cusp form on $E_{7,3}$ from any Hecke cusp form whose corresponding automorphic representation has no supercuspidal local components.

Ben Brubaker gave a general overview on the connection between quantum groups and Whittaker functions of metaplectic Eisenstein series. Daniel Bump then proceeded to report on recent joint work with Brubaker, Buciumas and Gray. Whittaker functions on the n-fold metaplectic cover of $GL(r)$ over a nonarchimedean local field were studied by Kazhdan and Patterson, who computed the scattering matrix of the intertwining integrals on the Whittaker models. It was shown in 2016 by Brubaker, Buciumas and Bump that this scattering matrix coincides with the R-matrix of a quantum group, a twist of quantum affine $U_{\sqrt{q}}(\widehat{\mathfrak{gl}}(n))$, where q is the residue cardinality. Moreover, they showed that the spherical Whittaker functions could be expressed as partition functions of solvable lattice models, whose internal structure is related to the quantum affine Lie superalgebra $U_{\sqrt{q}}(\widehat{\mathfrak{gl}}(n|1))$. In recent work, Brubaker, Buciumas, Bump and Gray proved that a second solvable lattice model has the same partition function using Yang-Baxter equations.

Automorphic forms on Kac-Moody groups

The keynote speaker was Manish Patnaik who reviewed the basic notions of Eisenstein series on loop groups, emphasising the different versions (positive, negative, ...) that exist in the Kac–Moody case whereas they coincide in the usual finite-dimensional Lie group case. He proved convergence results for these Eisenstein series in the number field and function field case, building on previous work by Garland. Another interesting aspect was the discussion of the affine version of the Weil representation appearing in the theta functions and a corresponding Siegel–Weil theorem.

Kyu-Hwan Lee presented his work with Carbone and Liu on Eisenstein series on more general Kac–Moody groups, focussing on the results they have obtained for the rank 2 hyperbolic case. These results include convergence, functional relations, constant term formulas and holomorphy results for cusp forms.

Alexander Braverman presented his work with David Kazhdan on the affine Tamagawa number formula. The goal of this work is to provide an affine generalization of Langlands calculation of the volume of the fundamental domain $G(\mathbb{Z})\backslash G(\mathbb{R}))$ for G a split semi-simple simply-connected Lie group. Their proposal is to do this by computing the cohomology of the moduli space of G-bundles for G a loop group over a function field.

Indefinite theta series, moonshine and black holes

The keynote speaker was Stephen Kudla, who reported on the recent flurry of activity on theta series associated with indefinite quadratic forms. There is an extensive theory of theta series for positive definite quadratic forms, and connections with representation theory through theta correspondences and minimal representations. Sparked by results of Zwegers in relation with mock modular forms, and generalizations in string theory due to Alexandrov, Banerjee,

Manschot and Pioline, a general theory of indefinite theta series is now emerging. Kudla explained his recent results with Jens Funke which shows that indefinite theta series can be constructed using the method of Kudla-Millson.

Roberto Volpato reported on his work with Natalie Paquette and Max Zimet on counting 1/4-BPS states in $\mathcal{N} = 4$ string theory. The generating function of such BPS-states is famously given by the reciprocal of the Igusa cusp form Φ_{10} for $Sp(4;\mathbb{Z})$. Their results combine consistency conditions on Φ_{10} with wall-crossing in string theory and has potential relevance for our understanding of Mathieu moonshine.

Katrin Wendland explained recent progress in constructing a module for Mathieu moonshine using K3 sigma models. She reported on her results on the refined elliptic genus, and the work of Song on the cohomology of chiral de Rham complex. The conclusion is that the generic space of states of K3-theories is modelled by this cohomology, and appears to agree with what is expected from a Mathieu moonshine module.

Martin Raum reported on recent work with Michael Mertens on the skew-Maass lift. This is a generalization of the classical Maass lift to skew-holomorphic Jacobi forms. Part of the motivation for this work comes from recent developments in umbral moonshine.

Christoph Keller reported on a series of works jointly with several people, including Belin, Maloney, Mühlmann, Castro, Gomes, Kachru and Paquette. This concerns an in-depth analysis of the spectrum of permutation orbifolds for genus 1 and 2 conformal field theories. They investigate relations with black holes, Siegel modular forms and emergent spacetime.

Miscellaneous

Thomas Creutzig explained his recent work with Davide Gaiotto, which gives a new class of vertex algebras related by S-duality in GL-twisted $\mathcal{N} = 4$ super-Yang-Mills. These vertex algebras arise in the intersection of pairs of 3d boundary conditions in the 4d theory.

Siddhartha Sahi presented his work on multivariate hypergeometric functions for tube domains. This generalizes the classical theory of hypergeometric functions to functions with matrix arguments. This theory is related to the theory of Jack- and Macdonald-polynomials.

Howard Garland presented work in progress on the cohomology of arithmetic groups and relations to an arithmetic generalization of Riemannian curvature. He offered speculations on applications to automorphic forms and higher-derivative corrections in string theory.

Hermann Nicolai explained a surprising connection between conjectured Kac–Moody symmetries of $\mathcal{N} = 8$ supegravity in $D = 4$ space-time dimensions and the standard model. Breaking supersymmetry (by an unknown mechanism) it is possible to assign quantum numbers under an $SU(3) \times U(1)$ group to the $\mathcal{N} = 8$ fermions that exactly match the corresponding quantum numbers in the standard model. This subgroup requires the extension to the conjectured Kac–Moody symmetry $K(E_{10})$ and cannot be accommodated in the standard R-symmetry $SU(8)$.

Outcome of the Meeting

Publications linked to the workshop

Modular graph functions can also be defined in principle for higher genus Riemann surfaces, with additional subtleties as was explained in the talk by Eric D'Hoker. Some of the ideas he sketched regarding higher string invariants were clarified in subsequent discussions and the results have now been written up and published online as [1]. Another preprint that was an immediate consequence of the workshop was [2] by Marie-France Vignéras who analysed certain supercuspidal p-adic representations.

Comments from workshop participants

"It is my view that mathematics has two very different main sources: physics and number theory. In automorphic forms, the historical development has been from number theory. Yet automorphic forms also appear in physics. So the Langlands program is a meeting ground for ideas coming from two different directions. I felt that the workshop was extremely helpful in getting number theorists and physicists together for a common cause. The format of mainly 40 minute talks with a few survey lectures worked very well."

- Daniel Bump

"Many thanks again for organizing the recent BIRS workshop, and for inviting me. I enjoyed my stay very much, and I learned a lot.

I had discussions with Birgit Speh and with Martin Raum which may or may not lead to new collaborations.... New ideas are abundant after the worskhop; particularly the discussions with Roberto Volpato and with Jeff Harvey have helped."

- Katrin Wendland

"Thank you organizing an inspiring conference. I found it useful to learn which examples and groups are useful. I intend to look into automorphic forms for $SO(4,4)$ and $SO(4,20)$."

- Birgit Speh

"I enjoyed the workshop a great deal. It was very exciting to hear from so many distinguished physicists about the way that automorphic forms appear in the study of string scattering amplitudes, and to hear from my colleagues in automorphic forms about their latest progress. And the connections to VOAs and mock modular forms were very striking; this is an area I would appreciate the chance to learn more about.

The lectures were quite interesting and generally of very high quality. Let me add that I have already watched both Friday lectures on-line, as I was not able to stay for them.

For specifics, Guillaume Bossard asked me some interesting questions, and I hope this will be the start of more discussions. I spent time talking to Martin Raum, with whom I am collaborating, and this was a helpful conversation in terms of moving our project forward. I spoke with Ben Brubaker and once again set in motion next steps on further joint work. I had a helpful conversation with Gordan Savin which followed up a prior conversation from our last meeting in Boston.

I had some helpful discussions with Daniel Persson, Axel Kleinschmidt and Dan Bump concerning future steps in facilitating collaborations between mathematicians and physicists."

- Solomon Friedberg

" I really enjoyed the workshop and would like to thank all of you for your hard work.

It was fantastic to have one half of the talks on physics and the other half on mathematics so that we could see the common grounds and interactions of the two disciplines. The organization of talks was very coherent, focusing on one or two themes each day. I could learn a lot each day.

I hope that there may be more workshops and conferences like this one in the future."

- Kuy-Hwan Lee

" First of all: congratulations on putting together a superb workshop in Banff. There was a good balance between time for talks and free time to discuss, especially since everyone was physically staying in the same place, sharing lunch and dinner.

I did not initiate any new collaborations, but got a lot done with present collaborators Boris Pioline and Michael Green on one collaboration, as well as with Piotr Tourkine on another collaboration.

*Having also attended last year's workshop at Simons in Stony Brook, I think this year's workshop had a more balanced set of topics and interests than last year's. The rather informal but efficient organization was perfect for me. Let me add that I view the Banff Center as an *ideal* place for such meetings, so any time you care to invite me again, I will be there !"*

- Eric D'Hoker

"My recent visit to BIRS was a great help in my research. I learned things from the talks of course and met several people whose work I knew but whom I had never met personally including Stephen Kudla and Marie-France Vigneras. Most importantly, I had conversations with Martin Westorholt-Raum and with Thomas Creutzig that had a direct bearing on a problem I was struggling with in a current research project. My conversations with them led me to a new approach and made me familiar with parts of the literature that I would have had difficulty finding on my own. This is exactly the sort of thing one hopes might happen at an interdisciplinary meeting like this. Thank you so much for providing such a pleasant atmosphere for these productive interactions."

- Jeff Harvey

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Chapter 24

Forest and Wildland Fire Management: a Risk Management Perspective (17w5145)

November 5 - 10, 2017

Organizer(s): Douglas G. Woolford (University of Western Ontario), David A. Brillinger (University of California at Berkeley), David L. Martell (University of Toronto), Frederic P. Schoenberg (University of California at Los Angeles), B. Mike Wotton (University of Toronto/Canadian Forest Service)

Modern wildland fire management poses many challenging interdisciplinary problems that link forest ecology, forest management, computer science, physics, mathematics, statistics, industrial engineering and operations research. Fire management was originally driven by a desire for fire exclusion. However, over the past century it has evolved from an activity to an emerging paradigm of risk management, focused on multiple competing values and risks.

Throughout its history in Canada, fire managers have always looked to the best science available for assistance in characterizing important and highly variable aspects of their fire environment and for providing key intelligence to inform their decision making. As wildland fire management has evolved, so too has the range of scientific disciplines needed to support decision making deepened to include many more quantitative disciplines. This BIRS workshop that brought together a mix of researchers and end-users was the ideal forum for furthering ongoing collaborative initiatives and for developing new ones.

The workshop focused on the design and implementation of science-backed, data-driven forest fire management decision support tools and their application by end-users (i.e., forest and wildland fire managers). Three main areas were targeted: (1) The science underpinning decision support tools for fire management, (2) Fire regime modelling, and (3) Bridging gaps between researchers and fire managers. Emphasis was placed on a data science based approach for addressing emerging questions at the forefront of fire science and fire management. Representatives from fire management agencies discussed their perspectives and needs as well as provided examples of operational decision support tools currently in use.

This report provides background on the field of quantitative wildland fire science and decision support for fire management with a particular emphasis on the outcomes of the 5-day workshop at BIRS and ongoing related initiatives.

Overview of the Field

Decision Support Tools for Fire Management

In order to make progress on fire management objectives, mathematical and statistical expertise will need to be brought to bear on the issues of uncertainty which complicate operational planning. There is a need to deliver "the right amount of right fire at the right place at the right time at the right cost". For example, what is the best strategy for an individual airtanker? Is it better to have it return to its original home base where it was deployed at the start of the day, the base from which it was dispatched to its most recent fire after completing service on that fire, or should it fly to the base nearest to that fire? Although progress has been made on this kind of problem using tools developed by operations researchers, there is a very urgent need to develop and test better predictive fire occurrence models, especially ones that incorporate the clustering of arrivals which can overwhelm agencies. There is also a need to further our understanding of the initial attack system, including estimating the probabilities for same day initial attack, and what factors contribute to initial attack success since the larger project fires that result when the initial attack system fails are a large drain on crews and equipment and incur very large costs.

One strength of a probability-based model is that the variability in the data will be reflected in the standard errors associated with predictions or fitted values. However, one of he many challenges faced by fire managers and the researchers that collaborate with them, is the difficulty of conveying such uncertainty to the users of the modelling product. Two types of uncertainties need to be addressed directly: 1) the standard error of the predicted probability and 2) the fact that predicted probabilities have an inherent uncertainty in the sense that although the model returns a probability, the event of interest is binary. How can these uncertainties best be incorporated in decision support tools aimed at informing fire managers?

Other key risk management issues to be discussed include harvest scheduling optimization, optimal allocation of scarce fire suppression resources, and examining effects of changes in fire detection management and technology.

Fire Regime Modelling

The economic and social impacts of wildfire in Canada raise important challenging statistical questions. Once an area has burned, does the most recent fire reduce the surrounding area's risk of burning and if so, to what degree and for how long and can such effects be replicated by active management of forest fuels? What ecological and forest management strategies are needed to maintain the current landscape mosaic and levels of biodiversity while protecting people, property and other values at risk? There is an interest in developing methods for simulating seasonal boreal forest fire regimes, comprising frequency, size, and severity, noting that there is a strong need to "backcast" characteristics of past fire regimes if there had not been any fire suppression. In this context, the focus involves shifts to modelling aggregate behaviour of fire regimes, say over an entire year and across a large study area (e.g., a province), rather than modelling at fine scale resolutions. Historical records of individual burn scars for many fire seasons are now becoming available in digital format. Fires can be viewed as realizations of a marked spatio-temporal point process (where the points are the ignitions and the marks are the characteristics of each individual fire, such as its duration, intensity, size, and so on). The fire ignition records combined with the burn scar data can be viewed as approximately independent, replicated point pattern data. Important topics for discussion include developing tools for exploring and visualizing these data, for investigating inhibitory impacts of burn scars on the future intensity of points in that area, and for determining whether such marks are separable, and if not, how to jointly model points and marks.

Bridging Gaps Between Researchers and Fire Managers

Many factors, including climate change, changing land use patterns, fuel build-ups, government fiscal realities and the fact that fire is a natural ecosystem process contribute to the emergence of more and increasingly complex and challenging fire loads. Hence, there is a growing and urgent need for decision support systems that fire managers can use to enhance their planning and decision-making processes. The development of such decision support systems calls for bringing experts from the mathematical and decision sciences together with ecologists, fire scientists and fire managers to develop collaborative team-based efforts to address such problems.

It is also recognized that it is simply not enough to produce advances in science. To have a significant impact,

potential end-users of research results must be able to interpret and trust the output from these models. Tools are needed to aid a manager who, with little lead time, must resolve complex decisions under considerable uncertainty and be responsible for the outcomes of such decisions. For example, fire managers need to know about the potential for periods of extreme fire activity–situations when the fire management system can become overwhelmed. To plan effectively, they need tools that capture the scientific understanding of what causes multiple occurrences of large (and expensive) fires and hence, allows them to quantify the probability that multiple regions will be overwhelmed at the same time, preventing one region from assisting another region through intra and inter-agency resource sharing. Making accurate forecasts of such events requires models for the stages of the "lifetime" of a fire together with models which can predict where fires will be ignited; such knowledge can lead to improved prediction as to how fire load changes both spatially and temporally.

Presentation Highlights

The objective of the workshop was to bring together researchers and end-users from a variety of disciplines (e.g., operations research, statistics, actuarial science, mathematics, computer science, ecology, forestry and the environmental and health sciences) to provide a forum to discuss fire management needs and to initiate interdisciplinary team-based collaborations aimed at addressing important problems in forest fire management. Several themes were spread across the 5 day workshop, an overview of highlights follows below.

Monday Morning: A High-Level Overview

In the opening session senior managers from two Canadian forest fire management organizations presented their agency's perspectives on the challenges they have experienced in recent years and the problems they would like researchers to address. Wally Born, the Executive Director of Wildfire Management Branch of the Province of Alberta's Agriculture and Forest ministry discussed some of their challenges moving forward from the Fort MacMurray wildfire in May of 2016. He placed wildfire management within the context of its modern role of having to balance multiple values on the forest landscape: 97% commitment of land with commercially viable forest for harvest, species as risk (with caribou as a species of major public interest), 500 forest communities in their protection zone and major industrial development and infrastructure. Some of the key questions he identified as being highly relevant to his organization were: What resources are needed over the next 10 years (and moving forward) to maintain the performance of the organization, particularly within increasing population and commitments of land; how much of what kind of fuel management of mitigation effort around communities will be needed to produce a measureable impact in reducing risk, suppression needs and costs; how do we set and define performance measures, in particular for the prevention program.

