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Contents

Five-day Workshop Reports

Chapter 1

Geometry and Swampland (22w5083)

January 24 - 28, 2022

Organizer(s): Mariana Grana (CEA/Saclay), Michela Petrini (Sorbonne University), Irene Valenzuela (Universidad Autonoma de Madrid)

Overview of the Field

Understanding the structure of quantum gravity is one of the most ambitious goals of fundamental physics. A concrete and particularly well developed framework to address specific questions of quantum gravity is string theory. String theory is consistently defined in ten dimensions, six of which should be curled up in some small internal compact manifold. The geometry of the internal compactication space is key in our understanding the prediction of string theory for the four-dimensional world we observe. The study of the internal compaction space has opened up far reaching connections between string theory and cutting-edge mathematics. The most famous example are Calabi-Yau manifolds. More recently, new very interesting connections have appeared in the context of Generalised Geometry, a generalized version of Riemannian geometry called generalized complex geometry, first developped by Hitchin.

Recently, there has been great interest in determining criteria which differentiate between effective low-energy field theories which can be consistently completed in the ultraviolet into quantum gravity, said to be in the 'Landscape', from theories which appear consistent but nonetheless defy such a coupling to quantum gravity, the so-called 'Swampland'. A number of such criteria, or Swampland Conjectures, have been proposed in the literature and attracted considerable interest in the high energy physics community. If confirmed, they have profound consequences for physics and cosmology, such as for the structure of large field inflation in early-time cosmology, or for the mechanism responsible for the observed late-time acceleration of the universe, to name but the most striking ones. On the other hand, the Swampland Conjectures translate, in the context of string theory, into conjectures regarding the structure of possible compactifications, or string geometries.

String theory is therefore in a unique position to quantitatively test - and possibly refine such general Swampland Conjectures. This Scientific Program of the workshop was to study these intriguing connections between general properties of quantum gravity and the geometry of string compactifications. What made this workshop rather unique is that it focused on the detailed interplay between the physics of the Swampland Conjectures and the mathematical structures underlying the effective field theories obtained in string theory.

Open Problems and goals of the workshop

The main goal of the workshop was to bring together leading physicists and mathematicians to explore the mathematical foundations of the recent Swampland Conjectures in the context of string theory. Specifically, we addressed the following points:

- Extend the analysis of the Swampland Conjectures to the topical realm of compactifications on generalized Calabi-Yau manifolds
- Achieve a better understanding of degeneration limits in moduli spaces of string compactifications and their relation to the Swampland Distance Conjecture
- Systematically explore the boundaries of the string landscape in the non-perturbative regime via geometrisation of dualities

Presentation Highlights in a virtual environment

The workshop started with two review talks, one on swampland and the other on generalized geometry. They served the purpose of introducing the swampland and generalised geometry communities with the ideas developed by the other, and generated fruitful discussions.

A virtual workshop is hardly comparable to a fully in-person meeting in terms of scientific interactions. However, we organised several events to foster interaction in the virtual environment which were a great success. On one hand, in addition to technical talks and reviews, we also scheduled an open discussion session everyday from Tuesday to Friday of one hour and a half long. This means that a big fraction (30%) of the workshop duration was dedicated to the discussion sessions on the central topics of the swampland programme. In the spirit of the workshop, there was also a discussion dedicated to exchanging tools between the two communities. The discussions started with a brief 20 min presentation of some invited discussion leader, to put in context the topic and highlight open questions, followed by an open discussion where everybody was invited to participate. The discussions were very lively and typically continued during the coffee break or via slack. They were a great success. On the other hand, we had several social events to encourage more informal interactions. First, during each coffee break we encourage people to move to Gather Town. Secondly, on Thursday after all the talks we had a 'Gather Town cocktail event" playing the role of the usual conference dinner of in-person events. We even had a conference speech from one of the key participants.

Scientific Progress Made

- The workshop served as a bridge between two communities: swampland and generalised geometry.
- There were many lively discussions about scale separation, since this is a clear point of overlap between both communities. The swampland community learnt that certain tools of generalised geometry can help to study the KK spectrum and determine if a given vacuum exhibits scale separation. On the other hand, the generalized geometry community understood the motivations for the no-scale separation swampland conjecture.
- There was also an active but constructive discussion about DGKT [1] which is a proposal for a scale-separated vacuum. Different researchers shared their views and pointed out interesting open problems present in the model. This might lead to future publications.
- The analysis of the asymptotics of the moduli space is both of interest for swampland and geometry. There were several presentations about studying the infinite distance limits and proposing new ways to go beyond the state-of-the-art.

- There was a very lively discussion about the status of de Sitter vacua in string theory (which are conjectured not to exist, according to swampland criteria). The strengths and weaknesses of different constructions were highlighted and discussed in depth.

Outcome of the Meeting

The workshop had a very high attendance, with order of 150 participants at every talk. This implies that most participants were attending all the talks, which is quite impressive given the virtual format. The discussion sessions were particularly lively, and went on during the breaks in Gather Town and, afterwards, in Slack. The feedback from all the community was extremely positive, we received comments like e.g. ”It was one of the most interesting workshops I have been to for years, content-wise”. It is quite fair to say that the workshop was a total success in terms of mixing both communities and having very productive discussions.

We also announced the launching of a mentoring program targeted at members of under-represented groups. This was very well received, and we collected a list of people willing to help in the organisation, as well as potential mentors and mentees.

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Bibliography

[1] O. DeWolfe, A. Giryavets, S. Kachru and W. Taylor, “Type IIA moduli stabilization,” JHEP **07** (2005), 066 doi:10.1088/1126-6708/2005/07/066 [arXiv:hep-th/0505160 [hep-th]].

Chapter 2

Predicting Pathways for Microplastic Transport in the Ocean (22w5073)

February 20 - 25, 2022

Organizer(s): Bruce Sutherland (University of Alberta), Michelle DiBenedetto (University of Washington), Alexis Kaminski (University of California, Berkeley), Ton van den Bremer (University of Delft)

Overview of the Field

Hundreds of millions of tons of plastic waste are produced each year. A large portion of this waste is improperly discarded and ultimately ends up in the environment; e.g. entering rivers through municipal waste disposal or entering the ocean directly through dumping [21]. It is estimated that about 20M tons of plastics enter into the ocean each year [4]. Despite the fact that most plastics produced are buoyant, and should be found floating on the ocean or washed up on shore, estimates based upon observations suggest that only a few percent (less than 300K tons) of the discarded plastics remain near the surface of the open ocean [34, 33]. This discrepancy has raised the question: where are the missing plastics?

The scientific community aiming to address this question has dominantly focused upon observations performed by oceanographers, marine biologists, sedimentary geologists and environmental scientists. However, the ultimate answer to this question must come through predictive models. It is for this reason that our workshop aimed to assemble researchers from across the spectrum of applied mathematicians, fluid dynamicists, physical oceanographers, engineers, and others to identify outstanding gaps in our understanding of the dynamics of plastic transport in the ocean.

The problem of predicting the ultimate fate of plastic pollution is vast. The density, size and shape of plastics vary greatly, they can be broken down into smaller pieces through stresses induced by turbulent flows, their density can change in time due to the accumulation of organic and inorganic deposits. As the particles evolve they are influenced by fluid dynamics processes over wide-ranging scales from ocean currents (hundreds to thousands of kilometers), eddies (tens to hundreds of kilometers), waves and fronts (tens to hundreds of meters), and turbulent mixing processes (millimeters to tens of meters). Our workshop participants had wide-ranging expertise in mathematical theory, numerical modelling and laboratory experiments, exploring dynamics from the microscopic to global scale. Ideas were shared through talks and lively themed discussions at our workshop, in which recent scientific advances where presented and pressing outstanding issues and possible solutions were brainstormed.

Recent Developments and Open Problems

Our workshop broke down the problem of microplastic transport by identifying four challenges in modelling microplastic transport. These were addressed through invited talks and by way of discussion groups involving all participants. Recent developments and open problems in each of these four challenge areas are discussed below.

Challenges modelling non-inertial and inertial particles

Non-inertial particles are so small that they are carried with the surrounding fluid like a passive tracer. In contrast, relatively large microplastics are inertial, meaning that the particles move differently from the surrounding fluid. Several factors influence the behaviour of inertial particles in fluid flow including their buoyancy, size, shape and the nature of the background flow, whether it is stationary, oscillating (due to waves), or turbulent.

A measure of the importance of particle buoyancy and size relative to the fluid viscosity is the particle Reynolds number, $\mathrm{Re}_p = w_s d_p/\nu$, in which w_s is the settling/rising velocity in stationary fluid (which depends on buoyancy), d_p is a measure of the particle size, and ν is the kinematic viscosity. A measure of the importance of particle inertia in turbulent flow is the Stokes number, $\mathrm{St} \equiv \tau_p/\tau_\eta$, in which $\tau_p \simeq w_s/d_p$ is the particle relaxation time and $\tau_\eta = (\nu/\varepsilon)^{1/2}$ is the dissipation time scale in which ε is the energy dissipation rate. If $\mathrm{Re}_p \ll 1$ and $\mathrm{St} \ll 1$, the particle is non-inertial and experiences Stokes drag.

With this as background, our workshop identified the following as some of the most significant open problems in modelling the motion of a plastic particle in fluid flow:

1. The wide range of length scales and densities of plastic particles corresponds to wide-ranging particle Reynolds numbers from $\mathrm{Re}_p \ll 1$ to $\mathrm{Re}_p \gg 1$, with microplastics on millimeter scales having $\mathrm{Re}_p \sim 1$. To simplify the equations of motion, existing theories have focused on the large and small Re_p regimes; few studies of weakly inertial particles have intermediate Re_p.

2. Microplastics in the turbulent ocean mixed layer have Stokes numbers in the range $\mathrm{St} \sim 0.001 - 0.01$; below surface waves, the range is $\mathrm{St} \sim 0.1 - 10$ [11]. As with the particle Reynolds number, most theories have focused upon the large and small St regimes. Inspired by the problem of microplastic transport, only recently have the dynamics of particles in wavy flow with $\mathrm{St} \sim 1$ been examined. Predominantly this has been through laboratory experiments with room for much more to be explored.

3. For mathematical convenience, most theories of particles in fluids assume they are spherical. However, the shapes of particles vary greatly from polygonal plates and shards to fibers. Further complicating matters, is that sufficiently long fibers and thin plastic sheets are flexible. And so it is necessary to take into account the elastic-plastic properties of the particles themselves.

4. While there have been great advances in the past decade developing numerical simulations that resolve the motion of individual and a small collection of particles in fluid flow, at present, most models are limited to assuming the particles are spherical. Although still numerically challenging, much could be learned about the evolution of a (possibly biofouled) particle in steady descent and in turbulence by devising methods to examine more complex particle shapes. These models could also explore in more detail the influence of background stratification on settling and particle aggregation.

Challenges modelling particle transformation

Plastics can transform by breaking up into smaller particles. It is also possible for organic or inorganic matter to accumulate on plastics, thus changing both their size and density. Our workshop identified the following open questions:

1. There have been some recent studies of particle break-up in turbulence, focusing upon fibers that are long compared to the dissipative scales of turbulence. It remains unclear how particles continue to break up when they are smaller than the dissipative scales (on the order of 1 millimeter).

2. The plasticity of a particle determines threshold stresses that lead to fracturing and break-up of particles. Observations show that plastics exposed to sunlight become more fragile, though this is expected to occur over relatively long times and only for buoyant plastics near the surface. The influence upon particle plasticity of sunlight is not well understood, with a pressing need to quantify the time-scale for particle transformation in this way.

3. The accumulation of microbes and other organic material on plastics is known as biofouling. All plastics in the ocean become biofouled on the scale of days. Over sufficiently long time (days, weeks or months depending on the size and relative density of the plastic), biofouled buoyant plastics can become more dense and sink. But at depth they can remineralize and become buoyant again. These dynamics have been observed in the field, but the microscopic processes are poorly understood, although laboratory experiments are beginning to provide insights.

4. A biofouled plastic has an ill-defined shape, with microbes and the accumulation of marine snow forming a semi-permeable coating around the otherwise solid plastic particle. The influence of this coating upon particle transport and settling remains poorly understood, as does the influence upon the degradation of the coating by turbulence and flow around an inertial particle.

Challenges modelling estuaries, coastal and submesoscale ocean processes

The energy containing scales in the ocean are predominately associated with mesoscale eddies and currents. For this reason, and for mathematical simplicity, theories have largely focused upon such motion (on the order of $100\,\mathrm{km}$ horizontal extent) for which the motion is geostrophic, being dominated by the Earth's Coriolis force. This regime is characterized by the Rossby number, $\mathrm{Ro} = UL/f$, being much smaller than 1. Here U and L, respectively, are the characteristic velocity and horizontal length scales of the flows, and f is the Coriolis parameter, a measure of the local angular speed of rotation about the vertical. Mesoscale processes describe, for example, eddies that develop in western boundary currents such as the Gulf Stream and Kuroshio current. The submesoscale has become increasingly studied in the past decade, facilitated by more powerful computational resources that permit high-resolution simulations of small-scale structures in large-scale flows. As discussed below, coastal and estuarine processes have not been so well studied in the context of particle transport.

Our workshop identified the following open questions

1. Great progress has been made in understanding surface convergence due to submesoscale processes, such as the formation of fronts between warm and cold water near western boundary currents. Simulations can also resolve convergence due to Langmuir circulations which form under the action of wind and waves that drive counter-rotating streamwise vortices forming windrows at the surface. However, global-scale ocean processes do not have the resolution to capture these processes. And so better parameterizations should be developed to predict the formation of fronts and windrows from coarse resolution models.

2. There has been significant progress in understanding Lagrangian transport by surface waves, as they are influenced by wind stress, Coriolis forces and the Stokes drift [31, 20]. Most of this work has focused upon the transport of passive particles ($\mathrm{Re}_p \ll 1$). Predictions of transport could be improved by drawing on results from experiments examining inertial particles of different shapes and density below waves.

3. One of the most pressing societal concerns is the wash-up of plastics on the coast, a process known as beaching. A relatively small number of physical oceanographers study near-coastal processes, particularly those in the surf-zone; this region is predominantly studied by civil and coastal engineers, whose focus has traditionally been more on sediment transport than buoyant particle transport. More work could be done examining the process of beaching of buoyant microplastics, as well as their removal from the coast by riptides.

4. As with the surf-zone, oceanographers have not focused much upon processes in rivers and estuaries, although these are the sites through which most plastics enter the ocean through municipal waste. More could be done, in particular, to study the transport and possible transformation of plastics when they encounter

strong turbulence associated with the tidal zone, and their possible interaction with suspended clay when they come into contact with sea water.

5. Coastal and estuarine processes can vary significantly depending upon several factors including topography, winds, tides and local currents. Likewise the source of microplastics (e.g. outlets from municipal waste and storm pipes) can be situated at different locations along the coast or upstream in rivers. This points to the need to develop regional models quantifying sources and sinks of microplastics.

Challenges modelling global transport

The ultimate goal is to predict globally where plastics ultimately deposit after being released into the ocean. However, numerical simulations of the global ocean are too coarsely resolved to capture the formation of fronts and other surface convergent phenomena, and they certainly cannot capture the rise and fall of the plastics themselves. Instead these processes must be parameterized being informed by theory, idealized high resolution simulations and semi-empirical models derived from the results of laboratory experiments [16, 22, 14, 27].

Our workshop identified the following open questions

1. While crude parameterizations have been developed for small-scale processes not captured by coarse-scale global simulations, there is great room for improvement, as described above. In some regions, the influence upon settling particles of density stratification (accounting for the change of temperature and salinity with depth) should be considered.

2. Limited observations remain a large obstacle to predicting particle transport. While estimates exist for the mass and types of produced plastic, the size, shape and density of plastics released at the source is unclear, with most observations reporting simply on number of plastic particles or their net mass. While it is not in the purview of our modelling community to perform such observations, we could provide better guidance about what information is needed to guide and prioritize our work.

Presentation Highlights

Being an entirely online workshop with participants primarily in Europe and North America, we chose to have only 12 talks, leaving room for discussion among all participants in Gathertown following the talks. Between Monday and Thursday, the invited opening talk gave an overview of research in each of the themed challenge areas described above. These were followed by two more focused research talks. A summary of the talks and group discussions was presented by the organizers on the Friday, followed by a general discussion of future directions.

Synopses of the invited summary talks are given below.

1. Michelle DiBenedetto: *Is shape important to plastic transport?*

   After providing an overview of the primary problem with predicting microplastic transport, namely where is the missing plastic, Dr DiBenedetto discussed progress in predicting the influence of shape upon particle settling below waves. Laboratory experiments showed that long elliptical particles became oriented in the waveward direction which can reduce the settling/rise rate of negatively/positively buoyant particles [18, 17].

   Such measurements are important for the interpretation of observations. Typically plastics are extracted by droguing nets near the ocean surface. Using the model by Kukulka [23], the distribution below the surface is assumed to be exponential. However, this model does not take into account the differential rise and fall of different-sized particles. Indeed, more careful observations show that the distribution changes significantly between $0.5\,\mathrm{m}$ from the surface and below. All this suggests that surface measurements underestimate the amount of plastic near the ocean surface by a factor of 3-13 [7]

   She also presented a case study of the X-Press Pearl nurdle spill off the coast of Sri Lanka in May 2021. Nurdles, or pre-production plastic pellets, spilled out a wrecked container ship onto the beaches of Sri Lanka. The shipwreck had also caught on fire, causing many of the nurdles to burn, melt, and agglomerate together. Differential transport was inferred between the pristine nurdles and the burned nurdles from observations of where they washed up on the beach [15]. This case study demonstrated how the physical properties of the plastic can affect their transport in the ocean.

2. Margaret Byron: *The influence of shape, size and density distribution on microplastic transport in environmental flows.*

   Dr Byron began by reviewing mechanisms for particle transformation though biofouling [25, 29, 24]. and fragmentation. Recent experiments suggest that plastics can act as a nucleation site for the growth of marine snow, which can double the settling velocity in part due to the effective increase in the particle size [8, 30]. Dr Byron presented a new consideration regarding the settling of biofouled plastic particles. It is usually assumed that the particles have uniform density. However, the very fact that marine snow and other organisms grow on the plastics means that the density of the biofouled plastics is non-uniform. Through laboratory experiments using particles composed of two elements with different densities, she showed that the particles reorient and possibly oscillate during their descent depending upon their length [2].

   Regarding particle break-up into smaller pieces, observations of the distribution of particle sizes, with most being smaller than $0.3\,\mathrm{mm}$ [1], suggests fragmentation occurs after plastics are released into the ocean. However, the processes leading to break-up remain unclear. Some plastics become more fragile over time with exposure to heat and ultraviolet radiation from the sun [36]. But this is a relatively long-time process. Turbulent processes, particularly those associated with breaking waves can lead to particle break-up. However, recent research has shown the bending, twisting and stretching forces leading to breaking become less pronounced for smaller particles over which there are smaller velocity changes over their extent [6, 19, 5, 35, 26]

3. Baylor Fox-Kemper: *Dispersion and dissipation: Turbulence statistics for the mesoscale to finescale with plastics on the move.*

   Dr Fox-Kemper demonstrated how submesoscale processes can result in convergence zones at the surface through Langmuir circulations and the formation of density fronts. These various mechanisms were recently discussed in a review paper [13].

   Through the observations of floating bamboo plates scattered over Langmuir cells, Chang et al [12] compiled the statistics of floating particles collecting at the convergence locations. Such surface convergence also occurs at density fronts between warm-cold and/or fresh-salty water. While eddies and currents may have scales on the order of tens of kilometers, the fronts can be just a few hundred meters wide, a length set by a balance of rotation, fluid inertia and turbulence [3, 9].

   What is clear in all these studies is to recognize that the dynamics at and below submesoscales differs qualitatively from mesoscales. The latter processes are dominated by currents and eddies that transport particles but do not lead to surface convergences observed at smaller scale [32, 10]. A major goal is to improve predictions of these convergence sites where floating plastics gather, and so can more efficiently be collected.

   Through a combination of theory and numerical simulations, the clustering statistics as a function of separation distance, r, were found to vary as $r^{1/3}$ about Langmuir cells, $r^{2/3}$ for submesoscale fronts and r^1 at the mesoscale [28]. The signal for convergence does not show up in an Eulerian frame, but in a Lagrangian frame, indicating the importance of the Stokes drift and the need to track fluid parcels in numerical simulations.

4. Erik van Sebille: *Whose plastic is that? Using Bayesian inference to attribute microplastic sources and sinks.*

   Dr van Sebille presented recent advances in predicting the fate of plastics in the global ocean using a combination of numerical simulations and statistics. Although the models are still in their early stages of development, they are already providing important insights, suggesting that once plastics are released near the coast, approximately half are deposited back on shore (beaching) on a time-scale of a month, with approximately 40% sinking over the course of about 80 days, and with about 10% remaining afloat [22]. Thus, except around islands and near western boundary currents (e.g. the Gulf Stream and Kuroshio), most plastics released into the ocean from estuaries drift less than 100km from release.

Scientific Progress Made

As intended, the significant progress was made through the cross-fertilization of ideas stemming from different disciplines. Standing out amongst the progress made was the strong emphasis on assessing plastic transport using Lagrangian transport models, and using Bayesian analysis to assess sources and sinks of microplastics.

Perhaps the greatest progress was made in identifying the numerous outstanding problems, and identifying the "low hanging fruit", being those problems that could potentially be solved with present computational and experimental resources so as develop informed parameterizations that could be used in coarse resolution global models.

One example of potential immediate benefits of the workshop to Canadians is the discussions spurred by Susan Allen, U. British Columbia, who is part of a team trying to improve regional modelling of the Salish Sea with their simulation "SalishSeaCast". Dr Allen raised specific points about what processes in SalishSeaCast were well-modelled and what processes were poorly constrained or inadequately modelled due to lack of observations or a poor understanding of the physical and biological processes involved. This spurred several focused discussions aiming to improve deficiencies in the regional model, resulting in the development of new collaborations.

Outcome of the Meeting

As evident from feedback following the meeting, participants expressed great excitement at the breadth and depth of topics covered both in talks and during Gathertown discussions.

The immediate benefits of the workshop include the wide-ranging new collaborations that have been created and past collaborations that have been reinvigorated by topics raised. New links have been forged between numerical modellers who have been developing different simulation methods, but see the potential for advancement through technology exchange. New interdisciplinary collaborative partnerships were formed between researchers focusing on fundamental fluid dynamics and applied mathematics on one hand, and researchers interested in biological and environmental applications on the other.

Finally, it is the intention of the co-organizers to prepare a review paper covering the content of lectures and discussions that took place during the workshop. This will be submitted to Physical Review Fluids, with due acknowledgement to the Banff International Research Station. The paper will thus reach out to the wider international community, broadening the base of collaborative activities already established by the workshop.

Participants

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Bibliography

[1] Cózar A. and 10 others. Plastic debris in the open ocean. PNAS, 111:10239–10244, 2014.

[2] B. R. Angle, M. J. Rau, and M. L. Byron. Effect of mass distribution on falling cylindrical particles at intermediate Reynolds numbers. In Proceedings of the ASME-JSME-KSME 2019 8th Joint Fluids Engineering Conference, Volume 5: Multiphase Flow, page V005T05A065, San Francisco, California, USA, July 28-August 1, 2019. ASME.

[3] A. S. Bodner, B. Fox-Kemper, L. P. Van Roekel, J. C. McWilliams, and P. P. Sullivan. A perturbation approach to understanding the effects of turbulence on frontogenesis. J. Fluid Mech., 883:A25, 2020.

[4] S. Borrelle and 19 others. Predicted growth in plastic waste exceeds efforts to mitigate plastic pollution. Science, 369:1515–1518, 2020.

[5] S. Bounoua, G. Bouchet, and G. Verhille. Tumbling of inertial fibers in turbulence. Phys. Rev. Lett., 121:124502, 2018.

[6] C. Brouzet, G. Verhille, and P. Le Gal. Flexible fiber in a turbulent flow: A macroscopic polymer. Phys. Rev. Lett., 112:074501, 2014.

[7] K. Brunner, T. Kukulka, G. Proskurowski, and K. L. Law. Passive buoyant tracers in the ocean surface boundary layer: 2. observations and simulations of microplastic marine debris. J. Geophys. Res., Oceans, 120:7559–7573, 2015.

[8] W. G. Burns, A. Marchetti, and K. Ziervogel. Enhanced formation of transparent exopolymer particles (TEP) under turbulence during phytoplankton growth. J. Plankton Res., 41:349–361, 2019.

[9] D. Calvert, G. Nurser, M. J. Bell, and B. Fox-Kemper. The impact of a parameterisation of submesoscale mixed layer eddies on mixed layer depths in the NEMO ocean model. Ocean Modelling, 154:101678, 2020.

[10] H. Cao, B. Fox-Kemper, and Z. Jing. Submesoscale eddies in the upper ocean of the Kuroshio extension from high-resolution simulation: Energy budget. J. Phys. Oceanogr., 51:2181–2201, 2021.

[11] M. Chamecki, T. Chor, D. Yang, and C. Meneveau. Predicted growth in plastic waste exceeds efforts to mitigate plastic pollution. R. Geophys., 57:1338–1371, 2019.

[12] H. Chang and 12 others. Small-scale dispersion in the presence of Langmuir circulation. J. Phys. Oceanogr., 49:3069–3085, 2019.

[13] E.A. D'Asaro and 18 others. Ocean convergence and the dispersion of flotsam. Proc. Nat. Acad. Sci., 115:1162–1167, 2018.

[14] C. de Lavergne, C. Vic, F. Roquet, A. F. Waterhouse, C. B. Whalen, Y. Cuypers, P. Bouruet-Aubertot, B. Ferron, and T. Hibiya. A parameterization of local and remote tidal mixing. J. Adv. Model. Earth Sys., 12:e2020MS002065, 2020.

[15] A. de Vos and 11 others. The M/V X-Press Pearl nurdle spill: Contamination of burnt plastic and unburnt nurdles along Sri Lanka's beaches. ACS Environ. Au, 2:128–135, 2022.

[16] P. Delandmeter and E. van Sebille. The Parcels v2.0 Lagrangian framework: new field interpolation schemes. Geosci. Model Dev., 12:3571–3584, 2019.

[17] M. H. DiBenedetto and N. T. Ouellette. Preferential orientation of spheroidal particles in wavy flow. J. Fluid Mech., 856:850–869, 2018.

[18] M. H. DiBenedetto, N. T. Ouellette, and J. R. Koseff. Transport of anisotropic particles under waves. J. Fluid Mech., 837:320–340, 2018.

[19] A. Gay, B. Favier, and G. Verhille. Characterisation of flexible fibre deformations in turbulence. EPL, 123:24001, 2018.

[20] C. Higgins, J. Vanneste, and T. S. van den Bremer. Unsteady ekman-stokes dynamics: Implications for surface wave-induced drift of floating marine litter. Geophys. Res. Lett., 47:e20220GL089189, 2020.

[21] J. R. Jambeck, R. Geyer, C. Wilcox, T. R. Siegler, M. Perryman, A. L. Andrady, R. Narayan, and K. L. Law. Plastic waste inputs from land into the ocean. Science, 347:768–771, 2015.

[22] M. L. A. Kaandorp, H. A. Dijkstra, and E. van Sebille. Closing the Mediterranean marine floating plastic mass budget: Inverse modeling of sources and sinks. Env. Sci. Tech., 54:11980–11989, 2020.

[23] T. Kukulka, G. Proskurowski, S. Mor' et-Ferguson, D. W. Meyer, and K. L. Law. The effect of wind mixing on the vertical distribution of buoyant plastic debris. Geophys. Res. Lett., 39:L07601, 2012.

[24] K. Kvale, A. E. F. Prowe, and A. Oschlies. A critical examination of the role of marine snow and zooplankton fecal pellets in removing ocean surface microplastic. Front. Mar. Sci., 6:808, 2020.

[25] M. Long, B. Moriceau, M. Gallinari, C. Lambert, A. Huvet, J. Raffray, and P. Soudant. Interactions between microplastics and phytoplankton aggregates: Impact on their respective fates. Mar. Chem., 175:39–46, 2015.

[26] T. B. Oehmke, A. D. Bordoloi, E. Variano, and G. Verhille. Spinning and tumbling of long fibers in isotropic turbulence. Phys. Rev. Fluids, 6:044610, 2021.

[27] V. Onink, C. E. Jongedijk, M. J. Hoffman, E. van Sebille, and C. Laufk" otter. Global simulations of marine plastic transport show plastic trapping in coastal zones. Environ. Res. Lett., 16:064053, 2021.

[28] J. Pearson, B. Fox-Kemper, R. Barkan, J. Choi, A. Bracco, and J. C. McWilliams. Impacts of convergence on structure functions from surface drifters in the Gulf of Mexico. J. Phys. Oceanogr., 49:675–690, 2019.

[29] A. Porter, B. P. Lyons, T. S. Galloway, and C. Lewis. Role of marine snows in microplastic fate and bioavailability. Env. Sci. Tech., 52:7111–7119, 2018.

[30] J. C. Prairie, Q. W. Montgomery, K. W. Proctor, and K. S. Ghiorso. Effects of phytoplankton growth phase on settling properties of marine aggregates. J. Mar. Sci. Eng., 7:265, 2019.

[31] K. Seshasayanan and B. Gallet. Surface gravity waves propagating in a rotating frame: The Ekman-Stokes instability. Phys. Rev. Fluids, 4:104802, 2019.

[32] J. R. Taylor. Accumulation and subduction of buoyant material at submesoscale fronts. J. Phys. Oceanogr., 48:1233–1241, 2018.

[33] E. van Sebille and 37 others. The physical oceanography of the transport of floating marine debris. Env. Res. Lett., 15:023003, 2020.

[34] E. van Sebille, C. Wilcox, L. Lebreton, N. Maximenko, B. D. Hardesty, J. A van Franeker, M. Eriksen, D. Siegel, F. Galgani, and K. L. Law. A global inventory of small floating plastic debris. Env. Res. Lett., 10:124006, 2015.

[35] G. Verhille. Deformability of disks in turbulence. J. Fluid Mech., 933:A3, 2022.

[36] C. P. Ward, C. J. Armstrong, A. N. Walsh, J. J. Jackson, and C. M. Reddy. Sunlight converts polystyrene into carbon dioxide and dissolved organic carbon. Env. Sci. Tech. Lett., 6:669–674, 2019.

Chapter 3

BIRS Workshop on Rate-induced Transitions in Networked Systems (22w5067)

March 27 - April 1, 2022

Organizer(s): Vítor V. Vasconcelos (University of Amsterdam, The Netherlands), Flavia Maria Darcie Marquitti (Universidade Estadual de Campinas, Brazil), Lisa C. McManus (University of Hawai'i at Mānoa, USA), Theresa W. Ong (Dartmouth College, USA)

Motivated by ecological systems as prototypical examples of complex adaptive systems (CAS), this workshop discussed frontiers in the multidisciplinary understanding of tipping points in adaptive structures. We set rate induced critical transitions (RIT)as the focal point or phase changes between different states. Taking biological examples, we found novel technological and social systems models that exhibited analogous properties. We also focused on the role of interconnectivity in maintaining adaptive capacity and aimed to identify a collection of problems under rapidly changing exogenous forcing in order to build interdisciplinary knowledge.

We had participants that were versed in critical transitions across many fields and challenged them to think of the adaptive capacity of their systems, the rate of change in exogeneous drivers, and the impacts of their interaction. By collating multidisciplinary problems liable to rate-induced transitions, we have (i) created updated definitions of resilience within this context, (ii) identified a gap of existing databases in a variety of fields that can inform early-warning signals under RIT, (iii) created a management framework for complex adaptive systems, and (iv) proposed the creation of a standardized set of guidelines for the future collection of data across fields.

Overview of the Field

Many of the world's processes are increasing their pace: the development and deployment of technology accelerates with economic growth [1], information flows increase with the automation of data analysis [2], data-driven models of world-markets lead to near-instantaneous transactions [3]. In the natural world, increasing emissions of CO_2 drive ecological change at all scales [4]. Socially, internet access expands and also disrupts ways of communicating and living [5]. Fortunately, all of these systems have a range of mechanisms to respond and adapt to change, generating resilience [6]. Schools now teach students the perils of the online world, and there is a growing effort to highlight and prevent the spread of false information [7]. Financial systems develop capital buffers and additional regulation and deregulation practices to prevent economic bubbles and market crashes [8]. Species adapt and migrate as temperatures and resource availability change [9]. Given that their elements are heterogeneous, interconnected and responsive, we label examples such as these as "complex adaptive systems". However, we do not expect unbounded adaptability from these systems. Indeed, we acknowledge and even exploit those limits, such as by treating bacterial infections through administering a large amount of antibiotics as opposed to a gradually increasing dosage.

The current theory of tipping points includes the study of transitions that occurs through bifurcation, noise, or external rates of change [13]. The climate system is a prototypical example exhibiting critical phenomena. E.g., when looking at the weather as a fast-process (stochasticity) and climate as a slow process, it can lead to Noise-induced tipping. But depending on which parameters of the system one varies, systems can exhibit different types of transitions [13]. Thompson & Sieber [17] distinguished between safe bifurcations (where an attracting state loses stability but is replaced by another "nearby" attractor), explosive bifurcations (where the attractor dynamics explore more of the phase space but still returns to the vicinity of the old attractor) and dangerous bifurcations (where the attractor dynamics after bifurcation are unrelated to its previous path). Identifying tipping points in these systems using time series requires the separation of time scales in the change in parameters, and between the slowest and faster decaying modes of the system.

Well-known bifurcation forms and noise-induced critical transitions hinge on the idea of escaping from attractors. However, adaptability challenges concepts of stability in such systems. Changing a control parameter is typically visualized as a shifting position on a static attractor, but in an adaptive system, there is a reshaping of the attractor itself such that it may revert to its original form. Rate-induced transitions occur when there is a critical value on the rate of change of a control parameter beyond which the system can no longer maintain the shape of the attractor, causing the system to tip into a different state. This type of transition is often related to a separation of time scales [15]. Early warning signals for the different types of transitions have also been studied [16, 14].

Applications of bifurcation theory and noise-induced transitions have permeated through many fields, but rate-induced transitions seems to be less well-known and have found fewer applications. A remarkable exception is the field of ecology. Rate-induced critical transitions have been applied to ecological models [11] and early warning signals for these transitions have been identified in salt marshes [12]. Salt marshes adapt to sea level rise by increasing sedimentation. The physical process of sedimentation, however, has a maximum rate and requires the presences of marshes. Finally, Siteur et al. [11] argue that the current resilience concept relies too much on bifurcation-type concepts and may not be sufficient to prevent system collapse.

In this workshop we aimed at extending and unifying the study of rate-induced critical transitions to a broad range of disciplines, with a focus on networked systems. Our goal was to frame rate-induced transitions in social and socio-technical, socio-ecological, and ecological systems as an intrinsic property of complex adaptive systems and further the study of critical transitions and their early-warning signals. As illustrated, the theory of rate-induced critical transitions plays a role in many different disciplines, such as physics, chemistry, engineering, ecology, epidemiology and the social sciences. We take as cornerstones examples originating from ecology, the field of prototypical complex adaptive systems and use it to extend the current concept of resilience and develop a new framework for systems management.

Field activity

While there have been a few recent workshops on the general topic of critical transitions and/or tipping points, these events are typically focused on either (i) presenting general analytical or numerical methods to identify early-warning signals or (ii) highlighting specific examples in fields such as ecology, medicine, and engineering. Overall, research relating to rate-induced transitions are typically not included, and if there is some discussion, it is a notably small fraction of talks. For example,the *"Predicting Transitions in Complex Systems"* Workshop at the Max Planck Institute for the Physics of Complex Systems (Dresden, Germany; April 2018 [1]) and the *"Anticipation of Critical Transitions in Complex Systems"* Workshop at the University of Muenster (Muenster, Denmark; Aug 2017 [2]) did not feature talks on rate-induced transitions. In the very recent *"Workshop on Critical Transitions in Complex Systems 2020"* (online workshop, July 2020 [3]), there were only two talks out of 31 total that discussed research on rate-induced transitions.

Workshops that focus specifically on integrating biology and complex system science show a similar pattern. Though there were a few recent workshops that mentioned the importance of understanding the rate of transition and feedback processes underlying these transitions, none have specifically studied the rate-induced aspect of these changes. A series of recent workshops at NIMBIOS focused on "long transients" in biological systems

[4], investigating how long-term transient dynamics emerge and how theory can be used to predict timing of sudden regime shifts [5]. However, it remains a challenge to understand how rate-changing feedbacks could affect the timescale of persistence of these transient states, inhibiting progress towards greater predictability. Another workshop was aimed at extending sustainability theory, emphasizing that "feedback" is often left out in forecasts despite its importance [6]. The limitations in these former workshops suggest the importance of our proposed work.

Motivation

Traditionally, tipping points in ecological systems are based on the idea of steady-state analysis. The flexibility to adapt to different conditions for living organisms can be attributed to behaviour, niche expansion, phenotypic plasticity, evolution, and most likely, a combination of these and other processes. Importantly, each of those mechanisms operates on its own timescale, generating typical and maximum rates for the adaptive response. Usually, the conditions of ecological systems change at a slower pace than adaptive capacity, allowing for a decoupling of parameter changes and ecological dynamics. However, the current anthropogenic disturbances are, on the one hand, eroding or destroying some of the mechanisms that support adaptive capacity and, on the other hand, increasing the rate at which the environment changes. Therefore, while time scale separation of external forces and adaptation has been possible so far, as the rates of external change increases, so does the importance of studying the systems' ability to cope with change. We need to focus more of our research efforts on the study of rate-induced tipping points and in distinguishing and identifying early-warning signals associated with these transitions.

Approach

In our workshop, we took a range of ecological processes, from coral reefs – the fastest-changing large-scale ecosystem on the planet – to tundra plant-root mutualisms – characterized by slow underlying processes – and collate the processes that contribute to the adaptability of these systems under changing environments. In parallel, we identified systems from different disciplines that exhibit analogous rates of change and adaptive properties, as well as the solutions and tools that have resulted from the study of these systems in transition.

To do so, our participants actively work in areas that included technology diffusion, the energy transition (green technology revolution), transitions to sustainable food systems, social networks, tipping-points in social norms, and prosocial computing in human-robot hybrid systems, all of which contain the key ingredients for rate-induced transitions. We invited participants versed in critical transitions and challenge them to think of the adaptive capacity of their systems, the rate of change in exogenous drivers, and the impacts of the interaction between both within their study systems.

We looked to emphasize the role of interconnectivity-based (network) properties on the elements of these systems. For example, diffusion networks across different marine communities can serve to distribute genes that have adapted to different environments; these connections are affected by changing ocean currents. Social capital, drivers of organizational success and livelihood adaptation are defined by the connections among individuals of a certain group. For example, online social networks enhance the number and interchangeability of those connections but weaken their strength. Large-scale coordination problems among humans, where tipping points are desired, can be slowed down by growing conflict but can also be influenced by interactions with artificial agents, which can divert information flows or simply create empathetic relations with humans. These examples illustrate the multiple, entangled disciplines that are relevant to our topic. Our interdisciplinary perspective piece (In prep.) will be framed around the goal of defining resilience in the context of rapid change and identifying early-warning signals that precede these transitions.

In preparation, we asked our participants to reflect on the key properties of their system that lead to rate-induced tipping. Namely, they filled out the form in the following subsection. This form was filled in for various subsystems and will serve as the cornerstone expert-based data to demonstrate the existence of rate-induced transitions in multiple fields and the important role of networks in those systems. The results of this form can be found in the "Presentation Highlights" section below.

To stimulate initial discussions and get our interdisciplinary scholars on the same page, we provided a set of references. These initial readings were divided into basic readings, followed by advanced and applied ones. The list illustrating the organization is in the section "Form: Initial literature" below. The literature was discussed by the groups, focusing on understanding of the differences between the different types of tipping points. The "Overview of the Field" section above presents the main points discussed by the group. Below, we provide the list sent to our participants:

Finally, we used the theoretical framing of rate-induced transitions to identify potential databases that can be used to identify early-warning signals across different systems. In doing so, we aimed to develop a standard set of guidelines for the future collection of data that will facilitate the analysis and identification of rate-induced transitions and their early-warning signals across disciplines.

By compiling this information and discussing it, we hope to raise awareness of the RIT field and find various novel applications of critical transition theory.

Initial goals

Given the current increasing pace of globally relevant processes across different fields, this workshop aimed to:

- take the examples of ecological processes studied under the context of rate-induced transitions and characterize the common properties of multidisciplinary problems that are expected to have rate-induced transitions,
- identify non-ecological models/examples that exhibit the same mathematical properties,
- identify multi-disciplinary examples of rate-induced critical transitions with a focus on networked systems,
- propose a definition for resilience based on the pace of adaptation and disturbance in a given system, and
- locate different databases in each field that could detect early-warning signals for rate-induced transitions, develop a standard set of guidelines for the collection of data in that context, and, ideally, propose the creation of specific equivalent datasets across fields to facilitate comparative studies.

Form: Systems with Rate-induced Transitions

The following is the list of questions we posed about each participant's system to understand whether it may exhibit rate-induced transitions and what elements contribute to the transition.

- System / System property:
- Sources of stress

  Statistically invariant/historical Sources of stress:

  Sources of stress/external element changing (at increasing rates):
- Adaptive/acclimation abilities to cope with changing elements

  What sources of variability of the environment contribute to adaptation:

  Adaptive ability to cope with changing elements deriving from networks:
- Is there evidence that your system has multiple stable equilibria? I.e., does the system change irreversibly after a large disturbance?

Form: Initial literature

Basic readings:

- Siteur, K., Eppinga, M. B., Doelman, A., Seiro, E., & Rietkerk, M. (2016). Ecosystems off track: rate-induced critical transitions in ecological models. Oikos, 125(12), 1689-1699.
- Neijnens, F. K., Siteur, K., van de Koppel, J., & Rietkerk, M. (2021). Early warning signals for rate-induced critical transitions in salt marsh ecosystems. Ecosystems, 24(8), 1825-1836.

- Ashwin, P., Wieczorek, S., Vitolo, R., & Cox, P. (2012). Tipping points in open systems: bifurcation, noise-induced and rate-dependent examples in the climate system. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 370(1962), 1166-1184.

Theory:

- Ashwin, P., Wieczorek, S., Vitolo, R., & Cox, P. (2012). Tipping points in open systems: bifurcation, noise-induced and rate-dependent examples in the climate system. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 370(1962), 1166-1184.
- Siteur, K., Eppinga, M. B., Doelman, A., Siero, E., & Rietkerk, M. (2016). Ecosystems off track: rate-induced critical transitions in ecological models. Oikos, 125(12), 1689-1699.
- Ritchie, P., & Sieber, J. (2016). Early-warning indicators for rate-induced tipping. Chaos: An Interdisciplinary Journal of Nonlinear Science, 26(9), 093116.
- Ashwin, P., Perryman, C., & Wieczorek, S. (2017). Parameter shifts for nonautonomous systems in low dimension: bifurcation-and rate-induced tipping. Nonlinearity, 30(6), 2185.
- Hahn J. Hopf bifurcations in fast/slow systems with rate-dependent tipping. arXiv preprint arXiv:1610.09418. 2016 Oct 28.

Applications:

- Neijnens, F. K., Siteur, K., van de Koppel, J., & Rietkerk, M. (2021). Early warning signals for rate-induced critical transitions in salt marsh ecosystems. Ecosystems, 24(8), 1825-1836.

Tipping-points/critical transitions:

- Suz, L. M., Bidartondo, M. I., van der Linde, S., & Kuyper, T. W. (2021). Ectomycorrhizas and tipping points in forest ecosystems. New Phytologist, 231(5), 1700-1707.
- Vasseur, D. A., DeLong, J. P., Gilbert, B., Greig, H. S., Harley, C. D., McCann, K. S., & O'Connor, M. I. (2014). Increased temperature variation poses a greater risk to species than climate warming. Proceedings of the Royal Society B: Biological Sciences, 281(1779), 20132612.
- Vanselow, A., Wieczorek, S., & Feudel, U. (2019). When very slow is too fast-collapse of a predator-prey system. Journal of theoretical biology, 479, 64-72.
- Scheffer, M., Van Nes, E. H., Holmgren, M., & Hughes, T. (2008). Pulse-driven loss of top-down control: the critical-rate hypothesis. Ecosystems, 11(2), 226-237.

Adaptability mechanisms:

- Valdovinos, F. S., Moisset de Espanés, P., Flores, J. D., & Ramos-Jiliberto, R. (2013). Adaptive foraging allows the maintenance of biodiversity of pollination networks. Oikos, 122(6), 907-917.

Environmental change:

- Pinek, L., Mansour, I., Lakovic, M., Ryo, M., & Rillig, M. C. (2020). Rate of environmental change across scales in ecology. Biological Reviews, 95(6), 1798-1811.

Networks for resilience:

- McManus, L. C., Vasconcelos, V. V., Levin, S. A., Thompson, D. M., Kleypas, J. A., Castruccio, F. S., & Watson, J. R. (2020). Extreme temperature events will drive coral decline in the Coral Triangle. Global Change Biology, 26(4), 2120-2133.

- McManus, L. C., Tekwa, E. W., Schindler, D. E., Walsworth, T. E., Colton, M. A., Webster, M. M., & Pinsky, M. L. (2021). Evolution reverses the effect of network structure on metapopulation persistence. Ecology, 102(7), e03381.

Presentation Highlights

We had a total of 19 talks that spanned multiple disciplines across three broad groups: ecological systems (e.g., forests and phytoplankton), social-ecological systems (e.g., infectious disease and agricultural systems) and social systems (e.g., manufacturing systems and political ideologies) - see Figure 3.1 for summaries. A common thread was the CAS perspective that was highlighted across all talks. The CAS characteristics that were particularly relevant included (1) the ability of each focal system to adapt in response to changing external conditions and (2) the existence of feedback loops and non-linearities.

Most participants presented work on a single system or a set of related systems that have the potential to exhibit rate-induced transitions (a handful of participants presented broad overviews of rate-induced transitions or a related topic). In general, participants working in ecological and social-ecological systems cited environmental seasonality (e.g., temperature, rainfall, fire) as the statistically invariant or historical stressor. For these systems, the sources of stress were largely related to climate change which affected the frequency, magnitude and overall ranges of cyclical environmental dynamics. Participants working in social systems cited general socio-economic disturbances or pulses as the major source of stress. Disturbances to social networks that modified the strength of edges/links between or within communities or removed nodes from the network was another major stressor. Even though few participants were actively working on rate-induced transitions before the workshop, it was evident that all participants were working on at least one system that had the potential to exhibit rate-induced transitions, even if participants had not yet formally addressed this question in their own work. The presentations served as the basis for both formal and informal discussions (in terms of addressing generalities and differences among systems), and was also the source for the initial material and ideas for the direction of the perspectives paper.

Advocating for JEDI as an Early Career Scholar Workshop

In this workshop, we focused on providing examples of strategies for early career scholars in promoting Justice, Equity, Diversity, and Inclusion (JEDI) in academic workspaces. The workshop began with definitions of JEDI and a review of the academic literature regarding common inequities in academic spaces towards particular identities (race, ethnicity, gender) and their various drivers (systemic bias, implicit bias, stereotype threat). As early career scholars, workshop participants are at critical points in their careers, where they are both the subject of significant academic JEDI barriers to success and also have new opportunities to enact change as they advance in their careers and take on more leadership roles and responsibilities.

Though always historically important and pressing, JEDI has become an increasingly dominant issue for triage in academic spaces as our workspaces become diverse and power structures shift in accordance. Today's early career scholars will be leaders in ushering in new policies to create more just, equitable, diverse and inclusive communities. The workshop provided examples from the literature of methods that new PIs can implement as they recruit and support students and staff. These include code of conducts, course design and departmental changes including the development of JEDI programs. The workshop organizer, Prof. Theresa Ong, provided an example of JEDI programming, the EEES Scholars Program at Dartmouth College. This program provides guidance for applicants of minoritized groups in the graduate school application process including CV building, formal and informal mentorship, and mock interviews. In our BIRS workshop, Prof. Ong talked through the challenges and

Presenter	System or System Property	Statistically invariant/historical stressor	Source of stress/changing external element	Sources of environmental variability	Adaptive ability derived from networks	Evidence of multiple stable equilibria?
Jude Kong [Eco]	Phytoplankton light intensity	Seasonal light variation	Increasing light			No
Wenying Liao [Eco]	Forest ecosystems	Seasonal variation	Increasing temperature, precipitation	Rapid evolution, phenotypic plasticity	Dispersal, root network for the redistribution of water/nutrients	Theoretically yes, but little empirical evidence
Mingzhen Lu [Eco]	Fynbos-forest system; savanna system; Tropical monodominant forest; Southern temperate forest	Fire (depletes nutrients), rainfall, herbivory, fungal associations	Climate change			Yes
Benton Taylor [Eco]	Forest regeneration	Hurricane/cyclone frequency; return interval of land clearing; rate of patch clearing; fire frequency	Climate change and increasing anthropogenic stressors			Yes
Senay Yitbarek [Eco]	Duckweed/rapid evolution/spatial structure/species interactions/trait distribution	Nutrient pollution; increasing floating plant biomass	Temperature, decreasing plant species richness	Variation in light, temperature and nutrients can contribute to rapid evolution	Genetic variation - short generation times and small size; plant-microbiome interactions - rapid evolution and host fitness	Yes
Victoria Junquera [SES]	Agricultural systems, rural regions	Weather, pests, regulations	Changing markets, changing land regulations (insecurity), income variability	Land use and/or livelihood changes	Social networks facilitate livelihood changes; networks as preconditions to crop booms	Yes
Juan Rocha [SES]	Ecological regime shifts; linked ecosystems		Climate, fires, erosion, agriculture, urbanization			Yes
Andrew Tilman [SES]	Livestock grazing pressure (Mongolian nomadic pastoralists)	Seasonal temperature, occasional droughts	Climate change, more extreme seasonality; increasing frequency of drought	Seasonal temperature and rainfall	Mongolians can move their herds to different locations or can move to the city	Yes
Luojun Yang [SES]	Infectious diseases, immunological and behavioral responses to infections	Pathogen evolution, demographic dynamics	Seasonality due to climate change, vaccination, policy intervention	Variability in immune responses and contact networks	Coupling and decoupling between disease transmission and information networks; clustering of susceptible individuals	Yes
Elisabeth Krueger [Soc]	Cities; Urban socio-technological development		Sea level rise and land subsidence; urban growth and infrastructure development	Seasonal water scarcity		
Flávio Pinheiro [Soc]	Countries, Regions, Organizations	"Market competition," technological shifts	Extreme socio-economic events, climate change	Technological and policy heterogeneity, changes in value chains		S-shaped curve of development suggests 2 basins of attraction
Fernando Santos [Soc]	Opinion consensus/polarization in social networks	Removing links between communities	Recommendation algorithms, strengthening links within communities/weakening between communities			Depending on network structure, polarization and consensus states can be equilibria
Peter Sloot [Soc]	Illegal cannabis production chain (Netherlands)		Law enforcement targeting certain links in the chain		Information dissipation (out-of-equilibrium dynamics) peaks at nodes with intermediate number of connections	

Figure 3.1: Summary of ecological systems (Eco), social-ecological systems (SES) and social systems (Soc).

opportunities in designing, advocating and supporting the successful implementation of these important bridge programs and how institutional structures impact advocacy strategies.

The workshop was specifically planned for Wednesday morning to allow for interpersonal connection and community building in the afternoon during our scheduled half day off. During the afternoon, many informal discussions were had across workshop attendees regarding specific institutional, programmatic, departmental barriers to JEDI and many impromptu brainstorming sessions emerged for co-developing strategies to implement change.

We found that the workshop brought together participants in mutual support of one another and the JEDI mission, as well as did a lot to build camaraderie and direct Early Career Scholars in obtaining information and strategies for implementing JEDI programming at their own institutions. Feedback from participants and BIRS staff was positive, with many expressing a desire to carry out outlined best practices at their own institutions.

Demographic information

At the close of our BIRS workshop participants, we distributed an anonymous demographic survey. We felt strongly that participants should self-identify themselves. Out of 30 total workshop participants, including 11 in-person attendees, 19 responded. Our participants were gender balanced with 57.9% identifying as men, 36.98% women, and 5.35% non-binary. Out of all survey respondents, 39% identified as White, Caucasian or European-American, while 66.6% identified as Asian, Asian-American, Black, Afro-American, Dutch, Hapa, and Latin American (participants could include more than one identity). Our participants are citizens in 10 different countries and were born in 13 different countries (see Fig. 3.2).

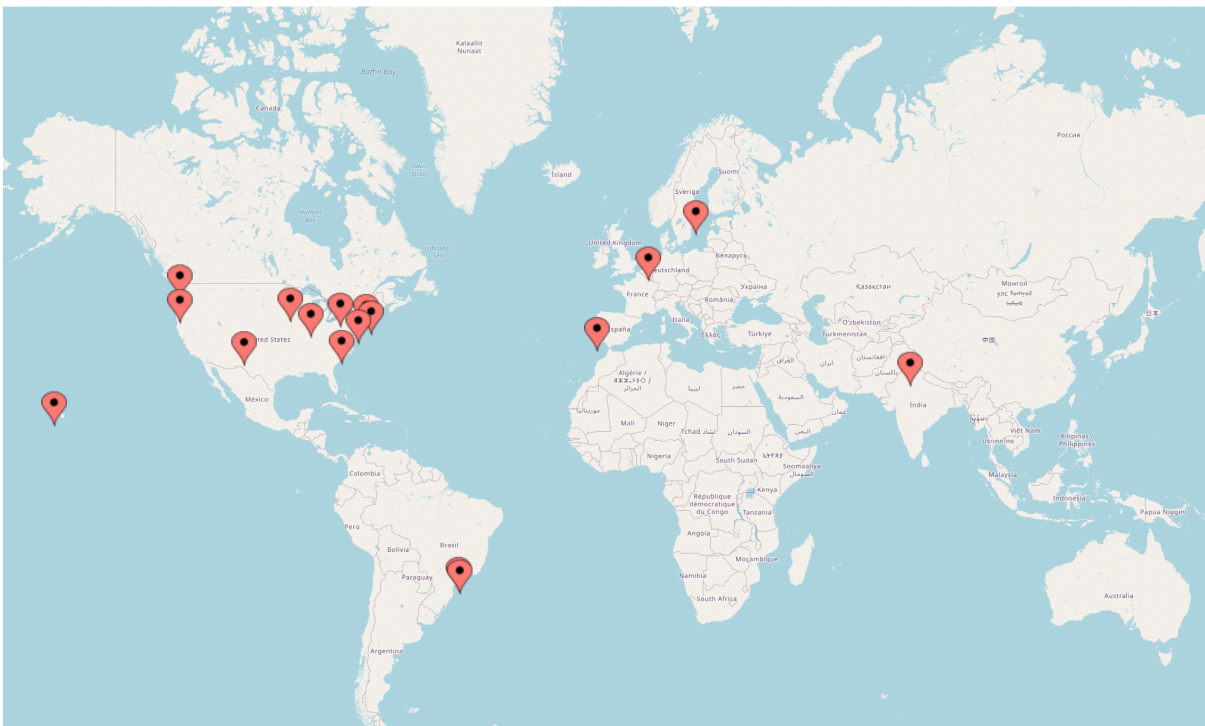
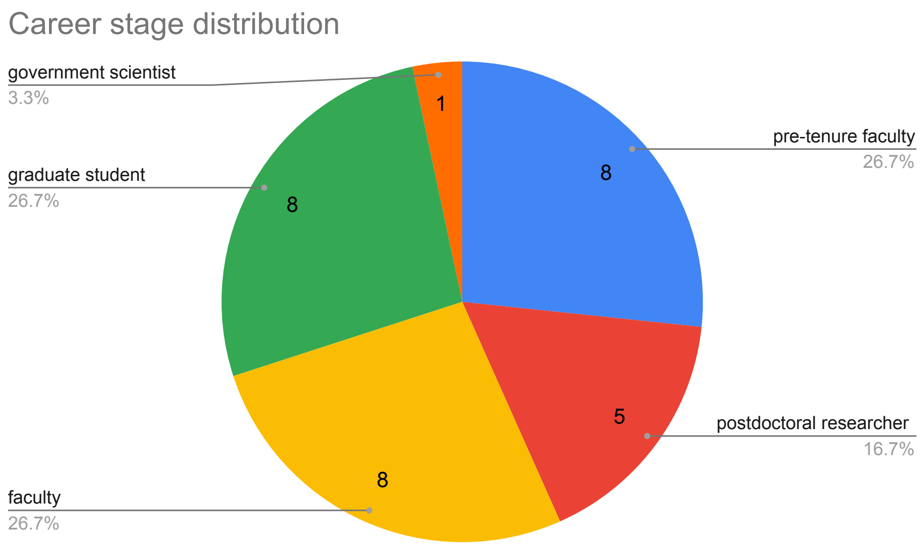


Figure 3.2: Map of workshop participant home institutions.

Regarding educational background, 15.8% of participants reported that the highest educational achievement of their parents/guardians was high school or similar secondary education or below (no formal schooling). An additional 26.3% had parents with a Bachelors but no graduate-level education. Overall, we feel that our BIRS workshop achieved high levels of demographic, background and career-stage diversity. We can see that there was an even distribution of faculty to graduate students, with a skew towards early career researchers (Fig. 3.3). Still, we acknowledge the continued need to strive for greater representation in coming years.



stage distribution.pdf

Figure 3.3: Career stage distribution of workshop participants.

Scientific Progress Made

During the workshop, organizers and invited participants discussed the readings, which were distributed prior to the meeting. The RIT literature has a strong connection with the tipping points literature. During the first day of the workshop, we discussed the three main types of tipping points that are already mathematically formalized: bifurcation (B), noise-induced (N) and rate-dependent (R) and their importance for complex adaptive systems. In the second phase of the workshop, the group explored and discussed the formalization of how these three types of tipping points are entangled. In the third phase, the group discussed how networked systems are affected by rate-induced phenomena, in the context three different (sets of) CAS: ecological, social, and social-ecological systems.

Combining types of tipping points

Complex adaptive systems are characterized by a multitude of positive and negative feedbacks between their elements. These resulting feedback loops can lead to multiple alternative regimes. Each (likely dynamic) regime has a basin of attraction, a set of state values or properties of the system that characterize it; these properties are typically associated with a stable state. External or environmental states affect these feedbacks in different ways, potentially causing different transitions (see Figure 3.4).

Three types of critical transitions are discussed in the literature, depending on how the environment of the system affects it. When environmental change increases the frequency and/or magnitude of perturbations, a critical environmental value can lead to a regime change induced by noise; this is known as a noise-induced transition (t_N). If the environmental state affects the strength of feedbacks, a critical value of the environment can occur, above which one of the feedbacks always dominates, leading to the disappearance of one of the current equilibria; this results in a change-induced transition or bifurcation tipping (t_B). If the changes in the environment move the basins of attraction but not their shape, small environmental changes lead to no regime shift. The system will only exhibit a transition when the movement of the basin of attraction outpaces the ability of the system to remain inside the basin of attraction of a current regime, leading to a rate-induced transition (t_R). Most changes in the external environment of a system will have all the three components (t_{B+R+N}). In Fig. 3.5, we show an example of how the different types of transitions can be combined in a simple example.

Hypotheses were raised about how the co-occurrence of the three types of bifurcations would anticipate or

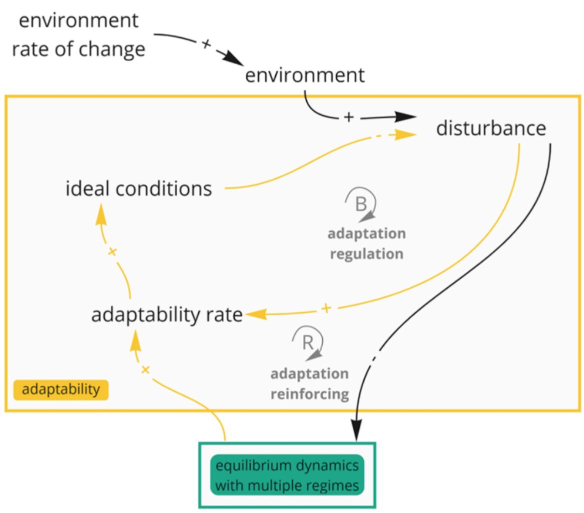


Figure 3.4: Positive and negative feedbacks affected by external/environmental changes in magnitudes or rates.

postpone a critical transition, both in time or parameter range. These hypotheses were formulated geometrically in terms of the shape of the attractor.

