# Banff International Research Station Proceedings 2021

BIRS

# Contents

# Five-day Workshop Reports

# Chapter 1

# Perspectives on Knot Homology (21w5105)

May 16 - 21, 2021

**Organizer(s):** Mina Aganagic (UC Berkeley), Slava Krushkal (University of Virginia), Ben Webster (University of Waterloo)

## Overview of the Field and Recent Developments

Quantum polynomial invariants of links and tangles, such as the Jones polynomial and the HOMFLY-PT polynomial, have played a central role in many areas of mathematics and physics over the last 3 decades. They have driven progress in low dimensional topology, representation theory, enumerative geometry, gauge theories, and string theory.

Knot homology is a far reaching generalization of polynomial invariants which is still being developed. Homological knot invariants are referred to as categorification of the polynomial invariants – polynomial invariants arise as the (graded) Euler characteristic of a homology theory. There are deeper structures which become manifest at the categorified level – just like in transition from the Euler characteristic to homology in basic algebraic topology. In particular, one can speak of maps between vector spaces, but not between numbers or polynomials. For the usual homology of spaces, this manifests as functoriality under continuous maps; the appropriate notion of a map between two knots is a surface cobordism in 4-space. In several cases, these give rise to maps on homology of the boundary links - a feature not available at the level of the Euler characteristic.

Many examples of homological invariants of knots and links were constructed over the last two decades, using a wide variety of methods: diagrammatic and geometric representation theory, gauge theory, and symplectic geometry. This reveals the richness of the subject, but is also a drawback. While there is a uniform mathematical construction of quantum link and three-manifold invariants, we are yet to discover a uniform approach to their homological generalization. String theory and quantum field theories have led to influential predictions about the structure of the homological invariants, and provide general outlines of how they should arise from physics. Physics of mathematical approaches to categorification, and mathematics of physical ones, are still being worked out.

Based on many recent developments, there is an expectation that the subject is reaching a turning point, not unlike what happened in the late 80s for polynomial knot invariants, where a unified view of the field finally emerged, through the works of Witten, Reshetikhin and Turaev, Drinfeld and others. The aim of the workshop was to bring together mathematicians and physicists working on the many different approaches to the subject, to capitalize on these developments.

# Outcome of the Meeting

The overall goal of the proposed workshop was to bring together researchers working on different aspects of knot homology in order to unify perspectives on the subject, explore the emerging connections between the mathematics and physics theories, and develop topological applications. Ideas from string theory, mentioned above, provide intriguing and far-reaching insight into the structure of knot homology. Yet most of these approaches are conjectural and have not been established rigorously, while others have been checked only for a restricted classes of links. One of the goals of the meeting therefore was to provide a forum for exchange of ideas by mathematicians and physicists working on knot homology.

There was a substantial interest in the workshop in the research community, with 85 registered participants. The list of speakers and participants features a representation of participants at all career stages; in particular a substantial number are recent PhDs.

The meeting took place online, but still it featured a very robust discussion. Frequently the breaks between the talks were completely used up by questions and exchanges between the speakers and other participants, both on Zoom and in Gather.town.

The workshop substantially contributed to the dissemination of most recent results in the subject, and to development of new collaborations between researchers working on different aspects of knot homology.

One participant told us that the conference had inspired him to start planning a conference on the Hilbert schemes of curves and their connections to knot homology.

# Progress in Research

Knot homology is a rapidly developing subject with many oustanding open problems. Below, we list progress on some problems presented at the conference:

- From topological perspective, an outstanding problem is the existence of an extension of knot homology to other 3-manifolds; if so its functoriality is expected to provide interesting invariants of 4-manifolds. Some of the Physics approaches suggest that an extension is possible; can this be formulated in mathematically rigorous terms for links in general 3-manifolds? Wedrich's talk during the conference addressed this question directly, presenting new invariants of 4-manifolds arising from the functoriality in Khovanov-Rozansky homology. Manolescu presented results on another application to 4-manifold topology: detection of exotic smooth structures on 4-manifolds using Rasmussen's invariant and its generalizations.

- The approach of Aganagic is related in physics terms to Wittens construction, and it is expected to be equivalent to Websters categorification of the Reshetikhin-Turaev invariants of links in 3-space. This question was at least provisionally answered during the workshop by the talks by Webster and Aganagić, which presented work in progress showing these invariants are the same.

- There is a long-standing conjecture that HOMFLY homology satisfies a version of Poincaré duality. Gorsky (joint with Hogancamp and Mellit) presented new work proving this result, using a similar approach the Hard Lefschetz theorem: constructing an $\mathfrak{sl}_2$ action, whose Weyl group gives the desired duality.

There were also new questions that arose from discussion at the conference. We list some interesting examples below:

- Willis' talk presented a construction of lifts of the action of $E$ and $F$ in $\mathfrak{sl}_2$ to the level of the annular Khovanov homotopy types. During the questions, Gorsky and Webster suggested these might be related to the correspondences between Grassmannians that lead to the action of $\mathfrak{sl}_2$ on the cohomology of Grassmannians. This is an intriguing direction for future research on annular Khovanov and Khovanov-Rozansky homotopy types.

# Participants

**Adnane, Mostafa** (UC Berkeley)
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**Agol, Ian** (UC Berkeley)
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**Anno, Rina** (Kansas State University)
**Baldridge, Scott** (Louisiana State University)
**Beliakova, Anna** (University of Zurich)
**Boden, Hans** (McMaster University)
**Borodzik, Maciej** (University of Warsaw)
**Brown, Jennifer** (UC Davis)
**Chae, John** (UC Davis)
**Cooper, Benjamin** (University of Iowa)
**Devaux, Steven** (Université de Montpellier)
**Dimofte, Tudor** (University of Edinburgh)
**Disney-Hogg, Alec Linden** (The University of Edinburgh)
**Dye, Heather** (McKendree University)
**Ekholm, Tobias** (Uppsala University)
**Elias, Ben** (University of Oregon)
**Fendley, Paul** (Oxford)
**Fenn, Roger** (University of Sussex)
**Garner, Niklas** (University of Washington)
**Glubokov, Andrey** (Purdue University)
**Gorsky, Eugene** (UC Davis)
**Grigsby, Elisenda** (Boston College)
**Gukov, Sergei** (California Institute of Technology)
**Haydys, Andriy** (University of Freiburg)
**He, Siqi** (Simons Center for geometry and physics)
**Hutchings, Michael** (University of California)
**Im, Mee Seong** (United States Naval Academy)
**Jankowska, Dorota** (University of Warsaw)
**Jiyuan, Han** (Purdue)
**Jordan, David** (University of Edinburgh)
**Kaiser, Uwe** (Boise State University)
**Kanstrup, Tina** (University of Massachusetts Amherst)
**Karimi, Homayun** (McMaster University)
**Kauffman, Louis** (University of Illinois, Chicago)
**Khovanov, Mikhail** (Columbia University)
**Kim, Ken** (LLNL University of California)
**Kivinen, Oscar** (University of Toronto)
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**Marino, Laura** (Universität Regensburg)
**Markiewicz, Maciej** (University of Warsaw)

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**McPhail-Snyder, Calvin** (UC Berkeley)
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**Niu, Wenjun** (UC Davis)
**Oblomkov, Alexei** (University of Massachusetts)
**Petkova, Ina** (Dartmouth College)
**Queffelec, Hoel** (CNRS)
**Rapcak, Miroslav** (UC Berkeley)
**Rasmussen, Jacob** (University of Cambridge)
**Rhea, Bakshi** (George Washington University)
**Robert, Louis-Hadrien** (Université du Luxembourg)
**Rozansky, Lev** (University of North Carolina)
**Rushworth, Will** (Syracuse University)
**Saxena, Vivek** (Rutgers University)
**Shende, Vivek** (Berkeley)
**Shirokova, Nadya** (Santa Clara University)
**Silvero, Marithania** (Universidad de Sevilla)
**Stošić, Marko** (Instituto Superior Técnico, Portugal)
**Sulkowski, Piotr** (University of Warsaw)
**Sun, Haoyu** (University of Texas, Austin)
**Tamas, Kalman** (Tokyo Institute of Technology)
**Tie, Jingzhi** (University of Georgia)
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**Tripp, Samuel** (Dartmouth College)
**Wagner, Emmanuel** (Institut de Mathématiques de Jussieu)
**Webster, Ben** (University of Waterloo)
**Wedrich, Paul** (Max Planck Institute for Mathematics and University of Bonn)
**Wildi, Arno** (University of Zurich)
**Willis, Michael** (Stanford University)
**Witten, Edward** (Institute of Advanced Study)
**Wong, C.-M. Michael** (Dartmouth College)
**Yozgyur, Ramazan** (University of Warsaw)
**Zhang, Melissa** (MSRI)
**Zibrowius, Claudius** (University of Regensburg)

# Chapter 2

# Quantum Foundations, Gravity, and Causal Order (21w5104)

May 30 - June 4, 2021

**Organizer(s):** Flaminia Giacomini (Perimeter Institute), Lucien Hardy (Perimeter Institute), Robert Mann (University of Waterloo)

## Overview of the Field

The goal of unifying the two pillars of physics, quantum theory and gravity, into a theory of quantum gravity, has eluded us for almost a century. On the one hand, this is partly due to the absence of experimental tests: the Planck length, the scale at which quantum gravity effects are expected, is too small to be tested by the current technology. On the other hand, whichever theory ultimately unifies general relativity with quantum mechanics will almost certainly introduce radically new concepts.

Recently, promising new directions are being explored at the interface of the field of quantum gravity and quantum information. These studies address the very conceptual foundations of quantum mechanics and general relativity, and introduce new tools towards the formulation of a theory of quantum gravity. For instance, attention has recently focused on the idea that, at the interface between gravity and quantum theory, the causal structure will be at the same time dynamical, as in general relativity, and subject to quantum fluctuations, as in quantum theory [1, 2, 3]. This led to the introduction of new mathematical formalisms, which can describe physics in settings with indefinite causal structures. Intuitively, this means that our conventional understanding of past, present, and future would not apply. For instance, there could be situations when an event A could happen both before and after a second event B. A possible concrete scenario in which this situation would manifest is when a gravitating mass is put in a Schrödingers cat state (i.e., when the mass is at the same time in two different locations).

A related idea receiving increasing attention is that of quantum reference frames [4, 5]. In all our physical theories, observations rely on the abstract notion of a reference frame, which refers to a coordinate system, i.e., to a grid built out of rods and clocks. However, when rods and clocks are considered as physical systems, exciting new phenomena arise due to the impact of both quantum theory and gravity (for example, clocks can display the time in a Schrödingers cat-like fashion). The impact of quantum theory on reference frames naturally leads to the notion of quantum reference frames, a field that is currently being extensively investigated using the tools of quantum information theory.

## Recent Developments and Open Problems

Recently, two experimental proposals [6, 7] to generate entanglement between two masses in a spatial superposition via Newtonian interaction has generated a lively debate on what could be inferred on the nature of the gravitational field if such entanglement is measured in the laboratory. Despite the fact that progress has been made, no consensus has been reached. Understanding this regime, which could be soon within the experimental reach, is crucial to make progress towards a unifying theory of quantum and gravity.

Recent results in the field of quantum coordinates [8] and quantum reference frames [9] have identified a possible path towards the understanding of physics at the interface of quantum theory and gravity by extending the Equivalence Principle to nonclassical spacetimes. However, still very little is known about how to incorporate gravitational effects with these new ideas and yet this is a matter of crucial importance. This is because the study of how physics is described from the point of view of such quantum reference frames, in particular at the interface of quantum theory and gravity, could shed light on how notions such as space, time, and causality should be modified in a quantum theory of gravity [10].

## Presentation Highlights

Perhaps the main highlight of the meeting was the blend of abstract theory with innovative experiment. From this perspective the meeting was very successful – experimentalists engaged with theorists in meaningful presentations and discussions on a variety of abstract topics that actually might be probed empirically in the foreseeable future.

Presentations grouped around themes. There were very interesting presentations and lively discussions about quantum reference frames, and their relationship to the causal structure of spacetime. These were presentations by Brukner, Castro-Ruiz, Smith, Chiribella, Allard-Guérin, Müller, Kissinger, and Wang on this subject, with many discussions generated afterward. Another theme involved a set of talks that took an operational perspective on the subject, asking how detectors would behave and respond in settings where causal order had a quantum mechanical character. Presentations by Fuentes, Henderson, Baccetti, Louko, Martin-Martinez, Ralph, and Weinfurtner engaged in this topic. A third theme of the meeting concentrated on how experiments are being developed (and in a few cases actually carried out) to probe these ideas. These included presentations by Morely, White, Walther, Bose, Sciarrino, Carney, and Wilson. Finally, there were presentations by Kempf, Surya, Dowker, Belenchia, Mazumdar, and Terno on how these ideas could affect our understanding of quantum gravity, and vice-versa.

## Scientific Progress Made

The workshop brought together researchers from the separate communities of quantum gravity, quantum information, and quantum foundations. These separate communities had never met to discuss their different approaches to the topics of the workshop. In the discussion sessions, the discussion was focused on how the new tools of indefinite causality and quantum reference frames can advance our understanding of physics at the interplay of quantum theory and gravity, both conceptually and mathematically. At the end of the workshop, shared knowledge had been created among the participants, who exchanged different ideas and perspectives.

The presence of experimental physicists has also been of key importance, as this allowed the most recent technological progress, especially in the field of quantum optics, quantum information and quantum communication, to be communicated to the community of theorists. This will likely contribute in the future to put forward new experimental proposals to test the interface of quantum mechanics and gravity.

## Outcome of the Meeting

The meeting took place in a virtual format. In this context, the workshop was extremely valuable in fostering interactions in a difficult period, due to the pandemic restrictions. A very good aspect of the online format was that it was more inclusive for young students, who would have not been able to participate otherwise. This significantly broadened participation in the meeting. The Gathertown facility got some use, but most participants tended to talk on Zoom during the break time instead of migrating to Gathertown.

Perhaps the most difficult aspect of the virtual format was that participants from Europe had little overlap with those in the Austral-Asian region. Participants in the former region could not easily participate in discussions and presentations with those in the latter region (and vice-versa) because the schedule meant that each would be awake in the middle of the night to do so. Yet despite these time zone difficulties, and zoom fatigue, all participants were extremely involved in the talks and discussion sessions as they were able.

## Participants

**Ali Ahmad, Shadi** (Dartmouth College)
**Allard Guerin, Philippe** (Perimeter Institute)
**Arabaci, Cemile** (University of Waterloo)
**Aspelmeyer, Markus** (University of Vienna and Austrian Academy of Sciences)
**Baccetti, Valentina** (RMIT)
**Belenchia, Alessio** (Universitaet Tuebingen)
**Blencowe, Miles** (Dartmouth College)
**Bose, Sougato** (University College London)
**Braun, Daniel** (U of Tuebingen Germany)
**Brukner, Caslav** (IQOQI Vienna Austria)
**Bunney, Cameron** (University of Nottingham)
**Butterfield, Jeremy** (University of Cambridge)
**Carney, Dan** (Berkeley National Lab)
**Castro Ruiz, Esteban** (ETH Zurich)
**Cepollaro, Carlo** (University of Milan)
**Chen, Lin-Qing** (University Libre de Bruxelles)
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**Christ, Margie** (University of Waterloo)
**Christodoulou, Marios** (University of Vienna)
**Costa, Fabio** (University of Queensland)
**Davis, Tamara** (University of Queensland)
**Dhumuntarao, Aditya** (University of Minnesota)
**Dowker, Fay** (Imperial College London)
**Foo, Joshua** (University of Queensland)
**Fuentes, Ivette** (University of Southampton)
**Galley, Thomas** (Perimeter Institute)
**Giacomini, Flaminia** (Perimeter Institute)
**Grimmer, Daniel** (University of Oxford)
**Guryanova, Yelena** (IQOQI Vienna)
**Hardy, Lucien** (Perimeter Institute Canada)
**Henderson, Laura** (University of Waterloo)
**Hoffreumon, Timothee** (Universite Libre de Bruxelles)
**Ipek, Selman** (University at Albany-SUNY)
**Jubb, Ian** (DIAS)
**Kakade, Kartik** (Indian Institute of Science Education and Research, Pune)
**Kempf, Achim** (University of Waterloo)
**Kissinger, Aleks** (University of Oxford)
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**Louko, Jorma** (University of Nottingham)
**MacLaurin, Colin** (University of Queensland)
**Mann, Robert** (University of Waterloo)

**March, James** (University of Warwick)
**Martin-Martinez, Eduardo** (University of Waterloo)
**Mazumdar, Anupam** (University of Groeningen)
**Morley, Gavin** (University of Warwick)
**Mueller, Holger** (UC Berkeley)
**Müller, Markus** (Austrian Academy of Sciences)
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**Papageorgiou, Maria** (University of Waterloo/University of Patras)
**Patterson, Everett Avison** (University of Waterloo)
**Ralph, Timothy** (University of Queensland)
**Rick Perche, Tales** (University of Waterloo)
**Ruep, Maximilian H.** (University of York)
**Russo, Andrea** (UCL)
**Sakharwade, Nitica** (Perimeter Institute)
**Sciarrino, Fabio** (Sapienza Universitá di Roma)
**Shalabi, Ahmed** (University of Waterloo)
**Shrapnel, Sally** (University of Queensland)
**Simao, Jose Diogo** (Friedrich-Schiller University Jena)
**Smith, Alexander** (Saint Anselm College)
**Suresh, Ardra** (Griffith University)
**Surya, Sumati** (Raman Research Institute)
**Terno, Daniel** (Macquarie University)
**Tiwari, Madhav** (Queen's University Belfast)
**Tjoa, Erickson** (University of Waterloo)
**Walther, Philip** (University of Vienna)
**Wang, Hui** (Dartmouth College)
**Weinfurtner, Silke** (The University of Nottingham)
**Wen, Robin** (University of Waterloo)
**White, Andrew** (University of Queensland)
**Wilson, Christopher** (University of Waterloo)
**Wood, Ben** (University of Warwick)
**Wood, Carolyn** (University of Queensland)
**Wurtz, Kelly** (Perimeter Institute)
**Zalel, Stav** (Imperial College London)
**Zych, Magdalena** (University of Queensland)

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[5] F. Giacomini, E. Castro-Ruiz, and Č. Brukner, “Quantum mechanics and the covariance of physical laws in quantum reference frames”, Nat. Commun. **10(1)** (2019): 1-13.

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[10] A. Belenchia, R. M. Wald, F. Giacomini, E. Castro-Ruiz, Č. Brukner, and M. Aspelmeyer, “Quantum superposition of massive objects and the quantization of gravity”, Phys. Rev. D **98(12)** (2018): 126009.

# Chapter 3

# Entropic Regularization of Optimal Transport and Applications (21w5120)

June 20 - 25, 2021

**Organizer(s):** Soumik Pal (University of Washington, Seattle), Aaron Palmer (University of British Columbia), Brendan Pass (University of Alberta)

## Overview of the Field

The mathematics of optimal transport (OT) has grown to become a unifying theme in many scientific disciplines, from solving purely mathematical problems of analysis, geometry, and partial differential equations to developing revolutionary new methods in economics, statistics, machine learning and artificial intelligence [4][3][2]. Much of these recent advances are due to remarkable leaps in computational methods in OT that hinge on entropy based regularizations [1]. Not only has this transformed the scope of applications, but the theory of entropy-regularized optimal transport has made important contributions in theoretical probability, statistical physics, and other areas of pure mathematics and engineering. With this breadth of theory and applications comes the challenge of organizing the different groups of researchers to allow an exchange of their most recent problems and results. Our BIRS meeting was aimed squarely at bringing together the disparate groups working on entropic regularizations of OT problems, irrespective of their home field.

## Recent Developments and Open Problems

There are three relevant asymptotics in this field: one is when the regularization parameter is vanishingly small [5][6], two is when the size of data is large [7], and three is the effect of large dimensions of the data [8]. Several recent developments have been focused on obtaining sharp rates of convergences in each of these three regimes. However, a complete understanding is still very far since many of these results have overly stringent assumptions that cannot be verified in practice. Moreover joint rates of convergence, although important in practice, have not been studied so far. Effects of the geometry of OT on statistical efficiency and machine learning algorithms is another burgeoning research area that was touched upon in this workshop.

# Presentation Highlights

We made a particular effort to invite junior researchers (below tenured professorship) with the hope that exposure to a variety of problems from different areas (all with optimal transport as a common feature) can help accelerate their research career. Among the thirty speakers throughout the week, 20 of them are junior researchers or students. We were heartened to see that some of the best presentations indeed came from the junior researchers. There were five women speakers and one with visual impairment.

# Outcome of the Meeting

Despite the remote format of our workshop, we believe that the interactions initiated by bringing together researchers from diverse backgrounds (including analysis, probability, geometry [9], statistics, machine learning [10] and computation) whose work intersects in entropic regularization of optimal transport, will have a significant impact on this new but rapidly developing field; in particular, we expect it to contribute to the resolution of some of the open problems highlighted above. As many of our participants are relatively early in their careers, we believe the meeting will play an important role in building their networks and stimulating their research.

# Participants

**Acciaio, Beatrice** (ETH Zürich)
**Backhoff, Julio** (Universität Wien – Mathematics)
**Beiglböck, Mathias** (University of Vienna)
**Borza, Samuël** (Durham University)
**Burr, Angel** (University of Washington)
**Chen, Yongxin** (Georgia Tech)
**Chen, Zhen-Qing** (University of Washington)
**Chen, Xiaohui** (U. Illinois at Urbana Champaign)
**Chen, Xiaohong** (Yale University)
**Conforti, Giovanni** (Ecole Polytechnique Paris – Mathematics)
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**Craig, Katy** (University of California, Santa Barbara)
**Elamvazhuthi, Karthik** (University of California, Los Angeles)
**Fan, Yanqin** (University of Washington)
**Fan, Jiaojiao** (Georgia Tech.)
**Fazel, Maryam** (University of Washington)
**Fox, Jeremy** (Rice University)
**Galeotti, Mattia** (Universitá di Bologna.)
**Galichon, Alfred** (New York University)
**Gao, Yuan** (Duke University)
**Ge, Yuxin** (Université Toulouse 3)
**Ghosal, Promit** (MIT)
**Ghoussoub, Nassif** (University of British Columbia)
**Greco, Giacomo** (TU Eindhoven)
**Han, Fang** (University of Washington, Seattle)
**Heitz, Matthieu** (UBC)
**Henry, Marc** (The Pennsylvania State University)
**Hiew, Joshua Zoen-Git** (University of Alberta)
**Hoffmann, Franca** (University of Bonn)
**Huang, Hui** (University of Calgary)
**Huesmann, Martin** (Universität Münster)
**Irfan, Talha** (UET Lahore)

**Jo, Min Jun** (UBC)
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**Kim, Young-Heon** (University of British Columbia)
**Kitagawa, Jun** (Michigan State University)
**Kobayashi, Forest** (University of British Columbia)
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**Leonard, Christian** (Université Paris Nanterre)
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**Liao, Wenyuan** (University of Calgary)
**Lim, Tongseok** (Purdue University)
**Liu, Lang** (University of Washington)
**Loewen, Philip** (University of British Columbia-Vancouver)
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**Maaz, Muhammad** (University of Toronto)
**Malagoda Gamage, Chamila** (Michigan State University)
**Mason, David M.** (University of Delaware)
**McCann, Robert** (University of Toronto)
**Murugan, Mathav** (University of British Columbia)
**Nam, Kyeongsik** (UCLA)
**Nenna, Luca** (Université Paris-Saclay)
**Nikolic, Dorde** (UCSB)
**Niles-Weed, Jonathan** (New York University)
**Nurbekyan, Levon** (University of California at Los Angeles)
**Nurbekyan, Levon** (UCLA)
**Obloj, Jan** (University of Oxford)
**Ohta, Shin-ichi** (Osaka University)
**Ottolini, Andrea** (Stanford University)
**Pal, Soumik** (University of Washington)
**Palmer, Aaron** (University of California, Los Angeles)
**Pass, Brendan** (University of Alberta)
**Peyré, Gabriel** (CNRS and École Normale Supérieure)
**Pooladian, Aram-Alexandre** (NYU)
**Rashid, Umar** (University of Engineering and Technology)
**Ruan, Yuanlong** (McGill University)
**Savare, Giuseppe** (Bocconi University)
**Scetbon, Meyer** (ENSAE)
**Schiebinger, Geoffrey** (University of British Columbia)
**Schroeder, Tobias** (Ruprecht Karl University of Heidelberg, Germany)
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**Simmons, David** (University of Washington)
**Somani, Raghav** (University of Washington, Seattle)
**Souza, Marcelo** (Purdue University)
**Tamanini, Luca** (Université Paris Dauphine)
**Tripathi, Raghav** (University of Washington, Seattle)
**Turnquist, Axel** (University of Texas at Austin)
**Vargas-Jimenez, Adolfo** (University of Alberta)
**von Renesse, Max** (Universität Leipzig)
**Warren, Micah** (University of Oregon)
**Wong, Ting-Kam Leonard** (University of Toronto)
**Xue, Wendao** (University of Washington)
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# Chapter 4

# Geometry via arithmetic (21w5051)

July 12 - 16, 2021

**Organizer(s):** Tim Browning (IST Austria), Brian Lehmann (Boston College), David McKinnon (University of Waterloo), Sho Tanimoto (Nagoya University)

## 1 Overview and Recent Developments

A central theme in mathematics is the analogy between number fields and function fields. There have been a succession of recent breakthroughs in this area, including:

*(i)* Work of Kebekus–Pereira–Smeets on the function field analogue of the "orbifold Mordell conjecture".