Tony Falco a senior manager from the Province of British Columbia's (BC) Wildfire Management Branch gave the group a detailed update on the unprecedented 2017 fire season in BC and the multiple challenges that the province had to deal with. Since that fire season was just wrapping up at the start of the workshop, Tony pointed out that there were still many after action reviews to be carried out; however he identified some of their ongoing research priorities as understanding fuels mitigation and maintenance fr risk reduction as well as general fuel typing issues with BC forest types.

This morning session was wrapped up by a panel discussion lead by Al Tithecott (former Director of Ontario Ministry of Natural Resources and Forestry's Aviation, Forest Fire, and Emergency Services (OMNRF-AFFES) branch). The panelists included Wally Born, Tony Falco as well as Rob McAlpine (Manager of Wildfire Response and Operations, OMNRF-AFFES). Discussion with the workshop group included greater elucidation on research needs identified in the formal presentations as well as other topics and challenges facing wildfire management organizations.

Monday Afternoon: Fire Operations

The afternoon session built on the high level perspective on wildfire management challenges presented by the senior operational managers in the morning. This session brought the discussion to a more operational level to provide examples of some specific research needs and solutions being developed currently. David Martell from the University of Toronto opened the afternoon, placing his decision support focussed research activity into a framework of data analytics familiar to many today. Examples of completed and ongoing research that fit into the framework of Descriptive, Predictive and

Prescriptive Analytics were presented and some of the opportunities and challenges of carrying out such research with operational agencies were discussed. Key among these was was the critical need for researchers to develop close collaborative relationships with fire managers to support not only the flow of the data they need for their analytics research but also, critically, to help them develop a sound understanding of the data and the caveats surrounding its observation and overall reliability. The session then included presentations from fire science specialists from the fire management agencies in Ontario highlighting how the needs of operational decision maker are changing. To be able to management wildfire in the future it is anticipated that a more defined risk management structure will be needed.

Keeping focused on bridging gaps between researchers and fire managers and with the objective of furthering collaborative research efforts, day one of the workshop wrapped up with a discussion of both the morning and afternoon sessions. Some key fire management agency needs and research opportunities were identified. This discussion focussed on the information and tools needed for risk analysis, risk reduction strategies and evaluating the effectiveness of the suite of potential mitigation, preparedness, response and recovery strategies that are available.

Tuesday Morning: Operations Research

The morning session of day two sought to highlight mathematical approaches and solution to problems using the Operations Research/Analytics methods that David Martell had spoken about on the previous day.

The session began with a keynote address by Mikael Rönnqvist of Laval University, who presented a talk entitled "Calibrated Route Finder: Improving the Safety, Environmental Consciousness, and Cost Effectiveness of Truck Routing in Sweden", describing how he and his colleagues had used operations research methods to develop their award-winning research on transportation logistics in the forest sector in Sweden. This talk provided some concrete examples of how modern analytics research methods can be exploited to develop decision support systems that had very significant demonstrated award-winning impacts on the forest resource sector and other sectors other than fire management.

Yu Wei from Colorado State University presented a talk entitled "Building operations research models to improve our ability to address uncertainties in wildland fire management" to share his current and past experience at bringing advanced mathematical programming methods to bear on fire management problems in the United States.

Matthew Thompson of the US Forest Service presented a talk entitled "Optimizing fire management strategies on the basis of risk and control opportunities" in which he described how he and his colleagues are exploiting analytics to develop spatially explicit decision support systems that fire managers can use when they develop and evaluate strategies for managing fires that have both beneficial and detrimental impacts and need not necessarily be managed using traditional fire exclusion strategies and tactics.

Jeremy Fried, also of the US Forest Service, presented a paper entitled "Modeling Stand Level Fuels Management Effectiveness and Economic Feasibility at Landscape Scale in the U.S.: a Forest Inventory Informed Approach." He was invited because of his extensive experience at modelling the use of forest harvest residue for biomass to produce energy and at the same time, achieve fuel management objectives.

Tuesday Afternoon: Game Theory, Artificial Intelligence and Applied Analytics

Kate Larson, a computer scientist from the University of Waterloo presented a talk entitled "The Strengths and Limitations of Game Theory for Fire Management", based in part on her research on resource sharing. Her talk was particularly timely because of the needs of Canadian forest fire management agencies to share so many fire management resources among themselves and with international fire management agencies in recent years.

Mark Crowley, another computer scientist from the University of Waterloo presented a talk entitled "Fighting Fire with AI: Using Artificial Intelligence to Improve Modelling and Decision Making in Wildfire Management" based on his collaborative research with colleagues in the United States. He was invited in part, because of the growing recognition of the need for fire management agencies to exploit new methods being developed by artificial intelligence researchers.

David Martell of the University of Toronto presented a talk entitled "Prescriptive analytics to inform forest and wildland fire management" in which he outlined some of what he considered to be emerging fire research needs. This was followed by Colin McFayden of the Ontario Ministry of Natural Resources and Forestry who served as a discussant and then facilitative an open discussion.

Wednesday Morning: Risk

Greater understanding of risk management and the use of risk analysis techniques were identified by fire managers earlier in the workshop as needed to address the complexity of current and future wildfire management. as being seen by operational fire management. Day three opened with a keynote speaker who has specialized in financial risk management. Matt Davison of the University of Western Ontario described some of the analytical techniques he has used in a variety of applied research settings and highlighted where these may be applicable in the environmental risk management area. Ideas like diversification, leverage and hedging were presented and discussion ensued about how they may be relevant in the wildfire context. For example, thinking about community protection in a diversification context may allow novel framing of the risk reduction potential of a suite or "portfolio" of mitigation and prevention solutions.

Talks that followed this keynote presented specific risk analysis and modelling studies. Jennifer Beverly of the University of Alberta presented highlights from several studies of landscape fire risk and summarized up lessons learns from these analyses for other researchers exploring these areas. Cordy Tymstra from Alberta's Agriculture and Forestry Wildfire Management Branch discussed enhancing situational awareness of spring wildfire danger in Alberta using methods similar to those used in biosurrvellience applications. The day ended with Cristina Vega-Garcia from the University of Lleida presenting results from several studies using fire occurrence prediction models for improved wildfire risk management in Spain.

Thursday Morning: Ecology, Mapping Fire Risk

Thursday's session began with a focus on ecological aspects of fire activity with Steve Cumming from Laval University presenting some work he had done on hierarchical modelling of the joint distribution of annual fire counts and fire size. Lori Daniels from the University of British Columbia presented a perspective on the 2017 wildfires situation in BC from her perspective as a fire ecologist and talked about potential solutions moving forward. Patrick James from the University of Montreal concluded the ecological session with some discussion of spatial modelling linking Spruce budworm defoliation within changes in the probability of fire ignition. One of the themes emerging from the discussion session after these talks was that the complex linkages between insect disturbance and subsequent fire activity are not well agreed upon the scientific community. The relatively recent and large scale Mountain Pine Beetle epidemic in BC and the expanding spruce budworm outbreak in eastern Canada provide research opportunities; newer more advanced methods of spatial data analysis may be useful to attempt to gain insight in this area.

Thursday morning ended with presentations on two ongoing different fire risk analysis and mapping projects that were both sponsored by the Canadian Safety and Security Program (Defense Research and Development Canada-Centre for Science) and being led by research scientists at the Canadian Forest Service: Xianli Wang discussed the development of a national assessment of landscape fire risk in Canada. Steve Taylor talked about machine learning methods that are being used to develop a set of national (Canadian) daily fire occurrence models.

Thursday Afternoon: Examples Modelling Fire Risk and Beginning the Wrap-up

Thursday afternoon sessions carried on the theme of landscape ecology that began earlier that day, with a presentation from Geoff Cary of Australian National University on perspectives on quantifying the mitigation of wildfire risk from both an Australian and international modelling point of view. Ellen Whitman from the University of Alberta presented work conducted related to landscape patterns of burn severity in Canada's boreal forest. Brett Moore from Alberta Agriculture and Forestry discussed projection of potential fire growth using ensemble weather forecasts to provide probabilistic estimates of potential wildfire locations from an operational viewpoint. The last of the talks was given by Frederic Schoenberg of the University of California, a statistician, who discussed his experience modelling using point process frameworks to talk about model building and the issues of missing variables and the inclusion of variables that had little real influence on the overall process being studied.

Thursday ended with a session led by Rob McAlpine (Manager of Wildfire Response and Operations, OMNRF-AFFES) who gave a closing discussion that emphasized future collaboration going forward. Participants discussed potential future research projects including working on various aspects of appropriate response, optimizing helicopter deployment, fire risk modelling and risk in general, quantifying uncertainty, as well as other areas. It was clear that some new teams were beginning to form and that collaborative research, both existing teams and new initiatives, would

continue beyond the workshop.

Friday Morning: The Connection to Banff and Parks Canada

Similar to a 2013 BIRS 5-day workshop "Managing Fire on Populated Landscapes" (Braun et al. 2013), we took advantage of the opportunity to engage Parks Canada. Jane Park, a fire and vegetation management specialist at Parks Canada who is stationed in Banff National Park was able to attend as a participant of the meeting. On Friday morning, there was a field trip to visit fuel management and prescribed burning operations sites in Banff National Park. Prescribed fires, which are fires that are intentionally set, planned and managed by fire specialist in order to help maintain forest health and biodiversity (Parks Canada 2018b). For an example including further information about recent prescribed burning during 2017 in Banff National Park see Parks Canada (2018a). This gave our BIRS workshop participants the unique chance to "get some dirt on their boots", which, in general, is not a common occurrence for quantitative researchers in the Mathematical and Statistical Sciences. Participants were able to see actual burn scars and burn sites up close and witness the forest succession that follows after such prescribed burn events.

Scientific Progress Made

Several members of the organizing committee are also involved as investigators/collaborators on a CANSSI (Canadian Statistical Sciences Institute) Collaborative Research Team grant. This BIRS meeting provided a forum for some members of that team, including both academic researchers and their graduate students and postdoctoral fellow to network and interact with other Canadian and international researchers who work in this area along with individuals from fire management agencies.

This BIRS 5-day workshop was a strategic workshop. It brought together researchers from a variety of disciplines (e.g., operations research, statistics, actuarial science, mathematics, computer science, forestry and the environmental and health sciences) and provided a forum to discuss fire management needs, quantitative methods for solving problems in the fire science and fire management context, and foster ongoing and initiate new team-based collaborations aimed at addressing important problems in these areas. Building on the momentum provided by this BIRS workshop, a tactical meeting of a smaller group of individuals (many of whom were at this BIRS meeting) is planned for February 2018 in London, Ontario, Canada. The targeted areas identified, in part, through the discussions leading up to and at this BIRS workshop include:

- **(i) Appropriate response, which is to minimize the total cost plus net loss on a fire-by-fire basis considering the impacts (positive and negative) and the cost of alternative approaches of fire response.**
- **(ii) Fire risk modelling, which includes the development and application of probabilistic fire growth models, quantifying socio/economic impact and accounting for risk tolerance and preference.**
- **(iii) Quantifying or accounting for uncertainty, which is prevalent in all fire management decisions. Uncertainty enters the decision systems in a) the state of the knowledge, b) data quality, c) human dimensions (biases/heuristics) and d) natural variability.**

Outcome of the Meeting

Fire management agencies have the difficult task of addressing two competing concerns: the importance of fire from an ecological perspective and the danger of fire from a human perspective. Faced with the need to deliver "the right amount of right fire at the right place at the right time at the right cost" forest fire managers must make difficult decisions on a regular basis.

There is a growing need for quantitative expertise in data science-based decision support models for wildland fire management. The organizing committee recognized this need and, in particular, the need to transfer knowledge from the mathematical and statistical sciences to managers. This BIRS workshop, which brought together a wide mix of researchers and end-users from across Canada along with international participants was another step forward in accomplishing such objectives.

The proposed workshop was timely in today's environment where interdisciplinary, collaborative teams–that include researchers from quantitative fields–are an absolute necessity for tackling high-impact applied problems. It highlighted the growing community of quantitative researchers and exposed some newer researchers from the computational, mathematical and statistical sciences to the wide variety of problems

Through presentations, panel and informal discussions that took place at this meeting, it is clear that any such models must not only account for a wide range of specific information, such as values at risk, the current fire load and the probability of fire occurrence and detection, but also incorporate information about the current locations of crews and equipment and their associated dispatch and travel constraints. Another message that was clearly conveyed was that it is not enough to produce advances in science. To have an impact, end-users must be able to interpret and trust the output from these models. Tools are needed to aid a manager who, with little lead time, must make decisions and be responsible for the outcomes of such decisions. For example, fire managers need to know about the potential for periods of extreme fire activity–situations when the fire management system can become overwhelmed. To plan effectively, they need tools that capture the scientific understanding of what causes multiple occurrences of large (and expensive) fires and hence, allows them to quantify the probability that multiple regions will be overwhelmed at the same time, preventing one region from assisting another region through intra and inter-agency resource sharing. Making accurate forecasts of such events requires models for the stages of the "lifetime" of a fire together with models which can predict where fires will be ignited; such knowledge can lead to improved prediction as to how fire load changes both spatially and temporally. Joint modelling methodology from the Statistical Sciences has the potential to take what are predominantly marginal models for individual components of wildland fire regimes and couple them together in a joint framework. One presenter (Cumming) discussed his preliminary forays into this arena and other participants and researchers are actively pursuing work in this area.

This workshop addressed fire management decision making from an operations research perspective by providing a forum for decision makers in fire management agencies to form tangible working relationships with ecologists, fire scientists, industrial engineers, mathematical modellers and statisticians. Participants engaged in discussions about the use of mathematical and statistical models to develop decision support tools. Examples of such tools currently in operational use by fire management agencies that result from interdisciplinary collaborations with quantitative researchers were on display. This BIRS meeting represents a culmination of a long-term initiative engaging quantitative researchers with other researchers working in wildland fire and individuals from fire management agencies. It is our opinion that the success of this workshop along with the new scientific initiatives and research collaborations that continue to follow are a direct result of the ongoing support of BIRS as well as other Canadian mathematical and statistical institutes, which is gratefully acknowledged.

Participants

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Chapter 25

Approximation algorithms and the hardness of approximation (17w5133)

November 12 - 17, 2017

Organizer(s): Julia Chuzhoy (University of Chicago), Jochen Koenemann (University of Waterloo), Mohammad R. Salavatipour (University of Alberta), Nisheeth Vishnoi (EPFL), David Williamson(Cornell University)

Overview of the Field

Most of the discrete optimization problems arising in the sciences, engineering, and mathematics are NP-hard. This means that there exist no efficient algorithms to solve them optimally, assuming the $P \neq NP$ conjecture. The area of approximation algorithms focuses on the design and analysis of efficient algorithms that find solutions of cost within a guaranteed factor of the optimal cost. The area of hardness of approximation focuses on proving lower bounds on the guarantees that any efficient approximation algorithm can obtain for given problems assuming that $P \neq NP$ (or a similar complexity assumption). Over the last two decades, there have been major advances in the design and analysis of approximation algorithms, and in the complementary topic of hardness of approximation, see Vazirani [53], or Williamson and Shmoys [56].

The long-term agenda of our area is to classify all of the fundamental NP-hard problems according to their approximability and hardness thresholds. This agenda may seem far-fetched, but remarkable progress has been made over the last two decades. Approximation guarantees and matching hardness thresholds have been established for several key problems; e.g.

- covering and partitioning (the set covering problem, Feige [19]),
- algebra (overdetermined system of equations, Hastad [25])
- graphs (clique, colouring, Zuckerman [58]),
- optimization (maximum cut, Goemans and Williamson [22], Khot et al. [30]),
- constraint satisfaction (maximum SAT problems, Hastad [25]).

More significant than these specific successes is the impact of novel techniques on related areas of mathematics. We provide some examples.

Combinatorial Optimization:

The method of iterative rounding has been developed in the area of approximation algorithms to give remarkably good results for problems beyond the reach of classical combinatorial optimization, see Jain [27], and Lau et al. [35]. The technique has recently yielded elegant new proofs for a number classic results in combinatorial optimization.