This formulation also set the basis for understanding systems' safe operating spaces in terms of the i) internal state change (exogenous, instantaneous shock), ii) external parameter change, and iii) rate of parameter change.

$$\frac{dx}{dt} = \left(1 - \frac{E[t]}{T_b}\right) - (x - E[t])^2$$

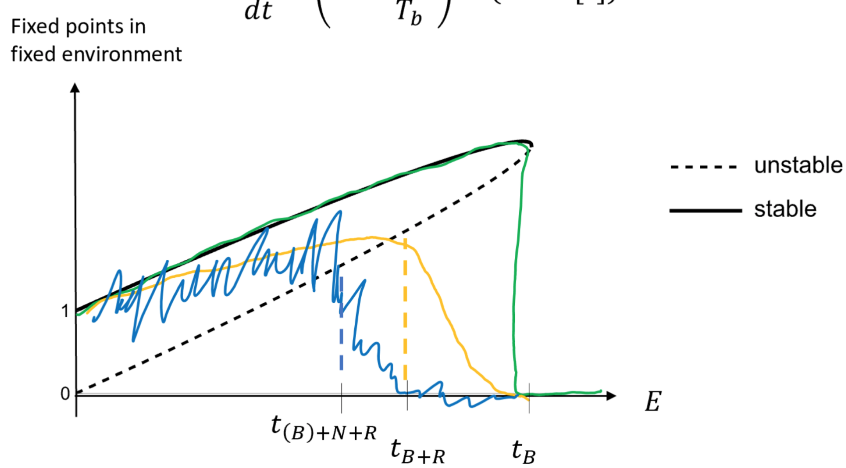


Figure 3.5: Tipping points and combinations. Bifurcation tipping point (t_B), bifurcation combined with rate induced tipping point (t_{B+R}), and bifurcation, noise and rate induced tipping point ($t_{(B)+N+R}$). Figure adapted from one of the presentations, based on the work of [18].

RIT in networked systems

The lag between external change and system adaptation can induce critical transitions. Systems of interest in social, ecological, social-ecological, and social-technological domains are often composed of agents organized in complex networks: individuals connected in social media platforms cross-validate information with their peers and counter misinformation, a case in which the system can tip into an undesirable state where fake news become prevalent, or a desirable state where truth prevails; coral reefs connected through current flows can receive species and genetic diversity such that the impact of changing temperature on one reef can be mitigated through re-colonization; urban dwellers receive resources (e.g., water, food, and energy) through infrastructure networks that are also adapted as demands change and grow while urbanization proceeds.

Pathogens of infectious diseases spread via contact networks of hosts, the topology of which affects the rate of transmission, the probability of outbreaks, and the emergence of new strains. From a pathogens perspective, a highly connected contact network of hosts is more likely to withstand the modification of edges/links due to perturbations such as enhanced human intervention strategy, ensuring the continuation of the spread of the disease at larger scales. However, when the removal of the connections happens at a very fast rate, and the whole region is immediately separated into individual isolated patches at the beginning of an infectious disease, the local infections will be more likely to be trapped within the region itself. A similar rate-induced transition can happen in scenarios when the intervention efforts are released at a fast versus at a slow rate. A fast recovery of the topology of contact networks relative to that of the pre-intervention scenario will enable the spread of disease from one local patch to another, the exchange of different dominant strains across patches, and further the reemergence of the disease at a global scale. A slow recovery of the topology of contact networks will result in local endemics. Finally, if the local endemics die out before regional connections are rebuilt, the chance of a global pandemic is less likely.

Questions raised during the workshop concerned the ability of networked system to cope with the challenges imposed by RIT: Can networks rewire fast enough to avoid RITs? Can network topologies reinforce environmental change and induce RITs? Can network properties inform interventions to prevent RITs?

These networked systems have the potential to affect the spread of disturbances that initially occur in different nodes: nodes can mutate or adapt based on neighborhood properties; links might be added, removed, swapped, or rewired; and specific network topologies can amplify or suppress diffusion processes. As a result, networks in the context of complex adaptive systems can strongly impact the change in rates imposed by external stressors and the rate at which systems can potentially adapt. Network structure can therefore determine whether a rate-induced regime shift can occur.

Different network topologies can represent different patterns of interaction, which can in turn affect the speed of environmental perturbations and how the system adapts. It is thereby instrumental to formalize interactions through networks and analyze their structural properties to understand how rates effectively lead to regime shifts. An example is the spread of computer viruses on networks. In random networks, the virus spreads only if the infection rate is above a critical threshold. In scale-free networks, on the other hand, the threshold is zero and viruses always spread. Network science already offers a set of metrics and distributions that can be used to describe interaction patterns and understand how different scales of interaction relate with the probability of regime-shifts:

- **Network connectedness and degree distribution; centrality measures.** Degree distribution is a basic network property. Random graphs are characterized by binomial degree distributions, determined by the probability that two nodes are connected. Scale-free graphs, on the other hand, have power law degree distributions with exponents in the range -3 to -2. Regime shifts in scale-free networks are usually very different from their analogues in random networks. Examples include explosive synchronization of oscillators and the spread of epidemics.

- **Modularity.** Modularity in networks creates potential mismatch between rates within and between groups. Such separation of time scales could trigger regime shifts in the system. The patterns of interaction within CAS are defined by the different possible network topologies representing the system.

- **Network motifs.** Network motifs are patterns of interconnections (or subgraphs) occurring in complex networks at numbers significantly higher than those in randomized networks. These recurrent simple building blocks of the complex network can be a result of evolutionary forces during the dynamical formation of the network. These relatively small groups can play an important role in biological and social networks, such as in financial trade patterns and on predator-prey trophic webs.

- **Percolation properties.** Percolation in networks is related to the formation of a long path in the network, such as in the formation of the giant component, making the network connected. As these long paths form, information can flow through the whole network. The percolation in networks can be increased by adding links with a given probability (bond percolation) or changing the node status with a given probability (site percolation). The percolation in networks is related to its robustness against failure of either nodes or links.

Other network properties and topologies might be important for understanding RITs. We believe that exploring the complexities of adaptive systems through network tools is an important step to understand the adaptive capacity of social, ecological and social-ecological systems to cope with perturbations that are happening faster and faster.

Outcome of the Meeting

Perspectives paper draft

The main outcome of the meeting is a perspectives paper that compiles the group's shared understanding of the necessary progress in the field of RITs in the following different dimensions: extending the range of applications to all CAS, focusing on the role of networks in RITs, and providing a novel management framework that explicitly considers the rate of external change. Below, we share our instructions regarding the future activities relating to the perspectives paper during and after the meeting. We believe that these helped to clarify the tasks at hand and the distribution of responsibilities.

Form: Instruction for perspectives piece

- In this meeting, the small working groups will discuss the proposed idea, as well as the outline and a proposed labor division.
- In a few months after the perspective paper is in good shape, we will call for another joint group meeting where the self-organized smaller working groups can pitch ideas.
- Everyone can then sign up for the different tasks. Next, the organizers (plus a small team) will work on the project to take it over the finish line.

Facilitated networking

Workshop participants, particularly early career scholars, reported high levels of satisfaction with networking opportunities, especially considering a dearth of conferences available during the COVID-19 pandemic. Our participants engaged with others at many career stages (full faculty, pre-tenure faculty, postdoctoral researchers, research professionals, and graduate students) across 17 institutions (Fig. 1). Graduate students worked in collaboration with faculty in note-taking and drafting our perspectives piece. During breakout writing sessions, graduate students were often paired with early career faculty and postdocs to guide writing on topics of shared interest.

Participants engaged with one another in both formal settings (group and breakout sessions) as well as informal meetings at meals and breaks. In-person participants also benefited from time spent together on hikes and other outdoor opportunities at Banff. Early career scholars also had a dedicated space to workshop advancing JEDI, detailed earlier in this report.

Inspired products

We are aware of the following products that were inspired by our workshop:

- Review paper on climate-change impacts on agroforestry futures from a RIT perspective submitted by the Ong Lab at Dartmouth College to the Journal of Experimental Biology, Summer 2022.
- Reimagining forest resilience through the lens of rate-induced transition theory Radcliffe Harvard Seminar Series, organized by Wenying Liao and Benton Taylor, March 2022.

- Reimagining forest resilience through the lens of rate-induced transition theory manuscript in preparation by Wenying Liao

Future collaborations

Many new research collaborations were inspired by the workshop proceedings including, but not limited to, the following:

- Forestry and agroforestry applications, Harvard University and Dartmouth College
- Socio-ecological systems thinking in regards to RIT

Future workshops

Many participants indicated a desire to continue conversations and develop theory in subsequent workshops. Some ideas include:

- Dartmouth College Conference Fund (2024-25) focused on RIT applications to agroforestry systems
- BIRS focused on spatial dimensions of RIT

Hybrid workshop format

Our workshop included 11 in-person and 29 virtual participants. Considering this distribution, we developed our workshop to maximize hybrid interactions. We fostered hybrid interactions by asking all in-person participants to also log into Zoom during group-wide discussions as well as in smaller breakout sessions. For all of our main talks, we assigned one of the workshop organizers to type a summary during the Q&A sessions in the chat for Zoom participants. Other workshop organizers were responsible for moderating questions from both in-person and for virtual participants.

Successes

When in-person participants also opened Zoom on their individual computers, virtual participants felt like they were one of many attendees rather than participating in isolation. Many virtual participants informed us that that they felt that they could easily contribute to the conversation. We found that we had consistent participation from virtual attendees throughout the workshop.

The chat summaries allowed our full group discussions to be quite effective. During Q&A sessions, we had active participation from both in-person and online participants. Many in-person participants noted that having written summaries of the Q&A helped reify ideas and frame their own contributions to the discussion. Virtual participants noted how the summaries helped clarify information from both speakers and commentators.

The smaller "disciplinary" working group meetings (in-person + online) were productive because all group members, most notably including the graduate students, felt comfortable sharing their ideas. Having a full group meeting soon after these small group meetings allowed people to address any outstanding questions from all perspectives.

Opportunities for improvement

We set up smaller "disciplinary" and "interdisciplinary" working groups, with fewer people split into three different physical and virtual rooms. After these meetings, we organized a final meeting with the whole group in the same room. We found that bringing everyone together for a final discussion, which lasted longer than the smaller groups, was a better use of time than our shorter "interdisciplinary" and split group sessions. For the "interdisicplinary" group sessions, participants took a long time to update their colleagues about discussions from "disicplinary" group sessions, while in the large group meeting, the communication flow was more direct and efficient.

In the future, we would recommend that for "interdisciplinary" group meetings, extra time should be allocated to ensure the group can come to consensus on topics of importance.

We also recommend increasing the number of full group discussions to allow for more streamlined interdisciplinary collaborations to take place.

The hybrid format posed some challenges in terms of timing for breaks and movement in physical space. Though virtual participants could conceptually log on immediately to new rooms, in-person participants had to find new spaces and Zoom links were sometimes confused in the process. We would advise establishing easy to follow links with informative titles such as breakout group IDs to help streamline future hybrid programming.

Other potential outcomes

During the workshop, we also set up a platform (in a shared live document) for people to devise their own tentative plans for future collaborations in specific related work. The instructions were organized in the following "Action Plan":

- Feel free to propose ideas for future collaborations
- Whoever proposed the idea, please write down a short summary of the project.
- Then, everyone should feel free to sign up (i.e., put your name down on the project) to work on that project after the workshop.
- The person that put down the idea will contact the others and organize the first (sub)group meeting.

Examples from our participants include potential work on:

- An opinion piece on RIT in planetary boundaries among three members;
- An article focusing on "Classical systems through the perspective of RIT: fisheries model", among five members;
- An article focusing on "Changing classical decision making from threshold focus to rate focus", among six participants;
- An article on "Population dynamics of decision making processes based on absolute value vs. a threshold on the rate of change";
- Studying "Discounting as a problem for political movement; the rate at which the risk is perceived changes that discounting parameter and how quickly politicians make decisions";
- "Dimensionality considerations: how many agents need to reach consensus affects the rate of adaptation (in a consensus-driven process); instances where top-down might lead to faster adaptation";
- Set up new projects/a new general discussion to look at "networks and scale" as a generalization of rate-induced transitions in networks, among six participants;
- Analysing how "SES: collective behavior, polycentric behavior, etc. disregard the speed at which you can achieve outcomes - those might be Nash equilibria, but it also depends on the speed of decision-making";
- Apply rate-induced transitions to "economic systems, since they focus on equilibrium dynamics"'
- 'Interplay between homophily and peer influence; polarization vs. non-polarization'
- Explore a 'Harvest model modified to show rate-induced transitions; separation of time scales', among three participants.

We expect that these future projects (and others not registered during the workshop) will generate many new and exciting research avenues.

Participants

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Campos, Amanda (Universidade de São Paulo)
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Junquera, Victoria (Princeton University)
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Krüger, Elisabeth (University of Amsterdam)
Levin, Simon (Princeton University)
Liao, Wenying (Harvard University)
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Bibliography

[1] C. Freeman, Economics of industrial innovation, Routledge, 2013.

[2] J. Van Dijck, T. Poell, and M. De Waal, The platform society: Public values in a connective world, Oxford University Press, 2018

[3] I. Aldridge, High-frequency trading: a practical guide to algorithmic strategies and trading systems, Vol. 604. John Wiley & Sons, 2013.

[4] G.T. Pecl, et al., Biodiversity redistribution under climate change: Impacts on ecosystems and human well-being, Science **355.6332** (2017), eaai9214.

[5] J. Van Dijck The network society, Sage, 2020

[6] D.R. Nelson, W.N. Adger, and K. Brown, Adaptation to environmental change: contributions of a resilience framework, Annu. Rev. Environ. Resour. **32** (2007), 395–419.

[7] D. Buckingham Media education: Literacy, learning and contemporary culture, John Wiley & Sons, 2013

[8] V.V. Acharya, L.H. Pedersen, T. Philippon, and M. Richardson, Measuring systemic risk, The review of financial studies **30**(1) (2017), 2–47.

[9] M. Lindner, M. Maroschek, S. Netherer, et al., Climate change impacts, adaptive capacity, and vulnerability of European forest ecosystems, Forest ecology and management **259**(4) (2010), 698–709.

[10] M. Lindner, M. Maroschek, S. Netherer, et al.,Complex adaptive systems, Daedalus **121**(1) (1992), 17–30.

[11] K. Siteur, M.B. Eppinga, A. Doelman, et al., Ecosystems off track: rate-induced critical transitions in ecological models, Oikos **125**(12) (2016), 1689-99.

[12] F.K. Neijnens, K. Siteur, J. van de Koppel, and M. Rietkerk, Early warning signals for rate-induced critical transitions in salt marsh ecosystems, Ecosystems, **24**(8) (2021), 1-12.

[13] P. Ashwin, S. Wieczorek, R. Vitolo, and P. Cox, Tipping points in open systems: bifurcation, noise-induced and rate-dependent examples in the climate system, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, **370**(1962) (2012), 1166-1184.

[14] P. Ritchie, and J. Sieber, Early-warning indicators for rate-induced tipping, Chaos: An Interdisciplinary Journal of Nonlinear Science, **26**(9) (2016), 093116.

[15] J. Hahn, Hopf bifurcations in fast/slow systems with rate-dependent tipping, arXiv, **preprint** (2016), arXiv:1610.09418.

[16] M. Scheffer, J. Bascompte, W.A. Brock, V. Brovkin, et al., Early-warning signals for critical transitions, Nature, **461**(7260) (2009), 53-59.

[17] J.M.T. Thompson and J. Sieber, Climate tipping as a noisy bifurcation: a predictive technique, IMA Journal of Applied Mathematics, **76**(1) (2011), 27-46.

[18] A. Synodinos, C. A. Aguilar-Trigueros, P. Gras, T. Heger et al., The rate of environmental change as an important driver across scales in ecology, ecoevorxiv, preprint (2021), 10.32942/osf.io/ajeyz.

Chapter 4

Women in Noncommutative Algebra and Representation Theory 3 (22w5033)

April 4 - 8, 2022

Organizer(s): Karin Baur (University of Leeds), Andrea Solotar (Universidad de Buenos Aires), Gordana Todorov (Northeastern University), Chelsea Walton (Rice University)

This report summarizes the objectives and scientific progress made at the 3rd workshop for women in noncommutative algebra and representation theory at the Banff International Research Station in Banff, Canada.

1 Objectives

The goals of this workshop were the following.

- To have accessible introductory lectures by participants in the themes of the workshop.
- To have each participant engaged in a stimulating research project and/ or be involved in a expansive research program in noncommutative algebra and/or representation theory.
- To have each participant provide or receive training toward this research activity (before and at the workshop) and to have made significant progress in such directions by the end of the workshop.
- To set-up mechanisms so that the collaborative research groups formed before/ at the workshop can continue research after the workshop, so that their findings will be published eventually.
- To provide networking opportunities and mentoring for its participants at and beyond the workshop

2 Introductory Talks

There were four “What is...?” talks given by participants of the workshop:

- Asilata Bapat: What is... the Bridgeland stability condition?
- Emily Gunawan and Emine Yildirim: What is ... a connection between cluster algebras and friezes?
- Maryam Khaqan: What is... moonshine?
- Florencia Orosz Hunziker: What is... a vertex algebra?

3 Scientific Progress Made

The group leaders are indicated by (*) below.

3.1 Verlinde-type Formulae for Fusion Coefficients

Group members: Georgia Benkart* (University of Wisconsin, Madison), Sarah Brauner (University of Minnesota), Laura Colmenarejo (North Carolina State University), Francesca Gandini (Kalamazoo College), Ellen Kirkman* (Wake Forest University), and Julia Plavnik (Indiana University).

Let $H = \Bbbk G$ be the group algebra of a finite group over a field of characteristic zero. Then H is a semisimple Hopf algebra. Denote the simple left H-modules by $S_1, \ldots, S_n$. For any fixed left H-module V and each simple module S_j, the tensor product $V \otimes_{\Bbbk} S_j$ is a left H-module so decomposes into a direct sum of simple left H-modules. Hence there are nonnegative integers $n_V(i,j)$ with

$$V \otimes_{\Bbbk} V_j = \sum_{i=1}^{n} n_V(i,j) S_i.$$

The matrix $N_V = (n_V(i,j))$ represents the $\Bbbk$-linear map $N_V : R(H) \to R(H)$ given by left tensoring with V acting on the representation algebra $R(H)$, expressed in terms of the basis of $R(H)$ that consists of isomorphism classes of the simple modules $[S_i]$. The character table of G, thought of as an $n \times n$ matrix S, simultaneously diagonalizes all matrices N_V, so $S^{-1} N_V S = D_V$, for a diagonal matrix D_V, whose entries also come from the character table of G. Solving for $N_V = S D_V S^{-1}$ gives a formula for the fusion coefficients in terms of data from the character table. Such a formula is called a "Verlinde formula", and such formulae occur in various contexts. As one such example, Witherspoon found such a matrix S related to a character table when H is a semi-simple almost cocommutative Hopf algebra (e.g. the Drinfeld Double of a semisimple Hopf algebra). As another example, if $H = \Bbbk G$, when $\Bbbk$ has characteristic p, then H may no longer be semisimple. Using simple composition factors of $\Bbbk G$-modules, instead of simple direct summands, the table of Brauer characters of G was used to study the maps N_V in work of Grinberg, Huang, and Reiner, who noted that the maps N_V can be considered for the modules of any Hopf algebra, since then the tensor product of two H-modules is again an H-module. More generally, onc can consider the fusion relations in a fusion category, and in the setting of a modular fusion category there is a symmetric matrix S, that is a representation of $\mathrm{SL}_2(\mathbb{Z})$, and has other remarkable properties, including the fact that it diagonalizes the fusion relations; in this case the category is braided, and the fusion algebra is semisimple, commutative, symmetric, among other special properties.

Our collaboration group is looking at some examples of Hopf algebras and their related matrices of fusion coefficients, searching for properties that extend the notions described above. In these examples the fusion relations are not always diagonalizable, so their Jordan form is considered, and we are interested in properties of matrices that place the N_V matrices into Jordan form. The N_V matrices' properties usually depend strongly on properties of V. Nevertheless, there is some interest among physicists in producing some sort of Verlinde formula in circumstances beyond the case where the tensor category is modular and the fusion algebra is semisimple and commutative. During our week at BIRS we computed various matrices related to the matrices N_V in several settings. Our work continues in our weekly virtual meetings that were initiated last fall. The group consists of researchers with interests in tensor categories, representation theory, algebraic combinatorics, commutative algebra, and noncommutative algebra, areas that already have come into play in our work.

3.2 Cluster Categories I

Group members: Karin Baur* (University of Leeds), Lea Bittmann (University of Vienna), Emily Gunawan (University of Oklahoma), Gordana Todorov* (Northeastern University), and Emine Yıldırım (Queen's University).

Cluster algebras were introduced by Fomin-Zelevinsky in 2002 in order to give a combinatorial framework for studying algebraic groups, and have since appeared in various fields including representation theory, triangulations of surfaces, Teichmüller theory, Poisson geometry, algebraic combinatorics and frieze patterns. The associated categories of representations, cluster categories, introduced by Buan-Marsh-Reineke-Reiten-Todorov have also had numerous applications throughout mathematics as described in Reiten's ICM 2010 talk on this subject.

During the week of April 4-8, we have looked at questions related to frieze combinatorics and their connections to triangulations of surfaces and to representation theory. Here, a frieze is an array of (possibly infinitely many) rows of integers, starting with a row of 0s and a row of 1s and satisfying the so-called diamond rule: any four entries

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfy $ad - bc = 1$. We consider periodic friezes, i.e. friezes for which all rows have a translational period. An example of a frieze of period 4 is here:

$$\begin{array}{ccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 3 & & 2 & & 2 & & 4 & & 3 & & 2 & & 2 & & 4 & \\
\cdots & & 5 & & 3 & & 7 & & 11 & & 5 & & 3 & & 7 & & \cdots \\
 & 18 & & 7 & & 10 & & 19 & & 18 & & 7 & & 10 & & 19 & \\
 & & 25 & & 23 & & 27 & & 31 & & 25 & & 23 & & 27 & & \\
 & \vdots & & & & \vdots & & & & & & \vdots & & & & \vdots &
\end{array}$$

Friezes have been introduced in the 70s by Coxeter and Conway, [5], [3], [4]. See for example [1] for a survey.

During the WINART3 workshop, our group has studied various types of infinite friezes. Finite friezes are known to arise from cluster algebras of Dynkin types A, D and E. Infinite friezes have been studied in the context of cluster algebras of Dynkin type $\widetilde{A}$ (affine type). A remarkable property of infinite periodic friezes is that their entries grow in a controlled way: if the frieze has period n then for any entry in the nth non-trivial row, the difference to the entry directly above it is a constant, independent of which entry in row n we choose, [2, Theorem 2.2]. This invariant of the frieze is called *growth coefficient* of the frieze. In the example above, the growth coefficient is 20: the first entry shown in row four is 25, the entry directly above it is 5, and so on. Continuing, the difference of an entry in row kn and an entry in row $kn-2$ is also always constant and can be given in terms of the growth coefficient.

Questions and Progress: The main objects we have studied during the week were triangulations of twice punctured disks: these give a geometric model for cluster algebras in type $\widetilde{D}$ (affine type). We were interested in the associated infinite friezes and in their growth coefficients. We found properties of these growth coefficients. In addition, we have considered the affine types $\widetilde{E}$ and determined certain associated friezes. We found that in both affine types, the growth coefficients behave well and are linked to band modules in the corresponding cluster category. In addition, we considered friezes arising from triangulations of a pair of pants, i.e. sphere with three boundary components. These provide new families of triples of infinite friezes which have different behaviour than the above tame types. A third direction we explored was a surgery and gluing construction on the surfaces we worked on. This allows us to go from triangulations of annuli to triangulations of pairs of pants and back, i.e. between triangulations of a sphere with two boundary components and a sphere with three boundary components.

3.3 Cluster Categories II

Group members: Ilke Çanakçı*(Vrije University Amsterdam), Francesca Fedele (Università degli Studi di Verona - Università degli Studi di Padova), Ana Garcia Elsener* (University of Glasgow - Universidad Nacional de Mar del Plata), Khrystyna Serhiyenko (University of Kentucky).

Cluster algebras are a class of commutative rings that were introduced by Fomin and Zelevinsky [7] in 2002. Their original motivation was coming from studying canonical bases in Lie Theory. Today, cluster algebras are connected to various fields of mathematics, including geometry, combinatorics, and representation theory of associative algebras. The study of cluster variables, the distinctive set of generators for a cluster algebra, can be simplified by working with their geometric model or their representation theoretic interpretation.

We focus on cluster algebras of Dynkin type A_n. Their geometric model consists of a triangulation of a disk with $n+3$ marked points, where the initial cluster variables are in bijection with the arcs of the triangulation and the remaining cluster variables correspond to the remaining arcs. The cluster variables can be computed combinatorially using a snake graph formula: each internal diagonal of the triangulated disk is associated to a snake graph and the corresponding cluster variable can be written down using its perfect matchings (also known as dimer covers).

Alternatively, the cluster variables can be computed homologically using representation theoretic means. Denoting by kA_n the path algebra of A_n, the cluster variables are in bijection with the indecomposable kA_n-modules. Moreover, the Caldero-Chapoton map [6] applied to each indecomposable in $\mathrm{mod}(kA_n)$ gives its corresponding cluster variable.

Our project. Starting from the above classic theory, our project aims to give a deeper understanding of the super cluster algebras of Dynkin type A_n, as studied combinatorially by Musiker, Ovenhouse and Zhang [8] in 2021. A super algebra is a $\mathbb{Z}_2$-graded algebra and it is generated by a set of even variables $\underline{x}$, which commute with each other, and a set of odd variables $\underline{\theta}$, which anticommute with each other and commute with the even ones. The geometric model of these algebras consists of an oriented triangulation (without internal triangles) of a disk with $n+3$ marked points, where the initial even variables are in bijection with the arcs of the triangulation and the remaining super cluster variables are in bijection with the remaining arcs. Moreover, the initial odd variables are associated to each triangle of the triangulation. As in the classic case, the super cluster variables can be computed combinatorially using a snake graph formula: each (oriented) internal diagonal in the disk is associated to a snake graph and the corresponding super cluster variable can be written down using its double dimer covers (obtained by superimposing two perfect matchings). Since each tile in a snake graph is obtained by gluing together two triangles, it has two associated odd variables which also play a role in the super snake graph formula.

We aim to give a representation theoretic interpretation of super cluster algebras of type A_n. Our proposal is to study the algebra $\widetilde{kA_n} := kA_n \otimes_k k[\epsilon]$, obtained by tensoring the path algebra of A_n with the dual numbers $k[\epsilon]$. We claim that the super cluster variables are in bijection with the indecomposable induced modules over $\widetilde{kA_n}$, that is the modules of the form $M \otimes_k k[\epsilon]$, where M is an indecomposable kA_n-module. In order to show this correspondence, we define a super Caldero-Chapoton map from the induced modules to the set of super cluster variables.

- We have worked out a bijection between the lattice of double dimer covers of a snake graph and the submodule lattice of the corresponding induced module. We are in the process of writing down a formal proof.
- We defined a super Caldero-Chapoton map and we understand the role of the odd variables combinatorially. We are exploring if these variables can be described in a representation-theoretic way.
- During the WINART3 workshop week we have also fully worked out the A_2 case, and we believe we understand the general case. We exchanged emails with technical questions with the super cluster algebra paper authors. We explored bibliography that will be used in our project and agreed on a writing plan that will take place in the next months. We will have zoom meetings periodically.

3.4 On the Monster Lie algebra

Group members: Darlayne Addabbo (University of Arizona), Lisa Carbone* (Rutgers University), Elizabeth Jurisich* (College of Charleston), Maryam Khaqan (Stockholm University), and Sonia Vera (Universidad Nacional de Cordoba).

This work concerns open questions about the structure of the Monster Lie algebra. Borcherds constructed the Monster Lie algebra $\mathfrak{m}$ to prove part of the Conway–Norton *Monstrous Moonshine Conjecture*. Let $\mathbb{M}$ denote the Monster finite simple group. A fundamental component of Borcherds' construction was Frenkel, Lepowsky and Meurman's Moonshine Module $V^\natural$, a graded $\mathbb{M}$-module with $\mathrm{Aut}(V^\natural) = \mathbb{M}$. $V^\natural$ is an example of a vertex operator algebra. The Monster Lie algebra $\mathfrak{m}$ is a quotient of the 'physical space' of the vertex algebra $V = V^\natural \otimes V_{1,1}$, where $V_{1,1}$ is a vertex algebra for the even unimodular 2-dim Lorentzian lattice $II_{1,1}$. We describe this construction in more detail below. The Monster Lie algebra $\mathfrak{m}$ also has a realization as the Borcherds (generalized Kac–Moody) algebra $\mathfrak{m} = \mathfrak{g}(A)/\mathfrak{z}$ where $\mathfrak{g}(A)$ is the Lie algebra with infinite generalized Cartan matrix A and $\mathfrak{z}$ is the center of $\mathfrak{g}(A)$:

$$A = \begin{array}{c|c|ccc|ccc|c}
 & \mathrm{c(-1)} & & \mathrm{c(1)} & & & \mathrm{c(2)} & & \\
\mathrm{c(-1)} & 2 & 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots \\
\hline
 & 0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\
\mathrm{c(1)} & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\
 & 0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\
\hline
 & -1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\
\mathrm{c(2)} & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\
 & -1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\
\hline
 & \vdots & & \vdots & & & \vdots & &
\end{array}$$

The numbers $c(j)$ are coefficients of q in the modular function $J(q) = j(q) - 744 =$

$$\sum_{i \geq -1} c(j) q^j = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

so $c(-1) = 1$, $c(0) = 0$, $c(1) = 196884, \dots$. If we set $I_0 = \{-1, 1, 2, 3, \dots\}$ and let $(i,k) \in I_0 \times \mathbb{Z}_{>0}$ for $1 \leq k \leq c(i)$. Then $\mathfrak{g}(A)$ has generators

$$e_{ik},\ f_{ik},\ h_{ik}$$

for $(i,k) \in I = \{(i,k) | i, k \in \mathbb{Z}, 1 \leq k \leq c(i)\}$ and simple roots α_{ik} for $(i,k) \in I_0 \times \mathbb{Z}_{>0}$, $1 \leq k \leq c(i)$. For $(i,k) = (-1,1)$ we write the generators as e_{-1}, f_{-1}, h_{-1}.

The Lie algebra $\mathfrak{g}(A)$ has defining relations:
(R1) $[h_{jk}, h_{i\ell}] = 0$,
(R2) $[h_{jk}, e_{i\ell}] = -(j+i)e_{i\ell}$,
(R3) $[h_{jk}, f_{i\ell}] = (j+i)f_{i\ell}$,
(R4) $[e_{jk}, f_{i\ell}] = \delta_{ji}\delta_{k\ell}h_{jk}$,
(R5) $(\operatorname{ad} e_{-1})^i e_{ik} = 0$ and $(\operatorname{ad} f_{-1})^i f_{ik} = 0$

for $(j,k), (i,\ell) \in I$. The h_{ik} are linearly dependent, so $\mathfrak{m} = \mathfrak{g}(A)/\mathfrak{z}$ has a two dimensional Cartan subalgebra $\mathfrak{h}$ with basis elements denoted h_1, and h_2.

The Monster Lie algebra has the usual triangular decomposition $\mathfrak{m} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{n}^\pm$ are direct sums of the positive (respectively negative) root spaces of $\mathfrak{m}$. We define the extended index set

$$E = \{(\ell, j, k) \mid (j,k) \in I^{\mathrm{im}},\ 0 \leq \ell < j\} = \{(\ell, j, k) \mid j \in \mathbb{N},\ 1 \leq k \leq c(j),\ 0 \leq \ell < j\}.$$

and set

$$e_{\ell,jk} := \frac{(\operatorname{ad} e_{-1})^\ell e_{jk}}{\ell!} \quad \text{and} \quad f_{\ell,jk} := \frac{(\operatorname{ad} f_{-1})^\ell f_{jk}}{\ell!},$$

for $(\ell, j, k) \in E$. The following non-trivial result gives an additional non-standard decomposition of $\mathfrak{m}$.

Theorem 3.1. *([9], [10]) Let* $\mathfrak{gl}_2(-1)$ *be the subalgebra of* $\mathfrak{m}$ *with basis* $\{e_{-1}, f_{-1}, h_1, h_2\}$. *Then*

$$\mathfrak{m} = \mathfrak{u}^- \oplus \mathfrak{gl}_2(-1) \oplus \mathfrak{u}^+$$

where $\mathfrak{gl}_2(-1) := \langle e_{-1}, f_{-1}, h_1, h_2 \rangle \cong \mathfrak{gl}_2$, $\mathfrak{u}^+$ *is a subalgebra freely generated by* $\{e_{\ell,jk} \mid (\ell, j, k) \in E\}$ *and* $\mathfrak{u}^-$ *is a subalgebra freely generated by* $\{f_{\ell,jk} \mid (\ell, j, k) \in E\}$.

To construct $\mathfrak{m}$ as a quotient of the 'physical space' of a vertex algebra V, we define $V = V^\natural \otimes V_{1,1}$, where $V_{1,1}$ is a vertex algebra for the even unimodular 2-dim Lorentzian lattice $II_{1,1}$. We have $\mathfrak{m} = P_1/R$ where

$$P_1 = \{\psi \in V^\natural \otimes V_{1,1} \mid L(n)\psi = \delta_{n0}\psi,\ n \geq 0\}$$

is the space of weight one primary vectors of the vertex algebra $V^\natural \otimes V_{1,1}$ and

$$R := \{v \in V \mid (u, v) = 0 \text{ for } u \in V\}$$

is the radical of the symmetric bilinear form $(\cdot, \cdot)$. It is an open question to construct specific elements of $V^\natural \otimes V_{1,1}$ that determine the generators of certain distinguished subalgebras of $\mathfrak{m}$ which are known only in terms of sets of generating root vectors.

Goals. The following are the main goals of this work. In the next subsection, we describe our current progress towards these goals.

- Find vertex operators that correspond to generators of the 'imaginary' $\mathfrak{gl}_2$ subalgebras $\mathfrak{gl}_2(\mathrm{im}_{(ik)})$ in $V^\natural \otimes V_{1,1}$ corresponding to imaginary simple root vectors $e_{ik},\ f_{ik},\ h_{ik}$ and imaginary simple roots α_{ik} for $(i, k) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, $1 \leq k \leq c(i)$.
- Find the vertex operators that correspond to the imaginary root vectors $\{e_{\ell,jk}\}$ and $\{f_{\ell,jk}\}$ which generate the free subalgebras $\mathfrak{u}^\pm$ of $\mathfrak{m}$ respectively.
- Clarify our understanding of how the action of $\mathbb{M}$ carries through $V^\natural \otimes V_{1,1}$ to $\mathfrak{m}$ as a quotient.
- Understand the role of the vectors in $V^\natural \otimes V_{1,1}$ that give rise to the co-dimension 1 free Lie algebra in $\mathfrak{n}^+$ and the structure they generate in $V^\natural \otimes V_L$.

Progress. During the months leading up to the WINART program, our WINART group members participated in a reading group lead by Prof. Carbone, which helped us to learn necessary background material for the project. Before our arrival at BIRS, we gave the vertex operators for the $\mathfrak{gl}_2$ subalgebra in $V^\natural \otimes V_L$ corresponding to the unique real simple root vector of $\mathfrak{m}$. This was previously also given in [11]. A main goal for our group during the WINART workshop was to begin identifying vertex operators for $\mathfrak{gl}_2$ subalgebras in $V^\natural \otimes V_L$ corresponding to imaginary simple root vectors $e_{ik},\ f_{ik},\ h_{ik}$ for $(i, k) \in I$ and simple roots α_{ik} for $(i, k) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, $1 \leq k \leq c(i)$. During the workshop, we made conjectures as to how to construct such vertex operators, and completed several preliminary calculations necessary for proving our conjectures. Upon returning from the WINART workshop, we proved the existence of certain vectors in $V^\natural$ that we need for our proposed construction and completed more computations in this direction.

3.5 Combinatorial models in representation theory: additive friezes

Group members: Asilata Bapat (Australian National University), Véronique Bazier-Matte (University of Connecticut), Eleonore Faber* (University of Leeds), Bethany Marsh* (University of Leeds), Kunda Kambaso (RWTH Universität Aachen), and Yadira Valdivieso (UDLAP University of the Americas Puebla).

Recently, many combinatorial models have arisen in the representation theory of algebras. Examples include the categorification of Coxeter–Conway friezes using cluster categories (first pointed out by Caldero–Chapoton [14]), the description of module categories via Dyck paths (Moreno Cañadas-Bravo Riós [21]) and the description of categories associated to gentle algebras via surface triangulations and ribbon graphs (see Baur–Coelho Simoes [12], Lekili–Polishchuk [19], Opper–Plamondon–Schroll [22]).

During the workshop we looked at a variant of Coxeter–Conway's frieze patterns, so-called *additive friezes*, and studied their representation-theoretic properties.

The notion of a (multiplicative) frieze pattern was introduced by Coxeter in [16] and further studied by Conway and Coxeter in the 1970s in [17, 18], where it was shown that integral friezes of finite rank correspond to triangulations of n-gons. After the introduction of cluster algebras in the 2000s, it was shown that the triangulations define a categorification of cluster algebras of type A [15] (see also [13]). Since then, friezes have been studied from various points of view in representation theory; in particular Morier-Genoud's survey [20] gives a good overview of recent developments.

While multiplicative friezes are well explained in the context of cluster categories, their additive counterparts (see [23] for definitions) have been studied much less from a representation theoretic point of view. In particular, we were interested in additive friezes of non-negative integers (AIFs), and how to enumerate them, fixing the rank n of the frieze.

Results: For small n, we could determine the number of AIFs using a computer program. Our model for additive friezes of rank n is the Auslander–Reiten quiver of $D^b(\mathrm{mod} kQ)$, where Q is a quiver of type A_n. We showed how to interpret additive friezes as elements of the dual of the Grothendieck group of $D^b(\mathrm{mod} kQ)$ and determined a suitable basis of the dual. Using cluster category methods, we were able to interpret additive friezes as points in an associahedron in some $\mathbb{R}^m$. The main task was to the find integer points that correspond to AIFs: we claim that these are integer points in a certain "symmetrization" of the associahedron. For small n we were able to verify this claim by direct computation and we are working on a proof for arbitrary n.

3.6 Hochschild (co)homology I

Group members: Hongdi Huang (Rice University), Monique Müller (Universidade Federal de São João del-Rei), María Julia Redondo* (Universidad Nacional del Sur), Fiorela Rossi Bertone* (Universidad Nacional del Sur), Pamela Suárez (Universidad Nacional de Mar del Plata).

The aim of this group is to consider a particular family of algebras, the gentle algebras, and study their deformations in terms of Maurer-Cartan elements.

The gentle algebras are a particular case of monomial algebras, and they can be described as path algebras kQ/I, with some particular conditions on the quiver Q and the monomial relations I.

It is well-known, see [24], that the deformations of an algebra are parametrized by the Maurer-Cartan elements, that is, elements f of degree one in $C^*(A)[1]$, the shifted Hochschild complex which has structure of DGLA, satisfying the equation

$$df + \frac{1}{2}[f, f] = 0.$$

The Hochschild complex $C^*(A)$ is obtained by applying the functor $\mathrm{Hom}_{A-A}(-, A)$ to the Hochschild resolution $C_*(A)$. When A is an algebra over a field k, the Hochschild resolution is a projective resolution of A in the category of A-bimodules. Usually, when dealing with computation of Hochschild cohomology, it is convenient to replace the Hochschild complex $C^*(A)$ by any other complex obtained from another projective resolution of A.

In the particular case of monomial algebras, Bardzell's complex $B^*(A)$ has shown to be an efficient replacement of Hochschild complex when dealing with computations. However, with this replacement we loose the DGLA structure needed to study deformations. One needs to consider L_∞-structures, which is a generalization of DGLA, in order to recover the connection with deformations, which is now given in terms of the generalized Maurer-Cartan equation.

Since Bardzell's complex is a retract of Hochschild complex, one can describe a L_∞-structure on $B^*(A)$ that induces a quasi-isomorphism of L_∞-algebras.

In [26] we have described explicitly this L_∞-structure on $B^*(A)$ for any monomial algebra A, using some comparison morphisms between $C^*(A)$ and $B^*(A)$ that have been introduced in [25].

Finally, in order to study deformations of gentle algebras, we have to compute the L_∞-structure of their Barzdell's complex, and describe their Maurer-Cartan elements. The problem with the generalized Maurer-Cartan equation is that it is a series and it could be divergent. So, the project for this group contemplates, in the case of gentle algebras, giving conditions under which the generalized Maurer-Cartan equation is convergent and, when possible, giving a description of the Maurer-Cartan elements.

We have started working a few weeks ago through Zoom meetings. We are already familiar with the problem and with the calculations we have to do using the comparison morphisms described in [25] and the L_∞-structure given in [26].

3.7 Hochschild (co)homology II

Group members: Dalia Artenstein (Universidad de la República), Janina Letz (University of Bielefeld), Amrei Oswald (University of Iowa), Sibylle Schroll* (University of Cologne), and Andrea Solotar* (Universidad de Buenos Aires).

Due to pandemic reasons and also due to personal reasons of some of the members of the group, the work of our group has been completely online.

Andrea Solotar explained in the first meeting the aim and scopes of our work: Our aim is to study a family of algebras called *string algebras* using their Hochschild cohomology and possibly the associated geometric models. String algebras are monomial special biserial algebras and as such they are an important testing ground for conjectures and ideas. Many other classes of well-studied algebras are part of this class of algebras, the most well-known ones being the so-called gentle algebras which relate with many other areas of mathematics. The Hochschild cohomology of an associative algebra endowed with the cup product and the Gerstenhaber bracket has a very rich structure. Andrea explained some results concerning gentle algebras that are part of unpublished work by Schroll and Solotar in collaboration with Cristian Chaparro Acosta and Mariano Suárez-Alvarez and which are important to the project on string algebras.

At a later stage, Andrea Solotar explained in detail the definition of string algebras and showed how to compute the Hochschild cohomology of a particular family of string algebras: namely, the above mentioned family of gentle algebras. For this, it is important to have a precise knowledge of how to use Bardzell's resolution for monomial algebras.

Some of the members of the team were not familiar with Hochschild cohomology, so this part of the project is taking longer than initially expected, but it is nevertheless important to spend time on it since it is a fundamental tool for our work.

At the end of the WINART3 week, Dalia Artenstein explained in her short talk the framework of our project and some preliminary examples that we have been discussing. Some interesting suggestions resulted from her talk.

Once all the members of our team acquire enough experience with the required methods of computation, we will be able to study a wide subfamily of non necessarily quadratic string algebras form the homological and the representation theoretic points of view.

3.8 Generalized Quantum Symmetry via Hopf algebroids

Group members: Bojana Femić (Mathematical Institute of the Serbian Academy of Sciences and Arts), Florencia Orosz Hunziker (University of Denver), Chelsea Walton* (Rice University), and Elizabeth Wicks* (Microsoft Corporation).

We seek to understand more about the representation categories of Hopf algebroids. Notions of Hopf algebroids naturally arise in various fields such as Poisson geometry, category theory, and algebraic geometry/topology says something about stable homotopy theory. From our perspective as non-commutative algebraists, bialgebroids and Hopf algebroids arise naturally in the study of symmetries.

What do we mean by this? A symmetry is a property-preserving transformation from an object A to itself. We will take A to be a $\Bbbk$-algebra here, where $\Bbbk$ is the ground field.

Classical symmetry: When A is a polynomial ring, its symmetries are fairly well-understood. For example, we have that the group $GL_2(\mathbb{C})$ acts on $\mathbb{C}[x, y]$ naturally by automorphisms, and the Lie algebra $\mathfrak{gl}_2(\mathbb{C})$ acts on $\mathbb{C}[x, y]$ naturally by derivations.

However, if we want to alter A (for example by a deformation that makes A noncommutative), we have to alter our acting object as well. This was one of the motivations to develop quantum symmetry.

Quantum symmetry: Let us suppose that we deform our original algebra as follows:

$$\mathbb{C}[x, y] \rightsquigarrow \mathbb{C}_q[x, y] := \frac{\mathbb{C}\langle x, y\rangle}{(yx - qxy)}, \quad q \in \mathbb{C}^\times.$$

In this case, groups or Lie algebras do not suffice to capture the symmetries of this algebra, since we cannot deform them in the same manner. However, there is a more general structure called a Hopf algebra that captures these symmetries. A Hopf algebra is defined to be a tuple $(H, m, u, \Delta, \epsilon, S : H \to H)$ where (H, m, u) is an algebra, (H, Δ, ϵ) is a coalgebra, and S is the antipode, such that these structures satisfy certain compatibility conditions. The Hopf algebras $\mathcal{O}_q(GL_2), U_q(\mathfrak{gl}_2)$ are Hopf algebras that act on $\mathbb{C}_q[x, y]$.

To be precise, we can say that a Hopf algebra H acts on an algebra A if and only if A is an algebra in the monoidal category $H\text{-mod}$. The category $H\text{-mod}$ is known to be monoidal with tensor product $\otimes_{\Bbbk}$.

Weak quantum symmetry: Now let us suppose that we want to alter A further. One natural operation is to take direct sums. We could imagine that if H is a Hopf algebra acting on A, then $H \oplus H$ acts on $A \oplus A$. For example,

$$\mathbb{C}[x, y] \rightsquigarrow \mathbb{C}[x, y] \oplus \mathbb{C}[x, y] \cong (\mathbb{C} \oplus \mathbb{C})[x, y].$$

However, $H \oplus H$ is no longer a Hopf algebra, so this does not fit into the quantum symmetry framework. We must alter our notion of acting object in order to allow for direct sums.

A natural candidate is the weak Hopf algebra. A weak Hopf algebra is defined to be a tuple $(H, m, u, \Delta, \epsilon, S : H \to H)$ where (H, m, u) is an algebra, (H, Δ, ϵ) is a coalgebra, and S is the antipode, such that these structures satisfy weaker compatibility conditions than a Hopf algebra. It is known that the direct sum of two weak Hopf algebras is again a weak Hopf algebra, and Hopf algebras are special cases of weak Hopf algebras.

In the previous example we generalized the base of the algebra: the base of $\mathbb{C}[x, y]$ is $\mathbb{C}$ while the base of $\mathbb{C}[x, y]^{\oplus 2}$ is $\mathbb{C} \oplus \mathbb{C}$. In both cases the base is commutative and also Frobenius separable. If we want to consider symmetries of algebras with any noncommutative base, we need to expand our notion of symmetry even further.

Towards Generalized Quantum Symmetry: We want to extend our algebra A as follows:

$$\mathbb{C}[x, y] \rightsquigarrow B[x, y],$$

where B is any $\Bbbk$-algebra. We think that the natural object to act on such an algebra is a Hopf algebroid with base B. We omit the definition of Hopf algebroid here, but remark that a weak Hopf algebra is a special case of a Hopf algebroid where the base is Frobenius separable.

Project: We want to continue the theme of understanding symmetries by understanding module categories over Hopf-type objects. In particular, we want to study categorical properties of $H\text{-mod}$ for H a Hopf algebroid. For Hopf algebras and weak Hopf algebras, it is well-known that such categories are monoidal, and work has been done to describe conditions under which the categories inherit desirable properties such as rigidity, braidedness, semisimplicity, and even modularity. We would like to extend similar conclusions to the Hopf algebroid case.

References

`For Section 1.2:`

[1] K. Baur. Frieze patterns of integers. *The Mathematical Intelligencer*, June 2021.

[2] Karin Baur, Klemens Fellner, Mark J. Parsons, and Manuela Tschabold. Growth behaviour of periodic tame friezes. *Revista Matemática Iberoamericana*, 35 (2):575–606, 2019.

[3] John H. Conway and Harold S. M. Coxeter. Triangulated polygons and frieze patterns. *Math. Gaz.*, 57(400):87–94, 1973.

[4] John H. Conway and Harold S.M. Coxeter. Triangulated polygons and frieze patterns. *Math. Gaz.*, 57(401):175–183, 1973.

[5] Harold S. M. Coxeter. Frieze patterns. *Acta Arith.*, 18:297–310, 1971.

`For Section 1.3:`

[6] Caldero, P., & Chapoton, F. (2006). Cluster algebras as Hall algebras of quiver representations. Commentarii Mathematici Helvetici, 81(3), 595-616.

[7] Fomin, S., & Zelevinsky, A. (2002). Cluster algebras I: foundations. Journal of the American Mathematical Society, 15(2), 497-529.

[8] Musiker, G., Ovenhouse, N., & Zhang, S. W. (2021). An Expansion Formula for Decorated Super-Teichmüller Spaces. Symmetry, Integrability and Geometry: Methods and Applications, 17(0), 80-34.

For Section 1.4:

[9] Jurisich, E., *Generalized Kac-Moody Lie algebras, free Lie algebras and the structure of the Monster Lie algebra*, J. Pure Appl. Algebra 126 (1998), no. 1-3, 233–266.

[10] Jurisich, E., *An exposition of generalized Kac-Moody algebras. Lie algebras and their representations*, (Seoul, 1995), 121–159, Contemp. Math., 194, Amer. Math. Soc., Providence, RI, (1996).

[11] Jurisich, E., Lepowsky, J. and Wilson, R. L. *Realizations of the monster Lie algebra*, Selecta Math. (N.S.) 1 (1995), no. 1, 129–161.

For Section 1.5:

[12] Karin Baur and Raquel Coelho Simões. A geometric model for the module category of a gentle algebra. Int. Math. Res. Not. IMRN 2021, no. 15, 11357–11392.

[13] Aslak Bakke Buan, Bethany Marsh, Markus Reineke, Idun Reiten and Gordana Todorov. Tilting theory and cluster combinatorics. Adv. Math. 204, no. 2, 572–618, 2006.

[14] Philippe Caldero and Frédéric Chapoton. Cluster algebras as Hall algebras of quiver representations. *Comment. Math. Helv.*, 81(3):595–616, 2006.

[15] P. Caldero, F. Chapoton and R. Schiffler. Quivers with relations arising from clusters (A_n case). Trans. Amer. Math. Soc. 358, no. 3, 1347–1364, 2006.

[16] H. S. M. Coxeter, Frieze patterns. *Acta Arith.* 18, 297–310, 1971.

[17] John H. Conway and Harold S. M. Coxeter. Triangulated polygons and frieze patterns. *Math. Gaz.*, 57(400):87–94, 1973.

[18] John H. Conway and Harold S.M. Coxeter. Triangulated polygons and frieze patterns. *Math. Gaz.*, 57(401):175–183, 1973.

[19] Yankı Lekili and Alexander Polishchuk. Derived equivalences of gentle algebras via Fukaya categories. Math. Ann. 376, no. 1-2, 187–225, 2020.

[20] Sophie Morier-Genoud. Coxeter's frieze patterns at the crossroads of algebra, geometry and combinatorics. *Bull. Lond. Math. Soc.*, 47(6):895–938, 2015.

[21] Agustín Moreno Cañadas and Gabriel Bravo Ríos. Dyck paths categories and its relationships with cluster algebras. Preprint arXiv:2102.02974 [math.RT], 2021.

[22] Sebastian Opper, Pierre-Guy Plamondon and Sibylle Schroll. A geometric model for the derived category of gentle algebras. Preprint arXiv:1801.09659 [math.RT], 2018.

[23] G. C. Shephard. Additive frieze patterns and multiplication tables. *Math. Gaz.*, 60(413):178–184, 1976.

For Section 1.6:

[24] M. Gerstenhaber, 'On the deformation of rings and algebras', *Ann. of Math.* (2) 79 (1964) 59–103.

[25] M. J. Redondo L. Román, 'Comparison morphisms between two projective resolutions of monomial algebras', *Rev. Un. Mat. Argentina* 59, no. 1 (2018) 1–31.

[26] M. J. Redondo, F. Rossi Bertone. *L_∞-structure on Bardzell's complex for monomial algebras*. J. Pure Appl. Algebra 226 (2021), 106935.

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Chapter 5

At the Interface of Mathematical Relativity and Astrophysics (22w5025)

April 25 - 29, 2022

Organizer(s): Edgar Gasperín (Instituto Superior Técnico, Lisbon), José Luis Jaramillo (University of Bourgogne-Franche Comte), Badri Krishnan (Radboud University), Juan A. Valiente Kroon (Queen Mary, University of London)

Overview of the Field

Mathematical relativity is concerned with the systematic study of the topological, geometric and analytic properties of generic solutions to the equations of the theory of general relativity —the Einstein field equations. This area of research has been of fundamental importance to explore the physical content of the relativistic theory of gravity formulated by Einstein. On the other hand, the detection of gravitational waves from the merger of black holes and neutron stars, has opened the remarkable possibility of verifying (or disproving) the predictions of mathematical relativity using observational data and numerical simulations.

General relativity is one of the pillars of mathematical physics. As such, it makes use of ideas and methods from diverse areas of mathematics —in particular differential geometry, topology and the theory of partial differential equations. Since its inception, general relativity has been a continuous source of challenging mathematical problems and conjectures which have sparkled the development of novel mathematical techniques and, even, whole research topics. A particular example of this interaction is the notion of mass in general relativity originally arising from physical considerations, which eventually led to one of the milestones of mathematical relativity —the proof of the positivity of the mass by Schoen & Yau [22] and, independently, Witten [25]. This result brings to the fore the intimate relation between relativity and geometric analysis and kick-started the rigorous study of geometric inequalities. Another example of this interaction is the use of ideas from differential topology pioneered by Penrose to study gravitational collapse [18] which eventually led to a number of singularity theorems —see e.g. [19]. The key ingredient of the singularity theorems, namely trapped surfaces, have led to a new formulation of all facets of black hole physics with important applications in gravitational wave astronomy, numerical relativity and quantum gravity [20]. A final example of current relevance of the interaction between general relativity and mathematics concerns Lorentzian geometry and the theory of hyperbolic partial differential equations in which the need to understand the physically crucial issues of the stability and the late-time behaviour of dynamic black holes has driven the development of novel mathematical ideas and techniques —see e.g. [7] and references within.

The detection of gravitational radiation from coalescing black holes and neutron stars [8, 9, 10, 12, 11] has led to the remarkable possibility of testing or disproving many of the predictions of mathematical relativity; see e.g.

[13, 14]. Research in the mathematical aspects of general relativity is driven to a large extent by the role of this theory in astrophysics —see e.g. [17]. Most of the central problems in the subject are deeply rooted on physical considerations. In the last couple of decades a natural bridge between mathematical relativity and astrophysics has been numerical relativity —i.e. the study of solutions to Einstein equations by means of computer simulations; see e.g. [1, 2] for introductory accounts to this fast-evolving subject. The need to obtain numerically stable and long running simulations, and extracting physical gauge invariant information, fostered further interactions between physicists, astrophysicists and mathematicians. The aftermath of the numerical resolution of the binary problem of black holes by Pretorious saw the conclusion of a longstanding challenge [21]. However, at the same time, it saw the split of the communities of mathematical and astrophysical relativity.

A departure point between the mathematically and astrophysically oriented community in general relativity has been the analysis of the Einstein field equations from a linear or a non-linear perspective. While some phenomena are well captured by the linearised equations, there are effects which have roots in the non-linearities of the Einstein field equations. A promiment example of the latter is the memory effect which can be split into linear and non-linear memory effects, these should ultimately result in a mesurable imprint in the wave form of gravitational waves generated by astrophysical events, hence putting in contact theory, numerics and observations.

Recent Developments and Open Problems

The last decade has seen spectacular developments in gravitational wave science in which the detection of the coalescence of astrophysical black holes and, more recently, neutron stars is becoming routine and, in an exciting more recent development, researchers have gained the technical ability of obtaining images of black holes and detecting their shadows. As such, the organising team of this workshop felt that it was time now to bring the communities of mathematical and astrophysical relativity together in a renewed and, hopefully, invigorating dialogue. Both communities enter this new era with a renewed technical and conceptual toolkit. On the mathematical side, there is a clear focus on the use of advanced techniques from the theory of partial differential equations: in particular, geometric analysis and asymptotic analysis. On the astrophysics side, research enjoys a spectacular period supported by three main efforts: the advent of gravitational wave astrophysics; the observational access to imaging the strong field regime in the neighbourhood of black horizons and finally, the consolidation of cosmology as a high precision subject. Mathematical relativity is at a crossroad. We believe that it would be healthy for the field to reassess its goals. In particular, this involves identifying fundamental astrophysical questions for which a coherent and rigorous mathematical formulation can be provided so that the mathematical community can start its investigation, and also evaluating research areas which do not have any potential of being tested observationally. It may well be that the mathematical insights thus obtained may lead to new questions for the theoretical and observational astrophysical community.

The stated aims of the workshop were:

i. To bring together researchers of mathematical relativity and relativistic astrophysics/gravitational wave science, aiming at identifying areas of potential fruitful interaction for both communities.

ii. Set an agenda, for the coming years, of research in mathematical relativity.

Presentations

The workshop was organised around thematic days with three presentations per day and ample time for discussion between talks. Given the time difference with Europe, where most of the online participants were based, the talks were scheduled in the morning, Banff time. At the begining of each day there was a brief presentation by the chair introducing the topic of the day, its context in the current research and the particular objectives. The schedule allows for 20 minutes after each talk for questions and discussions. At the end of the talks, around 10 minutes were devoted to a wrap up of the day and to provide a teaser for the session in the following day.

Additional time for (online) scientific discussions was not scheduled and let the individual participants to arrange them as necessary.

Day 1: Non-linear stability of black holes and other mathematical topics

A particular research problem has dominated the focus of the mathematical relativity community since the mid 2000s: the nonlinear stability of black holes —in particular, that of the Kerr spacetime. Besides the natural physical importance of the issue, this question has led to the development of new mathematical techniques and the reassessment of ideas and tools that lied buried in the literature. The collective work of a number of groups world-wide in almost over two decades have brought tantalisingly close the aim of a full proof of the nonlinear stability of the Kerr spacetime which, undoubtly will appear in the coming years.

Lars Andersson (Max Planck Institute for Gravitational Physics): Remarks on the Black Hole Stability problem

In this talk based on recent work with Thomas Bckdahl, Pieter Blue and Siyan Ma, I will review some aspects of our approach to the black hole stability problem. The use of a radiation gauge plays an important role and I will discuss some features of the resulting reduced system.

Rita Teixeira da Costa (Princeton): Mode stability for Kerr black holes

The Teukolsky master equations are a family of PDEs describing the linear behavior of perturbations of the Kerr black hole family, of which the wave equation is a particular case. As a first essential step towards stability, Whiting showed in 1989 that the Teukolsky equation on subextremal Kerr admits no exponentially growing modes.

Stefanos Aretakis (Toronto): Observational signatures for extremal black holes

We will present results regarding the asymptotics of scalar perturbations on black hole backgrounds. We will then derive observational signatures for extremal black holes that are based on global or localized measurements on null infinity. This is based on joint work with Gajic-Angelopoulos and ongoing work with Khanna-Sabharwal.

Bridge to Day 2

A key particular aspect in the discussions of Day 1 concerned the control of linear perturbations in black hole background spacetimes, as an intermediate stage in the full proof of black hole stability. This point constitutes the link between the discussion of Day 1 and Day 2, the latter devoted to quasi-normal modes, namely one of the key aspects of the linear perturbation theory.

Day 2: Quasinormal modes

Quasinormal modes constitute a paradigmatic example of 'meeting point' between mathematical and astrophysical relativity since, on the one hand they play a key role in the mathematical understanding and control of certain aspects of the (linear) stability problem and, on the other hand, the quasinormal frequencies are 'a priori' accessible through gravitational wave observations. Day 2 is devoted to cover a set of points in the broad spectrum of quasinormal modes that are of particular relevance in current research. In this sense, Day 2 starts talk addressing the capability to extract quasinormal modes frequencies from actual observational data, with a particular focus on so-called overtones. The second talk addresses a particular issue concerning such overtones, namely their potential instabilities under ultraviolet perturbations in the black hole environment. Finally, the day concludes with an analysis of integrability issues in quasinormal modes, uncovering a hidden (Darboux) symmetry.