*(ii)* Work of Lehmann–Sengupta–Tanimoto on the role that geometric techniques can have in formulating a more refined version of the Manin conjecture by helping to identify the right exceptional sets.

*(iii)* Work of Bilu, Bourqui and Chambert-Loir–Loeser on the motivic version of Manin's Conjecture, which predicts convergence of the space of rational curves in the Grothendieck ring of varieties $K_0(\mathrm{Var})$.

*(iv)* Work of Browning–Sawin–Vishe on the circle method and its application to study rational curves on low degree hypersurfaces, together with the development of a "geometric circle method", which operates at the level of cohomology groups.

*(v)* Work of McKinnon–Roth on the notion of a Diophantine approximation constant for all polarized projective varieties, together with the discovery that the geometric invariants of the variety can be used to give bounds on the approximation constant.

Our workshop focused on the analogy between Diophantine geometry and the theory of curves on varieties. Diophantine geometers study the existence and distribution of rational points on algebraic varieties defined over number fields, but as witnessed in the workshop, we are now seeing how ideas and techniques from number theory can be applied to study moduli spaces of rational curves. A primary objective of the workshop was to bring together researchers belonging to the different schools, who would otherwise rarely interact.

## 2 Presentation Highlights

We deliberately chose a balance of speakers from the number theoretic and geometric camps, but we also paid close attention to maintaining a balance of genders and seniority. In view of the wide range of time zones that the participants were spread across, and the danger of Zoom fatigue, we kept the schedule fairly light with at most three 45 minute talks per day, allowing ample time for discussion on Gathertown. Some of the highlights were as follows:

- Jason Starr's talk *From geometry to arithmetic to geometry* discussed geometric proofs of various results for low degree Fano hypersurfaces, originally obtained by arithmetic geometers.
- In his talk *Rigidity of moduli spaces*, Benson Farb discussed various miracle constructions of moduli spaces in algebraic geometry and in particular he showed moduli spaces and maps between them are rigid in the sense of homotopy theory by exhibiting plenty of examples.
- In her talk *Zeta statistics*, Margaret Bilu described some new advances in the theory of measures on the Grothendieck ring. Her work clarifies the relationships between several prior results by various authors and will play a key role in future formulations of Manin's Conjecture.
- Isabel Vogt's talk *Arithmetic and geometry of Brill–Noether loci of curves* described various geometric and arithmetic results regarding Brill–Noether loci of smooth projective curves, and in particular she talked about how Brill–Noether loci behaves for curves which are not general in moduli.
- Will Sawin ended the conference with a beautiful talk *The Geometric Manin's Conjectures* which outlined a new set of questions about the moduli spaces of curves on algebraic varieties, leading to a lot of discussion after the talk.

## 3 Outcome of the Meeting

We were initially rather cautious by the idea of running an online workshop, but the BIRS staff have accrued lots of experience and were really helpful throughout the process. The combination of using Zoom for the lectures and Gathertown for the discussions worked rather well, with most of the talks generating good feedback and interactions. With the cancellation of many workshops and local seminar series, there are fewer opportunities for young mathematicians to explain their work and so we invited a good proportion of junior participants to speak. Whilst the online format lacked the informal exchanges that are integral to a face-to-face workshop, the access to Gathertown and reduced schedule still led to many meaningful discussions.

# Participants

**Alfaraj, Abdulmuhsin** (University of Warwick)
**Ascher, Kenneth** (Princeton University)
**Banerjee, Oishee** (Hausdorff Center for Mathematics)
**Beheshti, Roya** (Washington Univeristy (St. Louis - MO))
**Bilu, Margaret** (IST Austria)
**Brandes, Julia** (University of Gothenburg)
**Browning, Tim** (Institute of Science and Technology Austria)
**Capuano, Laura** (Università degli Studi Roma Tre)
**Castravet, Ana Maria** (University of Paris-Saclay, Versailles)
**Chen, Qile** (Boston College)
**Ciurca, Tudor** (Cambridge)
**DeVleming, Kristin** (University of California San Diego)
**Ellenberg, Jordan** (University of Wisconsin)
**Faisant, Loïs** (Grenoble)
**Farb, Benson** (University of Chicago)
**Fontanari, Claudio** (Università degli Studi di Trento)
**Glubokov, Andrey** (Purdue University)
**Gounelas, Frank** (Technische Universität München)
**Grieve, Nathan** (Royal Military College of Canada, Carleton University and L'Université du Québec à Montréal (UQAM))

**Huang, Zhizhong** (IST Austria)
**Javanpeykar, Ariyan** (Universitat Mainz)
**Kebekus, Stefan** (Albert-Ludwigs-Universität Freiburg)
**Lehmann, Brian** (Boston College)
**Loeser, Francois** (Sorbonne Université)
**Loughran, Daniel** (University of Bath)
**Manzateanu, Adelina** (University of Leiden)
**Matei, Vlad** (University of Tel Aviv)
**McKinnon, David** (University of Waterloo)
**Mello, Jorge** (York University)
**Nakahara, Masahiro** (University of Washington)
**Park, Jun-Yong** (IBS Center for Geometry and Physics)
**Peyre, Emmanuel** (University of Grenoble)
**Pieropan, Marta** (Utrecht University)
**Poonen, Bjorn** (Massachusetts Institute of Technology)
**Riedl, Eric** (Notre Dame)
**Roth, Mike** (Queen's University)
**Santens, Tim** (KU Leuven)
**Satriano, Matthew** (University of Waterloo)
**Sawin, Will** (Columbia University)
**Sengupta, Akash** (Columbia University)
**Shen, Mingmin** (University of Amsterdam)
**Smeets, Arne** (Radboud Universiteit Nijmegen)
**Starr, Jason** (SUNY Stony Brook)
**Streeter, Sam** (University of Bath)
**Tanimoto, Sho** (Nagoya University)
**Tian, Zhiyu** (Beijing University)
**Tosteson, Philip** (University of Chicago)
**Tschinkel, Yuri** (Courant Institute NYU and Simons Foundation)
**Turchet, Amos** (Roma Tre University)
**Uppal, Harkaran** (University of Bath)
**Várilly-Alvarado, Anthony** (Rice University)
**Viada, Evelina** (Georg-August-Universität)
**Viray, Bianca** (University of Washington)
**Vishe, Pankaj** (Durham University)
**Vogt, Isabel** (University of Washington)
**Wang, Julie Tzu-Yueh** (Academia Sinica)
**Yasuda, Takehiko** (Osaka University)
**Yasufuku, Yu** (Nihon University)

# Chapter 5

# Totally Disconnected Locally Compact Groups via Group Actions (21w5151)

August 15 - 20, 2021

**Organizer(s):** Robert Guralnick (University of Southern California), Katrin Tent (Universität Münster), Donna Testerman (École Polytechnique Fédérale de Lausanne), George Willis (University of Newcastle, NSW)

## Overview of the Field

Group actions are a mathematical approach to analysing structures by studying the rearrangements of the elements of the structure that preserve its defining properties. Actions by totally disconnected locally compact (t.d.l.c.) groups were the focus of this workshop.

T.d.l.c. group actions are infinite but have finite sub-degrees and so the theory involves ideas relevant to infinite permutation groups as well as ideas relevant to finite permutation groups. While concepts from logic and topology come to the fore when studying infinite permutation groups, finite permutation groups are understood through combinatorial methods, with a large role played by representation theory.

This workshop brought together researchers on both finite and infinite permutation groups to share techniques and recent advances. The various subjects are advancing on many fronts due to several recent developments, in model theory, in analysis and topology, in number theory and combinatorics, in geometry, and via the numerous applications of the classification of finite simple groups. The workshop allowed a diverse group of researchers to gather and discuss advances and future directions.

## Recent Developments and Open Problems

Developing a general structure theory of t.d.l.c. groups is a major challenge and significant progress was reported by Colin D. Reid and his collaborators. In his talk based on the article [2], Reid outlined a theory of well-foundedness for a large class of groups that includes the elementary groups defined by Phillip Wesolek in [4] as well as many compactly generated simple groups which are not elementary. As with the elementary groups, Reid's class supports an ordinal rank which might be used for induction arguments. In her lecture, Alejandra Garrido described joint work with Reid and David Robertson that gives a method for producing new examples of compactly generated simple groups. Simon M. Smith spoke about his work with Reid in which they completely classify all groups acting on trees that have the Tits independence property [3].

In another significant development, André Nies explained current joint work with Alexander Melnikov on the question of when can a t.d.l.c. group can be called computable. This work can be expected to inform computation in t.d.l.c. groups which, since they are topological, is only possible up to approximation. The approximations will be by finite groups because t.d.l.c. group actions have finite sub-degrees, and there will be a link to finite permutation groups through these approximations.

# Presentation Highlights

One highlight was the opening lecture by Alex Lubotzky, in which he explained ground-breaking ideas concerning almost solutions and near solutions to equations in permutation groups, and connections with computer science and sofic groups, in a clear and accessible manner. The lecture by Anne Thomas also a pleasure to listen to. She gave a clear and intuitive explanation of recent results about global fixed points for groups acting on buildings that complement previous work.

Another highlight was the series of three lectures given by the early career researchers Waltraud Lederle, Rachel Skipper and Tianyi Zheng. Each of them gave an excellent talk about their work on aspects of Neretin's group. Although this work had been done independently, synergies and connections between the ideas were noticed in the course of the talks.

In addition, Scott Harper gave an outstanding talk on the spread of a finite group in which he presented recent joint work with Tim Burness and Robert Guralnick that solved a longstanding problem, see [1].

# Scientific Progress Made

The following talks and interactions during the workshop contributed to progress with several projects.

- Progress was made on a joint project of Cheryl Praeger, H. Dugald Macpherson and Simon M. Smith in a Gathertown meeting held across different time zones.
- The interaction between Martin Liebeck and David Craven may lead to progress on an important case of maximal subgroups of the finite exceptional groups.
- David Craven's talk Maximal subgroups of finite simple groups directly impacted on R. Guralnick's work with Gareth Tracey.
- Melissa Lee's talk Bases of primitive permutation groups and applications inspired G. Willis and Michal Ferov to refine an approach to computation in t.d.l.c. groups they are pursuing. This refinement may facilitate more efficient computation and transfer of results about finite permutation groups.

# Outcome of the Meeting

The online format allowed many more participants, coming from diverse regions and time zones, than would normally be present at such a workshop. However, the different time zones made it more difficult for participants to interact. Indeed, the clocks of organisers in Newcastle, Europe and Banff were 8 hours apart from each other. The difficulty of listening to the lectures is remediated by the availability of recordings.

Almost all the talks were high quality and the recordings will be a valuable resource. Many young people, including but not limited to those mentioned above, were exposed to a large international audience and had opportunities to make new research connections.

# Participants

**Allingham, David** (University of Newcastle)
**Aschbacher, Michael** (California Institute of Technology)

**Bamberg, John** (University of Western Australia)
**Barnea, Yiftach** (Royal Holloway, University of London)
**Bartosova, Dana** (University of Florida)
**Baumeister, Barbara** (Universität Bielefeld)
**Bischof, Sebastian** (University of Giessen)
**Borovik, Alexandre** (University of Manchester)
**Bors, Alexander** (Carleton University)
**Burness, Tim** (University of Bristol)
**Capdeboscq, Inna** (University of Warwick)
**Caprace, Pierre-Emmanuel** (UCLouvain)
**Carter, Max** (The University of Newcastle)
**Castellano, Ilaria** (University of Milano-Bicocca)
**Chang, Mun See** (University of St. Andrews)
**Chatterjee, Pralay** (Institute of Mathematical Sciences - India)
**Craven, David** (University of Birmingham)
**Devillers, Alice** (University of Western Australia)
**Ershov, Mikhail** (University of Virginia)
**Ferov, Michal** (The University of Newcastle)
**Freedman, Saul** (University of St Andrews)
**Garrido, Alejandra** (Universidad Autónoma de Madrid and ICMAT)
**Garzoni, Daniele** (University of Tel Aviv)
**Ghadernezhad, Zaniar** (Imperial College London)
**Giudici, Michael** (The University of Western Australia)
**Glubokov, Andrey** (Purdue University)
**Goffer, Gil** (Weizmann Institute)
**Gorazd, Roman** (The University of Newcastle)
**Grazian, Valentina** (University of Milano - Bicocca)
**Gruber, Jonathan** (École Polytechnique Fédérale de Lausanne)
**Guralnick, Robert** (University of Southern California)
**Hähndel, Paula** (Martin Luther University Halle-Wittenberg)
**Harper, Scott** (University of Bristol)
**Hudson, Scott** (University of South Wales)
**Kaplan, Itay** (Hebrew University of Jerusalem)
**Kechris, Alexander** (California Institute of Technology)
**Korhonen, Mikko** (SUSTech)
**Lederle, Waltraud** (UCLouvain)
**Lee, Melissa** (University of Auckland)
**Lee, Tae Young** (Rutgers University)
**Li, Cai Heng** (Southern University of Science and Technology)
**Liebeck, Martin** (Imperial College)
**Litterick, Alastair** (University of Essex)
**Lubotzky, Alex** (Hebrew University of Jerusalem)
**Macpherson, Dugald** (University of Leeds)
**Malicki, Maciej** (Institute of Mathematics of the Polish Academy of Sciences)
**Malle, Gunter** (Technische Universität Kaiserslautern)
**Mandal, Arunava** (IISER Mohali)
**Marchionna, Bianca** (University of Milano-Bicocca)
**Maroti, Attila** (Alfréd Rényi Institute of Mathematics)
**Marquis, Timothee** (Université Catholique de Louvain)
**McCallum, Rupert** (University of Tübingen)
**Mckemmie, Eilidh** (Hebrew University of Jerusalem)
**Monod, Nicolas** (EPFL)

**Morgan, Luke** (University of Primorksa)
**Moscatiello, Mariapia** (University of Bologna)
**Muller, Isabel** (Imperial College London)
**Nagnibeda, Tatiana** (University of Geneva)
**Nekrashevych, Volodymyr** (Texas A and M University)
**Neretin, Yury** (Universität Wien)
**Nies, Andre** (The University of Auckland)
**Norris, Miriam** (Kings College London)
**Nuez González, Javier de la** (University of the Basque Country)
**Panagiotopoulos, Aristotelis** (Carnegie Mellon University)
**Parker, Chris** (University of Birmingham)
**Pengelly, Rachel** (University of Birmingham)
**Pinto E Silva, João Vitor** (The University of Newcastle)
**Praeger, Cheryl** (University of Western Australia)
**Pyber, Laci** (Rényi Institute of Mathematics Budapest)
**Raja, C.R.E.** (Indian Statistical Institute)
**Ramagge, Jacqui** (Durham University)
**Reid, Colin** (University of Newcastle)
**Rekvenyi, Kamilla** (Imperial College London)
**Rizzoli, Aluna** (University of Cambridge)
**Robertson, David** (University of New England)
**Roney Dougal, Colva** (University of St Andrews)
**Rosendal, Christian** (University of Maryland)
**Saunders, Jack** (University of Western Australia)
**Schillewaert, Jeroen** (University of Auckland)
**Segal, Dan** (Oxford University)
**Segev, Yoav** (Ben-Gurion University)
**Seitz, Gary** (University of Oregon)
**Semal, Lancelot** (UCLouvain)
**Sercombe, Damian** (Ruhr-Universität Bochum)
**Shah, Riddhi** (Jawaharlal Nehru University)
**Shalev, Aner** (Hebrew University of Jerusalem)
**Simion, Iulian** (Babes-Bolyai University)
**Skipper, Rachel** (Ohio State)
**Smith, Simon** (University of Lincoln)
**Song, Shujiao** (Yantai University)
**Stewart, David** (Newcastle University)
**Sunic, Zoran** (Hofstra University)
**Tent, Katrin** (University of Muenster)
**Testerman, Donna** (École Polytechnique Fédérale de Lausanne)
**Thillaisundaram, Anitha** (University of Lincoln)
**Thomas, Anne** (The University of Sydney)
**Thomas, Adam** (Warwick University)
**Tiep, Pham** (Rutgers University)
**Tornier, Stephan** (The University of Newcastle)
**Tracey, Gareth** (University of Oxford)
**Trost, Alexander** (Ruhr University Bochum)
**van Beek, Martin** (University of Birmingham)
**Vdovina, Alina** (Newcastle University)
**Verret, Gabriel** (University of Auckland)
**Wales, David** (California Institute of Technology)
**Weigel, Thomas** (University of Milano-Bicocca)

**Weiss, Richard** (Tufts University)
**Willis, George** (University of Newcastle, New South Wales)
**Wilson, John** (University of Cambridge)
**Z, Shaopeng** (University of Maryland)
**Zalesskii, Pavel** (University of Brasilia)
**Zheng, Tianyi** (UC San Diego)
**Zordan, Michele** (Imperial College London)

# Bibliography

[1] Timothy C. Burness, Robert M. Guralnick and Scott Harper, The spread of a finite group, Ann. Math. (2) **193** (2021), 619–687.

[2] Colin D. Reid, A class of well-founded totally disconnected locally compact groups, `arXiv:2107.05329v1`.

[3] Colin D. Reid and Simon M. Smith (with an appendix by Stephan Tornier), Groups acting on trees with Tits' independence property (P), `arXiv:2002.111766v1`.

[4] Phillip R. Wesolek, Elementary totally disconnected locally compact groups, Proc. Lond. Math. Soc. (3), **110** (2015), 1387–1434.

# Chapter 6

# Nonlinear Potential Theoretic Methods in Partial Differential Equations (21w5100)

September 06 - 10, 2021

**Organizer(s):** Andrea Cianchi (Universita di Firenze, Italy), Giuseppe Mingione (Universita di Parma, Italy), Igor Verbistky (University of Missouri, USA), Jérôme Vétois (McGill University, Canada)

## Overview of the Field

The existence and regularity theory of solutions to partial differential equations of elliptic and parabolic type is not only a central topic in modern analysis, but has also crucial connections with differential geometry and mathematical physics. Though this area has very classical roots, it has grown tremendously over the last few years, thanks to the work of a number of researchers in various prestigious academic institutions worldwide. The introduction of innovative ideas has provided a sharp nonlinear counterpart of several major aspects of the standard linear theory. The lack of tools that genuinely pertain to the linear theory has required the development of original methods and techniques in the proof of fine properties, estimates and existence results for solutions to nonlinear elliptic and parabolic PDE. They have produced remarkable progress in such fields as Nonlinear Potential Theory, Geometric-Harmonic Analysis, and related Functional Inequalities.

## Presentation Highlights

Various recent developments in the existence and regularity theory of solutions to partial differential equations of elliptic and parabolic type have been presented on the occasion of this online workshop. The following themes are included among those addressed in the talks delivered by the participants:

• The opening talk by Xavier Cabre was devoted to his recent solution (joint with Figalli, Ros-Oton and Serra) of several open problems concerning boundedness and regularity of stable solutions to semilinear elliptic equations for dimensions up to $n = 9$. Sharp Morrey type estimates were obtained for stable solutions for dimensions $n > 9$. Several open problems were discussed.

• The topic of energy-critical Schrödinger-type elliptic equations has been addressed in two presentations by Bruno Premoselli and Frédéric Robert. This type of equations arises from problems in both Mathematical Physics and Conformal Geometry. The main difficulty in the energy-critical case lies in a potential lack of compactness of families of solutions, which generates an instability phenomenon. Premoselli presented an instability type result showing the existence of towering families of solutions (i.e., solutions with multiple concentrating peaks on top of each other) in the context of the standard energy-critical Schrödinger elliptic equation on locally conformally flat manifolds of dimension greater than or equal to 7, which turns out to be optimal for this type of solutions. Robert reported on results of compactness, stability and multiplicity of solutions to the energy-critical Hardy-Schrödinger equation on the Poincaré ball, an interesting new context for such equations, which recently emerged from a new functional inequality introduced by Kunnath Saandeep and Cyril Tintarev.

• Some interesting problems of Geometric Analysis have also been addressed during this workshop. Pengfei Guan reported on the role played by entropy functionals in the study of convergence of solutions to the Gauss curvature flow, as well as some new flows of similar type, i.e., flows by powers of the Gauss curvature and anisotropic Gauss curvature flows. Connor Mooney addressed another geometric problem, the Bernstein problem. This problem asks whether the hyperplanes are the only minimizing solutions to a given energy functional in the Euclidean space. The original Bernstein problem concerns the area functional. Mooney presented some new constructions of entire minimal graphs for a class of anisotropic energy functionals in dimensions 4 and 6, two dimensions where such constructions are not possible in the case of the area functional.

• A series of talks was concerned with nonlinear potential theory. Jana Björn presented new developments on fine nonlinear potential theory in metric spaces, which extends further her pioneering work in this direction. Iwona Chlebicka presented sharp nonlinear potential estimates, as well as various existence and regularity theorems, for scalar quasilinear elliptic equations with Orlicz growth of the coefficients and measure data, as well as for systems of such equations, based on her joint work with Giannetti, Youn and Zatorska-Goldstein. Nguyen Cong Phuc spoke on Choquet integrals and capacitary inequalities related to his recent solution of an old problem on strong capacitary inequalities posed by David Adams.