Metric Embeddings:

Structure-preserving embeddings between various geometric spaces have been studied intensively for decades, in fields such as differential geometry and functional analysis. There are many applications of metric embeddings in the area of approximation algorithms. Moreover, the interaction between these fields has increased recently (see [37], [8], [7]).

Analysis of Boolean Functions:

Recent progress on hardness of approximation has come with the development of new tools in the area of "analysis of Boolean functions". This area combines techniques from harmonic analysis, probability theory, and functional analysis to study basic properties of Boolean functions. One recently developed tool, the Invariance Principle, has led to fruitful connections between hardness of approximation and the geometry of Gaussian space, see Mossel et al. [42].

Objectives of the Workshop Proposal

The goals of the workshop were as follows:

(i) To bring together leading researchers in the fields of approximation algorithms and complexity theory, and to stimulate the exchange of ideas and techniques between the two groups.

(ii) To focus on a few key topics that could lead to deep new results in the areas of approximation algorithms, combinatorial optimization, hardness of approximation, and proof complexity. We describe a few topics below.

(a) The most famous problem in all of discrete optimization is perhaps the Traveling Salesman Problem (TSP). Yet despite the attention paid to this problem, its approximability remains poorly understood. The best known approximation algorithm for the symmetric case is a classic $3/2$-approximation algorithm due to Christofides from 1976. On the other hand, the known hardness-of-approximation results are very weak.

Over the last few years, there has been remarkable progress on several special cases of the TSP and on some closely related problems. Many of these advances are introducing new and very interesting connections between different areas such as probability, structural graph theory, coupled with technically difficult yet powerful new methods such as interlacing families of polynomials. In 2011 Oveis Gharan et al. [43] used properties of strongly Rayleigh measures together with an elaborate analysis of the structure of near-minimum cuts to obtain the first improvement on the $3/2$-approximation guarantee for a key special case of TSP called the graphic TSP. Since then, there has been a series of more work on this special case and related questions. The most recent result on this special case is a $7/5$-approximation algorithm of Sebo and Vygen [49] that hinges on a key probabilistic lemma of Momke and Svensson [41] coupled with an in-depth and novel analysis of structures that are well known in Combinatorial Optimization. An et al. [1] improved on a 20-year old $5/3$-approximation guarantee of Hoogeveen [26] for the s-t path TSP. Subsequently, Sebo [48] and Vygen [54] have improved on these results to obtain an 1.599-approximation guarantee, by using further probabilistic insights. More recently, Gottschalk and Vygen [23] and Sebo and van Zuylen obtained 1.566 and 1.529 approximations for the s,t-path TSP problem, respectively. Relying on (and extending) the major result by Marcus, Spielman, and Srivastava [39] that proves a conjecture of Kadison-Singer, Anari and Oveis Gharan [2] recently showed the existence of $O(\mathrm{poly}(\log\log)n)$-thin spanning trees. The result implies an $O(\mathrm{poly}(\log\log)n)$ upper bound on the integrality gap of the Held-Karp LP relaxation for the asymmetric TSP (improving the $O(\log n/\log\log n)$ bound from 2010 [9]). The LP is long conjectured to have an $O(1)$ integrality gap. The result of [2], however, does not imply an approximation algorithm, it only provides an estimate of the optimum value. Almost concurrently, Svensson [51] showed that for the

case of shortest path metrics of directed graphs (graphic ATSP), the integrality gap of the Held-Karp LP is $O(1)$ and provides an efficient algorithm for it. The two biggest open problems in this area remain to improve upon the $3/2$-approximation for TSP and to obtain a constant factor approximation for ATSP. By re-focusing attention on this problem, our goal is to continue the momentum from the past two workshops. Two notable new results in this area were found very recently, after our BIRS workshop: Vygen and Traub [55] recently presented a $1.5 + \varepsilon$ approximation for s,t-path TSP, nearly matching the performance ratio of Christofides' algorithm for metric TSP. Svensson, Tarnawski, and Vegh [52] presented a constant factor approximation for the asymmetric TSP problem; their bounds are relative to the standard ATSP LP relaxation, confirming the conjecture that it has constant integrality gap.

(b) The Unique Games Conjecture (UGC) which was posed in 2002 by Khot [29] and the implications of it have attracted a lot of attention over the last 13 years. The conjecture states that a certain type of constraint satisfaction problem is hard to approximate. If the conjecture is true, it shows that many of the approximation algorithms we have (in particular SDP based algorithms) are best possible ([45, 31, 30]). More specifically UGC implies near tight approximatibility thresholds for a large class of constraint satisfaction problems (CSPs) among others (see Raghavendra [46]). In a sense, UGC predicts that there is a "meta-algorithm" that is optimal for those problems and this meta-algorithm is based on SDP [10]. Refuting the conjecture would most likely require designing new algorithmic techniques that could potentially lead to improved approximation algorithms for many other problems. One component of the workshop will focus on this conjecture and surrounding issues in the complexity of optimization problems.

Lasserre hierarchy / Sum-of-Squares algorithms:

Use of semidefinite programming (SDP) relaxations and the lift-and-project strengthening of them has attracted a lot of attention in the field in the last decade or so. The Lasserre hierarchy is a systematic method of strengthening SDP relaxations by adding more constraints. In some instances these methods have been successfully applied to obtain improved approximation algorithms for some classical results (e.g. [15]). More importantly, although some results support the UGC, some recent works have cast more doubts on it using Lasserre SDP based algorithms. For example, Arora et al. [6] shown that the powerful Lasserre SDP hierarchy of algorithms could be used to obtain a subexponential-time algorithm for Unique Games (UG). More recently, Barak et al. [11] have used the connection between Lasserre algorithms and Sum-of-Squares (SOS) proof complexity, and have shown that the known hard instances of the UG problem can be analyzed by constant-degree SOS proofs, and thus be solved efficiently.

Extended formulations and their complexity:

Feasible solutions to instances of combinatorial optimization problems often naturally correspond to the vertices of certain polyhedra. One way of designing an efficient algorithm for a given optimization problem is therefore to find a compact description for the associated polyhedron, and to then apply an efficient LP algorithm. In ground-breaking work Yannakakis [57] first showed that for the TSP, every symmetric LP formulation must have an exponential number of constraints. Symmetry here means that for every permutation of cities there is a corresponding permutation of the variables that leaves the LP invariant. Fiorini et al. [21] recently resolved Yannakakis' main open problem and showed that TSP has no symmetric or asymmetric polynomial-sized formulation. In another breakthrough, Rothvoss [47] recently showed that no subexponential-size extended formulation can exist for the matching polyhedron either.

(c) Routing problems in graphs arise in many areas of computers science, from VLSI design to Robotics. They have also been extensively studied in the graph theory community. Two of the most basic graph routing problems are the Edge Disjoint Paths (EDP) problem and Congestion Minimization. In EDP, we have to route a maximum number of demand pairs from a given collection in a graph via disjoint paths. In Congestion Minimization, all demand pairs must be routed, while minimizing the maximum load on any edge. Both problems are still poorly understood: for EDP the best upper and lower bounds have ratios $O(n^{1/2})$ [32] and $\Omega(\log^{1/2} n)$ [4, 3], while the upper and lower bounds for congestion minimization have ratios $O(\log n/\log\log n)$ [44] and (roughly) $\Omega(\log\log n)$ [5], respectively. If one allows up to 2 paths to share an edge, a polylogarithmic approximation was recently shown [16, 17]. Graph routing problems are naturally closely related to network cuts and flows. The new techniques for graph decomposition introduced in [16] have lead to new results in several other areas, such as a polynomial bound for the Excluded Grid Theorem of Robertson and Seymour [14]. Another closely related topic is graph sparsification: given a graph G and a small subset T of its vertices, called terminals, we would like to "compress" G into a much smaller graph H that contains the vertices of T, so that H behaves similarly to G with respect to the terminals. Graph sparsifiers naturally arise in approximation algorithms, graph theory, and fixed parameter tractability, and they have been studied in all these communities, often independently. If we require that the sparsifier H

only contains the terminals, then there are known constructions that achieve quality (approximation factor) $O(\log k/\log\log k)$ for both the cut and the flow sparsifiers [40, 36, 13, 38, 18], and it is known that no better than $\Omega(\sqrt{\log n})$-quality is achievable for this setting [20, 38, 13, 18]. If H is allowed to contain additional vertices, better results (namely constant-quality) are known. Unfortunately, we still do not know whether it is possible to construct constant-quality cut and flow sparsifiers whose size only depends on k. Some of these recent results rely on graph decomposition techniques that were developed in the area of approximation algorithms for graph routing problems. Quality-1 cut sparsifiers were introduced under the name of mimicking networks by Hagerup et al. [24], and they have been used to provide kernels for various cut problems, such as, for example, minimum multiway cut [33]. However, the best current upper and lower bounds on the size of a mimicking network are $2^{2^{O(k)}}$ [24, 28, 12] and $2^{\Omega(k)}$ [34, 28], respectively, which show that even this very basic problem is still not well understood.

Presentation Highlights

Bimodular Integer Linear Programming

The first plenary talk was by Rico Zenklusen. He showed how one can solve any integer linear program (ILP) defined by a constraint matrix whose sub-determinants are all within $\{-2,-1,0,1,2\}$ in strongly polynomial time. This is a very nice extension of the well-known fact that ILP's with totally unimodular (TU) constraint matrix are solvable in strongly polynomial time. This result uses several techniques. They first reduce the problem to a particular parity-constrained ILP over a TU constraint matrix and then use Seymour's decomposition of TU matrices to break this ILP into simpler base problems. Then they show how these simpler problems can be solved using combinatorial optimization techniques. He also highlighted some open problems in this field and some possible extensions to larger classes of ILPs.

A Simply Exponential Upper Bound on the Maximum Number of Stable Matchings

In the 2nd plenary talk, Shayan Oveis Gharan highlighted their recent result on the number of stable matchings, which is a classical problem. They show that for any stable matching instance with n men and n women the number of stable matchings is at most C^n for some universal constant $C > 1$. The proof is based on a reduction to counting the number of down-sets of a family of posets that we call mixing.

Approximating spanners and distance oracles

Despite significant recent progress on approximating graph spanners (subgraphs which approximately- preserve distances), there are still several large gaps in our understanding. In the 3rd plenary talk, Michael Dinitz did a survey of some recent results, with a focus on low-stretch spanners (from SODA '16) and on spanners with demands (from SODA '17). He described the gaps and open problems which remain, including open questions on the power of linear and semidefinite relaxations for these kinds of network design problems. He also presented some recent results on approximation algorithms for optimizing distance oracles (the natural data-structure version of spanners). The talk mostly focused on the many interesting open problems remaining in approximation algorithms for optimizing data structures.

Approximation Schemes for Clustering Problems: Now With Outliers

Recent developments in local search analysis have yielded the first polynomial-time approximation schemes for the k-Means, k-Median, and Uncapacitated Facility Location problems (among others) in a variety of specific classes of metrics . An important extension of these problems is to the setting with outliers. That is, we we are given an additional parameter Z and may discard up to Z points/clients in the input. This is especially important in the setting of k-Means clustering where even a small fraction of outliers may cause a noticeable deviation in the centroids of near-optimum solutions. In the 4th plenary talk, Zac Friggstad started with a brief review of their recent work from last year in local search analysis for k-Means clustering in Euclidean metrics. Then he presented a more recent development: a general framework for adapting local search analysis for clustering problems to get approximations for their variants

with outliers. In particular, how they obtain the following results for clustering in doubling metrics (including constant-dimensional Euclidean metrics) and shortest path metrics of bounded genus graphs. 1) PTASes for uniform opening cost UFL with outliers and 2) bicriteria PTASes that open $(1+\varepsilon)k$ centres for k-Median and k-Means clustering with outliers. There is no violation on the given bound for outliers in any of these approximations.

The Paulsen problem, continuous operator scaling, and smoothed analysis

The Paulsen problem is a basic open problem in operator theory: Given vectors $u_1, ..., u_n$ in R^d that are eps-close to satisfying the Parseval's condition and the equal norm condition, is it close to a set of vectors $v_1, ..., v_n$ in R^d that exactly satisfy the Parseval's condition and the equal norm condition. Given $u_1, ..., u_n$, the squared distance (to the set of exact solutions) is defined as $\inf_v \sum_{i=1}^n normu_i - v_{i2}^2$ where the infimum is over the set of exact solutions. Previous results show that the squared distance of any eps-close solution is at most $O(poly(d, n, \varepsilon))$ and there are eps-close solutions with squared distance at least $Omega(d\varepsilon)$. The fundamental open question is whether the squared distance can be independent of the number of vectors n.

In the last plenary talk, Lap Chi Lau showed how they answer this question affirmatively by proving that the squared distance of any eps-close solution is $O(d^7\varepsilon)$. Their approach is based on a continuous version of the operator scaling algorithm and consists of two parts. First, they define a dynamical system based on operator scaling and use it to prove that the squared distance of any eps-close solution is $O(d^2n\varepsilon)$. Then, they show that by randomly perturbing the input vectors, the dynamical system will converge faster and the squared distance of an eps-close solution is $O(d^3\varepsilon)$ when n is large enough and eps is small enough. To analyze the convergence of the dynamical system, they develop some new techniques in lower bounding the operator capacity, a concept introduced by Gurvits to analyzing the operator scaling algorithm.

Scientific progress made and outcome of the meeting

The schedule of the workshop provided ample free time for participants to work on joint research projects. A number of new research projects were initiated during the workshop, while some other researchers used the opportunity to continue to work on projects started earlier. The research talks and the plenary talks were very well received.

Rico Zenklusen reports that during the workshop he and his colleagues (Chaitanya Swamy, André Linhares, and Neil Olver) made progress on a project on optimizing over the intersection of matroids. In particular, at BIRS he and Swamy spent time during the workshop on this project and they found an interesting extension and application of a technique we developed earlier. In particular, further discussions with Lap-Chi that he had at BIRS proved very helpful to find this connection. This revived a project on which they worked for some time already, but had trouble to identify a good way to fully exploit their techniques.

Also, Mohammad Salavatipour and his Ph.D. student Mirmahdi Rahgoshay (who participated at the workshop) started a project with Rico Zenklusen on a problem related to resource management on a network (with application to fire containment and spread of other harmful events). So far they have been able to improve the previously best known results (a 12-approximation by Zenklusen in SODA17) to an asymptotic approximation scheme that runs in quasi-polynomial time. The project is on-going with the goal of obtaining a true polynomial time approximation scheme.

Sam Hopkins (another participant) reports that he made significant progress on two projects as a result of the workshop. At the workshop he gave the first public talk on his recent work with Jerry Li on clustering via sum of squares proofs; discussions with colleagues afterwards and questions during the talk helped clarify a number of points in the new paper and its relationship to previous work. In particular Aravindan Vijayaraghavan and Sam had some substantial discussions on these matters. He also had the opportunity to continue an existing project with Tselil Schramm on integrality gaps for linear programming hierarchies, on which we made substantial progress.

Andreas Wiese reports that he and Fabrizio Grandoni continued their collaboration on a problem related to the Unsplittable Flow on a Path problem. Their discussions were very fruitful, in particular they were able to find the best possible approximation factor for an important special case. Moreover, they could simplify some of their argumentations which will yield a cleaner presentation in the paper they plan to write on the topic.

Fabrizio Grandoni reports: "I continued my collaboration with Parinya Chalermsook and Bundit Laekhanukit about the Group Steiner Tree and related problems. The discussions that we had during the workshop were very

fruitful and might eventually lead to some concrete progress on the problems that we are studying."

Tselil Schramm tells us: "During the workshop, Sam Hopkins and myself continued working on an ongoing project, trying to use the recent "pseudocalibration" technique to improve Sherali-Adams lower bounds for max-cut and other constraint satisfaction problems. I was also exposed to the Paulsen problem by Lap Chi Lau's excellent talk, and later began working on the problem (trying to improve on Lap Chi and coauthors' result)."

Also, Viswanath Nagarajan reports: "I continued my collaboration with Anupam Gupta on designing approximation algorithms for stochastic load balancing. The breakout times during the workshop were very useful in continuing our research discussions on this topic. We were able to come up with a counter-example for one of approaches. This has been useful for us in identifying an alternative approach, which is still work in progress. "

The above are only a few examples of the research progress made during or after the workshop, and there are other ongoing projects that started at the workshop.