Colin Capano (Max Planck Institute for Gravitational Physics): Observational evidence for quasi-normal modes from astrophysical black holes

The LIGO and Virgo interferometers have detected nearly 100 binary black hole mergers to date. The black holes formed by these mergers have provided our first opportunity to directly observe quasi-normal modes (QNMs) emitted by a perturbed Kerr black hole. However, detecting more than the dominant QNM is challenging. It has been claimed that an overtone of the dominant QNM can be detected at the merger of GW150914 (and other events); the detection of a sub-dominant angular mode has been claimed in the merger GW190521. Both of these claims remain controversial, with conflicting evidence presented by different groups. I will review these detection claims and the evidence for each.

Rodrigo Panosso Macedo (Southampton): Pseudospectrum and black hole quasi-normal mode (in)stability

Black hole spectroscopy is as a powerful approach to extract spacetime information from gravitational wave observed signals. However, quasinormal mode (QNM) spectral instability under high wave-number perturbations has been recently shown to be a common classical general relativistic phenomenon. I will discuss these recent results on the stability of QNM in asymptotically flat black hole spacetimes by means of a pseudospectrum analysis.

Carlos Sopuerta (Institute of Space Sciences, Barcelona): Symmetries in the dynamics of perturbed Schwarzschild Black Holes

There are two important physical processes around black holes that can be well described using relativistic perturbation theory: Scattering of electromagnetic and gravitational waves (and other fields) and quasinormal mode oscillations that take place, for instance, after the coalescence of a black hole binary. It is well-known that these physical processes can be described in terms of gauge-invariant master functions. We have analyzed the space of all the possible master functions for the case of non-rotating black holes and we find two branches of solutions. One branch includes the known results: In the odd-parity case, the most general master function is an arbitrary linear combination of the Regge-Wheeler and the Cunningham-Price-Moncrief master functions whereas in the even-parity case it is an arbitrary linear combination of the Zerilli master function and another master function that is new to our knowledge. The other branch is very different since it includes an infinite collection of potentials which in turn lead to an independent collection master of functions which depend on the potential. We also find that of all them are connected via Darboux transformations. These transformations preserve physical quantities like the quasinormal mode frequencies and the infinite hierarchy of Korteweg-de Vries conserved quantities, revealing a new hidden symmetry in the description of the perturbations of Schwarzschild black holes: Darboux covariance.

Bridge to Day 3

Black hole spectroscopy constitutes an emerging and promising major research program aiming at probing the spacetime dynamics and astrophysics of black holes from the analysis of the observed frequencies of quasinormal modes. However, as discussed in this session and without entering into the underlying reasons for such instability, such research program must address the potential instability of quasinormal overtones, that may affect some of the goals in the program. On the other hand, the integrability concepts discussed in this session, in particular from the passage of Darboux transformation to the infinite conserved quantities of Korteweg-de Vries equations, may offer inights into the degrees of freedom responsible for such quasinormal instabilities. In sum, the effort to reconstruct the bulk spacetime features from scattering-like data (such as a quasinormal mode frequencies), as well as the discussion of algebraic structures involving an infinite number of conserved quantities constitute the link between Day 2 and Day 3.

Day 3: BMS structures and black holes

The scientific thread in Day 3 session is the discussion of the role of asymptotic symmetries as asymptotic algebraic structures providing insights into geometric and astrophysical features of the bulk of the spacetime. In particular, focus is set on BMS symmetries at null infinity, an 'outer' spacetime 'boundary' corresponding to the far radiative wave zone. In the first talk such BMS structure is extended to the 'inner' boundary of black hole spacetimes with a stationary horizon, namely to so-called non-expanding horizons. In this setting physical charges and fluxes are defined (this will be revisited in last talk's of Day 5), and local degrees of freedom ('gravitational radiation') associated to perturbed non-expanding horizons are discussed. Interestingly, the second talk explores/suggests the possibility that the quasinormal mode ultraviolet instability discussed in Day 2 might be understood in terms of such local degrees of freedom associated with BMS symmetries both at null infinity and the black hole horizon. Finally, the last talk extends such BMS algebraic structures to larger asymptotic groups and, in particular, relates the asymptotic dynamics (namely asymptotic Einstein equations) to the representation properties of the resulting asymptotic symmetry.

Jerzi Lewandowski (Warsaw): Gravitational radiation through non-expanding horizons

It is well-known that blackhole and cosmological horizons in equilibrium situations are well-modeled by non-expanding horizons (NEHs). Multipole moments to characterize their geometry will be introduced. A 1-dimensional extension of the BMS group acts on NEH. These symmetries will be used to define charges and fluxes on NEHs, as well as perturbed NEHs (gravitational radiation). They have physically attractive properties. Also, a new quadrupole formula for gravitational radiation through cosmological horison in de Sitter spacetime will be presented.

Edgar Gasperín (CENTRA, Lisbon): Energy scales and black hole pseudospectra: the structural role of the scalar product

A pseudospectrum analysis has recently provided evidence of a potential generic instability of the black hole (BH) quasinormal mode (QNM) spectrum. Such instability analysis depends on the assessment of the size of the perturbations. This is encoded in the scalar product and its choice is not unique. In this talk, we will address the impact of the scalar product choice, founding it on the physical energy scales of the problem. Applications of the scalar product in the QNM problem will be discussed as well as further insights into potentially geometric structures in the QNM problem brought to the forefront by geometric structures in the QNM problem brought to the forefront by the expression for the energy flux at null infinity.

Roberto Olivieri (Paris Observatory): The Weyl-BMS group and the asymptotic gravitational dynamics

Asymptotic symmetries play an important role and have deep implications in our understanding of gravity. After a short review of asymptotic symmetries at null infinity in Einstein gravity, I will introduce the Weyl-BMS group, a recent extension of the original BMS group, and discuss its main properties. I will also show that the asymptotic Einsteins equations can be derived from the requirement that the Noether charges associated to the Weyl-BMS generators form a representation of the Weyl-BMS algebra.

Bridge to Day 4

The focus on BMS symmetries in Day 3 brings about, on the one hand, the relation with gravitational wave memory. Indeed, research in recent years has demonstrated the close relation between gravitational wave memory, BMS symmetries and scattering of infrared gravitons. On the other hand, the associated charges and fluxes make enter quasi-local quantities into the picture. These two points, namely gravitational wave memory and quasi-local quantities in general relativity, constitute the link to Day 4.

Day 4: Gravitational memory and quasilocal observables

Day 4 is devoted to the discussion of gravitational memory and extended new gravitational effects associated with sources that are not stationary outside at large distances, on the one hand, and to quasi-local quantities characterising geometrically physical quantities such as gravitational energy or the quasi-local evolution of black holes, on the other hand. Both (related) subjects are archetypical examples of the interaction between mathematical relativity and relativistic astrophysics.

Lydia Bieri (University of Michigan): Gravitational Radiation in General Spacetimes

Studies of gravitational waves have been devoted mostly to sources such as binary black hole mergers or neutron star mergers, or generally sources that are stationary outside of a compact set. These systems are described by asymptotically-flat manifolds solving the Einstein equations with sufficiently fast decay of the gravitational field towards Minkowski spacetime far away from the source. Waves from such sources have been recorded by the

LIGO/VIRGO collaboration since 2015. In this talk, I will present new results on gravitational radiation for sources that are not stationary outside of a compact set, but whose gravitational fields decay more slowly towards infinity. A panorama of new gravitational effects opens up when delving deeper into these more general spacetimes. In particular, whereas the former sources produce memory effects that are finite and of purely electric parity, the latter in addition generate memory of magnetic type, and both types grow. These new effects emerge naturally from the Einstein equations both in the Einstein vacuum case and for neutrino radiation. The latter results are important for sources with extended neutrino halos.

José M. M. Senovilla (University of the Basque Country): Pure gravitational energy inside an empty ball

Gravity manifests itself as curvature of spacetime, and its strength can be measured by considering the variations of radius, area and volume of small balls with respect to their counterparts in flat spacetime. These variations can actually be put in relation, via the Einstein field equations, with the energy density of matter at the ball's centre. In this talk I will also consider what happens when the matter energy density vanishes. The elementary geometric quantities still feel the effect of pure gravity, leading to variations that should be related to the gravitational strength or, in simple words, to the gravitational energy density. These variations now involve terms quadratic in the curvature that can be appropriately put in connection with the Bel-Robinson tensor. New definitions of quasi-local gravitational energy arise. Some basic examples will be discussed.

Daniel Pook-Kolb (Max Planck for Gravitational Physics): The ultimate fate of apparent horizons in a binary black hole merger

Apparent horizons are routinely used in numerical relativity to infer properties of black holes in simulations of dynamical systems. Advances in numerical methods allowed us to follow these objects into the interior of merging black holes, revealing how the two original horizons connect (non-smoothly) with the remnant horizon. However, this still left the question of their final fate open. In this talk, I will present our most recent results on axisymmetric head-on mergers, showing that the evolution of apparent horizons is much more intricate than previously thought: In the interior of the newly formed common horizon, the original horizons are individually annihilated by unstable horizon-like structures. This completes our picture of how two black holes become one and provides the analog of the famous pair-of-pants diagram of the event horizon now for the apparent horizon.

Bridge to Day 5

Day 5 does not follow uniquely from the discussion in Day 4 but gathers together, among other inputs, elements from the stability analysis of Day 1, the integrability notions of Day 2, the probe of bulk properties from asymptotic data in Day 3 (and Day 2), and quasi-local notions discussed in Day 4. In this sense, Day 4 culminates the week putting together some of the different elements discussed along the workshop.

Day 5: binary black hole mergers

Day 5 is devoted to the discussion of some of the most challenging aspects underlying the understanding of binary black hole mergers. In spite of the fact that numerical simulations of binary black hole coalescences have become routine calculations for a broad spectrum of groups worldwide, the (qualitative) understanding of the underlying analytic, geometric and physical mechanisms involved in this problem are far from being correctly understood. This session aims at bringing light into some partial aspects, from the tension between non-linear/linear aspects of the ringdown problem (first talk), the elucidation of the universality and simplicity properties of the binary black hole (merger) waveform (second talk) and, finally, the construction of a robust framework for the reconstruction of black spacetimes in a correlation 'Gravitational Wave Tomography' approach (third talk).

Luis Lehner (Perimeter and University of Guelph): Puzzles and/or insights in the RingDown regime of black hole collisions

Understanding the behavior of black hole relaxation to equilibrium is presenting new challenges at theoretical and practical levels. This talk will discuss some recent developments which are raising new (and revisiting old) questions on this topic.

J.L. Jaramillo (Dijon) and B. Krishnan (Radboud): Simplicity in binary black hole merger waveforms

Before the first successful numerical simulations of binary black hole mergers in 2005, it was considered plausible that the gravitational wave signal could have complicated modulations and even be chaotic. After all, general relativity is a non-linear theory and these non-linearities are especially important near the merger. However, the reality is that the signals are so far seen to be rather simple. This does not mean that the signals are trivial, rather that the complications due to e.g. precession, eccentricity etc. are contained in the deviations from a simple underlying model. In this talk, we will propose a reason for this simplicity based on the framework of "singularity theory" developed by Whitney, Arnold and Thom in the 1960s. We shall propose that certain radiative aspects of binary black hole mergers are similar to other common observed physical phenomena such as caustics and rainbows in optics, and this theory provides hints for deeper mathematical structures in binary black hole dynamics.

Abhay Ashtekar (Penn State): Imaging Horizon Dynamics via Gravitational Wave Tomography

When black holes merge (or form by gravitational collapse), we have a common dynamical horizon whose geometry changes dramatically as it settles down to the final Kerr horizon. This evolution can be invariantly characterized by the dynamics of a set of multipoles. Unfortunately, the causal structure of space-time prevents the outside observers from directly witnessing it. However, thanks to Einsteins equations, this dynamics is encoded in the profiles of gravitational waves observed at infinity. Using results presented in previous talks at this workshop, and from joint work with Neev Khera, I will present a transform that reconstructs the late time dynamics of the horizon geometry using gravitational waves at null infinity. Just as one monitors changes in the internal structure of objects from outside using electromagnetic tomography, one can image the horizon dynamics using gravitational waves at infinity.

Scientific progress made

In general terms the main scientific output from this workshop is the fact that two communities which have been partially disconnected in General Relativity in recent years, have been able to interact and get to know the progress and current open problems in each area and hence, identify potential points of interaction. We are confident that this first interaction will lead to eventual collaborations (actually this has already started), academic visits or further conferences. This workshop was, arguably, the first event with such specific aim and we expect it to be the first in a regular series of meetings. Moreover, the fact that the workshop had an hybrid format and the talks were recorded and are available on the BIRS website gives a more tangible outcome of the conference where such interaction happened.

As specific points identified in this workshop for boosting the interaction between mathematical and astrophysical relativists we can highlight:

- Disentangling of non-linear and linear mechanisms in the problem of black hole stability and devising of observational strategies to probe and assess them. This requires a close interaction between mathematical relativists (namely analysts) with astrophysicists.

- Assessment of the black hole spectroscopy program, specifically the evaluation of the actual astrophysical content of quasinormal frequencies, with a particular emphasis in the stability properties of overtones. This will involve the simultaneous and combined analysis from the mathematical and the data analysis perspective.

- Enlarging of the standard 'celestial mechanics' approach to the binary black hole problem to a (complementary) 'correlation approach'. This involves the enlarging of mathematical tools to include scattering techniques, in particular from inverse scattering theory. This entails a 'paradigm shift' in which new gravitational wave observables specifically adapted to scattering theory should be devised. This will require the close interaction between analysts/geometers and gravitational wave data analysts.
- Role of algebraic (asymptotic) spacetime structures in the characterization of gravitational local degrees of freedom and the construction of physical observables to be monitored in gravitational wave data.
- Systematic introduction of integrability tools in the analysis of the strong field regime of gravitational dynamics. This is closely related to the study of (asymptotic) spacetime symmetries and we expect a convergence between different communities with distinct approaches and expertise.
- Identification of 'singularity/catastrophe theory' concepts and tools as potentially relevant notions to understand some of the qualitative aspects of gravitational dynamics in the strong field regime.

Outcome of the Meeting

Originally, the workshop was envisaged as fully face-to-face with plenty of opportunities for spontaneous interaction between participants. In view of the still prevalent difficulties in international travel we have to settle for a hybrid workshop with a small but enthusiastic team based in Banff. The hybrid format opens the opportunity to open the workshop to a broader audience which is a good thing.

As two of the organisers (EGG & JAVK) are Mexican researchers working in Europe we made use of the opportunity to reach to the Mexican scientific community. For this we scheduled two events (Tuesday, Thursday) outside the scientific programme:

- **Tuesday:** "Equality, diversity and inclusion in Gravity research".
- **Thursday:** "Work perspectives in gravivation in Mexico" (in Spanish).

Besides some of the attendees of the purely scientific part of the conference the two round tables brought the interaction on these topics with PhD students and posdoc researchers from several institutions, among them, students from Instituto Superior Técnico (IST), Institut de Mathématiques de Bourgogne (IMB) and from the Gravitation division of the Mexican Physics Society (SMF).

In summary, as a characterising feature singularising the present meeting from others in the field, we can conclude the identification of a new (and very specific) window for scientific interaction between i) partial differential equation analysts, ii) theoretical physicists specialised in symmetries, representation theory and integrability, iii) numerical relativists and iv) gravitational wave data analysts has been identified in this workshop. New challenges has been proposed and the required tools to address them have been identified. This outlines a work program for the coming years where key topics not discussed in the present (such as spinors and twistor constructions, as a particular but significant instance) should be incorporated. We conclude that a rich and promising interdisciplinary field of research in the boundary between mathematical relativity and astrophysics has put forward in this BIRS meeting.

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Bibliography

[1] M. Alcubierre, Introduction to $3 + 1$ numerical Relativity, Oxford University Press, 2008.

[2] T. W. Baumgarte & S. L. Shapiro, Numerical Relativity: solving Einstein's equations on the computer, Cambridge University Press, 2010.

[3] E. Berti, V. Cardoso, & A. O. Starinets, Quasinormal modes of black holes and black branes, Class. Quant. Grav. **26**, 163001 (2009).

[4] K. D. Kokkotas & B. G. Schmidt, Quasinormal modes of stars and black holes, Living Rev. Rel. **2**, 2 (1999).

[5] R. A. Konoplya & A. Zhidenko, Quasinormal modes of black holes: From astrophysics to string theory, Rev. Mod. Phys. **83**, 793–836 (2011).

[6] Planck Collaboration, Planck 2018 results. I. Overview and the cosmological legacy of Planck, In `arXiv:1807.06205`, 2018.

[7] M. Dafermos & I. Rodnianski, Lectures on black holes and linear waves, in Evolution equations, edited by D. Ellwood, I. Rodnianski, G. Staffilani, & J. Wunsch, Clay Mathematics Proceedings, Volume 17, page 97, American Mathematical Society-Clay Mathematics Institute, 2010.

[8] B.P Abbott et al. (LIGO Scientific Collaboration & V. Collaboration), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. **116**, 061102 (2016).

[9] B.P. Abbott et al. (LIGO Scientific Collaboration & V. Collaboration), BinaryBlack Hole Mergers in the first Advanced LIGO Observing Run, Phys. Rev.X6, 041015 (2016)

[10] B.P. Abbott et al. (LIGO Scientific Collaboration & V. Collaboration), GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo dur-ing the First and Second Observing Runs

[11] A.H. Nitz et al, 1-OGC: The first open gravitational-wave catalog of binary mergers from analysis of publicAdvanced LIGO data Astrophys. J. 872, 195 (2019)

[12] T. Venumadhav et al, New Binary BlackHole Mergers in the Second Observing Run of AdvancedLIGO and Advanced Virgo Phys.Rev. D100, 023011 (2019)

[13] B.P. Abbot et al, Tests of General Relativity with GW170817 Phys. Rev. Lett. 123, 011102 (2019)

[14] B.P. Abbot et al, Tests of general relativity with GW150914 Phys. Rev. Lett. 116, 221101 (2016)

[15] Event Horizon Telescope collaboration, First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole, Astrophys. J. Lett. **875**, L1 (2019).

[16] P. Lalanne et al., Light Interaction with Photonic and Plasmonic Resonances, Laser & Photonics Reviews **12**, 1700113 (2018).

[17] L. Lehner & F. Pretorious, Numerical Relativity and Astrophysics, Ann. Rev. Astron. Astrophys. **52**, 661 (2014).

[18] R. Penrose, Gravitational collapse and space-time singularities, Phys. Rev. Lett. **14**, 57 (1965).

[19] R. Penrose, Techniques of differential topology in relativity, Society for industrial and applied Mathematics, 1972.

[20] A. Ashtekar and B. Krishnan Isolated and dynamical horizons and their applications Living Rev.Rel. 7 (2004) 10

[21] F. Pretorious, Evolution of binary black hole spacetimes, Phys. Rev. Lett. **95**(121101) (2005).

[22] R. Schoen & S. T. Yau, On the proof of the positive mass conjecture in general relativity, Comm. Math. Phys. **65**, 45 (1979).

[23] C. M. Warnick, On quasinormal modes of asymptotically anti-de Sitter black holes, Commun. Math. Phys. **333**(2), 959–1035 (2015).

[24] O. Dreyer et al, Black hole spectroscopy: Testing general relativity through gravitational wave observations Class.Quant.Grav. 21 (2004) 787-804

[25] E. Witten, A new proof of the positive energy theorem, Comm. Math. Phys. **80**, 381 (1981).

Chapter 6

Inverse Problems for Anomalous Diffusion Processes (22w5043)

May 8 - 13, 2022

Organizer(s): Diane Guignard (University of Ottawa), Barbara Kaltenbacher (University of Klagenfurt), William Rundell (Texas A&M University)

Overview of the Field

The roots of the modern theory of diffusion processes are to be found in the early decades of the nineteenth century. The work of Fourier in formulating his famous heat equation comes to mind. Fourier's arguments were from a macroscopic viewpoint but from a particle-diffusion perspective, the first systematic study was by two Scottish scientists: the chemist Thomas Graham and the botanist Robert Brown. Einstein's 1905 paper on the topic provided the now generally accepted explanation for Brownian motion and had far-reaching consequences for physics. There are two key pieces to this work: first, the assumption that a change in the direction of motion of a particle is random and that the mean-squared displacement over many changes is proportional to time; second, he combined this with the Boltzmann distribution for a system in thermal equilibrium to get a value on the proportionality, the "diffusivity" D in $\langle x^2 \rangle = 2Dt$, $D = \frac{RT}{6N\pi\eta a}$ where T is the temperature, R the universal gas constant, N is Avogadro's number and $\gamma = 6\pi\eta a$ where a is the particle radius and η the viscosity. In the same year, the first dialogues about "random walk" models based on the ideas of Lord Rayleigh some 20 years previous began.

The heat equation can be derived from such a model based on equal step lengths in equal time intervals with the coupling between time and space $dt \propto dx^2$ given by the diffusion constant. Indeed, the time and space steps can be drawn from a probability density functions $p(t)$ and $\psi(x)$ – provided that the mean of $p(t)$ and the variance of $\psi(x)$ are finite. Therefore, classical Brownian motion can be viewed as a random walk in which the dynamics is governed by an uncorrelated, Markovian, Gaussian stochastic process. On the other hand, when the random walk involves correlations, non-Gaussian statistics or a non-Markovian process (for example, due to "memory" effects) the diffusion equation will fail to describe the macroscopic limit. These considerations lead to so-called "anomalous diffusion" processes and the physically-motivated examples are numerous.

Issue with the fixed paradigms of Brownian motion should not lie only with the underlying assumptions; we should ask if the outcomes of the model are satisfactory from a physical perspective. Rayleigh's observation that in Brownian motion the particles have a high probability of being near their starting position can be seen in terms of the probability density function given by the Gaussian; a relatively slow diffusion initially but a very rapid decay of the plume in space. It has certainly been observed that many processes exhibit a very different effect.

If we break the above assumptions by either $p(t) \propto t^{-1-\alpha}$ for $0 < \alpha < 1$ or $\psi(x) \propto x^{-2-\beta}$ for $1 < \beta < 2$, then the model no longer generates classical differential equations but ones with fractional derivatives. These can be described as an operator product of an integer order derivative and an Abel fractional integral – another 1820's idea.

Partial differential equations remain the lingua franca of the physical sciences and we want to ensure any model will have such a translation. In the case of a time fractional derivative, the diffusion equation becomes $\partial_t^\alpha u - \triangle u = f$ and the governing function is no longer the exponential of the parabolic form, but a Mittag-Leffler function $E_{\alpha,\beta}(z)$. Such *sub-diffusion* equations have very different properties. The most basic of these is perhaps the nonlocality of the operator. The fractional derivative's value at time t depends on all previous values of the solution - a striking example of the non-Markovian behaviour of the model. Moreover, the long term decay of their solution is only linear in time and they have very limited smoothing behaviour in terms of the initial data and f – again in sharp contrast to the classical parabolic case. Needless to say, their analysis is correspondingly more complex. However, this slow decay can be a substantial advantage in inversion and many extremely (exponentially) ill-conditioned inverse problems become less so (polynomial) under the subdiffusion paradigm. This comes at a price. Numerical methods are needed to obtain computable approximations of the solutions which can be used, for instance, to determine if a mathematical model is accurately describing the underlying physical phenomena. Due to the complexity of problems involving fractional operators, new and often clever methods must be used to obtain efficient and reliable approximations.

A similar effect occurs in wave equations under damping. Incorporating a classical first order time derivative simulating velocity, the exponential decay of the solution makes it difficult to extract information from anything but very small times. Under fractional damping, the decay is only linear and vastly changes an inverse problem designed to recover important coefficients in the equation from time-trace data.

In the superdiffusion case, where we incorporate a spatial fractional derivative, the situation is even more complex and, unlike the subdiffusion case which is now becoming well-understood, remains more of an enigma. We should remark here that there are many definitions of fractional operators in space - in particular of elliptic operators. These include the "fractional Laplacian of order β" which in $\mathbb{R}^d$ can be viewed as that pseudo differential operator whose Fourier transform has symbol ξ^β or in a bounded domain Ω as that operator with the same eigenfunctions as $-\triangle$ but with eigenvalues raised to the power β. These have a rich mathematical theory but their direct connection to a physical process is more tenuous. On the other hand, they have useful mathematical properties that simulate many of the desired physical properties.

In both the sub and super-diffusion cases, there is the question of obtaining effective numerical methods as the parabolic paradigm no longer holds and the spatial effects involve the entire region. Again, the situation is that much has been done but there is even more to do especially in the super-diffusion models, and this aspect will be a central part of our workshop.

Objectives of the Workshop

The primary interest of the organisers was, on the one hand, the analysis of inverse problems for partial differential equations and on the other hand, the development and analysis of numerical methods for solving the underlying equations.

Of particular interest are reaction diffusion equations either of subdiffusion type $\partial u_t^\alpha - Lu = f(x,t,u)$ or containing a space fractional operator L^β as in $\partial u_t^\alpha - L^\beta u = f(x,t,u)$. These can be single equations or coupled systems and there may be unknown coefficients appearing in the elliptic operator or its fractional substitute, as well as in the nonlinear term $f(x,t,u)$. Such equations in classical derivative form have formed the basis of chemical reactions, combustion, mathematical ecology and epidemiology, just to name a few application areas. In each case, there is a considerable rationale for the inclusion of non-Markovian models and their differential equation counterparts. Materials with memory require such a mechanism: many biologists and epidemiologists have pointed out that species rarely diffuse with a Brownian-type motion and they have to incorporate "long space step" jumps in the model. This is precisely what fractional operators provide. All of the above mentioned applications have conditions under which the forced, classical exponential decaying solutions no longer capture the dynamics and their replacement with nonlocal operators provide a more realistic physical model.

Inverse problems play a major role in all of the above. Indeed, the central issue with complex epidemic modelling is the recovery of the various rate constants under high noise measurements and it is clear that such finite dimensional parameter identification problems are vastly more simple and model-restrictive than the more universal approach of treating the unknowns as functions in a prescribed class.

In recent years, the development of numerical methods for problems involving fractional operators has made considerable strides in reducing the computational cost of previous methods. This is especially true for the case of direct or forward problems. For instance, borrowing tools traditionally used in the context of parametric PDEs, the reduced basis method has been recently considered to efficiently approximate the solution of fractional diffusion problems. Computational methods for inverse problems, where recovery of the unknowns is frequently done by iterative methods, is more challenging. The strategy of using as much of the information as possible from previous iteration steps can make a significant difference yet is a considerable challenge from both an algorithmic and a convergence analysis perspective. Moreover, the performance of such methods in the context of ill-posed inverse problems is still to be explored. This aspect was seen as one of the main goals of this workshop.

Another goal was to seek the group's interest in tackling partial differential operators with space fractional derivatives of Riemann-Liouville or Caputo type rather than as "fractional powers" of existing classical elliptic operators – the so-called superdiffusion case. The rationale for this is a closer connection to the underlying physics and models based on the probabilistic approach noted in the overview section. The difficulties here are considerable from both an analysis and a numerical perspective. Many of the PDE results in standard use either don't hold or do so in restricted sense, or are simply unknown. The same can be said to some extent about the functional analysis of these operators and, compared to other definitions, there are significantly less numerical algorithms available with provable convergence properties. The blend of participants we had invited are ideally suited to making substantial progress on this problem.

While each of the organisers have research areas grounded in partial differential equations, they do so from slightly different perspectives that will be valuable for the proposed meeting.

Diane Guignard is a classically trained numerical analyst specialising in adaptive algorithms and both linear and nonlinear reduced models for PDEs. Barbara Kaltenbacher and William Rundell have worked in PDE inverse problems for their entire career: the former is an expert in regularisation techniques, the latter has taken a mathematical physics perspective. Both Kaltenbacher and Rundell have a long individual history of meeting organisation as well as in collaboration. They also have collaborated in many research papers central to the workshop topic.

Recent Developments and Open Problems

The first discussion session in the afternoon of Monday, May 9 was very lively and ran well over the allocated time (it was the last session of the day). There was general consensus that the traditional fractional powers of an operator had seen an enormous success from both a numerical and inverse problems perspective. There was also the sense that the area has become perhaps over-saturated and new directions are needed. To be avoided are trends in "artificial" new definitions of fractional operators; research needs to be tied more strongly to physical applications. Ricardo Nochetto and Andrea Bonito who have been at the forefront of current research in numerical methods for some time were very much in agreement with this consensus as was Masahiro Yamamoto representing a more theoretical viewpoint. Specific topics such as fractional partial differential equations with space-dependent fractional power or with different type of boundary conditions (other than homogeneous Dirichlet boundary condition), as well as fully nonlinear fractional partial differential equations were touched on during this discussion. Again, all participants of this session agreed that physical motivations for studying such problems, even if interesting by themselves from a mathematical point of view, are needed.

A second discussion round in the morning of Thursday, May 12 (chaired by Masahiro Yamamoto and Bangti Jin) dealt with similar questions as the first one, but putting more emphasis on time fractional models. Concerning the analysis, it was once more emphasized that fully nonlinear fractional partial differential equations are a still largely unexplored field with many crucial mathematical challenges and interesting applications. Also the importance of physically sound modeling was addressed - in particular in view of a certain trend to just "fractionalizing" classical models without caring about a justification that threatens the image of fractional calculus in the scientific community. In the numerics of fractional PDEs, many new approaches have been developed recently, including time adaptivity or structure and asymptotic preserving methods. However, the tools for analyzing these methods are to some extent still missing. The development of these tools is one of the key open tasks in this area and will certainly benefit from the interaction between numerics oriented researchers and people with expertise on the analytical side of fractional calculus. Finally, joining the space and the time fractional world - which was one of the main motivations of this workshop - remains an important and worthwhile aim.

Presentation Highlights

Juan Pablo Borthagaray: Linear and quasi-linear fractional operators in Lipschitz domains: regularity and approximation

In this talk, we discuss the formulation, regularity, and finite element approximation of linear and quasi-linear fractional-order operators in bounded, Lipschitz domains. We emphasize recent results about Besov regularity, a priori error estimates in quasi-uniform and graded meshes, and local error estimates for linear problems.

Abner Salgado: Time fractional gradient flows, theory and numerics

We consider a so-called time fractional gradient flow: an evolution equation aimed at the minimization of a convex and l.s.c. energy, but where the evolution has memory effects. This memory is characterized by the fact that the negative of the (sub)gradient of the energy equals the so-called Caputo derivative of the state. We introduce a notion of "energy solutions" for which we refine the proofs of existence, uniqueness, and certain regularizing effects provided in [Li and Liu,SINUM 2019]. This is done by generalizing, to non-uniform time steps the "deconvolution" schemes of [Li and Liu, SINUM 2019], and developing a sort of "fractional minimizing movements" scheme. We provide an a priori error estimate that seems optimal in light of the regularizing effects proved above. We also develop an a posteriori error estimate, in the spirit of [Nochetto, Savare, Verdi, CPAM 2000] and show its reliability.
This is joint work with Wenbo Li (UTK).

Andrea Bonito: (tutorial talk) The Dunford-Taylor Method and Fractional Diffusion

In the first part of the talk, we review numerical algorithms for the approximation of fractional elliptic operators with a particular attention on their analysis and implementations. Our main emphasis is on methods using the Dunford-Taylor representations of fractional diffusion problems, but other methods are discussed as well. The Dunford-Taylor representation consists of an improper integral, which is approximated an exponentially convergent sinc quadrature method. In turn, the integrand at each quadrature point is approximated using a standard finite element method. The method is easily parallelizable and consists of a straightforward modification of standard finite element methods for reaction-diffusion problems.

In the second part of the talk, we propose numerical methods for the discretization of the surface-quasi geostrophic (SQG) system. The latter is a nonlinear partial differential system of equations coupling transport and fractional diffusion phenomena. The time discretization consists of an explicit strong-stability-preserving three-stage Runge-Kutta method while a flux-corrected-transport method while the space discretization is based on the Dunford-Taylor representations discussed earlier. In the so-called inviscid case, we show that the resulting scheme satisfies a discrete maximum principle property under a standard CFL condition and observe, in practice, its second-order accuracy in space. The algorithm successfully approximates several benchmarks with sharp transitions (frontogenesis) and fine structures typical of SQG flows. In addition, theoretical Kolmogorov energy decay rates are observed on a freely decaying atmospheric turbulence simulation.

Bangti Jin: (tutorial talk) Tutorial on Recent Advances on Inverse Problems for Time-Fractional Diffusion

Diffusion type models involving a fractional-order derivative in time have received a lot of attention in the physical and engineering communities during the last few decades, due to their extraordinary capabilities for accurately describing anomalous transport processes. There have also been intensive research activities on inverse problems for such models, starting from the pioneering works of Cheng, Nakagawa, Yamamoto and Yamazaki (Inverse Problems, 2009). In this tutorial talk, I will describe basic facts about the direct problem of the canonical mathematical model and discuss some recent advances on related inverse problems. I will mainly discuss three classes of model inverse problems, i.e., backward problem, order determination and inverse coefficient problems, and describe representative recent results, the idea of proofs and some outstanding issues.

Eric Soccorsi: Inverse coefficient problem for time fractional diffusion equations

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with $C^{1,1,}$ boundary $\partial\Omega$. We consider an initial boundary value problem for a fractional diffusion equation on $\Omega \times (0, T)$, $T > 0$, with time-fractional Caputo derivative of order $\alpha \in (0, 1) \cup (1, 2)$. We prove that two out the three time-independent coefficients ρ (density), a (conductivity) and q (potential) appearing in the equation $\rho(x)\partial_t^\alpha u(x, t) + \nabla \cdot (a(x)\nabla u(x, t)) + q(x)u(x, t) = 0$ are recovered simultaneously from measurements of the solutions on a subset of $\partial\Omega$ at fixed time $T_0 \in (0, T)$.

Vanja Nikolic: Time-fractional Moore–Gibson–Thompson equations

In this talk, we will present several time-fractional generalizations of the JordanMooreGibsonThompson (JMGT) equations in nonlinear acoustics. Following the procedure described in Jordan (2014), these time-fractional acoustic equations are derived from four fractional versions of the MaxwellCattaneo law in Compte and Metzler (1997). Additionally to presenting the local well-posedness results, we will also discuss their limiting behavior as the fractional order tends to one, leading to the classical third order in time (J)MGT equation. The talk is based on joint work with Barbara Kaltenbacher (University of Klagenfurt).

Yikan Liu: Unique determination of orders and parameters in multi-term time-fractional diffusion equations by inexact data

As the most significant difference from parabolic equations, the asymptotic behavior of solutions to time-fractional evolution equations is dominated by the fractional orders, whose unique determination has been frequently investigated in literature. Unlike all existing results, in this talk we explain the uniqueness of orders and parameters (up to a multiplier for the latter) only by the inexact data near $t = 0$ at a single point. Moreover, we discover special conditions on unknown initial values for the coincidence of observation data. As a byproduct, we can even conclude the uniqueness for initial values or source terms by the same data. The proof relies on the asymptotic expansion after taking the Laplace transform and the completeness of generalized eigenfunctions.

Guanglian Li: Wavelet-based Edge Multiscale Parareal Algorithm for subdiffusionequations with heterogeneous coefficients in a large time domain

I will present in this talk the Wavelet-based Edge Multiscale Parareal (WEMP) Algorithm recently proposed in [Li and Hu, J. Comput. Phys., 2021] to efficiently solve subdiffusion equations with hetero-geneous coefficients in long time. This algorithm combines the advantages of multiscale methods thatcan effectively deal with heterogeneity in the spatial domain, and the strength of parareal algorithms forspeeding up time evolution problems when sufficient processors are available. Compared with the previ-ous work for parabolic problem, the main challenge in both the analysis and simulation arises from thenonlocality of the fractional derivative. To conquer this obstacle, an auxiliary problem is constructed oneach coarse temporal subdomain to uncouple the temporal variable completely. In this manner, the ap-proximation properties of the correction operator is proved. In addition, a new summation of exponentialsums is derived to generate single-step time stepping scheme, with the number of terms of $\mathcal{O}(|\log \tau_f|)$ independent of final time. Here, τ_f is the fine-scale time step size. We derive the convergence rateof this algorithm in terms of the mesh size in the spatial domain, the level parameter used in the multi-scale method, the coarse-scale time step size and the fine-scale time step size. Several numerical tests arepresented to demonstrate the performance of our algorithm, which verify our theoretical results perfectly.

Matti Lassas: Geometric inverse problems for the fractional diffusion equation

Given a connected compact Riemannian manifold (M, g) without boundary, $dimM \geq 2$, we consider a space-time fractional diffusion equation with an interior source that is supported on an open subset V of the manifold. The time-fractional part of the equation is given by the Caputo derivative of order $\alpha \in (0, 1]$, and the space fractional part by $(-\Delta_g)^\beta$, where $\beta \in (0, 1]$ and Δ_g is the Laplace–Beltrami operator on the manifold. The case $\alpha = \beta = 1$, which corresponds to the standard heat equation on the manifold, is an important special case. We construct a specific source such that measuring the evolution of the corresponding solution on V determines the manifold up to a Riemannian isometry.

Olena Burkovska: Identifying the fractional power and extent of interactions in nonlocal models

Nonlocal operators of fractional type are a popular modeling choice for applications that do not adhere to classical diffusive behavior; however, one major challenge in nonlocal simulations is the selection of model parameters. In this talk we propose an optimization-based approach to parameter identification for fractional models with an optional truncation radius. We formulate the inference problem as an optimal control problem where the objective is to minimize the discrepancy between observed data and an approximate solution of the model, and the control variables are the fractional order and the truncation length. For the numerical solution of the minimization problem we propose a gradient-based approach, where we enhance the numerical performance by an approximation of the bilinear form of the state equation and its derivative with respect to the fractional order. We present several numerical tests in one and two dimensions that illustrate the theoretical results and show the robustness and applicability of our method. This work is in collaboration with M. D'Elia and C. Glusa.

Tram Thi Ngoc Nguyen: From neural-network-based learning to discretization of inverse problems

We investigate the problem of learning an unknown nonlinearity in parameter-dependent PDEs. The nonlinearity is represented via a neural network of an unknown state. The learning-informed PDE model has three unknowns: physical parameter, state and nonlinearity. We propose an all-at-once approach to the minimization problem. (Joint work: Martin Holler, Christian Aarset)

More generally, the representation via neural networks can be realized as a discretization scheme. We study convergence of Tikhonov and Landweber methods for the discretized inverse problems, and prove convergence when the discretization error approaches zero. (Joint work: Barbara Kaltenbacher)

Ekaterina Sherina: Quantitative optical coherence elastography

Elastography is an imaging modality which can map the biomechanical properties of a given sample, and is interested in identifying the spatial distribution and values of its biomechanical parameters. It is typically implemented as an add-on, e.g. to ultrasound, magnetic resonance imaging, optical coherence tomography (OCT) etc. In this work, we consider optical coherence elastography (OCE), which is a promising emerging research field but still lacking high precision and reproducibility. We aim at a quantitative multi-faceted analysis of key factors such as data quality and properties of reconstruction methods required for the successful application of quantitative elastography. Mathematically, we deal with two inverse problems in OCE - a reconstruction of the mechanical displacement and a reconstruction of the Young's modulus (stiffness) from OCT data of a sample which undergoes a static compression. In this work, we propose, analyse and compare three reconstruction methods for the Young's modulus: uniaxial analysis, strain map based reconstruction facilitating a particle tracking improved optical flow (EOFM), and a novel image-based inverse reconstruction method (IIM). The quality of the proposed reconstruction methods with respect to samples of different mechanical properties is investigated by comparing their performance on twelve silicone elastomer phantoms with inclusions of varying size and stiffness.

Andrea Aspri: Phase-field approaches for reconstruction of elastic cavities

In this talk I will present some recent results on geometrical inverse problems related to the shape reconstruction of cavities and inclusions in a bounded linear isotropic medium by means of boundary measurements. We adopt the point of view of the optimal control, that is we rephrase the inverse problems as a minimization procedure where the goal is to minimize, in the class of Lipschitz domains, a mis-fit boundary functional or an energy-type functional with the addition of a regularization term which penalizes the perimeter of the cavity/inclusion to be reconstructed. The optimization problem is addressed by a phase-field approach, approximating the perimeter functional with a Modica-Mortola relaxation.

This is a joint work with E. Beretta, C. Cavaterra, E. Rocca and M. Verani.

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Chapter 7

Combinatorial Reconfiguration (22w5090)

May 8 - 13, 2022

Organizer(s): Marthe Bonamy (Laboratoire Bordelais de Recherche en Informatique), Nicolas Bousquet (Université Claude Bernard Lyon 1), Daniel Cranston (Virginia Commonwealth University), Takehiro Ito (Tohoku University), Moritz Mühlenthaler (Laboratoire G-SCOP, Grenoble INP), Naomi Nishimura (University of Waterloo), Ryuhei Uehara (Japan Advanced Institute of Science and Technology)

Overview of the Field

In the decade since the framework of combinatorial reconfiguration was proposed [13], the area has attracted the attention of researchers in a broadening range of related research areas. Unlike in conventional optimization problems, where the goal is to find a solution with the most desirable property or properties (such as minimum size or maximum weight), reconfiguration studies the relationship among feasible solutions. In a real-life setting, knowing the relationships among solutions makes it possible to determine whether and how it is possible to transform one solution step-by-step into another solution with minimal stoppage of a production line or minimal disruption to clients. Additional examples of reconfiguration range from classical puzzles and games (where a transformation might be the rotation of a Rubik's cube or the sliding of a tile) to well-studied classical optimization problems (where a transformation might change, add, or delete one element of a solution).

The study of reconfiguration problems brings together problems and techniques in a variety of fields in mathematics and computer science such as combinatorial game theory, enumeration, random sampling via Monte Carlo Markov chains, bioinformatics, discrete geometry, and statistical physics. Many algorithmic and combinatorial questions related to this framework have been introduced and studied in the last few years; in addition, many old conjectures can be reformulated in this setting. In particular, many classical combinatorial results, such as Vizing's theorem and the 5-color theorem, are based on reconfiguration arguments. References for the listed topics and further information on combinatorial reconfiguration can be found in two recent surveys on the topic [12, 16].

More formally, for a given classical problem, selected type of transformation step, and definition of feasibility, a reconfiguration graph for an instance of the problem is formed by creating a vertex in the graph for each feasible solution and an edge for each transformation from one solution to another by the application of a single step. The challenges in solving reconfiguration problems arise from the fact that the number of solutions is typically exponential in the size of the instance, requiring more nuanced approaches than simply generating and exploring the reconfiguration graph.

One natural reconfiguration problem is the reachability problem, where the goal is to determine whether there is a path connecting the vertices corresponding to two given solutions. Although it may not be surprising that the reachability problem is intractable (PSPACE-hard) for many classical problems, more surprising is the fact that there is not a clear correspondence between tractability of classical problems and tractability of reachability problems derived from those classical problems. When reachability proves to be intractable, further investigations are required to determine the dividing lines between classes of instances for which reachability is tractable and classes of instances for which it is not. In such cases, there is not always an obvious relationship between "easy" classes for classical problems and "easy" classes for reachability.

The idea of trying to solve the reachability problem simultaneously for all possible pairs of solutions results in the idea of determining whether or not a reconfiguration graph is connected. Once it has been determined that there exists a path between a pair of solutions, next steps include algorithms to find a path or to find a shortest such path. In turn, such endeavours raise the question of the diameter of the graph (the maximum length of the shortest path between any pair of vertices). Connectivity and diameter of the reconfiguration graphs are of particular interest in random sampling. Researchers have also considered the impact of varying the choice of the type of transformation step, and have studied additional properties of the reconfiguration graph, such as determining whether the reconfiguration graph is isomorphic to the instance [11, 1], and when the reconfiguration graph is Hamiltonian [4].

Questions that are now framed as reconfiguration problems have been asked about specific classical problems as far back as the 1800's [14] (for the 15-puzzle) by both computer scientists and mathematicians. Examples include Hirsch's conjecture in combinatorial optimization and the flip distance [15] between triangulations. Only recently has the reconfiguration framework been defined and become a focus of study; such work has brought together mathematicians and computer scientists who previously were working individually on reconfiguration of a variety of different classical problems. These collaborations have resulted in significant progress on the reconfiguration of classical problems, both individually and grouped in classes, and the understanding of the reconfiguration framework itself, including both restrictions and extensions.

Presentation Highlights

The presentations consisted of tutorials, invited talks, contributed talks, and mentoring sessions. The summaries of the presentations are lightly edited versions of material supplied by the speakers themselves.

Invited Tutorials

Takehiro Ito (Tohoku University): Invitation to Combinatorial Reconfiguration

I gave a broad introduction talk for combinatorial reconfiguration. Combinatorial reconfiguration studies reachability and related questions over the solution space formed by feasible solutions of an instance of a combinatorial search problem. The study of reconfiguration problems is motivated from a variety of fields such as puzzles, statistical physics, and industry. I briefly explained these backgrounds and applications. In particular, the application to power distribution networks was interesting also for researchers who study reconfiguration problems for a long time. I also explained trends of research on combinatorial reconfiguration, and tried to explain why reconfiguration problems are difficult (and hence interesting) to solve.

Catherine Greenhill (UNSW Sydney): Markov chains, mixing time and connections with reconfiguration

After outlining what I understood reconfiguration people to be interested in, I discussed some of the basic theory about Markov chains, the fact that irreducibility of a reversible Markov chain is equivalent to connectivity of the reconfiguration graph, and then discussed the canonical path method for analysing the mixing time of a Markov chain: this method uses paths through the reconfiguration graph in an intrinsic way.

Amer Mouawad (University of Bremen): Parameterized algorithms for reconfiguration problems

A <u>graph vertex-subset problem</u> defines which subsets of the vertices of an input graph are feasible solutions. We view a feasible solution as a set of tokens placed on the vertices of the graph. A <u>reconfiguration variant</u> of a vertex-subset problem asks, given two feasible solutions of size k, whether it is possible to transform one into the other by a sequence of token slides (along edges of the graph) or token jumps (between arbitrary vertices of the graph) such that each intermediate set remains a feasible solution of size k. Many algorithmic questions present themselves in the form of reconfiguration problems: Given the description of an initial system state and the description of a target state, is it possible to transform the system from its initial state into the target one while preserving certain properties of the system in the process? Such questions have received a substantial amount of attention under the so-called combinatorial reconfiguration framework. We consider reconfiguration variants of three fundamental graph vertex-subset problems, namely INDEPENDENT SET, DOMINATING SET, and CONNECTED DOMINATING SET [3]. We survey some older and more recent work on the parameterized complexity of all three problems when parameterized by the number of tokens k. The emphasis will be on positive results and the most common techniques for the design of fixed-parameter tractable algorithms.

Invited Talks

Jun Kawahara (Kyoto University): Invited talk: A ZDD-based solver for combinatorial reconfiguration problems

Joint work with Takehiro Ito, Yu Nakahata, Takehide Soh, Akira Suzuki, Junichi Teruyama, Takahisa Toda

In this talk, we introduce a solver for combinatorial reconfiguration problems based on a data structure called ZDD (Zero-suppressed binary Decision Diagram). A ZDD is a compact representation of a family of sets and provides various efficient operations such as taking the union and intersection of two families. It is known that given a graph, we can efficiently construct a ZDD representing the set of all the independent sets of the graph. Using this property, we propose a method for solving the independent set reconfiguration problem and other problems.

Jonathan Narboni (Labri): Vizing's conjecture holds

Contributed Talks

Daniel Cranston (Virginia Commonwealth University): Kempe Equivalent List Colorings

Joint work with Reem Mahmoud [6]

Mohar conjectured that if G is connected and k-regular, then each two of its k-colorings are k-Kempe equivalent. This was proved for $k = 3$ by Feghali, Johnson, and Paulusma [8] (with a single exception $K_2 \square K_3$, also called the 3-prism) and for $k \geq 4$ by Bonamy, Bousquet, Feghali, and Johnson [2]. We prove an analogous result for list-coloring. For a list-assignment L and an L-coloring ϕ, a Kempe swap is called L-valid for ϕ if performing the Kempe swap yields another L-coloring. Two L-colorings are called L-equivalent if we can form one from the other by a sequence of L-valid Kempe swaps. Let G be a connected k-regular graph with $k \geq 3$. We prove that if L is a k-assignment, then all L-colorings are L-equivalent (again with a single exception $K_2 \square K_3$). When $k \geq 4$, the proof is completely self-contained, so implies an alternate proof of the result of Bonamy et al.

Hiroshi Eto (Tohoku University): Reconfiguration of Regular Induced Subgraphs

Joint work with Takehiro Ito, Yasuaki Kabayashi, Yota Otachi, Kunihiro Wasa [7]

We introduced the problem of reconfiguring d-regular induced subgraphs in a graph, which generalizes the well-studied independent set reconfiguration problem. In the problem, we are given a graph G and d-regular induced subgraphs U_S and U_T of G, and we are asked to decide whether there is a reconfiguration sequence between U_S and U_T under the token jumping (TJ) or token sliding (TS) rule. Therefore, the problem is exactly the independent set reconfiguration problem when $d = 0$.

In the talk, we systematically study the complexity of the problem, in particular, on chordal graphs and bipartite graphs. Our results for $d \geq 1$ give interesting contrasts to known ones for $d = 0$. More specifically, on chordal graphs, both TS and TJ rules have the same complexity (i.e., PSPACE-complete) for $d \geq 1$, whereas they are different for $d = 0$: the problem under TS rule remains PSPACE-complete, while the problem under TJ rule can be solved polynomial time.

Henning Fernau (Universität Trier): Order Reconfiguration under Width Constraints

Joint work with Emmanuel Arrighi and Mateus de Oliveira Oliveira from Bergen, Norway and with Petra Wolf from Trier, Germany

We consider the following order reconfiguration problem: Given a graph G together with linear orders ω and ω' of the vertices of G, can one transform ω into ω' by a sequence of swaps of adjacent elements in such a way that at each time step the resulting linear order has cutwidth (pathwidth) at most ℓ? We show that this problem always has an affirmative answer when the input linear orders ω and ω' have cutwidth (pathwidth) of at most $\ell/2$. This result also holds in a weighted setting.

Using this result, we establish a connection between two apparently unrelated problems: the reachability problem for two-letter string rewriting systems and the graph isomorphism problem for graphs of bounded cutwidth. This opens an avenue for the study of the famous graph isomorphism problem using techniques from term rewriting theory.

As further avenues of research, we suggest to study this cutwidth (pathwidth) reconfiguration problem in more details. More generally, we propose to look into other reconfiguration problems based on swapping neighboring elements in a finite linear order. For instance, dynamic aspects of the One-Sided Crossing Minimization problem (rather famous in Graph Drawing) lead to another special case of this type of reconfiguration problems.

Sevag Gharibian (Paderborn University): Reconfiguration in the Quantum Setting

Based on joint work with Jamie Sikora [10], Johannes Bausch and James Watson [18], and Dorian Rudolph (draft in preparation)

In this talk, we give a gentle introduction to the quantum analogue of reconfiguration problems for SAT. This includes a brief primer on quantum computation and how one defines ”quantum SAT”, i.e. the local Hamiltonian problem. We then formalize the reconfiguration problem of Ground State Connectivity, which asks whether two ground states of a local Hamiltonian H are ”connected” through the ground space of H via a short sequence of 2-qubit quantum gates. We then cover various results on GSCON over the last years, including: (1) GSCON with poly-length sequences of gates is QCMA-complete, where QCMA is a quantum analogue of Merlin-Arthur (MA), (2) GSCON on 1D, translation-invariant chains is QMAexp-complete, where QMAexp is the quantum analogue of NEXP, and (3) GSCON with exponentially many gates and inverse polynomial promise gap is in P, since we show that any pair of ground states of a local Hamiltonian H are connected in this setting, even up to inverse exponential precision.

Guilherme Gomes (Google): Some results on Vertex Separator Reconfiguration

We present the first results on the complexity of the reconfiguration of vertex separators under the three most popular rules: token addition/removal, token jumping, and token sliding. We show that, aside from some trivially negative instances, the first two rules are equivalent to each other and that, even if only on a subclass of bipartite graphs, TJ is not equivalent to the other two unless NP = PSPACE; we do this by showing a relationship between separators and independent sets in this subclass of bipartite graphs. In terms of polynomial time algorithms, we show that every class with a polynomially bounded number of minimal vertex separators admits an efficient algorithm under token jumping, then turn our attention to two classes that do not meet this condition: $\{\mathrm{3P1}, \mathrm{diamond}\}$-free and series-parallel graphs. For the first, we describe a novel characterization, which we use to show that reconfiguring vertex separators under token jumping is always possible and that, under token sliding, it can be done in polynomial time; for series-parallel graphs, we also prove that reconfiguration is always possible under TJ and exhibit a polynomial time algorithm to construct the reconfiguration sequence.

Sajed Haque (University of Waterloo): Labelled Token Sliding Reconfiguration of Independent Sets on Forest

Arnott Kidner (Memorial University of Newfoundland): Gamma-Switchable Homomorphisms

Informally, a (m, n)-mixed graph is a mixed graph whose edges are assigned m colours and arcs are assigned n colours. For a permutation π that acts on the edge colours, arc colours, and arc orientations, we say switching at a vertex v with respect to π changes the edges/arcs incident with v with the action of π. We show that it is polynomial time decidable to determine whether; for a fixed permutation group, there admits a sequence of switches on a (m, n)-mixed graph such that the resulting graph admits a homomorphism to a simple target on 2 vertices. This is accomplished using the reconfiguration graph.

Kshitij Gajjar (NUS, Singapore) : Reconfiguring Shortest Paths in Graphs and Abhiruk Lahiri (Charles University): Revisiting shortest path reconfiguration

In the Shortest Path Reconfiguration (SPR) problem, we are given two shortest paths and the goal is to transform one shortest path to the other by changing one vertex at a time so that all the intermediate configurations are also shortest paths. SPR has several real-world applications like repaving roads in a systematic way and cargo container stowage on ships.

We presented our work in two separate talks. In the first talk, presented by Kshitij Gajjar, we showed that SPR can be solved in polynomial time for many graph classes, including circle graphs, bridged graphs, the Boolean hypercube and bounded diameter graphs. Most of our proofs are by providing a complete characterization of the shortest paths for the graph class. We also explored a generalization of SPR known as k-SPR and showed that k-SPR is PSPACE-complete, even for some graph classes (viz. line graphs) for which SPR is known to be solvable in polynomial time.

In the second talk, presented by Abhiruk Lahiri, we continued our discussion on shortest path reconfiguration. We presented an alternative proof for the shortest path reconfiguration of interval graphs. We also showed that there exists an intersection graph of triangles such that the reconfiguration sequence can have a size exponential in the number of vertices. These are the only few results we know of the shortest path reconfiguration problem on geometric graphs.

The talks are based on joint work with Agastya Vibhuti Jha, Manish Kumar and Abhiruk Lahiri [9].

Stephanie Maaz (University of Waterloo): Parameterized Complexity of Reconfiguration of Atoms

The work presented was motivated by the challenges arising when preparing arrays of atoms for use in quantum simulation. It tackles the problem of reconfiguring one arrangement of tokens (representing atoms) to another using as few moves as possible; because the problem is NP-complete on general graphs as well as on grids, it focuses on presenting the parameterized complexity for various parameters, considering both undirected and directed graphs, and tokens with and without labels. For unlabelled tokens, the presentation went over the fixed-parameter algorithms under the following parameters: the number of tokens, the number of moves, and the number of moves plus the number of vertices without tokens in either the source or target configuration. It also presented the proof of the problem's intractability under both labelled and unlabelled tokens when parameterizing by the difference between the number of moves and the number of differences in the placement of tokens in the source and target configurations.

Thomas Suzan (G-SCOP): Reconfiguration of digraph homomorphisms

For a fixed graph H, the H-Recoloring problem asks whether given two homomorphisms from a graph G to H one can be transformed into the other by changing the color of a single vertex in each step and maintaining a homomorphism to H throughout. This problem generalizes the reconfiguration of graph colorings by changing the color of a single vertex at a time and has been studied mainly in the context of undirected graphs. In general, the H-recoloring problem is PSPACE-complete, but there are some classes of undirected graphs for which a polynomial algorithm is known. The most general algorithmic result so far has been proposed by Wrochna in 2014, who introduced a topological algorithm that solves the H-Recoloring problem in polynomial time when H is square-free. We show that his topological approach can be generalized to the setting of digraph homomorphisms. In particular we show that

1. if H is loopless, the corresponding reconfiguration problem admits a polynomial-time algorithm if H does not contain a 4-cycle of algebraic girth 0 and that

2. if H is reflexive (that is, H has a loop on each vertex), the problem admits a polynomial-time algorithm if H contains no triangle of algebraic girth 1 and no 4-cycle of algebraic girth 0.

While the first result is based on the so-called monochromatic neighborhood property that also plays a crucial role in Wrochna's algorithm, for the second result we introduce the so-called push-or-pull property, which allows us to work with the same topological approach in the reflexive digraph setting.

Mentoring sessions

Hugo Akitaya (Tufts University):Reconfiguration of District Maps

Motivated by applications in gerrymandering detection, we study a reconfiguration problem on connected partitions of a connected graph G. A partition of $V(G)$ is connected if every part induces a connected subgraph. It is desirable to obtain parts of roughly the same size, possibly with some (additive) slack s. A Balanced Connected k-Partition with slack s, denoted (k, s)-BCP, is a partition of $V(G)$ into k nonempty subsets, of sizes $n_1, \ldots, n_k$ with $|n_i - n/k| \leq s$, each of which induces a connected subgraph. We present an overview of complexity and algorithmic results for the reachability and connectivity problems using flips (changing the membership of a single vertex) and recombinations (merging two adjacent districts and re-splitting them into connected parts).

Reza Bigdeli (University of Waterloo): Disconnecting the Triangulation Flip Graph of Points in the Plane by Forbidding Edges

The flip graph for a set P of points in the plane has a vertex for every triangulation of P, and an edge when two triangulations differ by one flip that replaces one triangulation edge by another.

The flip graph is known to have some connectivity properties:

(1) the flip graph is connected;

(2) connectivity still holds when restricted to triangulations containing some constrained edges between the points;

(3) for P in general position of size n, the flip graph is $\lceil \frac{n}{2} - 2 \rceil$-connected, a recent result of Wagner and Welzl [17].

We introduce the study of connectivity properties of the flip graph when some edges between points are forbidden. An edge e between two points is a flip cut edge if eliminating triangulations containing e results in a disconnected flip graph. More generally, a set X of edges between points of P is a flip cut set if eliminating all triangulations that contain edges of X results in a disconnected flip graph. The flip cut number of P is the minimum size of a flip cut set.

We give a characterization of flip cut edges that leads to an $O(n \log n)$ time algorithm to test if an edge is a flip cut edge and, with that as preprocessing, an $O(n)$ time algorithm to test if two triangulations are in the same connected component of the flip graph. For a set of n points in convex position (whose flip graph is the 1-skeleton of the associahedron), we prove that the flip cut number is $n - 3$.

Hany Ibrahim (University of applied science Mittweida): Edge Contraction and Forbidden Induced Graphs

I gave a talk about edge contraction and forbidden induced graphs, in which I raised some questions by the end of the talk. While being in the working session about "token sliding and minimum vertex separators", I think I was able to suggest a constructive idea which is reducing the order of the graph by deleting vertices which are irrelevant to the problem. That is, to label every vertex either essential or irrelevant (maybe a vertex is irrelevant if it don't belong to any minimum vertex separator). Removing such irrelevant vertices would enable a specific lattice structure to form and eventually an encoding for the lattice. The idea is without proof yet, but looks interesting.

Jeffrey Kam (University of Waterloo): Extension of subgraph reconfiguration

I gave a mentoring session talk on extending some results in subgraph reconfiguration. For most part of the talk, I describe how one could show the NP-hardness of reconfiguring trees with path-width k or less, for any fixed k. I also briefly mentioned the NP-hardness result for reconfiguring graphs with tree-width k or less. Towards the end, I suggested some future directions for the project, such as extending to the multi-token model and the idea of a global minimum running buffer (GMRB) with respect to a graph property. In various subgraph reconfiguration problems under the vertex-variant, the PSPACE-completeness results are derived from the Shortest Path Reconfiguration (SPR) problem. So, our extension towards the multi-token model under the vertex-variant will likely utilize results from the k-Shortest Path Reconfiguration problem.