• Robert McOwen spoke on his joint work with Maz'ya concerning elliptic equations in divergence form of Gilbarg-Serrin type with coefficients that are not Dini continuous. Some striking examples were presented of the equations whose coefficients do not have Dini mean oscillation, but with Lipschitz continuous solutions. These examples lie outside the class introduced by Kuusi and Mingione in 2012, and later considered by Dong and Kim, which has recently attracted a lot of attention. They are based on novel analytic methods.

• Nonlocal operators were one of the main themes of the conference. Tadele Mengesha spoke on his joint work with Armin Schikorra and Sasikarn Yeepo concerning Calderón-Zygmund theory, based on self-improving properties of nonlocal operators. New results include higher order integrability and improved Sobolev regularity for Hölder continuous kernels.

• The analysis of geometric properties of solutions to elliptic and parabolic PDEs and related inequalities of isoperimetric type is the subject of a successful direction of research. Symmetry properties of solutions to elliptic and parabolic PDEs, and geometric features of their level sets, such as convexity or starshapedness, have been investigated in a rich literature. The detection of extremals in variational problems in mathematical physics, and the ensuing isoperimetric inequalities, typically benefits from this geometric information. Approaches relying upon symmetrization methods as well as on mass transportation techniques are also critical in this connection. Paolo Salani reported on the preservation of concavity properties by heat flow. In particular, he showed that there exist concavities stronger than log-concavity that are preserved by the Dirichlet heat flow. Carlo Nitsch presented some recent results on optimal thermal insulation, namely on the best displacement of a prescribed volume of insulating material around a given conductor. He discussed various issues in connection with these problems and proved a conjecture according to which the ball surrounded by a uniform distribution of insulating material maximizes the heat content.

• Lisa Beck and Cristiana De Filippis gave talks about sharp regularity results for minimizers (and in some cases) stationary points of functionals with non-standard growth conditions and/or exhibiting nonuniform ellipticity. Lisa Beck concentrated on local Lipschitz continuity of minimizers of functionals with autonomous principal part. The results cover both operators and integrands with polynomial anisotropic growth and those having fast growth of exponential type. Cristiana De Filippis presented a solution to a few longstanding problems concerning nonuniformly elliptic operators and functionals. In particular, the classical issue of Schauder estimates for nonlinear, nonuniformly elliptic equations with unbalanced polynomial growth has been settled. Moreover, the local gradient Hölder continuity of minima of nondifferentiable functionals has been proved in the nonuniformly elliptic case. While in the uniformly elliptic case (i.e., p-Laplacean type problems) the result is nowadays standard, after the work of Frehse, Giaquinta & Giusti, Ivert, Manfredi from the 80s, nothing was known in the nonuniformly elliptic one.

• Verena Bögelein presented some new regularity results for solutions to nonlinear elliptic systems with a strong degeneration. Specifically, the leading vector field is assumed to lose any ellipticity property on a bounded domain. While for such cases Lipschitz continuity is known to be a standard fact, the higher regularity, and, in particular, the gradient Hölder continuity, is a more delicate issue. Here more intrinsic results are presented concerning the Hölder continuity of the so called stress tensor, obtained in an intrinsic way via delicate schemes of linearization and iterations.

• Domenic Breit, after reviewing the fundamentals of Nonlinear Calderón-Zygmund theory, presented new results concerning the fine Besov regularity of solutions for degenerate nonlinear problems and therefore higher fractional differentiability of solutions in terms of given data. As it is natural in this case, maximal regularity is obtained by means of the stress tensor, for which delicate schemes of comparison estimates must be developed. The results presented were of particular interest in the two dimensional case, where a complete treatment has been achieved.

• Lars Diening presented some recent and very fine results on the validity of Calderón-Zygmund type estimates for degenerate systems of p-Laplacean type when coefficients are degenerate/singular and in a new borderline class that allows to improve all the results known up to now. They in particular relate to the classical Muckenhoupt conditions on weights.

• The concluding talk by Camillo De Lellis addressed the Onsager problem for locally dissipative solutions to the Euler equations in Fluid Dynamics. This problem asks whether below a certain threshold regularity, there exist Hölder-continuous solutions to the Euler equations which dissipate the total kinetic energy. De Lellis reported on recent constructions of such solutions which, besides dissipating the total kinetic energy, also satisfy a suitable form of local energy inequality.

## Participants

**Alberico, Angela** (National Research Council of Italy (CNR))
**Armstrong, Scott** (Scott Armstrong)
**Beck, Lisa** (Augsburg University)
**Björn, Jana** (Linköping University)
**Bogelein, Verena** (Paris-Lodron-University Salzburg)
**Breit, Dominic** (Heriot-Watt University)
**Cabre, Xavier** (ICREA and Universitat Politécnica de Catalunya)
**Cao, Dat** (Minnesota State University, Mankato)
**Carozza, Menita** (Universitá del Sannio)
**Chlebicka, Iwona** (University of Warsaw)
**Cianchi, Andrea** (University of Firenze)
**Cîrstea, Florica** (University of Sydney)
**De Filippis, Cristiana** (Universitá di Torino)
**De Lellis, Camillo** (Camillo De Lellis)
**De Silva, Daniela** (Barnard College - Columbia University)
**Diebou Yomgne, Gael** (Rheinische Friedrich-Wilhelms-Universität Bonn)
**Diening, Lars** (Bielefeld University)
**Dong, Hongjie** (Brown University)
**Focardi, Matteo** (Universitá di Firenze)
**Frank, Rupert** (Caltech / University of Münich)
**Gazzola, Filippo** (Politecnico di Milano)
**Ghoussoub, Nassif** (University of British Columbia)
**Glubokov, Andrey** (Purdue University)
**Guan, Pengfei** (McGill University)
**Hänninen, Timo** (University of Helsinki)

**Kh.Balci, Anna** (Bielefeld University)
**Kristensen, Jan** (University of Oxford)
**Kuusi, Tuomo** (University of Helsinki)
**McOwen, Robert** (Northeastern University)
**Mengesha, Tadele** (The University of Tennessee, Knoxville)
**Mingione, Giuseppe** (University of Parma)
**Mooney, Connor** (University of California, Irvine)
**Nguyen, Phuc** (Louisiana State University)
**Nguyen, Quoc-Hung** (Chinese Academy of Sciences)
**Nitsch, Carlo** (Universitá di Napoli ”Federico II”)
**Premoselli, Bruno** (Université Libre de Bruxelles)
**Robert, Frédéric** (Université de Lorraine)
**Salani, Paolo** (Universitá di Firenze)
**Savin, Ovidiu** (Columbia University)
**Schwarzacher, Sebastian** (Charles University)
**Seesanea, Adisak** (Sirindhorn International Institute of Technology)
**Shaposhnikova, Tatiana** (Royal Institute of Technology)
**Singh, Gurpreet** (Dublin City University)
**Slavikova, Lenka** (Faculty of Mathematics and Physics, Charles University in Prague)
**Storn, Johannes** (Bielefeld University)
**Van Schaftingen, Jean** (Université catholique de Louvain)
**Verbitsky, Igor** (University of Missouri)
**Vetois, Jerome** (McGill University)
**Vladimir, Maz’ya** (Linkoeping University)
**Weng, Liangjun** (Albert-Ludwigs-Universität Freiburg)
**Weth, Tobias** (Goethe-University Frankfurt)

# Chapter 7

# Connecting Network Structure to its Dynamics: Fantasy or Reality? (21w5005)

September 26 - Oct 1, 2021

**Organizer(s):** Tomas Gedeon (Montana State University), Steve Haase (Duke University), Konstantin Mischaikow (Rutgers University)

## Overview of the Field

Systems biology hold the promise to revolutionize our understanding of life and our approach to human health. Central to this effort is the ability to use large sets of imprecise experimental data to build predictive mathematical models of cellular systems, in particular, as they evolve in time.

In spite of tremendous promise, we are facing important challenges preventing us from establishing clear links between the genotype and the cellular phenotype. Apart from the fact that most of the diseases cannot be linked to individual gene mutations, but rather depend on differential expression in dozens of genes, many phenotypes of interest (cell cycle maintenance, circadian rhythm, development) are time dependent. Since cells exhibiting these phenotypes respond to their environment in real time and responses may depend on the cellular history, the underlying genetic structure that is responsible for the observed cellular phenotype is a regulatory network, rather than a gene, or a collection of genes. Therefore the principal challenge in bridging the genotype-phenotype gap is the lack of methods that predict dynamics from the structure of the network.

In the related field of Synthetic Biology, the main goal is to develop rational designs of networks that can produce a desired set of outputs for a given set of inputs.

The major hurdle for both Systems Biology and Synthetic Biology fields is that the engineered (or reverse-engineered) networks are operating in parameter regimes that are largely unknown and experimental measurement of parameters is largely intractable. Moreover, parameters usually change depending on the growth environment of cells. Thus, the development of mathematically justifiable methodologies to predict dynamic phenotypes from network topology is critical to the future of both Systems and Synthetic Biology fields

## Recent Developments and Open Problems

In the last two decades more and more biological data comes in a form of a time series tracking behavior of systems like cell cycle, development and circadian rhythm. This led to development of models for these processes in the context of different organisms, as well as different modeling approaches. The central challenge in constructing systems biology models is lack of parameters needed to support traditional ODE models. This, coupled with the size of the modeled systems and thus the number of parameters, leads to severe under-parameterization of the models. There are two fundamentally different approaches to this issue:

1. development of traditional ODE based models, where the parameters are obtained by combing the existing literature for experimental data, with skilful interpretation of the data withing the model framework and deducing parameter values or at least reasonable parameter ranges. This is combined with careful combination of simulations with matching limited time series data to find reasonable set of parameters. These methods were represented at the conference by contributions by D. Thieffry and J. Tyson.

2. modeling methods which revert the traditional order of parameterization of the model first, and matching the data second paradigm. The idea is to find an approximation of all dynamics that the network is capable of, and use the comparison to data to help select the parameter regions that are capable of supporting the observed dynamics. The comparison is done on a qualitative level, which is different that the detailed comparison of ODE model output with experimental time series. This approach had been represented at the conference by presentations by M Gameiro and B. Cummins, who both talked about approach called *DSGRN*, and Michael Savageau who talked about *phenotype space modeling* of metabolic fluxes.

The set of open problems that the entire community is working on include

- What is the best way to extract biological insight from the modeling techniques we have?
- What is the best type of data that we can collect in order to validate the models?
- What are the next steps in model development and their analysis that will be most useful to biologists?

## Presentation Highlights

**Philip Benfey (Duke)** described dynamic process of plant root development and dynamics of the network responsible for development of two new cell types from a stem cell.

**Jay Dunlap (Dartmouth) and Jennifer Hurley (Rensselaer)** presented a series on talks on circadian rhythms in both Neurospora and in human cells. They presented a series of striking results about chronobiology thay may benefit from insights from mathematical modeling.

**Jan Skotheim (Stanford) and John Tyson (Virginia Tech)** discussed different aspects of cell cycle. Skotheim talked about how cell size triggers entry to the cell cycle and a related question of how cell maintains stable constant concentration of proteins as in grows. Tyson presented a comprehensive model of cell cycle where the increased mass of a cell triggers saddle node bifurcation that is associated to S/G1 transition. This model can explain several phenotypes that are observed after perturbing the cell.

**Dennis Thieffry (Paris) and Reka Albert (Penn State)** presented models that use Boolean networks. Thieffry described parameterization and interrogation of a large network in T-cell immune checkpoints. Albert described new mathematical approaches that allow her group to find attractors in large Boolean network with dozens to a hundred nodes.

**Marcio Gameiro (Rutgers) and Bree Cummins (Montana State)** disussed different aspects of DSGRN. Gameiro talked about a recent work in which they examined all three node networks for the potential to act like a bistable switch. The potential of DSGRN to examine the entire dynamics of a networks, and then compare such dynamics across many networks, was also used by Cummins. She described three methods which, when used together, can suggest structure of networks based on just experimental time series data.

**Michael Savageau (Davis)** discuss phenotype space modeling which is an approach to model complex metabolic networks by first finding all potential pairs of dominant fluxes that balance all metabolites, and then query the resulting collection for the observed phenotype.

## Scientific Progress Made

Important connection have been established between different modeling communities, especially between Boolean network community (Thieffry, Albert), switching systems community (Glass, Edwards, Perkins), DSGRN community (Cummins, Gameiro, Gedeon, Haase, Mischaikow) and phenotype space community (Savageau).

Several systems biologists commented that the conference open their eyes to mathematics that modeling community is doing and the range of tools that are already available. In particular, circadian community (Dunlap, Hurley) have expressed interest in learning more about modeling approaches that were presented.

Talks by systems biology community exposed mathematicians to new methodologies of collecting data, their advantages and limitations. At the same time, the fascinating problems that were presented call for further development of mathematical models.

## Outcome of the Meeting

This workshop provided an opportunity for the participants to step out of their usual domains of expertise and focus on the conceptual understanding and analysis of time dependent processes in cell biology. As indicated above in the presentation highlights there was a wide variety of topics presented during the lectures. However, we believe that the success of this workshop should be measure by the length and intensity of the ensuing discussion. These discussions lasted on the order of 2 hours per day after the lecture period and were brought to closure by the time constraints (Europe vs. California) of the participants.[1] Furthermore, the broad expertise proved to be a boon as opposed to a hindrance as the conversation ranged from the abstract – mathematical techniques and modeling – to the practical – what in principle do these techniques offer for specific questions of biological importance and what needs to be done to either implement or modify these techniques to address these questions.

These conversations has resulted in several groups that are already planning follow-up meetings and projects.

1. Mischaikow, Gameiro, Cummins, Gedeon and Albert plan a 3 day meeting in early 2022 to meet at Rutgers and discuss further how they can merge DSGRN and Albert's work on attractor finding to develop a powerful platform for large network modeling.

2. Haase, Benfey, Mischaikow and Gedeon plan to be part of a team that will apply for NSF-Simmons Center for Mathmatical Biology, so they can work on plant developmental models.

3. Jay Dunlap, Jen Hurley and Steve Haase started an intriguing discussion on similarities in time series profiles of genes that are engaged in circadian rhythm and cell cycle generation. This similarity extends to similarity between motifs of core oscillator that generate these rhythms in both cases. There is an intriguing possibility that the core oscillators may have come from the same ancestral network. The BIRS meeting was a catalyst for this emerging collaboration.

4. There is an ongoing discussion between L. Glass and the DSGRN community (Cummins, Gameiro, Gedeon, Mischaikow) that was started at the workshop about connections between DSGRN and earlier work of Glass and collaborators on switching systems.

5. Haase and Tyson began a conversation regarding endocycling phenotypes in specific mutants of yeast cells. This conversation continued with both the Tyson group and the Haase, Gedeon, Mischaikow group discussing interests in modeling the endocycling phenotype.

[1] The inability to continue these discussion was by far the greatest disappointment resulting from an online as opposed to in person conference.

## Participants

**Albert, Reka** (Pennsylvania State University)
**Andreas, Elizabeth** (Montana State University)
**Belton, Robin** (Montana State University)
**Benfey, Philip** (Duke University and HHMI)
**Cannon, William** (Pacific Northwest National Lab and UC Riverside)
**Carlson, Ross** (Montana State University)
**Chaves, Madalena** (INRIA Sophia Antipolis)
**Cummins, Bree** (Montana State University)
**Day, Sarah** (William and Mary)
**de Jong, Hidde** (INRIA Grenoble)
**Duncan, William** (Montana State University)
**Dunlap, Jay** (Dartmouth)
**Edwards, Roderick** (University of Victoria)
**Gameiro, Marcio** (Rutgers University)
**Gedeon, Tomas** (Montana State University)
**Glass, Leon** (McGill University)
**Haase, Steve** (Duke University)
**Hurley, Jennifer** (Rensselaer Polytechnic Institute)
**Jolly, Mohit Kumar** (Indian Institute of Science)
**Kaern, Mads** (University of Ottawa)
**Kalies, William** (Florida Atlantic University)
**Kelliher, Tina** (Geisel School of Medicine at Dartmouth)
**Mischaikow, Konstantin** (Rutgers University)
**Paulevé, Loïc** (CNRS, LaBRI, Bordeaux, France)
**Perkins, Theodore** (Ottawa Hospital Research Institute / University of Ottawa)
**Santillan, Moises** (Centro de Investigación y de Estudios Avanzados del IPN)
**Savageau, Michael** (University of California, Davis)
**Skotheim, Jan** (Stanford University)
**Thieffry, Denis** (Ecole Normale Supérieure)
**Tyson, John** (Virginia Polytechnic Institute / State University)
**Veliz-Cuba, Alan** (University of Dayton)
**Xing, Jianhua** (University of Pittsburg)
**Yeung, Enoch** (Santa Barbara)

# Chapter 8

# Cohomology of Arithmetic Groups: Duality, Stability, and Computations (21w5011)

## October 10 - 15, 2021

**Organizer(s):** Jeremy Miller (Purdue University), Jennifer Wilson (University of Michigan)

## Overview of the Field

The (co)homology of arithmetic groups is a rich subject with many applications to topology and number theory. It has connections to algebraic $K$-theory [59], automorphic forms, and more recently to Galois representations (e.g. Scholze [76]). The conference focused on topological and computer-assisted approaches to computations of these cohomology groups, especially those approaches making use Borel–Serre duality [16] and the study of Steinberg modules. Another major point of emphasis was examining the analogy between arithmetic groups and other families of groups such as mapping class groups.

### Homological stability and its generalizations

Since the work of Borel [14], homological stability has been an important tool for studying the homology of arithmetic groups. A family of groups $G_n$ is said to satisfy homological stability if the groups $H_i(G_n)$ do not depend on $n$ for $n \gg i$. Specifically, we say that the family $G_n$ satisfies homological stability with slope $k \in \mathbb{Q}$ if there is some $b \in \mathbb{Q}$ so that

$$H_i(G_n) \cong H_i(G_{n+1}) \qquad \text{for all } i \leq kn + b.$$

Many families of arithmetic groups satisfy homological stability, such as general linear groups of number rings [14, 25, 85]. Often computing the stable homology groups involves different techniques than proving that they stabilize. These calculations may use analytic techniques (e.g. Borel [14]) or $K$-theoretic the techniques (e.g. Weibel [87]). We do not know precise stable ranges for many families of arithmetic groups that are known to stabilize.

The first major advancement on this question in the last 35 years is the work of Galatius–Kupers–Randall-Williams [44] where they improve the rational stable range from slope $1/2$ to slope $2/3$ for general linear groups of number rings. This result was proven using the machinery of $E_k$-cells which is in the process of revolutionizing the field of homological stability. Here $E_k$ denotes the little disk operad in the sense of May [64].

The application of this technology to the problem of cohomology of arithmetic groups has only just begun; the focus until now has been using these ideas to study mapping class groups and general linear groups of fields. For example, Galatius–Kupers–Randall-Williams [46] established that the homology groups of mapping class groups exhibit secondary stability, a stability pattern outside the classical stable range. The question of secondary stability for families of arithmetic groups is a major goal of the field.

The (co)homology of congruence subgroups is a major topic in number theory and a subject where stability techniques have recently found applications. The cohomology of these groups do not stabilize in a classical sense but instead exhibit controlled growth (Putman [71]). This controlled growth is known as representation stability, a phenomena discovered by Church and Farb [30]. Again, this stability has been crucial for calculations such as Calegari's computation of the second homology group of congruence subgroups of $\mathrm{SL}_n(\mathbb{Z})$ [21, 22].

### Steinberg modules and the Church–Farb–Putman conjectures

Another major theme of the conference was dualizing modules for arithmetic groups. For the sake of concreteness, consider the groups $H^i(\mathrm{SL}_n(\mathcal{O}))$, the cohomology of groups of the special linear group of a ring of integers $\mathcal{O}$ in a number field. The classifying spaces of these groups are not closed manifolds so these groups do not satisfy Poincaré duality. However, Borel–Serre [16] showed that the groups $H^i(\mathrm{SL}_n(\mathcal{O});\mathbb{Q})$ satisfy a twisted form of Poincaré duality. There is an $\mathrm{SL}_n(\mathcal{O})$-representation $St_n(K)$ called the Steinberg module and a number $\nu_n$ such that

$$H^{\nu_n - i}(\mathrm{SL}_n(\mathcal{O});\mathbb{Q}) \cong H_i(\mathrm{SL}_n(\mathcal{O}); St_n(K)).$$

We will refer to these groups $H^{\nu_n - i}(\mathrm{SL}_n(\mathcal{O});\mathbb{Q})$ as the codimension $i$ cohomology groups. Church–Farb–Putman [31] conjectured that the codimension $i$ cohomology groups of arithmetic groups should exhibit various forms of stability. There has been significant recent progress studying these codimension $i$ cohomology groups by studying the Steinberg module [32, 33, 65, 4, 45].

An example of the interplay between ideas from the field of stability and computer computations is Kupers' new proof that $K_8(\mathbb{Z}) \cong 0$ [55]. His use of stability results [33] dramatically reduces the needed input from supercomputers.

### Comparison and interactions between arithmetic groups and other families of groups

There are many properties of arithmetic groups that are shared by other families of groups arising in topology and geometric group theory, such as mapping class groups $\mathrm{Mod}_g$ of an orientable surface, and the automorphism groups of free groups $\mathrm{Aut}(F_n)$. In particular, $\mathrm{Mod}_g$ and $\mathrm{Aut}(F_n)$ both satisfy homological stability and a form of twisted Poincaré duality. Church–Farb–Putman [31] conjectured that mapping class groups and automorphism groups of free groups should also have a form of stability for their high dimensional cohomology. In particular, they conjectured that the codimension $i$ cohomology should vanish for $n \gg i$. This was recently disproven by Chan–Galatius–Payne [22] in codimension 1. This was a shocking result. There is ongoing work to better understand the analogies between mapping class groups and general linear groups, and the extent to which the qualitative behaviour of these groups is similar or differs.

Another avenue of research involves leveraging information about one family to tell you about the other family. For example, there are natural maps $\mathrm{Mod}_g \to \mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$. It is an interesting question to see when cohomology classes pull back non-trivially along these maps and how characteristic classes for arithmetic groups compare to characteristic classes for these other families of groups. See for example the work of Tshishiku [83].

## Recent Developments and Open Problems

### High-dimensional cohomology of arithmetic groups

The Church–Farb–Putman conjecture [31] predicts that the codimension-$i$ rational cohomology of $\mathrm{SL}_n(\mathbb{Z})$ vanishes for $n$ sufficiently large compared with $i$. This is proven in codimensions at most 2 [58, 33, 20] with the codimension two case being work-in-progress.

This conjecture admits many generalizations. For example, the ring $\mathbb{Z}$ can be replaced by any number ring $\mathcal{O}$. By work of Church–Farb–Putman [32] and Miller–Patzt–Wilson–Yasaki [66], the codimension $i$ cohomology $H^{\nu_n - i}(\mathrm{SL}_n(\mathcal{O}); \mathbb{Q})$ does not vanish for large $n$ for all number rings $\mathcal{O}$. However, the question is open as to whether it always vanishes when the ring is a Euclidean domain. This is true in codimension $i = 0$ by work of Lee–Szczarba [58]. Kupers–Miller–Patzt–Wilson [56] proved the codimension-1 homology vanishes for the Gaussian integers and Eisenstein integers.

In a different direction, Brück–Patzt–Sroka [19] are working to prove the codimension-1 homology of the symplectic groups $\mathrm{Sp}_{2n}(\mathbb{Z})$ vanishes for large $n$. The codimension zero case for $\mathrm{Sp}_{2n}(\mathbb{Z})$ is work of Gunnells [51].