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Chapter 26

Nonlinear and Stochastic Problems in Atmospheric and Oceanic Prediction (17w5061)

November 19 - 24, 2017

Organizer(s): Youmin Tang (University of Northern British), Wansuo Duan (Institute of Atmospheric Physics, Chinese Academy of Sciences), Adam Monaha (University of Victoria), Shouhong Wang (University of Indianan), Siraj Islam (University of Northern British)

Overview of the Field

Advances on the theory of nonlinear dynamical systems, either deterministic or stochastic, as well as advances on the theory of nonlinear partial differential equations (PDEs) and their numerical treatment, have had a profound impact on the modeling and understanding of atmospheric and oceanic dynamics. With the rapid development of nonlinear and stochastic methods and their successful application in atmospheric and oceanic sciences, our capability of atmospheric and oceanic prediction has obtained significant improvement in the past decades. The success of numerical weather forecast is recognized as one of the most important achievements in science, technology and society in the 20th century.

The application of nonlinear and stochastic methods in atmospheric and oceanic prediction is ubiquitous. For example, the development of both control theory of nonlinear dynamic systems and estimation theory of stochastic systems makes it possible to optimally combine observations with numerical models (i.e., data assimilation), resulting in the revolutionary improvement in prediction initializations. The second example is the ensemble prediction that changes the traditional concept of prediction from single, deterministic prediction to multiple, probabilistic prediction, which deems from the Monte Carlo theory and the optimal unsteady theory of nonlinear dynamics systems. The ensemble prediction greatly addresses the uncertainties of predictions, and has been widely applied in operational prediction centers.

Recent Developments and Open Problems

Over past decades, significant progress has been made in applying nonlinear and stochastic methods onto atmospheric and oceanic prediction. A typical example is the ensemble prediction, which considers well the high degree of non-

linearity and stochastic forcing of the atmosphere and ocean systems. The development of nonlinear and stochastic methods make the optimal perturbations available, which is fundamental in the construction of ensemble predictions. The other example is the data assimilation, which optimally use observations to initialize prediction and to constrain model parameter. Our success in weather forecast and climate prediction achieved in recent years can be to a great extent attributed to the advance of data assimilation. However, some specific challenges still emerge in atmospheric and oceanic predictions inherent to nonlinear and stochastic theory, which can be summarized as following major scientific problems: (i) How can we estimate the model states for high-dimensional nonlinear and non-Gaussian systems, and which nonlinear and stochastic method will be best for the particular type of problem? (ii) What new nonlinear optimal methods or strategies need to be developed to detect the optimal prediction error growth? (iii) What new measurement metrics should be developed to optimally quantify the prediction utility for a nonlinear and stochastic system? (iv) How can we measure and differ the contribution of different error sources to predictability, and further reduce the errors? (v) How should nonlinear and stochastic methods complement the dynamical models used in the atmospheric and oceanic prediction community? (vi) How can full advantage be taken, in atmospheric and oceanic sciences, of the powerful tools that have been developed by mathematicians in the framework of the theory of dynamical systems (dynamic transitions, Lyapunov exponents and vectors, fluctuation-dissipation theorem, invariant and parameterizing manifolds).

It is clear that the atmospheric and oceanic predictions are cross-disciplinary and especially embodied in applications of mathematics in atmospheric and oceanic predictions. To push the collaboration of the researchers from atmosphere, ocean, and mathematics and explore the above major scientific problems in predictions and predictabilities of atmospheric and oceanic motions, The Banff International Research Station (BIRS) hosted the workshop "Nonlinear and Stochastic Problems in Atmospheric and Oceanic Prediction" during November 19 - 24, 2017. This workshop was co-convened by Prof. Youmin Tang (from Canada), Prof. Shouhong Wang (from America), Prof. Adam Mohana (from Canada), and Prof. Wansuo Duan (from China). And twenty-seven experts from China, Canada, US, and UK attended the workshop and presented talks on the problems mentioned above.

Presentation Highlights

According to the pre-mentioned major scientific problems, the workshop classified the talks in three themes: (i) nonlinear geophysical fluid dynamics, (ii) data assimilation, and (iii) predictability and prediction.

Nonlinear Atmosphere and Ocean Dynamics

The atmosphere and ocean experiences significant changes in their nonlinear dynamics. To understand these changes, their mechanisms, and their regional implications a quantitative understanding of processes in the coupled ocean and atmosphere system is required. The presentations delivered under this theme shed light on such nonlinear dynamics of several atmospheric/ocean systems such as El-Nino Southern Oscillation (ENSO), Madan Julian Oscillation (MJO), turbulent systems and large geophysical flows. For example, Dr. Hai Lin discussed ensemble integrations using a primitive equation atmospheric model to investigate the atmospheric transient response to tropical thermal forcings that resemble El Nino and La Nina. He showed that such response develops in the North Pacific within one week after the integration and the signal in the North Atlantic and Europe is established by the end of the second week. Dr. Entcho Demirov talked about the role of dominant patterns of ocean and atmospheric variability which define coherent variations in physical characteristics over large areas. He applied Bayesian Gaussian mixture models to define four dominant subseasonal weather regimes. Based on his analysis, he developed a computationally efficient stochastic weather generator for analysis and prediction of the Subpolar North Atlantic atmospheric decadal variability. Dr. Chun-Hsiung Hsia applied a semi-discretized Euler scheme to solve three dimensional primitive equations. With suitable assumptions on the initial data, he showed the long time stability of the proposed scheme.

Some interesting talks were delivered on dynamics of geophysical flows highlighting recent scientific progress. For example, Dr. Corentin Herbert discussed construction of a perturbative theory which describes the large-scale flow, relying on a technique known as stochastic averaging. Using direct numerical simulations in an idealized situation, he showed the validity and limitations of this approach by testing analytical predictions for the structure of the large-scale flow, such as the velocity profile or the eddy momentum flux profile. He further discussed the long term dynamics of large-scale geophysical flows and the abrupt transitions they undergo, using tools from large deviation theory. Dr. David Straub examined simulations of freely decaying turbulence in a non-hydrostatic Boussinesq model for which the base state stratification contains a tropopause. Dr. Xiaoming Wang discussed numerical algorithms that are able to capture

the long-time statisitcal properties such as the climate, of large dissipative turbulent systems and applications to certain prototype Geophysical fluid dynamics models. Dr. Shouhong Wang presented an overview of dynamic transition theory and its applications to various geophysical fluid flows.

Data Assimilations

In this theme, the difficulties of the assimilation approaches of ensemble Kalman filter (EnKF) and Particle filters (PF) were mainly discussed. These two approaches are both the numerical realization of the Bayes' Theorem. Nevertheless, the former adopts the linear approximation to the observational operator and assume that the observational errors obey Gaussian distribution in statics while the latter avoids these linear approximations and assumptions and is much realistic. The former has been widely used in predictions of atmosphere and ocean but there are still limitations and difficulties in realistic predictions. The latter is most advanced in theory but presently under exploration. Clear, in-depth investigations are urgently needed for these two approaches. The presentations in this workshop provided some progresses on these two approaches. For example, to reduce the huge computational resource of EnKF in high resolution coupled earth systems, Prof. Shaoqing Zhang considered the combination of stationary, slow-varying and fast-varying filters and designed a high efficiency approximates EnKF (Hea-EnKF) to dramatically enhance the computational efficiency. He showed that it is the improved representation on stationary and slow-varying background statistics that makes the Hea-EnKF while only requiring a small fraction of computer resources be better than the standard EnKF that uses finite ensemble statistics. This also makes the Hea-EnKF practical to assimilate multi-source observations into any high-resolution coupled model intractable with current super-computing power for weather-climate analysis and predictions. For PF, Dr. Nan Chen introduced a conditional Gaussian framework for prediction of nonlinear turbulent dynamical systems, which allows efficient and analytically solvable conditional statistics that facilitates the real-time data assimilation and prediction. Especially, this method is able to beat the curse of dimensions in traditional particle methods. In addition, a challenge of PF is to avoid the filter degeneracy with limited computational resources, where the filter degeneracy is a situation in which almost every particle has negligible weight and contributes little to representing the PDFs. The filter degeneracy is still an open question. Dr. Zheqi Shen focused on this question and tried to give several strategies to overcome this filter degeneracy phenomenon, include using a hybrid scheme of PF and EnKF with a tunable parameter, and using vector weights to achieve localization, and finally thought that it is great potential to develop the feasible PF for large geophysical model systems with nonlinear/non-Gaussian observational systems.

Predictability and Prediction

The "predictability" is a fundamental issue in both atmospheric and oceanic research, as well as numerical weather and climate prediction. It indicates the extent to which even minor imperfections in the knowledge of the current state or of the representation of the system limits knowledge of subsequent states. Study of predictability could provide useful information on reducing the prediction uncertainties for the atmospheric and oceanic motions. Study of predictability can include two aspects. The first is studying the reason and mechanism of the uncertainty of forecast result (e.g., forecast error) and the second is studying the way or methodology to reduce such kind of uncertainty. In the workshop, quite a few presentations focused on these two aspects and addressed progresses.

For the first aspect, it is important to estimate the predictability limit. In this field, Prof. Frank Kwasniok showed that the predictability of large-scale atmospheric flow is characterized by finite-time Lyapunov exponents and covariant Lyapunov vectors. Prof. Jianping Li considered this limitation and reported a nonlinear local Lyapunov exponent approach (NLLE). He explored the basic theory of the NLLE and then applied this approach to the predictability studies, and finally showed the usefulness of the NLLE in estimating the predictability time for weather and climate events and determining the sensitive area for targeting observations. Subsequently, Prof. Ruiqiang Ding presented another application of NLLE in ensemble forecast and reported the approach of orthogonal nonlinear local Lyapunov vector (NLLV). He used the NLLV to generate the initial perturbations of the ensemble forecast and demonstrated the applicability of NLLV in ensemble forecast. Singular vectors (SVs) is also an approach to estimating the limit of predictability. SVs have been successfully applied in operational weather forecast in ECWMF. In the presentation of Prof. Youmin Tang, he proposed the concept of climatological SVs. The competing aspect of this approach is without using tangent model in calculating SVs. Dr. Siraj Ul Islam investigated the error growth dynamics of SVs and showed that when the SVs are used as an initial perturbation, the forecast skill of key atmospheric variables over South Asian Monsoon region is significantly improved. Further, he demonstrated that the predictions with the singular vector have a more reliable ensemble spread, suggesting a potential merit for a probabilistic forecast. The linearity of SVs still

limits the forecast skill. If considering effect of nonlinearity in SVs, the forecast skill generated by the SVs may be greatly improved. In the workshop, Prof. Wansuo Duan focused on this limitation of linear singular vector (LSV) and presented an approach of conditional nonlinear optimal perturbation (CNOP) to estimate the maximum prediction error. Prof. Duan applied this approach to the predictability of ENSO and revealed the difference between CNOP and LSV. Then he used this approach to present the optimal observational array for dealing with the challenge of ENSO predictions due to diversities of El Nino. By hindcast experiments, the target observations determined by the CNOP were illustrated to be useful in distinguishing the types of El Nino events in predictions. In addition, Dr. Guodong Sun also showed the usefulness of the CNOP in dealing with the effect of model parametric errors on predictability and emphasized the nonlinear interaction of multiply physical parameters is beneficial to improve the simulation ability and prediction skill of the soil moisture.

For the second aspect of predictability study, the presentations in the workshop reported several interesting approaches to improving the forecast skill for weather and climate. For example, Prof. Fei Zheng suggested a new stochastic perturbation technique to improve the prediction skills of ENSO through using an intermediate coupled model. Similar idea was existed in the presentation of Dr. Aneesh Subramanian. Dr. Aneesh Subramanian compared the stochastic perturbation scheme representing model errors and the super-parameterization strategy for including the effects of moist convection through explicit turbulent fluxes calculated from a cloud-resolving model (CRM) embedded within a global climate model (GCM), finally showing that the combination of the two approaches helps improve the reliability of forecasts of certain tropical phenomena, especially in regions that are affected by deep convective systems.

Another aspect to predictability is to identify the most predictable mode. Prof. Timothy M. Delsole focused on the degree to which interactive ocean circulations are important for making useful predictions of the next decade and identified the most predictable patterns of global sea surface temperature in coupled atmosphere-ocean models. He suggested that interactive ocean circulation is not essential for the spatial structure of multi-year predictability previously identified in coupled models and observations, but the time scale of predictability, and the relation of these predictable patterns to other climate variables, is sensitive to whether the model supports interactive ocean circulations or not, especially over the North Atlantic.

Scientific Progress Made

The participants learned several new techniques and scientific progress to understand and resolve atmospheric and oceanic dynamics. The group discussion touched on most of the challenges and scientific issues related to nonlinear geophysical fluid dynamics, data assimilation, and predictability and prediction. Participants discussed various ways to make dynamical ensemble prediction more skillful on season to decadal time scales along with the discussion of deficiencies in numerical modeling. Such discussion helped participants to design better experiments or better data collection protocols. All these discussions were very productive especially for the early career young scientists learning new and advance scientific techniques from the leading scientist working in climate dynamics.

Outcome of the Meeting

This meeting brought together various mathematical and atmospheric backgrounds. The meeting was very timely and useful, and will surely have a strong impact on the further developments of predictability and data assimilation and related subjects. The organizers were in particular pleased to observe the many young people interested in the subject; some of them have already made substantial contributions (as witnessed by the talks given by junior researchers) and will surely continue to advance the subject.

The meeting provided a great overview of the field, with a number of excellent talks describing some recent exciting results in predictability and data assimilation. It is quite certain that some new interesting results will emerge as a result of the workshop.

Participants

Chen, Nan (New York University)
Delsole, Timothy (George Mason U.)
Demirov, Entcho (Memorial University)
Ding, Ruiqiang (Institute of Atmospheric Physics, Chinese Academy of Sciences)
Duan, Wansuo (Chinese Academy of Sciences)
Frediani, Maria E. B. (University of British Columbia)
Herbert, Corentin (Ecole Normale Supérieure de Lyon)
Hsia, Chun-Hsiung (National Taiwan University)
Islam, Siraj ul (University of Northern British Columbia)
Kieu, Chanh (Indiana University)
Kwasniok, Frank (University of Exeter)
Li, Jianping (Beijing normal university)
Lin, Hai (Environment and Climate Change Canada)
Liu, Ting (Second Institute of Oceanography)
Lu, Fei (Johns Hopkins University)
Mao, Yiwen (University of Victoria)
Monahan, Adam (University of Victoria)
Pimentel, Sam (Trinity Western University)
Shen, Zheqi (Second Institute of Oceanography)
Straub, David (McGill University)
Subramanian, Aneesh (University of California, San Diego)
Sun, Guodong (Chinese Academy of Sciences)
Tang, Youmin (University of Northern British Columbia)
Wang, Shouhong (Indiana University)
Wang, Xiaoming (Florida State Uni. & Fudan Uni.)
Zhang, Shaoqing (Ocean University of China)
Zheng, Fei (Chinese Academy of Sciences)

Bibliography

Two-day Workshop Reports

Chapter 27

Special Western Canada Linear Algebra Meeting (17w2668)

July 7 - 9, 2017

Organizer(s): Hadi Kharaghani (University of Lethbridge), Shaun Fallat (University of Regina), Pauline van den Driessche (University of Victoria)

History of this Conference Series

The Western Canada Linear Algebra meeting, or WCLAM, is a regular series of meetings held roughly once every 2 years since 1993. Professor Peter Lancaster from the University of Calgary initiated the conference series and has been involved as a mentor, an organizer, a speaker and participant in all of the ten or more WCLAM meetings. Two broad goals of the WCLAM series are a) to keep the community abreast of current research in this discipline, and b) to bring together well-established and early-career researchers to talk about their work in a relaxed academic setting.

This particular conference brought together researchers working in a wide-range of mathematical fields including matrix analysis, combinatorial matrix theory, applied and numerical linear algebra, combinatorics, operator theory and operator algebras. It attracted participants from most of the PIMS Universities, as well as Ontario, Quebec, Indiana, Iowa, Minnesota, North Dakota, Virginia, Iran, Japan, the Netherlands and Spain. The highlight of this meeting was the recognition of the outstanding and numerous contributions to the areas of research of Professor Peter Lancaster. Peter's work in mathematics is legendary and in addition his footprint on mathematics in Canada is very significant.