Rin Saito (Tohoku University): Reconfiguration of vertex disjoint shortest paths on split graphs

I introduced a new reconfiguration problem, called the Vertex-Disjoint Shortest Paths reconfiguration problem, which is a combination of the well-studied vertex-disjoint paths problem (studied from the 1970's) and the shortest path reconfiguration problem (introduced by Bonsma in 2010). I presented that the problem is PSPACE-complete for bipartite graphs, while I proposed a polynomial-time algorithm for split graphs.

Scientific Progress Made

The presentations as well as the open problem sessions provided a variety of starting points for discussions and collaborations among participants. Although most of the groups were in-person only or virtual-only, there were also several hybrid groups. Not only was there progress during the workshop itself, but also in subsequent meetings as groups continued to work on problems together.

The groups listed here are only the ones that were reported to the organizers; they can be viewed as a mere sampling of all the activity that the workshop fostered.

Because work listed here includes work in progress, details may be omitted.

Colouring reconfiguration

Collaborators: Daniel Cranston, Jeffrey Kam, Arnott Kidner, Ben Moore, Mortiz Mühlenthaler, Thomas Suzan

Question: Is 3-mixing in P for graphs embedded in a fixed surface Σ.

We mostly considered the torus, as the planar case was already solved by Johnson, van den Heuvel, and Cereceda [5]. We developed a better understanding of the planar algorithm, as Dan noted that if a graph has a basis for the cycle space that consists of 4-cycles, then G is 3-mixing. In the planar case, this happens to be sufficient, so we can restate the planar structure theorem in the following way:

Theorem: A graph G is 3-mixing if and only if G is bipartite, and contains a basis for its cycle space that consists of 4-cycles.

Given that a graph embedded on some surface Σ has a cycle space of "faces + generators for non-contractible cycles", the rephrasing suggests we should be looking at some sort of conditions on the generators for the cycle space to be sufficient for 3-mixing when embedded on some fixed surface. It is even interesting just in the torus, but we don't quite know how to do the torus yet. In particular, it's not clear when two vertices x, y of a graph embedded in the torus where $xy \notin E(G)$, can be identified and preserve the embedding.

There are some possible easier questions that also popped up:

Question: Is it true a planar graph G is C_5-mixing if and only if it has a cycle basis of 4,6 and 8 cycles?

Question: can a similar theorem to the 3-mixing result hold for C_4-free graphs?

Question: Is H-reconfiguration in P if each edge lies in at most one 4-cycle?

Question: Is H-reconfiguration PSPACE-complete if H contains a K_4?

As partial progress, consider the graph $H = (7, 2)-$circular clique plus an apex vertex. Since $(7, 2)$-circular recolouring reduces to C_7 recolouring by subdividing each edge of the input graph twice, we start with this, and then follow the original reduction for showing the 8-vertex wheel reconfiguration is PSPACE-complete, to seemingly show that H-recolouring is PSPACE-complete.

Cutwidth

Collaborators: Faisal Abu-Khzam, Nicolas Bousquet, Henning Fernau, Amer Mouawad, Naomi Nishimura, Vinicius dos Santos

The workshop provided an opporunity to determine a group of researchers interested in working on open problems presented by Henning Fernau. Meetings will take place after the workshop, when there is more time to schedule them in.

Graph colourings

Collaborators: Matthew Johnson, Carl Feghali

Although the authors had collaborated in the past, the workshop gave them an opportunity to reconnect in the examination of a new area of research.

Minimum st-Separator Reconfiguration

Collaborators: Siddharth Gupta, Clment Legrand-Duchesne, Tom C. van der Zanden, Guilherme C. M. Gomes, Reem Mahmoud, Amer Mouawad, Yoshio Okamoto, Vinicius F. dos Santos

The group started meeting during the workshop, and has been meeting regularly since then. The authors have results relating to both the token sliding and jumping models, considering both decision variant of the problem as well as the problem of determining a minimum length reconfiguration sequence.

Reconfiguring trees

Collaborators: Hugo Akitaya, Therese Biedl, Daniel Cranston, Jeffrey Kam, Théo Pierron, Vinicius dos Santos, Ryuhei Uehara

The problem is trying to transform one (unlabeled) tree into another by repeatedly deleting an edge and adding an edge (staying a tree at each step).

We consider two variants of the problem: “edge-flipping” (no constraint on the deleted/added edge pair) and “edge-sliding” (a single edge can be “slid” along one edge; if wx is an edge, then wy can be replaced by xy).

We studied reconfiguration problems, especially under the aspect of wanting approximation algorithms for the shortest reconfiguration path between two instances (a topic that appears to have been barely studied). We focused on the problem of reconfiguring trees under edge flips. We proved that this is APX-hard (even for caterpillars), but also showed that there exists an O(1)-approximation algorithm for reconfiguring between two caterpillars.

Shortest path reconfiguration

Collaborators: Nicolas Bousquet, Kshitij Gajjar, Abhiruk Lahiri, Amer Mouawad

The collaborators started to work on the reconfiguration of shortest paths from a parameterized point of view during the workshop. They continued to meet on a weekly regular basis since the end of the workshop. They were able to prove that the problem is $W[1]$-hard parameterized by $k + \ell$ on bounded degeneracy graphs (and by k on general graphs), k being the length of the path and ℓ, the length of the desired transformation. On the positive side, they show that the problem is FPT parameterized by k on nowhere dense graphs. The work is still in progress and many questions still have to be solved, in particular the complexity on restricted graph classes such as bounded treewidth graphs.

Other new connections

Many participants reported having made new connections at the workshop; a few samples are listed below:

- Sevag Gharibian reported new connections with work related to the quantum setting, notably Alexandre Cooper, Stephanie Maaz, Amer E Mouawad, and Naomi Nishimura.

- Arnott Kidner received useful feedback on his talk, such as the suggestion of looking at 'lights out' and the idea of posing the problem of Γ-switch equivalence as a game. Work has now started on the game variant of the problem, which has resulted in progress towards a result on Γ-switch equivalence on hypergraphs.

- Rin Saito reported that after his mentoring session, Ryuhei Uehara advised that the problem is probably solvable in polynomial time also for Ptolemaic graphs. He and his co-authors are planning to discuss the details with Ryuhei Uehara after the workshop.

- Jeffrey Kam reports that Ryuhei Uehara suggested that we should consider outerplanar graphs for the minimum running buffer problem, which eventually leads to the following problem: Is there a minor closed graph property where it is not closed under k-subdivision but has a valid and bounded global minimum running buffer?

Outcomes of the Meeting

The meeting not only achieved the intended goals, but also helped to shape ideas on how to conduct future meetings. We first discuss how the hybrid format facilitated our goals, and then discuss topics adapted and expanded from those listed in the proposal.

Learning from the challenges and opportunities resulting from a hybrid format

Our community benefited greatly from the hybrid format, allowing participation by those for whom in-person participation would be not be possible, due to financial constraints, family obligations, health issues, and other reasons. Due to the backlog in visa processing, three in-person participants had to cancel their travel plans at the last minute, but were still able to participate virtually. In addition, the ability to selectively participate, rather than commit an entire week to the workshop, facilitated the participation of those new to the area, whose work is currently only tangential to reconfiguration. This broaded exposure resulted in 77 participants, helping our community to thrive and grow.

BIRS was instrumental in ensuring that the in-person and virtual participants were able to interact constructively. Many participants provided feedback on the excellence of the infrastructure and staff support and how, in contrast to their experiences in other venues, it was easy for participants to hear and see others both during talks and during question periods and discussions. Our experience was so positive that we have chosen to continue to have hybrid meetings in the future.

The tools that we used to be able to communicate, including a Discord server and a shared Overleaf document, allowed collaborations and communications to take place. We were so pleased with the results that we've decided to continue to use them in the long term. We also made use of Gather Town as a social space for virtual participants, such as during coffee breaks.

The overwhelming success of the hybrid model, and in particular the infrastructure and support provided by BIRS, has influenced plans for future meetings of the research community.

We intend to retain the following features of the program structure:

- Introductions: Each participant, in-person or virtual, is given a chance to speak briefly about their topics of interest. Such a session makes it possible for each participant to have a chance to address the community, even if not later giving a presentation. Several participants mentioned how useful it was to find out whom else to contact in the future to work on problems of mutual interest.

- Tutorials: In order to welcome newcomers and broaden the community, we intend to continue to ask for in-depth presentations on various topics. The tutorials are aimed at non-experts in a particular subarea, serving not only to level the playing field for all participants, but also to add to the archive of videos, enriching the resources available to those who might wish to teach graduate courses in the area.

- Invited talks: New breakthroughs in the area are highlighted in invited talks.

- Contributed talks: We would like to continue the policy of allowing everyone interested to present talks. In future workshops, we might need to consider changing the amount of time per talk or to find other ways to ensure that we leave adequate time for other types of sessions.

- Mentoring sessions: Instead of limiting presentations to those with completed work, we invited participants to present work in progress for the purposes of getting feedback from more experienced researchers in the area.

- Open problem sessions: As participants may be looking for collaborators on problems of interest, we dedicate time for presentation of open problems, which later serve as inspirations for working sessions.

- Working sessions: Problems identified in the open problem sessions, talks, and/or tutorials are discussed in small groups. Participants are welcome to join any of the groups and to move between different groups on different days.

Strengthening the community

As we noted in the proposal, researchers in the area of combinatorial reconfiguration do not have many opportunities to gather together in one place, as they are not only located on several different continents, but attend different conferences in different fields in mathematics and computer science. Since the community is still quite young and developments are occurring rapidly, it is important to be able to share ideas and help shape not only what research will be done but also how we will strengthen and build the community.

We gave all participants a chance to introduce their interests, giving others a chance to link a face to a name, and know whom to contact for future discussions (and well as ones that took place during the workshop).

As discussed in detail in Section 7, new collaborations were formed both among in-person participants, among virtual participants, and in at least one instance a mixture of in-person and virtual participants. In addition, several of the participants reported that even though they did not begin collaborations at the workshop, the workshop helped them to find whom to contact in the future, as well as new directions of interest.

Broadening the community

Our goals were to welcome newcomers to the community and to support underrepresented and early career participants as well as to build bridges to closely-related fields. The hybrid format made it possible to participate without making the financial and time commitment required to attend the entire in-person workshop.

Several talks, both formally designated as tutorials and otherwise, provided background information in related fields at a level suitable for newcomers to the areas. Catherine Greenhill provided an invited tutorial on connections between Markov chains, mixing time, and reconfiguration. Amer Mouawad's invited tutorial discussed connections between parameterized complexity and reconfiguration. In addition, Sevag Gharibian's contributed talk on reconfiguration in the quantum setting provided a gentle introduction to the area.

All participants were given chances to interact with others, whether through introductions, talks, mentoring sessions, or working sessions, as well as informally during coffee breaks. The workshop was, for several of our participants, their first opportunity to present their work to an international audience.

One of the ways that our community supports diversity is through diverse ideas and approaches. One example is the advent of a programming challenge (discussed and promoted in the invited talk by Jun Kawahara), complementing the theoretical work in the area.

To continue the connections in the community, participants were invited to join a mailing list in the area and a satellite workshop at ICALP taking place this year. A Discord channel continues to be used for communication and collaboration.

The recordings of the presentations provide additional resources to help others learn about the community and the work being done.

Providing suport for early-career participants

The workshop was designed to promote, showcase, connect, and encourage all participants, with particular attention to early-career participants. All participants were encouraged to give presentations, including in mentoring sessions, contribute to open problem sessions, and form collaborations in working sessions.

Although details have been omitted here due to the work being still in progress, various participants reported on new directions of research based on either questions or comments on their presentations or in collaborations that followed in the working sessions.

Participants

Abu-Khzam, Faisal (Lebanese American University)
Akitaya, Hugo (Tufts University)
Banjak, Hussein (American University of Beirut)
Bartier, Valentin (ENS Lyon)
Biedl, Therese (University of Waterloo)
Bigdeli, Reza (University of Waterloo)
Bohm, Jessica (University of Waterloo)
Bonamy, Marthe (Laboratoire Bordelais de Recherche en Informatique)
Bousquet, Nicolas (Université Claude Bernard Lyon 1)
Brewster, Richard (Thompson Rivers University)
Cooper-Roy, Alexandre (University of Waterloo)
Cranston, Daniel (Virginia Commonwealth University)
Demaine, Erik (Massachusetts Institute of Technology)
Ding, Zhiqian (University of Waterloo)
Dyer, Martin (University of Leeds UK)
El Sabeh, Remy (American University of Beirut)
Eto, Hiroshi (Tohoku University)
Feghali, Carl (Ens Lyon)
Fernandes dos Santos, Vinicius (Federal University of Minas Gerais (Brazil))
Fernau, Henning (Universität Trier)
Gajjar, Kshitij (NUS, Singapore)
Gharibian, Sevag (Paderborn University)
Gomes, Guilherme (Google)
Greenhill, Catherine (UNSW Sydney)
Gupta, Siddharth (University of Warwick, UK)
Haque, Sajed (University of Waterloo)
Hoang, Duc A. (Kyoto University)
Hommelsheim, Felix (University of Bremen)
Ibrahim, Hany (University of applied science Mittweida)
Ito, Takehiro (Tohoku University)
jaafari, tala (AUB)
Johnson, Matthew (Durham University)
Kakimura, Naonori (Keio University)
Kam, Jeffrey (University of Waterloo)
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Bibliography

[1] S. Alikhani, D. Fatehi, S. Klavzar. On the Structure of Dominating Graphs. Graphs and Combinatorics 33: 665–672, 2017.

[2] Marthe Bonamy, Nicolas Bousquet, Carl Feghali, and Matthew Johnson. On a conjecture of Mohar concerning Kempe equivalence of regular graphs. Journal of Combinatorial Theory, Series B, 135:179–199, 2019.

[3] Nicolas Bousquet, Amer E. Mouawad, Naomi Nishimura, Sebastian Siebertz. A survey on the parameterized complexity of the independent set and (connected) dominating set reconfiguration problems. arxiv preprint, abs/2204.10526, 2022.

[4] M. Celaya, K. Choo, G. MacGilivary, K. Seyffarth. Reconfiguring k-colourings of complete bipartite graphs. Kyungpook Math. Journal, 56: 647–655, 2016.

[5] Luis Cereceda, Jan van den Heuvel, and Matthew Johnson. Finding paths between 3-colorings. Journal of Graph Theory, 67(1):69–82, 2011.

[6] Daniel W. Cranston, Reem Mahmoud. Kempe Equivalent List Colorings arxiv preprint, abs/2112.07439, 2021.

[7] Hiroshi Eto, Takehiro Ito, Yasuaki Kabayashi, Yota Otachi, Kunihiro Wasa Reconfiguration of Regular Induced Subgraphs Proceedings of WALCOM: Algorithms and Computation, pages 35–46, 2022.

[8] C. Feghali, M. Johnson, D. Paulusma. A Reconfigurations Analogue of Brooks' Theorem and Its Consequences Journal of Graph Theory 83:340–358, 2016.

[9] K. Gajjar, A. V. Jha, M. Kumar, A. Lahiri. Reconfiguring Shortest Paths in Graphs. Proceedings of the AAAI Conference on Artificial Intelligence, 36(9), pages 9758–9766, 2022.

[10] Sevag Gharabian, Jamie Sikora. Ground state connectivity of local Hamiltonians. arxiv preprint, abs/1409.3182, 2014.

[11] Ruth Haas, Karen Seyffarth. The k-Dominating Graph. Graphs and Combinatorics, 30: 609–617, 2014.

[12] Jan van den Heuvel. The complexity of change. Surveys in Combinatorics 2013, 409:127–160, 2013.

[13] Takehiro Ito, Erik D. Demaine, Nicholas J. A. Harvey, Christos H. Papadimitriou, Martha Sideri, Ryuhei Uehara, and Yushi Uno. On the complexity of reconfiguration problems. Theoretical Computer Science, 412(12-14):1054–1065, 2011.

[14] W.W. Johnson, W. E. Story. Notes on the "15" puzzle. American Journal of Mathematics 2:397–404, 1879.

[15] Anna Lubiw, Vinayak Pathak. Flip distance between two triangulations of a point set is NP-complete. Computational Geometry, 49:17–23, 2015.

[16] Naomi Nishimura. Introduction to reconfiguration. Algorithms, 11(4):52, 2018.

[17] Uli Wagner, Emo Welzl. Connectivity of Triangulation Flip Graphs in the Plane (Part I: Edge Flips). Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, pages 2823–2841, 2020.

[18] James D. Watson, Johannes Bausch, Sevag Gharabian. The Complexity of Translationally Invariant Problems beyond Ground State Energies. arxiv preprint, abs/2012.12717, 2020.

Chapter 8

New Interactions Between Statistics and Optimization (22w5094)

May 22 - 27, 2022

Organizer(s): Jonathan Niles-Weed (New York University), Lnac Chizat (École Polytechnique Fdrale de Lausanne), Rachel Ward (University of Texas at Austin), Francis Bach (École Normale Supérieure)

Introduction to the field

Machine learning is the process by which a computer autonomously extracts knowledge from data. After years of continual progress, machine learning is now able to carry out difficult tasks such as image classification, language translation, or genomic data analysis. It typically works as follows: first a human designs a computer program with many free parameters, and then an algorithm adjusts these parameters so that the program achieves a good performance on training data. The design and analysis of such algorithms is done by researchers in optimization. But the actual goal is to solve the task for new data that does not belong to the training set. Understanding when and why good performance on training data also leads to good performance on unseen data is the task of researchers in statistics.

While optimization and statistics are both relevant to advance machine learning, they are distinct research communities, with different tools and languages. In the past, important breakthroughs have occurred when these two communities have interacted, an example being the introduction of the Stochastic Gradient Descent algorithm, used to train most machine learning programs in today's large scale applications. In this spirit the objective of the BIRS workshop "New Interactions Between Statistics and Optimization" was to facilitate new interactions and collaborations between the statistics and optimization communities, with the intention of sparking new ideas at the interface of the two fields.

Overview of the workshop

The workshop brought in twenty-two experts in statistics, optimization and related applied fields to present the latest developments and explore new directions in the field. There were fourteen talks and one open problem session during the workshop. The talks covered the following themes:

1. Implicit statistical regularization associated to common optimization algorithms

2. Representation costs associated with neural networks architectures
3. Statistical analysis of overfitting methods in high dimension
4. Statistics within optimization

These specialized topics were complemented by a talk by Lydia Liu who invited us to take a broader perspective and consider different statistical settings and models in order to take into account the interactions between algorithms and the environment in which they are deployed. She presented the various work on this topic completed as part of her PhD thesis.

Presentation Highlights

Implicit regularization

Statisticians have traditionally considered optimization as a black box: they study an estimator characterized as the unique minimizer of a well-posed optimization problem, or an approximation thereof. In this viewpoint, the optimization algorithm plays no statistical role and the task of optimization theory is confined to providing, for each problem, the most efficient and reliable algorithms.

However, in modern practice the choice of the algorithm strongly interferes with the statistical properties of the resulting estimator. A classical occurrence of this interference is via the practice of early stopping, i.e. stopping an iterative optimization algorithm before it comes close to a minimizer. Indeed, the optimization path, that is the sequence of intermediate estimators generated by an iterative optimization algorithm is specific to each algorithm, and therefore the "early stopped" estimator is algorithm dependent. This phenomenon has been studied for decades under the name iterative regularization.

More recently, practitioners have realized that, for supervised learning, simply minimizing (to global optimality) the empirical risk without any regularization sometimes led to state of the art performance in many machine learning tasks, but also that this performance highly depends on the algorithm. This phenomenon is possible because the empirical risk has, in overparameterized settings, an infinity of minimizers with very different statistical properties. The way in which each optimization algorithm selects a specific minimizer is called the implicit bias of the algorithm.

Several talks in the workshops have presented the latest advances in the understanding of algorithmic regularization.

Matus Telgarsky: Two implicit bias proof techniques The workshop opened with this topic of implicit bias. Matus started with a review and classification of the various implicit bias analyses that have been proposed for the gradient descent (GD) algorithm for linear models in classification. He in particular proposed a classification into strong and weak implicit bias guarantees: the former guarantees that the algorithm returns an estimator characterized as being optimal for some auxiliary optimization problem while the latter only attempts at directly providing statistical guarantees, bypassing "optimization" completely. The former is a stronger characterization but the latter is the one that matters from the statistical perspective and is more broadly available.

Loucas Pillaud-Vivien: The role of stochasticity in learning algorithms The talk of Loucas pertained to the difference between the implicit bias of the stochastic gradient descent (SGD) and GD method. While there is no difference in the basic case of linear regression, he presented a setting – the so-called diagonal two-layer linear neural networks – where SGD is provably biased towards sparser solutions than GD.

Nadav Cohen: Generalization in Deep Learning Through the Lens of Implicit Rank Minimization This talk also dealt with GD in a regression setting but for a more complex class of models, starting from matrix factorization, to tensor factorization and concluding with hierarchical tensor factorization. In all these settings, Nadav has shown that the GD algorithm exhibits an implicit bias towards low rank solutions (this is theoretically guaranteed in the early phases of training, and often empirically observed in later phases).

Arthur Jacot: Regimes of Training in Deep Neural Networks: a Loss Landscape Perspective In the same context of deep linear neural networks, Arthur presented an analysis with a different point of view: instead of focusing on the optimization dynamics, he described the loss landscape in which this algorithm navigates. He in particular has shown that depending on the scale of the initialization, the properties of this landscape wildly differ, going from a well-behaved strongly convex landscape for large scales to a landscape full of saddle points for small initializations.

Representation costs

For classification with the logistic loss, it has been shown in various contexts that GD selects a classifier for which the ratio of the margin over the ℓ_2-norm of the parameters is maximized. This observation naturally leads to the statistical (or approximation theoretical) question of characterizing which kind of functions can be represented by a certain parameterized model when the parameters span a ℓ_2-ball in parameter space. Equivalently, one can ask, given a target function to learn, what is the smallest norm of parameters with which it can be represented? The importance of this question is that it indicate which types of solutions are favored by particular architectures, and therefore clarifies the inductive bias of modern machine learning procedures. This line of inquiry has undergone recent progress, much of which was presented at the workshop.

Rebecca Willett: Linear layers in nonlinear interpolating networks Prior work on representation costs has mostly focused on the setting of ReLU (nonlinear) networks with a single hidden layer, or multi-layer networks, all of whose layers are linear. Both settings are quite far from the multi-layer, nonlinear networks used in practice. To begin to bridge this gap, Rebecca presented recent work on the representation costs for multi-layer networks whose first $L-1$ layers are linear and whose last layer uses a nonlinear ReLU activation. The results reveal that the structure encouraged by this architecture is more complicated than that revealed in prior work on the one-layer or multi-layer linear case, and that this structure includes an interplay between "sparsity-inducing" properties, which encourage simple solutions, and "alignment-inducing" properties, which encourages collapse to low-dimensional subspaces.

Suriya Gunasekar: A convolution property and proof using polynomial representation Most neural networks used for image classification employ convolutional layers with multiple channels (e.g., for inputs which are three-channel RGB images). Even when the number of layers is small, the convolutional structure makes characterizing the representation costs implied by the network theoretically challenging. Suriya presented a key step of a recent proof connecting representation costs in such networks to certain semi-definite programs (SDP). The proof is based on polynomial representations of discrete convolution operators, and is used to show that an auxiliary SDP representation of the network possesses rank-one solutions.

Alberto Bietti: How can kernels help us understand deep architectures? Exactly answering the representation cost question is generally out of reach for more complicated models. A work-around in order to understand the effect of compositionality and of certain computational structure (convolution, pooling, patch extraction, etc) is to study a kernel method with the same structure. This approach was presented by Alberto who has shown how kernel methods which include these kinds of structure where able to better represent functions that are stable under certain groups of transformations.

Benign overfitting

In classical supervised learning theory, one fixes a learning problem and studies how the prediction error decreases as a function of the number n of observed samples. In this context, methods that interpolate or overfit often lead to a sub-optimal, or even sometimes increasing, error when the observations are noisy. Benign overfitting refers to apparently paradoxical observation that in various settings – which are typically high-dimensional – overfitting leads to predictions that are almost optimal. This phenomenon, which was first discovered empirically, has led to a fruitful line of works that studies the performance of overfitting predictors in high-dimensional regimes.

Ohad Shamir: The Implicit Bias of Benign Overfitting Though originally observed in the context of deep learning, benign overfitting is now recognized to be a more general feature of high-dimensional estimation tasks, even in simple models such as linear regression. The wealth of such examples has led to the informal understanding that overfitting may essentially always be benign in sufficiently high-dimensional settings. Ohad presented results complicating this understanding, and showing that for linear regression and binary classification, overfitting is only benign in some very specific scenarios. In effect, this reveals a type of "implicit bias" for the benign overfitting phenomenon: overfitting implicitly biases certain types of solutions, and the overfitting is benign only if those solutions correspond to the optimal ones.

Theodor Misiakiewicz: Kernels in high-dimension: implicit regularization, benign overfitting and multiple descent The talk of Theodor consisted in a tutorial on the analysis of benign overfitting for high-dimensional kernel ridge regression. He first presented a general heuristic that one can replace the high-dimensional (potentially random) features by Gaussian random variables with matching first two moments, as long as the dimension d and the number of samples n scale polynomially $\log d \asymp \log n$. For this Gaussian model, one can derive the test error rather conveniently and observe various phenomena such as benign overfitting and non-monotonicity of the error. He finally presented various specific contexts where these results have been rigorously proved.

Statistics within optimization

In most of the talks presented above, although optimization practice inspired statistical theory and vice-versa, there was still a clear distinction between the two fields; with an auxiliary or implicit optimization problem at their interface. In this paragraph, we discuss the works presented during the workshop where this distinction is more blurry.

Nati Srebro: Early Stopping, Regularization, Interpolation and Uniform Convergence Nathi presented the general program through which one can combine optimization and statistics to better understand supervised machine learning. Until now, research works were typically following this line of thoughts: (i) one identifies a complexity measure R over the space of predictors such that, in certain contexts, the optimization algorithm (say, GD or SGD) converges to the minimizer of R over 0-training loss predictors, (ii) one then studies the generalization guarantees of this estimator via uniform convergence. He presented recent theoretical analyses that directly combine these two steps (optimization and statistics) and avoid resorting to uniform convergence. This leads to finer guarantees and is a "weak implicit bias" analysis, in terms of the classification proposed by M. Telgarsky in his talk.

Tomas Vakeviius: A tutorial on offset Rademacher complexity The talk of Tomas was dealing with the fundamental statistical problem of regression without distributional assumption where the goal is to compete with the best linear predictor. In this context, bounding the test loss with the Rademacher complexity is suboptimal, and this can be improved with localized Rademacher complexity for convex hypothesis classes. However, if one wishes to use non-convex hypothesis class, a more recent applicable tool is the offset Rademacher complexity. Tomas presented that analysis of Audibert's star estimator with this tool.

Varun Kanade: Statistical Complexity of Mirror Descent Considering the mirror descent algorithm for least-squares regression, Varun proved that there exists an iteration number for which the estimator satisfies the condition to apply offset Rademacher complexity analysis. This leads to precise statistical bounds on the behavior of mirror descent with the entropy mirror map for sparse linear regression. This analysis is one of the first that captures the effect of early stopping for non-euclidean optimization geometries.

Damien Scieur: Average-case analysis in optimization For the final talk of this workshop, Damien presented a new kind of analysis in optimization that consist in looking at the best average run-time of algorithms (instead of their best worst-case run-time which is more standard). When specialized to the case of quadratic problems, this idea invokes tools from random matrix theory and orthogonal polynomials theory to derive the optimal algorithm given a distribution over optimization problems. It in particular sheds a new light on Polyak momentum which is shown to be optimal in certain contexts.

Broader impacts

Statistics and optimization in machine learning are employed in the service of creating large-scale systems with implications for the rest of society. An important question, therefore, is to understand the implications of the techniques and objectives used in machine learning when it is deployed in practice, and how it interacts with broader values and goal.

Lydia Liu: Social Dynamics of Machine Learning for Decision Making An underappreciated aspect of machine learning systems is that they are often used simultaneously across many contexts or by many strategic agents, and that their use is dynamic (i.e., agents interact with these systems on multiple occasions over time). As a result, optimality or efficiency guarantees that hold at the level of a single instance may fail to correctly capture the behavior of machine learning systems as actually deployed. Drawing on techniques from microeconomics and algorithmic game theory, Lydia presented several insights from this perspective, with a particular focus on long-term impact. Her work reveals that several strategies designed to address short-term failures of fairness in machine learning systems can provably have negative consequences for long-term welfare.

Recent Developments and Open Problems

The open problem session held on the second day of the meeting focused on the following topics.

- Find a class of statistical problems where the choice of a specific architecture matters on the optimization and statistical aspects. Anything that justifies certain choices of architectures and informs practice. Matus mentionned [1] as an example of work in this direction.

- Prove a non-asymptotic and quantitative version of the implicit bias result for wide two-layer neural networks in [2].

- Memory constrained first-order optimization: suppose one F be a convex and 1-Lipschitz function defined . this is an old-standing open question on which recent progress has been made which increased the lower bound [3].

- How small the initialization needs to be to obtain a certain statistical performance? (ε-loss, $\alpha(\varepsilon)$).

Outcome of the Meeting

The Covid pandemic crisis led to a fragmentation of the mathematics, statistics, and computer science research communities, with far fewer opportunities for learning and engagement between disciplines. Moreover, in our experience, online talks and conferences barely solve this porblem and fail to promote genuine interaction. The organizers and participants are grateful to the support from BIRS, which allowed us to hold a vibrant and stimulating meeting.

Another important outcome in the meeting was the interaction between younger and more senior researchers. The workshop had a healthy mix of participants from different career levels, and several junior faculty members present at the workshop expressed gratitude for the opportunity to advertise their work and to have in-depth discussions with senior faculty members. Due to the Covid pandemic, several of the graduate student participants had never had the opportunity to attend an in-person workshop before, and their experience was very positive.

Comments on hybrid format We were fortunate to receive permission from the BIRS staff to have a larger in-person component of our workshop than had been authorized earlier in the pandemic. As a result, all of our speakers were able to present in-person, and a lively open problem session was held in the BIRS auditorium. Though substantially smaller than a typical, pre-Covid BIRS meeting, this critical mass of in-person participants was crucial to sparking the discussions and connections that occurred during the workshop. Moreover, the intimate size of the meeting meant that participants (from early-career students and postdocs to senior faculty) had the opportunity to meet and mingle informally at meals and at social events.

We also had a number of participants who joined virtually via zoom, though their overall number. At this point in the pandemic, we observed that interest in virtual workshops—no matter how well integrated—is waning, and that it was challenging to meaningfully integrate the few participants who elected to participate remotely.

Participants

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Bietti, Alberto (NYU)
Bruna, Joan (New York University)
Chizat, Lénaïc (Université Paris Sud)
Cohen, Nadav (Tel Aviv University)
Flammarion, Nicolas (EPFL)
Gabrie, Marylou (Ecole Polytechnique)
Gunasekar, Suriya (MSR)
Huang, Haoxiang (New York University)
Jacot, Arthur (EPFL)
Kanade, Varun (University of Oxford)
Liu, Lydia (Berkeley / Cornell)
Misiakiewicz, Theodor (Stanford)
Needell, Deanna (UCLA)
Niles-Weed, Jonathan (New York University)
Pillaud-Vivien, Loucas (EPFL)
Robeva, Elina (UBC)
Rudi, Alessandro (INRIA - Paris)
Sarkar, Purnamrita (UT Austin)
Scieur, Damien (Samsung SAIL Montreal)
Shamir, Ohad (Weizmann Institute of Science)
Shi, Bobby (UT Austin)
Simchowitz, Max (MIT)
Singh, Aarti (Carnegie Mellon University)
Srebro, Nati (Toyota Technological Institute - Chicago)
Sur, Pragya (Harvard)
Telgarsky, Matus (UIUC)
Vanden-Eijnden, Eric (Courant Institute)
Vaškevičius, Tomas (EPFL)
Villar, Soledad (Johns Hopkins)
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Willett, Rebecca (University of Chicago)
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Bibliography

[1] I. Safran, G. Vardi, and J.D. Lee, On the Effective Number of Linear Regions in Shallow Univariate ReLU Networks: Convergence Guarantees and Implicit Bias, to appear in COLT (2022).

[2] L. Chizat and F. Bach, Implicit bias of gradient descent for wide two-layer neural networks trained with the logistic loss, In Conference on Learning Theory, (2020), 1305–1338.

[3] A. Marsden, V. Sharan, A. Sidford, and G. Valiant, Efficient Convex Optimization Requires Superlinear Memory, arXiv preprint arXiv:2203.15260 (2022).

Chapter 9

Cross-community Collaborations in Combinatorics (22w5107)

May 29- June 3, 2022

Organizer(s): Marthe Bonamy (Laboratoire Bordelais de Recherche en Informatique), Natasha Morrison (University of Victoria), Jozef Skokan (London School of Economics)

Overview

In recent years some of the most exciting breakthroughs in Combinatorics on longstanding conjectures have resulted from innovative applications of established techniques to areas where they not necessarily used before. We would like to harness the power of collaboration and bring together open-minded participants with different areas of expertise to produce novel research in a number of globally studied areas including. We aspired to create new productive long-term bonds between members of the global community.

A large focus of the workshop was on the training and career enhancement of junior researchers. This was achieved through their fostering new collaborations with world-leading members of the global community during our focused small group work sessions. This gave junior participants opportunities to learn about and work in areas outside of their PhD/postdoctoral focus, gaining invaluable skills and knowledge. They were able to forge meaningful relationships with senior members of the community outside their home institution.

Workshop objectives

1. A primary objective of the workshop was to stimulate and foster genuinely new (and productive) collaborations amongst participants in topical areas that are not necessarily what they would usually work on and to create *new* long-term bonds between members of the global community.

2. Another key objective of the workshop was the training and career enhancement of junior participants. We have deliberately decided to make the workshop small - 21 people, to not be intimidating for more junior researchers and allow them to flourish. We aspired to a very welcoming and comfortable environment and for them to be able to develop meaningful relationships with senior members of the community.

3. We were committed to ensuring our final participant list is diverse and supports those under-represented in the mathematical sciences. Systematic barriers to inclusion are all too present in our field and we do not wish to enhance the problem.

In the sections below we will detail the scientific progress made during the workshop, and explain how we met each of these objectives.

Open Problems

One of the key goals of our workshop was to foster new and exciting collaborations amongst members of the combinatorics community that did not typically work together. We invited all participants to submit well thought out open problems in advance, and begun the workshop with an open problem session where these problems would be presented. In this section we summarise the problems that were suggested for the workshop.

Are trees eventually Turán-good?

For a pair of graphs G and F, say that G is F-free if G does not contain a subgraph isomorphic to F. Let $\mathcal{N}(H,G)$ denote the number of copies of a graph H in G, that is, the number of subgraphs of G isomorphic to H, and let

$$\mathrm{ex}(n,H,F) = \max\{\mathcal{N}(H,G) \mid G \text{ is an } n\text{-vertex } F\text{-free graph}\}$$

be the maximum number of subgraphs isomorphic to the target graph H in an n-vertex F-free graph.

Turán's theorem states that any K_{r+1}-free graph contains at most about $(1-\frac{1}{r})\binom{n}{2}$ edges, and furthermore the unique extremal graph (up to isomorphism) is the so-called Turán graph, $T_r(n)$: the complete r-partite graph with parts as balanced as possible. In 2015 Alon and Shikhelman [15] introduced the generalized Turán problem $\mathrm{ex}(n,T,F)$ which is the maximum number of subgraphs isomorphic to the "target graph" T in an n-vertex F-free graph (avoiding the "forbidden" graph F). Their paper sparked broad interest and results are known for many familes of T and F.

One difficulty in precisely determining $\mathrm{ex}(n,H,F)$ is identifying a potential extremal graph. In many cases when the problem is tangible, the extremal graph turns out to be the Turán graph. Let F be a graph with chromatic number $k+1$ and say that a graph H is F-Turán-good if for n sufficiently large, $\mathrm{ex}(n,H,F) = \mathcal{N}(H,T_k(n)))$. That is, the Turán graph $T_k(n)$ is an n-vertex F-free graph containing the maximum possible number of copies of H. The term Turán-good was recently introduced by Gerbner and Palmer [16], but the study of this phenomenon goes back much further to work of Györi, Pach and Simonovits [30]. See [16] for a comprehensive summary of what is known so far about F-Turán-good graphs.

Gerbner and Palmer [16, Conjecture 20] conjectured that for every graph H there exists $r_0 = r_0(H)$ such that H is K_{r+1}-Turán-good for every $r \geq r_0(H)$. This conjecture is known to hold for complete multipartite graphs, paths and C_5.

Problem 9.0.1 *Does the Gerbner-Palmer conjecture hold for trees? That is, if T is a tree, is there $r = r(T)$ such that for large enough n, the K_{r+1}-free graph on n vertices containing the maximum number of subgraphs isomorphic to T is the Turán graph $T_r(n)$?*

Rainbow saturation

Some standard terminology: a rainbow copy of a graph H is an edge-coloured graph is a copy of H whose edges are coloured with distinct colours.

Let $\mathrm{sat}_{\mathrm{rbw}}(n,H)$ be the minimum number of edges in an edge-coloured graph on n vertices which does not have a rainbow copy of H which is maximal with respect to this property. Girão, Lewis and Popielarz [1] consider this parameter (as well as a variant where there is another parameter t and the edges are coloured using colours from $[t]$). Among other things, they ask the following (in fact, they conjecture it is true).

Problem 9.0.2 *Is it true that for every graph H there is a constant $c = c(H)$ such that $sat_{rbw}(n,H) \leq cn$ for every $n \geq 1$?*

As background, define $\mathrm{sat}(n,H)$ to be the minimum number of edges in an n-vertex graph G which does not have a copy of H and is maximal with respect to this property. The above question is a natural extension of the following known fact: for every graph H there is a constant $c = c(H)$ such that $\mathrm{sat}(n,H) \leq cn$ for every $n \geq 1$.

Oriented chromatic number

The oriented chromatic number of an oriented graph $\vec{G}$, denoted $\chi_o(\vec{G})$ is the least integer k so that there is a graph homomorphism from $\vec{G}$ to some tournament of order k. In other words, it is the least k so that the vertices of $\vec{G}$ can be k-coloured with no arcs within a colour class and all arcs between two colour classes oriented in the same direction. It is conjectured (see [5]) that if $\vec{G}$ is an orientation of a 3-regular graph, then $\chi_o(\vec{G}) \leq 7$ and there are examples of 3-regular oriented graphs that require 7 colours. Duffy [6] (see also [5] for preliminary results) has shown that if G is an orientation of a connected 3-regular graph, then $\chi_o(\vec{G}) \leq 8$.

Problem 9.0.3 *Let $\vec{\mathcal{G}}(n, d)$ be a model for a random orientated d-regular graph, obtained by first selecting a random (undirected) d-regular graph uniformly at random and then orienting the edges uniformly at random. For $\vec{G} \sim \vec{\mathcal{G}}(n, 3)$ what is the range for $\chi_o(\vec{G})$, with high probability?*

If $\vec{G} \sim \vec{\mathcal{G}}(n, 3)$, then by deterministic results, $\chi_o(\vec{G}) \leq 8$. A first moment calculation shows that with high probability, $\chi_o(\vec{G}) > 4$. Indeed, considering local structures, with positive probability, $\vec{G}$ contains a directed 5-cycle which has oriented chromatic number 5.

In the case of 2-regular graphs, an orientation of a 2-regular graph can have oriented chromatic number in $\{2, 3, 4, 5\}$, but one can show (see [7]) that for a random oriented 2-regular graph, the oriented chromatic number is either 4 or 5 with high probability, each occurring with positive probability.

An Upper Bound on the DP-Chromatic Index

DP-colourings or correspondence colourings are a generalisation of list colourings introduced by Dvok and Postle [8]. Given a graph G, a cover $\mathcal{H}$ of G has the following properties: each vertex v of G has an associated set $L(v)$ of vertices in $\mathcal{H}$ where the induced graph on $L(v)$ is complete, and all the sets $L(v)$ are disjoint (this is analogous to the list of colours allowed on v). In addition, for each edge uv of G there is an matching between $L(u)$ and $L(v)$. An $\mathcal{H}$-colouring of G is an independent set in $\mathcal{H}$.

The DP-chromatic number $\chi_{DP}(G)$ of G is the smallest k such that G admits a DP-colouring for any cover $\mathcal{H}$ where all of the sets $L(v)$ have size k. One can easily see that $\chi_{DP}(G) \geq \chi_l(G)$ where $\chi_l(G)$ is the list-chromatic number of G. However, the two are not always equal.

One can similarly define the DP-chromatic index $\chi'_{DP}(G)$ of a graph G: it is the DP-chromatic number of the line graph $Line(G)$ of a G, i.e. the graph with vertex set $E(G)$ and two vertices adjacent if and only if the corresponding edges of G share an endpoint.

The famous edge-list colouring conjecture says that the list-chromatic index of a graph is equal to the chromatic index. Vizing made a weaker conjecture, that the list-chromatic index is at most $\Delta(G) + 1$. Bernshteyn and Kostochka asked whether this holds for the DP-chromatic index.

Problem 9.0.4 (Problem 1.13 from [9]) *Is $\chi'_{DP}(G) \leq \Delta(G)+1$? Or does there exist a graph G with $\chi'_{DP}(G) \geq \Delta(G) + 2$?*

This is stronger than Vizing's original conjecture, however introducting DP-colourings has proved useful in proving results about list-colourings as it is more 'local'. For example, Dvok and Postle originally introduced DP-colouring to answer a long-standing question of Borodin that every planar graph without cycles of lengths 4 to 8 is 3-list-colorable [8].

Arc-doubling in eulerian digraphs

The square of a digraph D without parallel arcs is the digraph D' that is obtained from D by adding the uw whenever $uv, vw \in A(D)$ for some $v \in V(D) - \{u, w\}$, omitting parallel arcs.

The following conjecture is due to Mody [19] where the property is proved for tournaments.

Conjecture 1 *Let D be an eulerian digraph without arcs in opposite direction and without parallel arcs. Then the square of D contains at least twice as many arcs as D.*

This conjecture would imply Seymour's second neighbourhood conjecture for eulerian digraphs. For some results on Seymour's second neighbourhood conjecture, see [20].

Update: Conjecture 1 also appears as [41, Conjecture 6.15] attributed to Seymour and/or Jackson.

A Turán Problem for Simplicial Spheres

Given a set of vertices V, say a collection S of subsets of V is a k-complex (short for k-dimensional homogeneous simplicial complex) if it is closed under subset inclusion and every maximal subset (called a facet) consists of $k+1$ vertices. Observe that a $(k+1)$-uniform hypergraph gives rise to a k-complex by taking the down-closure of its edges. Linial proposed a topological version of Turán's problem for k-complexes [11, 12]. The following remains open:

Problem 9.0.5 *Let $k \geq 3$ and let S^k denote the k-sphere. How many facets can an n-vertex k-complex $\mathcal{S}$ have while containing no homeomorphic copy of S^k? In other words, if such an $\mathcal{S}$ has $O(n^{k+1-\lambda})$ facets, what is the optimal value of $\lambda = \lambda(S^k)$?*

From [10], $\lambda(S^1) = 1$ and from [13], $\lambda(S^2) = \frac{1}{2}$. It is conjectured that $\lambda(S^k)$ is also the correct value of the "universal exponent" λ_k, a lower bound for all $\lambda(S)$ that depends only on the dimension of S.[14] show $\lambda_k > 0$ for all k, but another interesting problem is to determine more precisely λ_k for $k \geq 2$.

Counting tournament-free orientations of $G(n, p)$

An orientation $\vec{G}$ of a graph G is an oriented graph obtained by assigning an orientation to each edge of H. Given a fixed oriented graph $\vec{H}$ and a graph G, we can define $D(G, \vec{H})$, the number of $\vec{H}$-free orientations of G. In this problem description, we will consider the case $G = G(n, p)$, the binomial random graph.

Let $C_k^{\circlearrowright}$ denote the strongly-connected orientation of the cycle C_k. For $\vec{H} = C_k^{\circlearrowright}$, some initial estimates on $\log D(G(n, p), C_k^{\circlearrowright})$ were obtained by [21]. The correct order of growth of this function was later determined up to polylogarithmic factors in [22] and [23], who showed that $\log D(G(n, p), C_k^{\circlearrowright}) = \widetilde{\Theta}(n / p^{1/(k-2)})$ with high probability when $p \gg n^{-1+1/(k-1)}$, where $\widetilde{\Theta}(\cdot)$ is analogous to $\Theta(\cdot)$ notation but with polylogarithmic factors omitted.

Although the upper bounds in [22] and [23] only deal with forbidding cycles (or families of oriented graphs), the lower-bound construction described in Proposition 7.6 of [22] works for any oriented graph $\vec{H}$ and suggests the following problem.

Problem 9.0.6 (Question 7.7 of [22]) *Let $\vec{H}$ be a strongly connected tournament with $k := v(H) \geq 4$ and let $p \gg n^{-2/(r+1)}$. Prove that, with high probability as $n \to \infty$,*

$$\log D(G(n, p), \vec{H}) = \tilde{\Theta}\left(\frac{n}{p^{(k-1)/2}}\right)$$

As mentioned, the lower bound is known in all cases, but a matching upper bound is only known when the tournament is a cycle, that is, when $v(H) = 3$.

Maximum twin-width of an n-vertex graph

The twin-width of a graph is a new graph parameter, which was recently introduced by Bonnet, Kim, Thomassé and Watrigant [24] and already found numerous applications. Informally, twin-width measures the complexity of reducing the graph to a one vertex graph by iterative identification of vertices with similar neighborhoods. See e.g. [25, Section 2.1] for the formal definition.

Some of the very basic extremal questions about twin-width remain open. Ahn, Hendrey, Kim and Oum [25] explored the maximum twin-width of n-vertex graphs and proved that it is at most $n/2 + O(\sqrt{n \ln n})$.

Conjecture 2 *Every n-vertex graph has twin-width at most $(n-1)/2$.*

The bound in Conjecture 2 is tight for conference graphs.

Extremal problems for circulant graphs

The area of 'extremal problems for regular graphs' asks questions like: 'Which d-regular graph on n vertices has the most independent sets?' (answer by Kahn and Zhao: a union of $K_{d,d}$'s). The d-regular graph with the fewest indpendent sets is a union of K_{d+1}'s (easy to show) but if we ask for the minimizer of the number of independent sets or the independence number over d-regular triangle-free graphs, the question becomes much harder (Shearer's upper bound on $R(3, k)$ comes from a lower bound on the minimum).

For background reading on this area see [26, 27, 28, 32]. The solution to some problems of this form also can be used to identify computational thresholds in approximate counting and sampling problems [29]. The type of techniques used in this area include the entropy method, inductive arguments, and the 'occupancy method' combining statistical physics tools and linear programming.

Define a *circulant graph* $G(n, S)$ on n vertices with difference set $S \subset \mathbb{Z}$ as the graph with vertex set $[n]$ and $(i, j) \in E(G)$ iff $i - j \equiv x \mod n$ for some $x \in S$. (For instance, the cycle C_n is obtained as $G(n, \{1\})$).

It is not too hard to show that for any fixed set S and any $\lambda \geq 0$, the limit

$$f(\lambda, S) := \frac{1}{n} \log Z_{G(n,S)}(\lambda)$$

exists and can be computed in terms of eigenvalues of a set of discrete recurrence relations involving the set S.

From the function $f(\lambda, S)$ you can read off some interesting graph parameters including the asymptotic growth rate of the number of independent sets of $G(n, S)$ and the limiting independence ratio of $G(n, S)$ (size of max independent set divided by n). We can associate $f(\lambda, S)$ to the infinite 'circulant' graph[1] $G(\mathbb{Z}, S)$ with vertex set $\mathbb{Z}$ and edges (i, j) when $i - j \in S$.

The following open-ended problem is to explore some extremal problems for independent sets (or other homomorphism counts or for other partition functions) for different classes of circulant graphs.

Problem 9.0.7 *Pose and solve some extremal problems for (infinite) circulant graphs.*
For instance,

- *For a given d, which d-regular difference set S maximizes (minimizes) $f(\lambda, S)$?*
- *For a given d and g, which d-regular difference set S of girth at least g maximizes (minimizes) $f(\lambda, S)$?*
- *Which triangle-free difference set S minimizes the independence ratio?*
- *Which d-regular difference set maximizes (minimizes) the asymptotic growth rate of the number of q-colorings of $G(n, S)$?*

One hope is that the connection to eigenvalues will make solving some of these problems tractable and give a good understanding of the types of structures that lead to more or fewer independent sets (or other (weighted) homomorphisms) in a graph.

Monochromatic paths in multipartite hypergraphs

This is a problem on monochromatic partitions for hypergraphs. As in the classical Ramsey problem, one is given a graph G whose edges are coloured with two colours by an adversary, and one wishes to find certain monochromatic subgraphs. Instead of just one monochromatic copy, as in Ramsey's theorem, in monochromatic partitioning problems we want to find a collection of such copies that together cover the whole vertex set of the host graph.

Gerencsér and Gyárfás [34] observed that in any 2-colouring of the edges of K_n there are two disjoint monochromatic paths, of different colours, that together cover the vertex set of K_n. The same is true for 2-colourings of the edges of the complete 3-uniform hypergraph $\mathcal{K}^{(3)}$, and tight paths [35].

For complete bipartite graphs, it is known that there is always a partition into 3 monochromatic paths. In fact, for a certain class of colourings 3 monochromatic paths are needed, while for all other colourings 2 paths are

[1] Maybe there's another term for a difference graph on $\mathbb{Z}$?

enough. (See [33].) It would be nice to know what happens in the complete 3-partite 3-uniform hypergraph $\mathcal{K}^{(3)}_{3\times n}$. There are 2-colourings of the edges of this hypergraph which cannot be partitioned into less than 4 monochromatic tight paths, but are these colourings 'worst possible'?

Problem 9.0.8 *Find p such that for any 2-colouring of the edges of $\mathcal{K}^{(3)}_{3\times n}$ there is a partition into p monochromatic paths.*

One could also look for a asymptotic cover, which is probably easier.

Repeated patterns

We say two copies of a graph H in an edge-colouring of K_n are colour-isomorphic if there is an isomorphism between these copies preserving the colours. Given $n,\ k \geq 2$ and a fixed graph H, define $f_k(n, H)$ to be the smallest integer C such that there is a proper edge-colouring of K_n with C colours containing no k vertex-disjoint colour-isomorphic copies of H.

The main question on this topic is naturally the following: given a graph H and an integer $k \geq 2$, determine $f_k(n, H)$. Among other results, Conlon and Tyomkin [36] showed that if H is a non-bipartite graph, then $f_2(n, H) = n$ for odd n, and $n - 1 \leq f_2(n, H) \leq n + 1$ for even n.

Problem 9.0.9 *Let H be a non-bipartite graph and let n be even. Determine $f_2(n, H)$.*

A natural starting point is to consider colour-isomorphic triangles.

Problem 9.0.10 *Determine $f_2(n, K_3)$.*

Answering a question of Conlon and Tyomkyn [36] conjectured in a stronger form way by Ge, Jing, Xu and Zhang [37], Janzer [38] proved that for any positive integers $k, r \geq 2$, we have

$$f_r(n, C_{2k}) = \Omega(n^{\frac{r}{r-1}\cdot\frac{k-1}{k}}). \tag{9.0.1}$$

Conlon and Tyomkyn showed that if H contains a cycle, then there exists r such that $f_r(n, H) = O(n)$. In view of this, one may consider the following problem.

Problem 9.0.11 *Given a graph H that contains a cycle, determine the smallest r such that $f_r(n, H) = O(n)$.*

Note that (9.0.1) shows that for an even cycle C_{2k} we have $r \geq k$. In [37], it is proved that $f_3(n, C_4) = O(n)$.

Partitioning Geometric Graphs

A geometric graph $G = G(P, E)$ is a graph drawn in the plane where the points P are in general position and edges E are straight line segments. A partition of a graph G is a set of edge-disjoint subgraphs of G such that each edge of G is in exactly one subgraph.

Problem 9.0.12 *Is there a constant $c < 1$ such that every complete geometric graph can be partitioned into at most cn plane subgraphs?*

The case $c = 1$ is easy, as we can simply take stars centered at every vertex. Bose, Hurtado, Rivera-Campo and Wood [39] showed that each geometric drawing of the complete graph can be partitioned into $n - \sqrt{\frac{n}{12}}$ plane subgraphs which is the current best result. If the pointset lies on a circle, then the graph can be partitioned into $\frac{n}{2}$ plane subgraphs but not less, since there are $\lfloor \frac{n}{2} \rfloor$ edges which pairwise cross.

If we ask about packing plane subgraphs in general graphs, edge colorings in planar graphs H are a special case. From a straight-line drawing of H, slightly extend all line segments, so that they cross at the vertices of H, and nowhere else.For maximum degree $\Delta = 4$ or $\Delta = 5$ it is conjectured to be NP-hard to decide whether a planar graph is Δ or $\Delta + 1$-edge colorable, while it was shown that planar graphs with $\Delta \geq 7$ are Δ-edge colorable.

$(F, \overline{F}^b)$-free graphs

Problem 9.0.13 *For $t \in \mathbb{N}$, are there $2^{O(n \log n)}$ bipartite graphs on n vertices that contain neither P_t nor the bipartite complement $\overline{P_t}^b$ as induced subgraph?*

This is related to the notion of adjacency labelling scheme (with labels of size $O(\log n)$). Relevant papers include the place where the conjecture (for all forests) was posed [2], the place where the conjecture was proved for star forests [4], and a theorem on $(F, \overline{F}^b$-free bipartite graphs [3].

Presentations

In this section we give details on the talks at the workshop. We invited a small number of senior researchers to give talks on powerful current methods or exciting recent results. Towards are goal of training younger researchers, we encouraged any non-senior researcher who wanted to speak to volunteer give a talk and had presentations from Natalie Behague, Florian Hoersch, JD Nir, Mahsa Shirazi and Corrine Yap.

Plenary talks

Speaker: Sergey Norin
Title: Burning Large Trees
Abstract: The burning number $b(G)$ of the graph G is the minimum k such that $V(G)$ can be covered by vertex sets of subgraphs $G_1, \ldots, G_k$ such that G_i has radius at most $i-1$. The burning number conjecture of Bonato, Janssen and Roshanbin states that $b(G) \leq \lceil \sqrt{n} \rceil$ for any connected n vertex graph. We will show that $b(G) \leq (1+o(1))\sqrt{n}$ by considering continuous and fractional variants of the problem.

Speaker: Richard Montgomery
Title: On the Erds-Gallai cycle decomposition conjecture
Abstract: In the 1960's, Erds and Gallai conjectured that the edges of any n-vertex graph can be decomposed into $O(n)$ cycles and edges. In 2014, Conlon, Fox and Sudakov made the first general progress on this by showing an n-vertex graph can always be decomposed into $O(n \log \log n)$ cycles and edges. I will discuss how to improve the $\log \log n$ in this bound to the iterated logarithm function, and the tools and methods involved. This is joint work with Matija Buci.

Speaker: Will Perkins
Title: The statistical physics perspective in combinatorics
Abstract: I'll introduce some objects, concepts, and questions from statistical physics, then explain how one can look at problems in extremal and enumerative combinatorics from this perspective. I'll describe methods from statistical physics for enumerating independent sets in graphs and how these can be combined with combinatorial tools like graph containers.

Speaker: Maya Stein
Title: Towards a Posa Seymour conjecture for hypergraphs
Abstract: A central problem in extremal graph theory is to study degree conditions that force a graph G to contain a copy of some large or even spanning graph F. One of the most classical results in this area is Dirac's theorem on Hamilton cycles. An extension of this theorem is the Posa-Seymour conjecture on powers of Hamilton cycles, which has been proved for large graphs by Komlos, Sarkozy and Szemeredi.

Extension of these results to hypergraphs, using codegree conditions and tight (powers of) cycles, have been studied by various authors. We give an overview of the known results, and then show a codegree condition which is sufficient for ensuring arbitrary powers of tight Hamilton cycles, for any uniformity. This could be seen as an approximate hypergraph version of the Posa-Seymour conjecture. On the way to our result, we show that the same codegree conditions are sufficient for finding a copy of every spanning hypergraph of bounded tree-width which admits a tree decomposition where every vertex is in a bounded number of bags.

This is joint work with Nicolas Sanhueza-Matamala and Matias Pavez-Signe.

Scientific Progress and Ongoing Collaborations

In this section we summarise some of the more tangible progress that has been made as a consequence of the workshop. We note, in addition, that a number of other collaborations and research visits been planned by participants who met at our workshop, so more progress is to come!

Turán-good graphs

During the workshop, Morrison, Norin, Rzążewski and Wesolek solved Problem 9.0.1. Moreover, they proved the conjecture of Gerbner and Palmer [16] holds for all graphs.

Theorem 9.0.14 *Let H be a graph and $r \geq 300v(H)^9$. Then H is K_{r+1}-Turán-good.*

Theorem 9.0.14 follows from a more technical result, which also implies that for $r \geq 300v(H)^9$ Turán graphs always maximize the number of copies of H among K_{r+1}-free graphs on any given number of vertices, i.e., the requirement that the number of vertices is large compared to r is unnecessary.

Twin-width

Behague, Johnston, Hörsch, Morrison, Nir, Norin, Rzążewski and Shirazi made progress in a number of directions towards proving Conjecture 2.

They were able to show that the conjecture holds for the Erdős-Rényi random graph $G(n, 1/2)$ and for all graphs on up to at most 14 vertices. They also obtained bounds on the twin-width of $G(n, p)$ for a wide range of p. This work is ongoing.

Rainbow saturation

As a consequence of discussions at this workshop, Behague, Johnston and Morrison, along with Odgen (a masters student of Morrison) initiated a collaboration which resulted in them solving Problem 9.0.2 fully. They have proved the following theorem.

Theorem 9.0.15 *For every graph H there exists a constant $c = c(H)$ such that*

$$\mathit{sat}_{\mathit{rbw}}(n, H) \leq cn.$$

This result is currently being written up.

Monochromatic paths in multipartite hypergraphs

During working sessions at the workshop, Bowtell, Skokan, Stein, Wesolek made progress towards resolving Problem 9.0.8. They can obtain two disjoint monochromatic loose paths covering all but $o(n)$ of the vertices in a 3-partite 3-graph. In addition, they can cover all vertices (or all but a small constant and for all n) by 3 such paths. They are currently working on extending their initial argument using absorbing and explore other generalisations, i.e. from loose paths in 3-partite 3-graphs to ℓ-paths in k-partite k-graphs for some more values of ℓ and k.

Training and career enhancement of junior participants

The workshop was carefully designed to maximise the benefit to less senior researchers (details of how we did this can be seen in our original proposal). As organisers, we designed the working groups in such a way as to ensure that all junior researchers were in some group with more senior participants, so that they could network and learn from their expertise. We also planned group social activities in the evenings to facilitate networking in a more relaxed setting. We believe we were very successful in these goals, as we have received several very positive emails from participants after the workshop thanking us for the invitation and telling us what they gained from the experience.

One PhD student wrote: "The workshop was one of the most valuable experiences I've had in my mathematical career thus far. On the professional side of things: I did not know most of the participants beforehand but through the workshop made many new professional and personal connections. I have begun two new collaborations that we plan to continue beyond the workshop. Through the talks and the problem presentations, I learned a lot about the breadth of problems which people in combinatorics are working on.

As a graduate student, I was not sure what to expect, but the organizers fostered a wonderfully collaborative atmosphere among all the participants. The fact that all participants, both junior and senior, were encouraged to contribute problems and give talks played a big part. I felt like I was able to make meaningful contributions to the mathematical conversations, which is not always the case at other mathematical conferences/events, and I think that was because of how welcoming the organizers and other participants were. The setting of Banff and the many informal social activities were, dare I say, equally as valuable as the research time itself. Whether during a hike, or mealtimes, or coffee breaks, I got to know most of the other participants and received great mentorship and advice about my future academic career. Overall, I am incredibly grateful that I had the opportunity to participate in this workshop."