### Improved ranges for homological stability

Homological stability for arithmetic groups is a classical topic. Most of the classical techniques proved slope-$1/2$ stable ranges. With recent developments in the theory of $E_k$-cells and Steinberg modules, Kupers–Miller–Patzt [57] proved slope-1 stability for $\mathrm{GL}_n(\mathcal{O})$ for a few special number rings $\mathcal{O}$. It is natural to ask what sharp stable ranges are for homological stability and how this depends on arithmetic properties of the number ring. A sharp stable range is not currently known for $\mathrm{GL}_n(\mathcal{O})$ for any number ring $\mathcal{O}$.

### Cohomology growth as the index increase

Let $\Gamma$ denote a finite index subgroup of an arithmetic group. There are many conjectures concerning how to bound the size of homology of $\Gamma$ in terms of its index. The most famous example of this is the Bergeron–Venkatesh conjectures [11] concerning torsion growth. The asymptotic behavior of the Betti numbers is conjecturally very sensitive to which homological degree one considers. Gerbelli-Gauthier [48] recently proved a result about bounding the rational cohomology away from the middle dimension.

### Connections with $K$-theory

There are many important connections between the cohomology of algebraic groups. Borel used his computation of the cohomology of arithmetic groups to compute the rational $K$-theory of number rings. Dwyer used homological stability for $\mathrm{GL}_n(\mathbb{Z})$ with twisted coefficients to prove finiteness results for $A$-theory [36]. The theory of Steinberg modules has led to many low degree calculations of $K$-groups. Recently, the flow of information has been reverse. Hebestreit–Land–Nikolaus [53] used results on symplectic $K$-theory to compute the stable homology of symplectic groups.

## Presentation Highlights

### Avner Ash: Homology of arithmetic groups and Galois representations

Ash's talk explored how Galois representations can help in the understanding and computation of the homology of congruence subgroups of $\mathrm{GL}_n(\mathbb{Z})$. He also described work [2] joint with Darrin Doud where they proved $\rho = \sigma_1 \oplus \sigma_2$ is an $n$-dimensional odd mod $p$ Galois representation with $\sigma_1$ and $\sigma_2$ irreducible odd Galois representations that are attached to Hecke eigenclasses in the homology of the predicted congruence subgroups, with predicted weights, then $\rho$ is attached to a Hecke eigenclasses in the homology of the predicted congruence subgroup of $\mathrm{GL}_n(\mathbb{Z})$, with predicted weight. By "predicted", Ash meant in the sense of Serre-type conjecture of Ash, Doud, Pollack, and Sinnott [3, 6]. The result required the additional assumption that $p$ is greater than $n + 1$ and that the Serre conductor of $\rho$ is square-free.

### Peter Patzt: Rognes' connectivity conjecture and the Koszul dual of Steinberg

Patzt's talk described how a homotopy equivalence between certain $E_k$-buildings both proves Rognes' connectivity conjecture for fields and computes the Koszul dual of Steinberg. Rognes' connectivity conjecture [74] states that the common basis complex is highly connected. This is relevant as the equivariant homology of this complex appears in a rank filtration spectral sequence computing the homology of the K-theory spectrum. The Steinberg modules appear in various contexts, importantly as the dualizing modules of special linear groups of number rings. They can be put together to form a ring. When considered equivariantly over the general linear groups of fields, one can show that this ring is Koszul and we compute its Koszul dual. Results in this talk included joint work with Miller and Nagpal [65], and Miller and Wilson [67].

## Alexander Kupers: On homological stability for $\mathrm{GL}_n(\mathbb{Z})$

Kupers explained what is known about homological stability for the general linear groups of the integers. In particular, he discussed a recent result from joint work with Miller and Patzt [57] that improves the homological stability range to slope 1. It builds on machinery developed with Galatius and Randal-Williams [44, 46, 45], and is closely related to homology with coefficients in the Steinberg module.

## Mathilde Gerbelli-Gauthier: Growth of cohomology in towers and endoscopy

Gerbelli-Gauthier discussed her results [48] concerning how fast Betti numbers grow in a congruence tower of compact arithmetic manifolds. The dimension of the middle degree of cohomology is proportional to the volume of the manifold, but away from the middle the growth is known to be sub-linear. She discussed this question from the point of view of automorphic forms, and outlined how the phenomenon of endoscopy can be used to explain the slow rates of growth and to compute upper bounds.

## Nathalie Wahl: Stability in the homology of classical groups

General linear groups, symplectic groups, unitary groups, and orthogonal groups have long been known to satisfy homological stability under appropriate conditions [12, 23, 26, 40, 42, 60, 69, 75, 85, 86]. Wahl reported on joint work with David Sprehn [77]. They improved the earlier known homological stability ranges for $\mathrm{Sp}_{2n}(\mathbb{F})$, $\mathrm{O}_{n,n}(\mathbb{F})$ and $\mathrm{U}_{2n}(\mathbb{F})$ over any field $\mathbb{F}$ other than $\mathbb{F}_2$, following a strategy of Quillen for general linear groups $\mathrm{GL}_n(\mathbb{F})$. Under more restricted assumptions, they deduced a stability theorem for the orthogonal group $\mathrm{O}_n(\mathbb{F})$. She presented these results, focusing on what these groups have in common, and presenting a less well-known strategy of Quillen that gives a slope 1 stability range for $\mathrm{GL}_n(\mathbb{F})$.

## Andrew Putman: The Steinberg representation is irreducible

In joint work with Andrew Snowden [73], Putman proved that the Steinberg representation of $\mathrm{GL}_n$ (or, more generally, a connected reductive group) over an infinite field is irreducible. For finite fields, this is a classical theorem of Steinberg [78, 79, 80] and Curtis [35].

## Nathan Broaddus: Level structures and images of the Steinberg module for surfaces with marked points

The moduli space $M$ of complex curves of fixed topology is an orbifold classifying space for surface bundles. As such the cohomology rings of $M$ and its various orbifold covers give characteristic classes for surface bundles. Broaddus discussed the Steinberg module which is central to the duality present in these cohomology rings. He explained current joint work with T. Brendle and A. Putman [18] on surfaces with marked points which expands on results of N. Fullarton and A. Putman [43] for surfaces without marked points. They showed that certain finite-sheeted orbifold covers $M[l]$ of M have large nontrivial $\mathbb{Q}$-cohomology in their cohomological dimension.

## Frank Calegari: The stable cohomology of $\mathrm{SL}(\mathbb{F}_p)$

Let $p$ be a prime. One can make sense of various compatible algebraic representations of $\mathrm{SL}_N(\mathbb{F}_p)$ as $p$ is fixed and as $N$ varies (for example, the standard representation, or the adjoint representation, or the trivial representation).

It turns out that the cohomology groups of these representations are stable as N gets large. Calegari discussed a conjectural description of the stable cohomology. He also discussed how this relates to a conjectural computation of $H_i(\mathrm{SL}_N(\mathbb{F}_p), \mathbb{F}_p)$ for $i$ fixed and $N$ going off to infinity. The conjecture was only asserted for primes sufficiently large compared to the homological degree.

## Dan Yasaki: Cohomology of Congruence Subgroups, Steinberg Modules, and Real Quadratic Fields

Given a real quadratic field, there is a naturally defined Hecke-stable subspace of the cohomology of a congruence subgroup of $\mathrm{SL}_3(\mathbb{Z})$. Yasaki and Ash [7] investigated this subspace and made conjectures about its dependence on the real quadratic field and the relationship to boundary cohomology.

## Melody Chan: The top-weight rational cohomology of $A_g$

Chan reported on joint work with Madeline Brandt, Juliette Bruce, Margarida Melo, Gwyneth Moreland, and Corey Wolfe [17]. They identified new top-weight rational cohomology classes for moduli spaces $A_g$ of abelian varieties, by using computations of Voronoi complexes for $\mathrm{GL}(g, \mathbb{Z})$ of Elbaz-Vincent–Gangl–Soulé [38, 39].

## Paul Gunnells: Modular symbols over function fields

Modular symbols, due to Birch and Manin [62, 63], provide a very concrete way to compute with classical holomorphic modular forms. Later modular symbols were extended to $\mathrm{GL}(n)$ by Ash and Rudolph [5], and since then such symbols and variations have played a central role in computational investigation of the cohomology of arithmetic groups over number fields, and in particular in explicitly computing the Hecke action on cohomology. A theory of modular symbols for $\mathrm{GL}(2)$ over the rational function field was developed by Teitelbaum [82] and later by Armana [1]. In his talk, Gunnells described how to extend this construction to $\mathrm{GL}(n)$ and showed how it can be used to compute Hecke operators on cohomology. This talk concerned joint work with Dan Yasaki [52].

## Mark McConnell: Binary Quadratic Forms and Hecke Operators for $\mathrm{SL}(2, \mathbb{Z})$

Robert MacPherson and McConnell [61] developed an algorithm for computing the Hecke operators on the cohomology $H_d$ of arithmetic subgroups of $\mathrm{SL}(n)$ defined over any division algebra, for all $d$ and all $n$. It extends Voronoi's notion of perfect forms by introducing tempered perfect forms. To find the tempered perfect forms, their code must compute the facets of a large convex polytope of $n(n+1)/2$ dimensions, which is slow even for $n = 3$ or $4$. McConnell reported on recent work, in the classical case of $\mathrm{SL}(2, \mathbb{Z})$, where they have succeeded in identifying the tempered perfect forms directly. The story comes down to binary quadratic forms in the spirit of Lagrange and Gauss, together with some modern class field theory. This talk described joint work with Erik Bahnson and Kyrie McIntosh [10].

## Oscar Randal-Williams: $E_\infty$-algebras and general linear groups

Randal-Williams discussed joint work with S. Galatius and A. Kupers [45] in which they investigate the homology of general linear groups over a ring $A$ by considering the collection of all their classifying spaces as a graded $E_\infty$-algebra. Randal-Williams explained diverse results that they obtained in this investigation, which can be understood without reference to $E_\infty$-algebras and which seem unrelated to each other. He then explained how the point of view of cellular $E_\infty$-algebras unites them.

## Bena Tshishiku: Unstable cohomology of arithmetic groups and geometric cycles

Tshishiku [84] constructed unstable cohomology classes of nonuniform arithmetic subgroups of $SO(p, q)$ using ideas of Millson–Raghunathan [68] and more recent work of Avramidi and Nguyen–Phan [8, 9]. The classes he constructed are dual to maximal periodic flats in the locally symmetric space. One motivation for this result is to produce characteristic classes for certain manifold bundles that are not in the algebra generated by the stable (generalized Morita–Miller–Mumford) classes.

## Benson Farb: Rigidity of moduli spaces

Algebraic geometry contains an abundance of miraculous constructions. Examples include "resolving the quartic"; the existence of 9 flex points on a smooth plane cubic; the Jacobian of a genus g curve; and the 27 lines on a smooth cubic surface. In this talk Farb explained some ways to systematize and formalize the idea that such constructions are special: conjecturally, they should be the only ones of their kind. Farb stated a few of these many (mostly open) conjectures. They can be viewed as forms of rigidity (à la Mostow and Margulis) for various moduli spaces and maps between them.

## Matthew Emerton: Cohomology of Shimura varieties via categorical Langlands

"Categorical Langlands" refers to a perspective on the local Langlands correspondence in which the *category* of smooth representations of a p-adic groups is related to the *category* of coherent sheaves on an appropriate moduli stacks of Galois representations. This perspective probably arose first in the geometric Langlands program, but it also relates to the number-theoretic technique known as Taylor–Wiles–Kisin patching [88, 81, 54] (which is used to prove modularity theorems). In his talk, Emerton explained a conjectural formula for the (singular, or étale) cohomology of Shimura varieties in terms of categorical Langlands.

## Tony Feng: The Galois action on symplectic K-theory

Feng talked about some connections between the cohomology of arithmetic groups, $K$-theory, and number theory. One reason for these connections is the fact that there is a natural Galois action on the cohomology of symplectic groups of integers, which turns out to provide Galois representations important in the Langlands correspondence. The same mechanism leads to a Galois action on a symplectic variant of $K$-theory of the integers. In joint work with Soren Galatius and Akshay Venkatesh [41], Feng computed this Galois action and they found that it also enjoys a certain universality.

## Benjamin Brück: High-dimensional rational cohomology of $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{Sp}_{2n}(Z)$

By results of Lee–Szczarba [58] and Church–Putman [33], the rational cohomology of $\mathrm{SL}_n(\mathbb{Z})$ vanishes in codimensions zero and one, that is,

$$H^{n(n-1)/2-i}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q}) = 0$$

for $n \geq i+2$ and $i = 0$ or $1$. Recall that $n(n-1)/2$ is the virtual cohomological dimension of $\mathrm{SL}_n(\mathbb{Z})$. Brück spoke about work in progress on two generalisations of these results: The first project is joint work with Miller, Patzt, Sroka, and Wilson [20]. They showed that the rational cohomology of $\mathrm{SL}_n(\mathbb{Z})$ vanishes in codimension two, that is,

$$H^{n(n-1)/2-2}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q}) = 0$$

for $n \geq 4$. The second project is joint with Patzt and Sroka [19]. Its aim is to study whether the rational cohomology of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{Z})$ vanishes in codimension one, that is, whether

$$H^{n^2-1}(\mathrm{Sp}_{2n}(\mathbb{Z};\mathbb{Q}) = 0$$

for $n \geq 2$.

## Orsola Tommasi: Stability results for toroidal compactifications of Ag

In her talk, Tommasi discussed the geometry of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension $g$ and its compactifications. As is well known, in degree $k < g$ the rational cohomology of $\mathcal{A}_g$, which coincides with the cohomology of the symplectic group, is freely generated by the odd Chern classes of the Hodge bundle by a classical result of Borel [14, 15]. Work of Charney and Lee [27] provides an analogous result for the stable cohomology of the minimal compactification of $\mathcal{A}_g$, the Satake compactification. However, for most geometric applications it is more natural to work with the toroidal compactifications of $\mathcal{A}_g$. Tommasi reported on joint work with Sam Grushevsky and Klaus Hulek [49, 50] on the toroidal compactifications of $\mathcal{A}_g$, and described stability results for the perfect cone compactification and the matroidal partial compactification and their combinatorial features.

### Fabian Hebestreit: The stable cohomology of symplectic groups over the integers

Hebestreit reported on joint work with M. Land and T. Nikolaus [53] in which they compute the stable part of the cohomology of both symplectic groups and orthogonal groups with vanishing signature over the integers at regular primes, in particular at the prime 2. Their approach is by identifying the stable cohomology with that of a certain Grothendieck–Witt space, whose homotopy type can be analyzed using recent advances in hermitian $K$-theory.

### Jordan Ellenberg: Legendre symbols and secondary stability

Mark Shusterman and Ellenberg were investigating the problem of controlling sums of Legendre symbols $\left(\frac{f}{g}\right)$ as $f$ and $g$ range over squarefree polynomials of degree $m$ and $n$ over $\mathbb{F}_q$, with $m$ and $n$ growing while the finite field $\mathbb{F}_q$ stays the same. This can be expressed as a problem about the trace of Frobenius acting on the étal cohomology of a space whose complex points are a $K(\pi, 1)$ for a certain finite-index subgroup of a colored braid group; it seems to me that the behavior we expect to see for these averages would follow from a good result on secondary homological stability for these subgroups. The question is whether the assembled topological might of this workshop can help figure out whether such a statement is true and provable with current methods.

## Scientific Progress Made

The objectives of the workshop were to disseminate the latest research developments in the cohomology of arithmetic groups, and to bring together researchers working on the cohomology of arithmetic groups and related objects from different vantages. The conference brought together mathematicians with backgrounds in areas including algebraic number theory, computational algebra, combinatorial algebraic geometry, arithmetic geometry, geometric group theory, homological and representation stability, algebraic K-theory, locally symmetric spaces, low dimensional topology. The organizers made a particular effort to invite junior researchers.

The organizers requested that all speakers make (the majority of) their talks accessible to graduate students and researchers in other areas. The workshop was an unusual opportunity for mathematicians working on cohomology of arithmetic groups to learn about the perspectives, goals, and toolkits of other mathematicians working on the subject from other subfields. The conference brought together mathematicians on several continents.

Although the primary goals of the workshop were educational, some research was completed during the week. For example, Alexander Kupers, Jeremy Miller, and Robin Sroka proved a slope-1 stability result for the symplectic groups $\mathrm{Sp}_{2n}(\mathbb{Z})$. This grew out of discussion following Kupers' talk and Wahl's talk. See Section 8 for more details on the outcomes of the workshop.

## Outcome of the Meeting

### Participants

The workshop had 140 confirmed participants, including participants from North and South America, Europe, and Asia. Additional participants watched the livestream or recordings of the lectures. Talks typically had 13 in-person participants in addition to 35-60 participants attending virtually on Zoom. A list of registered participants is given below.

Miller, Jeremy (Purdue University)
Wilson, Jennifer (University of Michigan)
Brück, Benjamin (ETH Zürich)
Chen, Eric (Princeton University)
Gadish, Nir (University of Michigan)
Gerbelli-Gauthier, Mathilde (Institute for Advanced Study)
Himes, Zachary (Purdue University)
Kupers, Alexander (University of Toronto)
Putman, Andrew (University of Notre Dame)
Scalamandre, Matthew (University of Notre Dame)
Sroka, Robin (McMaster University)
Tshishiku, Bena (Brown University)
Yasaki, Dan (The University of North Carolina at Greensboro)
Adem, Alejandro (University of British Columbia)
Ash, Avner (Boston College)
Aumonier, Alexis (Københavns Universitet)
Bao, Chengyang (University of Chicago)
Bauman, Alex (University of Michigan)
Bellovin, Rebecca (no affiliation given)
Belmont, Eva (UC San Diego)
Berglund, Alexander (Stockholm University)
Berkove, Ethan (Lafayette College)
Bernard, Calista (University of Minnesota)
Berrick, Jon (Yale-NUS College)
Bianchi, Andrea (Københavns Universitet)
Bibby, Christin (Louisiana State University)
Boggi, Marco (Universidade Federal de Minas Gerais)
Boyd, Rachael (Max Planck Institut für Mathematik)
Brendle, Tara (University of Glasgow)
Broaddus, Nathan (The Ohio State University)
Bruce, Juliette (University of California, Berkeley)
Bustamante, Mauricio (Universidad Católica de Chile)
Calegari, Frank (University of Chicago)
Chan, Melody (Brown University)
Coulangeon, Renaud (Institut de Mathematiques de Bordeaux)
Cremona, John (University of Warwick)
Das, Ronno (University of Copenhagen)
Djament, Aurélien (CNRS et université de Lille)
Dotto, Andrea (University of Chicago)
Dutour Sikiric, Mathieu (Rudjer Bosković Institute)
Edson, Nzaganya (Stellenbosch University)
Elbaz-Vincent, Philippe (Université Grenoble Alpes)
Ellenberg, Jordan (University of Wisconsin)
Emerton, Matthew (University of Chicago)
Farb, Benson (University of Chicago)
Feng, Tony (Massachusetts Institute of Technology)
Fennell, James (NSA-retired)
Fisher, Sam (University of Oxford)
Fu, Yu (University of Wisconsin)
Galatius, Søren (Københavns Universitet)
Ganapathy, Karthik (University of Michigan)
Gangl, Herbert (Durham University)
Garkusha, Grigory (Swansea University)
Glubokov, Andrey (Purdue University)
Grayson, Daniel R. (University of Illinois Urbana-Champain)
Gunnells, Paul (University of Massachusetts)
Hainaut, Louis (Stockholm University)
Harman, Nathan (University of Michigan)
Hebestreit, Fabian (University of Bonn)
Hoang, Anh (University of Minnesota)
Hogben, Leslie (Iowa State University)
Howe, Sean (University of Utah)
Iwasa, Ryomei (University of Copenhagen)
Jain, Yajit (Brown University)
Jiménez Rolland, Rita (Universidad Nacional Autónoma de México)
Kammeyer, Holger (Heinrich-Heine-Unversität Düsseldorf)
Karn, Trevor (University of Minnesota)
Kechagias, Nondas (University of Ioannina)
Koziol, Karol (University of Michigan)
Krannich, Manuel (University of Cambridge)
Kumar V, Pravin (Indian Institute of Science Education and Research) Lewis, Tudur (University of Glasgow)
Land, Markus (University of Copenhagen)
Laudone, Robert (University of Michigan)
Le, Daniel (Purdue University)
Li, Shiyue (Brown University)
Lin, Milton (Johns Hopkins University)
Lindell, Erik (Stockholm University)
Linskens, Sil (University of Bonn)
Lipnowski, Michael (McGill University)
Litt, Daniel (University of Georgia)
Maity, Saikat (University of Calcutta)
Malestein, Justin (University of Oklahoma)
Malin, Connor (University of Notre Dame)
Margalit, Dan (Georgia Institute of Technology)
Mathew, Akhil (University of Chicago)
Mawyer, Farley (York College)
McConnell, Mark (Princeton University)
Mejia, Andres (University of Pennsylvania)

Meyer, Anja (University of Manchester)
Minahan, Daniel (Georgia Institute of Technology)
Mishra, Pratyush (North Dakota State University)
Mondal, Arghya (Chennai Mathematical Institute)
Morris, Dave (University of Lethbridge)
Nariman, Sam (Purdue University)
Ørsnes Jansen, Mikala (Københavns Universitet)
Pacheco-Tallaj, Natalia (Massachusetts Institute of Technology)
Page, Aurel (Inria - Institut de Mathématiques de Bordeaux)
Palmer-Anghel, Martin (Mathematical Institute of the Romanian Academy)
Patzt, Peter (Copenhagen University/ University of Oklahoma)
Pedersen, Jonathan (University of Toronto)
Pelletier, Hector (École polytechnique fédérale de Lausanne)
Peng, Keyao (Grenoble Alpes University)
Piterman, Kevin (Universidad de Buenos Aires)
Pol, Luca (Universität Regensburg)
Powell, Geoffrey (Universit d'Angers)
Raksit, Arpon (MIT)
Randal-Williams, Oscar (University of Cambridge)
Reiner, Victor (University of Minnesota)
Sam, Steven (UC San Diego)
Scheiwiller, Lukas (University of Michigan)
Schwermer, Joachim (Universität Wien)
Sengun, Haluk (University of Sheffield)
Serrano Garcia, Higinio (CINVESTAV, México)
Sharifi, Romyar (University of California, Los Angeles)
Sierra, Ismael (University of Cambridge)
Soroko, Ignat (Florida State University)
Souli, Arthur (University of Glasgow)
Spitler, Ryan (Rice University)
Stark, Christopher (NSF)
Stavrou, Andreas (University of Cambridge)
Stevens, Glenn (Boston University)
Stoll, Robin (Stockholm University)
Studenmund, Daniel (Binghamton University)
Sytilidis, Filippos (University of Oxford)
Thalagoda, Kalani (UNC Greensboro)
Tommasi, Orsola (University of Padova)
Tosteson, Philip (University of Chicago)
Vespa, Christine (Universit de Strasbourg)
Vogtmann, Karen (University of Warwick)
Wahl, Nathalie (University of Copenhagen)
Wawrykow, Nicholas (University of Michigan)
Westerland, Craig (University of Minnesota)
Wong, Tian An (University of Michigan-Dearborn)
Wortman, Kevin (University of Utah)
Xicotencatl, Miguel A. (Centro de Investigación y de Estudios Avanzados)
Yao, Zijian (University of Chicago)
Yost-Wolff, Calvin (University of Michigan)
Yun, Claudia (Brown University)
Zou, Foling (University of Michigan)

## Participant Feedback

The conference was well-received. Following the conference we issued an online survey. We received 15 responses and participant feedback was almost entirely positive.