Presentation Highlights

The conference began with a presentation from Prof. Chi-Kwong Li surveying the numerical range and connections to dilation. During this lecture the following conjecture was stated: If $A \in M_3$ has no reducing eigenvalues, then there is a $T \in B(H)$ such that the numerical range of T is contained in that of A, and T has no dilation of the form $I \otimes A$. This statement led to interesting discussion among various participants after the lecture. Prof. Ion Zaballa from Spain gave an inspiring lecture on reducing matrix polynomials, which is a topic of interest to our guest of honour, Prof. Peter Lancaster. The morning talks were followed by two insightful lectures given by graduate students, Jane Breen (University of Manitoba) and Xavier Martinez-Rivera (Iowa State University). Both presented recent work, one on clustering in Markov chains, and the other on a signed principal rank characteristic sequence that has connections to the famous principal minor assignment problem. In the afternoon the focus of the workshop switched to matrix theory,

especially the spectra of matrices. Prof. David Watkins, a former doctoral student of Prof. Peter Lancaster, discussed fast and stable computation of certain matrices, taking sparsity into account. This talk was based on a paper that had won a SIAM outstanding paper prize. The first day concluded with an interesting lecture on the localization of the spectrum of a matrix. This topic is classical when considering the numerical range or Gersgorin's disc theorem. In this presentation, Prof. Michael Tsatsomeros, adapted a formulation for drawing the numerical range to constructing a new object called the envelope of A, which does a better job enclosing the spectrum of a matrix. The lecture was highlighted by some fascinating movies constructing the various regions associated with the envelopes of certain matrices.

On Sunday, the first three lectures were concerned with combinatorial matrix theory, with two of them delivering interesting connections to the classical topic of Hadamard matrices. The other, by Prof. Colin Garnett, provided insights on the patterns that allow all possible spectra (SAPs), and proved the $2n$ conjecture for irreducible patterns of order 6, and order 7 over the reals. Participants suggested ways in which the techniques presented might be applicable for reducible patterns. After the break, the next lecture was concerned with Bezout equations of stable matrix functions. This topic has deep roots with the research of Prof. Peter Lancaster and his work was featured prominently throughout the talk.

Our conference concluded with a heart-warming lecture delivered by the distinguished mathematician, friend, and colleague, Prof. Richard Guy. Prof. Guy provided a wonderful perspective on the life and celebrated career of Prof. Peter Lancaster.

Many thanks to BIRS and PIMS for the opportunity and funding for this very successful special WCLAM.

Some Open Problems and Questions

We list here a sample of the questions and unresolved problems that led to interesting discussions and constructive thought.

- **(i)** **(Chi-Kwong Li, College of William and Mary) If $A \in M_3$ has no reducing eigenvalues, then there is a $T \in B(H)$ such that the numerical range of T is contained in that of A, and T has no dilation of the form $I \otimes A$.**
- **(ii)** **(Sarah Plosker, Brandon University)What graphs other than chains exhibit PST? What if we do not require that the state transfer be perfect? What about weighted chains?**
- **(iii)** **(Colin Garnett, Black Hills State University) Is it possible to randomly select a pattern that does not satisfy the algebraic condition set forth with large enough n that is in fact spectrally arbitrary? What can we say about the $2n$ conjecture over R versus over C?**
- **(iv)** **(Behruz Tayfeh-Rezaie, Institute for Research in Fundamental Sciences (IPM)) Classify all Hadamard matrices with only two distinct types for 4-tuples of rows. Find another infinite family of Hadamard matrices with only two distinct types for 4-tuples of rows.**
- **(v)** **(Michael Tsatsomeros, Washington State University) Can the envelope of a matrix be refined by considering more eigenvalues and/or eigenvectors of its Hermitian part?**

Special Occasion Associated with the Meeting

One other important feature of our meeting was a birthday celebration to honour the 88th birthday for Prof. Peter Lancaster. The occasion was held in the BIRS lounge on Saturday evening with all participants being involved. There was a cake brought by Peter's daughter and the event included reading well-wishes from friends and colleagues of Peter's that could not attend the meeting at BIRS.

Participants

Adm, Mohammad (University of Regina)
Breen, Jane (University of Manitoba)

Craigen, Robert (University of Manitoba)
Fallat, Shaun (University of Regina)
Guy, Richard (The University of Calgary)
Hassani Monfared, Keivan (University of Calgary)
Holzmann, Wolfgang (University of Lethbridge)
Kaashoek, Rien (Vrije Universiteit)
Kharaghani, Hadi (University of Lethbridge)
Kim, In-Jae (Minnesota State University)
Kirkland, Stephen (University of Manitoba)
Lancaster, Peter (University of Calgary)
Li, Chi-Kwong (College of William and Mary)
Martinez-Rivera, Xavier (Iowa State University)
Masgreghi, Javad (Laval University)
McDonald, Judith (Washington State University)
Nasserasr, Shahla (Nova Southeastern University)
Pereira, Rajesh (University of Guelph)
Plosker, Sarah (Brandon University)
Raouafi, Samir (University of Regina)
Sasani, Sara (University of Lethbridge)
Suda, Sho (Aichi University of Education)
Tayfeh-Rezaie, Behruz (Institute for Research in Fundamental Sciences (IPM))
Tsatsomeros, Michael (Washington State University)
Van den Driessche, Pauline (University of Victoria)
Vander Meulen, Kevin (Redeemer University College)
Watkins, David (Washington State University)
Zaballa, Ion (Euskal Herriko Unibertsitatea)

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Chapter 28

Modeling and Simulation: Practical Engineering Applications (17w2689)

September 1 - 3, 2017

Organizer(s): Zhangxing John Chen (University of Calgary), Jesse Zhu (Western University), Zhenghe Xu (University of Alberta)

Overview of the Field of Modeling and Simulation

Computational Science has been called a third branch of science, along with theory and experiment. In truth, it is part of theory and part of experiment, but it is different from either. Many theoretical problems can only be solved using a high performance computer. Modeling and simulation, the major component of computational science, is much more than simply elaborating pure theory. It is a critical tool used to analyze important and varied phenomena such as fluid flow and transport, chemical processing, air pollution, water contamination, nano material design, information, computational chemistry, and phase interfaces. In September 2011, a workshop on Modelling and Simulation was held at the Banff International Research Station (BIRS) in Banff, Alberta. The major focus was the presentation of theoretical modelling and simulation work by researchers and software developers from around the world. It also highlighted activities in the high performance computing area.

Recent Developments and Open Problems of Modeling and Simulation

Since 2011, the rapid development of computer power and sophisticated computational techniques has resulted in the application of high performance computing to modelling and simulation with unprecedented accuracy. Their scope has also impacted a wide range of important science and engineering challenges. These challenges are comprehensive emerging issues requiring a thorough understanding of the underlining principles of physics, chemistry, mathematical modelling, numerical solution techniques and computing infrastructure. Their understanding will have profound implications and applications in mathematics, science, engineering, medicine and industry. Rapid economic developments and social issues have significantly increased research in modelling and simulation in these areas. For example, the need to study and understand the complex physical and chemical processes occurring in and around the earth, such as groundwater contamination, oil and gas reservoir production, discovering new oil reserves, ocean hydrodynamics, CO2 storage and sequestration, and air quality control, is all vital to our living environment, economic development, natural resource management and national security. The study of these problems through laboratory experiments, modelling theories and simulation techniques requires interdisciplinary collaborations between engineers, mathematicians, com-

putational scientists and researchers working in industry, government laboratories and academy. Bringing together these experts for a unified focused effort will advance predicting, understanding and optimizing many complex phenomena. The goal of this two-day BIRS workshop on September 1-3, 2017 was to hold lectures that emphasize the rapid development of modelling and simulation theories and their practical engineering applications [1, 2, 3].

Workshop Presentation Highlights

The workshop themes include: (A) Mathematics of Multiphase Fluid Flow and Transport; (B) High-Quality Discretization of Flow and Transport; (C) Computational Modeling of Multiscale Phenomena; (D) Parallel Computing; (E) Nonlinear Effects on Propagation Properties of Numerical Models; (F) Interfacial Phenomena in Mineral and Material Processing; (G) Spatially Explicit Carbon and Water Cycle Modelling; (H) Air Pollution and Industrial Wastes.

24 presentations in this workshop have dealt with these topics. Speakers have been carefully selected to ensure that a range of modeling and simulation techniques can be explored. This diversity is necessary in order to address various phenomena arising from emerging issues in the energy and environment sectors [4, 5].

Some of the major presentations are summarized as follows:

The presentation given by Dr. Jesse Zhu covered research related to aspects of particle technology from particle formation, characterization, particle flow to many mathematical and practical applications. His fundamental studies included development of fluidization theory and identification of new regimes, the expansion to liquid-solid fluidization, inverse fluidization and untrafine powder fluidization, and the detailed studies on many fluidized bed reactors.

The presentation given by Dr. Jinyu Sheng covered many modeling and simulation problems on physical oceanography, modelling and prediction of extreme marine events, interaction of ocean waves and currents, air-sea interaction, and tidal and storm-induced circulation.

The presentation given by Dr. Liu Zheng covered his research on mathematical, chemical and biomolecular fundamentals of bioprocessing engineering with special interests in protein conformational transition, nanostructured enzyme catalysts, and formulation of natural product with biological functions.

The presentation given by Dr. Frank Cheng covered mathematical problems in corrosion science and engineering of pipelines, including coating failure mode and effect analysis, cathodic protection shielding, high-voltage alternating current interference, CO2 corrosion, microbial corrosion, flow-accelerated corrosion, and under-deposit pitting corrosion.

The presentation given by Dr. Haibo Niu covered his research related to the fate/transport process of contaminants in marine environment, with a focus on oil spills and development of mathematical models to predict the fate/behaviours of oil to support emergency response.

The presentation given by Dr. Johnny Chen covered modeling and simulation issues in electrochemical energy storage and conversion technologies due to decreasing global fossil fuel supplies and increasing environmental concerns. In order to provide sustainable energy infrastructures and resources for future generations, significant improvements to the current state of these technologies is imperative.

The presentation given by Dr. John Chen covered his scientific research on laboratory experiments, mathematical modeling, and the supporting analyses for recovery of oil and gas resources. In particular, his team work has been focused on lab, modeling and simulation studies of fluid flow and transport in unconventional oil and gas reservoirs (heavy oil, oil sands, and tight and shale oil and gas reservoirs). He shared his industrial collaboration experience with the participants.

The presentation given by Dr. Charles Xu covered his modeling approach in development of highly efficient and cost-effective hydrothermal liquefaction (HTL) processes for energy recovery from various solid wastes/residues and development of catalytic processes and novel catalysts for the production ”green” fuels, chemicals from renewable feedstock, and air emissions control.

The presentation given by Dr. Qiao Sun covered her current research activities around building a modeling framework for mechanical systems using wind turbine as an example. The goal is to be able to combine physics based and data driven models to assist machine health condition assessment and failure prediction. A system level model that includes component models based on underlying physical principles can provide the key to a solution with the much

needed fault prediction capabilities. In recent years, her focus is to establish an integrated wind turbine system model that can represent the digital copy of an actual system particularly in terms of its health condition.

The presentation given by Dr. Ying Zheng covered her research related to aspects of chemical reaction Engineering, catalyst synthesis and catalytic processes, her recent interests on catalytic process intensification and her fundamental mathematical studies on new material synthesis, catalysis, reaction mechanism and mass/heat transfer in heterogenous catalysis, reactor and process design.

The presentation given by Dr. Hongbo Zeng covered his research interests in colloid and interface science, functional materials and nanotechnology, with a special focus on intermolecular and surface interactions in soft matter (e.g., polymers, biopolymers, biological systems, surfactants, and emulsions) and mathematical and engineering processes, and development of functional materials with engineering/bioengineering/ environmental applications.

The presentation given by Dr. Laurence Yang covered his current interest focused on cyber-physical-social systems (CPSS) design, data analytics on parallel and distributed (cloud) platforms. The booming growth and rapid development in embedded systems, wireless communications, sensing techniques and emerging support for cloud computing and social networks have enabled researchers and practitioners to create a wide variety of Cyber-Physical-Social Systems (CPSS) that reason intelligently, act autonomously, and respond to the users needs in a context and situation-aware manner. It, as a novel emerging paradigm, has gained popularity within the research community and industry due to the fact that it enables deep fusion among humans, computers, and things.

The presentation given by Dr. Dong Liang covered numerical problems in development of conservative high-order characteristics methods for atmospheric aerosol transports; development of efficient Moving-cut HDMR approach for aerosol chemical process; study of energy laws of electromagnetic waves in metamaterials, and development of energy-conserved S-FDTD schemes for computational electromagnetics; development of efficient mass conservative domain decomposition parallel computing methods for environmental computation and for contamination flow in porous media; and development of optimal control approaches for environmental pollution in atmosphere and in groundwater.

The presentation given by Dr. Huaxiong Huang covered numerical solutions for partial differential equations, mathematical models and scientific computing for problems in science and engineering, and modeling real world problems from industry and medicine.

The presentation given by Dr. Jing Chen covered development of regional and global carbon cycle models using these traits as inputs; development of instruments and methods for the ground measurement of these traits; development of a distributed hydrological model using these traits and other inputs; development of canopy radiative transfer models for remote sensing applications; development of biosphere stable isotope models for global carbon cycle research; development of atmospheric inversion and global carbon assimilation systems for estimating the global carbon cycle using atmospheric CO2 concentration; and development of a long-term forest carbon cycle model with consideration of both disturbance and non-disturbance effects.

The presentation given by Dr. Nancy Chen covered her research in modeling the primary, secondary, and tertiary recovery processes; simulating complex recovery process in unconventional heavy oil bitumen and tight shale formations; and optimizing reservoir development strategies to maximize oil recovery or net present values.

The presentation given by Dr. Yaushu Wong covered numerical solutions for partial differential equations, mathematical model and scientific computing for problems in science and engineering, and data sciences and its applications to real world problems.

The presentation given by Dr. Xianguo Li covered his research centred in thermofluid sciences, including fluid flow, heat and mass transfer, thermodynamics, combustion, energy and power systems.

The presentation given by Dr. Gemma Lu covered her current research focused on the interfacial and surface science, biomimetic materials, nanocomposites/nanomaterials/nanotechnology, and microfluidic systems applicable in energy, environmental and biomedical fields such as sustainable energy, oil recovery enhancement, wastewater and tailings treatment, environmental sensing, CO2 capture and utilizations, and soil remediation.

In addition to the regular workshop presentations, there were also considerable time for interactions between participants. The workshop organizers coordinated round-table sessions where questions and answers were shared. Due to the high-level and interdisciplinary nature of the participants invited, those who attended this proposed workshop were exposed to novel ideas and will build new relationships for future collaborations. The workshop presentations, discussion at the round-table sessions and possible future collaborations among the participants will support significant progress of mathematics, science and engineering modelling and simulation.

Scientific Progress Made in Modeling and Simulation

The objective of this workshop is to bring together the worlds top active researchers (and their more junior counterparts) who study energy and environmental modeling and simulation to discuss past, recent, and prospective advances in this area. The speakers have summarized important advances from the past two decades and have discussed the current understandings, the state-of-the-art techniques, and the current major challenges. Each session of this workshop has provided a vehicle for participants to learn novel techniques and new advances in this area of work. The content has been academic in nature while addressing the many significant applications for industry [6, 7].

The ultimate goal of the workshop is to expose workshop participants (in particular, junior researchers) to the latest developments in the field of modeling and simulation, while emphasizing the impact of this field on science, engineering, and industry.

The study of the diverse topics presented in the workshop through laboratory experiments, mathematical theory, and computational techniques requires interdisciplinary collaboration between engineers, mathematicians, computational scientists, and researchers working in industry, government laboratories, and academy. The collaborative work of researchers in this workshop will create meaningful progress in predicting, understanding, and optimizing many complex phenomena. The rational for this two-day BIRS workshop is to hold lectures that pull together the major ideas and recent research results, chart future directions, and address newly emerging issues for energy and environment modeling and simulation. It is anticipated that the participants have left the workshop knowing the future research directions and the needed potential applications.