Equity, Diversity and Inclusion

In order to achieve our third objective, we spent a lot of time and consideration before the workshop on ensuring our final participant list was diverse and strongly includes those from groups under-represented in the mathematical sciences.

More details about our nomination process can be found in our EDI statement, we present a summary here. We solicited nominations for PhD students and postdocs to invite, both from invited participants and other members of the community. This resulted in us being able to invite many wonderful young researchers that were not already personally known to us the organisers. This resulted in all participants meeting new colleagues and future collaborators at the workshop. We asked in particular for nominations of those under-represented in the mathematical sciences and for nominations of participants that were based at different institutions to the nominator (to try to ensure diversity and minimise nepotism).

As noted in our EDI statement, we wrote that "We are aiming for at least 50% of our final participants to identify as female and for at least 20% to be visible minorities." We are happy to confirm that we achieved these targets, despite COVID causing a number of participants to drop out close to the workshop and us having to issue new invites at the last minute.

We also wrote: "We will select a diverse and representative subset of the more senior participants to give longer talks". We did do this, but unfortunately two of our plenary speakers dropped out at the very last minute, and several other senior participants did not feel prepared to talk at such short notice, so the final cohort of plenary speakers was not as diverse as it had been planned to be.

Conclusion

We believe that the workshop was a great success, both scientifically and for the career enhancement of more junior researchers. We are very happy to have been able to facilitate such a positive impact on the more junior members of our community, especially given how disruptive the previous few years have been. We very much hope to be able to repeat this success with a similar event in the future.

Participants

Behague, Natalie (University of Victoria)
Boettcher, Julia (London School of Economics and Political Science)
Bonamy, Marthe (Laboratoire Bordelais de Recherche en Informatique)
Bowtell, Candida (University of Birmingham)
Collares, Mauricio (Federal University of Minas Gerais)
Gunderson, Karen (University of Manitoba)
Hoersch, Florian (TU Ilmenau)
Johnston, Tom (University of Bristol)
Letzter, Shoham (UCL)
Montgomery, Richard (University of Warwick)
Morrison, Natasha (University of Victoria)
MOTA, GUILHERME (USP)
N. Shirazi, Mahsa (University of Regina)
Nir, JD (University of Manitoba)
Norin, Sergey (McGill University)
Perkins, Will (Georgia Tech)
Rzążewski, Paweł (Warsaw University of Technology)
Skokan, Jozef (LSE)
Stein, Maya (Universidad de Chile)
Wesolek, Alexandra (Simon Fraser University)
Yap, Corrine (Rutgers University)

Bibliography

[1] Giro, A., Lewis, D. & Popielarz, K. Rainbow saturation of graphs. J. Graph Theory. **94** pp. 421-444 (2020)

[2] Lozin, V. & Zamaraev, V. The structure and the number of P7-free bipartite graphs. European Journal Of Combinatorics. **65** pp. 143-153 (2017)

[3] Scott, A., Seymour, P. & Spirkl, S. Pure pairs. IV. Trees in bipartite graphs. ArXiv Preprint ArXiv:2009.09426. (2020)

[4] Atminas, A. Classes of graphs without star forests and related graphs. ArXiv Preprint ArXiv:1711.01483. (2017)

[5] Duffy, C., MacGillivray, G. & Sopena, . Oriented colourings of graphs with maximum degree three and four. Discrete Math.. **342**, 959-974 (2019), https://doi-org.uml.idm.oclc.org/10.1016/j.disc.2018.10.008

[6] Duffy, C. Colourings of oriented connected cubic graphs. Discrete Math.. **343**, 112021, 6 (2020), https://doi-org.uml.idm.oclc.org/10.1016/j.disc.2020.112021

[7] Gunderson, K. & Nir, J. The oriented chromatic number of random graphs of bounded degree. (2022)

[8] Dvok, Z. & Postle, L. Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. Journal Of Combinatorial Theory, Series B. **129** pp. 38-54 (2018), https://www.sciencedirect.com/science/article/pii/S0095895617300795

[9] Bernshtein, A. & Kostochka, A. On differences between DP-coloring and list coloring. Matematicheskie Trudy. **21**, 61-71 (2018)

[10] Mader, W. Homomorphieeigenschaften und mittlere Kantendichte von Graphen. Math. Ann.. **174** pp. 265-268 (1967)

[11] Linial, N. What is high-dimensional combinatorics?. RandomApprox. (2008)

[12] Linial, N. Challenges of high-dimensional combinatorics. Lovsz's Seventieth Birthday Conference. (2018)

[13] Brown, W., Erds, P. & Ss, V. On the existence of triangulated spheres in 3-graphs, and related problems. Period. Math. Hungar.. **3** pp. 221-228 (1973)

[14] Long, J., Narayanan, B. & Yap, C. Simplicial homeomorphs and trace-bounded hypergraphs. ArXiv Preprint ArXiv:2011.08167. (2020)

[15] Alon, N. & Shikhelman, C. Many T copies in H-free graphs. J. Combin. Theory Ser. B. **121** pp. 146-172 (2016)

[16] Gerbner, D. & Palmer, C. Some exact results for generalized Turn problems. European J. Combin.. **103** pp. Paper No. 103519, 13 (2022), https://doi-org.uml.idm.oclc.org/10.1016/j.ejc.2022.103519

[17] Cutler, J., Nir, J. & Radcliffe, A. Supersaturation for subgraph counts. Graphs Combin.. **38**, Paper No. 65, 21 (2022), https://doi-org.uml.idm.oclc.org/10.1007/s00373-021-02454-y

[18] Hei, D., Hou, X. & Liu, B. Some exact results of the generalized Turn numbers for paths. (arXiv,2021), https://arxiv.org/abs/2112.14895

[19] Mody, N. Squaring an Eulerian Digraph and the Seymours Second Neighborhood Conjecture. Unpublished Manuscript. (2012), https://www.academia.edu/16700143/Squaring_an_Eulerian_Digraph_and_the_Seymours_Second_Neighborhood_Conjecture

[20] Verheijen, T. Seymour's Second Neighborhood Conjecture. (2007), https://www.math.ru.nl/OpenGraphProblems/TimV/aboutthiswebsite.html

[21] Allen, P., Kohayakawa, Y., Mota, G. & Parente, R. On the number of orientations of random graphs with no directed cycles of a given length. The Electronic Journal Of Combinatorics. pp. P1-52 (2014)

[22] Collares, M., Kohayakawa, Y., Morris, R. & Mota, G. Counting restricted orientations of random graphs. Random Structures & Algorithms. **56**, 1016-1030 (2020)

[23] Campos, M., Collares, M. & Mota, G. Counting C_k-free orientations of $G(n,p)$. In preparation

[24] Bonnet, ., Kim, E., Thomass, S. & Watrigant, R. Twin-width I: tractable FO model checking. 2020 IEEE 61st Annual Symposium On Foundations Of Computer Science (FOCS). pp. 601-612 (2020)

[25] Ahn, J., Hendrey, K., Kim, D. & Oum, S. Bounds for the Twin-width of Graphs. ArXiv Preprint ArXiv:2110.03957. (2021)

[26] Zhao, Y. Extremal regular graphs: independent sets and graph homomorphisms. The American Mathematical Monthly. **124**, 827-843 (2017)

[27] Davies, E., Jenssen, M., Perkins, W. & Roberts, B. Independent sets, matchings, and occupancy fractions. Journal Of The London Mathematical Society. **96**, 47-66 (2017)

[28] Csikvri, P. Extremal regular graphs: the case of the infinite regular tree. ArXiv Preprint ArXiv:1612.01295. (2016)

[29] Davies, E. & Perkins, W. Approximately Counting Independent Sets of a Given Size in Bounded-Degree Graphs. 48th International Colloquium On Automata, Languages, And Programming (ICALP 2021). (2021)

[30] Gyori, Ervin and Pach, Janos and Simonovits, Miklos. On the maximal number of certain subgraphs in K_r-free graphs. Graphs Combin.. pp. P31–37 (1991)

[31] Perarnau, G. & Perkins, W. Counting independent sets in cubic graphs of given girth. Journal Of Combinatorial Theory, Series B. **133** pp. 211-242 (2018)

[32] Sah, A., Sawhney, M., Stoner, D. & Zhao, Y. A reverse Sidorenko inequality. Inventiones Mathematicae. **221**, 665-711 (2020)

[33] Pokrovskiy, A. Partitioning edge-coloured complete graphs into monochromatic cycles and paths. J. Combin. Theory, Ser. B, **106** (2014), 7097.

[34] Gerencsr, L. & Gyrfs, A. On Ramsey-type problems.. Annales Univ. Etvs Section Math., 10:167170, 1967.

[35] Stein, M. Monchromatic paths in 2-edge-coloured graphs and hypergraphs.. ArXiv:2204.12464.

[36] Conlon, D. & Tyomkyn, M. Repeated patterns in proper colourings. ArXiv E-prints. pp. earXiv:2002.00921 (2020)

[37] Xu, Z., Zhang, T., Jing, Y. & Ge, G. Color isomorphic even cycles and a related Ramsey problem. ArXiv E-prints. pp. arXiv:2004.01932 (2020,4)

[38] Janzer, O. Rainbow Turn number of even cycles, repeated patterns and blow-ups of cycles. ArXiv E-prints. pp. arXiv:2006.01062 (2020,6)

[39] Bose, P., Hurtado, F., Rivera-Campo, E. & Wood, D. Partitions of complete geometric graphs into plane trees. Computational Geometry. **34**, 116-125 (2006)

[40] Bustamante, S., Corsten, J., Frankl, N., Pokrovskiy, A. & Skokan, J. Partitioning Edge-Colored Hypergraphs into Few Monochromatic Tight Cycles. SIAM Journal On Discrete Mathematics. **34**, 1460-1471 (2020,1), https://doi.org/10.1137

[41] Sullivan, B. A Summary of Problems and Results related to the Caccetta-Haggkvist Conjecture. (2006)

Chapter 10

Poisson Geometry, Lie Groupoids and Differentiable Stacks (22w5035)

June 5 -10, 2022

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Overview of the Field

Poisson Geometry is an amalgam of three classical theories: it is *Foliation Theory* inside which *Symplectic Geometry* and *Lie Theory* interact with each other. Geometrically, a Poisson structure on a space M is, first of all, a (possibly singular) foliation of M; hence M is partitioned into leaves. Secondly, the leaves are endowed with symplectic structures. Thirdly, transversal to the leaves, we have Lie groups/algebras. Often the space M has some additional structure which leads to further connections. For example, M could be an algebraic variety in which case Poisson brackets on M are often intimately related to *Representation Theory* and *Noncommutative Geometry*. Altogether, one of the strengths of Poisson Geometry is its potential to provide often unexpected interplays between diverse fields.

Groups typically arise as the symmetries of some given object. The concept of a groupoid allows for more general symmetries, acting on a collection of objects rather than just a single one. Groupoid elements may be pictured as arrows from a source object to a target object, and two such arrows can be composed if and only if the second arrow starts where the first arrow ends. Just as Lie groups (as introduced by Lie around 1900) describe smooth symmetries of an object, Lie groupoids (as introduced by Ehresmann in the late 1950's) describe smooth symmetries of a smooth family of objects. That is, the collection of arrows is a manifold G, the set of objects is a manifold M, and all the structure maps of the groupoid are smooth. Ehresmann's original work was motivated by applications to differential equations, but since then Lie groupoids have appeared in many other branches of mathematics and physics, such as Algebraic Geometry (Grothendieck), Foliation Theory (Haefliger), Noncommutative Geometry and Index Theory (Connes-Skandalis). Nowadays, one can find many other applications of Lie groupoids, such as in geometric mechanics, equivariant differential geometry, higher gauge theory, orbifold theory, exterior differential systems, Ricci flows, and generalized complex geometry.

Motivated by quantization problems, Karasev and Weinstein introduced the symplectic groupoid of a Poisson manifold in the late 1980's, as a way to "untwist" the complicated behavior of the symplectic foliation underlying the Poisson manifold. Moreover, the infinitesimal symmetries corresponding to Lie groupoids are described by Lie algebroids, and at the same time it was realized that Lie algebroids can be characterized as vector bundles with fiberwise linear Poisson structures. Once these connections between Lie groupoid theory and Poisson geometry were established, the two fields exploded and became inseparable.

Recent days have also seen rapid developments on shifted Poisson and symplectic structures on (derived) differentiable or algebraic stacks. A differentiable stack is, roughly speaking, a Lie groupoid up to Morita equivalence, and the stack represented by a symplectic groupoid of a Poisson manifold naturally has a 1-shifted symplectic structure. There have also been remarkable recent advances in other geometries, such as Dirac geometry and generalized complex geometry, that generalize Poisson geometry and have Lie groupoids and Lie algebroids at their cores. Many basic concepts and constructions in these geometries can be rephrased using the language of differential stacks, and such reformulations put these geometric structures in vastly new perspectives and establish further connections with other fields of mathematics such as algebraic geometry, deformation theory and high category theory.

Recent Developments and Open Problems

This BIRS workshop was centred around Poisson geometry, Lie Groupoids and differentiable stacks. In spite of all the progresses made so far and some amazing recent advances, deep and rich interconnections between these areas remain to be discovered and the workshop brought together different groups of people and diverse viewpoints. Among the major open problems that remain, where Lie groupoids should play a fundamental role, the workshop addressed the following specific topics:

- **Integrations of Poisson and Dirac structures**: Originally conceived as a framework for Dirac's theory of second class constraints in geometric mechanics, Dirac geometry has emerged as a flexible generalization of Poisson geometry with far-reaching applications [11, 15, 32, 39, 49]. Generalizing the fact that symplectic groupoids are integrations of Poisson manifolds, the global objects integrating Dirac manifolds are presymplectic groupoids [16]. While a fundamental result established in [25, 26] describes the precise obstruction for a Lie algebroid to have an integration into a Lie groupoid, explicit integrations of given Poisson or Dirac structures into symplectic and presymplectic groupoids remain interesting and desirable, as such integrations have applications in the problems of normal forms and linearizations around their leaves [1, 24, 27, 28, 29], in quantization [8, 43], as well as in applications to topological field theory through Poisson sigma models [20]. Explicit integrations for a large class of Poisson and Dirac structures originated from the theory of quantum groups have recently been given in [18, 59].

- **Generalized complex geometry and mirror symmetry**: Mirror symmetry suggests deep relations between complex manifolds with their symplectic "mirror" manifolds. Generalized complex structures treat symplectic and complex structures on equal footing [39, 45], as suggested by mirror symmetry and other physical dualities. They have a corresponding global object given by Lie groupoids with a multiplicative structure consisting of a symplectic form and a complex structure, satisfying certain compatibility relations [23], but this is not the full picture. The complete description of global counterparts of generalized complex structures is more intricate and has been only recently obtained in [5]. There also have been proposals to explain the origin of a monoidal structure on the Fukaya category via symplectic groupoids [62].

- **Multiplicative structures on Lie groupoids and stacks**: generalizing the previous cases, the study of multiplicative structures (differential forms, multivector fields, connections, and so on) and their infinitesimal versions [14, 16, 46, 43] has provided new insights into both classical problems of differential geometry and the geometry of moduli spaces (stacks). This study includes for example Poisson group(oid)s [54], closely connected with integrable systems and quantum groups, holomorphic structures on Lie groupoids [47], and Riemannian structures on stacks [36, 37]. Additionally, one can recast Cartan's work on Lie pseudogroups in the language of multiplicative forms on Lie groupoids, showing that the classical Spencer operator appears as the linearization data of the Cartan Pfaffian system [30].

- **Shifted symplectic geometry**: Shifted symplectic geometry is the study of symplectic structures on spaces known as derived stacks, which are generalizations of smooth manifolds and algebraic varieties. A novel aspect of symplectic structures on stacks is that they have an integer grading: usual symplectic structures are simply 0-shifted in this grading, but alternative degrees -1, 1, and 2 have been much studied over the past 10 years since the foundational work of Pantev, Toen, Vaquie, and Vezzosi [61]. They are very closely related to and tie together all the subjects listed in this proposal. For example, the symplectic form on the space G of arrows of a symplectic groupoid over the Poisson manifold M gives rise to a 1-shifted symplectic structure on the stack M/G (see [34] for a more general picture). Similarly, by work of Pym and Safronov, Dirac structures may be interpreted as 2-Lagrangians in a 2-shifted symplectic manifold. Finally, recent work [5] of Bailey and Gualtieri shows that generalized complex structures may be interpreted as holomorphic 1-shifted symplectic stacks. Perhaps most importantly, -1-shifted symplectic structures are at the heart of what is called the Batalin-Vilkovisky formalism for classical and quantum field theory in physics; much of the current research in the subject aims to make use of this relationship to make progress on current problems in geometry and physics, such as the problem of deformation quantization (after Kontsevich) of derived stacks which was solved in [19] for the case of nonzero shifts but remains open for shift zero. Another important open question is to develop more explicit forms of results such as the -1-shifted Darboux theorem of [12, 13]. This result asserts the existence of local algebraic models for the Chern-Simons potential, which are in turn used to construct various refined Donaldson-Thomas/3-manifold invariants, but are currently very difficult to calculate.

- **Higher Lie groupoids and higher gauge theory**: Higher gauge theory, as developed by Baez and coauthors (see e.g. [4]), is an extension of gauge theory that describes parallel transport not only for point particles but also for higher-dimensional objects; in particular, it treats horizontal lifts of surfaces, rather than just paths. From a physics perspective, it is motivated by string theory, but has also been applied to other fields, such as loop quantum gravity. Just as ordinary gauge theory concerns fiber bundles with structure Lie groups, higher gauge theory deals with bundles with gauge structure given by higher, or categorified, versions of Lie groups and groupoids. Particular types of higher groupoids, known as double Lie groupoids, arise naturally in Poisson geometry: for example, the integration of Poisson Lie group(oid)s often leads to double symplectic groupoids [52]. More general Lie 2-groupoids (or stacky groupoids) [66] can be regarded as groupoid structures on differentiable stacks, i.e., as models for groupoid structures on singular quotients. The general treatment of higher groupoids usually involves simplicial and homotopical methods, which is linked to the fact that, from a Lie theoretic perspective, higher Lie groupoids are thought of as global versions of L_∞-algebroids, though a precise connection remains elusive.

Presentation Highlights

Paired lectures

The morning sessions on Monday–Thursday consisted of "paired lectures", in which two researchers were invited to coordinate lectures related to specific topics where recent advances have suggested significant opportunities for future developments.

Ana Bălibanu and **Ioan Mărcuţ** delivered a pair of lectures on the problem of desingularizing the symplectic foliation of a Poisson manifold to make it regular (i.e. such that all symplectic leaves have the same dimension). The goal, given a Poisson manifold M, is to find a Poisson manifold $\widetilde{M}$ whose leaves are equidimensional, and a proper Poisson map $\tilde{M} \to M$ that is an isomoprhism over the locus where M is regular. Mărcuţ explained how, in the context of Poisson manifolds of compact type(s) [27, 28], a sequence of blowups along closed submanifolds can be used to desingularize the foliation; in the case where M is the dual of a Lie algebra of compact type, this yields a desingularization of the coadjoint orbits. Bălibanu explained the complex algebraic counterpart of this construction (the Grothendieck–Springer alteration for complex semi-simple Lie algebras, which is generically a covering rather than an isomorphism) and gave an overview of the basic theory of symplectic singularities and their versal deformations, following Namikawa [57].

Andrew Harder and **Mykola Matviichuk** spoke about holomorphic log symplectic manifolds: these are holomorphic Poisson manifolds that have an open dense symplectic leaf whose symplectic form has logarithmic poles on the boundary. Harder explained his work [42] concerning the properties of the mixed Hodge structure on the cohomology ring of the open leaf, and its role in the study of semi-stable degenerations of compact hyprekähler manifolds, where log symplectic structures naturally appear on the irreducible components of the singular fibre. Matviichuk discussed his recent work [56] with Pym and Schedler on local normal forms and deformations of log symplectic structures, giving a conjectural condition for them to be "holonomic" (meaning that the Poisson cohomology sheaves governing the deformation are locally finite-dimensional), and sketching a proof that this property holds for Hilbert schemes of log Calabi–Yau surfaces, based on a novel construction of the corresponding symplectic groupoid.

Francis Bischoff and **Charlotte Kirchhoff-Lukat** spoke about applications of Fukaya categories to problems in generalized complex and Poisson geometry. Bischoff explained his recent work [7] with Gualtieri, giving a general proposal for quantizing holomorphic Poisson manifolds using the generalized Kähler metrics and the Fukaya category of the symplectic groupoid and fully realizing it in the case of Poisson structures generated by torus actions. Kirchhoff-Lukat explained her work in progress on the Lagrangian Floer theory of two-dimensional real log symplectic manifolds (known as Radko surfaces), in which one has to modify the usual construction of Floer homology by allowing disks that intersect the boundary of the symplectic leaves in a controlled fashion.

Miquel Cueca and **Chris Rogers** spoke about various aspects of higher Lie theory. Cueca gave physical motivations and explained different approaches to describe higher Lie groupoids and their infinitesimal counterparts by means of simplicial methods and graded geometry, with concrete focus on the case of higher cotangent bundles [33]. Rogers explained joint work in progress with Jesse Wolfson concerning a homotopy-theoretic toolkit for constructing explicit integrations and differentiations in higher Lie theory, enjoying good geometric properties. Their work improves the earlier results of E. Getzler's [35] and A. Henriques' [44], proving that every finite-type Lie n-algebra integrates to a finite dimensional Lie n-group. More important, they propose an inverse to this construction, which was missing in those earlier works. The construction of the inverse builds upon the work of A. Beilinson on Chern-Weil theory, and the work of J. Pridham on the cosimplicial Dold-Kan correspondence.

Research talks

The workshop also included several sessions of research talks, covering a wide variety of topics related to Poisson geometry and stacks, and grouped loosely by theme.

On Monday afternoon, the focus was on representations and cohomology for Poisson structures and Lie algebroids. **Maria Amelia Salazar** discussed a definition of relative cohomology for a Lie subalgebroid, and its application to the construction of characteristic classes of representations. **Florian Zeiser** explained his calculation, joint with Hoekstra and Mărcuţ, of the Poisson cohomology of all 3-dimensional Lie algebras. **Linhui Shen** described his work with Casals, Gorsky, Gorsky, Le and Simental, in which they construct cluster structures on braid varieties of complex simple groups of ADE (confirming a conjecutre of Leclerc) and use them to construct and quantize Poisson structures on these varieties.

Tuesday afternoon concerned the application of Lie algebroids to the study of Poisson and generalized complex structures. **Aldo Witte** spoke about his joint work with Cavalcanti and Klaasse on so-called "elliptic symplectic structures" (related to the log symplectic structures from Harder and Matviichuk's morning session above). In paricular he explained a connected sum procedure that enables to construction of many examples of elliptic symplectic structures on non-complex manifolds. **Marco Gualtieri** and **Yucong Jiang** gave a pair of talks on the theory of generalized Kähler (GK) manifolds, giving a description of the latter in terms of holomorphic Manin triples, extending earlier work Bischoff–Gualtieri–Zabzine to cover arbitrary GK manifolds.

On Thursday afternoon, the focus was on (higher) categorical structures. **Daniel Alvarez** described joint work with Bursztyn and Cueca in which they apply Pantev–Toën–Vaquié–Vezzosi's theory of shifted symplectic structure to elucidate the problem of groupoid integrations of various Poisson-like structures, such as Poisson homogeneous spaces (building on earlier work to Bursztyn–Iglesias–Lu) and quasi-Poisson manifolds. **Frank Neumann** discussed his work with Szymik concerning the characteristic map on the Hochshild cohomology of differential graded categories, interpreting it as an edge map in a spectral sequence and providing concrete examples illustrating various (in)finite dimenisionality phenomena. **Cristian Ortiz** explained a version of Morse–Bott theory for groupoids, which yields a Morse-style complex that computes the cohomology of the associated quotient stack.

Finally, Friday morning feaatured talks about geometric structures on Lie algebroids and Lie groupoids. **Clarice Netto** decribed a notion of Courant–Nijenhuis algebroids and outlined several examples related to Kähler geometry and Poisson–Nijenhuis structures. **Joel Villatoro** discussed an extension of the theory of Lie groupoids and

Lie algebroids to the category of diffeological spaces, which enables integration of Lie algebroids to groupoids in diffeological spaces even when a groupoid in manifolds cannot be found. The conference closed with a talk by **Reyer Sjamaar**, who explained joint work with Lin, Loizides and Song that generalizes the index-theoretic "quantization commutes with reduction" theorem to the context of transversaly symplectic Riemannian foliations.

Scientific Progress Made

In order to stimulate progress, the schedule included afternoon discussion sessions led by the "paired speakers" from the morning sessions. These discussions produced some of the most directly visible scientific progress at the meeting, which we now summarize.

Desingularization: In the first discussion session, Bălibanu and Mărcuţ set forth a number of directions for future investigation, such as possible generalizations of the Grothendieck–Springer alteration based on symplectic groupoids, extending the blowup construction for Poisson submanifolds to the context of Dirac structure, and generalizations of the notion of symplectic singularity in which we ask that the resolution has a Poisson structure instead of (pre)symplectic structure. The discussion led to refinements of several of these questions and ideas to address them; for instance, Matviichuk immediately produced nontrivial examples of non-symplectic Poisson resolutions from elliptic curves.

Log symplectic manifolds: Harder and Matviichuk higlighted a number of open problems, including the problem of constructing/reversing toric/semi-stable degenerations of (log) symplectic varieties to obtain new examples, and ways to relax the smoothness hypothesis often imposed in log symplectic geometry, e.g. allowing singularities of the manifold itself (a logarithmic generalization of symplectic singularities) and allowing the boundary to by divisorial log terminal instead of normal crossings. By clarifying the key features of these problems, the discussion opened some exciting new avenues for interactions between Poisson geometry, birational geometry, and hyperhähler geometry.

Fukaya categories: Bischoff and Kirchhoff-Lukat led a discussion outline open problems around the appearance of Fukaya categories in Poisson geometry. A particularly active discussion centred around an idea of Kirchoff-Lukat to relate her logarithmic Fukaya category of surfaces to the ordinary Fukaya category of the symplectic groupoid; this led to some progress in understanding the exact mechanisms of such a correspondence, which is likely to shed significant light on the connection between Fukaya categories and quantization.

Higher Lie theory: In the final discussion session, Cueca discussed the problem of defining cotangent bundles of Lie 2-groupoids. These objects must be VB Lie 2-groupoids that carry a 2-shifted symplectic structure, so that the corresponding Lie 2-algebroid can be obtained from a particular 2-shifted lagrangian. These cotangent bundles are also related to coadjoint orbits of Lie 2-algebras, toric symplectic groupoids, and higher Hamiltonian actions. Other topics of discussion were the monoidal properties of the Dold-Kan functor and the internal Hom in the category of higher VB-groupoids. Meanwhile, Rogers gave more details on the integration procedure for Lie n-algebras explained in his talk, discussing the existence of the Lie group cover that allows the integration and explaining conceretely how it works for Lie 2-algebras, by integrating abelian pieces and cocycles. Roger also explained that the integration functor respects the structure of ICFO (Incomplete Category of Fibrant Objects) and the sense in which this is the adjoint to the differentiation defined by Pridham. There were particularly active discussions centred around the relationship with the Van Est map and possible extensions to Lie n-algebroids.

Outcome of the Meeting

For several participants, this workshop was the first in-person scientific meeting since the beginning of the COVID-19 pandemic, and consequently there were many opportunities for new discussions and collaborations that had been sorely lacking in recent years. We thank BIRS for this opportunity, and for the chance to host the meeting at an

increased capacity, which made it possible for a much larger number of junior researchers to attend and network in a way that had not been possible remotely. We are also grateful for the flexibility of the staff at BIRS, who gracefully accommodated last-minute changes to the in-person attendance list and provided support for talks and discussion sessions that involved remote participants.

Participants

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Bibliography

[1] A. Alekseev and E. Meinrenken, Linearization of Poisson Lie group structures. J. Symplectic Geom. 14 (2016), 227–267.

[2] I. Androulidakis and G. Skandalis, The holonomy groupoid of a singular foliation, J. Reine Angew. Math., 626 (2009), 1–37.

[3] C. Arias Abad and M. Crainic, The Weil algebra and the Van Est isomorphism. Ann. Inst. Fourier (Grenoble) 61 (2011), 927–970.

[4] J. Baez and U. Schreiber, Higher gauge theory. In: Categories in Algebra, Geometry and Mathematical Physics, eds. A. Davydov et al, Contemp. Math. **431**, AMS, Providence, Rhode Island, 2007, pp. 7–30

[5] M. Bailey and M. Gualtieri, Integration of generalized complex structures. ArXiv:1611.03850.

[6] K. Behrend and P. Xu, Differentiable stacks and gerbes, J. Symplectic Geom. 9 (2011), 285–341.

[7] F. Bischoff and M. Gualtieri, Brane quantization of toric Poisson varieties, Comm. Math. Phys. 391 (2022), n. 2, 357–400.

[8] F. Bonechi, N. Ciccoli, J. Qiu and M. Tarlini, Quantization of Poisson manifolds from the integrability of the modular function. Comm. Math. Phys. 331 (2014), 851–885.

[9] R.L. Bryant, Bochner-Kaehler Metrics, J. of Amer. Math. Soc. 14 (2001), 623–175.

[10] R.L. Bryant, Notes on Exterior Differential Systems, preprint arXiv:1405.3116.

[11] H. Bursztyn, A brief introduction to Dirac manifolds, Geometric and topological methods for quantum field theory, 4–38, Cambridge Univ. Press, Cambridge, 2013.

[12] O. Ben-Bassat, C. Brav, V. Bussi and D. Joyce, A 'Darboux theorem' for shifted symplectic structures on derived Artin stacks, with applications, Geom. Topol. 19 (2015), no. 3, 1287–1359.

[13] C. Brav, V. Bussi and D. Joyce, A Darboux theorem for derived schemes with shifted symplectic structure, J. Amer. Math. Soc. 32 (2019), no. 2, 399–443.

[14] H. Bursztyn and A. Cabrera, Multiplicative forms at the infinitesimal level, Math. Annalen 353 (2012), 663–705.

[15] H. Bursztyn and M. Crainic, Dirac geometry, quasi-Poisson actions and D/G-valued moment maps, J. Diff. Geom. 82 (2009), 501–566.

[16] H. Bursztyn, M. Crainic, A. Weinstein and C. Zhu, Integration of twisted Dirac brackets, Duke Math. J. 123 (2004), 549–607.

[17] H. Bursztyn and T. Drummond, Lie theory of multiplicative tensors. Math. Annalen 375 (2019), 1489-1554.

[18] H. Bursztyn, D. Iglesias-Ponte and J.-H. Lu, Dirac geometry and integration of Poisson homogeneous spaces, arXiv:1905.11453. To appear in J. Differential Geom.

[19] D. Calaque, T. Pantev, B. Toën, M. Vaquiè and G. Vezzosi, Shifted Poisson structures and deformation quantization, J. Topology, 10 (2017), 483-584.

[20] A.S. Cattaneo and G. Felder,Poisson sigma models and symplectic groupoids Quantization of singular symplectic quotients, Progr. Math., vol. 198, Birkhaeuser, Basel, 2001,pp. 61–93.

[21] A. Connes, Noncommutative Geometry, Academic Press, San Diego, 1994.

[22] M. Crainic, Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes, Comment. Math. Helv. 78 (2003), no. 4, 681–721.

[23] M. Crainic, Generalized complex structures and Lie brackets, Bulletin of the Brazilian Mathematical Society, 42 (2011), 559–578.

[24] M. Crainic and R.L. Fernandes, A geometric approach to Conn's linearization theorem. Annals of Math 173 (2011), 1119–1137.

[25] M. Crainic and R.L. Fernandes, Integrability of Lie brackets, Annals of Math. 157 (2003), 575–620.

[26] M. Crainic and R.L. Fernandes, Integrability of Poisson brackets, J. Diff. Geom. 66 (2004), no. 1, 71–137.

[27] M. Crainic, R. L. Fernandes, D. Martinez-Torres, Poisson manifolds of compacts types (PMCT I), J. Reine Angew. Math. 756 (2019), 101–149.

[28] M. Crainic, R. L. Fernandes, D. Martinez-Torres, Regular Poisson manifolds of compacts types (PMCT II), Astérisque, 413 (2019).

[29] M. Crainic and I. Marcut, A normal form theorem around symplectic leaves. J. Diff. Geom. 92 (2012), 417–461.

[30] M. Crainic, A. Salazar and I. Struchiner, Multiplicative forms and Spencer operators. Mathematische Zeitschrift 279 (2015), 939–979.

[31] M. Crainic and O. Yudilevich, Lie Pseudogroups á la Cartan, preprint arXiv:1801.00370.

[32] T. Courant, Dirac structures, Trans. Amer. Math. Soc. 319 (1990), 631–661.

[33] M. Cueca, The geometry of graded cotangent bundles, J. Geom. Phys. 161 (2021), 104055.

[34] M. Cueca and C. Zhu, Shifted symplectic higher Lie groupoids and classifying spaces. Arxiv: 2112.01417.

[35] E. Getzler, Lie theory for nilpotent L_∞-algebras, Ann. of Math. (2) 170, 1 (2009) 271–301.

[36] M. del Hoyo and R.L. Fernandes, Riemannian Metrics on Lie Groupoids. J. Reine Angew. Math., 735 (2018), 143?173.

[37] M. del Hoyo and R.L. Fernandes, Riemannian metrics on differentiable stacks, Mathematische Zeitschrift 292 (2019), Issue 1?2, pp 103?132.

[38] C. Debord and J.-M. Lescure : Index theory and groupoids. In: Geometric and topological methods for quantum field theory, 86-158, Cambridge Univ. Press, Cambridge, 2010

[39] M. Gualtieri, Generalized complex geometry, Annals of Math. 174 (2011), 75–123.

[40] M. Gualtieri, S. Li and B. Pym, The Stokes groupoids. J. Reine Angew. Math. 739 (2018), 81-119

[41] A. Haefliger, Homotopy and integrability, Lecture Notes in Mathematics 197 (1971), 133–163.

[42] A. Harder, Mixed Hodge structures in log symplectic geometry, arXive 2005.11367.

[43] E. Hawkins, A groupoid approach to quantization. J. Symplectic Geom. 6 (2008), 61–125.

[44] A. Henriques, Integrating L_∞-algebras, Compos. Math. 144 (2008), no. 4, 1017–1045.

[45] N. Hitchin, Generalized Calabi-Yau manifolds, Quart. J.Math. 54 (2003), 281-308.

[46] D. Iglesias, C. Laurent-Gengoux and P. Xu, Universal lifting theorem and quasi-Poisson groupoids, J. Eur. Math. Soc. 14 (2012), 681–731.

[47] C. Laurent-Gengoux, M. Stienon, P. Xu, Integration of holomorphic Lie algebroids. Math. Annalen. 345 (2009), 895–923.

[48] J.-M. Lescure, D. Manchon, S. Vassout, About the convolution of distributions on groupoids, J. Noncommut. Geom. 11 (2017), 757–789.

[49] D. Li-Bland, E. Meinrenken, Dirac Lie groups, Asian Journal of Mathematics 18 (2014), 779–816

[50] D. Li-Bland, E. Meinrenken, On the Van Est homomorphism for Lie groupoids. Enseign. Math. 61 (2015), 93–137.

[51] J. Lott, Dimensional reduction and the long-time behavior of Ricci flow, Comment. Math. Helv. 85 (2010), 485–534.

[52] J.-H. Lu and A. Weinstein, Groupoïdes symplectiques doubles des groupes de Lie-Poisson, C. R. Acad. Sci. Paris Ser. I Math. 309 (1989), no. 18, 951–954.

[53] K. Mackenzie, Lie groupoids and Lie algebroids in differential geometry Cambridge Univ. Press, 2005.

[54] K. Mackenzie and P. Xu, Lie bialgebroids and Poisson groupoids. Duke Math. J. 73 (1994), 415-451.

[55] E. Martinez, Lagrangian Mechanics on Lie Algebroids, Acta Appl. Math. 67 (2001), 295–320.

[56] M. Matviichuk, B. Pym and T. Schedler, A local Torelli theorem for log symplectic manifolds, arXive 2010.08692.

[57] Y. Namikawa, Poisson deformations of affine symplectic varieties, Duke Math. J., 156 (2011), n.1, 51–85.

[58] I. Moerdijk. Orbifolds as groupoids: an introduction. In: Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math. 310 (2002), 205–222.

[59] V. Mouquin, Local Poisson groupoids over mixed product Poisson structures and generalised double Bruhat cells, arXiv:1908.04044.

[60] V. Nistor, A. Weinstein, and P. Xu, Pseudodifferential operators on differential groupoids, Pacific Journal of Mathematics 189 (1999), 117–152.

[61] T. Pantev, B. Toën, M. Vaquié and G. Vezzosi, Shifted symplectic structures, Publ. Math. Inst. Hautes E'tudes Sci. 117 (2013) 271–328

[62] J. Pascaleff, Poisson geometry, monoidal Fukaya categories, and commutative Floer cohomology rings. Preprint arXiv:1803.07676.

[63] M. Pflaum, H. Posthuma, X. Tang, The localized longitudinal index theorem for Lie groupoids and the van Est map, Adv. Math. 270 (2015), 223–262.

[64] A. Weinstein, Symplectic groupoids and Poisson manifolds. Bull. Amer. Math. Soc. 16 (1987), no. 1, 101–104.

[65] A. Weinstein, Lagrangian mechanics and groupoids. Mechanics day (Waterloo, ON, 1992), 207-231, Fields Inst. Commun., 7, Amer. Math. Soc., Providence, RI, 1996.

[66] C. Zhu, n-groupoids and stacky Lie groupoids. Int. Math. Res. Not. IMRN 2009, no. 21, 4087–4141.

Chapter 11

Arithmetic Aspects of Algebraic Groups (22w5161)

June 12 - 17, 2022

Organizer(s): Alex Lubotzky (Weizmann Institute of Science), Dave Morris (University of Lethbridge), Mikhail Ershov (University of Virginia), Gopal Prasad (University of Michigan),

Overview of the Field

Classical results and long-standing problems concerning algebraic groups over global fields are the foundation of the arithmetic theory of algebraic groups, but this workshop also considered more recent developments that extend local-global principles, finiteness results, and other aspects of the classical theory to algebraic groups over more general fields of arithmetic nature. This includes function fields of curves over local fields or global fields, and, in some cases, even arbitrary finitely generated fields. All aspects of the subject have close connections with number theory and arithmetic geometry, and many results have important applications in other areas, from group theory to geometry and combinatorics.

Algebraic groups over global fields

The origins of the arithmetic theory of algebraic groups can be traced back to the analysis of finite-index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ in connection with the classical theory of modular forms [Fricke, Klein, Poincaré] and the work of Gauss, Hermite, and later Minkowski and Hasse, on the theory of integral and rational quadratic forms. These results subsequently developed into the theory of arithmetic and S-arithmetic groups and the general local-global approach that has been successfully used in many situations. The theory was complemented by approximation results and results on the structure of the groups of rational points of algebraic groups over global fields. It has been an area of active research for over 50 years, with foundations laid by Borel, Harish-Chandra, and Serre in the 1960s, and with many new applications discovered in the last decade. Nevertheless, some critical question in the theory remain open, and one of the objectives of the workshop was to discuss the recent progress, and look for new approaches. Some specific topics that were discussed in the workshop include:

Congruence Subgroup Problem, its generalizations and applications. While it was known already to Fricke and Klein that $\mathrm{SL}_2(\mathbb{Z})$ has numerous finite-index subgroups that do not contain any congruence subgroups, it was not proved until the 1960s that if $n \geqslant 3$, then every infinite, normal subgroup of $\mathrm{SL}_n(\mathbb{Z})$ has finite index and contains a suitable congruence subgroup. This solution of the Congruence Subgroup Problem (CSP) for $\mathrm{SL}_n(\mathbb{Z})$ was followed by a period of very active research that generalized the result to S-arithmetic subgroups of many other simple algebraic groups over global fields. However, some important cases still remain open; in particular, the CSP has not been solved for arithmetic subgroups of the group $G = \mathrm{SL}_{1,D}$ of norm 1 elements in a central division K-algebra D — even when D is a quaternion algebra). In 2017, G. Prasad and A. Rapinchuk proposed a new approach that provides a simpler and shorter proof of many of the known cases, and one can expect this approach to lead to further progress on this long-standing problem. A few years earlier, in 2013, Y. Shalom and G. Willis proposed a generalization of the CSP, by formulating a precise conjecture on the homomorphisms of S-arithmetic groups into general locally compact groups. Very recently, the techniques developed to investigate the CSP were used by N. Avni, A. Lubotzky, and C. Meiri to establish a new form of rigidity, which can be called first-order rigidity: if G is a connected, isotropic, almost-simple algebraic group over $\mathbb{Q}$, such that $\mathrm{rank}_{\mathbb{R}} G \geq 2$, and Λ is any finitely generated group that satisfies precisely the same first-order axioms as some finite-index subgroup Γ of $G(\mathbb{Z})$, then Λ is isomorphic to Γ.

Bounded generation. An abstract group is said to be "boundedly generated" if it is a product of finitely many cyclic subgroups. This seemingly simple property has surprisingly strong and diverse consequences for abstract groups, and the implications are particularly striking for S-arithmetic groups. For example, an abstract boundedly generated group that satisfies one additional natural condition is known to have only finitely many inequivalent irreducible complex representations in each dimension. (This is a form of homomorphism rigidity.) For an S-arithmetic subgroup of an absolutely almost simple simply connected algebraic group, bounded generation implies the congruence subgroup property. And bounded generation has been used to estimate Kazhdan constants and to analyze the actions of arithmetic groups; it also played a significant role in the works of Shalom-Willis mentioned above. These applications illustrate that proving bounded generation of S-arithmetic groups is an important problem, with far-reaching consequences. In 1983, D. Carter and G. Keller proved that $\mathrm{SL}_n(\mathcal{O})$ has bounded generation for any $n \geqslant 3$ and any ring of algebraic integers $\mathcal{O}$. Since then, the list of S-arithmetic groups known to have bounded generation has been steadily increasing — with important results obtained in the last few years.

A very important development in this area occurred in the year prior to the workshop. In a remarkable paper by Corvaja, A. Rapinchuk, Ren and Zannier, it was shown that if a linear group over a field of characteristic zero is boundedly generated by semi-simple elements, it must be virtually solvable. This implies that infinite S-arithmetic subgroups of absolutely almost simple anisotropic algebraic groups over number fields are never boundedly generated. A whole series of other "boundedness" conditions are also being investigated (and applied). For example, Chevalley groups over the rings of integers in function fields are not boundedly generated, but in the past year it was shown that such groups in the higher rank case possess *bounded elementary generation*, that is, can be written as products of finitely many root subgroups. Another related condition is "bounded generation by the conjugacy class." (This means that the normal subgroup generated by an arbitrary element is a product of finitely many copies of the conjugacy class of that element and its inverse.) While this property is known, for example, for $\mathrm{SL}_n(\mathcal{O})$ (where $n \geqslant 3$ and $\mathcal{O}$ is a ring of algebraic integers), its general consequences have not been explored yet. For example, it is not known whether this property for an S-arithmetic subgroup of an absolutely almost simple simply connected algebraic group implies the congruence subgroup property.

New local-global principles. Local-global principles are at the heart of the arithmetic theory of algebraic groups: they provide a uniform perspective on the analysis of many phenomena, and serve as an important technical tool in the investigation of others. They are often expressed in terms of the injectivity of certain global-to-local maps for Galois cohomology, but others are of a different nature. In the last decade, a number of new local-global principles have been established. In particular, while the norm principle for finite extensions of global fields has been investigated for a long time, an understanding of the local-global principle for the products of norms from several extensions was achieved only (relatively) recently. This is known as the multinorm principle, and, surprisingly, it often holds when the usual norm principle fails for each of the individual extensions. This principle was used in the investigation of the Margulis-Platonov conjecture for anisotropic inner forms of type A_n, and also in the analysis of another local-global principle that governs the existence of an embedding of an etale algebra with an involutive automorphism into a simple algebra with involution. This embedding principle is important for understanding the maximal tori of simple groups of classical types over global fields. The genus of a given simple algebraic group G is the collection of isomorphism classes of simple groups with the same maximal tori as G, and it is reasonable to hope that these results will make it possible to determine the cardinality of the genus of any simple group of classical type. (This would have applications to the analysis of isospectral and length-commensurable locally symmetric spaces in differential geometry.) In addition, the multinorm principle can be stated as the vanishing of the Tate-Shafarevich group of an associated "multinorm torus," and this torus can be realized as a maximal torus of an isotropic inner form of type A_n, so a natural generalization would provide useful conditions that ensure the triviality of the Tate-Shafarevich group of maximal tori in other simple isotropic algebraic groups of classical types.

Algebraic groups over more general fields

Results obtained in the last decade have lead to the realization that many classical results over global fields should be considered in the broader context of finitely generated fields. For example, the genus is expected to be finite over arbitrary finitely generated fields, and this has already been established for inner forms of type A_n (and also for the related notion of genus of a finite-dimensional division algebra). Very recently, the finiteness of the genus was established for the spinor groups of quadratic forms and some other groups of classical types over 2-dimensional arithmetic fields. The general case, which remains wide open, is the subject of active investigation.

It was discovered that the study of the genus of a simple algebraic group G over a finitely generated field K is closely related to the problem of analyzing the K-forms of G that have good reduction at a divisorial set of places V of K. This is a totally new area of research at the meeting ground of the theory of algebraic groups and arithmetic geometry. Previously, simple algebraic groups over $\mathbb{Q}$ with good reduction at all primes were considered by B.H. Gross, but more general situations have never been analyzed. In addition to the finiteness of the genus, another important consequence of the finiteness of the number of forms with good reduction is the properness of the map $H^1(K, \overline{G}) \to \prod_{v \in V} H^1(K_v, \overline{G})$ for the Galois cohomology of the adjoint group $\overline{G}$. (Over global fields, the properness is known for all groups.)

Yet another important finiteness property in the classical theory is the finiteness of the class number of a global field; its generalization to algebraic groups states that for any algebraic group over a global field, the number of double cosets of the adele group modulo its subgroups of integral and principal adeles is finite (and the analog for regular schemes of finite type over $\mathbb{Z}$ or its localizations). One can introduce adele groups for algebraic groups over any field with respect to any set of discrete valuations subject to some mild assumptions, and then formulate a condition that naturally generalizes both finiteness results. The question is when does this condition hold? At this point, the affirmative answer is known only for reductive split groups over the function fields of curves over finitely generated fields with respect to the set of geometric places. The argument uses a variant of strong approximation for split groups. While strong approximation has been fully investigated for the groups over global fields, its analysis for anisotropic groups over the function fields of curves over general fields is only beginning.

Theorems on homomorphism rigidity and the congruence subgroup problem have been proved for split simple groups over rings other than the rings of S-integers of global fields. (For example, there is now a fairly explicit description of all finite-dimensional complex representations of the group $SL_n(\mathbb{Z}[X])$.) These results need to be extended to more general isotropic groups; in particular, this work is expected to give a proof of the Borel-Tits conjecture on abstract homomorphisms of the groups of rational points of simple isotropic groups. And there is the following very general conjecture of Prasad-Rapinchuk: if G is an absolutely almost simple algebraic group over an arbitrary field K, then any finite quotient of $G(K)$ is solvable.

Presentation Highlights

Abid Ali (Rutgers University)
Integrality of unipotent subgroups of Kac-Moody groups

Let G be a Kac-Moody group over $\mathbb{Q}$. There are several approaches to defining its "$\mathbb{Z}$-form" $G(\mathbb{Z})$, including the functorial definition proposed by Tits, and in most cases it is unknown whether different definitions yield isomorphic groups. In this talk the speaker discussed some recent progress on finding a relationship between different $\mathbb{Z}$-forms of G. The talk was based on a joint work with Lisa Carbone (Rutgers), Dongwen Liu (Zhejiang) and Scott S Murray (Toronto) which generalizes Chevalley's fundamental theorem on the integrality for finite dimensional semisimple Lie groups.

Nir Avni (Northwestern University)
Distributions of words in unitary groups

Let G be a compact group, and let w be a nontrivial word in the free group F_r, so w defines a function from G^r to G. Let μ_w be the image of Haar measure under this map. (Thus, μ_w is the distribution of a random value of w when the variables are i.i.d.) The talk presented recent work (and provided historical background) on the Fourier coefficients of this measure in the case where G is a unitary group U_d.

For each character χ of U_d, the corresponding Fourier coefficient of the measure μ_w is the expectation $\mathbb{E}\,\chi\big(w(x_1,\ldots,x_r)\big)$. The maximum value of $\chi\big(w(x_1,\ldots,x_r)\big)$ is $\chi(1)$, and it is conjectured that the expected value is substantially smaller than this: $\forall w \neq 1, \exists \varepsilon > 0, \forall d, \forall \chi, \mathbb{E}\,\chi\big(w(x_1,\ldots,x_r)\big) < \chi(1)^{1-\varepsilon}$.

This talk was based on a joint work with Itay Glazer, which proved that the conjecture is true for the fundamental representations of U_d. In more elementary terms, this means that the expected value of the kth coefficient of the characteristic polynomial of $w(x_1,\ldots,x_r)$ is bounded by $\binom{d}{k}^{1-\varepsilon}$.

Rony Bitan (Afeka Academic College)
$\tau(G) = \tau(G_1)$: An equality of Tamagawa numbers

Given a smooth, geometrically connected and projective curve C defined over a finite field k, let $K = k(C)$ be the function field of rational functions on C. The Tamagawa number $\tau(G)$ of a semisimple K-group G is defined as the covolume of the discrete group $G(K)$ (embedded diagonally) in the adelic group $G(A)$ with respect to the Tamagawa measure. The Weil conjecture, recently proved by Gaitsgory and Lurie, states that if G is simply connected then $\tau(G) = 1$.

This talk was based on a work in progress (joint with Gunter Harder, Ralf Kohl and Claudia Schoemann) whose aim is to prove, without relying on the Weil conjecture, the following fact: Let G be a quasi-split inner form of a split semisimple and simply-connected K-group G_1. Then $\tau(G) = \tau(G_1)$. This theorem can serve as a part of an alternative proof to the Weil conjecture.

Mikhail Borovoi (Tel Aviv University)
Galois cohomology of a real reductive group

The goal of this talk was to describe the first Galois cohomology set $H^1(R, G)$ of a connected reductive group G over the field of real numbers $\mathbb{R}$ by the method of Borel and Serre and by the method of Kac. The talk was based on the recent preprints [6] and [7].

Vladimir Chernousov (University of Alberta)
New evidence that cohomological invariants might determine Albert algebras/groups of type F_4 uniquely up to isomorphism

In this talk, based on a joint work with A. Lourdeaux and A. Pianzola, the speaker provided a sketch of a proof that Albert algebras arising from the first Tits construction are determined uniquely up to an isomorphism by the Rost cohomological invariant g_3.

Uriya First (University of Haifa)
Sheaves on simplicial complexes and 2-query locally testable codes

This talk was based on joint work with Tali Kaufman [12], which shows that if arithmetic groups with certain properties exist, then one can construct good 2-query locally testable codes. The device that enables this is a novel notion of sheaves on simplicial complexes. The latter are typically taken to be quotients of an affine building by an arithmetic group, e.g., Ramanujan complexes.

Julia Hartmann (University of Pennsylvania)
Bounding cohomology classes over semi global fields

This talk was based on a joint work with David Harbater and Daniel Krashen [14]. It provides a uniform bound for the index of cohomology classes in $H^i(F, \mu_\ell^{\otimes i-1})$ when F is a semiglobal field (i.e., a one variable function field over a complete discretely valued field K). The bound is given in terms of the analogous data for the residue field of K and its finitely generated extensions of transcendence degree at most one. An explicit bound is obtained in example cases when the information on the residue field is known.

Chen Meiri (Technion)
Conjugacy width in higher rank orthogonal groups

It is known that conjugacy classes of elements in orthogonal groups over number fields have finite width. (This means that every element of the group is a product of a bounded number of elements of the conjugacy class.) This talk was based on joint work with Nir Avni which presents evidence that the same is true for conjugacy classes of elements in higher rank arithmetic groups of orthogonal type. It is shown that the question whether a conjugacy class of such a group has a finite width can be viewed as a congruence subgroup problem on a non-standard saturated model of the group.

Alexander Merkurjev (UCLA)
Classification of special reductive groups

An algebraic group G over a field F is called special if for every field extension K/F all G-torsors (principle homogeneous G-spaces) over K are trivial. Examples of special groups are special linear groups, general linear groups, and symplectic groups. A. Grothendieck classified special groups over an algebraically closed field. In 2016, M. Huruguen classified special reductive groups over arbitrary fields. This talk was based on the recent preprint [18] where it is shown how to improve the classification given by Huruguen.

Raman Parimala (Emory University)
Pencils of quadrics and hyperellliptic curves

This talk discussed a weak Hasse principle for a smooth intersection of two quadrics in $\mathbb{P}^5$ and connections to period index questions for the associated hyperelliptic curves.

Eugene Plotkin (Bar Ilan University)
Bounded generation and commutator width of Chevalley groups and Kac-Moody groups: function case

This talk was based on a recent joint work with B. Kunyavskii and N. Vavilov [17] which provides new results on bounded elementary generation and bounded commutator width for Chevalley groups over Dedekind rings of arithmetic type in positive characteristic. In particular, Chevalley groups of rank greater than 1 over polynomial rings and Chevalley groups of arbitrary rank over Laurent polynomial rings (in both cases the coefficients are taken from a finite field) are boundedly elementarily generated. The speaker presented rather plausible explicit bounds and discussed applications to Kac-Moody groups and various model theoretic consequences and certain conjectures which look quite tempting.

Igor Rapinchuk (Michigan State University)
Algebraic groups with good reduction and applications

Techniques involving reduction are very common in number theory and arithmetic geometry. In particular, elliptic curves and general abelian varieties having good reduction have been the subject of very intensive investigations over the years. The purpose of this talk, based on joint papers with V. Chernousov and A. Rapinchuk, was to report on recent work that focuses on good reduction in the context of reductive linear algebraic groups over higher-dimensional fields.

Zinovy Reichstein (University of British Columbia)
Hilbert's 13th Problem for algebraic groups

The algebraic form of Hilbert's 13th Problem asks for the resolvent degree $rd(n)$ of the general polynomial $f(x) = x^n + a_1x^{n-1} + \ldots + a_n$ of degree n, where $a_1, \ldots, a_n$ are independent variables. Here $rd(n)$ is the minimal integer d such that every root of $f(x)$ can be obtained in a finite number of steps, starting with $\mathbb{C}(a_1, \ldots, a_n)$ and adjoining an algebraic function in $\leq d$ variables at each step. It is known that $rd(n) = 1$ for every $n \leq 5$. It is not known whether or not $rd(n)$ is bounded as n tends to infinity; it is not even known whether or not $rd(n) > 1$ for any n. Recently Farb and Wolfson defined the resolvent degree $rd_k(G)$, where G is a finite group and k is a field of characteristic 0. In this setting $rd(n) = rd_C(S_n)$, where S_n is the symmetric group on n letters and $\mathbb{C}$ is the field of complex numbers. This talk was based on a recent preprint [21] which defines $rd_k(G)$ for any field k and any algebraic group G over k. Surprisingly, Hilbert's 13th Problem simplifies when G is connected. In particular, the speaker explained why $rd_k(G) \leq 5$ for an arbitrary connected algebraic group G defined over an arbitrary field k.

Jinbo Ren (Institute for Advanced Study)
Applications of Diophantine Approximation in Group Theory

An abstract group Γ has the property of "bounded generation" if it is equal to a product of finitely many fixed cyclic groups. Being a purely combinatorial property of groups, bounded generation has a number of interesting consequences and applications in different areas including Kazhdan's constants computation, semi-simple rigidity, the Margulis-Zimmer conjecture and the Serre's congruence subgroup problem.

This talk was based on a recent joint work Corvaja, A. Rapinchuk and Zannier [11] where the following result is proved: if a linear group $\Gamma \subset \mathrm{GL}_n(K)$ over a field K of characteristic zero is boundedly generated by semi-simple (diagonalizable) elements then it is virtually solvable. As a consequence, one obtains that infinite S-arithmetic subgroups of absolutely almost simple anisotropic algebraic groups over number fields are *never* boundedly generated. The proof relies on the subspace theorem (a far-reaching generalization of Roth's Fields medal work) from Diophantine approximation and properties of generic elements.

David J Saltman (Center for Communications Research - Princeton)
Cyclic Matters

This work was motivated by the problem of describing cyclic Galois extensions and differential crossed product algebras in mixed characteristic, with the the goal of lifting from arbitrary characteristic p rings to suitable characteristic 0 rings. The first step was the construction of Artin-Schreier like polynomials and extensions in mixed characteristic, where the group acts by $\sigma(x) = \rho x + 1$ ($\rho^p = 1$). This leads to Azumaya algebras A defined by $xy - \rho\, yx = 1$. Of course, a generalization to degrees higher than p is desirable. This leads to an Albert like criterion for extending cyclic Galois extensions of rings and to the definition and study of almost-cyclic Azumaya algebras, generalizing A above.

George Tomanov (Université Claude Bernard Lyon 1)
Actions of maximal tori on homogeneous spaces and applications to number theory

During the last decades long-standing conjectures in number theory have been reformulated and, subsequently, some of them successfully solved using the homogeneous dynamics approach. The approach is based on the description of the closures of orbits for the natural action of subgroups H of an algebraic group G on the homogeneous space G/Γ where Γ is an arithmetic subgroup of G. The closures of such orbits are well-understood when H is unipotent and considerably less when H is a torus. This talk described some recent results on the action of maximal (split or non-split) tori on G/Γ and related applications.

Charlotte Ure (University of Virginia)
Symbol Length in Brauer Groups of Elliptic Curves

Elements in the Brauer group of an elliptic curve E may be described as tensor products of symbol algebras over the function field of E by the Merkurjev-Suslin Theorem. The symbol length is the smallest number n so that every element in the Brauer group can be expressed as a tensor product of at most n symbols. This talk was based on a recent joint work with Mateo Attanasio, Caroline Choi, and Andrei Mandelshtam [1] which describes bounds on the symbol length of E. In particular, it is shown that the symbol length in the prime torsion for a prime q of a CM elliptic curve over a number field is bounded above by $q + 1$.

Kirill Zaynullin (University of Ottawa)
The canonical dimension of a semisimple group and the unimodular degree of a root system

This talk was based on a recent preprint [27], which provides a short and elementary algorithm to compute an upper bound for the canonical dimension of a split semisimple linear algebraic group. The key tools are the classical Demazure formula for the characteristic map and the elementary properties of divided-difference operators. Using this algorithm, one can confirm all previously known bounds by Karpenko and Devyatov as well as produce new bounds (e.g., for adjoint simple groups of type F_4 or E_6, and for some semisimple groups).

Open Problems

The workshop featured a problem session on Tuesday evening, during which 7 open problems were discussed. We are very grateful to Asher Auel for taking detailed notes from the problem session.

Problem 1 (Andrei Rapinchuk). *Groups with bounded generation.*

An abstract group Γ has bounded generation (BG) if there exist $\gamma_1, \dots, \gamma_d \in \Gamma$ with $\Gamma = \langle\gamma_1\rangle \cdots \langle\gamma_d\rangle$, which means $\Gamma = \{\, \gamma_1^{n_1}\gamma_2^{n_2} \cdots \gamma_d^{n_d} \mid n_1, n_2, \dots, n_d \in \mathcal{Z} \,\}$.

What are some examples? Finitely generated nilpotent groups. What else? Carter and Keller showed that $\Gamma = \mathrm{SL}_n(\mathcal{Z})$ for $n \geq 3$ has BG, see [8]. This fact can be rephrased in the terminology of elementary linear algebra. It is a basic fact that, over a field, every invertible matrix can be reduced to the identity matrix by elementary row operations. The same is true for matrices with integer entries. (Furthermore, for a matrix with determinant 1, the only necessary row operation is adding a multiple of one row to another row, so we see that the original matrix is a product the elementary matrices, which are unipotent.) What Carter and Keller proved is that every matrix in $\mathrm{SL}_n(\mathcal{Z})$ (for fixed $n \geq 3$) can be reduced to the identity in a bounded number of steps.