One open-ended survey question asked participants whether or how they benefitted from the conference. Here is a summary of their responses:

- Many participants commented that they benefitted from the breadth of the workshop. They learned about current research and met participants from outside of their own subfield.
- Several participants, including graduate students, commented that they found the talks accessible and informative.
- Two respondents commented that they had begun new projects or new collaborations as a direct result of the workshop. Others commented that the workshop gave them new ideas for projects or directions they may pursue in future.

Below are some excerpts (quoted word-for-word) from the responses to this question.

*"I thought it [the workshop] went well. It is really difficult to bring two fields together in one conference, but as far as I could tell the number theorists and topologists actually talked to each other!"*

*"All the talks I 'went' to were excellent and helped me keep current with research in my field."*

*"Although I am almost out of the field, many [speakers] thankfully reviewed some backgrounds which benefit me a lot. I work on compute motivic cohomology of algebraic groups, and they do have some connection with representation theory. And I am also impressed a lot with all kinds of moduli spaces which play important roles."*

*"The talks were interesting and opened up my horizons in mathematics. This may give me potential new directions in my research."*

*"I started a new collaboration with Jeremy Miller, Peter Patzt, and Jenny Wilson on a conjecture with Rognes. I also started working on a project involving the low-dimensional cohomology of congruence subgroups of* $\mathrm{SL}(n, \mathbb{Z})$*, inspired by Calegari's talk. I learned many other new things as well, but these are the most concrete and easily-described benefits to me personally."*

*"I learned a lot from the many of the talks. I also received useful feedback on my own work after having given my talk."*

*"I made new professional connections. I got new ideas for projects, one of which I am currently pursuing with another participant of the conference."*

*"Yes, I really enjoyed the opportunity to listen to mathematicians with very different perspectives. It's not very often that I go to conferences with number theorists, for example, and I wasn't sure how much I'd be able to follow, but all the speakers did a great job in addressing a broad audience. I learned a lot, and plan to rewatch several of the talks."*

*"Discussed some problems I'd been thinking about with other participants and got some enlightening perspectives, and generally learned a lot from the talks!"*

*"The workshop was very helpful for me, both for continuing the work on my current research projects and for getting a better overview of the bigger picture. The talks, but most of all the exchange with the other participants were very beneficial. I found it in particular very interesting to see what type of questions people with a slightly different background (especially Voronoi tessellations & number theory) were interested in in relation to the topics of the meeting. Without this workshop, I would most probably not have met them or attended one of their talks."*

*"It was an opportunity to see the whole spectrum of work being done in this area. I did not know so many different topics were being explored. I have been working for some years now in one area (subgroups of* $\mathrm{SL}(n, \mathbb{Z})$*), but I have been branching out to get back to another area I worked on in my postdoc days (subgroups of* $\mathrm{Sp}(4, \mathbb{Z})$*). The talks at this conference about Sp were especially helpful, because they showed me some things I did not expect about the way that field was going."*

*"As a beginning graduate student, the talks helped me get an overview of what problems people are working on and what they consider interesting and within current reach."*

*"I found the talks informative and engaging. The workshop suggested new research problems that would be interesting to pursue."*

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# Chapter 9

# Topology and Entanglement in Many-Body Systems (21w5502)

## October 11 -15, 2021

**Organizer(s):** Sven Bachmann (UBC), Fernando Brandao (Caltech), Martin Fraas (UC Davis)

After being postponed in the hope of the possibility of an in-person participation followed by a months-long hesitation about the possibility of a hybrid workshop, the meeting was moved to fully online about one and a half month before its starting date. The participation increased from 21 to a final 48 registered members.

## Overview of the Field

The topic of the workshop was interpreted in a broad way, which was reflected in the diversity of the participants. These included physicists and mathematicians working on quantum information, quantum matter, quantum dynamics, and operator algebras. The workshop showcased intense exchanges between these fields, in particular the fact that the boundary between mathematics and physics is more psychological than real. Quite generally, the workshop addressed the the emerging understanding of correlated (more precisely, entangled) phases of quantum matter at zero temperature that is based on methods of (algebraic) topology and operator algebras. What started in the 1980's as a beautiful but isolated case of topological matter, namely the quantum Hall effect, is now just one example of a large class of topological states. The talks revolved around one central question: how do we classify topological states of matter in a physically meaningful and mathematically stable way?

## Presentation Highlights

The opening presentation by Amanda Young [10] discussed the topic where it all started, namely the quantum Hall effect, but explained a proof of a famous conjecture (that is about as old as the subject) of the existence of a spectral gap for a model of the fractional quantum Hall effect. The following talk by Anurag Anshu [1] was about a breakthrough proof of another long standing conjecture of what is known as an area law for the entanglement entropy, and it used new ideas in polynomial approximations of projectors. Tuesday's talks took place in the Canadian evening to accommodate for Japanese speakers. The first talk by Yoshiko Ogata [7] offered a very general C*-algebraic construction of an index characterizing topological phases with the presence of a group symmetry, one of the outstanding results of the past years in mathematical physics. The second talk by Kohtaro Kato [9] offered a completely new understanding of topological phases that relies only on the entanglement present in the state, in a perfect match with the title of the workshop. Wednesday's presentations concentrated on the stability of properties of topologically ordered phases. In the first talk, Bruno Nachtergaele [6] addressed the stability of the spectral gap above the ground state energy under perturbations, a key property in any form of classification of phases. Michael Levin's presentation [5] followed with the stability of gaps and small energy splitting for specific models but under a large class of long range perturbations; the methods involved a new twist on cluster expansions used in statistical mechanics. Another fascinating property of topological phases is the bulk-boundary correspondence which took center stage in Thursday's first talk by David Pérez-García [2], who also provided an excellent overview on tensor networks and their use in quantum mechanics. The last three talks of the workshop came back to the classification problem. Jeongwan Haah [3] presented an ambitious and general classification of a certain class of quantum dynamics known as cellular automata, or quantum circuits in information theory. On Friday, quantum cellular automata were again at the heart of Michael Walter's talk [8], which featured a proof of a complete classification of a very general class of automata in one dimension. To close the workshop, Nikita Sopenko [4] presented a very ambitious classification scheme for quantum states which relied on the construction of new chain complexes of operators in quantum lattice systems.

All ten talks contained results of groundbreaking nature on current topics relevant to physics, mathematics and quantum information theory. The growing understanding of entanglement structures of ground states of quantum systems, both in the presence and absence of group symmetries, was very explicit throughout the workshop.

## On the online format

The online format was accepted only reluctantly both by the organizers and the participants. The result was positively surprising for all. As a participant put it: 'I felt like I learned a lot. It was honestly one of the best online programs I have participated in.' One aspect probably contributing to an active participation was the light schedule with only two talks per day which avoided Zoom fatigue. The Q&A sessions immediately following the talks and being as long as the talks themselves turned out to be rich, lively, and they gave the participants a real opportunity to exchange ideas and raise interesting issues.

## Scientific Progress and Outcome of the Meeting

The online format certainly made it difficult (in fact impossible) to start new active collaborations – which would be one of the goals of a BIRS workshop otherwise. However, the meeting went far beyond a mere sequence of talks and its major outcome is the renewed affirmation of a community of scientific goals across members coming from a broad spectrum of research fields and institutions. Given (i) the very well crafted talks and (ii) the real possibility of asking questions and participating in discussions, all member of the workshop learned new results and had a chance to understand the new techniques being developed to understand topological quantum matter. The fusion of topology, algebra and analysis displayed during the workshop has only started to provide new insights and understanding of physics and information theory at low temperatures.

## Participants

**Alhambra, Alvaro** (MPQ)
**Anshu, Anurag** (UC Berkeley)
**Artymowicz, Adam** (Caltech)
**Avron, Yosi** (Technion)
**Bachmann, Sven** (The University of British Columbia)
**Bols, Alex** (Copenhagen University)
**Cirac, Juan Ignacio** (Max-Planck-Institut für Quantenoptik)
**De Roeck, Wojciech** (KU Leuven)
**Du, Zhiqian** (UC Davis)
**Elgart, Alexander** (Virginia Tech)
**Fidkowski, Lukasz** (University of Washington)
**Fraas, Martin** (UC Davis)

**Furlotte, Justin** (University of British Columbia)
**Glubokov, Andrey** (Purdue University)
**Gosset, David** (University of Waterloo)
**Graf, Gian Michele** (ETH Zurich)
**Haah, Jeongwan** (Microsoft Research)
**Hannani, Amirali** (KU Leuven)
**Hingorani, Rahul** (UC Davis)
**Jackson, Thomas** (UC Davis)
**jappens, tijl** (KU Leuven)
**Kapustin, Andrey** (Caltech)
**Kato, Kohtaro** (Nagoya University)
**Lange, Markus** (SISSA)
**Lemm, Marius** (University of Tübingen)
**Levin, Michael** (University of Chicago)
**Moon, Alvin** (Copenhagen)
**Nachtergaele, Bruno** (University of California, Davis)
**Ogata, Yoshiko** (The University of Tokyo)
**Oshikawa, Masaki** (University of Tokyo)
**Pérez-García, David** (Complutense University of Madrid - ICMAT)
**Porta, Marcello** (SISSA, Trieste)
**Prosniak, Oskar** (KU Leuven)
**Ragone, Michael** (UC Davis)
**Ranard, Daniel** (Massachusetts Institute of Technology)
**Raussendorf, Robert** (University of British Columbia)
**Schuch, Norbert** (University of Vienna)
**Schulz-Baldes, Hermann** (Universität Erlangen-Nürnberg)
**Sopenko, Nikita** (Caltech)
**Tauber, Clément** (Université de Strasbourg)
**Vadnerkar, Siddharth** (UC Davis)
**Walter, Michael** (University of Amsterdam)
**Warzel, Simone** (Technical University of Munich)
**Wild, Dominik** (Max-Planck Institute of Quantum Optics)
**Witteveen, Freek** (University of Amsterdam)
**Yang, Bowen** (Caltech)
**Young, Amanda** (Techincal University of Munich)
**Zhang, Carolyn** (University of Chicago)

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[4] Anton Kapustin and Nikita Sopenko. Hall conductance and the statistics of flux insertions in gapped interacting lattice systems. Journal of Mathematical Physics, 61(10):101901, 2020.

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# Chapter 10

# Statistical Aspects of Non-linear Inverse Problems (21w5009)

## November 1 - 5, 2021

**Organizer(s):** François Monard (University of California Santa Cruz), Gabriel P. Paternain (University of Cambridge), Angkana Rüland (Heidelberg University)

## Main Objectives of the Workshop

One of the main objectives of the workshop "Statistical aspects of non-linear inverse problems" was to bring together international experts working at the interface of inverse problems, statistics, numerics, the analysis of partial differential equations and to foster the interaction between these fields. Due to the ongoing pandemic, the workshop was held in an online format. As a response to this, the number of initially envisioned talks was reduced, with a final programme covering three talks per day at the intersection of the European and North American working time zones. In spite of the reduced interaction possibilities due to the online format, the workshop was very successful in connecting the different communities, with many lively, exciting and community-overarching discussions, raising important mathematical questions and offering various different perspectives. The speakers included both highly renowned experts in their various fields and young researchers working at the intersection of the fields. Despite the reduced number of talks, the workshop essentially covered all the topics from the original proposal.

## Main Topics of the Workshop

Overall, the talks and topics of the workshop can be grouped into the following closely connected and synergetic research areas with many talks at the interface of several of these fields. In chronological order in each area (with intersections of the areas for many talks) these can be grouped as follows:

- Statistical and probabilistic aspects of inverse problems. Richard Nickl gave an overview on various questions and challenges (consistency, uncertainty quantification and computability) in the study of Bayesian nonlinear inverse problems and recent break-throughs in this field [6, 4, 5]. Judith Rousseau discussed estimation aspects and Bayesian nonparametric methods for a deconvolution problem and a nonparametric hidden Markov model. Considering the example of Radon transforms in different settings, Plamen Stefanov used microlocal defect measures to analyze how the power spectrum of noise is transformed by the action of an FIO, as a way to describe the noise content in a reconstruction after inverting a linear, noisy inverse problem [8]. Motivated by seeking to provide improved computer-based speech, Samuli Siltanen showed how Bayesian inversion of a blind deconvolution problem and MCMC can be used to provide significantly improved glottal inverse filtering. With problems from fluid dynamics as a starting point, Nathan Glatt-Holtz provided statistical sampling and consistency results for these, and presented a unified framework for a large family of Hamiltonian-based MCMC sampling schemes. Hanne Kekkonen presented consistency results for Bayesian inference in a parabolic inverse problem.

- Deterministic PDE-based (non-linear) inverse problems. Mikko Salo presented a unified framework on instability mechanisms in inverse problems covering general geometries, low regularity and microlocal regularity settings [3]. Barbara Kaltenbacher discussed all-at-once formulations of inverse problems in a minimization framework incorporating regularization. In his talk, Giovanni S. Alberti presented general stability, uniqueness and recovery results for various nonlinear inverse problems with finite measurements in which the unknowns lie in finite-dimensional submanifolds, the latter possibly generated by recent neural-network models satisfying certain injectivity assumptions. Jan Bohr presented range characterization results for non-linear X-ray transforms [1] and stability results for nonlinear-inverse problems including global and local conditions for log-concave approximations.

- Numerical analysis of statistical inverse problems and machine learning. Bambdad Hosseini presented a rigorous framework for the solution of nonlinear PDEs based on Gaussian processes generalizing kernel methods. Christoph Schwab presented multilevel approximation of Gaussian random fields on Riemannian manifolds, and possibilities of covariance compression and estimation [2]. Youssef Marzouk discussed a dimension reduction technique for Bayesian inverse problems with nonlinear forward operators by approximating the likelihood function by a ridge function. Motivated by progress in computational physics and chemistry, Robert Scheichl presented a sampler based on low-rank approximation in tensor-train format. Sven Wang presented non-asymptotic computational guarantees for Langevin-type MCMC algorithms with polynomial scaling in the relevant quantities, with applications to an inverse problem for Schrödinger's equation [7].

## Outcome of the Meeting

All in all, the combination of the presentations and discussions at the workshop led to an excellent mix of backgrounds and many novel perspectives on these problems. It successfully brought together many international experts working at the intersection of these areas whose different expertise led to exciting perspectives and questions. Several inspiring discussions following the talks reflected that new connections were created between expert scientists, paving the way for longer-term outcomes such as novel interdisciplinary projects and collaborations.

## Participants

**Abraham, Kweku** (University of Cambridge)
**Agapiou, Sergios** (University of Cyprus)
**Alberti, Giovanni S.** (University of Genoa)
**Altmeyer, Randolf** (University of Cambridge)
**Beretta, Elena** (NYU Abu Dhabi)
**Bohr, Jan** (University of Cambridge)
**Bubba, Tatiana** (University of Bath)
**Caddel, Brynn** (University of California Santa Cruz)
**Caro, Pedro** (Basque Center for Applied Mathematics)
**Castillo, Ismaël** (Sorbonne Université)
**Covi, Giovanni** (TK)
**Cristina, Butucea** (Paris)
**Dashti, Masoumeh** (University of Sussex)

**de Hoop, Maarten** (Rice University)
**Deo, Neil** (University of Cambridge)
**Flynn, Steven** (University of Bath)
**Frederiksen, Christian** (Tulane University)
**Garcia Ferrero, Maria Angeles** (BCAM Bilbao)
**Giordano, Matteo** (University of Oxford)
**Glatt-Holtz, Nathan** (Tulane University)
**Glubokov, Andrey** (Purdue University)
**Helin, Tapio** (LUT University)
**Hoffmann, Marc** (Paris)
**Hohage, Thorsten** (University of Göttingen)
**Hosseini, Bamdad** (California Institute of Technology)
**Ilmavirta, Joonas** (University of Jyväskylä)
**Kaltenbacher, Barbara** (University of Klagenfurt)
**Kato, Kengo** (Cornell University)
**Kay, Jonathan** (University of California Santa Cruz)
**Kekkonen, Hanne** (Delft University of Technology)
**Krometis, Justin** (Virginia Tech)
**Krupchyk, Katya** (University of California, Irvine)
**Lassas, Matti** (University of Helsinki)
**Mammen, Enno** (Heidelberg)
**Marzouk, Youssef** (Massachusetts Institute of Technology)
**Monard, Francois** (University of California Santa Cruz)
**Nelsen, Nicholas** (California Institute of Technology)
**Nickl, Richard** (University of Cambridge)
**Nisbet, Alex** (Tulane University)
**Oksanen, Lauri** (University of Helsinki)
**Paternain, Gabriel** (University of Cambridge)
**Railo, Jesse** (University of Cambridge)
**Ray, Kolyan** (Imperial College London)
**Reich, Sebastian** (Universität Potsdam)
**Reiß, Markus** (Universität zu Berlin)
**Rousseau, Judith** (Oxford University)
**Rüland, Angkana** (University of Heidelberg)
**Saksala, Teemu** (North Carolina State University)
**Salo, Mikko** (University of Jyväskylä)
**Santacesaria, Matteo** (University of Genoa)
**Scheichl, Robert** (Heidelberg University)
**Schmidt Hieber, Johannes** (University of Twente)
**Schmidt-Hieber, Johannes** (University of Twente)
**Schwab, Christoph** (ETHZ)
**Siltanen, Samuli** (University of Helsinki)
**Somersalo, Erkki** (Case Western Reserve University)
**Spokoiny, Vladimir** (WIAS and Humboldt University Berlin)
**St-Amant, Simon** (University of Cambridge)
**Stefanov, Plamen** (Purdue)
**Strauch, Claudia** (Aarhus University)
**Szabo, Botond** (Vrije Universiteit Amsterdam)
**Teckentrup, Aretha** (University of Edinburgh)
**Trabs, Mathias** (Hamburg)
**Tsybakov, Alexandre** (Institut Polytechnique de Paris)
**Uhlmann, Gunther** (University of Washington and HKUST)

**van de Geer, Sara** (Eidgenössische Technische Hochschule Zürich)
**Wang, Sven** (MIT)
**Wang, Yiran** (Emory University)
**Xi, Chen** (Fudan University)
**Zech, Jakob** (Universität Heidelberg)
**Zhou, Hanming** (UC Santa Barbara)
**Zou, Joey** (University of California Santa Cruz)

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# Chapter 11

# Quantum Field Theories and Quantum Topology Beyond Semisimplicity (21w5121)

October 31 - November 5, 2021

**Organizer(s):** Thomas Creutzig (University of Alberta), Nathan Geer (Utah State University), Christoph Schweigert (Universität Hamburg)

## Overview of the Field

The interaction between quantum field theory and topology has many exciting facets, ranging from index theorems, higher categorical structures to quantum topology which is by now a well-established subfield of mathematics. This field has connections to a wide range of topics, including quantum groups, subfactors, logarithmic conformal field theory, Khovanov homology, loop groups, topological quantum field theory, Vassiliev invariants, and quantum computing.

A principal feature of quantum topology is its interplay between algebra and low-dimensional topology. A central algebraic notion in this context is the notion of a modular tensor category. These are braided tensor categories where the braiding obeys a non-degeneracy condition.

While semisimple modular tensor categories have been intensely used in quantum topology for more than two decades, non semisimple modular tensor categories have come to the fore more recently; we have seen important progress, both for their theory and their applications. The quintessential example of a modular category is the one of modules over the restricted or the small quantum group associated to a simple Lie algebra. These categories are non semisimple, and many series of examples are by now known. Another source of modular tensor categories – semisimple and non semisimple are representation categories of vertex algebras obeying certain finiteness conditions.

A vertex operator algebra, VOA for short, is the chiral algebra of a two dimensional conformal field theory. VOAs have numerous applications in mathematics, ranging from monstrous moonshine, topology and geometry to probabilistic processes as SLE. (In the latter example, the relevant tensor category is known to be non semisimple.)

## Recent Developments and Open Problems

In 2013, Beem, Rastelli and collaborators realized that sectors of protected local operators of four dimensional supersymmetric quantum field theories are described by VOAs [14]. Over the last eight years this has grown into an active interaction between theoretical physicists and mathematicians and numerous exciting new insights in the theory of vertex algebras as well as physics have been made. The VOAs are meaningful invariants of the higher dimensional quantum field theories and more refined structure is encoded in representation theoretic data, as e.g. Higgs and Coulomb branches and categories of line operators. In particular physics considerations resulted in many deep conjectures [14, 15, 35, 21, 17, 18, 19, 33] that are by now partially proven and have led to a substantial advancement of the area of vertex operator algebras [1, 2, 3, 44, 4, 24, 25].

Non-modular representation categories of non-rational VOAs and their meaning in topology play a central role here. A highlight is surely the recent proposal by Feigin and Gukov that VOAs together with their representation categories provide new highly interesting invariant of four-manifolds with boundary [34].

Nonsemimple modular categories (and their "unrolled" cousins) have features that correspond to interesting and useful applications in topology and quantum field field theory. While a satisfactory mathematical formulation of general quantum field theories does not et exist, topological quantum field theories, conformal quantum field theories and various sectors of supersymmetric quantum field theories have proved amenable to rigorous mathematical analysis. Indeed, these types of QFT's have been at the heart of the highly successful interactions between mathematics and physics of the past few decades, see for example [36, 53, 47, 56, 46].

In recent years, many mathematicians have been working on these aspects. We mention here a few more fields that are involved: non-semisimple quantum topology was pushed by a group headed by Christian Blanchet, Francesco Constantino and Nathan Geer, see [10, 11, 12, 22, 23, 28, 37, 38, 39]. Vertex algebras as a source of non semisimple categories have been pioneered in particular by the work of Drazen Adamovic, Thomas Creutzig, Antun Milas, David Ridout, Simon Wood and others, for example see [16, 5, 55]. In particular first examples of non semisimple and non finite vertex tensor categories were recently realized [9, 26, 27]. Finite, but non semisimple tensor categories was developed by Pavel Etingof, Dmitri Nikchysh, Victor Ostrik, Christoph Schweigert, Jürgen Fuchs and many others [30, 29, 31, 54].

One central feature of QFT's is that they can be viewed from many different perspectives — say at different energy scales, or using different fundamental degrees of freedom. A single physical object, such as partition function, may then be formulated in many different but equivalent ways. This provides an overwhelming advantage for any mathematical invariant realized in QFT, often leading to the prediction of many unexpected properties and relations to other structures. In particular, quantum topology coming from non-semi-simple algebra is ripe for further development and synthesis with other quantum field theories.

## Presentation Highlights

The conference focused on the interactions of non-semi-simple quantum topology, Conformal Field Theories (CFTs) and mathematical physics interpretations of these fields. In particular, the organizers invited participants in five major areas and setup the conference so participants could share and investigate relations between these areas. Here is a list of the participants organized into their particular area:

1. New 3d TQFTs from mathematical physics
   Miranda Cheng, Tudor Dimofte, Boris Feigin, Sergei Gukov, Du Pei

2. Non-semi-simple quantum topology:
   Anna Beliakova, Christian Blanchet, Francesco Constantino, Nathan Geer, Andrey Glubokov, Rinat Kashaev, Cristina Palmer-Anghel, Bertrand Patureau-Mirand, Nicolai Reshetikhin, Vladimir Turaev

3. Logarithmic Conformal Field Theories
   Jürgen Fuchs, Azat Gainutdinov, Simon Lentner, Ingo Runkel, Christoph Schweigert, Yorck Sommerhauser, Lukas Woike, Ana Ros Camacho

4. Vertex Operator Algebras
   Drazen Adamovic, Tomoyuki Arakawa, Thomas Creutzig Terry Gannon, Shashank Kanade, Robert McRae, Antun Milas, Andrew Linshaw, David Ridout, Simon Wood, Jinwei Yang

5. Supersymmetric quantum field theories and algebraic structures
   Christopher Beem, Davide Gaiotto, Jörg Teschner, Julia Plavnik, Emily Cliff, Matthew Young.