Outcome of the Workshop

The Banff International Research Station is a beautiful location for learning and building relationships. Its common areas have supported our goals to have researchers engaged in discussion throughout the workshop event. We have brought researchers from around the world to share their perspectives, test ideas, and create new connections both intellectually and socially while exploring the latest developments in modeling and simulation. This workshop has promoted, enhanced, and stimulated cross-continental research interactions and collaborations in mathematical sciences and will shape changes in the research work completed with modeling and simulation.

Participants

Chen, Zhangxing John (University of Calgary)
Chen, Guohua (Hong Kong Polytechnic University)
Chen, Jingming (University of Toronto)
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Chen, Nancy Shengnan (University of Calgary)
Cheng, Frank (University of Calgary)
Huang, Huaxiong (York University)
Le, X. Chris (University of Alberta)
Li, Xianguo (University of Waterloo)
Liang, Dong (York University)
Liao, Wenyuan (University of Calgary)
Liu, Zheng (Tsinghua University)
Lu, Gemma (University of Calgary)
Niu, Haibo (Dalhousie University)
Sheng, Jinyu (Dalhousie University)
Sun, Qiao (University of Calgary)
Wong, Yaushu (University of Alberta)
Xu, Charles (Western University)
Yang, Laurence T. (St Francis Xavier University)
Zeng, Fanhua (University of Regina)

Zeng, Hongbo (University of Alberta)
Zhao, Boxin (University of Waterloo)
Zheng, Ying (University of Edinburgh)
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Chapter 29

Transdisciplinary Approaches to Integrating Policy and Science for Sustainability (17w2688)

October 6 - 8, 2017

Organizer(s): Gabriela Alonso Yáñez (University of Calgary), Kathleen Halvorsen (Michigan Technological University), Marcella Ohira (Inter American Institute for Global Change Research)

Overview

In Canada and in many other countries, there is a clear commitment to environmental sustainability through meaningful, effective collaboration between discipline specialists, resource-users, communities, and funding agencies. Recognising that effective cross sector collaboration and transdisciplinary (TD) integration is rare, the workshop overarching goal was to train future leaders in transdisciplinary approaches for policy and science for sustainability in the Americas. The workshop enabled 25 participants to be part of several educational and training activities designed to teach elements for successful teamwork; multi-disciplinary method and methods to engage stakeholders; project management and criteria for teamwork evaluation; science communication to different audiences; and revising and finalizing transdisciplinary team proposals.

Recent Developments and Open Problems

A growing body of scholarship identifies obstacles that hinder truly transdisciplinary collaborations. For instance, the lack of active and sustained interaction of scientists with stakeholders, the absence of collaboration for framing problems and setting shared agendas and goals, and a shortage of effective communication pathways among researchers and policy-makers serve to negatively impact the transfer, dissemination, and use of scientific knowledge ([1], [2], [3], [4]). This is particularly relevant for teams currently addressing global socioecological problems, which feature high diversity of team membership and high task-interdependency.

Presentation Highlights

(i) Members of First Nations communities led several discussions and support participants in better understanding the perspectives and views of non-academic actors in transdisciplinary projects.

(ii) The Improvisational workshop to help participating scientists better communicate their findings.

(iii) Parks Canada staff led discussions on innovative strategies for protecting natural resources informed by a socioecological approach.

(iv) In collaboration with the Calgary Public Library and the Latin American Research Center, the seminar included a multidisciplinary seminar on Global Change and cross sector collaboration with representatives from various sectors of Canada's First Nations, Academics, Calgary legislative and private sectors.

(v) Session on traditional land and indigenous ways of knowing including a visit to two Sacred sites (Sundial Hill Medicine Wheel and Women's Buffalo Jump) of the Black Foot First Nation led by First Nations contrasting with Latin American indigenous communities.

Scientific Progress Made

Recent research concludes that training interventions offer a promising opportunity to improve performance of international TD teams by teaching skills and competencies necessary to enable multi-cultural and multi-institutional collaborations ([5], [6]). These training interventions are particularly relevant for a variety of audiences working on practicing and promoting TD "namely, those who make policy, who fund science, who carry out research and who organize and implement institutional systems" ([7], p. 62). Professional development designed specifically to improve and sustain TD work such as the one that took place in BIRS Banff are just beginning to emerge ([8], [9], [10]). However, these capacity building initiatives are critical for addressing policy relevant global change issues of the 21st century.

Outcome of the Meeting

Four transdisciplinary project proposals developed at the Seminar are now in the early stages of advancement and implementation. They include 25 participants from 14 countries representing universities, various tiers of government, NGOs, and the private sector.

Group 1. One group project concerns work to identify and characterize features of community governance that affect water provision in four locations in the Latin American countries of Chile, Brazil, Colombia and Argentina. The team's main goal is to gather information about the ways in which people organize themselves and make decisions about water provision at a local scale in four different Latin American sites. Knowledge about water provision at local scale is central to plan for strategies to address extreme droughts in the region.

Group 2. Another team is focusing on developing a framework to manage temperate grassland ecosystems in Argentina, Brazil, Canada and Uruguay. Working in close collaboration with scientists, government staff, NGOs and farmers, the team plans to create a framework that identifies and considers the ways in which various social groups and sectors currently adopt sustainable practices in temperate grasslands to design policy strategies that responds to local needs.

Group 3. By researching influences of livelihood and governance mechanisms on resource management, this team hopes to provide concrete recommendations that could help strengthen conditions in which people address stressors affecting coupled-human natural systems, particularly in smallholder contexts. Through this project, they aim to assess the influence that smallholder livelihoods and governance mechanisms have on the management of both Peruvian small-scale shark fisheries and the Bolivian smallholder agriculture. In each, they will identify main livelihoods and sources of income within target communities, reliance on common-pool resources, main formal and non-formal governance mechanisms and develop recommendations to enhance the management of common-pool resources in regards to local livelihood and governance structures. Their project will use mixed methods, combining quantitative and qualitative methodologies specific to each case study.

Group 4. This project will carry out a comparative case study in two countries in the LAC region: Colombia and the Dominican Republic. In addition to the case studies, secondary information comes from another three countries

in the region: Jamaica, Cuba and Guatemala. This will allow a better understanding of the general situation in the region, in order to identify the best solutions and alternatives for cocoa producers to adapt to the adverse effects of climate change. Besides capacity building for cocoa producers to address climate change, the project will contribute to the literature and practice of the measure and value of ecosystem services In addition; an important outcome is the modelling of future climatic scenario within the cocoa production cycle. Using systems dynamics to integrate the different variables involved, such as the production, climate, social and economic variables. This will allow integrating an innovative transdisciplinary perspective to the results of the process a perspective.

The outcomes of these projects go beyond developing reports and scientific journal articles, and include direct involvement with communities in an effort to co-produce knowledge that is applicable to real-world current situations.

As a result of the seminar, the IAI has offered to provide US$ 5,000 to three teams, which received positive proposal reviews from the seminar steering committee. The four teams are in the process of submitting their transdisciplinary proposals to national and international funding agencies to raise additional funds.

For further information about the seminar program, participants, lectures and outcomes, please access

`http://www.iai.int/?p=24364&lang=en`

Participants

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Chapter 30

Retreat for Young Researchers in Stochastics (17w2695)

October 20 - 22, 2017

Organizer(s): Martin Barlow (UBC), Chris Hoffman (U. Washington), Mike Kouritzin (U. Alberta)

Overview

This was the third annual meeting of the PIMS Postdoctoral Training Centre in Stochastics (PTCS). The Retreat offers an opportunity for young probabilists or users of probabability from Western Canada and Washington state to interact, communicate their recent results and ongoing research programs and initiate new collaborations. The participants were 9 postdocs, either supported by PTCS or members of the PTCS network, 5 junior faculty at PIMS Universities, and 9 senior faculty from PIMS Universities; many of those in the last group are advisors for PTCS postdocs.

Presentation Highlights

The range of topics presented covered many of the active areas in modern probability, and included stochastic differential equations and stochastic partial differential equations, stochastic perturbations of dynamical systems, ergodic theory, compressed sensing and epidemic models in mathematical biology.

Noah Forman (PTCS, UW) presented his work on dynamic random trees. These arise in models in population genetics. The main result gives an extension to random trees of the classic characterization of exchangeable random variables on $\mathbb{N}$ as a mixture of independent sequences. The lifting of this result to the tree case requires a number of pruning and re-grafting operations on random trees. This work raises several interesting problems concerning the limiting object, which is the continuum random tree of Aldous.

Eric Foxall described an epidemic model (the 'stochastic logistic model') on a network. His talk presented the simplest case, that of a complete graph, and gave a detailed analysis of the structure and limits of the model near the critical point. The complete graph has little geometric structure, and a long term goal of this research is to look at networks with more interesting and realistic geometry.

The talks by Gerrardo Barrera Vargas and Zhonwei Shen looked at small stochastic perturbations of dynamical systems. This area has quite classical roots – the pioneering work of Ventsel and Freidlin was in the late 1960s, but our understanding of the connections between 'chaotic dynamics' and 'stochastic dynamics' is very incomplete. The research group of Y. Yi at U. Alberta is now very active in this area, and this workshop was helpful in building links between this group and the probability groups at UBC and UW.

Shirou Wang described new work in ergodic theory, which gives that the entropy map upper semi-continuous for a class of dynamical systems.

Wenning Wei described her work with S. Tang (Fudan U.) on Hölder regularity in space of solutions to Backwards Stochastic PDE. (BSPDE). BSPDE's arise for example in optimal control problems for (forward) SPDE's. For a particular class of linear BSPDE's they are able to show the solution has sufficient smoothness in space to give the BSPDE a classical interpretation. A key idea is to consider spatial regularity into the space of L^2 functions of time, which is a novel perspective and gives new results, even for the deterministic setting.

Simone Brugiapaglia discussed his recent work with Ben Adcock (SFU) and Casie Bau (Grad student now at UBC) on sparsity-based approaches for high dimensional function approximation.

Participants

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Bibliography

Focused Research Group Reports

Chapter 31

Extensions of index theory inspired by quantum scattering theory (17frg668)

June 18 - 25, 2017

Organizer(s): Alan Carey (Australian National University), Fritz Gesztesy (Baylor University)

Overview of the Field

In this section we describe the background to the meeting as it stood prior to the results announced by the participants.

The part of quantum theory that is relevant to this proposal is scattering theory. We approach scattering theory via the spectral shift function theory of Krein, Koplienko, and Potapov–Skripka–Sukochev [12, 13, 14, 15].

On the geometric side following from [6] the work of Carey, Grosse and Kaad has introduced a new spectral invariant the 'homological index' [9, 7]. This homological index provides the appropriate framework for a major extension of older ideas of R. Carey and J. Pincus [10] on almost commuting operators. It is expressed as a pairing of a new cyclic homology theory developed in [9] with its dual cohomology theory. An open question is whether we can relate the homological index to the spectral shift function. In addition we want to know if it is possible to use cyclic theory to investigate the latter and conversely use the spectral shift function to calculate the homological index.

One major obstacle is the lack of computable examples. We expect to be able to understand these spectral functions in the case of scattering for supersymmetric quantum Hamiltonians, Dirac-type operators on non-compact manifolds and their perturbations. In particular, our aim is to connect the spectral properties of the operators under investigation to invariants such as spectral flow, the Fredholm index, Witten index and generalisations. Previous work in this direction [1] uses the de Rham complex viewed à la Witten as a supersymmetric system. This work, though it shows connections to spectral geometry, is incomplete and the conjectures made there about extensions to more general situations were never published.

The work on the homological index in examples [7] shows that there is a deeper extension of index theory to the non-Fredholm case not understood in the 80s. In addition, our previous work has already led to a remarkable new insight into a question first raised by Atiyah-Patodi-Singer on the relationship of spectral flow for paths of Dirac operators on odd-dimensional manifolds to the Fredholm index for a Dirac operator on the suspension of the manifold. We summarise this aspect now because during the BIRS meeting progress was made.

This line of research started in [2] where it was shown that, when both exist, spectral flow is given by Krein's spectral shift function (even in semi-finite von Neumann algebras). Related work for operators with essential spectrum appeared in [11] in 2011 (a long paper in Advances in Math.) Two main results were obtained in this latter paper, the first being a trace formula and the second, a generalised Pushnitski formula [15].

The focus then however, was on Fredholm theory inspired by the fact that spectral flow relates directly to the topology of the underlying manifold on which the path of operators is defined. The surprising discovery made in [8] is that the trace formula due originally to Pushnitiski [15] and generalised in [11] is in fact an operator trace identity that has as a corollary, in the Fredholm case, the relationship between spectral flow and the Fredholm index first discovered by Atiyah-Patodi-Singer. The results of [8] do not depend on a Fredholm assumption and thus go far beyond the results of Robbin and Salamon [16].

The main aim of this focussed research group proposal is to investigate generalisations of [11] to situations involving non-Fredholm operators inspired by the main result of [8]. One specific question is whether the spectral shift function can be really thought of as a generalisation of spectral flow to non-Fredholm situations. As an adjunct to this we will investigate the Witten index as a method of calculating this generalised spectral flow.

In [11] and [6] it is assumed that we have self adjoint operators D and perturbations A such that for complex z not in the spectrum of D the product $A(D-z)^{-1}$ is trace class (i.e., one employs a relatively trace class condition). This rules out geometric examples. In work underway we have shown that the relatively Schatten class condition such as, $A(D-z)^{-n}$, $n = 2, 3, \dots,$ is trace class, is what is needed for the geometric case (and PDEs in general).

Our primary technical objective is to relax this (severe) relatively trace class perturbation restriction so as to have very widespread applications in mathematical physics and geometry. There are some papers on the arxiv already on graphene and some unpublished work of Moore and Witten that contain examples of great interest. Moreover condensed matter theorists have speculated about generalisations of the Robbin-Salaman result for the study of spectral flow in their models.

Thus the study of examples in two and three dimensions is already important and it will assist to provide insight into general theory and applications. We anticipate interesting connections with condensed matter theory where spectral flow has played a role in the study of many model Hamiltonians.

Recent Developments and Open Problems

2.1 The spectral shift function. **For a pair of self-adjoint not necessarily bounded operators A_0 and A_1 on a Hilbert space $\mathcal{H}$ such that their difference, $A_1 - A_0$, is trace class, there exists a unique function $\xi \in L^1(\mathbb{R})$ (known as the Krein–Lifshitz spectral shift function) satisfying the fundamental trace formula, $\mathrm{Tr}(\phi(A_1) - \phi(A_0)) = \int_{\mathbb{R}} \xi(x)\phi'(x)dx$, whenever ϕ belongs to a class of suitably admissible functions.**

More precisely, we are considering the following situation. Take a path of self adjoint operators $A(t),\ t \in \mathbb{R}$ on a Hilbert space $\mathcal{H}_0$ and form $\mathbf{D_A} = (d/dt) + \mathbf{A}$ acting on $\mathcal{H}_+ = L^2(\mathbb{R}; dt; \mathcal{H}_0)$. Here $(\mathbf{D_A}u)(t) = u'(t) + A(t)u(t)$ $u \in L^2(\mathbb{R}; dt; \mathcal{H}_0),\ \ t \in \mathbb{R}$, and $\mathbf{A}$ is a direct integral of the family $\{A(t)\}_{t\in\mathbb{R}}$ (with asymptotes $A(t) \underset{t\to\pm\infty}{\longrightarrow} A_\pm$ in the norm resolvent sense) over $\mathbb{R}$. If we let $\mathcal{D}_+ = \mathbf{D_A}$, $\mathcal{D}_- = \mathcal{D}_+^*$ and $\mathcal{D}$ be the 2×2 matrix-valued operator introduced above acting on $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_+$, then under a relatively trace class assumption on $A_\pm$ we can compute the relationship between spectral flow for the path $A(t)$, the spectral shift function for the pair $A_\pm$.

It is important to note that the spectral shift function also exists when no Fredholm condition is assumed and hence no standard notion of spectral flow exists. This raises the interesting question of what the spectral shift function is measuring geometrically when the Fredholm condition is absent.