For $\mathrm{SL}(n, \mathcal{Z})$, the $\gamma_1, \dots, \gamma_d$ are elementary matrices, so are unipotent. For a long time, it was an open question whether such $\gamma_1, \dots, \gamma_d \in \mathrm{SL}_n(\mathcal{Z})$ can be chosen to be semi-simple elements, but it was recently proved that this is impossible, see [11]. More generally, the expectation is that if a group has no unipotent elements, then it usually should not have BG. As an example of this, it was recently shown that if Γ is boundedly generated by semisimple elements, then Γ is virtually solvable, i.e., has a solvable subgroup of finite index. Therefore, if $\Gamma \subset \mathrm{GL}_n(\mathbb{C})$ is an anisotropic group, i.e., if every element is semisimple, then Γ has BG if and only if Γ is finitely generated and virtually abelian, i.e., has an abelian subgroup of finite index.

A profinite group Δ has bounded generation (BG) if there exist elements $\gamma_1, \dots, \gamma_d \in \Delta$ such that $\Delta = \overline{\langle\gamma_1\rangle} \cdots \overline{\langle\gamma_d\rangle}$ where the overline means the topological closure.

There exist many S-arithmetic groups $\Gamma = G(\mathcal{Z})$ with the congruence subgroup property (CSP), which (roughly speaking) means that $\widehat{\Gamma} = \prod_p G(\mathcal{Z}_p)$, where the hat $\widehat{\ }$ means the profinite completion, and the product is over all primes. See the survey [19], and the references within, for more details on the CSP. It is known that this implies that $\widehat{\Gamma}$ has BG as a profinite group. (On the other hand, if the original group Γ is anisotropic, then we know from above that Γ does not have BG.)

Question. Given an abstract group Γ whose profinite completion $\widehat{\Gamma}$ has BG, can one find $\gamma_1, \dots, \gamma_d \in \Gamma$ such that $\widehat{\Gamma} = \overline{\langle\gamma_1\rangle} \cdots \overline{\langle\gamma_d\rangle}$.

We know that there exist such $\gamma_1, \dots, \gamma_d$ in $\widehat{\Gamma}$ (because we assume $\widehat{\Gamma}$ has BG as a profinite group), but the question is whether these elements can be chosen to be in the original group Γ, instead of in the profinite completion.

The easiest case might be to take an integral quadratic form q. If q has Witt index ≥ 2 over $\mathbb{R}$, then $\mathrm{Spin}(q)(\mathbb{Z})$ is known to have CSP (this was proved by M. Kneser); otherwise, one can consider the group of points $\mathrm{Spin}(q)\big(\mathbb{Z}[1/s]\big)$ over a suitable localization. This would be a good test case.

Problem 2 (Peter Abramenko). *Generation by elementary matrices.*

Following P.M. Cohn [10], we call a (not necessarily commutative) ring R with 1 a GE_n ring (n a natural number > 1) if $\mathrm{GL}_n(R)$ is generated by elementary and invertible diagonal matrices, i.e., if $\mathrm{GL}_n(R) = \mathrm{GE}_n(R)$.

For commutative R this is equivalent to $\mathrm{SL}_n(R) = \mathrm{E}_n(R)$. We will restrict to (commutative) integral domains in the following. It is clear that fields and Euclidean domains are GE_n rings for all n. GE_n properties of S-arithmetic rings are also well known (but also not relevant to this problem). A. Suslin [22] studied the question of when GE_n properties of a base ring A carry over to (Laurent) polynomial rings over A. In particular, he obtained the following:

Theorem 1. If A is a field or Euclidean domain, and ℓ, m and n are natural numbers with $\ell \leq m$, then $R = A[t_1, \ldots, t_m; t_1^{-1}, \ldots, t_\ell^{-1}]$ is a GE_n ring for all $n > 2$.

This leaves the question when these rings are also GE_2. A general answer was given by H. Chu [9]. Among his results for integral domains S are the following:

Theorem 2. If $R = S[t]$ is a GE_2 ring, then S is a field.

Corollary. If A is a field, $m > 1$, and $\ell < m$ or A is any integral domain that is not a field, m is any natural number and $\ell < m$, then $R = A[t_1, \ldots, t_m; t_1^{-1}, \ldots, t_\ell^{-1}]$ is not a GE_2 ring.

Theorem 3. If $R = S[t, t^{-1}]$ is a GE_2 ring, then S is a Bezout domain.

Corollary. If A is a field and $\ell = m > 2$ or A is any integral domain which is not a field and $\ell = m > 1$, then $R = A[t_1, \ldots, t_m; t_1^{-1}, \ldots, t_m^{-1}$ is not a GE_2 ring.

It is worth noting that for Laurent polynomial rings the situation is more complicated than for polynomial rings as described in Theorem 2. Namely, Chu also proved:

Theorem 4. If S is a valuation domain (but not a field), then $R = S[t, t^{-1}]$ is still a GE_2 ring.

So the most interesting questions in this context which (to the best of our knowledge) are still open after many decades are the following two:

Question 1. Is $\mathcal{Z}[t, t^{-1}]$ a GE_2 ring, i.e., is $\mathrm{SL}_2\big(\mathcal{Z}[t, t^{-1}]\big) = \mathrm{E}_2\big(\mathcal{Z}[t, t^{-1}]\big)$?

Obviously, the latter group is finitely generated. So a weaker variant of this question would be:

Question 1′. Is $\mathrm{SL}_2\big(\mathcal{Z}[t, t^{-1}]\big)$ finitely generated?

Question 2. Is it true for some/all/no fields F that $R = F[t_1, t_2, t_1^{-1}, t_2^{-1}]$ is a GE_2 ring?

Problem 3 (Eugene Plotkin and Boris Kunyavskii). *Matrix word maps.*

Let $w(x, y) \in F_2$ be a nontrivial word in the free group on x, y. Let $G = \mathrm{PSL}_2(\mathbb{C})$. Then w defines a map $w : G \times G \to G : (g_1, g_2) \mapsto w(g_1, g_2)$.

Question. Is w always surjective? In other words, for any $a \in \mathrm{PSL}_2(\mathbb{C})$, does the equation $w(x, y) = a$ always have a solution?

The answer is believed to be "yes". This has been checked by computer for "short words" and it's also true if w is a commutator or belongs to the second commutant subgroup in the derived series. However, nobody knows what happens if the word lies deeper in the derived series. For more details, see [16].

On the other hand, the answer is "no" for $G = \mathrm{SL}_2(\mathbb{C})$. A counterexample can be obtained by taking $w(x) = x^n$, where n is even. In general, if G is a connected, semisimple algebraic group over $\mathbb{C}$, then the power map $x \mapsto x^n$ cannot be surjective on $G(\mathbb{C})$ unless n is relatively prime to the order of the center of G.

One might want to generalize to any adjoint algebraic group G, but there are counterexamples in general, which requires a slight modification of the question. The only group which might possess exactly the same property is $\mathrm{PSL}(n, \mathbb{C})$.

Problem 4 (Uriya First). *Extensions of torsors.*

Let F be a field, e.g., $F = \mathbb{C}$. Let G, H_1, H_2 algebraic groups over F and consider morphisms $H_1 \to G$ and $H_2 \to G$.

Question. Is there a G-torsor $T \to X$ over an F-variety X that is extended from H_1 but not from H_2?

As an example, for $\mathrm{O}_n \to \mathrm{GL}_n$ and $\mathrm{Sp}_n \to \mathrm{GL}_n$, the question is equivalent to the existence of a locally free module E on X such that E has a regular quadratic form but not a regular symplectic form. This is known to be true for small n, e.g., [5], and also when n is divisible by 4 (unpublished).

Of course, if there is a morphism $H_1 \to H_2$ compatible with the morphisms to G, then every G-torsor extended from H_1 is also extended from H_2. The general expectation is that, if there is no such morphism, then the question has a positive answer for some F-variety X.

If one bounds the complexity of the possible X, then this becomes harder. For example, for $\mathrm{PGL}_p \to \mathrm{PGL}_p$ the identity map and $\mathcal{Z}/p\mathcal{Z} \rtimes \mu_p \to \mathrm{PGL}_p$ and taking $X = \mathrm{Spec}(F)$, then this question is equivalent to whether there exists a noncyclic p-algebra. Similarly, for $G \to G$ the identity map and $\{1\} \to G$ the inclusion of the trivial subgroup, the question has a positive answer over $X = \mathrm{Spec}(F)$ if and only if G is not a special group.

At the opposite extreme, the question should be easiest to answer if one takes "$X = BG$," and the question is open even in the topological category.

If we restrict to affine X, then, by taking Levi subgroups of H_1, H_2 and replacing G with $G/\mathrm{rad}_u(G)$, we can reduce to the case where G, H_1, H_2 are reductive (at least if F is perfect).

Past work has addressed special cases of this problem using topological methods, by choosing X to be an appropriate finite dimensional algebraic approximation of the classifying space $BG(\mathbb{C})$ of the complex Lie group $G(\mathbb{C})$. While the first use of such approximations is Raynaud's [20] study of stably free modules, this technique has been developed in the past decade by Antieau and Williams [2, 3, 4] with dramatic results on the purity problem for torsors. The results in [5] and [23] use similar techniques to address the above question. These methods usually require careful analysis of topological obstruction invariants tailored to the specific choice of the groups H_1, H_2, G. Also, they are oblivious to unipotent radicals, e.g., if $H_1 = B_2$, $H_2 = T_2$, $G = GL_2$, then we cannot use such methods. Is there a way to address this problem in general (rather than treating special cases separately), and more generally, in the presence of unipotent radicals?

Problem 5 (Chen Meiri). *Local-global property for commutators.*

Let $\mathcal{O}$ be a ring of S-integers with infinitely many units and consider $\mathrm{SL}_2(\mathcal{O})$.

Question. If $g \in \mathrm{SL}_2(\mathcal{O})$ is locally a commutator, then is g a commutator?

Here, "locally" means in the profinite completion. For carefully chosen p, there are counterexamples when $\mathcal{O} = \mathcal{Z}[\frac{1}{p}]$. Are there any counterexamples when $\mathcal{O}$ is the ring of integers in $\mathbb{Q}(\sqrt{D})$ where D is a square-free positive integer?

Since $\mathcal{O}$ has infinitely many units, we know that $\mathrm{SL}_2(\mathcal{O})$ has the congruence subgroup property, so "locally" is equivalent to checking modulo all congruence subgroups.

One can ask the same question for $\mathrm{SL}_2(\mathcal{Z})$, or the free subgroup $F_2 \subset \mathrm{SL}_2(\mathcal{Z})$. Khelif [15] proved that the answer is "yes" for the free group (though here the congruence subgroup property does not hold), and the same methods apply to $\mathrm{SL}_2(\mathcal{Z})$, see [13]. However, for a general free product of finite cyclic groups $C_n * C_m$, the question is open.

Problem 6 (Dave Morris). *Normal subsemigroups.*

Let G be a simple algebraic group over a field K of characteristic 0. A subset $N \subset G(K)$ is a normal subgroup if and only if N is nonempty, closed under multiplication, closed under inverses, and closed under conjugation from G. We have general classification results for all normal subgroups.

Question. Classify the normal subsemigroups (so not assumed to be closed under inverses).

In fact, this classification should reduce to the classical one, as conjectured in [26]:

Conjecture. Every normal subsemigroup is a subgroup.

Maybe one expects the conjecture to also hold for arithmetic groups such as $\mathrm{SL}_n(\mathcal{Z})$ for $n \geq 3$?

The question can be rephrased in different ways, as the following are equivalent:

- every normal subsemigroup is a subgroup,
- for every $x \in G(K)$ there exist $y_1, \ldots, y_n$ such that $x^{-1} = x^{y_1} \cdots x^{y_n}$ (where $x^y = y^{-1}xy$ is the conjugate of x by y),
- for every $x \in G(K)$, there exist $y_1, \ldots, y_n$ such that $1 = x^{y_1} \cdots x^{y_n}$,
- there does not exist a nontrivial bi-invariant partial order on $G(K)$, i.e., $x < y \Rightarrow gx < gy$ and $xg < yg$ for all $g \in G(K)$ (and "nontrivial" means there exist some x and y such that $x < y$).

The conjecture was verified when K is algebraically closed or a local field, and when G is a split classical group. But it is open for $K = \mathbb{Q}$.

Problem 7 (Andrei Rapinchuk). *How to classify algebraic groups?*

Let K be an arbitrary field and L/K a fixed quadratic extension. Can one classify all simple groups over K that are split over L?

Specifically, say that G is $\underline{L/K\text{-admissible}}$ if G has a maximal K-torus T that is anisotropic over K but splits over L. (For example, $\mathbb{C}/\mathbb{R}$-admissible tori are compact.) Can we classify these groups?

It would be especially interesting to work out the case of types E_6, E_7, E_8.

Something is special about $\mathbb{C}/\mathbb{R}$, which is that there is a unique nonsplit central simple algebra, which makes the classification nice, see [6, 7].

This notion of L/K-admissible groups was introduced by Boris Weisfeiler (or Vesfeler) [24], [25], and there is a theory of the admissible tori in G, including elementary moves that allow one to move from one admissible torus to another.

Outcome of the Meeting

We feel that the hybrid format worked very well for this meeting. For many participants this was the first in-person meeting in more than 2 years, and they enjoyed the opportunity to fully interact with their colleagues. At the same time, a number of mathematicians we invited could not travel for various reasons, so it was very helpful to have the virtual option as well. Several participants, including both in-person and online participants, indicated that thanks to this workshop, they were able to either start a new project or make progress on an existing project. We are very grateful to BIRS for allowing us to run the workshop at increased in-person capacity and for accommodating last minute changes.

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Bibliography

[1] Mateo Attanasio, Caroline Choi, Andrei Mandelshtam, and Charlotte Ure *Symbol length in Brauer groups of elliptic curves*, preprint, https://arxiv.org/abs/2107.10886

[2] Benjamin Antieau and Ben Williams, *The topological period-index problem over 6-complexes*, J. Top. **7** (2014), 617–640.

[3] Benjamin Antieau and Ben Williams, *Unramified division algebras do not always contain Azumaya maximal orders*, Invent. math. **197** (2014), no. 1, 47–56.

[4] Benjamin Antieau and Ben Williams, *Topology and purity for torsors*, Doc. Math. **20** (2015), 333–355.

[5] Asher Auel, Uriya A. First, and Ben Williams, *Azumaya algebras without involution*, J. Eur. Math. Soc. **21** (2019), 897–921.

[6] Mikhail Borovoi and Dmitry A. Timashev, *Galois cohomology of real semisimple groups via Kac labelings*, preprint, https://arxiv.org/abs/2008.11763

[7] Mikhail Borovoi and Dmitry A. Timashev, *Galois cohomology and component group of a real reductive group*, preprint, https://arxiv.org/abs/2110.13062

[8] David Carter and Gordon Keller, *Bounded Elementary Generation of* $\mathrm{SL}_n(\mathcal{O})$, Am. J. Math. **105** (1983), no. 3, 673–687.

[9] H. Chu, *On the* GE_2 *of graded rings*, Journal of Algebra **90** (1984), 208–216.

[10] Paul M. Cohn, *On the structure of the* GL_2 *of a ring*, Publ. Math. IHÉS **30** (1966), 5–53.

[11] Pietro Corvaja, Andrei S. Rapinchuk, Jinbo Ren, and Umberto M. Zannier, *Non-virtually abelian anisotropic linear groups are not boundedly generated*, Invent. Math. **227** (2022), 1–26.

[12] Uriya A. First and Tali Kaufman, *On good* 2*-query locally testable codes from sheaves on high dimensional expanders*, preprint, https://arxiv.org/abs/2208.01778

[13] Amit Ghosh, Chen Meiri, and Peter Sarnak, *Commutators in* SL_2 *and Markoff surfaces I*, preprint, https://arxiv.org/abs/2110.11030

[14] David Harbater, Julia Hartmann, and Daniel Krashen, *Bounding cohomology classes over semiglobal fields*, preprint, https://arxiv.org/abs/2203.06770

[15] A. Khelif, *Finite approximation and commutators in free groups*, J. Algebra **281** (2004), no. 2, 407–412.

[16] E. Klimenko, B. Kunyavskii, J. Morita, and E. Plotkin, *Word maps in Kac-Moody settings*, Toyama Mathematical Journal **37** (2015), 25–53. https://u.cs.biu.ac.il/~plotkin/papers/wordmaps.pdf

[17] Boris Kunyavskii, Eugene Plotkin and Nikolai Vavilov, *Bounded generation and commutator width of Chevalley groups: function case*, preprint, https://arxiv.org/abs/2204.10951

[18] Alexander Merkurjev, *Classification of special reductive groups*, https://www.math.ucla.edu/~merkurev/papers/special-1.pdf

[19] M.S. Raghunathan, *The congruence subgroup problem*, Proc. Indian Acad. Sci. (Math. Sci.) **114** (2004), no. 4, 299–308.

[20] Michèle Raynaud, *Modules projectifs universels*, Invent. math. **6** (1968), 1–26.

[21] Zinovy Reichstein, *Hilbert's 13th Problem for Algebraic Groups*, https://arxiv.org/abs/2204.13202

[22] A. Suslin, *On the structure of the special linear group over polynomial rings*, Math. USSR Izvestija **11** (1977), no. 2, 221–238.

[23] Uriya First and Ben Williams, *Involutions of Azumaya algebras*, Doc. Math. **25** (2020) 527–633.

[24] Boris Weisfeiler, *Semisimple algebraic groups which are split over a quadratic extension*, Math. USSR Izvestiya, 5 (1971) 57–72. https://doi.org/10.1070/IM1971v005n01ABEH001007

[25] Boris Weisfeiler, *Monomorphisms between subgroups of groups of type G_2*, J. Algebra **68** (1981), 306–334.

[26] Dave Witte, *Products of similar matrices*, Proc. Amer. Math. Soc. **126** (1998), 1005–1015.

[27] Kirill Zainoulline, *The canonical dimension of a semisimple group and the unimodular degree of a root system*, https://arxiv.org/abs/2108.07835.

Chapter 12

Modern Breakthroughs in Diophantine Problems (22w5162)

June 19 - 24, 2022

Organizer(s): Michael Bennett (University of British Columbia), Nils Bruin (Simon Fraser University), Samir Siksek (Warwick University), Bianca Viray (University of Washington)

Overview of the Field and Recent Developments

The subject of Diophantine equations is currently experiencing a rapid succession of breakthroughs. These include:

(i) The work of Rafael von Känel, Benjamin Matschke, Hector Pasten, and others, proving powerful results on classical Diophantine equations by associating solutions to points on modular or Shimura curves.

(ii) Recent successes in making the Chabauty-Kim method effective, explicit and practical, due to Balakrishnan, Dogra, Müller, and others.

(iii) Progress on Manin's conjecture and other quantitative questions by a new generation of analytic number theorists, including Browning, Loughran, Schindler, Tanimoto and many others.

(iv) The introduction of the notion of Campana points which interpolate between rational and integral points, and which give rise to a host of new Diophantine problems.

(iv) Applications of modularity over number fields to the asymptotic Fermat conjecture and other Diophantine problems due to Bennett, Dahmen, Freitas, Kraus, Sengun, Siksek and others.

Whilst these and other successes constitute dramatic progress on problems of tremendous historical importance, there has also been a divergence of methods and approaches, and the subject is undergoing a period of fragmentation. A primary objective of the workshop was to reverse this fragmentation by bringing together researchers belonging to disparate Diophantine traditions, and who would otherwise rarely interact.

Presentation Highlights

Stephanie Chan: Integral points in families of elliptic curves

Given a family of elliptic curves, it is natural to ask how often does they have integral points, and how many integral points there are on average. In this talk Chan gave beautiful answers for two natural families, the congruent number curves, and the cubic twists of a Mordell curve. For example, fix a non-square $k \neq 0$, and consider

$$E_B \;:\; Y^2 \;=\; X^3 + kB^2.$$

The family $\{E_B \;:\; B \in \mathbb{N}\}$ consists of the cubic twists of the Mordell curve $Y^2 = X^3 + k$. Let

$$E_B(\mathbb{Z}) \;=\; \{(X,Y) \in \mathbb{Z}^2 \;:\; Y^2 = X^3 + kB^2\}.$$

Chan sketched proofs of the following results

$$\#\{1 \leq B \leq N \;:\; E_B(\mathbb{Z}) \neq \emptyset\} \;\ll_k\; N \cdot \left(\frac{\log\log N}{\log N}\right)^{1/2}$$

and

$$\sum_{\substack{1 \leq B \leq N \\ B \text{ cubefree}}} \#E_B(\mathbb{Z}) \;\ll_k\; N.$$

For details see [8], [9].

Levent Alpoge: Integers which are(n't) the sum of two cubes

Thanks to Fermat we have a complete description of which integers are sums of two rational squares. Alpöge sketched the proofs of the following beautiful theorem.

Theorem 12.0.1 (Alpöge, Bhargava and Schnidman) *When ordered by their absolute values, a positive proportion of integers are the sum of two rational cubes, and a positive proportion of integers are not.*

The problem of representing an integer n as the sum of two rational cubes is equivalent to deciding if the elliptic curve

$$E_{d,n} \;:\; y^2 = x^3 - dn^2$$

has rational points, for $d = 432$. As torsion is rare in these families, the problem translates into determining for which values of n is the rank of $E_{432,n}$ positive. The main ingredient is the following estimate for the average size of 2-Selmer group of $E_{d,n}$.

Theorem 12.0.2 (Alpöge, Bhargava and Schnidman) *Fix $d \neq 0$ and let n range over integers satisfying any finite set (or even "acceptable" infinite sets) of congruence conditions. Then*

$$\operatorname{avg}_n \# \operatorname{Sel}_2(E_{d,n}) = 3.$$

For details see [2].

Hector Pasten: On Vojta's conjecture with truncation of rational points

In [36], Vojta proposed a far-reaching generalization of the abc conjecture. Vojta's conjecture is a Diophantine approximation statement in varieties of any dimension and involves truncated counting functions (these are a generalization of the logarithm of the radical of an integer). Pasten sketched the proof of the first unconditional result towards Vojta's conjecture with truncated counting functions in varieties of arbitrary dimension. A striking application is the following corollary, which can be thought of as a subexponential version of the abc conjecture.

Corollary 12.0.3 (Pasten) *Let $\varepsilon > 0$. There is a number $\kappa_\varepsilon > 0$ effectively depending on ε such that the following holds: Let a, b, c be coprime positive integers with $a + b = c$. Suppose $a < c^{1-\eta}$ for some $\eta > 0$. Then*

$$c < \exp\left(\eta^{-1} \cdot \kappa_\varepsilon \cdot R^{(1+\varepsilon)(\log_3^* R)/(\log_2^* R)}\right),$$

where $R = \operatorname{rad}(abc)$.

For details see [29].

Abbey Bourdon: Sporadic points of odd degree on $X_1(N)$ coming from $\mathbb{Q}$-curves

We say a degree d point x on a curve C is isolated if it does not belong to an infinite family of degree d points parametrized by a geometric object-either $\mathbb{P}^1$ or a positive rank abelian subvariety of the curve's Jacobian. We say x is sporadic if there are only finitely many points on C of degree at most d. Every sporadic point is isolated, but the converse need not hold. It was known from recent work of Bourdon, Ejder, Liu, Odumodu, and Viray [5] that Serre's uniformity conjecture implies that there are only finitely many elliptic curves with j-invariant in $\mathbb{Q}$ which give rise to an isolated point of any degree on $X_1(N)$. On the other hand, by recent work of Bourdon and Najman, an analogous finiteness result on non-CM $\mathbb{Q}$-curves would actually imply Serre's Uniformity Conjecture. The talk highlighted unconditional results in that direction for $\mathbb{Q}$-curves giving rise to sporadic points of odd degree. For details see [7].

Pip Goodman: Determining cubic and quartic points on modular curves

Let C be a curve over $\mathbb{Q}$, and let $C^{(d)}$ denote the d-th symmetric power. The $\mathbb{Q}$-points of $C^{(d)}$ correspond to degree d rational divisors on C. In particular, if we can determine $C^{(d)}(\mathbb{Q})$ then we know all degree d points on C. Wetherell (unpublished) and Siksek [34] have extended Chabauty's method to determine $C^{(d)}$, under a suitable condition on the rank. One difficulty is that $C^{(d)}(\mathbb{Q})$ might be infinite. For example, if $\rho : C \to D$ has degree d and $D(\mathbb{Q})$ is infinite then $\rho^*(D(\mathbb{Q}))$ is an infinite subset of $C^{(d)}(\mathbb{Q})$. Previous work on symmetric Chabauty focuses on the case where the only infinite source of rational points on $C^{(d)}$ comes from a single degree d map $\rho : C \to D$. In practice, this has turned out to be severely limiting. For example, if C is a modular curve then one often has many maps to modular curves of smaller level, elliptic curves and to quotients by Atkin–Lehner involutions. In this talk Goodman, reporting on joint work with Box and Gajović, consider the most general case where an infinite family within $C^{(d)}$ has the form

$$P + \rho_1^*(C_1^{d_1}(\mathbb{Q})) + \cdots + \rho_r^*(C_r^{d_r}(\mathbb{Q}))$$

where P is a fixed rational divisor of degree d_0, the maps $\rho_i : C \to C_i$ have degrees e_i and

$$d = d_0 + d_1 e_1 + \cdots + d_r e_r.$$

They use their new variant of Chabauty to determine the cubic points on $X_0(N)$ for $N = 53, 57, 61, 65, 67, 73$ and the quadratic points on $X_0(65)$, thereby answering questions posed by Zureick-Brown. For details see [4].

Adela Gherga: Efficient resolution of Thue–Mahler equations

A Thue–Mahler equation has the form

$$F(X,Y) \ = \ a \cdot p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}, \qquad \gcd(X,Y) = 1, \qquad n_i \geq 0$$

where $F \in \mathbb{Z}[X,Y]$ is an irreducible binary form of degree ≥ 3, a is a non-zero integer, and $p_1, \ldots, p_r$ are distinct primes. The talk highlighted a new algorithm for solving Thue–Mahler equations that makes heavy use of a newly developed "Dirichlet sieve". This allows for the resolution of Thue–Mahler equations of large degree or with a large number of primes. For example, the algorithm determines all solutions to $P(X^4 - 2Y^4) \leq 100$ with $\gcd(X,Y) = 1$, finding that there are precisely 49 solutions. Here $P(m)$ denotes the largest prime divisor of m. This work links with algorithms due to Gherga, Bennett, Rechnitzer, von Känel, Matschke to determine all elliptic curves over $\mathbb{Q}$ with a given set of bad primes, by reducing the problem to the resolution of cubic Thue–Mahler equation. For details see [3], [17].

Isabel Vogt: Geometry of curves with abundant points

Let X be a curve over a number field k, and let $d \geq 2$. When does X possess infinitely many degree d points? It is known by classification results of Harris–Silverman and Abramovich–Harris that if this happens with $d = 2, 3$, then the curve X admits a non-constant map of degree at most d to either $\mathbb{P}^1$ or an elliptic curve. For $d \geq 4$ the analogous statement is false by work of Debarre and Fahlaoui. Vogt sketched joint work with Kadets that extends the classification of Harris–Silverman and Abramovich–Harris to larger values of d. For a curve X/k we define the arithmetic degree of irrationality $\mathrm{a.irr}_k\, X$ to be the smallest integer k such that X has infinitely many closed points of degree d. We define the geometric degree of irrationality $\mathrm{a.irr}_{\overline{k}}\, X$ to be the minimum of the values $\mathrm{a.irr}_L\, X$ as L ranges over finite extensions of k.

Theorem 12.0.4 (Kadets and Vogt) *Suppose X/k is a nice curve. Then the following statements hold:*

1. *If $\mathrm{a.irr}_k\, X = 2$, then X is a double cover of $\mathbb{P}^1$ or an elliptic curve of positive rank;*
2. *If $\mathrm{a.irr}_k\, X = 3$, then one of the following three cases holds:*
   (a) *X is a triple cover of $\mathbb{P}^1$ or an elliptic curve of positive rank;*
   (b) *X is a smooth plane quartic with no rational points, positive rank Jacobian, and at least one cubic point;*
   (c) *X is a genus 4 Debarre-Fahlaoui curve;*
3. *If $\mathrm{a.irr}_{\overline{k}}\, X = d \leqslant 3$, then $X_{\overline{k}}$ is a degree d cover of $\mathbb{P}^1$ or an elliptic curve;*
4. *If $\mathrm{a.irr}_{\overline{k}}\, X = d = 4, 5$, then either $X_{\overline{k}}$ is a Debarre-Fahlaoui curve, or $X_{\overline{k}}$ is a degree d cover of $\mathbb{P}^1$ or an elliptic curve.*

For details see [22].

Diana Mocanu: The modular approach to Diophantine equations over totally real fields

Freitas, Kraus and Siksek [16], [15] have related solutions to the Fermat equation $X^p + Y^p + Z^p = 0$ over totally real fields to solutions to a certain S-unit equations using modularity and level lowering. Mocanu extends this to the generalized Fermat equations $X^p + Y^p = Z^2$ and $X^p + Y^p = Z^3$ where the S-unit equations are replaced by equations of the form $\alpha + \beta = \gamma^2$ and $\alpha + \beta = \gamma^3$ where α, β are S-units. Under certain class-field-theoretic assumptions Mocanu can control solutions to these equations. A sample theorem is the following.

Theorem 12.0.5 (Mocanu) *Let $d \equiv 5 \pmod 8$ be a rational prime, and write $K = \mathbb{Q}(\sqrt{d})$. There is a constant B_K such that for all primes $p > B_K$, the equation $a^p + b^p = c^2$ has no non-trivial primitive solutions $(a, b, c) \in \mathcal{O}_K^3$ with $2 \mid b$.*

For details see [25].

Open Problems

The organizers thank Alex Best for transcribing the open problems.

Abbey Bourdon: Two Problems on Isolated Points

Let C be a nice curve over a number field k. For the sake of simplicity, assume there exists $P_0 \in C(k)$; for a more general setup, see [5, §4]. We say a closed point $x \in C$ of degree d is **sporadic** if there are only finitely many points of degree at most d. More generally, we say x is **isolated** if it does not belong to an infinite family of degree d points parametrized by $\mathbb{P}^1$ or a positive rank abelian subvariety of the curve's Jacobian. Precisely, to x we can associate the k-rational effective divisor

$$D = P_1 + \cdots + P_d$$

where $P_1, \ldots, P_d$ are the points in the Gal_k-orbit corresponding to x. Thus x gives a k-rational point on the dth symmetric power of C, denoted $C^{(d)}$. With this identification, we can study the image of x under the natural map to the curve's Jacobian

$$\Phi_d : C^{(d)} \to \mathrm{Jac}(C)$$

which sends the effective divisor D of degree d to the class $[D - dP_0]$. We say x is **isolated** if the following conditions are both satisfied:

1. There is no other point $y \in C^{(d)}(k)$ such that $\Phi_d(x) = \Phi_d(y)$.

2. There is no positive rank abelian subvariety $A \subset \mathrm{Jac}(C)$ such that $\Phi_d(x) + A \subset \mathrm{im}(\Phi_d)$.

Any sporadic point is isolated (though the converse need not hold), and any curve has only finitely many isolated points. See [5, Theorem 4.2].

Recent investigations [5, 6, 7, 14, 33] have sought to characterize the elliptic curves producing sporadic and isolated points on $X_1(N)$. We say $j \in X_1(1) \cong \mathbb{P}^1$ is a **sporadic** (resp., **isolated**) j**-invariant** if it is the image of a sporadic (resp., isolated) point on $X_1(N)$ for some positive integer N. If one assumes Serre's Uniformity Conjecture, then there are only finitely many isolated j-invariants in $\mathbb{Q}$ [5, Corollary 1.7], and the subset of non-CM j-invariants in $\mathbb{Q}$ corresponding to isolated points of odd degree has been identified explicitly [6, Theorem 2]. As a first step in the case of even degree, define $J_{\mathrm{isog}}(\mathbb{Q})$ to be the set of j-invariants associated to elliptic curves over $\mathbb{Q}$ with a nontrivial rational cyclic isogeny. This set contains all known examples of isolated j-invariants in $\mathbb{Q}$: those corresponding to CM elliptic curves plus $j = -3^2 \cdot 5^6/2^3$, $3^3 \cdot 13/2^2$, and $-7 \cdot 11^3$. See [5, 6, 27, 21]. By work of Lemos [23], Serre's Uniformity Conjecture holds for all non-CM elliptic curves over $\mathbb{Q}$ possessing a nontrivial cyclic $\mathbb{Q}$-isogeny. Thus, by [5, Corollary 1.7], there are only finitely many isolated j-invariants in $J_{\mathrm{isog}}(\mathbb{Q})$.

Question 1. Can the set of isolated j-invariants in $J_{\mathrm{isog}}(\mathbb{Q})$ be computed explicitly? Are there any isolated j-invariants in $\mathbb{Q}$ which lie outside this set?

We note that there are similarities between this question and the methods used to prove Theorem 2 in [6]. There, an essential observation was that if $x \in X_1(n)$ is a point of odd degree with $j(x) \in \mathbb{Q}$ and $j(x) \neq 3^3 \cdot 5 \cdot 7^5/2^7$, then there exists $y \in X_0(p)(\mathbb{Q})$ with $j(x) = j(y)$ for some odd $p \mid n$; see [6, Thm. 3]. If one follows the approach of [6], it will be necessary to perform a more sophisticated analysis of the possible combinations of simultaneously non-surjective Galois representations associated to elliptic curves $E/\mathbb{Q}$. Partial progress can be made using work of Morrow, Daniels, and González-Jiménez [26], [10] on fiber products of modular curves in combination with results obtained via formal immersions as in work of Darmon and Merel [13, Thm. 8.1] and Lemos [23, Prop. 2.1]. An analysis of certain "entanglement" modular curves was also necessary in [6] and similar computations may be required for Question 1. For more on entanglement modular curves, see [7, 12, 11].

Instead of studying isolated or sporadic points associated to elliptic curves with j-invariant in $\mathbb{Q}$, one could more generally hope to understand isolated points corresponding to $\mathbb{Q}$-curves. Here, by $\mathbb{Q}$-curve, I mean an elliptic curve isogenous (over $\overline{\mathbb{Q}}$) to its Galois conjugates. This class contains all elliptic curves with j-invariant in $\mathbb{Q}$, as well as any curve in the corresponding geometric isogeny class, though there are others not of this form (the so-called "strict" $\mathbb{Q}$-curves). A key motivation for studying sporadic points associated to $\mathbb{Q}$-curves is the following: If all non-CM $\mathbb{Q}$-curves giving rise to a sporadic point on $X_1(N)$ belong to only finitely many geometric isogeny classes—even as we allow N to range over all positive integers—then Serre's Uniformity Conjecture holds. See [7, Theorem 1.3]. We have such a finiteness result for odd degree, where one can show all non-CM $\mathbb{Q}$-curves corresponding to a sporadic point of odd degree on $X_1(N)$ belong to the $\overline{\mathbb{Q}}$-isogeny class of the elliptic curve 162.c3 with j-invariant $-3^2 \cdot 5^6/2^3$ [7, Theorem 1.4]. However, it is unknown whether this isogeny class contains any sporadic j-invariants besides $-3^2 \cdot 5^6/2^3$. This inspires the following question, which also appears as Question 2 in [7]:

Question 2. Does there exist a non-CM $\overline{\mathbb{Q}}$-isogeny class containing infinitely many sporadic j-invariants?

Note that the answer to Question 2 is yes if there exists an elliptic curve producing a sporadic point of sufficiently low degree [7, Proposition 8.1], but the only known examples satisfying this condition are CM elliptic curves.

Nathan Grieve: Approximation sets for properly intersecting divisors

Consider a polarized projective variety (X, L) defined over a number field $\mathbf{K}$. Let $D_1, \ldots, D_q$ be a collection of nonzero effective and properly intersecting Cartier divisors on X. Fix a finite set of places S, of $\mathbf{K}$, and let $N = q \cdot \#S$.

Expanding on the viewpoint of Schmidt [32], inside of $\mathbb{R}^N$, there is an approximation set

$$Approx(X, L; D_1, \ldots, D_q; S) \subseteq \mathbb{R}^N.$$

In defining such approximation sets, a key point is a concept of density of rational points with respect to the subspace topology on $X(\mathbf{K})$ that is induced by the linear sections of the complete linear series $|L|$ and powers thereof (cf. [32, p. 706] and [18, Definition 3.1]).

Arguing as in [32, p. 708], the compactness of such approximation sets follows from the Ru-Vojta Arithmetic General Theorem (see for instance [31, p. 961], [18, Theorem 1.1]). However, it remains an interesting problem to determine defining inequalities of such approximations sets. Such a result would make progress towards a general form of [32, Theorem 2] which, in particular, would treat the case of properly intersecting divisors.

Hector Pasten: Büchi's problem

Observe: $1, 4, 9, 16, \ldots$ have differences 3, 5, 7, which have differences 2, 2, 2, etc.

You can also take non-consecutive squares such as $0,\ 49,\ 100, \ldots$ which have difference $49,\ 51$, and then difference 2. But it seems harder to construct long sequences like this.

The problem is to find how long such a sequence can be, there are infinitely many known examples of length four.

This is known as Büchi's problem; show that there exists a uniform M (i.e. constant) such that every sequence of $\geq M$ squares with second differences equal to 2 is trivial (i.e. the squares are consecutive). One expects that $M = 5$ (via heuristic and also from known evidence), but any bound would be interesting.

What is known: A theorem of Vojta shows that the Bombieri–Lang conjecture implies a positive answer to Büchi's problem [35]. Pasten also shows, conditionally on the abc-conjecture, that Büchi's problem admits a positive answer [28]. The challenge is to find something unconditional in this direction.

Nils Bruin remarks that Vojta's approach is via surfaces, showing eventually the surface classifying these is of general type, and so the Bombieri–Lang conjecture is applicable. However, Pasten's approach is less geometric.

Stanley Xiao: One of the cuboid conjectures

An **perfect Euler brick** is a rectangular prism with side lengths a, b, $c \in \mathbb{N}$ such that all face diagonals (d, e, f) are natural numbers and also the space diagonal g is a natural number. No example of a perfect Euler brick is known, however, it is easy to construct examples of bricks where only the edges and face diagonals are natural numbers (e.g. $(a, b, c, d, e, f) = (44, 117, 240, 125, 255, 267)$.

If we do not insist on the prism being rectangular, and allow non-right angles, examples are also known.

There are conjectures known as the cuboid conjectures, suggested 10 years ago, which appear on the Wikipedia page for the Euler brick. Together the three cuboid conjectures, imply there is no perfect Euler brick. The first conjecture is easy (for experts on invariant theory of quadratic forms), the second seems harder and the third seems to be 99% of the work.

The second cuboid conjecture is as follows:

Conjecture 2. For any two positive coprime integer numbers $p \neq q$ the tenth-degree polynomial

$$\begin{aligned} Q_{pq}(t) =& t^{10} + \left(2q^2 + p^2\right)\left(3q^2 - 2p^2\right)t^8 \\ &+ \left(q^8 + 10p^2q^6 + 4p^4q^4 - 14p^6q^2 + p^8\right)t^6 \\ &- p^2q^2\left(q^8 - 14p^2q^6 + 4p^4q^4 + 10p^6q^2 + p^8\right)t^4 \\ &- p^6q^6\left(q^2 + 2p^2\right)\left(-2q^2 + 3p^2\right)t^2 \\ &- q^{10}p^{10} \end{aligned}$$

is irreducible over the ring of integers $\mathbb{Z}$.

Conjecture 2 may be possible using the expertise we have, though it is not completely clear what the motivation for this conjecture is. John Voight suggests looking at Runge's method.

Drew Sutherland: Modular curves arising in the classification of Galois images

Mazur's vertical uniformity problem asks for the determination of possible ℓ-adic images of Galois representations of elliptic curves $E/\mathbb{Q}$. i.e. given a prime ℓ what are the possibilities for the image of

$$\rho_{E,\ell^\infty} \; : \; G_{\mathbb{Q}} \to \mathrm{GL}(E[\ell^\infty]),$$

as E ranges over all elliptic curves over $\mathbb{Q}$. For $\ell = 2$ the answer is known due to Rouse–Zurieck-Brown. For $\ell = 11, 17$ the answer is known due to Balakrishnan et. al.

Why don't we know more? We need to determine the rational points on certain modular curves such as

$$X_{ns}^+(25), \qquad X_{ns}^+(27), \qquad X_{ns}^+(131), \qquad X_{ns}^+(\ell) \text{ for } \ell \geq 19.$$

There are two other curves that interesting that we would like to know the rational points on (given by their LMFDB labels of the form $L.I.g.n$ where L is the level, I the index, g the genus, and n the curve number). The first is known as $49.147.9.1$, and is a degree 7 cover of $X_{ns}^+(7)$ (which is genus 0), the CM points are above $j = 0$, and the plane model has been computed, of degree 21. The gonality is at least 3, and the Jacobian is geometrically irreducible and of rank 9. The corresponding modular form is
https://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/2401.2.a.f/

The second is known as $49.196.9.1$, it is a degree 7 cover of $X_s^+(7)$ the CM points are above $j = 0$, and the plane model has been computed, of degree 14. The gonality is at least 5, and the Jacobian decomposes as the product of a dimension 3 and dimension 6 piece corresponding modular forms are
https://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/2401.2.a.b/
and
https://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/2401.2.a.c/
The rank of the Jacobian is 9.

Models for these curves can be found at
https://github.com/AndrewVSutherland/ell-adic-galois-images/tree/main/models.

It is conjectured that there are no non-CM non-cuspidal points, the problem is to prove this. Once this is done, the only remaining obstruction to determining the possible ℓ-adic Galois images will the non-split Cartans.

Seeing as there are not many rational points, quadratic Chabauty might be tricky, but still a good option.

Hector Pasten: Zariski density of rational points on general type surfaces of irregularity 2

Let X be a surface of general type defined over a number field k. Suppose that the irregularity of X is $q = 2$ and that the albanese map is surjective. Prove that $X(k)$ is not Zariski dense in X.

Some context for the problem is given by the following cases:

- If $q > 2$, Faltings's theorem on subvarieties of abelian varieties implies that $X(k)$ is not Zariski dense in X.

- If $q = 2$, but the albanese map is not surjective, then its image is a curve to which one can apply Faltings's theorem. We deduce that $X(k)$ is not Zariski dense.

Thus, it seems that the proposed problem is the next natural case of the Bombieri-Lang conjecture for surfaces.

Natalia Garcia Fritz: Finding differentials for which a divisor is integral

Given a smooth projective surface $X/\mathbb{C}$, and $D = \sum_{j=1}^{q} D_j$ a reduced divisor on X formed by different irreducible curves D_j, we want to find a non-trivial section $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$ such that every D_j is an ω-integral curve, with $\mathcal{L}$ of low degree on each D_j (or at least degree independent of q). In that way we will be able to prove more instances of Vojta's conjecture or Campana's conjecture for surfaces in the function field case. One can consider Hasse-Schmidt differentials instead of $S^r\Omega^1_{X/\mathbb{C}}$ to cover more cases.

Here are some particular examples:

- In $\mathbb{P}^2$, with D_j in the quadratic family of lines $t^2x + sty + s^2z = 0$, one can choose $\omega \in H^0(\mathbb{P}^2, \mathcal{O}(4) \otimes S^2\Omega^1_{\mathbb{P}^2/\mathbb{C}})$ which locally looks like $dxdx - ydxdy + xdydy$.
- In $\mathbb{P}^2$, with D_j lines, we can choose $\omega \in H^0(\mathbb{P}^2, \mathcal{O}(3) \otimes (HS^2_{\mathbb{P}^2/\mathbb{C}})_3)$ which locally looks like $dyd_2x - dxd_2y$.
- In $\mathbb{P}^2$, if we consider D_j in the family $cx^k - (c+1)y^k = c(c+1)z^k$ with $c \in \mathbb{C} \setminus \{-1, 0\}$, $k > 2$ an integer, we can choose $\omega \in H^0(\mathbb{P}^2, \mathcal{O}(k+3) \otimes S^2\Omega^1_{\mathbb{P}^2/\mathbb{C}})$ which locally looks like $x^{k-1}ydxdx + (1 - x^k - y^k)dxdy + xy^{k-1}dydy$.

Problems:

1. Find suitable conditions on D to make this work in more generality
2. Find systematic approach to construct other explicit examples.

Outcome of the Meeting

We were fortunate to attract around 38 in person participants for the workshop and another 31 online participants. For many this had been their first face-to-face event in over 2 years. Our foremost priority was to be useful to younger participants, whose careers must have suffered the most during the past two years. We did this in three ways:

- Prioritise talks given by younger participants to allow them to advertise their work.
- With the aim of kick-staring collaborations and new research, we encouraged speakers to suggest open problems during their talks.
- We kept the schedule light (9.00–12.00, 13.30–3.00) to allow ample time for discussions and collaboration.

It is clear from the talks and discussions that the subject remains a very active field. Whilst breakthroughs continue to be made on older problems such as Vojta's conjectures, there are also some newer areas have become prominent in the last few years and provide excellent opportunities for active research and further breakthroughs. These include the following:

- The application of methods from arithmetic statistics to study Diophantine problems in families [2], [8], [9].
- Low degree points on curves, both from the theoretical [22] and from the computational [4], [30] perspectives.
- The recently formulated concepts of sporadic and isolated points [4], [5], [6], [7] on modular curves.
- Quadratic Chabauty used to determine rational points on curves where the Jacobian Mordell–Weil rank equals the genus [1].
- Applications of modularity of elliptic curves over number fields to Fermat-type equations of signatures (p, p, p), $(p, p, 2)$, $(p, p, 3)$ [24], [25].

One of the surprises of the workshop was a talk (based on [20]) by Avinash Kulkarni reporting on how machine learning was successfully used to aid in the computation of the periods of projective hypersurfaces. The use of machine learning in computational arithmetic geometry is certainly an avenue worthy of further exploration and experimentation.

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Bibliography

[1] N. Adžaga, S. Chidambaram, T. Keller, O. Padurariu, Rational points on hyperelliptic Atkin-Lehner quotients of modular curves and their coverings, arxiv:arXiv:2203.05541.

[2] L. Alpöge, M. Bhargava and A. Schnidman, Integers expressible as the sum of two rational cubes, preprint.

[3] M. A. Bennett, A. Gherga and A. Rechnitzer, Computing elliptic curves over $\mathbb{Q}$, Math. Comp. **88** (2019), 1341–1390.

[4] J. Box, S. Gajović, P. Goodman, Cubic and Quartic Points on Modular Curves Using Generalised Symmetric Chabauty, International Mathematics Research Notices **mab358** (2022).

[5] A. Bourdon, Ö. Ejder, Y. Liu, F. Odumodu, and B. Viray, On the level of modular curves that give rise to isolated j-invariants, Adv. Math. **357** (2019), 106824, 1–33.

[6] A. Bourdon, D. Gill, J. Rouse, and L. D. Watson, Odd degree isolated points on $X_1(N)$ with rational j-invariant, arxiv:2006.14966.

[7] A. Bourdon and F. Najman, Sporadic points of odd degree on $X_1(N)$ coming from $\mathbb{Q}$-curves, arxiv:2107.10909.

[8] S. Chan, Integral points on cubic twists of Mordell curves, arxiv:2203.11366.

[9] S. Chan, The average number of integral points on the congruent number curves, arxiv:2112.01615.

[10] H. Daniels and E. González-Jimenéz, Serre's constant of elliptic curves over the rationals, to appear in Exp. Math., arxiv:1812.04133.

[11] H. Daniels, Á. Lozano-Robledo, and J. S. Morrow, Towards a classification of entanglements of Galois representations attached to elliptic curves, arxiv:2105.02060.

[12] H. Daniels and J. S. Morrow, A group theoretic perspective on entanglements of division fields, to appear in Trans. Amer. Math. Soc. Ser. B., arxiv:22008.09886.

[13] H. Darmon and L. Merel, Winding quotients and some variants of Fermat's last theorem, J. Reine Angew. Math. **490** (1997), 81–100.

[14] Ö. Ejder, Isolated points on $X_1(\ell^n)$ with rational j-invariant, Res. Number Theory **8** (2022), no. 1, Paper No. 16, 7.

[15] N. Freitas, A. Kraus and S. Siksek, Local criteria for the unit equation and the asymptotic Fermat's Last Theorem, Proceedings of the National Academy of Sciences **118** (2021), No. 12.

[16] N. Freitas and S. Siksek, The Asymptotic Fermat's Last Theorem for Five-Sixths of Real Quadratic Fields, Compositio Mathematica **151** (2015), 1395–1415.

[17] A. Gherga and S. Siksek, Efficient resolution of Thue–Mahler equations, arxiv:2207.14492.

[18] N. Grieve, On arithmetic inequalities for points of bounded degree, Res. Number Theory **7** (2021), no. 1, Paper No. 1, 14.

[19] N. Grieve, Files from BIRS Workshop: 22w5024, https://www.birs.ca/workshops/2022/22w5024/files/.

[20] K. Heal, A. Kulkarni, and E. C. Sertöz, Deep learning Gauss-Manin connections, Adv. Appl. Clifford Algebr. **32(2)**:Paper No. 24, 41, 2022.

[21] M. van Hoeij, Low degree places on the modular curve $X_1(n)$, arxiv:1202.4355.

[22] B. Kadets and I. Vogt, Subspace configurations and low degree points on curves, arXiv:2208.01067.

[23] P. Lemos, Serre's uniformity conjecture for elliptic curves with rational cyclic isogenies, Trans. Amer. Math. Soc. **371** (2019), no. 1, 137–146.

[24] P. Michaud-Jacobs, On some generalized Fermat equations of the form $x^2 + y^{2n} = z^p$, Mathematika **68** (2022), no. 2, 344–361.

[25] D. Mocanu, Asymptotic Fermat for signatures $(p, p, 2)$ and $(p, p, 3)$ over totally real fields, Mathematika **68** (2022), Issue 4, 1233–1257

[26] J. S. Morrow, Composite images of Galois for elliptic curves over $\mathbf{Q}$ and entanglement fields, Math. Comp. **88** (2019), no. 319, 2389–2421.

[27] F. Najman, Torsion of rational elliptic curves over cubic fields and sporadic points on $X_1(n)$, Math. Res. Lett. **23** (2016), no. 1, 245–272.

[28] H. Pasten, Powerful values of polynomials and a conjecture of Vojta, J. Number Theory **133** (2013), no. 9, 2964–2998.

[29] H. Pasten, On the arithmetic case of Vojta's conjecture with truncated counting functions, arxiv:2205.07841.

[30] B. Banwait, F. Najman and O. Padurariu, Cyclic Isogenies of Elliptic Curves over a Fixed Quadratic Field, arxiv:2206.0889.

[31] M. Ru and P. Vojta, A birational Nevanlinna constant and its consequences, Amer. J. Math. **142** (2020), no. 3, 957–991.

[32] W. M. Schmidt, Vojta's refinement of the subspace theorem, Trans. Amer. Math. Soc. **340** (1993), no. 2, 705–731.

[33] H. Smith, Ramification in division fields and sporadic points on modular curves, arxiv:1810.04809.

[34] S. Siksek, Chabauty for symmetric powers of curves, Algebra & Number Theory **3** (2009), No. 2, 209–236.

[35] P. Volta, Diagonal quadratic forms and Hilberts tenth problem, Hilberts tenth problem: relations with arithmetic and algebraic geometry (Ghent, 1999), Contemp. Math., **270**, Amer. Math. Soc., Providence, RI, 2000, pp. 261–274.

[36] P. Vojta, A more general abc conjecture, Internat. Math. Res. Notices (1998) **21**, 1103–1116.

Chapter 13

Derived Categories, Arithmetic, and Reconstruction in Algebraic Geometry (22w5108)

July 3 - 8, 2022

Organizer(s): Laure Flapan (Michigan State), Katrina Honigs (Simon Fraser U.), Martin Olsson (UC Berkeley), Adam Topaz (U Alberta)

Overview of the Field

The workshop focused on various topics which all in some way concern the problem of reconstruction in algebraic geometry. The basic problem is to understand various properties of an algebraic variety, such as its birational class, whether it has a rational point, etc., from certain invariants which may be simpler to understand.

The main foci of the workshop were threefold:

(i) Derived Torelli theorems.

(ii) Derived categories, cohomology, and arithmetic applications.

(iii) Anabelian geometry and reconstruction results.

(i) It is well-known that the derived category of coherent sheaves is not a faithful invariant of algebraic varieties in the sense that one can have two non-isomorphic algebraic varieties with equivalent derived categories. This is, in fact, the origin of the subject of Fourier-Mukai transforms and derived equivalences. At the same time, the derived category appears to be a strong invariant. Recent conjectures of Lieblich and Olsson [9] predict that in many cases the derived category, together with some additional information related to numerical Chow groups, should suffice to recover the birational equivalence class of an algebraic variety. This is also closely related to an older conjecture of Orlov that varieties with equivalent derived categories should have isomorphic Chow motives.

(ii) Related to (i) is the basic question of what information is encoded in the derived category, or better the ∞-category, of coherent sheaves on a variety. A lot is known, especially in regards to cohomology. For example, one can show that certain twisted versions of any of the standard cohomology theories (Betti, de Rham, étale, crystalline) are preserved under derived equivalence. More recently, in the work of Bhatt, Morrow, and Scholze [3] it was shown how to relate topological Hochschild homology, obtained from the ∞-category of perfect complexes on a smooth projective variety over a perfect field of positive characteristic, to the de Rham-Witt complex, and therefore also crystalline cohomology. These results lead to numerous questions related to understanding the arithmetic information encoded in the derived category and/or the cohomology of algebraic varieties.

(iii) Inspired by work on derived categories, Kollár, Lieblich, Olsson, and Sawin [8] have obtained several new results about recovering the isomorphism class of a variety of dimension ≥ 2 purely from the Zariski topological space, and, in some cases, the Zariski topological space along with its divisor class group. Topaz has also obtained results which recover function fields of higher-dimensional algebraic varieties from data encoded in certain cohomological structures [12][13]. These results are closely related to work of Zilber on curves and their Jacobians [14], and to work of Bogomolov-Korotaev-Tschinkel [4], Bogomolov-Tschinkel [5][6], Pop [10], Pirutka-Cadoret [7], and others, from anabelian geometry. There are some further connections as well with the newly developed theory of exodromy due to Barwick, Glasman and Haine [2]. One of the main goals of the workshop was to bring together researchers working around such reconstruction results from these different points of view.

Discussion of open problems

An important component of the workshop was discussion around open problems in the field. To facilitate discussion around this the workshop included two scheduled events. Following the practice at many conferences a "problem session" was held on Tuesday (second day) of the workshop wherein participants shared problems, ideas, speculations, etc., about important open problems in the field. This was preceded by a session "organized discussion" at the end of Monday (first day) wherein participants were broken into small groups and discussed problems they might highlight for the entire group in Tuesday's problem session. This organized discussion served two purposes: (1) It focused Tuesday's problem session; (2) It helped foster interactions among participants and especially made it easier for junior participants to engage with the group. This approach appears to have worked well. Appended to this report is a writeup of the problems discussed during the problem session.

Hybrid format

This workshop was delivered in a hybrid format with 20 in-person participants and 14 hybrid participants. While the technological facilities could allow substantially more remote participants the organizers deemed it important to retain the workshop atmosphere for both in-person and remote participants and therefore extended invitations to participants who would take an active part (of course, with the lectures posted online this did not present an obstacle for others to view the research presentations). It should also be noted that for various pandemic-related reasons several participants (and one organizer) had to change their travel plans at the last minute and switch from in-person attendance to remote attendance.

Presentations

The meeting included presentations in various formats including three survey talks, one-hour research presentations, and 10-15 minutes "lightning talks" by more junior participants. While most presentations were in-person the workshop also included several talks delivered remotely. With this format, all in-person participants were given an opportunity to speak.

Survey Lectures

1. Lieblich, Max (U Washington): Reconstruction.

   Abstract: I will discuss various general types of reconstruction or characterization results, ranging from basic algebraic geometry to new results obtained in joint work with Kollár, Olsson, and Sawin, and with Alper and de Jong. In each case, one attempts to extract a complete algebraic invariant – for example, a ring, a group, a Hodge structure, an abelian category, a tensor triangulated category, a motive, a topos, an abstract projective structure, a poset – from a geometric object. Each type of invariant has successes and failures.

2. Stix, Jacob (Goethe-Universität Frankfurt), An invitation to anabelian Geometry.

   Abstract: This will be a survey talk about anabelian geometry.

3. Hassett, Brendan (Brown University): Derived categories and rational points.

   Abstract: This is a survey of the relationship between derived equivalence, the existence of rational points, and other arithmetic properties. Given smooth varieties X and Y over a field k, assume to be derived equivalent over k, how are the k-rational points of X and Y related? We summarize what is currently known for K3 surfaces as well as some important recent results of Addington-Antieau-Honigs-Frei in higher dimensions. (joint with Tschinkel)

One-hour research lectures

Talks indicated as "Zoom lecture" were delivered by remote participants via Zoom.

1. Ballard, Matthew (University of South Carolina), Generation in prime characteristic/a GUT for flops.

   Abstract: A double feature talk. During the first half, Ill discuss how, when, and where does the Frobenius pushforward generate the derived category. This is joint with Pat Lank. In the second half, Ill introduce a general construction which extracts integral kernels form flips and show how it gives an equivalence for stratified Mukai flops. This is joint with Nitin Chidambaram and David Favero.

2. Bragg, Daniel (University of California Berkeley), A Stacky Murphys Law for the Stack of Curves.

   Abstract: We show that every Deligne-Mumford gerbe over a field occurs as the residual gerbe of a point of the moduli stack of curves. Roughly speaking, this means that the moduli space of curves fails to be a fine moduli space in every possible way. This is joint work with Max Lieblich.

3. Cadoret, Anna (Sorbonne Universit), Degeneracy locus of ℓ-adic local systems - an anabelian approach.

   Abstract: Let X be a smooth variety over a number field k. I will review the general heuristic underlying our expectation that the set of k-rational points in the degeneracy locus of a p-adic local system whose geometric monodromy is semisimple (perfect?) are not Zariski-dense and recall some of the motivations for this question. This heuristic relies on a geometric conjecture and (a weak form of) the Bombieri-Lang conjecture. In the second part of the talk, I will give an hint of the proofs of the geometric conjecture when X is a curve (joint with A. Tamagawa) and a product of 2 curves.

4. Frei, Sarah (Rice University), Symplectic involutions of hyperkahler fourfolds of Kummer type.

   Abstract: The middle cohomology of hyperkahler fourfolds of Kummer type was studied by Hassett and Tschinkel, who showed that a large portion is generated by cycle classes of fixed-point loci of symplectic involutions. In recent joint work with Katrina Honigs, we study symplectic fourfolds over arbitrary fields which are constructed as fibers of the Albanese map on moduli spaces of stable sheaves on an abelian surface. We have extended the results of Hassett and Tschinkel and characterized the Galois action on the cohomology. We do this by giving an explicit description of the symplectic involutions on the fourfolds. This has natural consequences for derived equivalences between Kummer fourfolds.

5. Gaulhiac, Sylvain (University of Alberta), Towards tempered anabelian recovery of lengths in Berkovich geometry.

   Abstract: In the framework of non-archimedean Berkovich geometry, questions of anabelian type are best answered using the so-called tempered fundamental group, introduced by Yves Andr. It is now known that in many cases, the tempered group of a curve determines its skeleton as a graph. This graph also has a natural metric. Does the tempered fundamental group determine the length of each edge? If the answer is positive in some cases for algebraic curves due to some work of Lepage, it remains unknown otherwise, even for the most simple curves : annuli. I will present a partial result in this direction, using some interesting methods of splitting radius of torsors and resolution of non-singularities.

6. Haine, Peter (UC Berkeley), Galois-theoretic reconstruction of schemes and Exodromy.

   Abstract: The classical theorem of Neukirch and Uchida says that number fields are completely determined by their absolute Galois groups. In this talk, well explain joint work with Clark Barwick and Saul Glasman generalizing this reconstruction result to schemes. Given a scheme S we construct a category $Gal(S)$ that records the Galois groups of all of the residue fields of S (with their profinite topologies) together with ramification data relating them. Well explain why the construction $S \mapsto Gal(S)$ is a complete invariant of normal schemes over a number field. The category $Gal(S)$ also plays some other roles. For example, just like how there is a monodromy equivalence between representations of tale fundamental group and local systems, there is an equivalence between representations of the category $Gal(S)$ and constructible sheaves. This invariant also gives rise to a new definition of the tale homotopy type.