The conference also had many early career participants: Andrew Ballin, Matthieu Faitg, Julian Farnsteiner, German Garces, Naoki Genra, Brandon Gill, Li Han, Naoki Genra, Aniket Joshi, Bryan Kettle, Christian Korff, Niklas Garner, Wenjun Niu, Shigenori Nakatsuka, Florencia Orosz Hunziker, Tomas Prochazka, Miroslav Rapcak, Andrew Riesen, Iordanis Romaidis, Mathew Rupert, Andrew Schopieray, Devon Stockall, Matthias Traube, Thomas Voss and Yang Yang.

We list these participants and their areas because it shows diversity of the research of the participants. One of the very positive outcomes of the conference is that diverse groups were able to interact and gain new perspectives of their research by comparing it with different areas of mathematics. The organizers are very pleased with this outcome and optimistic it will lead to interesting future research. It also contained educational components for the early career participants.

The conference had three main components: 1) introductory talks, 2) research level talks and 3) discussion sections. All three of this built upon each other. The introductory talks gave an overview of a field while also highlighting some open problems. The research talks built upon these talks and were given by leading researchers in the field. Then the discussion sections allowed for time to exchange views and ideas about the topics of the research talks. This also gave a place for junior researchers to ask questions and share ideas. There were five graduate students and one postdoc attending the conference in person. All six gave additional in person talks in the evenings. Next we discuss each of these main components in more detail.

## Introductory talks

To provide background and training for junior researchers, we had five introductory talks, by:

- Sergei Gukov (California Institute of Technology)

  Title: 3d Spin$^c$ TQFT from quantum groups at generic q: an overview.

  Summary: This talk gave an overview over his recent work on new $q$-series invariants of 3-manifolds labeled by Spin$_c$ structures. These invariants have unexpected properties and are related to the characters of logarithmic vertex algebras. In a specific limit, they can be related to the CGP theory defined in [11, 22], which was also the topic of the talks by Costantino, Gainutdinov and Runkel.

  Abstract: This talk will be a broad survey of recent work on new q-series invariants of 3-manifolds labeled by Spin-C structures. While the original motivation for studying these invariants is rooted in topology, they exhibit a number of unexpected properties and connections to other areas of mathematics, e.g. turn out to be characters of logarithmic vertex algebras. When q tends to a root of unity, this Spin-C TQFT can be related — via a version of the Fourier transform that physicists may naturally call ”bosonization” and that exchanges 0-form and 1-form symmetries — to the BCGP theory, due to Blanchet, Costantino, Geer and Patureau-Mirand.

- David Ridout (University of Melbourne)

  Title: Relaxed modules and logarithmic CFT

Summary: The talk reviewed the representation theory of admissible-level affine vertex operator algebras beyond highest weight modules, focusing on so-called relaxed highest-weight modules. He explained why consistency of constructions in conformal field theories can be only achieved if this class of modules is included. Many following talks and the discussions referred to these lectures and Adamovic's talk was about the precise realization of relaxed-highest weigh modules and their extensions.

Abstract: The paradigm of rational (or log-rational) conformal field theory is intimately entwined with highest-weight theory for the associated vertex operator algebras. However, there are many natural examples of VOAs for which the consistency conditions of CFT require one to look beyond the highest-weight module category. I will review some recent work on examples, including the admissible-level affine VOAs of $sl(2)$ and describe the central role played by the so-called relaxed highest-weight modules.

- Tomoyuki Arakawa (RIMS, Kyoto University)

  Title: 4D/2D duality and VOA theory: an overview.

  Summary: The talk reviewed the duality relating four-dimensional supersymmetric gauge theories to vertex algebras, discovered by Beem and collaborators, from the point of view of vertex algebras. In particular, he explained why the corresponding VOA necessarily has non-semisimple representations, unless the four-dimensional gauge theory has a zero Higgs branch. Connections between the principal block of small quantum groups and of admissible level affine vertex algebras was pointed out, generalizing an observation stated in the lecture of David Ridout. Correspondences of quantum group and VOA categories was then later the topic of the talks by Rupert and Lentner.

  Abstract: The 4D/2D duality discovered by Beem et. al associates a VOA to any 4D $N = 2$ SCFT. The corresponding VOA has non-semisimple representations unless the 4D theory has zero Higgs branch. In this talk I overview the 4D/2D duality especially in the viewpoint of VOA theory.

- Nicolai Reshetikhin (University of California at Berkeley and Tsinghua University, Beijing)

  Title: Quantum groups at roots of unity and topological invariants

  Summary: The talk gave an overview of quantum groups at a root of unity and their relation to topological invariants, emphasizing open problems. He emphasized two aspects: quantum groups with large center and the construction of corresponding invariants of 3-manifolds with flat connections and the relation of these invariants with conformal field theories and vertex algebras. These structures are the quantum topology versions of physical QFTs discussed by Dimofte. This was also related to the research talks by Costantino and Feigin.

  Abstract: This talk is an overview of structures and developments related to quantum groups at a root of unity and their relation to topological invariants. The subject was developing actively over the last 30 plus years, but there are still many open problems. The study of quantum groups with large center and the construction of corresponding invariants of 3-manifolds with flat connections are some of the important developing directions in this area. The other, just emerging, is the relation of these invariants with conformal field theories and VOA.

- Tudor Dimofte (University of Edinburgh, on leave from University of California Davis)

  Title: Non-semisimple and derived QFT's for quantum groups at a root of unity

Summary: This talk discussed aspects of a three-dimensional topological QFT $T(G, k)$ whose braided tensor category of line operators is (conjecturally) isomorphic to the derived category of modules for the quantum group $U_q(g)$ at a $2k$-th root of unity. This work is a derived generalization of the non-semisimple TQFT defined in [11, 22] and a generalization of classic Chern-Simons theory with compact group $G$. This talk was related to many of the talks of the conference because it laid out a conjectural framework to relate many of the re-normalized quantum invariants, VOAs and QFTs discussed in the workshop. The preprint corresponding to this talk has just appeared [20].

Abstract: I will discuss aspects of a 3d topological QFT $T(G, k)$ whose braided tensor category of line operators is (conjecturally) isomorphic to the derived category of modules for the quantum group $U_q(g)$ at a 2k-th root of unity — and whose state spaces and partition functions provide a derived generalization of associated non-semisimple TQFT's recently constructed by Costantino, Geer, and Patureau-Mirand. The field theory $T(G, k)$ is a topological twist of a 3d N=4 Chern-Simons-matter theory, which generalizes classic Chern-Simons theory with compact group $G$, at level k, extending it to a non-semisimple and derived setting. More so, $T(G, k)$ admits chiral boundary conditions supporting the Feigin-Tipunin VOA's based on $g$, generalizing the appearance of the chiral WZW model in Chern-Simons theory. Supersymmetric localization allows for simple calculations of some observables/invariants when $G = SU(n)$.

## Discussion Sections

We had three intensive discussion sessions, chaired by Christian Blanchet (Paris) and Jörg Teschner (Hamburg). These discussion sessions were very useful because they supplemented and gave a place for participants to continue discussing the topics of the normal talks of the conference. In particular, they allowed participants to discuss how results of the talks were related to each other. These discussions contained interactions which led to smaller discussions which are likely to lead to collaborative work.

The discussion by Christian Blanchet focused on renormalized invariants and TQFT and what future directions this area can go. In particular, at the beginning of the discussion Christian listed the following four topics:

- Renormalized invariants arising from new non-semi-simple categories (including super Lie algebras),
- Q-series invariants,
- Asymptotics,
- Renormalized invariants with new target spaces, including using the work of [48].

Within each of these topics Christian had questions for particular researchers. For example, he asked Sergei Gukov about the Q-series invariants [41, 42, 43]. These invariants are q-series with integer coefficients, which are predicted by physics but only well defined for certain 3-manifolds. The discussion lead to interesting questions about how the Q-series invariants are related to renormalized invariants of [22, 13]. Partial relationships have been considered in [40]. The discussion talked about how to consider this work and what needed to be done next.

Also, Nathan Geer and Cristina Anghel were asked about the invariants recently defined in [7, 8] this lead to a discussion of possible physical interpretations and relations with the super-group Chern-Simons theories given in [6, 51, 52]. This discussion sparked questions from people in both physical QFTs and the VOA area. The organizes were told these questions have been followed up with further smaller group questions that will likely lead to new collaborations.

Finally, in Christians discussion session questions were asked about the notion of a derived modular functor that had been recently developed in work by Lentner, Mierach, Sommerhäuser, Schweigert [45] and Schweigert, Woike, [48, 49, 50]. Lukas Woike (Copenhagen) was spontaneously ask to give a presentation on the derived Verlinde formula that relates various structures of an $E_2$-algebra on the Hochschild complex of a non-semisimple modular tensor category.

In Jörg Teschner's discussion sections the various appearances of VOAs and their representation categories in three and four dimensional quantum field theories were discussed. Jörg started the discussions with an introduction and then experts as Dimofte, Gaiotto and Creutzig continued with explanation. Topics of particular interest were the derived categorical structure that physics requires and thus has to be understood better mathematically. Another important topic was the appearance of the Feigin-Tipunin algebras (most important VOAs with non semi simple representation theory) as limits of families of VOAs that are localized at corners of topological boundary conditions in the gauge theory. This was also the subject of Boris Feigin's research talk, a key part of Dimofte's lecture and will be explained more in coming work of Feigin and Lentner.

## Research talks

The conference provided opportunities for those conducting cutting-edge researchers in all fields to present their results. Here is a brief description of the talks presented at this workshop, presented in the order they were given. We refrain from giving references and refer instead to the material available on the BIRS webpage.

Due to the pandemic, travel restrictions applied so that only seven particpants, all from Alberta, could participate directly in Banff. All other participants were online participants, based in Australia, Asia, Europe and Northern America. The daily program took up to seven hours; due to the different time zones, this was a major challenge to many participants. It was generally felt that a similar workshop with everybody present in Banff could be very helpful in the future.

Francesco **Costantino** (Toulouse University) presented between the non-semisimple invariants associated to the unrolled version of quantum $sl_2$ at roots of unity and the BPS series invariants that were the topic of Gukov's talk.

Simon **Wood** (Cardiff University) explained new duality structures in the representation categories of $C_2$-cofinite vertex algebras, called Grothendieck-Verdier dualities. He illustrated these structures in the case of Heisenberg and lattice vertex algebras.

Drazen **Adamovic** (University of Zagreb) discussed non-semisimple categories of modules for affine vertex (super)algebras. He focussed on the construction of projective modules and methods based on free field realizations and quantum Hamiltonian reductions.

Boris **Feigin** (Higher School of Economics, Moscow) Vertex algebras "with big center", logarithmic theories and bundles of vertex algebras. Vertex algebras with a large center can be realized as semiclassical limits of deformable families of VOAs. These algebras appear at the corners of topological boundary conditions in $N = 4$ super Yang-Mills theory in four dimensions. Boris Feigin explained this construction and a supposedly equivalent realization as cohomologies of certain bundles of vertex algebras over flag manifolds.

Rinat **Kashaev** (Université de Genève) explained how the the reciprocal of the Alexander polynomial of a knot can be viewed as a universal quantum invariant that is associated to the Hopf algebra of regular functions on the group of affine linear transformations of the complex plane. This fact is consistent with the Melvin-Morton-Rozansky conjecture about the relation of the colored Jones polynomials to the reciprocal of the Alexander polynomial that was proven by Bar-Nathan and Garoufalidis.

Miroslav **Rapcak** (UC Berkeley) explained recent developments concerning the representation theory of the $W_\infty$ algebra. Using its charge-conjugation automorphism in the language of affine Yangians, he showed that there is a new class of affine Yangian modules with non-diagonalizable action of Cartan generators which has striking connections to Pandharipande-Thomas invariants.

Jürgen **Fuchs** (Karlstads universitet) explained the relation between chiral and local conformal field theory, a topic that was also important in Davide Gaiotto's talk. In particular he showed that pivotal module categories provide a source of symmetric Frobenius algebras which are natural candidates for the bulk and boundary algebras in full conformal field theories. The bulk algebra, as well as more general defect fields, can be expressed as certain coends whose structure morphisms give in particular bulk-boundary OPEs. A complete set of OPEs is obtained that passes all genus zero consistency checks.

Cristina Palmer-Anghel (Université de Genève) constructed a unified topological model for the coloured Jones polynomials and the coloured Alexander polynomials which are both based on the quantum group $U_q(sl(2))$. To this end, she considered certain homology classes given by Lagrangian submanifolds in configuration spaces such that the $N$-th coloured Jones and the $N$-th coloured Alexander invariants come as different specialisations of a state sum of Lagrangian intersections in configuration spaces. She also presented a globalised model without state sums.

Anna **Beliakova** (Universität Zürich) explained her recent construction of quantum invariants of smooth 4-dimensional 2-handlebodies (i.e. of 4-balls with finitely many 1- and 2-handles attached) whose algebraic input datum is a (possibly non-semisimple) unimodular ribbon category $\mathcal{C}$. Whenever $\mathcal{C}$ is factorizable, the underlying invariant only depends on the boundary and the signature of the four-dimensional 2-handlebody. In the example provided by the category of finite-dimensional representations of the small quantum $sl_2$ at a root of unity $q$ of order $r \cong 0 \mod 8$, the new invariants depend on the interior of the handlebody.

Du **Pei** (Harvard University) explained how Coulomb branches of supersymmetric gauge theories which are frequently non-compact lead to topological field theories that are not semisimple. He presented several case studies where regularization provided insight into these novels topological field theories.

Ingo **Runkel** (Universität Hamburg) described three-manifold invariants that are defined via surgery presentations where the algebraic input is a possibly non-semisimple ribbon category together with a modified trace on a tensor ideal. He discussed how the invariant depends on the choice of the tensor ideal. If the ribbon category is modular and the ideal is that of projective objects, the theory comes from an universal construction taht defines a topological field theory on so-called admissible bordisms. If the input category is even semisimple, the construction was shown to produce the Reshetikhin-Turaev TQFT.

Azat **Gainutdinov** (CNRS, Université de Tours) explained how the famous Reshetikhin-Turaev-Witten construction of three-dimensional topological field theories which is based on semisimple modular tensor categories was generalized in the mid-nineties by Lyubashenko to non-semisimple modular tensor categories. Many important examples of such categories come from two-dimensional Logarithmic Conformal Field Theories and as representation categories of small quantum groups. Azat Gainutdinov showed that the non-semisimple topological field theory presented by Ingo Runkel provides mapping class group representations that (projectively) agree with those defined by Lyubashenko.

Robert **McRae** (Tsinghua University) showed how to obtain non-semisimple modular tensor categories as representation categories of vertex operator algebras that are $C_2$-cofinite and rigid with duals given by the contragredient module. He discussed strategies to prove rigidity of the categories in general and how rigidity is inherited under vertex operator algebra constructions that are known to preserve $c_2$-cofiniteness, such as tensor products, extensions, and finite solvable orbifolds.

Simon **Lentner** (Universität Hamburg) and Matthew **Rupert** (Utah State University) gave a combined talk explaining Kazhdan-Lusztig conjectures for logarithmic vertex algebras. These conjectures state that certain vertex operator algebras have the same monoidal category of representations as small quantum groups, which are in fact nonsemisimple modular tensor categories. They introduced the underlying concepts and placed the conjecture in a broader algebraic context, relating it in particular to the Andruskiewitsch-Schneider program. In the second part, they explained the proof that the category of representations of the small quantum group $u_q(sl_2)$ is characterized by its abelian structure and the existence of certain induction functors. These induction functors are readily available for the category of representations of the vertex operator algebras in question from its two free field realizations, allowing for a proof of the logarithmic Kazhdan Lusztig conjecture in this case.

Antun **Milas** (State University of New York at Albany) Characters of vertex algebras and Schur indices of rational and non-rational vertex algebras. Some of these characters agree with Schur indices of certain Argyres-

Douglas gauge theories and with $\hat{Z}$-invariants of plumbed three-manifolds.

Davide **Gaiotto** (Perimeter Institute) in the concluding talk A (non) chiral algebra wishlist, explained expectations on two-dimensional structures triggered by $S$-duality in higher-dimensional supersymmetric gauge theories. $S$-duality is a generalization of electro-magnetic duality in four-dimensional $N = 4$ supersymmetric Yang-Mills theory. Mathematically it translates to the quantum geomtruc Langlands correspondence, that is equivalences of spaces of conformal blocks of associated VOAs. Davide Gaiotto explained his expectations and observations on these dualities for full conformal field theories. This included also structures of full, local conformal field theories that had been discussed in the talk by Fuchs [32]. The talk was followed by a long discussion of Gaiotto and Teschner about these correspondences.

# Scientific Progress Made and Outcome of the Meeting

This conference focused on creating connections between different research areas and educating junior mathematicians. To make the conference successful the team used straightforward but not commonly used techniques: we invited a handful of research groups and asked a member in each group to give a introductory/pedagogical talk. Building on this talk, other members of the group gave more advanced talks. Finally, we had discussion sections to discuss possible collaborations and to answer questions from junior participants. This created many interactions between several of the research groups. It is likely these interactions will lead to future research.

This meeting was very timely and the interaction was maybe as lively as an event with many online participants can be. In particular there is great interest in the topic and in interdisciplinary collaborations. A concrete outcome is that a new subject within the area of vertex algebras is emerging with derived vertex tensor categories.

# Participants

**Adamovic, Drazen** (University of Zagreb, Faculty of Science,)
**Arakawa, Tomoyuki** (RIMS, Kyoto University)
**Ballin, Andrew** (UC Davis)
**Beem, Christopher** (Oxford University)
**Beliakova, Anna** (University of Zurich)
**Blanchet, Christian** (Université de Paris)
**Cheng, Miranda** (Harvard University)
**Cliff, Emily** (Université de Sherbrooke)
**Costantino, Francesco** (Toulouse University)
**Creutzig, Thomas** (University of Alberta)
**Dimofte, Tudor** (University of Edinburgh, on leave from University of California Davis)
**Faitg, Matthieu** (University of Hamburg)
**Farnsteiner, Julian** (University of Hamburg)
**Feigin, Boris** (Higher School of Economics - Moscow)
**Fuchs, Jürgen** (Karlstad University)
**Gainutdinov, Azat** (CNRS, Université de Tours)
**Gaiotto, Davide** (Perimeter Institute)
**Gannon, Terry** (University of Alberta)
**Garces, German** (University of Alberta)
**Garner, Niklas** (University of Washington)
**Geer, Nathan** (Utah State University)
**Genra, Naoki** (The University of Tokyo)
**Gill, Brandon** (University of Alberta)
**Glubokov, Andrey** (Purdue University)
**Gukov, Sergei** (California Institute of Technology)
**Han, Li** (Capital Normal University, China)

**Joshi, Aniket** (University of Alberta)
**Kanade, Shashank** (University of Denver)
**Kashaev, Rinat** (Université de Genève)
**Kettle, Bryan** (University of Alberta)
**Korff, Christian** (University of Glasgow)
**Lentner, Simon** (University of Hamburg)
**Linshaw, Andy** (University of Denver)
**McRae, Robert** (Tsinghua University)
**Milas, Antun** (State University of New York at Albany)
**Nakatsuka, Shigenori** (IPMU)
**Niu, Wenjun** (UC Davis)
**Orosz Hunziker, Florencia** (University of Denver)
**Palmer-Anghel, Cristina** (Université de Genève)
**Patureau-Mirand, Bertrand** (Bretagne-Sud University)
**Pei, Du** (Harvard University)
**Plavnik, Julia** (Indiana University)
**Prochazka, Tomas** (Ludwig Maximilian University of Munich)
**Rapcak, Miroslav** (UC Berkeley)
**Reshethikin, Nicolai** (University of California at Berkeley)
**Ridout, David** (University of Melbourne)
**Riesen, Andrew** (University of Alberta)
**Romaidis, Iordanis** (Hamburg University)
**Ros Camacho, Ana** (Cardiff University)
**Runkel, Ingo** (U Hamburg)
**Rupert, Matthew** (Utah State University)
**Schopieray, Andrew** (University of Alberta)
**Schweigert, Christoph** (U Hamburg)
**Sommerhauser, Yorck** (Memorial University of Newfoundland)
**Stockall, Devon** (University of Alberta)
**Teschner, Joerg** (University of Hamburg and DESY)
**Traube, Matthias** (Hamburg University)
**Turaev, Vladimir** (Indiana University)
**Voss, Thomas** (Hamburg University)
**Woike, Lukas** (University of Copenhagen)
**Wood, Simon** (Cardiff University)
**Yang, Jinwei** (University of Alberta)
**Yang, Yang** (University of Hamburg)
**Young, Matthew** (Utah State University)

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# Chapter 12

# Women in Inverse Problems (21w5035)

December 5 - 10, 2021

**Organizer(s):** Liliana Borcea (University of Michigan), Chrysoula Tsogka (University of California Merced)

## Overview of the Field

The objective of this workshop was to bring together women in the broad and vibrant field of Inverse Problems. Both established as well as early career researchers came together and discussed their recent research achievements. The workshop facilitated professional networking and created mentoring opportunities for women researchers. Our ultimate goal is to help broaden female participation in research careers in particular in the field of Inverse Problems, as well as to create new research collaborations.

Recent years have seen tremendous growth in the volumes of observational and experimental data that are being collected, stored, processed, and analyzed. This opens the door for new formulations, methods and developments for inverse problems at unprecedented scales. These new capabilities come also with unprecedented challenges. This workshop was focused on women contributions to modern mathematical and computational aspects of inverse problems.

## Reformulation of the workshop for online version

Although this workshop was originally planned to be in person, amidst the Covid-19 pandemic we were offered the option to chose between in person and online format. We opted for the online version given the quarantine requirements that were in place in Canada at the time of our decision. To organize a successful virtual meeting we followed the following principals:

- Our workshop schedule was condensed to three hours a day so that every participant could attend the workshop and not miss other important commitments during the same week in their usual working environment (e.g., teaching, committee meetings, etc). The workshop schedule was communicated well in advance and the hours were chosen early in the day so that the time be reasonable for participants from Europe.

- Offer the opportunity to every participant to give a short presentation. All participants gave a one slide presentation so as to introduce themselves and their research.