2.2 Supersymmetry and graded spaces

We are given a Hilbert space $\mathcal{H}$ equipped with a $\mathbb{Z}_2$ grading γ, (that is, $\gamma^* = \gamma, \gamma^2 = I$), and a self adjoint unbounded

operator $\mathcal{D}$ such that $\mathcal{D}\gamma + \gamma\mathcal{D} = 0$. Then we may write,

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}.$$

In addition, we are given self-adjoint operators H_0 and H_1 (think of $H_0 = \mathcal{D}_+\mathcal{D}_-$ and $H_1 = \mathcal{D}_-\mathcal{D}_+$) acting on a separable graded Hilbert space $\mathcal{H}$, and ask for which scalar functions f, the difference $f(H_1) - f(H_0)$ lies in the n^{th} Schatten–von Neumann ideal $\mathcal{L}^n(\mathcal{H})$. This question has been topical in perturbation theory for over 60 years starting in 1953 for the case where the difference $V := H_1 - H_0$ belongs to the trace class. In fact, Krein proved that

$$\xi(\lambda; H_1, H_0) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \Im(\ln(D_{H_1/H_0}(\lambda + i\varepsilon))) \text{ for a.e. } \lambda \in \mathbb{R}, \tag{31.0.1}$$

or, equivalently, in terms of the perturbation determinant,

$$\det((H_0 - zI)(H_1 - zI)^{-1}) = \exp\bigg(\int_{\mathbb{R}} \frac{\xi_{H_0,H_1}(\lambda)}{(\lambda - z)} d\lambda \bigg), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{31.0.2}$$

Based on what we know about the relationship between Krein's spectral shift function and standard spectral flow, we have conjectured that there is a generalised spectral flow formula, for a relatively Schatten class perturbation condition in the non-Fredholm case.

2.4 Details on the higher Schatten conjectures.

Recalling the notation of subsection 2.2, we remark that Atiyah, Patodi, and Singer studied operators of the form $\mathbf{D_A}$ that we defined there. This class of operators form models for more complex situations. They arise in connection with Dirac-type operators (on compact and noncompact manifolds), the Maslov index, Morse theory (index), Floer homology, Sturm oscillation theory, etc.

One of the principal aims of such a study is to compute the Fredholm index of $\mathbf{D_A}$. The most general published computation of this kind to date is contained in [11] under the assumption that $A_+ - A_-$ is of trace class relative to A_-. Note that prior to [11] and [15] it was required to assume that operators in the path $A(t)$ had discrete spectrum.

Invoking the spectral shift function $\xi(\lambda; A_+, A_-)$ and the perturbation determinant associated to the pair (A_+, A_-), $D_{A_+/A_-}(z) = \det_{\mathcal{H}}((A_+ - zI)(A_- - zI)^{-1})$ corresponding to the pair (A_+, A_-), the following was established in [11],

$$\begin{aligned} \operatorname{Index}(\mathbf{D_A}) &= \operatorname{SpFlow}(\{A(t)\}_{t=-\infty}^{\infty}) = \xi(0; A_+, A_-) \\ &= \pi^{-1} \lim_{\varepsilon \downarrow 0} \Im\big(\ln\big(\det_{\mathcal{H}}\big((A_+ - i\varepsilon)(A_- - i\varepsilon)^{-1}\big)\big) \\ &= \xi(0_+; H_1, H_0). \end{aligned}$$

We conjectured that these results will generalise when we have relatively Schatten class perturbations.

Work underway

We have a complete theory for the case of one dimension that has just been published. With $A_- = \frac{d}{idt}$, $A_+ = A_- + F$ where F is a self adjoint $N \times N$ matrix acting on $\mathcal{H} = L^2(\mathbb{R}; \mathbb{C}^N)$, we were able to establish the following:

(i) **A Pushnitksi formula (cf. [15]), implying that**

$$\xi_{A_-,A_+} = \xi_{H_1,H_2},$$

where H_1, H_2 are as above acting on $\mathcal{K} = L^2(\mathbb{R}, \mathcal{H})$;

(ii) **explicit formulas for ξ_{A_-,A_+} for these examples;**

(iii) **the trace formula:**

$$2z\mathbf{Tr}_{\mathcal{K}}[(z + H_1)^{-1} - (z + H_2)^{-1}] = \mathbf{Tr}_{\mathcal{H}}[g_z(A_-) - g_z(A_+)],$$

where $g_z(x) = x(z + x^2)^{-1/2}$ (a smoothed version of the sign-function for $z < 0$);

(iv) an abstract framework for relatively Hilbert-Schmidt perturbations $A_+ - A_-$ in which the Pushnitski formula and hence the result mentioned in (i) could be proved;

(v) the existence of a class of examples of pseudo-differential operators in any dimension to which the method developed in [11] applies.

The main gap

To utilise the methods we have employed previously, we need to study scattering for Dirac-type operators in two and three dimensions. The existing literature is not adequate for our purposes. The objective then is to extend the one-dimensional analysis described above point by point to higher dimensions.

Presentation Highlights

(i) Alan Carey summarized the state of the art of spectral flow as it pertained to our meeting.

(ii) Galina Levitina explained the proof of the principal trace formula and the generalisations of Pushnitski's work, applicable to higher dimensions.

(iii) Roger Nichols and Fritz Gesztesy described in detail the proof of absence of singular continuous spectrum for flat space, massless Dirac-type operators based on the strong limiting absorption principle in all space dimensions ≥ 2. In addition, a global limiting absorption principle (for the entire real line) for free, massless Dirac-type operators, again in all space dimensions ≥ 2, was presented.

(iv) Jens Kaad explained the recent work on the homological index and the resolvent version of the principal trace formula (PTF).

Scientific Progress Made

As a result of Galina Levitina's presentation we can announce an end point to the work begun in [11] so that the relationship between the Witten index and generalised spectral and between the associated spectral shift functions has been completely settled [5].

In interactions between the participants, the outstanding example of two-dimensional systems using Dirac-type massless Hamiltonians was resolved in the affirmative. In fact, a first attempt prior to our Banff meeting required the limitation of space dimension ≥ 3, but due to interactions at the meeting this could be extended to the important two-dimensional case (with potential interest in connection with graphene applications). Thus, a proof of the global limiting absorption principle in all space dimensions ≥ 2 emerged during the meeting and will result in paper [4]. The latter also contains essential self-adjointness results of Dirac-type operators which are of interest in their own right. No doubt, without the opportunity afforded by BIRS, this research would not have been done as it involved essential contributions by nearly every participant at the meeting.

In addition, a proof of the limiting absorption principle for all nonzero energies for massless Dirac-type operators with matrix-valued potentials (applicable to electromagnetic potentials decaying at infinity like $|x|^{-\rho}$, with $\rho > 1$) in all dimensions ≥ 2, has now been established [3].

A plan was formulated for publications from the meeting and we have identified at least three lengthy papers (or even monographs), reporting our results being completed over the next six months.

Outcome of the Meeting

The principal outcomes were,

(i) the finalisation of the proof of the principle trace formula as an operator trace identity (cf. [5]),

and,

(ii) establishing the limiting absorption principle for certain Dirac-type operators on flat space in all dimensions ≥ 2 (cf. [3], [4]).

Participants

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Chapter 32

Stochastic lattice differential equations and applications (17frg671)

September 10 - 17, 2017

Organizer(s): Xiaoying Han (Auburn University), Hakima Bessaih (Department of Mathematics, University of Wyoming), Maria J. Garrido-Atienza (University of Sevilla), Bjorn Schmalfuss (Institut for Mathematics)

Introduction

Lattice dynamical systems arise naturally in a wide range of applications where the spatial structure has a discrete character, such as image processing, pattern recognition, and chemical reaction theory. For some cases, lattice dynamical systems arise as discretization of partial differential equations, while they can be interpreted as ordinary differential equations in Banach spaces which are often simpler to analyze.

In particular, lattice systems have been used in biological systems to describe the dynamics of pulses in myelinated axons where the membrane is excitable only at spatially discrete sites. For example, dynamics of excitable cells can be described by the following lattice system:

$$\frac{\partial v_n}{\partial t} = d(v_{n+1} - 2v_n + v_{n-1}) + f(u_n, v_n), \tag{32.0.1}$$

$$\frac{\partial u_n}{\partial t} = g(u_n, v_n), \tag{32.0.2}$$

where v represents the membrane potential of the cell, u comprises additional variables such as gating variables, chemical concentrations, necessary to the model, the subscript n indicates the nth cell in a string of cells, and d is the coupling coefficient $d = 1/R$, where R is the inter-cellular resistance. Typical models that can be fit in the scheme of (32.0.1) - (32.0.2) include Beeler-Reuter, Hodgkin-Huxley, FitzHugh-Nagumo, and many other models.

Lattice systems have also been used in fluid dynamics to describe the fluid turbulence in shell models. For example, the GOY and Sabra models:

$$\left(\frac{\partial}{\partial t} + \nu k_n^2\right) u_n = i(a_n u_{n+1} u_{n+2} + b_n u_{n-1} u_{n+1} + c_n u_{n-1} u_{n-2})^* + f_n, \tag{32.0.3}$$

where u represents the complex modes, n denotes the "shell index" that runs from 1 to ∞, and $*$ stands for complex conjugation.

Most systems in natural sciences and engineering are affected by uncertainty. From modeling point of view, to take into account the uncertainty, random effects have to be included. These random effects are considered not only as compensations for the defects in some deterministic models, but rather essential phenomena. Stochastic lattice systems come into play to describe systems with discrete spatial structure and random spatio-temporal forcing, i.e., noise. More specially, the term $f(u_n, v_n)$ ca be described as follows:

$$f(u_n, v_n) = \sigma(u_n, u_n) dB_n^H(t) \tag{32.0.4}$$

where $B_n^H(t)$ is Brownian (or fractional) Brownian motion of Hurst parameter H.

Summary of activities held at the BIRS station

Our research team was scheduled to meet on September 11-18, 2016 but that date got re-scheduled because of time conflicts with some members of the team. The meeting was held a year later: September 10-17, 2017. Hence, part of the research proposed initially was already finished before the meeting, see [1].

The team had lively discussion during their stay at the BIRS station and brainstorming ended up very positive and we have already a few new projects lined up. We will inform BIRS when manuscripts are ready and will acknowledge BIRS support. In particular, we have 3 defined projects outlined below:

Problem I. Synchronization of lattice equations

The synchronization of systems in physics and other sciences is an important issue. For an overview we refer to A. Pikovsky et al. First time that synchronization was considered from the point of view of physics goes back to C. Huygens. He observed in 1665 that two pendulum clocks which hang at the same wall or another structure oscillate synchronized after some time. Similar phenomena can be observed in other systems. To forecast synchronization in physical systems it is necessary to formulate mathematical models and to find mathematical tools which allow to interpret these phenomena in the sense of dynamical systems. An appropriate model for our physical system should contain random parameter. Hence the theory of random dynamical systems seems to provide the right tools to deal with the systems mentioned above.

The subgroup of the Focused Research Team by Bessaih, Garrido-Atienza and Schmalfuß dealt to formulate a research plan for the following system. We consider the following lattice system

$$du_i(t) = (\nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + f_i(u_i))\, dt + \sigma_i h_i(u_i) d\omega_i(t),\ i \in \mathbb{Z}. \tag{32.0.5}$$

We consider any of these equation as a nonlinear model of a physical grid. Any grid is coupled with the neighbor elements of the grid by a linear operator. The subsystems of the grid is perturbed by a noise $d\omega_i$ (either Brownian or fractional Brownian motion).

Consider now two or a finite number of parallel grid. The mathematical model is a system of equations like (32.0.5) where maybe the nonlinear parts can vary form equation to equation. These models also contain an operator which is responsible for the coupling of the systems of the different grids. In the case of two systems we consider

$$\begin{aligned} du =& (\nu A u - \Lambda u - \kappa K_1(u - v) + F(u))dt + d\omega \\ dv =& (\nu A v - \Lambda v - \kappa K_2(v - u) + G(v))dt + d\hat{\omega} \end{aligned} \tag{32.0.6}$$

which is now a system of stochastic evolution equations in the space of square integrable functions such that $u,\, v \in l_2$. Here A denotes a (generalized) linear operator responsible for the interaction of neighbored elements in each of the grids. The coupling between different grids is given by the operators K_i.

The first step we had to overcome is to ensure that the system of equations generates a random dynamical system ϕ^κ which depends on the intensity κ of the coupling.

One of our goals is to proof the following theorem:

Theorem 32.0.1 ***Assume appropriate assumptions such that*** (32.0.6) ***generates a random dynamical system on*** $l_2 \times l_2$***. Assume that*** K_j ***is appropriate.***

- ***Then for*** $\kappa > \kappa_0$ ***the random dynamical system has a random attractor*** $\mathcal{A}^\kappa$***.***
- ***We have***

$$\lim_{\kappa\to\infty} d_{l_2\times l_2}(\mathcal{A}^\kappa, \mathcal{A}^\infty) = 0$$

where $\mathcal{A}^\infty = \mathcal{A}^0 \times \mathcal{A}^0$ ***to be the attractor of the (random) dynamical system generated by***

$$du = (\nu Au - \Lambda u + \frac{1}{2}(F(u) + G(v))dt + \frac{1}{2}d(\omega - \hat{\omega}).$$

- ***For*** $\kappa > \kappa_0$ ***and the additional assumption*** $F = G$ ***we have synchronization on the level of trajectories, namely***

$$\lim_{t\to\infty} \|u(t) - v(t)\|_{l_2} = 0.$$

To interpret this theorem we note at first that a (random) attractor is a small often finite dimensional set in a infinite dimensional phase space which is in our case $l_2 \times l_2$. This set contains essential information about the long term behavior of a system. The second statement of the above theorem allows us to conclude that for large intensity parameter the motion of the coupled system takes place in a neighborhood of the product of the attractor of an averaged system. The third statement teaches us that the long time states of the subsystems u, v are synchronized. We are convinced that there are many applications for our results.

Problem II. Dynamics of lattices generated by partial differential equations with nonlocal terms

T. Caraballo proposed and discussed with the organizers some details about the idea of studying the asymptotic behaviour of the lattice dynamical system generated by the discretization of this problem:

$$\begin{cases} \dfrac{du}{dt} - a(l(u))\Delta u = f \quad t > 0, x \in \mathbb{R} \\ u(0) = u_0 \in L^2(\mathbb{R}). \end{cases} \tag{32.0.7}$$

We consider $l(u) = \int_{\mathbb{R}} u(x)dx$, the function a is continuous and bounded by two positive constants:

$$0 < m \le a(s) \le M \quad \forall s \in \mathbb{R}.$$

A tentative discretized problem could be

$$\begin{cases} \dfrac{du_i}{dt} = a(\tilde{l}(\overrightarrow{u}))(u_{i-1} - 2u_i + u_{i+1}) + f \quad t > 0, x \in \mathbb{R} \\ u_i(0) = (u_0)_i, \end{cases} \tag{32.0.8}$$

where, for instance, we could use the trapezium formula, $l(u) \approx \tilde{l}(\overrightarrow{u})$ to discretize the nonlocal term, although other alternatives could be considered as well.

The main objectives could be:

(i) To prove that (32.0.8) can be rewritten like a system of ordinary differential equations (lattice)

$$\begin{cases} \dfrac{d\overrightarrow{u}}{dt} = a(\tilde{l}(\overrightarrow{u}))A\overrightarrow{u} + f = F(\overrightarrow{u}) \\ u(0) = u_0 \in l^2, \end{cases} \tag{32.0.9}$$

where $\overrightarrow{u} = (u_i)_{i\in\mathbb{Z}}$.

(ii) To prove existence and (eventually) uniqueness of solution in l^2 for the problem (32.0.9). As far as we know, this is a new problem still not analyzed in the literature.

(iii) To study the existence of attractors.

(iv) To analyze more general frameworks (considering more general terms like $f(u)$, $N > 1$).

During the discussions, it was emphasized that, before carrying out this program, it would be necessary a deeper study about the PDE model with nonlocal term and all of its dynamical aspects. In fact, there exists an extensive literature already published on this model with interesting applications to some problems of the applied sciences. This means that this is a long term program which cannot be developed in a short stay, but that it is worth being analyzed in future.

Problem III. Neural oscillator networks models

Han proposed a stochastic lattice dynamical system arising from neural oscillator networks. The system reads:

$$\dot{\theta}_i = f_i + z(\theta_i)(\sum_{j\neq i} a_{ji} g(\theta_j) + \varepsilon_i I(t)). \tag{32.0.10}$$

Here the unknown variables θ_i are the states of the phase oscillators, i.e., angles parametrized by numbers in the interval $[0, 1]$ with periodic boundary conditions. The coupling matrix a_{ji} describes the network structure, and the function $z(\theta)$ measures how big an effect an input has on a neuron in state θ. $I(t)$ in an external input and ε_i is the intensity of external forcing current at the ith neuron.