7. Huang, Jesse (University of Alberta), Homotopy Path Algebras.

   Abstract: In this talk, I will define a basic class of algebras, "homotopy path algebras", and explain the relation between a homotopy path algebra and entrance/exit paths on an appropriately stratified classifying space that naturally gives a cellular resolution of the diagonal bimodule. An earlier result of mirror symmetry due to Bondal-Ruan and certain Berglund-Hübsch-Krawitz mirrors can be recovered as an application. I will also discuss some results on minimal cellular resolutions of diagonal bimodules. This is based on joint work with David Favero.

8. (Zoom lecture) Markman, Eyal (University of Massachusetts Amherst), Rational Hodge isometries of hyper-Kahler varieties of $K3[n]$-type are algebraic.

   Abstract: Let X and Y be compact hyper-Kahler manifolds deformation equivalence to the Hilbert scheme of length n subschemes of a K3 surface. A cohomology class in their product XxY is an analytic correspondence, if it belongs to the subalgebra generated by Chern classes of coherent analytic sheaves. Let f be a Hodge isometry of the second rational cohomologies of X and Y with respect to the Beauville-Bogomolov-Fujiki pairings. We prove that f is induced by an analytic correspondence. We furthermore lift f to an analytic correspondence F between their total rational cohomologies, which is a Hodge isometry with respect to the Mukai pairings, and which preserves the gradings up to sign. When X and Y are projective the correspondences f and F are algebraic.

9. Sankar, Soumya (Ohio State University), Curve classes on conic bundle threefolds and applications to rationality.

   Abstract: Conic bundles are a geometrically rich class of varieties. In the 70's, Beauville showed that over an algebraically closed field, the group of algebraically trivial curve classes on a conic bundle threefold is isomorphic to the Prym variety of a double cover naturally associated with it. In joint work with Sarah Frei, Lena Ji, Bianca Viray and Isabel Vogt, we study curve classes on (geometrically standard and geometrically ordinary) conic bundle threefolds over arbitrary fields of odd characteristic. We then use the description of these classes to study the rationality of such varieties. Indeed, Hassett-Tschinkel and Benoist-Wittenberg introduced an obstruction to rationality, namely the intermediate Jacobian torsor obstruction, closely related to the structure of the group of curve classes on threefolds. We show that this obstruction is insufficient to characterize rationality.

10. (Zoom lecture) Kaushal Srivastava, Tanya (IIT Gandhinagar), Counting Twisted Fourier Mukai partners of an ordinary K3 surface.

    Abstract: The talk is based on joint work with Sofia Tirabassi. I will be discussing tame twisted K3 surface over an algebraically closed field of positive characteristic and counting its untwisted FM partners. On the way to the counting results, we will also discuss that every tame twisted Fourier Mukai partner of a K3 surface of finite height is a moduli space of twisted sheaves over it.

11. (Zoom lecture) Zilber, Boris (Oxford University), Arithmetic geometry through the eyes of model theory.

    Abstract: I am going to discuss a progress in an ongoing project (since approx 2000) which aims to formalise the notion of an analytic covering space of a complex algebraic variety in such a way that the formal cover is unique up to abstract isomorphisms (categorically axiomatised). It turned out deeply dependent on and related to both arithmetic geometry and transcendental number theory. Model-theoretic geometry presents aspects of both in an explicit and predictive format.

Lightning talks

There were four lightning talks given by the most junior of the workshop participants. The talks were given by

1. Andrew Kwon (University of Pennsylvania)
2. Martin Lüdtke (Rijksuniversiteit Groningen)
3. Jack Petok (Dartmouth College)
4. Libby Taylor (Stanford University)

Outcome of the Meeting

The workshop was structured to be forward-looking with plentiful time for informal discussion of research directions. This was further facilitated in a conscious way with structured activities included the organized discussions, the problem session, and of course the shared meals and coffee breaks. Together with the wonderful facilities at BIRS the workshop created a creative environment reinvigorating existing and beginning new collaborations. This workshop also provided an important forum for researchers working in areas (derived categories of coherent sheaves, anabelian geometry, and algebraic topology) that don't normally interact a lot to learn more about the connections between these fields and to establish professional connections to aid in future research.

Appendix. Problem List

The following is a list of problems discussed as part of the problem session, moderated by Max Lieblich on July 5, 2022, for the BIRS meeting: Derived Categories, Arithmetic, and Reconstruction in Algebraic Geometry.

The following summary is based on a notes of Brendan Hassett and Martin Olsson.

1. Given two K3 surfaces X and Y derived equivalent over a field K. Is it possible that $X(K) = \emptyset$ but $Y(K) \neq \emptyset$?

   Remark. Given X and (Y, α), where the latter is twisted by a Brauer class, there are counterexamples by Ascher-Dasaratha-Perry-Zhou.

   Remark. Given X and Y hyperkähler fourfolds of $K3^{[2]}$ type, there are counterexamples by Addington-Antieau-Frei-Honigs.

   General question: When is the existence of a K-point a derived invariant? How does this vary with K.

   The original question is known over a finite field K.

2. Let X be a smooth projective variety over a finite field k. What is meaning of the trace of Frobenius on the ℓ-adic Mukai lattice
$$\oplus_{n\in\mathbb{Z}} H^*(X, \mathbf{Q}_\ell(n))[2n]?$$
   This is a module over $\mathbf{Q}_\ell[\beta^{\pm 1}]$ where $\deg(\beta) = 2$ with cyclotomic action.

   Remark. See work of Toen and Vezzossi [11].

3. Is $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for smooth threefolds?

4. Is there a smooth projective X over k algebraically closed that is an "algebraic $K(\pi, 1)$" that is not obviously derived from curves or abelian varieties?

   Remark: Ball quotients don't seem to work as the cohomology of the group is different from its profinite completion.

5. Let C be a curve over an algebraically closed field. Is there a non-model-theoretic proof that the abstract Jacobian determines the $\mathbb{Z}$-scheme C.

   Related: Is there a version of the Rabinovich theorem without model theory.

6. What can one say about fields K for which $\mathrm{Gal}_K \simeq \mathrm{Gal}_{\mathbf{Q}}$.

   Is there a valuation w/divisible value group whose residue field is $\mathbf{Q}$?

7. It is known that in general one cannot reconstruct K from its Galois group Gal_K, where $K/\mathbf{Q}_p$ finite. One can have $\mathrm{Gal}_K \simeq \mathrm{Gal}_{K'}$ but $K \not\simeq K'$. Is there nonetheless some geometric object associated to K that is determined by the Galois group?

8. Given A and B abelian varieties over a field K. We know that A and B are derived equivalent if and only if

   $$A \times \hat{A} \simeq B \times \hat{B}.$$

   Suppose now that T and T' are torsors under an abelian variety A. What ensures that $D(T) \simeq D(T')$?

   Remark: Article of Antieau-Krashen-Ward covers genus one [1].

   How does this interact with translation by $H^1(A)$? Can we reduce to the case $T = A$?

9. Many hyperkähler manifolds can be constructed as moduli spaces of objects of Bridgeland stable objects in Kuznetsov components of Fano varieties. Are all hyperkählers like this? What is the geometric relationship between the Fano and the hyperkähler manifold?

   Example: Cubic fourfolds and varieties of lines, hyperkähler of $K3^{[2]}$-type.

   Blow up $\mathbf{P}^N$ along embedded hyperkähler?

10. If X and Y admit a filtered derived equivalence, are they birational? This means $D(X) \simeq D(Y)$ such that the induced

    $$\mathrm{CH}_{num}(X) \to \mathrm{CH}_{num}(Y)$$

    respects the codimension filtration.

    Evidence: assuming the standard conjectures then filtered derived equivalence implies an isomorphism of motives $M_X \simeq M_Y$.

11. Let X be a smooth projective geometrically rational variety over K. Suppose that X admits a full strong exceptional collection of line bundles over the base field. Does X have a K point?

    Orlov conjectures that X should be rational when there is a full exceptional collection, over an algebraically closed field. Elagin-Lunts say that there is a stratification by rational varieties.

    Ballard has obtained results for toric varieties.

12. Can the Kollár-Lieblich-Olsson-Sawin results be extended to weaker situations, e.g., X admitting an ample family of invertible sheaves?

    What about characteristic p? Can only hope to do this up to purely inseparable maps.

13. Recall a theorem of Voevodsky: normal varieties over finitely generated fields of characteristic 0 are determined by their étale topoi.

    What about characteristic p?

    Voevodsky says it goes through, without proof.

- field should have transcendence degree at least one;
- only recover things up to purely inseparable ambiguity?

It was suggested this should be accessible via “Bogomolov program” after passing to field extensions; this approach can only work in higher dimensions.

Participants

Ballard, Matthew (University of South Carolina)
Barwick, Clark (Edinburgh)
Bogomolov, Fedor (New York University)
Bragg, Daniel (University of California Berkeley)
Brakkee, Emma (University of Amsterdam)
Cadoret, Anna (Sorbonne Université)
Favero, David (University of Alberta)
Flapan, Laure (Michigan State University)
Frei, Sarah (Rice University)
Gaulhiac, Sylvain (University of Alberta)
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Rapinchuk, Igor (Michigan State University)
Sankar, Soumya (Ohio State University)
Srinivasan, Padmavathi (ICERM)
Stix, Jakob (Goethe-Universität Frankfurt)
Taylor, Libby (Stanford University)
Tevelev, Jenia (University of Massachusetts)
Tirabassi, Sofia (Stockholm University)
Topaz, Adam (University of Alberta)
Torres, Sebastian (IMSA-Miami / ICMS-Sofia)
Voloch, Felipe (University of Canterbury)
Zilber, Boris (Oxford University)

Bibliography

[1] Benjamin Antieau, Daniel Krashen, and Matthew Ward, Derived categories of torsors for abelian schemes, Adv. Math. **306** (2017), 1–23. MR 3581296

[2] Clark Barwick, Saul Glasman, and Peter Haine, Exodromy, arxiv:1807.03281, 2018.

[3] B. Bhatt, M. Morrow, and P. Scholze, Topological hochschild homology and integral p-adic hodge theory, Publ. Math. IHES **129** (2019), no. 3, 199–310.

[4] F. Bogomolov, M. Korotiaev, and Y. Tschinkel, A Torelli theorem for curves over finite fields, Pure Appl. Math. Q. **6** (2010), no. 1, Special Issue: In honor of John Tate. Part 2, 245–294. MR 2591191

[5] Fedor Bogomolov and Yuri Tschinkel, Reconstruction of function fields, Geom. Funct. Anal. **18** (2008), no. 2, 400–462. MR 2421544

[6] ———, Reconstruction of higher-dimensional function fields, Mosc. Math. J. **11** (2011), no. 2, 185–204, 406. MR 2859233

[7] Anna Cadoret and Alena Pirutka, Reconstructing function fields from Milnor K-theory, Algebra Number Theory **15** (2021), no. 9, 2261–2288. MR 4355474

[8] J. Kollr, M. Lieblich, M. Olsson, and W. Sawin, Topological reconstruction theorems for varieties, (2020).

[9] M. Lieblich and M. Olsson, Derived categories and birationality, arXiv:2001.05995, 2020.

[10] Florian Pop, On the birational anabelian program initiated by Bogomolov I, Invent. Math. **187** (2012), no. 3, 511–533. MR 2891876

[11] Bertrand Toën and Gabriele Vezzosi, Trace and Künneth formulas for singularity categories and applications, Compos. Math. **158** (2022), no. 3, 483–528. MR 4434529

[12] Adam Topaz, A torelli theorem for higher-dimensional function fields, `arXiv:1705.01084`, 2017.

[13] ———, Recovering function fields from their integral ℓ-adic cohomology with the galois action, arXiv:1910.03563, 2019.

[14] B. Zilber, A curve and its abstract Jacobian, Int. Math. Res. Not. IMRN (2014), no. 5, 1425–1439. MR 3178604

Appendix A

Building Networks: Women in Complex & Nonlinear Systems (22w5062)

September 18 - 23, 2022

Organizer(s): Nina Fefferman (University of Tennessee), Nancy Rodriguez (University of Colorado), Alexandria Volkening (Purdue University), Heather Zinn Brooks (Harvey Mudd College)

Overview

The purpose of this workshop was to bring together researchers with expertise in a wide range of mathematical tools and applications related to complex and nonlinear systems. We had three primary objectives related to building connections and collaborations, helping establish supportive professional communities, and highlighting the work of underrepresented scholars.

Our first goal was to **connect diverse perspectives to initiate new collaborations**. Mathematicians working on different problems in complex systems often use related techniques; nevertheless, because the applications involved are so broad, researchers who would benefit from talking to each other may not have the opportunity to interact. For example, many of the mathematical techniques used to understand intracellular transport, traffic flow, and content spread on social networks are similar, but researchers working on these topics often publish in different journals and attend different conferences. One first goal was, therefore, to connect mathematicians, physicists, and other scientists to build new collaborations between people working across a spectrum of application areas. To achieve this goal, our workshop featured talks from a diverse group of scholars who are representative of the interdisciplinary nature of complex systems. Crucially, the workshop also included structured breakout sessions where groups of attendees (in-person and virtual) discussed open problems and generated numerous research directions as a group. These groups have continued to meet and work on the research projects that started at BIRS.

Our second goal was to **empower early-career researchers and build mentorship networks**. We designed our workshop to increase the visibility, professional connections, and supportive mentorship community of all of our participants. Through breakout sessions, we provided organic opportunities for interaction between experts in the field, early-career faculty, postdocs, and graduate students. We were especially delighted to hear research ideas and projects proposed by early-career participants in our breakout discussions. Throughout the talks and breakout sessions in our workshop, we heard from many young researchers who actively shared their ideas, asked questions, and provided their expertise—it was clear that they felt empowered and valued.

The final goal our of our workshop was to **highlight and support the contributions of underrepresented scholars in complex systems**. Peer networks are both research-productive and critical to the success of underrepresented scholars. In particular, previous Women in Research networks (e.g., AWM research networks) have been very successful, resulting not only in many publications and long-term collaborations, but also in the revitalization of their participants. Alongside our BIRS workshop, we established an AWM Women in Complex and Nonlinear Systems research network. The BIRS workshop was the first time this network met, and the consensus was a clear revitalization of all participants. The time that we had at BIRS was a perfect way to kick off the AWM network, which we expect to continue to grow through future activities (e.g., building on the success of our kick-off meeting at BIRS, several of us are leading an AWM workshop at the 2024 SIAM Annual Meeting). This network will continue to be open to scientists of all gender identities and strive to support the research of underrepresented groups, increase their visibility, and build lasting communities. We are committed to including a broad group of women (binary, non-binary, and transgender) and their allies, representative of the complex-systems community. We especially strive to support researchers with intersectional experiences, including scholars of color, LGBTQIA scientists, and others who have been historically underserved.

Workshop Structure

We organized a five-day hybrid workshop with 27 participants (13 in-person participants and 14 virtual participants) representing a broad range of research areas from experimental physics, network science, fluid dynamics, collective dynamics, epidemiology and quantum information. Each morning we heard from three of the participants (with equal representation from virtual and in-person participants) and then had structured breakout groups in the afternoon. Below we give more details of our daily workshop activities:

- **Monday:** We kicked off our workshop with a wonderful and inspiring talk given by Karen Daniels. Karen's talk was followed by two excellent talks given by mid-career researchers.
  - Karen Daniels (North Carolina State University): **Building networks (In fact, I'm actually building networks)**
  - Daphne Klotsa (University of North Carolina, Chapel Hill): **A touch of non-linearity: mesoscale swimmers and active matter in fluids**
  - Katie Newhall (University of North Carolina, Chapel Hill): **Effective thermal equilibrium induced by crosslinking proteins in polymer chromosome model**

We began the afternoon by opening up the floor to the in-person and virtual participants to share any ideas about possible projects. We had one scribe who wrote down all relevant information about the discussion. After a vibrant and lively discussion a total of nine possible research directions where discussed. These were posed by participants from recent graduates to senior researchers. See section page 4 ("Additional Research Ideas Generated") for a full list of these research directions.

- **Tuesday:** Tuesday's three talks—by faculty and postoc fellows—were also extremely stimulating and engaging.
  - Moumita Das (Rochester Institute of Technology): **Soft mechanics and fracture properties of cartilage and cartilage-inspired soft network materials**
  - Tahra Eissa (University of Colorado, Boulder): **Learning efficient representations of environmental priors in neuronal networks**
  - Mari Kawakatsu (University of Pennsylvania): **Diversity and structure in complex social systems: case studies in political polarization & emergent hierarchies**

  In the morning we sent out a poll to participants to rank their top three projects from our discussion on Monday. Poll responses were due by noon. After lunch, we ranked the top three projects and assigned participants to those three. We had about an equal number of participants for each of the three projects. Please see section page 3 ("Collaborative Research Projects") for a description of these projects. For the last one and a half hours of the day, participants broke our into research groups. Each group had a separate working space and had significant participation from virtual members.

- **Wednesday:** On Wednesday we had three amazing talks and took the afternoon off to hike and enjoy the beauty that Banff has to offer. Our speakers, spanning faculty to postdoctoral-fellow career stages, are listed below:
  - Irina Popovici (US Naval Academy): **A rigorous approach to the dynamics of self-propelled swarms via a novel central manifold approximation technique**
  - Natalia Komarova (University of California, Irvine): **Evolutionary modeling of cancer: Protective effect of aspirin in colorectal carcinogenesis**
  - Alice Schwarze (Dartmouth College): **Connecting dynamics on and of networks to data - motif-based and mean-field approaches**

- **Thursday:** We began our Tuesday morning with two excellent one-hour talks delivered virtually:
  - Laura Miller (University of Arizona): **Using computational fluid dynamics to understand muscle driven movement by soft tissues and bodies: Case studies in tubular hearts and jellyfish**
  - Maria D'Orsogna (California State University, Northridge): **A mathematical model of reward-mediated learning in drug addiction**

  As a result of discussions at Banff, we noticed there was high interest in quantum information shared among the in-person participants. We thus added a talk on this topic to the schedule on Thursday. Our final talk of the workshop was given by Namrata Shukla from Banaras Hindu University, India.

- **Friday:** Most workshops participants were traveling on Friday, so we did not hold talks on this day. Several participants gathered at BIRS and worked on their projects or on organizing future mini-symposiums to continue building on the momentum initiated at BIRS.

Collaborative Research Projects

As a product of the discussions that we held among virtual and in-person participants on Monday, we generated many research ideas and selected three for further collaboration. Here we discuss these three ongoing projects.

- **Library and abortion deserts**: Inspired by food deserts, we want to better understand how access to libraries and books is spatially distributed. Libraries are a resource to which, in theory, everyone in the United States has access. There are records of where U.S. public libraries are located and how they are used (e.g., zip codes, number of annual visits, number of books in circulation, number of events hosted). How has access to library resources changed over the last few decades? How does this relate to changes in population? In an ongoing collaborative project that we initiated at BIRS, we are applying topological data analysis techniques, as well as other approaches, to better how access to libraries has changed. Our project builds on this **data set**.

  As a related project that illustrates how similar mathematical techniques can be applied to a broad range of complex-systems applications, we also want to look at access to abortion clinics. While it involves very different time scales, this project has some relationship to our above project on library access. It also includes different feedback dynamics that we must consider, and there are many questions that we have: Can we better understand how birthing centers and number of midwives are spatially distributed? How might medical specialties or doctors themselves change as a result of new abortion legislature? We are treating this project carefully, since we acknowledge that this application could attract (potentially critical) attention.

- **What makes a biological system both robust and resilient?** A system is robust if it is not easily damaged, and it is resilient if, when damaged, it can easily 'bounce back'. Robustness and resiliency depend on diversity, heterogeneity, and redundancy. We are interested in better understanding broad features of complex systems that promote robustness and resiliency from a network perspective. For example, there are studies of robustness for power grid networks, and rich work on resiliency across different fields. In neuroscience, how does one build a system that supports complex behaviors, but is flexible? Robustness could also mean that large parameter changes do not lead to qualitative dynamical changes or structural changes. Are there features of network topology that support robustness?

- **Are there interesting trends in networks-based models of influence and social power?** In social psychology, there are conceptual models of how people choose to act: socially derived or internally psychologically derived. There has not been a lot of work that connects models from social psychology with "complex contagion" models of social setting and network structure. It might be of interest to model either a specific system or build more conceptual models to connect these two perspectives. One application that we are interested in is related to work climate and diversity in academic departments. Some reasonable mathematical frameworks that could be used include ordinary differential equations, agent-based models, and game-theoretic perspectives.

Additional Research Ideas Generated

During our time at BIRS, our in-person and virtual participants held lively, stimulating conversations on possible research projects. We had one scribe who documented all of the ideas generated during these conversations. Here we give a brief description of each of the nine potential projects generated. We selected the three collaborative projects that we eventually pursued from this list:

- Library deserts (pitched by Alexandria Volkening): Inspired by food deserts, can we study library access across the United States?
- Abortion clinic access (pitched by Heather Zinn-Brooks): With the recent U.S. supreme court decision to overturn Roe versus Wade, access to abortion clinics will be made difficult in some states. Can we better understand abortion access across the United States?
- Robustness and resilience (pitched by Moumita Das): Can we study robustness and resilience in general biological networks?
- Fitting data to co-evolving dynamics (pitched by Alice Schwarze): Can we study the interplay between co-evolving models of opinion dynamics and game-theoretic models?
- Network-based models of influence and social power (pitched by Nina Fefferman): Can we develop network models to study influence and social power?
- Departmental service in universities (pitched by Nancy Rodriguez): Can we do a massive data-collection effort to obtain information on equity of service across academic departments in the United States?
- Nested hierarchical networks (pitched by Namrata Shukla): Can we study the flow of information from larger graphs to subgraphs and vice versa?
- Approaching failure transition by becoming more or less consistent (pitched by Karen Daniels): Can we develop toy models that lead to failure in these two (seemingly contradictory ways) of becoming more or less consistent, by changing parameters?
- Network dynamics under constraints (pitched by Mari Kawakatsu): Can we study dynamics on networks, but with constraints?

WICANS Norms

It is important for our research and mentorship network to enable healthy and fruitful collaboration. To ensure this, we had a discussion at BIRS about some important norms that we aim to follow. With regards to any publications that come out of this work, it has been established that everyone in the research group will be an author of the paper. Moreover, we discussed the problem that women's papers are cited less frequently, and we pledged to cite each others work when it is relevant. Furthermore, we are ambassadors of each others' work and will share any new papers and results from members of our network to other colleagues and networks.

Statistics

Our workshop included active participation virtually and in person, with an emphasis on providing unstructured time for collaboration building. Five speakers presented their research virtually, and six speakers shared their work in person. The workshop participants spanned career stages, with seven full professors, six associate professors, seven tenure-track assistant professors, six postdoctoral researchers (including several in the first months of their positions), and one professional outside academia. Our workshop drew participants from four countries, and included five researchers from primarily undergraduate institutions. Based on our identifications, there were 27 women participants and five of these participants also belonged to underrepresented minority communities.

Conclusion

Our workshop accomplished the three goals that we set out to achieve. We connected women from various disciplines though our research project groups. We also empowered early-career researchers by giving them a forum to share their work and share their research ideas, in an environment that valued their expertise and served as a supportive community. A forum was also provided to underrepresented scholars. Most importantly we enjoyed a wonderful and inspiring week at BIRS, with great science, great conversations, and wonderful hikes. Our participants reported leaving their time at BIRS feeling re-energized. We are so grateful to have had the opportunity to spend a week at BIRS and are indebted to all of the BIRS staff who made the week a success.

Participants

Bañuelos, Selenne (IPAM, UCLA & Cal. State Univ. Channel Islands)
Barbaro, Alethea (TU Delft)
Brooks, Heather Zinn (Harvey Mudd College)
Cook, Keisha (Clemson University)
D'Orsogna, Maria Rita (Calfiornia State University Northridge)
Daniels, Karen (North Carolina State University)
Das, Moumita (Rochester Institute of Technology)
Eissa, Tahra (University of Colorado Boulder)
Espanol, Malena (Arizona State University)
Fefferman, Nina (University of Tennessee)
Feffermen, Nina (University of Tennessee)
Hill, Kaitlin (St. Mary's University)
Hill, Kaitlin (St. Mary's University)
Hoffmann, Franca (Rheinische Friedrich-Wilhelms-Universität Bonn)
Jilkine, Alexandra (University of Notre Dame)
Kawakatsu, Mari (University of Pennsylvania)
Klotsa, Daphne (University of North Carolina)
Komarova, Natalia (University of California Irvine)
Miller, Laura (University of Arizona)
Newhall, Katie (University of North Carolina at Chapel Hill)
Pasha, Mirjeta (Arizona State University)
Popovici, Irina (US Naval Academy)
Rodriguez, Nancy (University of Colorado at Boulder)
Schwarze, Alice (Dartmouth College)
Shukla, Namrata (Banaras Hindu University, India)
Silber, Mary (University of Chicago)
Towers, Sherry (Institute for Advanced Sustainability Studies)
Tymochko, Sarah (UCLA)
Volkening, Alexandria (Purdue University)

Appendix B

Noncommutative Geometry and Noncommutative Invariant Theory (22w5084)

September 25 - 30, 2022

Organizer(s): Jason Bell (University of Waterloo), Chelsea Walton (Rice University)

We are grateful to BIRS for supporting this five-day workshop, which brought together 22 in-person researchers (with an additional 22 virtual participants). The participants included a mix of PhD students, postdocs, and senior researchers from North America, Europe, and Asia. As a group, the participants had diverse research interests, which include noncommutative geometry, representation theory, the study of Hopf algebra actions, quantum groups, and other areas.

Overview of the field

Noncommutative algebra is a rich and diverse field that has influences rooted in algebraic and differential geometry, representation theory, algebraic combinatorics, mathematical physics, and other areas. Rapid developments in noncommutative algebra, especially in noncommutative algebraic geometry, influence many other mathematical disciplines. We briefly give an overview of some of the main trends that shape our field at this time, with an emphasis on the areas represented during the workshop.

In terms of the scope of the meeting, the main focus involved the following four connected sub-areas, for which we now give a quick overview.

(a) **Noncommutative Invariant Theory**

Classical Invariant Theory enjoys a long history, beginning with seminal work of Cayley, Hilbert, E. Noether, and others. Here, one examines subrings of polynomials that remain invariant under group actions. Much of this theory can be extended to the quantum setting where the polynomial ring is replaced by a suitable noncommutative analogue (typically an Artin-Schelter regular algebra or the quantized coordinate ring of a variety). One can also work more generally with actions of Hopf algebras instead of restricting to group actions. The key motivating question in this field is then: how much of the classical theory can be lifted to this general setting? This field has been a hugely active area of study and has undergone rapid advancement over the past decade, with many important results being proved in this time (see, for example, [27, 28, 29]). Key recent results in this area of research include work of Ferraro, Kirkman, Moore, and Peng [31], which looks at obtaining Noether's bound for noncommutative rings, and a noncommutative version of Knorrer's periodicity theorem by Conner, Kirkman, Moore, and Walton [25] (see also the work of Mori and Ueyama [46]). Moreover, noncommutative Kleinian singularities were defined recently in work of Chan, Kirkman, Walton, and Zhang [19, 20] via their development of a noncommutative McKay correspondence. Also, a version of Auslander's theorem that arises in the McKay correspondence has been studied in the noncommutative setting by several authors, including the work of Bao, He, and Zhang [6]; the work of Gaddis, Kirkman, Moore, and Won [32]; the work of Crawford [26]; and the work of Buchweitz, Faber, and Ingalls [12].

(b) **Artin-Schelter Regular Algebras: classification and applications**

Artin-Schelter regular (often abbreviated as AS regular) algebras are in some natural sense noncommutative analogues of polynomial algebras, and for this reason they play an integral role in both noncommutative invariant theory (see, for example, [38]) and in noncommutative geometry. In the three-dimensional case, these algebras were classified by Artin and Schelter [2] and Artin, Tate, Van den Bergh [5], and the resulting algebras have since appeared within many different contexts, including questions in noncommutative invariant theory and questions about the classification problem for Artin-Schelter regular algebras of dimension. In this case, some partial progress has been made by Lu, Palmieri, Wu, and Zhang [45], using techniques involving the A_∞-Koszul Dual.

New examples of Artin-Schelter regular algebras have also been given by Chirvasitu, Kanda, and Smith [21] and by Cassidy and Vancliff [14], in which the latter authors use skew Clifford algebras, with the aim of creating new AS-regular algebras. Further new interesting directions have also recently emerged. In particular, the noncommutative Zariski cancellation problem is now an active area of research with work of Bell and Zhang [8]; Lezama, Wang, and Zhang [42]; Bell, Hamidizadeh, Huang, and Venegas [7]; and Tang, Venegas, and Zhang [54], who have considered the classical Zariski cancellation problem in the more general context of Artin-Schelter regular algebras. Recent work of Walton and Zhang [56] looks at Artin-Schelter regularity of the quadratic dual of the Fomin-Kirillov algebras, which has important connections to algebraic combinatorics and provides a new approach to resolving a twenty-year-old conjecture of Fomin and Kirillov.

(c) **Universal Quantum Groups**

Given an algebra, it is natural to ask is whether there is a universal object that controls the symmetries of the algebra. In the context of Hopf algebra coactions, this entity is a universal quantum group and these have proved to be profoundly useful in both the purely algebraic and C^*-algebraic settings. Many of these quantum groups that arise can in fact be realized as deformations of coordinate rings of affine algebraic groups, and can thus be shown to possess good algebraic properties. Recent work in this direction includes that of Chirvasitu, Walton, Wang [24]; that of Huang, Walton, Wicks, and Won [37]; and of Chakraborty and Saurabh [15], which studies the Gelfand-Kirillov dimension of cosemisimple universal quantum groups.

(d) **Noncommutative projective geometry and application of geometric methods in noncommutative algebra**

Noncommutative projective geometry first emerged as a distinct discipline during the late 80s, as part of the work of Artin, Schelter, Stafford, Tate, Van den Bergh, Zhang and others [4, 5, 2, 3, 57], who applied techniques from algebraic geometry to understand the Sklyanin algebras and related rings. The field has grown significantly since these beginnings and there have been many striking applications of geometric methods in noncommutative algebra over the past ten years, including work of Sierra and Walton [53] proving that the enveloping algebra of the Witt Lie algebra is non-noetherian; work of Chan and Ingalls [17, 18] on the "noncommutative minimal model program," work of Chirvasitu, Kanda, and Smith [21, 22, 23], and work of Rogalski, Sierra, and Stafford [48, 49, 50, 51] on the theory of birationally commutative surfaces. The chief open problem in the area is Artin's Conjecture [1], which gives a proposed birational classification of all noncommutative surfaces, and important recent progress towards this conjecture has been made by Faber, Ingalls, Okawa, and Satriano [30].

Presentation Highlights

The lectures delivered over the week of the workshop were uniformly of high quality, and many of the participants shared their experiences with us. We now include some of the highlights from these presentations. In particular, we are indebted to Lucas Buzaglo, Fabio Calderón, Jason Gaddis, Hongdi Huang, Frank Moore, Van Nguyen, Manny Reyes, Kent Vashaw, Padmini Veerapen, and Xingting Wang for sharing their thoughts with us about several presentations from the workshop.

- The opening talk of the workshop was delivered by James Zhang, who gave a particularly inspiring survey of open problems. The stated criteria for problems to be included in the list (aside from being of personal interest to the speaker) were: that the problems have significant consequences if they are answered, that they be motivated by fields of mathematics outside of noncommutative algebra, and that they have a strong influence on the development of the subject. It is noteworthy that most of the problems listed were proposed after 2000.

  After an opening discussion of important sources of noncommutative algebra, the first groups of problems centred on topics related to Hopf algebras. First came several problems about the classification of Hopf algebras of small GK-dimension. The next group of questions were clustered around homologically-flavoured questions related to Hopf algebras. This included the Brown-Goodearl conjecture of whether an affine noetherian Hopf algebra has finite injective dimension, along with several variants thereof. It also included the Etingof-Ostrik conjecture about finite generation of Ext-algebras of finite tensor categories, which includes as a special case the finite generation of the Ext-algebra of a Hopf algebra.

  At this point the talk shifted to questions related to Artin-Schelter (AS) regular algebras. The first set of questions related to actions of Hopf algebras on AS regular algebras. This included questions about when the associated Auslander map is an isomorphism, about classification of Hopf algebras acting on a given AS regular algebra, and classification of AS regular algebras acted upon by a given Hopf algebra. Then questions related to the Ozone group (recently defined by Chan, Gaddis, Won, and Zhang) of a AS regular algebra satisfying a polynomial identity were discussed.

  The survey concluded by recounting Artins Conjecture [1] about the classification of finitely generated division algebras of transcendence degree 2 and providing a long list of questions that were not able to be discussed in the allotted time.

  This excellent talk will serve as a rich source of kindling to fuel a wide range of future research within the noncommutative algebra community for many years to come.

- Ellen Kirkman also gave an excellent talk, titled "Homological Regularities," based on her recent joint work with Robert Won and James Zhang [40]. Her talk gave an overview of homological measures for noncommutative graded algebras, especially those pertaining to regularity. The various measures highlighted were "Tor-regularity," "Castelnuovo-Mumford (CM) regularity," "AS regularity," "concavity," and weighted versions, with the three latter invariants being introduced by Kirkman, Won, and Zhang [40].

  In particular, Tor-regularity, first studied by Jrgensen, is used to study how far an algebra is being Koszul. One can also consider local cohomology modules and Castelnuovo-Mumford (CM) regularity to study

Koszulity. These measures have a powerful application in finding bounds on degrees of generators of invariant rings in noncommutative invariant theory, particularly for actions of semisimple Hopf algebras acting on Artin-Schelter regular algebras.

A new measure, called “AS regularity”, can also be defined in terms of Tor-regularity and CM-regularity, equal to zero precisely when the algebra is AS-regular. In fact, there is an example of a 3-Koszul Artin-Schelter regular algebra that has Tor-regularity 1 and has CM-regularity -1, motivating the definition of AS regularity as being the sum of the two measures. The next measure introduced in the talk, “concavity,” is given in terms of CM-regularity, and is zero for an noetherian Artin-Schelter regular algebra precisely when the algebra is Koszul. The name pertains to how far the corresponding noncommutative space is from being flat. Concavity can be used to show when certain noetherian Artin-Schelter regular algebras can be an invariant ring from an action of a semisimple Hopf algebra.

Lastly, weighted versions of both Tor-regularity and CM-regularity were used to refine the measures above. One application is that it provides a measure of how far a Veronese subring is from being Koszul. We expect that there will be numerous other applications of the measures introduced in Kirkman-Won-Zhang’s work to understand noncommutative graded algebras in the near future.

- Xingting Wang gave a talk, “Twists of graded Poisson algebras and related properties,” which described his joint work with Xin Tang and James Zhang [55]. Wang began by defining when a set of graded derivations of a graded Poisson algebra forms what he calls a “Poisson Twisting System.” The talk presented the implications of the existence of such systems and gave methods for their construction for multivariate Poisson polynomial algebras.

  The first result presented made use of the notion of divergence of a smooth Poisson algebra, with an explicit formula given for it in the multivariate Poisson polynomial case; this led to the introduction of “semi-Poisson derivations, which are in turn a generalization of the classical notion of Poisson derivations. Importantly, the authors show that these new objects, much like their classical counterparts, form a Lie algebra. From here, Wang presented necessary and sufficient conditions on a set of graded derivations of a multivariate Poisson polynomial algebra to form a Poisson twisting system, which yield new Poisson algebras and which the authors call the “twisted Poisson algebras associated to multivariate Poisson polynomial algebras.

  Using the theory of semi-Poisson derivations, Wang then defined the notion of “rigidity for graded Poisson algebras, which in some sense measures how far an algebra is from being unimodular (that is, having trivial modular derivation) and showed implications of rigidity taking certain values. Wang ended the talk with a specific example: the Poisson polynomial algebra on three variables (together with a unimodular Poisson structure induced by a cubic polynomial) and illustrated how the concepts introduced in the talk relate to the Poisson centre, rigidity, and the Ozone group.

- Michael Wemyss gave an enlightening talk, “Local forms of noncommutative functions,” which discussed his recent joint work with Gavin Brown [9]. Wemyss’ talk focused on how one can adapt singularity theory to the noncommutative setting. In the commutative case, when studying isolated singularities, one often works with the completed local ring, invokes the Cohen Structure theorem, and then works with a multivariate power series algebra. A primary goal in this context is to study the classification problem among elements in this ring, or to determine when such a classification is impossible. Simple singularities are classified according to the classical ADE classification.

  Working in the more general noncommutative setting, one can now consider the formal noncommutative power series ring; i.e., the completion of the free associative algebra, On first glance, one might naively assume that this algebra is highly pathological. Surprisingly, Weymss makes the case that the process of completion introduces a huge group of units, and since these units are typically non-central, one can use inner automorphism to make changes of variables, which enable Brown and Weymss to obtain a striking analogue of the commutative theory. The classification problem in this setting is to determine when two elements of the completed free algebra are isomorphic; that is, when their corresponding Jacobi algebras are isomorphic. The Jacobi algebra is a local ring and by studying the growth rate of the filtration corresponding to its Jacobson radical, one obtains a new invariant, which is better behaved than Gelfand-Kirillov dimension in this setting.

Wemyss proposes that a classification is both possible and is ADE, and he uses the theory of preprojective algebras to directly relate the general problem to the classical ADE classification. Wemyss also argued that these "noncommutative function algebras" arise naturally through a theory of contractions and he defines a contraction algebra. Weymss finished by stating conjectures regarding how the classification of contraction algebras should correspond to that of Jacobi algebras. These problems will undoubtedly lead to further investigations and should reveal additional surprising connections between noncommutative algebra, algebraic geometry, deformation theory, and representation theory.

- Wendy Lowen gave a talk entitled "Enriching the nerve construction," based on her joint with Arne Mertens [44], which links noncommutative geometry and higher category theory, via the nerve functor. Her talk looked at quasi-categories over a monoidal category (e.g., the category of modules over a commutative ring), which are in some sense relaxations of dg-categories. The chief underlying question investigated during this talk was: for linear categories can we define a nerve taking values in modules rather than sets?

  The solution, starting with a monoidal category, is to construct a certain quiver on that category, which leads to the theory of templicial objects. From here, one can define a nerve functor from a category over the monoidal category into the category of templicial objects. The main theorem presented showed that this construction is compatible with the classical setting, and several concrete examples with diagrams were given during this talk, which provided additional motivation and provided greater understanding of the ideas involved in this work.

- Daniel Chan gave an inspiring talk on the noncommutative minimal model program for orders on surfaces. In the setting of classical algebraic geometry, the minimal model program (MMP), introduced by Mori in the late '80s, is a significant organizing paradigm, which has since had a revolutionary impact on the study of threefolds and higher-dimensional algebraic geometry. Mori's original motivation was to construct a birational model of a complex projective variety which is as simple as possible (or, in some natural sense, minimal), and has its origins in the work of Enriques and Castelnuovo giving a birational classification of surfaces.

  A Noncommutative version of this program was initiated by Chan and Ingalls [17] for orders on surfaces over a field (geometric surfaces) and by Chan et al. [16] in higher dimensions. Chan discussed joint work with Ingalls [18] on the MMP for orders on arithmetic surfaces X. This noncommutative theory can also be viewed purely algebro-geometrically, as algebraic geometry enriched by a Brauer class β, and Chan adopted this point of view during his talk. In this abstract setting, an arithmetic surface is taken to be a normal, separated, integral two-dimensional excellent scheme X which is quasi-projective over a noetherian affine scheme and has finite residue fields. Chan and Ingalls show in the case of prime index Brauer class that terminal resolutions exist. In addition, Chan and Ingalls give a classification of terminal singularities and Castelnuovo contractions where surprising new phenomena show up. Whereas for both the commutative theory of arithmetic surfaces and for orders over geometric surfaces, the theory closely mimics the classical situation arising in the work of Castelnuovo and Enriques, in this broader setting we find that if $\beta \neq 0$, regularity of X is neither a sufficient nor necessary condition for being terminal, and Castelnuovo contractions may or may not correspond to blowing up closed points even when X is regular.

- Charlotte Ure gave the talk “Twisting Comodules and Preregular Forms,” which gave an overview of her interesting joint work with Hongdi Huang, Van Nguyen, Kent Vashaw, Padmini Veerapen, and Xingting Wang [34]. Given a graded algebra A, there are numerous ways to change (or “twist”) the multiplicative structure to produce a new algebra defined on the same underlying vector space. The most common such technique is via a graded automorphism and is commonly called the Zhang twist [57]. In the case when H is a Hopf algebra, one may also twist the multiplication structures of H using a 2-cocycle defined on H. Ure’s talk analyzed the effect performing such twists has on several important constructions from (co)representation theory and classification problems. More precisely, Manin introduced the universal quantum group of A, which is a Hopf algebra that universally (right) coacts on an algebra A. The authors’ first result is that the twist of the universal quantum group of A by a 2-cocycle is the universal quantum group of a cocycle twist of A by the inverse of the original 2-cocyle.

  Additionally, Dubois-Violette showed that every N-Koszul Artin-Schelter regular algebra can be expressed as a derivation-quotient algebra. Motivated by this result, the authors show that the Zhang twist of such a derivation-quotient algebra can again be expressed as a derivation-quotient algebra with respect to a twisted version of the subspace used to define the original algebra. There were several other similar results discussed in the talk, which illuminated the behaviour of fundamental constructions under twisting operations. These new results will certainly be useful to researchers in the area.

- Robert Won, gave a presentation, titled “PI skew polynomial rings and their centers, which discussed currently ongoing work with Kenneth Chan, Jason Gaddis, and James Zhang. This talk gave interesting new results in noncommutative invariant ring theory, with a view towards understanding the centre of a skew polynomial ring S that is an Artin-Schelter regular algebra of finite global dimension and finite Gelfand-Kirillov dimension. This is approached via the study of the Ozone group of S, a subgroup of the full graded automorphism group consisting of those that are inner. The authors show that the Ozone group has a nice description involving only diagonal automorphisms. By its definition, the Ozone group fixes the centre of S and so properties possessed by the centre can be reinterpreted in the setting of noncommutative invariant theory.

  Kirkman, Kuzmanovich and Zhang [39] earlier had proved several invariant properties under group actions. Won’s talk developed a parallel theory for the invariant theory of skew polynomial rings in relation to the Ozone group. In particular, the centre of such an algebra is regular if and only if the Ozone group is generated by reflections; and the centre is Gorenstein if and only if the Ozone group, modulo the normal subgroup generated by reflections, acts with trivial homological determinant.

  To give sufficient conditions for these desirable properties to hold, the authors introduce new algebraic invariants, including “Ozone Jacobians, “Ozone arrangements, and “Ozone discriminants. These notions coincide precisely with the corresponding notions of the Jacobian, arrangement, and the discriminant when the center is regular, introduced by Kirkman and Zhang [41]. These new invariants are then used to characterize explicitly when the Auslander map is an isomorphism, when the centre is Gorenstein, and when the centre is regular. These are among the most desirable algebraic properties in the context of noncommutative invariant theory.

  These new Ozone invariants will inevitably be extremely useful in both theoretical and computational settings when understanding Artin-Schelter regular algebras satisfying a polynomial identity.

- Lucas Buzaglo, a PhD student at the University of Edinburgh, gave a fascinating talk on the universal enveloping algebras of Krichever-Novikov algebras [13]. The theory of enveloping algebras of finite-dimensional Lie algebras is today well-understood, due to classical work of Dixmier and others. For infinite-dimensional Lie algebras, however, the theory of their enveloping algebras is murkier and little is known outside of some important families. A fundamental question in this vein is whether the enveloping algebra of an infinite-dimensional Lie algebra is necessarily non-noetherian. For years, the best hope for a potential counterexample was the enveloping algebra of the positive part of the Witt Lie algebra; this was shown to be non-noetherian, however, by Sierra and Walton [53] via methods from noncommutative projective geometry.

An important class of Lie algebras comes from taking an irreducible complex affine variety V of positive dimension and looking at its vector fields. Bugzalo shows that enveloping algebras for all Lie algebras arising in this manner are necessarily non-noetherian. Remarkably, the question reduces completely to the case when V is a curve, which can in turn be handled by reducing to the work of Sierra and Walton via a sequence of reductions making use of clever faithful flatness arguments.

- Van Nguyen gave the talk "Tensor representations of finite-dimensional Hopf algebras," which gave an overview of her joint work with Benkart, Biswal, Kirkman, and Zhu [10, 11]. Her talk focused on two distinct ways of studying tensor representations of finite-dimensional Hopf algebras. The first method involved studying McKay matrices associated to finite-dimensional Hopf algebras, which encode the relations for tensoring simple modules with a particular representation of the Hopf algebra. The second part of the talk focused on the study of the centralizer algebra of tensor representations of quasi-triangular Hopf algebras.

  The main motivation for studying McKay matrices is to find a relationship between them and an appropriate analogue of the character table for finite-dimensional Hopf algebras. Such a relationship was studied by Witherspoon in the late '90s for finite-dimensional almost cocommutative Hopf algebras, where it was shown that the characters provide eigenvectors for the McKay matrix. Several interesting results concerning McKay matrices were presented during the lecture. Of particular interest are certain identities involving the trace of the action of the Hopf algebra on the simple modules. This gives rise to the notion of left and right eigenvectors for the McKay matrix and allows the authors to recover a result of Grinberg, Huang, and Reiner as a special case.

  The study of centralizer algebras mainly focused on the case where the Hopf algebra is the Drinfeld double of a Taft algebra, which is an example of a non-semisimple quasitriangular Hopf algebra. The main result in this direction is that there is an injective algebra homomorphism from some Temperley-Lieb algebra to the centralizer algebra of a tensor power of a 2-dimensional module. This is an isomorphism for small values of the tensor power, giving a full description of the centralizer algebra. There will undoubtedly be more interesting future work done in further understanding how eigenvectors of the McKay matrix relate to the character theory of Hopf algebras.

- Dan Rogalski gave the talk "Results on infinite-dimensional weak Hopf algebras," which discusses his joint work with Rob Won and James Zhang [52]. This was a very clear presentation, which discussed the advancement of a program to prove analogues of the properties of finite-dimensional Hopf algebras in the setting of infinite-dimensional weak Hopf algebras. In particular, to prove: (1) the existence of (unique) integrals, which are realized as hom spaces in the category of modules for the Hopf algebra; (2) the Gorenstein property; and (3) the existence of a Nakayama automorphism and associated winding automorphism.

  Weak Hopf algebras can be motivated from the direction of monoidal categories. Hopf algebras occur precisely, via Tannakian reconstruction, from tensor categories with fibre functors to the category of vector spaces. On the other hand, when the category of vector spaces is replaced by bimodules for a finite-dimensional semisimple algebra, one reconstructs a weak Hopf algebra whose representation theory gives the original tensor category. Motivation for studying weak Hopf algebras also arises from recent work of Huang, Walton, Wicks, and Won [37], in which they arise from considering natural algebraic structures co-acting on a path algebra modulo relations. Yet another motivation for the study of weak Hopf algebras comes from their flexibility: while Hopf algebras are not closed under direct sum, weak Hopf algebras are, and so these algebras have become increasingly important objects of study within noncommutative algebra.

On the other hand, the algebraic definitions of weak Hopf algebras are sometimes difficult to work with. Therefore, generalizations of (1), (2), and (3) above are desirable. For Noetherian infinite-dimensional Hopf algebras, a driving research question was the Brown-Goodearl conjecture, which states that an infinite-dimensional Noetherian Hopf algebra has finite injective dimension, and is Artin-Schelter Gorenstein. A partial analogue of this conjecture was recently proven in the weak Hopf setup by Rogalski, Won, and Zhang [52]. In particular, if a Noetherian weak Hopf algebra is finite over an affine centre, then it is proved to have finite injective dimension and to decompose as a direct sum of Artin-Schelter Gorenstein algebras.

Work in progress on the existence of integrals for infinite-dimensional weak Hopf algebras was also discussed. This approach builds on work of Lu, Wu, and Zhang [42], where integrals for infinite-dimensional Gorenstein Hopf algebras were given via Ext groups (as opposed to usual Hom groups in the finite-dimensional setting).

The novel results presented in this talk will serve to significantly advance the theory in an interesting direction. Since Hopf algebras have had major applications in representation theory, algebraic geometry, and topology, I expect that these results will motivate similar applications for weak Hopf algebras.

- Evelyn Lira-Torres, a PhD student, gave a talk called "Quantum Riemannian Geometry on the Fuzzy Sphere," which was based on her joint work with Majid [43]. This talk was unique in that it was at the interface of quantum physics and noncommutative algebra. The quantum spacetime hypothesis states that spacetime is not a continuum, and if one accepts this hypothesis, spacetime can then be effectively described via the use of a noncommutative coordinate algebra. In this talk, the speaker followed this line of thought and looked at the "fuzzy sphere," which is defined to be an algebra A which is the enveloping algebra of the angular momentum Lie algebra factored out by an ideal generated by a single relation. The goal of this talk was to explore the quantum Riemannian geometry of the fuzzy sphere, and this talk gave an interesting glimpse into connections between noncommutative algebra and mathematical physics.

- Jason Gaddis spoke on "Pointed Hopf actions on quantum generalized Weyl algebra," in joint work with Robert Won [GW99]. Gaddis' talk dealt with Hopf actions of pointed Hopf algebras on $\mathbb{Z}$-graded algebras. A special case of this had been looked at earlier by Cuadra, Etingof, and Walton [28], who showed that actions of semisimple Hopf algebras on Weyl algebras (which are $\mathbb{Z}$-graded) necessarily factor through group algebras. In light of the work of [28], it is natural to ask what happens for Hopf algebra actions on generalized Weyl algebras.

  The main result presented in this talk was the classification of inner-faithful actions of generalized Taft algebras on quantum generalized Weyl algebras which respect the $\mathbb{Z}$-grading. Towards the main result, other interesting questions were studied, notably <u>which cyclic subgroups of automorphism groups of generalized Weyl algebras are in fact realizable as restrictions of an inner-faithful action to the group of group-like elements?</u>

  Gaddis closed by computing the invariants for inner-faithful actions of Taft algebras on generalized Weyl algebras under Taft algebra actions and showed generically that these invariant rings are commutative rings whose associated graded rings are Kleinian singularities.

A complete list of talks can be found at the BIRS website.

Scientific Progress Made and Outcome of the Meeting

As mentioned at the beginning of this report, the workshop was a hybrid meeting, which had 22 in-person participants with various research interests connected to noncommutative algebra. As a result of this meeting, researchers working in different areas of noncommutative algebra were able to engage in often-fruitful discussions and many new projects were started during this workshop. This meeting had several younger researchers, postdocs, and graduate students and this meeting was particularly useful in allowing them to network and create new joint projects.

Additionally, several participants noted that this was a great workshop with many inspiring talks and new opportunities for collaboration, and many were appreciative of the return for face-to-face discussions. The program had several lectures highlighting recent progress on difficult problems and we include some testimonials from participants about research programs that they started at Banff and thoughts on results disseminated during the lectures.

First, several key advances in the field were disseminated during the workshop. Notably, Dan Rogalski discussed his recent proof of the Brown-Goodearl conjecture for module-finite weak Hopf algebras, with Robert Won and James Zhang [52]. In particular, they show for an affine weak Hopf algebra that is a finite module over its centre, one obtains several desirable features: finite injective dimension and Artin-Schelter regularity.

Hongdi Huang, Van C. Nguyen, Charlotte Ure, Kent B. Vashaw, Padmini Veerapen, and Xingting Wang recently began a sequence of collaborations [34, 35, 36] that began through an AIM-square. The authors all gave talks during the conference, with several of them discussing key parts of their recent accomplishments. Padmini Veerapen noted that the fact all of her collaborators were present allowed them to meet "on some nights and made good progress."

Padmini Veerapen added further remarks about how invaluable the BIRS environment and opportunity to meet with others was to her, saying, "I'm currently working on three projects and I was able to make good progress on all three while being at BIRS. With my teaching load, it can be hard to get research work done during the fall semester." In addition to the above-mentioned work with Huang and others, which is her first major project, she noted about the second project, "my collaborator and I spent a lot of time discussing background material with Dan Rogalski—I know that [for] some of the stuff Dan told us, it would have taken 2–3 months to figure this out on my own from books." While about her third project she said, "I was excited by the work of Kenta Ueyama and asked him if he would be willing to add something to a survey of twists paper that I'm writing and he happily accepted! I've only heard of Kenta and never met him but have always been super impressed by this work."

Charlotte Ure noted that she had a great and productive time at the workshop, saying, "I had many useful conversations with the other participants and got some new ideas for my current research projects. In particular, I've been working on a question related to the period-index problem on Jacobians that I was able to discuss. Additionally, as I'm usually collaborating over Zoom with Hongdi Huang, Van Nguyen, Padmini Veerapen, Kent Vashaw, and Xingting Wang, it was very productive to come together in person. We received some input and new questions after our talks that we've been discussing since."

Hongdi Huang noted that this workshop provider her with "a great chance for us to communicate mathematics in person, which is beneficial for our continuing collaboration and promoting our friendship as well." She also observed that there were many great and inspirational talks, and added, "during this conference, my collaborators ... obtained some new ideas on how to develop our future research direction. We also learned some new tools which are very useful to our current project."

James Zhang said, "I enjoyed the talks and the conversations with many participants. During the workshop Dan Rogalski and I were wondering if there are higher-Koszul AS algebras with finite Gelfand-Kirillov dimension. Now we might have some ideas to show that there is no N-Koszul AS regular algebra with finite (and large enough) Gelfand-Kirillov dimension for global dimension up to 8. I feel that in-person discussions are extremely helpful."

Ryan Kinser said that he "made several connections which contributed positively to my research program and career development during the meeting. One was the opportunity to speak with senior colleagues about their experiences writing and reviewing NSF research grant proposals. I received guidance which led to a higher quality proposal and a clearer idea of what is likely to be funded. Another was initial discussions with Matt Satriano about future collaboration (also with Jenna Rajchgot) to prove a 2009 conjecture of Anders Buch on equivariant K-classes of 'quiver cycles'. We plan to further collaborate on this, and explore the possibility of submitting a Research in Teams proposal to Banff in 2023."

Xin Tang wrote the following: "Though I have watched and been influenced by many recorded talks in BIRS, it is the first time that I am finally visiting BIRS. I feel lucky to be one of the participants, and I really enjoyed the 'view'. There are many great talks during the workshop, especially several of which are coming from an impressive collaborative group. I liked the talk 'Some Open Questions in Noncommutative Algebra' by James Zhang, which gives a wonderful survey of the current status. Milen Yakimov's talk 'Azumaya Loci of Root of Unity Quantum Cluster Algebras' is fascinating as his many other talks. These are just some examples, and there is much more to be said about many other illuminating talks. I have also had the chance to have extensive discussions with my collaborators about our projects concerning Poisson algebras. Motivated by the energetic talk 'Local Forms of Noncommutative Functions' by Michael Wemyss, we are led to consider a small addition in our current ongoing project. Everything is so convenient and well-organized. It had been a terrific week for me, and I hope that I will be able to visit BIRS again."

Jason Gaddis said, "I want to begin by thanking the organizers for the opportunity to attend, and the staff at BIRS for facilitating a very successful meeting. It was refreshing to be back in person. I very much appreciate the number of interesting talks and the ability to converse and share ideas with colleagues. One recurrent theme that stuck out to me was the idea of twisting in its various forms. The talks by Padmini Veerapen and Charlotte Ure both touched on the idea of graded ('Zhang') twists versus 2-cocycle twists in the setting of coalgebras. Xingting Wang discussed graded twists of Poisson algebras, while Kenta Ueyama talked on twisted Segre products. This is related to other material in the literature on graded twists and twisted tensor products, for example the work of Conner and Goetz."

He added, "In considering these topics, one question that came up in conversation was how to properly define a twisted tensor product of Poisson algebras. An Ore extension of automorphism type can be defined as a graded twist of a polynomial ring, but can also be defined as a twisted tensor product. The construction discussed by Wang leads to the first realization of a Poisson Ore extension as a graded Poisson twist. The question is whether there is an appropriate realization of the second type for Poisson Ore extensions. I believe that further discussions will lead to interesting collaborations on the subject."

Finally, a collaboration between Matt Satriano and Colin Ingalls also started, motivated by Satriano's question about an analogue of the wild automorphism question of Rogalski, Reichstein, and Zhang [47] in the setting of vector fields on a projective variety. Here one asks if X is a projective variety with a section s of the tangent bundle of X with the property that there are no proper subvarieties Y of X with $s|_Y$ restricting to a map from Y to its tangent bundle, then X must be an abelian variety. Ingalls and Satriano made significant progress towards a resolution of this problem during the week and they feel this will ultimately result in a joint publication along with Rahim Moosa.

We look forward to seeing the future progress made from the collaborations begun at this BIRS workshop and future research directions shaped by the presentations given.

Participants

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Bibliography

[1] M. Artin, Geometry of quantum planes, Azumaya algebras, actions, and modules (Bloomington, IN, 1990), 1–15, Contemp. Math., 124, AMS, Providence, RI, 1992.

[2] M. Artin and W. Schelter, Graded algebras of global dimension 3, *Adv. in Math.* **66** (1987), no. 2, 171–216.

[3] M. Artin and J. T. Stafford, Noncommutative graded domains with quadratic growth, *Invent. Math.,* **122** (1995) 231–276.

[4] M. Artin, J. Tate, and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, *The Grothendieck Festschrift,* Vol. I, 33–85, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990.

[5] M. Artin, J. Tate, and M. Van den Bergh, Modules over regular algebras of dimension 3, *Invent. Math.* **106** (1991), no. 2, 335–388.

[6] Y.-H. Bao, J.-W. He and J.J. Zhang, Pertinency of Hopf actions and quotient categories of Cohen-Mcaulay algebras. J. Noncommut. Geom. **13** (2019), no. 2, 667710.

[7] J. Bell, M. Hamidizadeh, H. Huang, and H. Venegas, Noncommutative analogues of a cancellation theorem of Abhyankar, Eakin, and Heinzer. Beitr. Algebra Geom. **62** (2021), no. 2, 295315.

[8] J. Bell and J. Zhang, Zariski cancellation problem for noncommutative algebras. Selecta Math. (N.S.) **23** (2017), no. 3, 17091737.

[9] G. Brown and M. Wemyss, Local Normal Forms of Noncommutative Functions. Available online at arXiv:2111.05900.

[10] G. Benkart, R. Biswal, Rekha, E. Kirkman, V. Nguyen, J. Zhu, Tensor representations for the Drinfeld double of the Taft algebra. J. Algebra **606** (2022), 764797.

[11] G. Benkart, R. Biswal, Rekha, E. Kirkman, V. Nguyen, J. Zhu, McKay matrices for finite-dimensional Hopf algebras. Canad. J. Math. **74** (2022), no. 3, 686731.

[12] R.-O. Buchweitz, E.M, Faber and C. Ingalls, A McKay correspondence for reflection groups. Duke Math. J. **169** (2020), no. 4, 599669.

[13] L. Buzaglo, Enveloping algebras of Krichever-Novikov algebras are not noetherian. Available online at arXiv:2112.08828.

[14] T. Cassidy and M. Vancliff, Skew Clifford algebras. J. Pure Appl. Algebra **223** (2019), no. 12, 50915105.

[15] P. S. Chakraborty and B. Saurabh, Gelfand-Kirillov dimension of the quantized algebra of regular functions on homogeneous spaces. Proc. Amer. Math. Soc. **147** (2019), no. 8, 32893302.

[16] D. Chan, K. Chan, L. de Thanhoffer de Vólcsey, C. Ingalls, K. Jabbusch, S. Kovács, R. Kulkarni, B. Lerner, B. Nanayakkara, S. Okawa, and M. Van den Bergh, The minimal model program for b-log canonical divisors and applications, 2017. Available online at arXiv:1707.00834.