- Offer the opportunity to both senior and early career researchers to give research talks.

| | Monday December 6 | Tuesday December 7 | Wednesday December 8 | Thursday December 9 | Friday December 10 |
|---|---|---|---|---|---|
| **08.45-09.00** | Opening Remarks | Opening Remarks | Opening Remarks | Opening Remarks | Opening Remarks |
| **09.00-09.50** | Talk (everyone -1 slide intro) | M. Cheney *Passive Source Localization* | C. Schönlieb *Machine learned regularization for inverse problems: what we know and what we do not know* | N. Petra *hIPPYlib-MUQ: An Extensible Software Framework for Large-Scale Bayesian Inverse Problems Governed by PDEs* | E. Ullmann *Inverse Problem techniques in the estimation of Rare Events with PDE-based models* |
| **09.50-10.00** | Break | Break | Break | Break | Break |
| **10.00-10.50** | S. Minkoff *Using Extended Source Inversion to solve an Acoustic Transmission Inverse Problem* | A. Gelb Empirical Bayesian inference using a support informed prior | M. Kilmer *Parametric Level-sets Enhanced To Improve Reconstruction (PaLEnTIR)* | J. Mueller *The D-bar method and pulmonary imaging with electrical impedance tomography* | B. Kaltenbacher *Some inverse problems for wave equations with fractional derivative attenuation* |
| **10.50-11.00** | Break | Break | Break | Break | Break |
| **11.00-12.00** | L. Borcea *Data driven reduced order modeling for solving inverse wave scattering problems* | E. Qian *Balanced truncation for Bayesian inference*<br>G. Stuart Oil Spill Source Location using Bayesian Techniques | Panel I *L. Borcea, D. Calvetti, C. Schönlieb, Y.Ou* | A. Ma Solving Multilinear Systems Iteratively<br>Y. Yang *Computational algorithms that implicitly regularize inverse problems.* | Panel II *M. Espanol, T.Vdovina, C. Tsogka, E. Resmerita* |

Figure 12.1: Schedule for the "Women in Inverse Problems" workshop (all times are in Banff, Alberta time (MT/UTC-7)). Everyone participated in the one minute slide presentations. Talkes 1-4 (in green) on Tuesday and Thursday were reserved for Early Career Scientists. Talks by established researchers are listed in blue. Panels I and II covered topics such as "Future directions in inverse problems" and "Women in Inverse Problems - community, challenges , open discussion", respectively.

- Interactive Q&A sessions. We organized two panel sessions: the first one was on "Future directions on Inverse Problems" and the second one on "Women in Inverse Problems - community and challenges". For the second one we asked the participants to propose questions they would like to be discussed. Both panels were very successful.

## Scientific Progress Made

Some highlights from the discussion on the future directions on Inverse Problems are the following:

- It is important to work on inverse problems beyond mathematical physics, towards less traditional fields, e.g.: mathematical biology, public health, social and political sciences.
- Machine learning and in particular deep learning (DL) offers interesting opportunities for inverse problems – high quality solutions through data-adaptivity & computationally efficient algorithms. In general, there is a lack of understanding of these approaches and a need for more mathematical scrutiny.
- Uncertainty quantification plays an important role in applications and it is of prime interest to model the different sources of errors (simplification, reduction, discretization).
- Reproducibility of results is crucial for credibility. Large scale computing has a pivotal role and there is a need for robust and efficient algorithms to be developed, with interfaces for increased accessibility.

## Presentation Highlights

The schedule consisted of one hour one-slide introductions, ten talks by established researchers, four talks by early career scientists, and two panels, with two ten minute breaks as shown in Figure 12.1. In addition, to continue the fruitful discussions, the participants met on gather town after the last talk of the day. Below we give details about every activity.

1. One-slide introductions. We opened the workshop with one hour long one-slide presentations. This gave the opportunity to all participants to introduce themselves and to give a short overview of their research. The introduction revealed a vast geographic diversity, as shown in Figure 12.2.

2. Talks by established researchers.

    - Susan Minkoff (Professor, Department of Mathematical Sciences, University of Texas at Dallas), "Using Extended Source Inversion to solve an Acoustic Transmission Inverse Problem"
    - Liliana Borcea (Peter Field Collegiate Professor of Mathematics, University of Michigan), "Data driven reduced order modeling for solving inverse wave scattering problems"
    - Margaret Cheney (Yates Chair and Professor of Mathematics, Colorado State University ), "Passive Source Localization"
    - Anne Gelb (John G. Kemeny Parents Professor, Department of Mathematics, Dartmouth College), "Empirical Bayesian inference using a support informed prior"
    - Carola-Bibiane Schönlieb (Professor of Applied Mathematics, University of Cambridge), "Machine learned regularization for inverse problems: what we know and what we do not know"
    - Misha Kilmer (William Walker Professor of Mathematics, Tufts University), "Parametric Level-sets Enhanced To Improve Reconstruction (PaLEnTIR)"
    - Noemi Petra (Associate Professor, University of California, Merced), "hIPPYlib-MUQ: An Extensible Software Framework for Large-Scale Bayesian Inverse Problems Governed by PDEs"
    - Jennifer Mueller (Professor Laureate, College of Natural Sciences, Colorado State University), "The D-bar method and pulmonary imaging with electrical impedance tomography"
    - Elisabeth Ullmann (Associate Professor, Department of Mathematics, Technical University of Munich), "Inverse Problem techniques in the estimation of Rare Events with PDE-based models"
    - Barbara Kaltenbacher (Professor, Department of Mathematics, University of Klagenfurt), "Some inverse problems for wave equations with fractional derivative attenuation"

3. Talks by early career researchers.

    - Elizabeth Qian (von Kármán Instructor, Department of Computing and Mathematical Sciences, Caltech), "Balanced truncation for Bayesian inference"
    - Georgia Stuart (Peter O'Donnell, Jr. Postdoctoral Fellow, Oden Institute for Computational Engineering and Sciences, The University of Texas at Dallas), "Oil Spill Source Location using Bayesian Techniques"
    - Anna Ma (Visiting Assistant Professor, Department of Mathematics, University of California, Irvine), "Solving Multilinear Systems Iteratively"
    - Yunan Yang (Simons-Berkeley Research Fellow, Simons Institute for the Theory of Computing), "Computational algorithms that implicitly regularize inverse problems"

4. Panel I: Future directions in inverse problems. Panelists: Liliana Borcea (University of Michigan), Daniela Calvetti (Case Western Reserve University), Carola-Bibiane Schönlieb (University of Cambridge), and Yvonne Ou (University of Delaware). The aim of this panel was to discuss research problems the community (panelists) believe would be most interesting/challenging during the next decade. Each panelist prepared a few slides (5-10 minutes) about themselves and their research and addressed the following questions:

    - Which are your main research projects/research goals in the next five years?
    - Where do you think that the field (of Inverse Problems) should go?
    - What are the challenges in this field?
    - Anything else you want to add?

    The presentations were followed by a Q&A session.

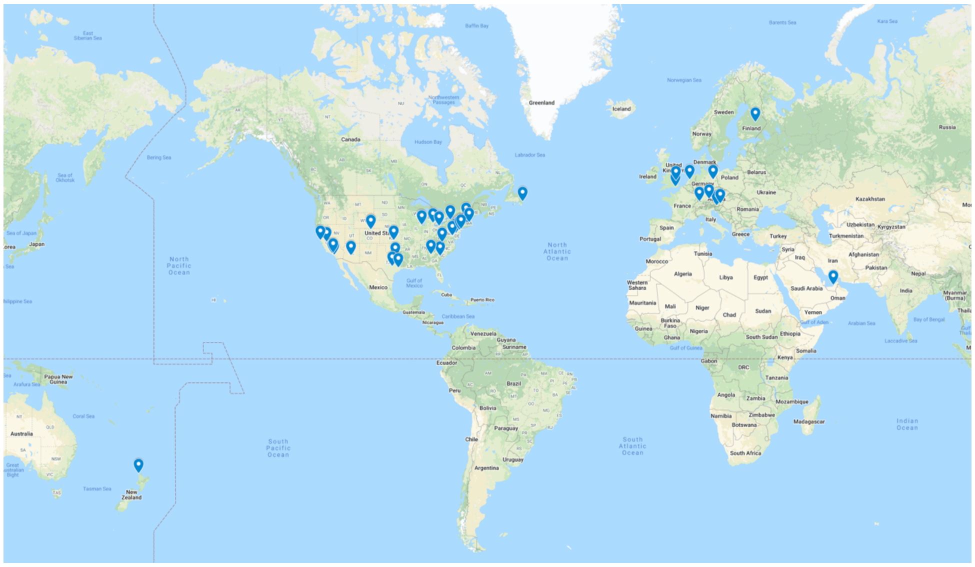

Figure 12.2: Geographic location of the “Women in Inverse Problems” participants.

5. Panel II: Women in Inverse Problems - community, challenges, open discussion. Panelists: Chrysoula Tsogka (University of California, Merced), Elena Resmerita (Alpen-Adria Universitaet), Malena Espanol (Arizona State University), and Tetyana Vdovina (Exxon Mobil). The goal of this panel was to discuss challenges that the Women in Inverse Problems community is facing. The topics were proposed by the participants via a google document form. These included:
   (a) Scientific career and family planning, work-life balance
   (b) How to write a successful grant proposal?
   (c) How to find and maintain a mentor through mid-career?
   (d) Path to tenure (“dos and don’ts”)
   (e) How to get a job in industry? How is research life in industry?
   (f) How to manage travels/conferences/workshops?

Since the panel was scheduled for one hour, a poll was created to order the topics based on interests. Based on the poll, the most interesting topics for the participants were a), b), and d). We therefore focused on these.

## Outcome of the Meeting

Below we list the outcomes of the workshop.

- All talks (except two) have been recorderd and are available on the BIRS website. We hope that these talks will be visited and watched by interested researchers in the field.
- Special issue of Inverse Problems on Women in Inverse Problems. The Guest Editors for this issue will be Elena Beretta (NYU Abu Dhabi), Noemi Petra (University of California, Merced), and Chrysoula Tsogka (University of California, Merced). The aim of this issue is to bring together and promote high-quality research by female researchers within the field of inverse problems. The usual scope of the journal will apply and papers will be refereed according to the usual high standards of the journal. The issue will be open to submissions until 30 September 2022 and the participants (by invitation) can submit manuscripts through ScholarOne Manuscripts. Accepted submissions will be published as they are ready.

- WIP: Women in Inverse Problems AWM Research Network. The goal of this research network is to keep the Women in Inverse Problems (WIP) community connected. The WIP community can forward this website to their students, postdocs and colleagues who can add themselves to the network list by clicking on the link provided in this page. For easier communication, a listserv has also been created, namely `wip at awm-math dot org`.
- AWM Workshop at SIAM Conference on Optimization (OP23). Organizers: Noemi Petra (University of California, Merced), and Chrysoula Tsogka (University of California, Merced). The events include: WM Research Mini-Symposia, Poster Session, Mentoring luncheon, and Career Panel. The organizers of the workshop will also join the AWM SIAM Committee, a subcommittee of the AWM Meetings Committee.
- Feedback from participants. In additional to the formal BIRS feedback, we also set up a google form and asked the following questions from the participants:
  - We would like to get some constructive feedback on the workshop (related to organization, content, everything). We would like to hear what worked but also what didn't, how would you change things, etc. Please feel free to share your thoughts. This will help the community organize new events perhaps better.
  - What future activities do you think the "Women in Inverse Problems" community could (or should) organize in the future?
  - There were multiple topics proposed by the community for panel II and we were able to touch only on 3-4. Obviously there is a need to create a channel where the community could ask questions and ask and offer support. One way to address this is to create a "slack workspace". Would you be interested in joining a WIP slack workspace? Or do you have any other suggestion to keep this momentum going?

  We have received several thank you notes and congratulations for organizing "an excellent meeting", for inviting a "great set of participants", for keeping it at "smallish size", several participants thought that the "talks were high quality", "enjoyed the panels and panelists", etc. We have received numerous great ideas for future plans. Probably the most repeated suggestion was "a follow-on conference in person", which we hope will be able to make it happen as soon as possible.
- Finally, following the feedback we received from the WIP community, we launched a WIP slack workspace, where the community can continue to grow and network.

**Acknowledgements** We want to thank Noemi Petra for her valuable help in the organization of the workshop including the preparation of this report.

# Participants

**Alghamdi, Amal** (Denmark Technical University)
**Arnold, Andrea** (Worcester Polytechnic Institute)
**Beretta, Elena** (NYU Abu Dhabi)
**Boiger, Romana** (Paul Scherrer Institut)
**Borcea, Liliana** (University of Michigan)
**Cakoni, Fioralba** (Rutgers University)
**Calvetti, Daniela** (Case Western Reserve University)
**Cheney, Margaret** (Colorado State University)
**Chung, Julianne** (Virginia Tech)
**Dashti, Masoumeh** (University of Sussex)
**de Wiljes, Jana** (University of Potsdam)
**Dubinkina, Svetlana** (VU Amsterdam)
**Espanol, Malena** (Arizona State University)
**Freitag, Melina** (University of Potsdam)

**Gelb, Anne** (Dartmouth College)
**Graff, Marie** (University of Auckland)
**Jadamba, Baasansuren** (Rochester Institute of Technology)
**Kaltenbacher, Barbara** (University of Klagenfurt)
**Khatri, Ratna** (U.S. Naval Research Laboratory)
**Kilmer, Misha** (Tufts)
**Kokkinaki, Amalia** (University of San Francisco)
**Krupchyk, Katya** (University of California, Irvine)
**Kuske, Rachel** (Georgia Tech)
**Li, Wei** (DePaul University)
**Liao, Wenjing** (Georgia Tech)
**Ma, Anna** (University of California Irvine)
**Malcolm, Alison** (Memorial University)
**Minkoff, Susan** (University of Texas at Dallas)
**Moskow, Shari** (Drexel University)
**Mueller, Jennifer** (Colorado State University)
**Muller, Kaitlyn** (Villanova University)
**Needell, Deanna** (UCLA)
**Ou, M. Yvonne** (University of Delaware)
**Pasha, Mirjeta** (Arizona State University)
**Pathiraja, Sahani** (University of Potsdam)
**Petra, Noemi** (University of California - Merced)
**Qian, Elizabeth** (Caltech)
**Resmerita, Elena** (Alpen-Adria Universität)
**Rutter, Erica** (University of California Merced)
**Sabat Landman, Malena** (University of Cambridge)
**Schönlieb, Carola** (University of Cambridge)
**Stuart, Georgia** (University of Texas at Austin)
**Taghizadeh, Leila** (Technical University of Munich)
**Tarvainen, Tanja** (University of Eastern Finland)
**Terzioglu, Fatma** (NC State University)
**Thi Ngoc Nguyen, Tram** (University of Graz)
**Titi, Aseel** (Wichita State University)
**Tsogka, Chrysoula** (University of California Merced)
**Ullmann, Elisabeth** (Technical University of Munich)
**Vdovina, Tetyana** (Exxon Mobil)
**Xiao, Jingni** (Rutgers University)
**Yang, Yunan** (ETH Zürich)
**Zheglova, Polina** (Memorial University)

# Chapter 13

# Women in Operator Algebras II (21w5199)

December 5 - 10, 2021

**Organizer(s):** Sarah Reznikoff (Kansas State University), Maria Grazia Viola (Lakehead University)

## Overview

The workshop Women in Operator Algebras II was held at the Banff International Research Station from December 6 to December 10, 2021. This was the second event in the Women in Operator Algebras series (the first workshop in the series was held at BIRS in November 2018), and it took place in hybrid form because of the COVID 19 pandemic.

The workshop had forty-two participants from 15 countries (Australia, Canada, China, Czech Republic, France, Germany, India, Italy, Korea, Denmark, Netherlands, New Zealand, Norway, USA, UK). Most of the researchers participated online, while seven researchers from the USA and Canada were on location.

The focus of the workshop was to have women in operator algebras conduct pioneering research in small groups and to build a network of women in the field to drive future research collaborations and offer career support and advice. We believe we have accomplished both goals.

Eight innovative and interesting research projects were proposed by the group leaders for the workshop, dealing with topics related to textiles, topological full groups, the Baum-Connes conjecture, entropy of operator-valued measures, inductive limits of spectral triples, and properties related to the classification of $C^*$-algebras, such as the dynamic asymptotic dimension of groupoids or the diagonal dimension of $C^*$-algebras of groupoids.

To avoid inviting only researchers we already knew, and to offer opportunities to graduate students and early researchers to initiate new collaborations, we issued only 17 invitations to group leaders and a few senior researchers, and filled the remaining spots through an application process. A call for applications was sent out using social media posts, an email lists for women in operator algebras, and email lists from major conferences in operator algebras. The application included a series of questions, a ranked list of 2-3 research projects of interest, a brief statement describing how the applicant's expertise fitted with the projects and how they would benefit from participating, and a CV.

Before the workshop, the participants were divided into research groups, according to their expertise and interests. Under the guidance of the group leaders, the members of the group started to read background material and formulate research questions and strategies weeks before the workshop took place.

On the first day of the workshop, the group leaders described the research project and some problems on which their group would be working during the week. During the week the group members met on zoom or/and in person, discussing ideas, background material, approaches to the questions under investigations, methods, and proofs. On the last day of the workshop, the various groups gave a final presentation describing the work done during the week, the results reached, further directions of research, and future plans to carry on the collaborations.

To foster a network of women in operator algebras and offer career support and advice to women in the field, we had a two-hour event in the middle of the week, organized in conjunction with the Women in Operator Algebras Mentor Network. The event was a discussion of difficult situations and issues experienced by the workshop participants or network members in their working environment, and how to find support and solutions when facing similar problems.

The hybrid format worked well in facilitating collaborative work, with all participants actively involved in zoom or in person in discussions and exchange of ideas. Probably the only component that was slightly missing from the workshop was the social exchanges that usually take place at conferences. Although we tried to implement the use of Gathertown to foster interactions between participants in different research groups, the different time zones and the unfamiliarity of many of the participants with Gathertown made its use unpractical.

Offering a brief demonstration on the use of Gathertown to all workshop participants on the first day of the workshop may have encouraged a greater use of the platform.

# Scientific progress made during the workshop: reports from the research groups

Operator algebras—the study of $*$-algebras of bounded linear operators on Hilbert space—is a major branch of functional analysis that originated with von Neumann in the early part of the 20th century as a mathematical framework for modeling phenomena in quantum mechanics. It has since become a fundamental part of modern theoretical physics, as the natural setting for quantum information theory and quantum computing. In addition, the field has profound connections to several other areas of mathematics such as group theory, knot theory, dynamical systems, ergodic theory, and conformal field theory.

According to the feedback that we have received from the participants, the workshop was very successful and progress was made by each group on some of the research question on which they were working. All groups have expressed their intention of continuing the collaboration and have made concrete plans to meet in person or on zoom in the first few months of 2022.

Details about the progress made and the plans to collaborate further can be found in the following reports submitted by the research groups.

## Project: Finite nuclear dimension of $\mathrm{C}^*$-algebras of étale groupoids

**Group Members**: Kristin Courtney, Anna Duwenig, Magda Georgescu, Astrid an Huef , Maria Grazia Viola

The classification of unital, separable, simple C$^*$-algebras of finite nuclear dimension which satisfy the universal coefficient theorem was recently completed by "many hands" [22]. Finite nuclear dimension [23], a generalisation to C$^*$-algebras of covering dimension of a topological space, is a key hypothesis in the classification program. So studying the nuclear dimension of classes of C$^*$-algebras is an interesting problem.

In [11], Guentner, Willett and Yu defined dynamic asymptotic dimension for an action of a discrete group on a locally compact space. Motivated by the transformation-group groupoid, they extended their definition to principal étale groupoids. They found a bound on the nuclear dimension of the reduced groupoid C$^*$-algebra involving the dynamic asymptotic dimension of the groupoid and the topological covering dimension of its unit space. During the workshop, we sought to understand the Guentner–Willett–Yu theorem and its proof, with the broad aim of extending their results to non-principal groupoids and/or twisted groupoids.

The proof of the Guentner–Willett–Yu theorem simplifies significantly if the unit space of the groupoid is compact. We felt that there should be a method of passing from the compact to the non-compact case involving only the groupoid and not functions on the groupoid. We now have a promising strategy for this: given an étale groupoid $G$ with non-compact unit space, we have defined a groupoid $\tilde{G}$ with compact unit space such that the reduced/full C$^*$-algebra of $\tilde{G}$ is isomorphic to the smallest unitisation of the reduced/full C$^*$-algebra of $\tilde{G}$. We think that the dynamic asymptotic dimension of $\tilde{G}$ is equal to that of $G$ and we have devised a strategy for proving this. Since the nuclear dimension of a C$^*$-algebra and its smallest unitisation coincide [23], this will be a nicer way of passing from the compact to the non-compact case.

A twist $E \to G$ is a groupoid extension of $G$ by the circle $\mathbf{T}$. Through remarkable representation and rigidity results, for example [3, 14, 20], we know that many C$^*$-algebras, including all classifiable C$^*$-algebras, have a twisted groupoid model. Hence a description of dynamic asymptotic dimension in the twisted groupoid setting promises significant structural insight into this large and well-studied class of C$^*$-algebras. We have developed an approach for showing that for a twist $E \to G$ where $G$ is principal, the nuclear dimension of the reduced C$^*$-algebra of the twist is bounded and our bound is the same as the one found by Guentner, Willett and Yu.

A possible application is to non-principal groupoids $H$ with closed orbits and continuously varying stability subgroups. The full C$^*$-algebras of such groupoids are isomorphic to the full C$^*$-algebras of a certain twist $E \to G$ with $G$ principal [17, 8]. If this isomorphism descends to the reduced C$^*$-algebras and the twist is over an étale groupoid, then our conjecture in the previous paragraph would apply. More generally, we have defined a notion of dynamic asymptotic dimension modulo stability groups which we plan to investigate.

During the week of the workshop, we established a lively and productive collaboration. Our immediate plans are to fill in the proof outlines/strategies we mentioned above. For this, we have allocated subgroups to certain tasks and the whole group meets on Zoom on a monthly basis to report progress and to plan. We have developed a long list of questions about nuclear dimension of C$^*$-algebras and dynamic asymptotic dimension of groupoids on which future work could be based.

## Project: The Baum-Connes conjecture and mapping class groups

**Group members**: Sara Azzali, Sarah Browne, Indira Chatterji, Maria Paula Gomez-Aparicio, Sanaz Pooya, Hang Wang

**Introduction** The Baum–Connes conjecture, introduced by Paul Baum and Alain Connes in the 80s, gives a way of computing the K-theory of the reduced C*-algebra of a locally compact group, using the equivariant K-homology of the universal space classifying the proper actions of the group [4, 5].. The conjecture has been proven for a large class of groups that includes all almost-connected Lie groups, amenable group, and hyperbolic groups, those including $\mathrm{SL}(2,\mathbb{Z})$. But the conjecture (in particular the question about surjectivity) remains open for many cases of discrete groups, including $\mathrm{SL}(n,\mathbb{Z})$, for $n > 2$, as well as mapping class groups of higher genus surfaces. The difficulty in many cases involves a rigidity property in representation theory and no real progress has been made in the last 20 years after Lafforgues spectacular breakthrough in 2001 which remains poorly understood [15].

Mapping class groups are orientation preserving homeomorphisms of surfaces, modulo the ones homotopic to the identity. Despite their several hyperbolic features encoded in the curve complex, the surjectivity of the Baum-Connes map is still open in full generality. The injectivity was proven by Hämenstadt in [12] as a consequence of boundary amenability. Some particular cases of mapping class groups like braid groups or genus one surfaces are known, by work of Oyono-Oyono [19], Schick [21], Chabert-Echteroff [7] (see also the recent [2]).

**The project** Our project is centered about Vincent Lafforgue's paper *K-théeorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes* [15] and its relation with mapping class groups.

Lafforgue's works represented a very important progress. He showed the Baum-Connes conjecture for large classes of locally compact groups with property (T): all semi-simple real or reductive p-adic Lie groups, and for discrete cocompact subgroups of a rank 1 Lie groups.

The main tool is Lafforgue's bivariant KK-functor for Banach algebras, $KK^{ban}$ and the main point is the equality of classes $\gamma = 1$ in $KK^{ban}$, which is realised by an homotopy.