To align with the topic of the focused group research, Han proposed to study the dynamics of the above lattice system where $I(t)$ is a random input, such as, Brownian motion or fractional Brownian motion. The group had a lively discussion on what the exact meaning of the model was, and what useful information could be obtained by studying such a system. In the end the group concluded that deeper understanding is needed to further consider this system.

During the rest of their stay, the subgroup formed by T. Caraballo and X. Han started an extensive literature review on the system (32.0.10), and gained an overview of system (32.0.10) as well as lattice models for neural networks in general. They initiated a detailed plan to study the stability and attractivity of system (32.0.10) and its generalizations. The plan involves T. Caraballo, X. Han and one of X. Han's current Ph.D. student. Two weeks after the closure of this focused research group, T. Caraballo visited X. Han at Auburn University, when the three of them were able to carry out the plan and obtain necessary analysis. Primitive calculations indicate that system (32.0.10) is stable and attractive in a proper sequence space under certain assumptions. More solid results are expected next year.

Participants

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Research in Teams Reports

Chapter 33

Derivative Free and Black Box Optimization (17rit681)

June 4 - 11, 2017

Organizer(s): Warren Hare (University of British Columbia), Charles Audet (Polytechnique Montréal)

Overview of the Field

Derivative-free optimization (DFO) is the mathematical study of the optimization algorithms that do not use derivatives. While a DFO algorithm was used to test one of the worlds first computers (the MANIAC in 1952), it was not until the 1990s that DFO algorithms were studied mathematically.

Blackbox optimization (BBO) is the study of optimization problems where the objective function is a blackbox. That is, no analytic description of the function is available, but given an arbitrary input the blackbox returns a function value. As BBO naturally arises whenever a computer simulation is involved in an optimization problem, BBO is one of the most rapidly expanding areas of applied optimization. BBO can naturally be approached by DFO.

DFO algorithms have principally fallen into one of two categories: direct search methods and model-based methods. Direct search methods work from an incumbent solution and examine a collection of trial points; if improvement is found, then the incumbent solution is updated, otherwise a search radius parameter is decreased and a new collection of trial points is examined. Model-based methods approximate the objective function with a model function, and use the gradients or even second derivatives of the model function to help guide optimization. (Note that while DFO studies algorithms that do not use derivatives, this does not mean that the objective function is nondifferentiable – for example the objective could be a computer simulation using numerical integration.) It was not until very recently that researchers began mixing direct search and model-based methods to create hybrid methods with improved performance.

Workshop Principle Objective

Derivative-free and blackbox optimization have made massive advances over the past decade and, in our opinion, represent one of the most rapidly expanding fields of nonlinear optimization research. We also feel that DFO and BBO represent one of the most important areas in nonlinear optimization for solving future applications in real-world problems. The foundational concepts in DFO and BBO have become sufficiently mature that it is now possible to teach them at a senior undergraduate level. The principle objective of this workshop was to provide Drs. Audet and Hare a week of focused activity to finalize a book titled "Derivative-free and blackbox optimization".

This book is designed to provide a clear grasp of the foundational concepts in DFO and BBO, in order to push these areas into the mainstream of nonlinear optimization.

The book is targeted at two broad audiences, and hope that both will find value. The first audience is individuals interested in entering (or better understanding) the fascinating world of DFO and BBO. The second audience is practitioners who have real-world problems to solve that cannot be approached by traditional gradient based methods.

The book does not present the absolute state-of-the-art in modern algorithms and theory. Instead it focuses on the foundational material required to understand and appreciate the state-of-the-art in DFO and BBO. To this end, in addition of studying several optimization methods, this book includes an elementary introduction to parts of nonsmooth analysis that have proved useful in conceiving and analyzing the methods within. The book also presents rigorous convergence theory for the algorithms in a way suitable for students in the mathematical sciences or in engineering.

In the past, practitioners faced with BBO problem have often fallen back on ad hoc methods, resulting in a plethora of papers publishing incremental improvements to solution quality. The methods covered in the book have proven convergence results, mathematically supported stopping criterion, and a track-record of practical success. Yet, for all of this, the methods are nonetheless easy to use and elegant in their simplicity. We hope the book can provide better and more consistent starting point for future applications.

Workshop Side Objectives

Development of the book mentioned in the principle objective has prompted several new research directions for Drs. Audet and Hare. The side objective of this workshop was to continue discussion and collaboration of these questions.

Outcome of the Meeting

A final version of "Derivative-free and blackbox optimization" was submitted to Springer on June 9th, 2017 (the last day of the workshop). The preface of the final version includes the much deserved line

> Special thanks to the Banff International Research Station (BIRS) for hosting us during the final stages of completing this book.

It is therefore clear that the principle objective of the workshop was accomplished.

The side objective of the workshop was addressed through several discussions about current projects and students. Notable progress was made in how to approximate a subdifferential using numerical directional derivatives and analysis of the quality of regression gradient approximation. Both problems are works in progress, but likely to result in papers in the near future.

Participants

Audet, Charles (Professor)
Hare, Warren (University of British Columbia)

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Chapter 34

The Structure of Finite Algebras and the Constraint Satisfaction Problem (17rit685)

August 6 - 13, 2017

Organizer(s): Andrei Bulatov (Simon Fraser University), Marcin Kozik (Jagiellonian University)

The Constraint Satisfaction Problem (CSP) appears in computer science in many forms. In the most general version a CSP instance consists of variables and constraints, and the goal is to evaluate the variables so that that all the constraints are satisfied. In a version of CSP particularly amenable to theoretical approach each constraint takes a form of a pair: a tuple of variables and a list of allowed evaluations. This restriction immediately makes the problem complete for NP. However, by further restricting the constraints allowed to appear in instances, we arrive at a very general framework capturing many natural problems in computer science.

The list of allowed evaluations in a constraint can be viewed as a relation of appropriate arity. In the parametrized (a.k.a non-uniform) version of CSP the set of relations allowed to appear in a constraint is restricted and called the language of the CSP. The famous dichotomy conjecture of Feder and Vardi [3] postulates that every finite language defines a CSP which is NPC or solvable in a polynomial time. The classical theorem of Schaefer [6], classifying the CSPs on two-element sets, is one of the most prominent results supporting the conjecture.

For a few years now the strongest partial results supporting the dichotomy conjecture use the algebraic approach pioneered in [5, 4]. It relies on a Galois correspondence [1] pairing a CSP language with an algebra. The algebra captures the computational complexity of the associated CSP up to a LOGSPACE reduction, but the connection goes deeper. The structure of the algebra is intrinsically connected with the types of obstacles the algorithm needs to overcome in order to verify an existence of a solution.

Many structural properties of algebras are captured by identities i.e. universally quantified equalities of operations. For example the identity $f(x,y) \approx f(y,x)$ states that an operation f is binary and commutative. Among the first, and most important, results of the algebraic approach is the fact that an algebra satisfies identities which cannot be satisfied by projections (such algebras are called Taylor algebras) or the associated CSP is NPC.

This year two independent papers introduced two algorithms solving CSPs for all the Taylor algebras and thus confirming the dichotomy conjecture. In each case the algorithm requires strong structural properties of the algebra to function correctly. In the paper of Andrei Bulatov [2] the structure is described by a colored graph defined on the elements of an algebra; in the other publication Dmitriy Zhuk [8] relies on the more "global" properties of being an absorbing subuniverse or a center. The goal of the workshop was to study properties of Taylor algebras using the tools developed for these proofs.

Specifying the problem

In both proofs one of the very first steps of the analysis is to refine the Taylor algebra. This reduces the number of operations, and at the same time (via the Galois correspondence) allows new relations into the constraint language. As long as the reduction produces a Taylor algebra the transformation has no impact on the correctness of the proof.

The first obstacle to comparing the approaches is the fact that each paper performs a different reduction. More precisely, starting with the same language, each proof allows a different set of extra relations, which makes the approaches incompatible without impacting their correctness. In order to overcome this obstacle the work during the meeting was focused on so called Taylor minimal algebras.

Taylor minimal algebras are the algebras corresponding to maximal tractable languages i.e. languages such that each new relation is either trivially definable (primitively positively definable) in the languages or makes in NPC. The fact that every Taylor algebra can be reduced to a minimal Taylor algebra requires some proof [7], but is true.

More importantly such an algebra cannot be further reduced without loosing the property of being a Taylor algebra and thus no reductions performed by Bulatov or Zhuk can modify it. This allows to perform a direct comparison of the structural results used in both proofs (albeit in the restricted case of Taylor minimal algebras).

The goal of the meeting can be concisely described as understanding the structure of Taylor minimal algebras by means of the proofs of the CSP dichotomy conjecture.

Results obtained during the meeting

Currently, there are three approaches to analyzing the structure of finite algebras that are relevant to the complexity of the CSP. Two of them are used in the dichotomy proofs mentioned above: Bulatov's local underlaying structure of the algebra captured by a directed graph (on the elements of the algebra) where each edge is a semilattice, affine or majority edge, and Zhuk's more global structure based on so-called centers and affine fragments of algebras. The third approach has been developed by Barto and Kozik and relies on the concept of absorbing subuniverse i.e. a set of elements of the algebra closed with respect to all the operations of the algebra and such that for some operation f the value $f(a_1, \ldots, a_n)$ is in the set as long as at most one a_i is outside.

During the meeting we were able to connect (to some extent) the three approaches. Most the results fall in the scope of at least one of the following broad groups:

- Refining, expanding, and reproving, using different techniques, some of the existing structural results. E.g. proving through colored graphs results that previously required using absorbing subuniverses, and the other way round.
- Comparing the constructions from the three approaches in Taylor minimal algebras. In particular, it turns out that these constructions are equivalent for the absorbing subuniverses where f can be taken binary or ternary and are not equivalent in general.
- Obtaining characterizations of algebras whose directed graph has only restricted set of colors. These characterizations can be obtained in different languages including Maltsev conditions (through identities) and through the structure of absorption.

Further research directions

The meeting initiated a line of research devoted to developing a systematic understanding of the structure of Taylor minimal algebras. The tools used to prove the CSP dichotomy conjecture are applicable and provide a guideline. The long–reaching goal is to construct a semblance of a structural theory for Taylor minimal algebras with a hope of eventually lifting it to all finite Taylor algebras.

Participants

Barto, Libor (Charles University)
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Chapter 35

Nikol'skii inequalities and their applications (17rit679)

December 3 - 10, 2017

Organizer(s): Feng Dai (University of Alberta), Andriy Prymak (University of Manitoba), Vladimir Temlyakov (University of South Carolina), Sergey Tikhonov (ICREA, CRM, UAB)

During our stay in Banff, we have studied the following questions:

- Remez and Nikol'skii's inequalities for trigonometric polynomials and spherical harmonics.
- Marcinkiewicz type inequalities for finite dimensional vector spaces of continuous functions.

Before discussing the obtained results, let us give a brief overview of the research field.

Overview of the Field

In the recent paper [4], a close connection between Remez and Nikol'skii's inequalities was shown. Here we restrict ourselves only to the questions related to Remez's inequalities.

Let X_N denote a linear subspace of functions in $L^p(\text{Ø}g)$ with $0 < p \le \infty$ and $\emptyset g$ being a probability space. The general form of the Remez inequality for a function $f \in X_N \subset L_p(\Omega)$, $0 < p \le \infty$, reads as follows: for any measurable $B \subset \Omega$ with measure $|B| \le b < 1$,

$$\|f\|_{L_p(\Omega)} \le C(N, |B|, p)\|f\|_{L_p(\Omega\setminus B)}.$$

Applications of Remez type inequalities include many different results in approximation theory and harmonic analysis; see [4] for more details and references.

For trigonometric polynomials $\mathcal{T}(Q)$ with frequencies from $Q \subset \mathbb{Z}^d$ (here $X_N = \mathcal{T}(Q)$ and $\Omega = \mathbb{T}^d$) the following result is well known [4]: For $d \ge 1$ and

$$Q = \Pi(\mathbf{N}) := \{\mathbf{k} \in \mathbb{Z}^d : |k_j| \le N_j, \quad j = 1, \dots, d\},$$

where $N_j \in \mathbb{N}^d$, for any $p \in (0, \infty]$, we have that $C(N, |B|, p) = C(d, p)$ provided that

$$|B| \le \frac{C}{\prod_{j=1}^d N_j}.$$

The investigation of the Remez-type inequalities for the hyperbolic cross trigonometric polynomials with

$$Q = \left\{ \mathbf{k} \in \mathbb{Z}^d : \quad \prod_{j=1}^{d} \max\{|k_j|, 1\} \le N \right\} \tag{35.0.1}$$

has been recently initiated in [4]. It turns out that for such polynomials the problem of establishing the optimal Remez inequalities has different solutions when $p < \infty$ and $p = \infty$. If $p < \infty$, then

$$C(N, |B|, p) = C(d, p)$$

provided that

$$|B| \le \frac{C}{N}.$$

The known result in the case $p = \infty$ reads as follows.

Theorem 35.0.1 ***[4] There exist two positive constants $C_1(d)$ and $C_2(d)$ such that for any set $B \subset \mathbb{T}^d$ of normalized measure***

$$|B| \le \frac{C_2(d)}{N(\log N)^{d-1}}$$

and for any $f \in \mathcal{T}(Q)$, where Q is given by (35.0.1), we have

$$\|f\|_\infty \le C_1(d)(\log N)^{d-1} \sup_{\mathbf{u} \in \mathbb{T}^d \setminus B} |f(\mathbf{u})|. \tag{35.0.2}$$

It is worth mentioning that this result is sharp with respect to the logarithmic factor. This is because the following statement is false.

There exist $\delta > 0$, A, c, and C such that for any $f \in \mathcal{T}(N)$ and any set $B \subset \mathbb{T}^d$ of measure $|B| \le (cN(\log N)^A)^{-1}$ the Remez-type inequality holds

$$\|f\|_\infty \le C(\log N)^{(d-1)(1-\delta)} \sup_{\mathbf{u} \in \mathbb{T}^d \setminus B} |f(\mathbf{u})|.$$

Scientific Progress Made

We obtain a nontrivial Remez inequality for the hyperbolic cross trigonometric polynomials with no logarithmic factor in (35.0.2).

Theorem 35.0.2 ***For $d \ge 2$, let***

$$\alpha_d = \sum_{j=1}^{d} \frac{1}{j} \qquad \textit{and} \qquad \beta_d = d - \alpha(d).$$

There exist two positive constants $C_1(d)$ and $C_2(d)$ such that for any set $B \subset \mathbb{T}^d$ of normalized measure

$$|B| \le \frac{C_2(d)}{N^{\alpha(d)}(\log N)^{\beta(d)}}$$

and for any $f \in \mathcal{T}(Q)$, where Q is given by (35.0.1), we have

$$\|f\|_\infty \le C_1(d) \sup_{\mathbf{u} \in \mathbb{T}^d \setminus B} |f(\mathbf{u})|.$$

The proof is based on the the discretization inequality and the following Marcinkiewicz-type inequality for the hyperbolic cross polynomials. The latter result is interesting by itself and it reads as follows.

Theorem 35.0.3 ***For*** $d \geq 1$***, let***

$$\alpha(d) = \sum_{j=1}^{d} \frac{1}{j} \qquad \textit{and} \qquad \beta_d = d - \alpha(d).$$

There exists a set Λ ***of at most*** $C_d N^{\alpha(d)} (\log N)^{\beta(d)}$ ***points in*** $[0, 2\pi)^d$ ***such that for all*** $f \in \mathcal{T}(N)$***,***

$$\|f\|_\infty \sim \max_{\omega \in \Lambda} |f(\omega)|.$$

Note that Theorem 35.0.3 for $d = 1$ is well known. More about Marcinkiewicz-type discretization can be found in [1]-[3].

Outcome of the Meeting

Currently we (F. Dai, A. Prymak, S. Tikhonov, and V.N. Temlyakov) are working on the Marcinkiewicz-type discretization problems. We have made significant progress on this problem during our discussions in Banff. We are currently preparing for a joint paper, which will include several interesting results in this direction. We believe that the meeting served as an excellent start for the new collaboration on various problems in approximation theory related to polynomial inequalities and discretizations.

Participants

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