**Mapping class groups and the Baum–Connes assembly map** The injectivity of the Baum-Connes assembly map for mapping class groups was proven by Hämenstadt in [12] as a consequence of boundary amenability. Some particular cases of mapping class groups like braid groups or genus one surfaces are known, by work of Oyono-Oyono [19], Schick [21], Chabert-Echteroff [7] (see also the recent [2]). The surjectivity of the Baum-Connes map is still open in full generality.

**The WOA 2 week** During the WOA 2 meeting, we have learned about

1. the Rips complex, which is used in showing that the $\gamma$ element acts as identity on the left hand side of Baum–Connes.
2. Banach KK-theory, in which one can perform the homotopy from $\gamma$ element to the trivial representation.
3. mapping class group reading Brendle–Childers's introduction to mapping class groups written for "Office Hours with a geometric group theorist."
4. the construction of $\gamma$ element in Kasparov–Skandalis in the case of group acting properly on bolic spaces [13].

We have identified a sequence of steps and related questions. We plan to develop Lafforgue's methods and approach MCG groups.

**Plans for the near future** As short term goal, we plan to write a survey about Chapter 2 in Lafforgue's paper. Our long term goal is to investigate the strong bolicity property of Lafforgue in order to know if there is a way of relaxing it so that mapping class groups satisfy it.

We applied to a Research in Pairs program at the IHP in Paris and our application got accepted hence, the part of our group that is based in Europe is going to meet in person for two weeks in May 2022, and we will meet with the others online to continue working on our project.

## Project: The $f$-divergence, entropy, and related measures of two operator-valued measures:

**Group members**: Sarah Plosker, Hui Tan, Kateryna Tatarko, Elisabeth Werner, Runlian Xia

**Main Goal** Our main goal is to generalize $f$-divergence and related concepts used to compare two classical (scalar-valued) measures to the setting of operator-valued measures.

**Positive Operator-valued Measures and Quantum Random Variables:** Positive, operator-valued measures (POVMs) arise as natural objects of study in quantum mechanics. Let $\mathcal{H}$ be a finite dimensional or separable Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$, $\mathcal{T}(\mathcal{H})$ the Banach space of all trace-class operators (all operators in $\mathcal{B}(\mathcal{H})$ that have a finite trace under any orthonormal basis), and $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ the convex subset of all positive, trace-one trace-class operators $\rho$ (called states or density operators). We use $X$ to denote a locally compact Hausdorff space and $\mathcal{O}(X)$ to denote the $\sigma$-algebra of Borel sets on $X$.

Then a positive operator-valued measure (POVM) $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})_+$ is an ultraweakly countably additive function. That is, for every countable collection $\{E_k\}_{k\in\mathbb{N}} \subset \mathcal{O}(X)$ of disjoint Borel sets one has

$$\nu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \nu(E_k),$$

with the sum converging in the ultraweak topology.

**$f$-divergence in the Classical Setting:** An $f$-divergence is a function that measures the difference between two probability distributions. Let $(X,\mu)$ be a measure space and let $P = p\mu$ and $Q = q\mu$ be probability measures on $X$ that are absolutely continuous with respect to the measure $\mu$. Let $f : (0,\infty) \to \mathbb{R}$ be a convex or a concave function and define $f^* : (0,\infty) \to \mathbb{R}$ of $f$ by

$$f^*(t) = tf(1/t), \ \ t \in (0,\infty). \tag{13.0.1}$$

The $f$-divergence $D_f(P,Q)$ of the measures $P$ and $Q$ is then defined by

$$\begin{aligned} D_f(P,Q) &= \int_{\{pq>0\}} f\left(\frac{p}{q}\right) qd\mu + f(0)\, Q\left(\{x \in X : p(x) = 0\}\right) \\ &+ f^*(0)\, P\left(\{x \in X : q(x) = 0\}\right), \end{aligned} \tag{13.0.2}$$

provided the expressions exist, where $f(0) = \lim_{t\to 0^+} f(t)$ and $f^*(0) = \lim_{t\to 0^+} f^*(t)$. We write

$$D_f(P,Q) = \int_X f\left(\frac{p}{q}\right) qd\mu. \tag{13.0.3}$$

An important example of an $f$-divergence is the relative entropy $D_{KL}(P\|Q)$: For $f(t) = t\ln t$ (with $^*$-adjoint function $f^*(t) = -\ln t$), the $f$-divergence is Kullback-Leibler divergence or relative entropy from $P$ to $Q$

$$D_{KL}(P\|Q) = \int_X p \ln \frac{p}{q} d\mu. \tag{13.0.4}$$

**Outcome and further research** During the event at BIRS, we came up with two possible extensions of the concept of $f$-divergence to the setting of POVM. It is now our goal to compare these two definitions, investigate in more detail their properties and in particular establish inequalities related to these notions. In the literature there already exist notions of $f$-divergence for quantum channels. We want to explore how our notions fit in this context.

## Project: (Inductive) Limits of spectral triples

**Group members**: Carla Farsi, Therese-Marie Landry, Nadia Larsen and Judith Packer

Start with an inductive limit of unital $C^*$-algebras

$$\mathcal{A} \ = \ \left(\lim A_j\right) = A_1 \to A_2 \to A_2 \dots$$

with embeddings $\varphi_j : A_j \to A_{j+1}$. In addition assume that each $A_j$ is endowed with a 'compatible' group action of a group $G_j \cong G$ (with each $G_j$ isomorphic to a fixed finite abelian group $G$, $|G| = |\hat{G}| = n$), such that the fixed point subalgebra $(A_j)^{G_j} \cong A_{j-1}$.

- If the above cover is 'regular' in the sense of Aiello et. al [1]. Then:

$$\mathcal{A} \ = \ \left(\lim A_j\right) \hookrightarrow A_0 \otimes \text{UHF}(n^\infty);$$

- If in addition to regularity we assume that $A_j \cong A_{j+1}$, for all j, then Aiello et. al prove that there exists an inductive limit spectral triple on

$$\mathcal{A} \ = \ \lim A_j.$$

We want to investigate if the second of the above two results of Aiello et al. generalize when we remove the hypotheses $A_j \cong A_{j+1}$. The first one is already stated and proved in [1, Section 1]. But for completeness we will start by reviewing it below.

Given an inductive limit of unital $C^*$-algebras

$$\mathcal{A} \;=\; \left(\lim A_j\right) = A_0 \to A_1 \to A_2 \to A_2 \ldots$$

with embeddings $\varphi_j : A_j \to A_{j+1}$, assume that each $A_j$ is endowed with a 'compatible' group action of a finite abelian group $G_j \cong G$ (isomorphic to a fixed group $G$, $|G| = n$), such that the fixed point subalgebra $(A_j)^{G_j} \cong A_{j-1}$.

Also assume that the above cover is 'regular' in the sense of Aiello et. al [1], i.e., there exists a section $\sigma$ that brings each eigenspace of the $G$-action on $B$ to a unitary $\sigma : \hat{G} \to B_k \cap U(B)$.

**Theorem 13.0.1** *(Aiello et al. [1, Theorem 1.5] ) In the above situation we have:*

$$\mathcal{A} \;=\; \left(\lim A_j\right) \hookrightarrow A_0 \otimes \mathit{UHF}(n^\infty),\ \textit{where}\ n = |\hat{G}|.$$

Next, resume looking at the situation of 2 algebras in the chain, that is, $\varphi : A \to B$. Now note that there is an expectation: $E_{B,A} : B \to A$, given by $E_{B,A}(b) := \sum_{g\in G} gb$. Endow $A$ and $B$ with the traces $\tau_A$ and $\tau_B$ such that $\tau_B = \tau_A \circ E_{B,A}$. By the GNS construction, $\tau_J$ induces a GNS representation on $\mathcal{H}_J$, for $J = A, B$.

**Proposition 13.0.2** *(See [1, Propositions 1.8 and 2.4]) With A, B, $E_{B,a}$, $\tau_A$, and $\tau_B$ as above,*

1. *The GNS representation associated to $\tau_B$ is unitariy equivalent to the representation $\widetilde{\pi_A}$ obtained from the representation $\pi_A$ associated to $\tau_A$ by [1, Proposition 1.8]:*

$$\widetilde{\pi_A}(b) := [\pi_A\big(M(b)_{hk}\big)]_{h,k\in\hat{G}} \in M_{\hat{G}}\big(\mathcal{H}_A\big) \cong \mathcal{B}(\mathcal{H}_A \otimes \mathbb{C}^n).$$

2. *The above map extends to an isomorphism of the above Hilbert spaces implemented by the unitary operator:*

$$v_i : \mathcal{H}_B \to \mathcal{H}_A \otimes \mathbb{C}^n$$

*given by:*

$$\xi \to \sum_{k\in\hat{G}}^{\oplus} \sigma(k)^{-1}\xi_k.$$

Now consider a spectral triple over $B$

$$\big(\mathcal{L}_B, \pi_B : B \to \mathcal{B}(\mathcal{H}_B), D_B\big), \tag{13.0.5}$$

where $\mathcal{L}_B$ is a dense subalgebra of $B$, and $\pi_B : B \to \mathcal{B}(\mathcal{H}_B)$ is as in Proposition 13.0.2.

We then want to use Proposition 13.0.2 to transport the spectral triple in Equation (13.0.2) to a spectral triple over the Hilbert space $\mathcal{H}_A \otimes \mathbb{C}^n$.

Moreover, we also want to, whenever possible, translate the spectral triple in Equation (13.0.5) to be a spectral triple of type

$$\big(\mathcal{L}_A, \widetilde{\pi_A} : A \to \mathcal{B}(\mathcal{H}_A \times \mathbb{C}^n), D_A\big). \tag{13.0.6}$$

In [1], Aiello, Guido and Isola concentrate on the case where each $A_j \cong A_\theta$, for some rational $\theta \in [0,1]$. We call this the *stable* case. Our aim is to generalize the results of [1] where $A_j \cong A_{\theta_j}$ to the *non-stable case*, i.e. the case of varying $\{\theta_j\} \subset [0,1]$ which need not be rational.

## Project: "Dubious textiles"

**Group Members**: Samantha J. Brooker, Priyanga Ganesan, Elizabeth Gillaspy, Ying-Fen Lin, Julia Plavnik

**The research project and its connections with other branches of mathematics** The overall goal of this research project is to leverage $C^*$-algebraic techniques and invariants to enhance our understanding of 2-dimensional shifts of finite type (2D SFTs). We hope to build on the successful connections that have been established between 1-dimensional shifts of finite type and Cuntz–Krieger algebras (equivalently, graph $C^*$-algebras) by a variety of researchers (notably Matsumoto-Matui and Carlsen-Rout, among many others). It is now well established that important equivalence relations among 1D SFTs (flow equivalence, two-sided conjugacy, one-sided conjugacy) are mirrored by equivalence relations among the associated $C^*$-algebras (stable isomorphism, diagonal-preserving stable isomorphism, gauge-intertwining stable isomorphism).

In the symbolic dynamics community, 2-dimensional shift spaces are generally understood to be much more diverse and complicated than 1D SFTs, and most researchers have focused their attention on specific subclasses of 2D SFTs. Johnson and Madden established in 1999, however, that every 2D SFT is conjugate to one which arises from a textile system. Subsequently, Kang, Pask and Webster established that higher-rank graphs of rank 2 (or 2-graphs) give rise to textile systems. However, not all textile systems arise from 2-graphs – only those which are LR (in the notation of Nasu, who introduced the concept of textile systems in 1995). While we know (thanks to the 2000 paper of Kumjian and Pask) how to compute the $C^*$-algebras of higher-rank graphs, it is not clear how to associate a $C^*$-algebra to a 2D SFT which is not LR.

A first focus of our investigation is to understand how conjugacy for 2D SFTs manifests at the $C^*$-algebra level. In their 1999 paper, Johnson and Madden established that any conjugacy for textile systems (and hence for any 2D SFTs) can be written as a sequence of in- and out-splittings, amalgamations (the inverse of in/outsplitting), and inversion (switching horizontal and vertical directions). Eckhardt et al (2021) established that insplitting for 2-graphs gives an isomorphism of $C^*$-algebras. However, we quickly discovered at BIRS that textile system insplitting and 2-graph insplitting are not the same. Worse, the result of a textile-system in- or out-split will never be an LR textile system. In other words, many 2-graph SFTs are conjugate to SFTs which do not arise from 2-graphs.

**Progress made during the week** Our main focus during the week at BIRS was to understand the relevant constructions: textile systems, 2D SFTs, 2-graphs, and in/outsplitting. After realizing the incompatibility of 2-graph insplitting and textile system insplitting, we began work on a proof that 2-graph in- and out-splitting does yield a conjugate 2D SFT.

**Future research** In addition to finalizing the proof referenced above, we also plan to pursue the following lines of research:

- Find more classes of 2D SFTs which have $C^*$-algebraic models. We hope that Spielberg's "categories of paths" will be useful here.
- Identify which conjugacy classes of 2D SFTs have 2-graph representatives. For these classes, we have a $C^*$-algebra model which we hope will shed light on the dynamics.

**Plans to continue the research** We are planning to meet in person at the University of Montana in March 2022 to continue working on this project. We are also planning to maintain the momentum by regular Zoom meetings. Due to geographical constraints, we cannot all meet at the same time (even over Zoom) so we are planning to have two standing biweekly meetings which everyone can attend at least one of.

## Project: Connections between ample groupoid C*-algebras, topological full groups and inverse semigroups

**Group members**: Lisa Orloff Clark, Becky Armstrong, Eun Ji Kang, Mahya Ghandehari, Dilian Yang

**Overview of Project:** Given an action of the integers on the Cantor set, Giordano, Putnam and Skau define a corresponding full group in [10]. Building on the results of [10], Matui considers a more general setting in [16]: given any Hausdorff effective étale groupoid with compact unit space, he defines a corresponding topological full group. He shows that this group is a groupoid invariant for certain classes of groupoids. The construction was generalised to non-effective groupoids by Nekrashevych in [18].

Let $G$ be an ample groupoid with compact Hausdorff unit space $G^{(0)}$. Then the topologoical full group $[[G]]$ is the collection of all compact open bisections $B \subseteq G$ such that $s(B) = r(B) = G^{(0)}$. The set $[[G]]$ is a group with identity $G^0 \in [[G]]$. The product and inverse in $[[G]]$ are given by:

$$BD := \{bd : b \in B, d \in D \text{ such that } (b,d) \in G^{(2)}\} \quad \text{and} \quad B^{-1} := \{b^{-1} : b \in B\}.$$

. We consider the following two questions:

1. The group $[[G]]$ sits inside of the inverse semigroup $S_G$ of all compact open bisections of $G$. Bice shows that the groupoid of ultrafilters of $S_G$ is isomorphic to $G$ itself giving a kind of duality [6, Proposition 1.2] (see also [9]). Can we view the set of ultrafilters of $[[G]]$ as a subset of the ultrafilters of $S_G$?

2. From the inverse semigroup point of view, we also see that there is a representation of the group algebra of $[[G]]$ into the Steinberg algebra of $G$. Similarly for the C*-algebras? What are the properties of these representations?

**Progress made during the week:** We considered the second question first and learned that the natural representation of the group algebra of $[[G]]$ is typically not injective. We are in the process of developing a combinatorial proof that considers the number of orbits in the unit space. In terms of the first question, we conducted a detailed investigation of some examples. We learned that singletons $\{g\}$ where $g \in [[G]]$ are in fact filters themselves but are almost never ultrafilters. Work is ongoing.

**Future plans:** We have set aside the week of 20 February to meet daily via zoom to continue working on this project.

## Project: Diagonal dimension of C$^*$-algebras

**Group Members**: Anshu Nirbhay, Dawn Archey, Camila F. Sehnem, Marzieh Forough, Ja A Jeong, Karen Strung

This project set out to study the so-called "diagonal dimension" of certain C$^*$-algebras containing C$^*$-diagonals. The diagonal dimension is a refinement of the completely positive approximation property that is designed to keep track of a prescribed C$^*$-diagonal subalgebra. It was introduced by Li, Liao and Winter.

A canonical example of a C$^*$-algebra with a C$^*$-diagonal is given by the inclusion of $C(X) \subset C(X) \rtimes_\alpha G$, where $C(X) \rtimes_\alpha G$ is the crossed product associated to a minimal free action of a discrete group $G$ on a compact metric space $X$. In the case that $G = \mathbb{Z}$, Li, Liao and Winter showed that the diagonal dimension of $C(X) \rtimes_\alpha \mathbb{Z}$ can be estimated using the dimension of $X$ and the Rokhlin dimension of the homeomorphism $\alpha$.

The original plan of the project was to extend this result to the case of a crossed product by a Hilbert $C(X)$-bimodule. When the Hilbert $C(X)$-bimodule is left and right full, it has the following form: As a right Hilbert $C(X)$-bimodule, it has the structure of a module of sections $\Gamma(\mathscr{V})$ of a line bundle $\mathscr{V}$ over $X$. A left module structure is given by defining $f\xi = \xi f \circ \alpha$, where $\alpha$ is a homeomorphism. Using the notion of Rokhlin dimension for a Hilbert C$^*$-correspondence, we were quickly able to obtain the result. It also became clear that both the diagonal dimension and the nuclear dimension of $C(X) \rtimes \mathbb{Z}$ depends only on the dimension of $X$ and the Rokhlin dimension of $\alpha$.

The crossed product of $C(X)$ by a Hilbert bimodule can also be realised as a twisted groupoid C$^*$-algebra for some twist over the transformation groupoid $X \times \mathbb{Z}$. This led us to conjecture that neither the nuclear dimension nor the diagonal dimension can distinguish between a principal étale groupoid and a twist over that groupoid. By the end of the workshop, the group decided to focus on proving this conjecture.

The group are continuing to collaborate and have had two zoom meetings since the end of the conference. Our research continues to focus on how the nuclear and diagonal dimensions relates the C$^*$-algebras of groupoids and their twists. We hope to meet as a group, or some subset of our group, in person in the summer of 2022.

# Acknowledgement

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# Participants

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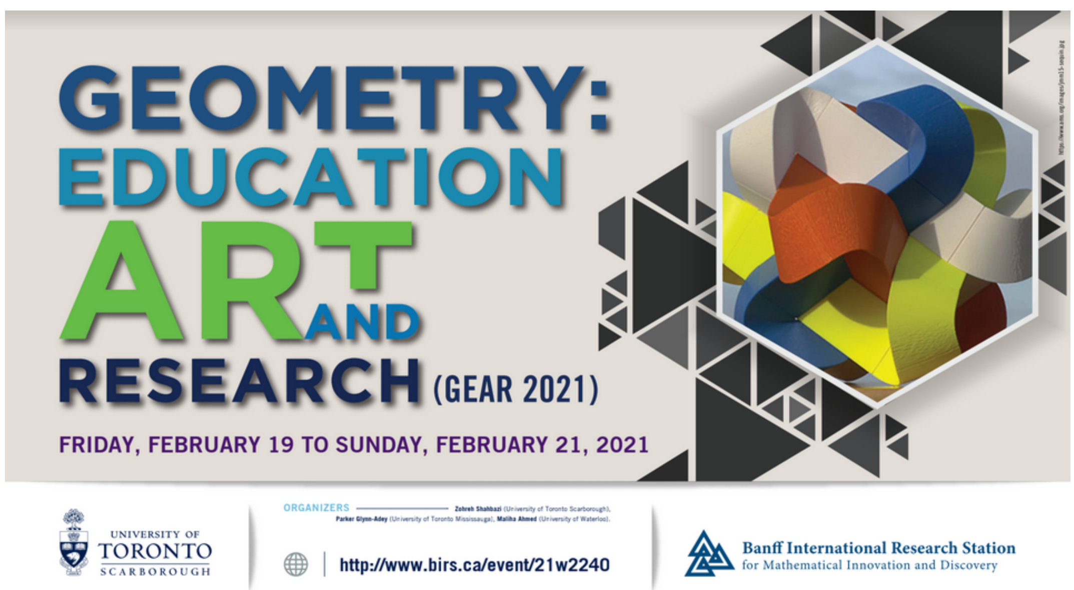
# Two-day Workshop Reports

# Chapter 14

# Geometry: Education, Art, and Research (21w2240)

February 19 - 21, 2021

**Organizer(s):** Zohreh Shahbazi (University of Toronto Scarborough), Maliha Ahmed (University of Waterloo), Parker Glynn-Adey (University of Toronto Mississauga)




Our goal for the Geometry: Education, Art, and Research online conference (GEAR2021) was to create a dialogue between researchers, educators, and artists. We aimed to provide a platform to present recent discoveries in geometry to the researchers, educators, artists, and the public via the arts and modern pedagogical approaches.

During the opening program on February 19th, each of the presenters and speakers took a few minutes to introduce themselves and briefly explained what they planned to present during their sessions. The format of the opening provided participants with the opportunity to learn more about each other's work and helped to foster future collaboration. There were 66 confirmed participants in total from various international institutes and organizations.

- The online format of the conference enabled participants from all over the world to easily attend. All sessions were recorded and the videos are available from BIRS website.
- We used a Slack channel to encourage participants to engage in ongoing discussions. Participants actively used the Slack channel and provided each other with interesting perspectives and additional information.
- Two graduate students presented their posters in breakout rooms on Zoom. Maliha Ahmed, one of the GEAR 2021 organizers, is currently a graduate student.
- We used a Google Jamboard to collect participants' creative work in research and geometry teaching. We used this material to create an art poster summarizing the conference. The poster is attached to this report.

GEAR2021 included seven parallel 50-minute workshops and two mini-courses. Each course included two 90-minute parts. In addition, we had eight 20-minute parallel talks, each with a 10-minute follow up question and answer period. In total, we had 17 hours of workshops, mini-courses, and talks. Video footage of the event is available on the BIRS website. The workshops and mini-courses were presented by artists, researchers and educators who work in the field of geometry from various public and private organizations and institutes. The topics covered included: Islamic geometric patterns [3], visualizing geometric spaces [1], origami [4], visual data analysis, dance and rotation groups [5], and wallpaper patterns [2]. As a result, presenters created an engaging and interesting experience which allowed us to identify common interests for exciting collaborations in the future. Presenters encouraged participants to publish in pedagogy and arts journals such as the Journal of Mathematics and the Arts and For the Learning of Mathematics. A review of GEAR2021 will be published in the Journal of Mathematics and Arts.

Additionally, two GEAR2021 presenters (Henry Segerman and Jayadev Athreya) are organizing a week-long graduate workshop for the summer of 2021. The Illustrating Mathematics workshop at the Park City Math Institute (PCMI) graduate summer school aims to bridge research and outreach in mathematics. GEAR2021 participants were encouraged to contribute to this event.

Our event concluded with a panel discussion with three panelists: Henry Segerman (Associate Professor of Mathematics at Oklahoma State University), Brian P. Katz (Professor of Mathematics Education at the California State University Long Beach), and Susan Gerofsky (Associate Professor of Curriculum and Pedagogy at the University of British Colombia).

The panelists shared their experiences using research and/or art to provide accessible and interesting content in their geometry courses. Participants discussed their approaches to bring research and teaching together using educational and artistic tools or models, differences that they recognized in their students' learning and attitudes when they teach concepts in an illustrative way by employing physical or artistic tools, and actions that they take to elevate public understanding and interest in mathematical developments.

The GEAR 2021 organizers are grateful for the opportunity to host such an exciting and interdisciplinary event at BIRS. We look forward to future collaborations, and an in-person meeting at BIRS sometime in the future.

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