[17] D. Chan and C. Ingalls, The minimal model program for orders over surfaces. Invent. Math. **161** (2005), no. 2, 427–452.

[18] D. Chan and C. Ingalls, The minimal model program for arithmetic surfaces enriched by a Brauer class. Available online at arXiv:2108.03105.

[19] K. Chan, E. Kirkman, C. Walton, J. J. Zhang, McKay Correspondence for semisimple Hopf actions on regular graded algebras, I. J. Algebra **508** (2018), 512538.

[20] K. Chan, E. Kirkman, C. Walton, J. J. Zhang, McKay Correspondence for semisimple Hopf actions on regular graded algebras, II. J. Noncommut. Geom. **13** (2019), no. 1, 87114.

[21] A. Chirvasitu, R. Kanda, and P. Smith, New Artin-Schelter regular and Calabi-Yau algebras via normal extensions. Trans. Amer. Math. Soc. **372** (2019), no. 6, 39473983.

[22] A. Chirvasitu, R. Kanda, and P. Smith, Feigin and Odesskii's elliptic algebras. J. Algebra **581** (2021), 173225.

[23] A. Chirvasitu, R. Kanda, and P. Smith, Maps from Feigin and Odesskii's elliptic algebras to twisted homogeneous coordinate rings. Forum Math. Sigma **9** (2021), Paper No. e4, 50 pp.

[24] A. Chirvasitu, C. Walton and X. Wang, On quantum groups associated to a pair of preregular forms. Available online at arXiv:1605.06428.

[25] A. Conner, E. Kirkman, F. Moore, C. Walton, Noncommutative Knrrer periodicity and noncommutative Kleinian singularities. J. Algebra **540** (2019), 234273.

[26] S. Crawford, Actions of small groups on two-dimensional Artin-Schelter regular algebras. J. Algebra **560** (2020), 609–666.

[27] J. Cuadra, P. Etingof and C. Walton, Finite dimensional Hopf actions on Weyl algebras. Adv. Math., **302** (2016), 25–39.

[28] J. Cuadra, P. Etingof and C. Walton, Semisimple Hopf actions on Weyl algebras. Adv. Math., **282** (2015), 47–55.

[29] P. Etingof and C. Walton, Semisimple Hopf actions on commutative domains. Adv. Math., **251** (2014), 47–61.

[30] E. Faber, C. Ingalls, S. Okawa, and M. Satriano, On stacky surfaces and noncommutative surfaces. Available online at arXiv:2206.13359.

[31] L. Ferraro, E. Kirkman, F. W. Moore, and K. Peng, On the Noether bound for noncommutative rings. Proc. Amer. Math. Soc. **149** (2021), no. 7, 27112725.

[32] J. Gaddis, E. Kirkman, F. W. Moore, and R. Won, Auslander's theorem for permutation actions on noncommutative algebras. Proc. Amer. Math. Soc. **147** (2019), no. 5, 18811896.

[33] J. Gaddis and R. Won, Hopf actions on quantum generalized Weyl algebras. Available online at arXiv:2102.00320.

[34] H. Huang, V. Nguyen, C. Ure, K. Vashaw, P. Veerapen, X. Wang, Twisting Manin's universal quantum groups and comodule algebras. Available online at arXiv:2209.11621.

[35] H. Huang, V. Nguyen, C. Ure, K. Vashaw, P. Veerapen, X. Wang, A cogroupoid associated to preregular forms. Available online at arXiv:2112.09098.

[36] H. Huang, V. Nguyen, C. Ure, K. Vashaw, P. Veerapen, X. Wang, Twisting of graded quantum groups and solutions to the quantum Yang-Baxter equation. Available online at arXiv:2112.09098.

[37] H. Huang, C. Walton, E. Wicks, and R. Won, Universal quantum semigroupoids. J. Pure Appl. Algebra **227** (2023), no. 2, Paper No. 107193.

[38] E. Kirkman, Invariant theory of Artin-Schelter regular algebras: a survey. Recent developments in representation theory, 2550, Contemp. Math., **673**, Amer. Math. Soc., Providence, RI, 2016.

[39] E. Kirkman, J. Kuzmanovich, and J Zhang, Shephard-Todd-Chevalley Theorem for skew polynomial rings. Algebr. Represent. Theory **13** (2010), no. 2, 127–158.

[40] E. Kirkman, R. Won, and J. Zhang, Weighted homological regularities. Available online at arXiv:2204.06679.

[41] E. Kirkman and J. Zhang, The Jacobian, reflection arrangement and discriminant for reflection Hopf algebras. Int. Math. Res. Not. IMRN 2021, no. 13, 98539907.

[42] O. Lezama,Y.-H. Wang, and J. Zhang, Zariski cancellation problem for non-domain noncommutative algebras. Math. Z. **292** (2019), no. 3-4, 12691290.

[43] E. Lira-Torres and S. Majid, Quantum gravity and Riemannian geometry on the fuzzy sphere. Lett. Math. Phys. **111** (2021), no. 2, Paper No. 29, 21 pp.

[44] W. Lowen and A. Mertens, Linear quasi-categories as templicial modules. Available online at arXiv:2005.04778.

[45] D.-M. Lu, J. H. Palmieri, Q.-S. Wu, and J. Zhang, Regular algebras of dimension 4 and their A_∞-Ext-algebras. Duke Math. J. **137** (2007), no. 3, 537584.

[46] I. Mori and K. Ueyama, Noncommutative Knrrer's periodicity theorem and noncommutative quadric hypersurfaces. Algebra Number Theory **16** (2022), no. 2, 467504.

[47] Z. Reichstein, D. Rogalski, and J. Zhang, Projectively simple rings. Adv. Math. **203** (2006), no. 2, 365407.

[48] D. Rogalski, S. J. Sierra and J. T. Stafford, Classifying orders in the Sklyanin algebra, Algebra Number Theory **9** (2015), no. 9, 2055–2119.

[49] D. Rogalski, S. J. Sierra and J. T. Stafford, Noncommutative blowups of elliptic algebras, Algebr. Represent. Theory **18** (2015), no. 2, 491–529.

[50] D. Rogalski, S. J. Sierra and J. T. Stafford, Ring-theoretic blowing down. I. J. Noncommut. Geom. **11** (2017), no. 4, 14651520.

[51] D. Rogalski, S. J. Sierra and J. T. Stafford, Some noncommutative minimal surfaces. Adv. Math. **369** (2020), 107151, 48 pp.

[52] D. Rogalski, Daniel, R. Won, J. Zhang, A proof of the Brown-Goodearl conjecture for module-finite weak Hopf algebras. Algebra Number Theory **15** (2021), no. 4, 971997.

[53] S. Sierra and C. Walton, The universal enveloping algebra of the Witt algebra is not noetherian. Adv. Math. **262** (2014), 239260.

[54] X. Tang, H. Venegas, and J. Zhang, Cancellation problem for AS-regular algebras of dimension three. Pacific J. Math. **312** (2021), no. 1, 233255.

[55] X. Tang, X. Wang, and J. Zhang, Twists of graded Poisson algebras and related properties. Available online at ArXiv:2206.05639.

[56] C. Walton and J. Zhang, On the quadratic dual of the Fomin-Kirillov algebras. Trans. Amer. Math. Soc. **372** (2019), no. 6, 39213945.

[57] J. Zhang, Twisted graded algebras and equivalences of graded categories. Proc. London Math. Soc. (3), **72** (1996), no. 2, 281311.

Two-day Workshop Reports

Appendix C

Canadian Math Kangaroo Contest Meeting (22w2255)

May 6 - 8, 2022

Organizer(s): Rossitza Marinova (Concordia University of Edmonton), Tzvetalin Vassilev (Nipissing University)

The Canadian Math Kangaroo Contest (CMKC) meeting took place in May 2022. The format of the meeting was hybrid with a total number of participants 28, of which 16 in-person and 12 virtual. The meeting was generously supported by BIRS and CMKC. BIRS provided the venue for the workshop, CMKC supported travel and meals of participants. Contest representatives from a number of Canadian cities (Calgary, Edmonton, Lethbridge, Regina, Brandon, Ottawa, Montreal, Toronto, Thunder Bay, Orillia) and two guests from United States and Puerto Rico attended the workshop in-person.

Overview of the Field

Math Kangaroo is an annual international math competition for school students. This is one of the world's largest math competition, with about six million participants worldwide. The main purpose of Math Kangaroo is to introduce participants to math challenges in an enjoyable way, thus, inspiring their further interest and advancement in mathematics. It provides participating students with great and valuable experience in competitive math.

The Canadian Math Kangaroo Contest (CMKC) non-profit organization administers the competition in Canada. CMKC aims to dispel the myth that mathematics is inaccessible and difficult by using an all-inclusive and open mathematics competition to create a positive environment that emphasizes the practical and fun nature of mathematics. Since joining the International Association Kangaroo without Borders in 2006, the CMKC has formed hosting partnerships with multiple universities and prestigious organizations across the country.

Since its inception, the CMKC has found many ways to expand the reach and scope of its unique program, and to build new, complementary programs based upon the competition. Over the past 17 years, the CMKC has grown significantly from its humble beginnings with approximately 300 participants at 3 locations to over 6500 students participating every year nationwide in more than 50 locations in 9 provinces across Canada. Thousands of students are involved in various training and learning activities on a regular basis.

Math Kangaroo contest is unique to Canada. Students can participate in Math Kangaroo independently of their home school's involvement. Math Kangaroo Centres in Canada typically are universities. Almost all other competitions run through schools. Math Kangaroo is still one of the very few math contests available for Canadian elementary students. During the last three years, 2020-2022, the contest was offered in online format, due to the Covid-19 restrictions.

While the reputation, the merit, and the quality of inspired learning are at a very high level, the atmosphere on the contest day is unique compared to most of the other contests. The Canadian Math Kangaroo program contributes to the science, engineering and education communities through its activities that revolve around the contest but go far beyond its organization.

Recent Developments and Open Problems

The workshop was intended for mathematicians and mathematics educators involved in the CMKC. The purpose of the meeting was to evaluate the current state and chart the future course for Math Kangaroo in Canada. Specifically, the CMKC-BIRS meeting aimed to achieve several objectives, as listed below.

Designing new contest problem proposals

These presentations and discussions aimed to encourage CMKC members to submit problem proposals for future competitions. Therefre, the meeting planned introduction to the KSF curricula, presentation of proposal examples by topic (arithmetic and algebra, numbers, geometry, logic), and tutorial on software tools for vector graphics. Proposed Math Kangaroo problems are expected to be creative, interesting, original, and provoking mathematical and logical thinking.

Hosting the Annual Meeting of Kangaroo Sans Frontiers (KSF)

In 2021, the CMKC team applied for hosting the annual meeting of the international association KSF in Canada, either in 2026 or in 2027. CMKC invited two KSF board members who represent United States and Puerto Rico in KSF. They hosted the KSF Annual Meeting in 2014 and 2019 and could share their experiences with the CMKC team. Thus, the Canadian team could learn important information about what the organization of the meeting would involve.

Programs for teachers and students

CMKC team has always sought ways to serve the community through its programs. The organizations offers online classes for students and publishes mathematical resources on a regular basis. Other content includes development of videos on math topics posted on the Math Kangaroo eLearning YouTube channel [1]. The 2022 CMKC-BIRS workshop gave opportunities for the group to have discussion and plan delivery of programs for teachers, including teacher workshops delivery and mathematical material creation.

Analyzing results from past contests

Important future work the team plans to undertake is studying and analyzing the results from past contests. This research involves examining available data from the past nine years (2014-2022) and getting insights into challenges students face with particular math concepts. The outcomes of such study can be used for improving the training programs.

Development of math learning and testing tools

Self-testing educational software tools can assist students in learning mathematics. Collaborative research involving online and adaptive learning software such as the QuizMASter tool [2] has the potential to contribute to the development of mathematical training mobile / web application.

Presentation and Discussion Highlights

The workshop consisted of several presentation and discussion sessions on topics of interest to the organization and the workshop participants.

- Friday, May 6:
  The evening session consisted of informal discussion on past contests and math training programs.

- Saturday, May 7:
  The morning session started with presentations of the Math Kangaroo curricula for all contest levels, namely: 1-2; 3-4; 5-6; 7-8; 9-10; 11-12. This introductory session was followed by a presentation by Valeria Pandelieva, who gave an overview of the competition, including its history in the world and in Canada. She also spoke about Math Kangaroo problems and topics, providing specific examples.

  The afternoon session included four presentations with discussions. Joanna Matthiesen and Luis Caceres gave a talk on what hosting of the KSF Annual Meeting involves. Fushua Lin gave a presentation with a title Adaptive QuizMASter, followed by demonstration of the QuizMASter. Gautam Srivastava led a tutorial on software tools for vector graphics, covering LaTeX TikZ, GeoGebra, and draw.io. This tools are needed for creating good quality graphics for the competition and math training materials. Agnes Fung presented the CMKC website most recent updates.

  On Saturday evening, the participants discussed what new programs CMKC can bring to teachers and students in near future. Everyone agreed that training programs for teachers are of very high priority.

- Sunday, May 8:
  The last two sessions on Sunday were used for demonstrating additional examples of problem proposals and plans for analyzing of results from past contests. Ildiko Pelczer presented ideas of problem proposals in various contexts: transformation; exchanges; processes; folding; equations; and configurations. Rossitza Marinova concluded the workshop with a talk on plans for how the CKMC team can work on analyzing results from past contests.

Outcome of the Meeting

The two-day BIRS workshop facilitated discussions and decisions on how to further improve the organization of the Math Kangaroo contest and accompanying programs.

The meeting is another significant milestone for the Canadian Math Kangaroo Contest team. Representatives from various provinces and countries exchanged ideas and discussed issues. The major meeting outcomes include planning future work for:

- Improving the quality of the mathematical content offered by the Math Kangaroo contest.
- Fostering of wider involvement from the CMKC community in proposing problems.
- Collaborating for research into competitive mathematics, such as math training tools and data analysis.

Sharing information and ideas is crucial for maintaining a program of such scope, diversity, quality and continuity. The CMKC meeting at BIRS facilitated efficient collaboration, coordination, and knowledge transfer among Math Kangaroo national and regional organizers.

Participants

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Bibliography

[1] Canadian Math Kangaroo eLearning, Videos, `https://www.youtube.com/channel/UCCldTuqySeICLcuRPEquxfaw/videos` (2021-2022).

[2] Dutchuk, M., Muhammadi, K.A., Lin, F. QuizMASter–A multi-agent game-style learning activity, International Conference on Technologies for E-Learning and Digital Entertainment, Lecture Notes in Computer Science, **5670** (2009). Springer, Berlin, Heidelberg, 263–272.

Appendix D

Alberta-Montana Combinatorics and Algorithms Days (22w2245)

June 3 - 5, 2022

Organizer(s): Hadi Kharaghani (University of Lethbridge), Ryan Hayward (University of Alberta), Mark Kayll (University of Montana), Robert Woodrow (University of Calgary)

Overview

The first weekend in June 2022 saw the inaugural *Alberta-Montana Combinatorics and Algorithms Days* hosted at the Banff International Research Station (BIRS). One purpose of the event was to bring together faculty and students from three Alberta universities (in Calgary, Lethbridge, and Edmonton) and the University of Montana (Missoula).

Combinatorics is the branch of mathematics concerned with finite sets: their properties, structures, and number. Studying the classic Rubik's Cube reveals the number of possible positions (it's $43,252,003,274,489,856,000$). Understanding the cube's structure leads to efficient algorithms for solving it (an Algorithm being a sequence of well-defined instructions for solving a problem, answering a question, or even playing a game). In 2010, a group of researchers working with Google proved that every one of that staggering number of positions could be solved in no more than 20 moves.

The fields of Combinatorics and Algorithms became inextricably linked at the dawn of the computer age due to computers themselves being finite structures. This meeting was conceived to offer regional researchers in these fields an opportunity to share their recent successes, tackle open problems together, and expose their students to the latest methods and developments.

Demographics (briefly)

The workshop attracted participants from a broader geographical region than originally envisioned; to wit, there were speakers from Calgary, Edmonton, Lethbridge, Missoula, Winnipeg, and Toronto. Likewise, the 'vertical' representation went even deeper than the organizers had originally hoped. Of the participants, there were two undergraduate students (one of whom spoke), five graduate students (four of whom spoke), and two postdoctoral fellows ((both of whom spoke). Thus, the total of sixteen talks was rounded out by nine lectures given by junior and senior faculty.

Presentation highlights

Here follows a précis of each of the talks in their original chronological order.

Ting Han Wei (University of Alberta) opened the scientific program with a lovely introduction to the (highly combinatorial) game of Go. Starting from the basics, he brought the audience to the state-of-the-art for solving 'small' (i.e., up to 6×6) boards. Solving the full board (19×19) may be beyond the abilities of humans ever to solve.

Thomas Pender (University of Lethbridge) began his talk by grounding the gathering on elementary Hadamard matrix theory. He quickly worked toward modern generalizations including orthogonal designs; this included some attractive constructions of these objects.

In her talk, Anastasia Halfpap (University of Montana) presented strong new results on extremal combinatorics. These particularly showcased examples and theory in situations where two (or more) structures of completely different character can achieve extremality in Turán-type problems.

In the first invited (full-hour) talk, Mark Kayll (University of Montana) introduced a graph family generalizing the well-known Kőnig-Egerváry graphs. This lecture—based in part on [6]—considered both theoretical and algorithmic aspects of these 'Egerváry' graphs.

Kris Vasudevan (University of Calgary) walked his audience through deep connections between graph theory and neuroscience, particularly brain disorder studies.

The second invited lecture featured Joy Morris (University of Lethbridge), who discussed pursuit games on generalized Petersen graphs. Among many results, Dr. Morris presented joint work [8] with her 16-year old daughter Harmony Morris (thus pushing the vertical integration mentioned above to a level still deeper than earlier indicated).

Cory Palmer (University of Montana) used a Star Wars lens to introduce his talk on enumerating stable matchings in complete bipartite graphs $K_{n,n}$ (with linear preference rankings at each vertex). In perhaps the most striking scientific announcement of the workshop, Dr. Palmer walked us through his recent improvement (with Dömötör Pálvölgyi) from 131072^n to 3.55^n on the best-known upper bound for the maximum number of stable matchings; see [9].

The lecture by Bobby Miraftab (University of Lethbridge) introduced a new graph decomposition (due to Stavropoulos) generalizing tree-decompositions. These so-called 'median-decompositions' exhibit connections with chordality and hyperbolicity in graphs.

In the third invited lecture, Michael Cavers (University of Toronto, Scarborough) spoke on reconfiguring vertex colourings of graphs, a formal way to capture the notion of two colourings being incrementally close. This far-reaching concept has applications to the infamous 15-puzzle, to change-ringing (in bell choirs), and to the so-called Glauber dynamics Markov chain (among others). See [2] as an example of one of Dr. Cavers' many cited references.

Vlad Zaitsev (University of Lethbridge) presented several new results and proofs concerning optimal constant-weight ternary codes. His work (with Hadi Kharaghani and Sho Suda) generalizes a result first proved twenty years ago; see [7].

Closing out the main program for the day, Ramin Mousavi (University of Alberta) presented on a variation of the classical Steiner tree problem. His work (with Zachary Friggstad) addresses an important special case of the 'Directed Steiner Tree' problem, namely when the underlying graph is planar or, more generally, excludes a fixed minor.

After a break for dinner, Rob Craigen (University of Manitoba) gave a plenary talk sharing his career-long experience and insights on Mathematics Education in Canada. This was followed by a lively discussion fueled by all the resulting questions and comments on a topic so close to many of the hearts at the workshop.

Professor Craigen continued to hold the floor to open the second morning, when he spoke on circulant partial Hadamard matrices. Among other things, his presented work suggests a possible resolution of the 'circulant Hadamard matrix conjecture'.

In the workshop's penultimate invited lecture, Ryan Hayward (University of Alberta) gave a compelling introduction and overview of the board game Hex, on which he has published two recent books: [4] and [5]. Unlike Go and Chess, Hex offers a game that lends itself to proofs concerning strategy and even solvability.

The workshop's final invited lecture was delivered by Zachary Friggstad (University of Alberta). Entitled *Prize-Collecting Walks and Branchings in Directed Graphs*, his talk was exemplary in bringing together the workshop themes (combinatorics and algorithms). Here we saw combinatorics (from a digraph perspective), linear programming, and a combinatorial algorithm (to replace an LP with a prohibitive number of variables and constraints). As these ideas were also applied to the design of approximation algorithms for vehicle routing problems, the audience also got to see a good blend of theory and applications. The manuscript [3]—by Prof. Friggstad and his collaborators—documents the combinatorial algorithm mentioned above.

Davoud Abdi (University of Calgary) closed the scientific program with his presentation of exciting developments in poset theory related to the Cantor-Schröder-Bernstein Theorem. He introduced NE-free posets, classified them, and sketched a new proof of a 'Thomassé Conjecture' for countable NE-free posets; see [1].

Meeting outcomes

This workshop was originally scheduled for May 2020 (as BIRS 20w2245), then postponed to September 2021, and finally to 3–5 June 2022. (Phew! We finally met!)

The original conception was to bring together Alberta and Montana researchers in the fields of Combinatorics and Algorithms **in a student-friendly environment**. The organizers hoped this would foster collaboration, community, and networking opportunities, especially for junior researchers.

Anecdotally, the event was successful in these goals.

One participating graduate student, 'Terry Doe', related a telling story to a workshop organizer. A few weeks before this BIRS event, s/he had attended another conference, where s/he decided "I guess math conferences are not for me." Fast-forward those weeks to see a dramatic about-face. As BIRS was unfolding, Terry completely reevaluated her/his position and expressed the hope that this same workshop would run again in 2023. "I don't know if it's that I like being around Canadians or what, but I feel so welcome here!"

At meals and other breaks, the organizers heard from many of the other attendees how much they were enjoying the workshop. There was uniformly excellent quality in the scientific program. And the relaxed atmosphere of Banff was conducive both to casual conversation and to mathematical inquiry. The organizers know of several instances of near-future collaborative visits initiated exclusively because of this BIRS workshop.

Two constructive critiques were voiced: (i) schedule fewer talks per day to allow for informal problem sessions; (ii) try for a one–two days longer event to allow for more sustained collaborative interaction.

These suggestions were welcomed by the organizers, who overall heard nothing but praise for a well-planned and structured two-day workshop.

Participants

Abdi, Davoud (University of Calgary)
Cavers, Mike (University of Toronto Scarborough)
Chen, Xinyue (University of Alberta)
Craigen, Robert (University of Manitoba)
Friggstad, Zachary (University of Alberta)
Halfpap, Anastasia (University of Montana)
Hayward, Ryan (University of Alberta)
Holzmann, Wolfgang (University of Lethbridge)
Jamshidian, Mahya (University of Alberta)
Kayll, Mark (University of Montana)
Kharaghani, Hadi (University of Lethbridge)
Miraftab, Babak (University of Lethbridge)
Morris, Joy (University of Lethbridge)
Mousavi, Ramin (University of Alberta)
Palmer, Cory (University of Montana)
Pender, Thomas (University of Lethbridge)
Sands, Bill (University of Calgary)
Van't Land, Caleb (University of Lethbridge)
Vasudevan, Kris (University of Calgary)
Wei, Ting-Han (University of Alberta)
Wood, Ryan (University of Montana)
Woodrow, Robert (University of Calgary)
Zaitsev, Vlad (University of Lethbridge)

Bibliography

[1] Davoud Abdi, Claude Laflamme, Atsushi Tateno, and Robert Woodrow, An example of Tateno disproving conjectures of Bonato-Tardif, Thomassé, and Tyomkyn, arXiv:2205.14679 [math.CO], 2022.

[2] Michael Cavers and Karen Seyffarth, Reconfiguring vertex colourings of 2-trees, *Ars Math. Contemp.* **17** (2019), 653–698.

[3] Sina Dezfuli, Zachary Friggstad, Ian Post, and Chaitanya Swamy, Combinatorial algorithms for rooted prize-collecting walks and applications to orienteering and minimum-latency problems, arXiv:2111.07414 [cs.DS], 2021.

[4] Ryan B. Hayward (with Bjarne Toft), Hex: the Full Story, CRC Press, Boca Raton, 2019.

[5] Ryan B. Hayward, Hex: a Playful Introduction, Vol. **54** in Anneli Lax New Mathematical Library, MAA Press, Providence, 2022.

[6] P. Mark Kayll and C.E. Larson, Egerváry graphs: Deming decompositions and independence structure, arXiv:2205.10598 [math.CO], 2022.

[7] Hadi Kharaghani, Sho Suda, and Vlad Zaitsev, On a class of optimal constant weight ternary codes, arXiv:2109.14995 [math.CO], 2019.

[8] Harmony Morris and Joy Morris, On generalised Petersen graphs of girth 7 that have cop number 4, arXiv:2009.00699 [math.CO], 2020.

[9] Cory Palmer and Dömötör Pálvölgyi, At most 3.55^n stable matchings, arXiv:2011.00915 [math.CO], 2020.

Appendix E

Canadian Abstract Harmonic Analysis Symposium (CAHAS) 2022 (22w2235)

June 17 - 19, 2022

Organizer(s): Brian E. Forrest (University of Waterloo), Keith F. Taylor (Dalhousie University), Volker Runde (University of Alberta)

Abstract Harmonic Analysis

Abstract harmonic analysis is the study of spaces and algebras associated with locally compact groups G, most prominently, but not exclusively, the group algebra $L^1(G)$ and the Fourier algebra $A(G)$. It has evolved out of classical Fourier analysis where, from the abstract point of view, the abelian groups $\mathbb{Z}$ and $\mathbb{R}^N$ are studied. There is strong interplay between abstract harmonic analysis and the theories of Banach algebras, operator algebras, quantum groups, and operator space theory.

The Canadian Abstract Harmonic Analysis Symposium

Abstract harmonic analysis has been strongly represented in Canada over the past decades.

The Canadian Abstract Harmonic Analysis Symposium (CAHAS) is a series of meetings in the area, which started in 1997 at the University of British Columbia with a meeting in the honor of Edmond Granirer on the occasion of his retirement. Ever since, CAHAS has been ongoing on an annual basis, albeit in varying formats: sometimes it took place as a section at a CMS meetings; sometimes, when other major events in closely related areas took place in Canada, it was subsumed under those events; in 2009, CAHAS was held in the form of a week long international conference at the University of Alberta with almost 80 participants to celebrate the 65^{th} birthday of Anthony To-Ming Lau, who had been a major leader of the field for decades; and in 2014, CAHAS took place as a two-day workshop at BIRS.

There has always been an emphasis at CAHAS meetings to allow junior researchers to get exposure to the community. CAHAS 2022 was no exception.

On June 30, 2020, Anthony To-Ming Lau retired. For this reason, it had been planned to hold a CAHAS meeting at BIRS in May 2020 to bring together his large "extended mathematical family" and celebrate his contributions to abstract harmonic analysis. The Covid-19 pandemic put that plan on hold. CAHAS 2022—the first time a meeting of the series was held in hybrid format—was the attempt to recreate the planned meeting to the extent possible.

Scientific Progress and Presentation Highlights

The presentations at the meeting covered a wide range of topics. There were, e.g., several talks on quantum groups and other "quantized" mathematics (Anderson-Sackeney, Crann, Viselter, Lee), wavelets (Hollingsworth, Milad, Potter), and Fourier algebras and their kin (Choi, Sawatzky, Spronk, Thamizhazhagan, Turowska).

The highlights were the presentations by Yemon Choi and Hannes Thiel.

Let G be a locally compact group, let $A(G)$ denote its Fourier algebra, and let $\mathrm{AM}(A(G))$ stand for the so-called amenability constant of $A(G)$, as introduced by the late Barry E. Johnson in [4]. Johnson showed, for finite G, that $\mathrm{AM}(A(G)) = 1$ if G is abelian and that $\mathrm{AM}(A(G)) \geq \frac{3}{2}$ if G is non-abelian. For general locally compact G, it is easy to see that $\mathrm{AM}(A(G)) = 1$ if G is abelian. For (not necessarily finite) non-abelian G, it was shown in [3] that $\mathrm{AM}(A(G)) \geq \frac{2}{\sqrt{3}}$ ([3]). Reporting on his work in [1], Choi showed at the meeting that $\mathrm{AM}(A(G)) \geq \frac{3}{2}$ for any non-abelian locally compact group G. What is remarkable about Choi's result is that the proof is very close in spirit to Johnson's original approach.

Let $\mathcal{O}_2$ denote the Cuntz algebra. Then it is well known—apparently due to George Elliot—that $\mathcal{O}_2$ is isomorphic to its tensor square $\mathcal{O}_2 \otimes \mathcal{O}_2$ ([6]). For general $p \in [1, \infty)$, generalizations $\mathcal{O}_2^p$ of $\mathcal{O}_2^p$ acting on L^p-spaces can be defined ([5]); it is natural to ask if $\mathcal{O}_2^p \cong \mathcal{O}_2^p \otimes \mathcal{O}_2^p$ for $p \neq 2$. In his talk, based on joint work with Choi and Eusebio Gardella, Thiel showed that $\mathcal{O}_2^p \cong \mathcal{O}_2^p \otimes \mathcal{O}_2^p$ if and only if $p = 2$.

Impact of the Hybrid Format

One challenge of the 2-day workshops at BIRS is that it can be difficult for researchers outside of Western Canada or the Northwestern United States to justify the time and travel costs associated with getting to Banff for what is, in essence, at most one and a half days of talks. This is especially true for those whose travel plans force them to miss a significant part of the second day.

The hybrid format provided a very elegant—and far less challenging than expected—way around this obstacle. Eventually 17 of the meeting's 40 attendees participated remotely from abroad; six of those—Yemon Choi (University of Lancaster; United Kingdom), Hun Hee Lee (Seoul National University; South Korea), Serap Öztop-Kaptanoğlu (University of Istanbul; Turkey), Hannes Thiel (Kiel University; Germany), Lyudmila Turowska (Chalmers University; Sweden), and Ami Viselter (University of Haifa; Israel)—gave presentations. Thanks to the excellent IT support by BIRS staff, the technical difficulties of handling a hybrid meeting turned out to be surprisingly minor.

In order to give postdocs, graduate students, and untenured faculty a better chance to connect with the community, it was decided that people from those groups should be given preference when it came to in-person participion. Indeed, of 13 in-person participants, there were one untenured assistant professor (Wiersma), three postdocs (Hollingsworth, Thamizhazhagan, Vati), and four graduate students (Anderson-Sackaney, Kazemi, Sawatzky, Vujicic).

Overall, the hybrid format made the meeting a very pleasant experience for both organizers and participants.

Outlook

As stated in Section 2, the Canadian Abstract Harmonic Analysis Symposium has been held (mostly) annually in a 2-day format. A 2-day conference is difficult to organize without the backing of, e.g., a large learned society or research institute behind it: few people will travel large distances for a 2-day meeting, funding is difficult to obtain, and the organizational overhead is almost the same as for a longer meeting.

There is currently some discussion in Canada's abstract harmonic analysis community about holding CAHAS bi-annually with BIRS as its base, but in hybrid format: this would increase the circle of potential participants and, at the same, time decrease the burden of legwork on the organizers. BIRS will likely be contacted in this matter very soon.

Participants

Anderson-Sackaney, Ben (University of Waterloo)
Choi, Yemon (Lancaster University)
Crann, Jason (Carleton University)
Daws, Matthew (University of Central Lancashire)
Elgun-Kirimli, Elcim (Acibadem University)
Forrest, Brian (University of Waterloo)
Galindo, Jorge (Universitat Jaume I)
Gardella, Eusebio (Chalmers University)
Ghandehari, Mahya (University of Delaware)
Hollingsworth, Kris (University of Minnesota)
Hu, Zhiguo (University of Windsor)
Ilie, Monica (Lakehead University)
Kazemi, Soroush (Carleton University)
Lee, Hun Hee (Seoul National University)
Lin, Ying-Fen (Queen?s University Belfast)
Loy, Rick (Australian National University)
Milad, Raja (Dalhousie University)
Neufang, Matthias (Carleton University)
Ng, Chi-Keung (Nankai University)
Oztop-Kaptanoglu, Serap (University of Istanbul)
Potter, Tom (Dalhousie University)
Runde, Volker (University of Alberta)
Sangani Monfared, Mehdi (University of Windsor)
Sawatzky, John (University of Waterloo)
Spronk, Nico (University of Waterloo)
Stokke, Ross (University of Winnipeg)
Thamizhazhagan, Aasaimani (University of Winnipeg)
Thiel, Hannes (Kiel University)
Todorov, Ivan (University of Delaware)
Turowska, Lyudmila (Chalmers University)
Ulger, Ali (Koc University)
Vati, Kedumetse (University of Alberta)
Viselter, Ami (University of Haifa)
Vujicic, Aleksa (University of Waterloo)
Wiersma, Matthew (University of Winnipeg)
Willis, George (University of Newcastle, New South Wales)
Zadeh, Safoura (University of Bristol)
Zhang, Yong (University of Manitoba)
Zhu, Yihan (University of Windsor)

Bibliography

[1] Y. CHOI, An explicit minorant for the amenability constant of the Fourier algebra. `arXiv:2012.14413`.

[2] Y. CHOI, E. GARDELLA, and H. THIEL, Rigidity results for L^p-operator algebras and applications, `arXiv:1909.03612`.

[3] B. E. FORREST and V. RUNDE, Norm one idempotent cb-multipliers with applications to the Fourier algebra in the cb-multiplier norm. *Canad. Math. Bull.* **54** (2011), 654-662.

[4] B. E. JOHNSON, Non-amenability of the Fourier algebra of a compact group. *J. London Math. Soc.* (2) **50** (1994), 361–374.

[5] N. C. PHILLIPS, Analogs of Cuntz algebras on L^p-spaces. `arXiv:1201.4196`.

[6] M. RØRDAM, A short proof of Elliott's theorem: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. *C. R. Math. Rep. Acad. Sci.* Canada **16** (1994), 31-36.

Appendix F

Multitaper Spectral Analysis (22w2230)

June 24 - 26, 2022

Organizer(s): David J. Thomson (Queen's University), Charlotte Haley (Argonne National Laboratory), David Riegert (Trent University)

Overview of the Field

The multitaper method was developed by David Thomson of Bell Labs in the 1970's and 1980's culminating in a landmark 1982 paper "Spectral Estimation and Harmonic analysis" [30]. The method had stemmed from the author's own experimentation and data analysis in communications, especially from the careful inspection of millimeter waveguide measurements using the state-of-the-art in frequency domain statistical analysis of time series, the (windowed) periodogram. Connecting the estimation of spectra with the mathematical development of the prolate spheroidal wave functions of Slepian, Landau, and Pollak [26, 25, 27], as well as the computational methods that made them easy to compute, was the major catalyst for the development of the multitaper method, which went on to inspire forty years of innovation in applied problems from communications systems to seismology to medical applications. The astounding quantitative power of the multitaper quickly caught fire in the applied community when a junior geophysicist at Scripp's Institute of Oceanography, Alan Chave, invited Thomson to give a talk shortly after the 1982 paper was published. Thomson, together with Chave, expanded these methods to obtain simple error estimates using the newly developed theory of the bootstrap. Chave used these to study the electrical conductivity of the earth, among other things. Other early adopters included Frank Vernon, a seismologist who is now principal investigator on the USArray, a transportable seismic array placed in the continental US capable of such sensitive measurements as to detect not only terrestrial seismic activity, but through the use of spectral methods, can be used for nuclear treaty verification, and can detect small changes due to solar acoustic oscillations in the background seismic hum. (Linda Hinnov, Lou Lanzerotti)

Thomson went on to develop the theory for nonlinear and nonstationary time series and published a landmark 1995 climate paper that showed using a technique called complex demodulation, that the central england temperature series can be seen to contain signatures of anthropogenic climate forcings in the slope of the phase of the complex demodulate. Michael Mann, of the "hockey stick graph", and Jeffrey Park developed the multitaper singular value decomposition which allowed for the multivariate analysis of numerous climate time series in a similar technique to complex demodulation of a single series, to map similar shifts in sea surface temperatures, with the added ability to recognize spatial variations in ocean currents. Advanced methods, such as quadratic inverse theory [32], characterization of numbers of false detections of oscillations [34], and multitaper estimators of autocorrelation [21, 35], higher-order spectra [31, 3] and others, were to come. Additional significant scientific contributions were the applications to solar physics data, starting with [36], and climate change [33].

Meanwhile, the developments of Slepian, Landau, and Pollak became widely known in the mathematical community, with the extensions of the Cartesian and discrete versions to the sphere by F. Alberto Gr unbaum. Now the study of simultaneous "duration and bandwidth limiting" is a subject of a 2012 book [19]. Generalized Slepian sequences on the sphere are currently being applied in the geophysics community [24].

While these significant developments are all relevant to applied problems, this illustrates one particular feature of the multitaper method: one does not necessarily know that multitaper is necessary unless one analyzes real data. Parametric methods for time series, for example, generally revolve around simple statistical assumptions like stationarity or Gaussianity, and statisticians may rely too heavily on asymptotic unbiasedness of conventional estimators. But when the multitaper method is used on real data, it becomes clear that one never approaches the ground truth fast enough with a periodogram, all sample sizes are too small and no time series are collected with enough precision to rival the provable statistical advantages that one obtains with the multitaper method. This is why, forty years after its initial publication, multitaper is well-established as a gold standard for the analysis of data for which real scientific conclusions are desired.

Recent Developments

The most recent developments in the field as represented by those in attendance were in the form of mathematical, statistical, and applied contributions. Broadly grouping together the presentations, applied mathematical contributions advancing the study of discrete prolate spheroidal functions and their analogues in higher dimensions and dyadic spaces were the presentations of Grunbaum [13, 6, 16, 18, 17, 24] and the 2012 book of Hogan and Lakey entitled "Duration and Bandwidth Limiting" [19].

On the statistical front, new results were presented by Peter Craigmile and colleagues on the development of a novel semiparametric estimation method for the spectrum [29], and a fully parametric method of Adam Sykulski [28, 14] and others. New theoretical results concerning the optimality of the multitaper bias and variance were presented by Jose Luis Romero [1]. Chave and Haley also brought novel results which made use of the newly-developed missing-data Slepian sequences [5, 15].

Recent applied contributions not mentioned in the highlights to come were the developments of Proloy Das and collegues on the subject of nonstationary and nonlinear time series in the medical context [9, 8, 7] where state space models, modeling dynamics, and Bayesian techniques were drawn in to complement a multitaper paradigm for nonlinear dynamics in electroencephalogram data. Numerous novel results from the seismology laboratories of Frank Vernon, Jeff Park [12, 10, 11, 4] were presented as well as geophysical results from Linda Hinnov (to be discussed later) and Frederik Simons.

This is not an exhaustive list of the recent developments presented in the meeting and those not mentioned here are not particularly more important than the others, but an effort has been taken to give as broad an overview of the strong points of the meeting as is possible, while respecting space constraints. In the next section, we give three presentation highlights.

Presentation Highlights

Connection between prolate spheroidal wave functions and zeros of the zeta function

Published within the last three months of this meeting is the remarkable contribution of Fields Medalist Alain Connes and his long-time collaborator, Henri Moscovici. The paper [6] makes an exciting relation between the zeros of the Riemann zeta function, important in number theory, and the prolate spheroidal functions, important to multitaper spectrum analysis, but more generally relevant to the study of functions which are as time and band limited as possible given physical constraints. The paper outlines a self adjoint extension of the prolate second-order differential operator in the interval $[-\lambda, \lambda]$ to the real line. It is found that the spectrum of this operator has negative eigenvalues in Hilbert space related to the squares of the zeros of the zeta function. A historical overview of this remarkable result was given in [13] and presented during the meeting.

Nonstationarity

Textbook examples of time series ordinarily assume that the processes involved are wide sense or weakly stationary, or can be broken into blocks that are approximately stationary. Unfortunately, most geophysical and climate time series carry diurnal cycles, solar cycles, tides, and signatures of the Earth's rotation, solar rotation, or both, and characterization of these oscillations is central to the main reasons for their study. For many of these processes, it is appropriate, then, to convert both temporal indices of the autocorrelation function to frequency. A landmark work of Loéve [20] describes a theoretical decomposition for such spectra involving two frequency indices, however no nonparametric estimation technique was put forth. Recent work has focused on the modeling and description of cyclostationary and generalized almost cyclostationary processes [23], but only with the multitaper technique can one obtain a consistent estimation procedure for these spectra. While these estimations unfortunately produce large numbers of false detections, one can produce summaries of multitaper Loéve spectra by either summing along lines of constant frequency offset (obtained by rotating the Loéve spectrum by $-45°$), or by counting numbers of detections above some significance level along some offset frequency. In this applied talk, numerous motivational examples (i) from seismology (with additional cycles introduced by tidal frequencies), (ii) from astronomy in which a gravitationally-lensed black hole is observed having significant 1 cycle per day offset coherences, (iii) signatures of solar rotation producing nonstationary signatures in temperature data from Marseille going back as far as 1897, and (iv) barometric pressure data from Piñon flat observatory which shows short period signatures of solar pressure (or possibly gravity) modes with high significance, were presented along with numerous open theoretical questions. It is useful to apply multitaper Loéve spectra with or without summarizing the offset coherences for the purposes of sanity checking any sufficiently long time series with or without known nonstationarities, as these data may contain surprising features.

Point Process Spectrum

A point process generates discrete events at particular locations in space. The main questions regarding point processes center around characterizing the first order properties (number of events per unit area) and second order properties (the relationships between locations of pairs of events). The spectrum of a point process is a convenient representation of second order features, but has not been widely used beyond the development of a periodogram-like estimator due to Bartlett in 1963 [2]. Some interesting new developments in this area [22] return to the fundamental problem of an analogous spectral representation for a spatial point process, and estimators with good statistical properties. It turns out that (i) the formation of a direct spectral estimator using tapering is possible in a straightforward way, that (ii) there is a necessary first-order bias correction to remove a constant contribution at all wavenumbers which is not standard in the ordinary spectral analysis of time series, (iii) there are no particularly fast methods (such as the Fast Fourier transform) for computing spectrum estimates of realizations of point processes, and finally, (iv) when isotropy of the random process is assumed, one can determine the analogous mean and variance properties of tapered spectrum estimators. Results on simulated processes show significant improvements when tapers are introduced, and inspire the next generation of scientific investigation of point processes using these updated methods.

Scientific Progress Made

As outlined above in the research highlights, there was the identification of open problems in nonstationary time series, the very recent mathematical discoveries of Fields Medalist Alain Connes and the intruiging connections with pure mathematics, and finally the presentation of Sofia Olhede on the (very) underdeveloped and underrated aspects of point process models.

Connecting applied researchers with theoretical and other appiled researchers also presented some serendipitous findings. Linda Hinnov's presentation on stratigraphy and the strain in the Earth's mantle provided a very large and very complex dataset for discussion. It is a particularly difficult dataset to access and understand without the expertise of a domain expert. Several members of the audience went to download the publically available data and correspond with Linda about the analysis. Had the workshop schedule been more flexible, participants agreed that they would have liked to see a presentation from some on-the-fly analyses in the form of a case study.

While a google scholar search for papers on the subject of multitaper from the last 5 years returns 5,670 results, this group was responsible for approximately 50-60 of them. Of these, this organizer was able to identify 45 journal articles from the last five years authored by those in attendance. In addition to this, there were tens of works in progress, and several ArXiV manuscripts presented ahead of publication. The rapid dissemination of recent and in-progress results was made at this meeting, resulting in numerous opportunities for new contacts and collaborations.

Outcome of the Meeting

The participants would like to meet again in person in two years time to discuss the rapidly-evolving aspects of this field, both mathematical and statistical, as described above. We believe that the major success of this meeting is the confluence of applied researchers from diverse fields, applied mathematicians and statisticians bringing open problems together with half-baked solutions, and putting forth these challenges in an open forum.

Participants agreed that this was a very stimulating meeeting for one held entirely remotely. The meeting also had numerous talks from early career researchers and underrepresented minorities. Numerous early career people privately thanked the organizers for the opportunity to present their work with widely respected researchers whose books and papers they had studied very closely. Despite the fact that the schedule was held in the mountain time zone, participants from North America as well as Europe (and one participant returning from Korea!) attended the talks as late as midnight in their respective time zones, and all talks were well attended.

Participants

Abreu, Luis Daniel (University of Vienna)
Aiello, Emily (Cytel)
Anitescu, Mihai (Argonne National Lab)
Boteler, Claire (Dalhousie University)
Burr, Wesley (Trent University)
Caicedo Vivas, Joan Sebastian (University of Delaware)
Chandna, Swati (Birkbeck - University of London)
Chave, Alan (Woods Hole Oceanographic Institution)
Craigmile, Peter (The Ohio State University)
Das, Proloy (Massachusetts General Hospital)
Dodson-Robinson, Sally (University of Delaware)
Frazer, William (Yale University)
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Bibliography

[1] Luís Daniel Abreu and José Luis Romero. MSE estimates for multitaper spectral estimation and off-grid compressive sensing. IEEE Transactions on Information Theory, 63(12):7770–7776, 2017.

[2] Maurice Stevenson Bartlett. The spectral analysis of point processes. Journal of the Royal Statistical Society: Series B (Methodological), 25(2):264–281, 1963.

[3] Y. Birkelund, A. Hanssen, and E. J. Powers. Multitaper estimators of polyspectra. Signal Processing, 83:545–559, 2003.

[4] Ross C Caton, Gary L Pavlis, David J Thomson, and III Vernon, Frank L. Methods for the robust computation of the long-period seismic spectrum of broad-band arrays. Geophysical Journal International, 222(3):1480–1501, 05 2020.

[5] Alan D Chave. A multitaper spectral estimator for time-series with missing data. Geophysical Journal International, 218(3):2165–2178, 2019.

[6] Alain Connes and Henri Moscovici. The uv prolate spectrum matches the zeros of zeta. Proceedings of the National Academy of Sciences, 119(22):e2123174119, 2022.

[7] Proloy Das and Behtash Babadi. A bayesian multitaper method for nonstationary data with application to eeg analysis. In 2017 IEEE Signal Processing in Medicine and Biology Symposium (SPMB), pages 1–5. IEEE, 2017.

[8] Proloy Das and Behtash Babadi. Dynamic bayesian multitaper spectral analysis. IEEE Transactions on Signal Processing, 66(6):1394–1409, 2017.

[9] Proloy Das and Behtash Babadi. Multitaper spectral analysis of neuronal spiking activity driven by latent stationary processes. Signal Processing, 170:107429, 2020.

[10] William D. Frazer, Adrian K. Doran, and Gabi Laske. Benchmarking Automated RayleighWave Arrival Angle Measurements for USArray Seismograms. Seismological Research Letters, 93(2A):763–776, 11 2021.

[11] William D. Frazer and Jun Korenaga. Dynamic topography and the nature of deep thick plumes. Earth and Planetary Science Letters, 578:117286, 2022.

[12] William D Frazer and Jeffrey Park. Seismic evidence of mid-mantle water transport beneath the yellowstone region. Geophysical Research Letters, 48(20):e2021GL095838, 2021.

[13] F. Alberto Grnbaum. Serendipity strikes again. Proceedings of the National Academy of Sciences, 119(26):e2207652119, 2022.

[14] Arthur P Guillaumin, Adam M Sykulski, Sofia C Olhede, and Frederik J Simons. The debiased spatial Whittle likelihood. arXiv preprint arXiv:1907.02447, 2019.

[15] C. L. Haley. Missing-data multitaper coherence estimation. IEEE Signal Processing Letters, 9 2021.

[16] Jeffrey A Hogan and Joseph D Lakey. Spatio–spectral limiting on redundant cubes: A case study. In Excursions in Harmonic Analysis, Volume 6, pages 97–115. Springer, 2021.

[17] Jeffrey A Hogan and Joseph D Lakey. Spatio-spectral limiting on boolean cubes. Journal of Fourier Analysis and Applications, 27(3):1–26, 2021.

[18] Jeffrey A Hogan and Joseph D Lakey. Spatio-spectral limiting on discrete tori: adjacency invariant spaces. Sampling Theory, Signal Processing, and Data Analysis, 19(2):1–38, 2021.

[19] Jeffrey A Hogan, Joseph D Lakey, and Joseph D Lakey. Duration and bandwidth limiting: prolate functions, sampling, and applications. Springer, 2012.

[20] Michel Loéve. Fonctions aléatoires du second ordre. Processus stochastique et mouvement Brownien, pages 366–420, 1948.

[21] L. T. McWhorter and L. L. Scharf. Multiwindow estimators of correlation. IEEE Trans. on Signal Processing, 46:440–448, 1998.

[22] Tuomas A. Rajala, Sofia C. Olhede, and David J. Murrell. What is the Fourier transform of a spatial point process?, 2020.

[23] Peter J Schreier and Louis L Scharf. Statistical signal processing of complex-valued data: the theory of improper and noncircular signals. Cambridge university press, 2010.

[24] Frederik J Simons, FA Dahlen, and Mark A Wieczorek. Spatiospectral concentration on a sphere. SIAM review, 48(3):504–536, 2006.

[25] D. Slepian and H. O. Pollak. Prolate spheroidal wave functions, Fourier analysis and uncertainty -I. Bell System Tech. J., 40(1):43–64, 1961.

[26] David Slepian. Estimation of signal parameters in the presence of noise. IRE Trans. on Information Theory, 3:68–89, 1953.

[27] David Slepian. Prolate spheroidal wave functions, Fourier analysis, and uncertainty V: the discrete case. Bell System Tech. J., 57:1371–1429, 1978.

[28] Adam M Sykulski, Sofia C Olhede, Arthur P Guillaumin, Jonathan M Lilly, and Jeffrey J Early. The debiased whittle likelihood. Biometrika, 106(2):251–266, 2019.

[29] Shuhan Tang, Peter F. Craigmile, and Yunzhang Zhu. Spectral estimation using multitaper Whittle methods with a LASSO penalty. IEEE Transactions on Signal Processing, 67(19):4992–5003, 2019.

[30] D. J. Thomson. Spectrum estimation and harmonic analysis. Proceedings of the IEEE, 70(9):1055–1096, 1982.

[31] D. J. Thomson. Multi-window bispectrum estimates. In Proc. IEEE Workshop on Higher-order spectral analysis, pages 19–23, Vail, Colorado, 1989. IEEE Press.

[32] D. J. Thomson. Quadratic-inverse spectrum estimates: applications to paleoclimatology. Phil. Trans. R. Soc. Lond. A, 332:539–597, 1990.

[33] D. J. Thomson. Dependence of global temperatures on atmospheric CO_2 and solar irradiance. Proceedings of the National Academy of Sciences, 94:8370–8377, 1997.

[34] D. J. Thomson and C. L. Haley. Spacing and shape of random peaks in non–parametric spectrum estimates. Royal Society Proceedings Series A, 470:21, 2014.

[35] D. J. Thomson, L. J. Lanzerotti, F. L. Vernon, III, M. R. Lessard, and L. T. P. Smith. Solar modal structure of the engineering environment. Proceedings of the IEEE, 95:1085–1132, 2007.

[36] D. J. Thomson, C. G. Maclennan, and L. J. Lanzerotti. Propagation of solar oscillations through the interplanetary medium. Nature, 376:139–144, 1995.

Appendix G

Mathematical Challenges in Computational Chemistry: Multiscale, Multiconfigurational Approaches, Machine Learning (22w2262)

July 08 - 10, 2022

Organizer(s): Sergey Gusarov (National Research Council Canada), Alexander E. Kobryn (National Research Council Canada), Stanislav Stoyanov (Natural Resources Canada), Valera Veryazov (Lund University, Sweden)

The meeting took place in July 2022 right after the largest international computational chemistry event WATOC 2022 in Vancouver, BC. The connection to the WATOC allowed us to invite to our symposium leading scientists from around the world (Canada, Japan, Sweden, US). The format of the meeting was in-person with the total number of participants 15 (during the meeting one person participated by video conferencing from the BIRS hotel room because of the positive COVID-19 test on the arriving day).

Overview of the Field

In the last two decades theory and modeling turned to become one of the major topics of applied chemistry along with analytic, synthetic, and other chemistry fields. This made possible because of significant improvements in methodology, numerical methods, and computer software and hardware. Much experimental research started to include computational modeling. The role of computer simulation in modern chemistry cannot be overestimated and the use of effective modeling and simulation plays a critical role in practical applications by providing insights into experiments and helping in system optimization. Specifically, simulations are more and more often used to substitute dangerous and expensive experiments with calculations. At the same time, the impressive progress of modern experimental research in material science and biology necessitates further developments and continuous extension of the applicability and accuracy of nowadays computational chemistry methods. The fast but accurate qualitative and quantitative modeling of large biological molecules, nanoparticles, and interfaces becomes the main focus of the research which requires significant computational efforts and is not always achievable at the current technology level. Most of the computational chemistry problems are about solving the Schrödinger equation for electrons in molecules or the Newton equations of motion for a system of classical particles. Consequently, the mathematics should play the central role in the new developments. The primary purpose of this workshop was to analyse the current needs and expectations of computational chemistry based on the experience provided by top leading scientists and discuss them with the methodology and computational software developers. The following sections have their names after the workshop sessions and comprise both the topics suggested in the initial presentations and topics brought to the surface during the round-table discussions and interpersonal talks.

Multiscale approach

It is convenient to exhibit the modern computational chemistry methods on length and time scale diagram, Figure G.1, where each category of approaches (e.g., ab initio or molecular mechanics) are approximately illustrated by rectangular box positioned according to its applicability (lower left corner) and computational cost (upper right corner). In the case of logarithmic time and length scale the boxes form almost linear hierarchical structure with overlapping regions where the corresponding methods could be applied to model the system of interest. These regions have a very important meaning in the methodology development and practical applications as they allow to verify/estimate the accuracy of coarser methods compare to their more accurate but computationally expensive counterpart. Typically, this could be done by averaging the detailed information from more precise approach and following comparison with the results of higher scale. In the ideal case, we would like to get accurate and detailed information for very large objects which is practically impossible. For example, in this hierarchy the chemical properties are a special interest in modern nano- and bio- sciences but they are only accessible within quantum chemical (ab-initio) and partially semi-empirical approaches which are limited by their polynomial (cubical in the case of LDA and GGA DFT and higher for more accurate approaches) scaling factors in respect to the size of system. Moreover, the application of quantum chemical approaches to the large systems will result in the huge amount of data unavailable to keep with the modern level of hardware. This led to in the principal restriction of modern computational chemistry which might result in the future competitive gap between experimental and computational chemistry.

A very attractive strategy to resolve these restrictions is to use machine learning (ML) methods enabling large-scale exploration of chemical space based on quantum chemical calculations. However, despite being fast and accurate for atomistic chemical properties, the modern ML models do not explicitly capture the electronic degrees of freedom of a molecule, which limits their applicability for reactive chemistry and chemical analysis. So, the new consistent descriptors are needed which are based on deep analysis of the structure of the Schrödinger equation. We will discuss the recent achievements in this area and analyse the pros and cons. Also, all other alternative developments are welcome to discuss during the workshop. For example, the general theory of multiscale techniques is intensively developing in the math community. This could provide a way to optimize the solution to Schrödinger equation, however these general math approaches should be translated to the practical language.

New methods in quantum chemistry

The session was moderated by Valera Varyazov, who also delivered a small talk followed by a brief overview by Victor Hugo Malamace da Silva. In conversation it was noticed that many breakthroughs in quantum chemistry have been inspired and driven by mathematical ideas. Use of a wavefunction in the form of determinant to provide a permutation symmetry of the electron wavefunction (the base of the Hartree-Fock theory) is a famous example of such influence. Cholesky decomposition is another example of a mathematical idea, which is used to reduce the amount of computed integrals in many computational codes. Unfortunately, such influence comes to applications with a significant delay (Cholesky decomposition is known from the beginning of 20th century, suggested to be used in 1977, and implemented about 40 years later). Furthermore, from the point of mathematics, quantum chemistry is a huge underdeveloped area: the solved equations can have multiple solutions, or be unstable. Extensive use of numerical approaches and approximations is not always justified. Although we do not have an immediate solution for the better use of mathematical ideas in quantum chemistry, we have to spot the problem with a hope for a change. Advances in the development of new hardware is another game changing factor for quantum chemistry. The “old” paradigm was based on the idea of limited resources (CPUs, memory, storage), and so promotes the reuse of computed data and batching of all calculations. With new hardware architectures the design of computational codes can be significantly revised and simplified at the same time.

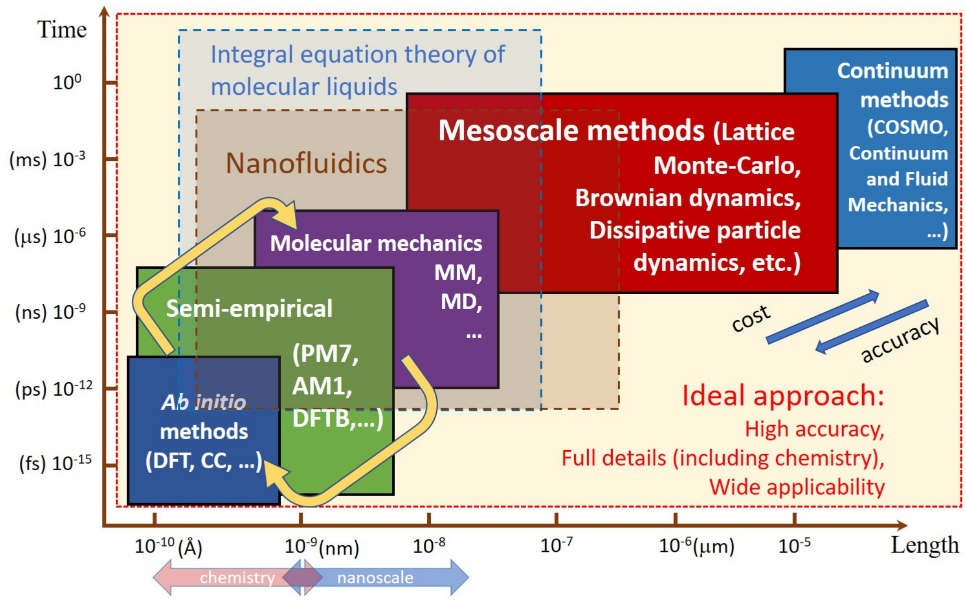

Figure G.1: Length-Time scales diagram.

New computational science ideas

The session, moderated by Stanislav R. Stoyanov, focused mainly on the highly promising and novel applications of ML in the field of computational chemistry. The opening presentation, delivered by Olga Lyubimova, started with an overview of ML, continued with an introduction to molecular featurization and representation, and culminated with a digest of recently proposed chemical descriptors for ML treatment. A heated discussion ensued on the trustworthiness and acceptance of computational chemistry results from ML compared to those from the traditional quantum chemistry-based results. Dr. Lyubimova effectively addressed the concerns, explaining that the mathematics behind ML is not only complex but also robust. Noting her initial cautious attitude towards ML and being trained as a quantum computational chemist, she shared that after taking ML training and carefully and comprehensively reviewing the literature on this novel topic, she started gradually gaining understanding and building confidence in the predictive capability of ML in computational chemistry. The selection of an ML method to employ was noted as a major challenge and an area for improvement towards automation. The consensus was that while quantum chemistry was based on solid physical and mathematical foundation it was very expensive computational and needed transformative performance improvements. In this context, mathematically sound ML approaches could be the necessary tools to speed-up computational chemistry research. It is noteworthy that the representatives with mathematical geology expertise were much more comfortable than computational chemists using ML, e.g., for image analysis, likely because the latter field did not have a solid physics-based predictive and interpretive framework comparable to the quantum theory. Several participants pointed out that while the complete replacement of quantum chemistry with ML would be premature, ML can effectively help address optimization and algorithm selection problems, thus helping accelerate quantum chemistry-based calculations. Moreover, hybrid density functional selection that is typically made based on the users experience or published recommendations could potentially be determined by using an ML algorithm, as it reflects the percentage of exchange and correlation included in the functional. Another area for improvement by using ML would be the correction for the dispersion interactions that were often done ad hoc, as these important interactions were not accounted by density functional theory, the most widely used quantum chemical method.

New ideas in the areas of mathematical libraries, algorithms, compilers, and hardware were discussed in brief because these were to a large extent covered in the two morning sessions. The importance of unified data formats and their key role in enhancing the communication between computational codes towards automation were also discussed, mainly in the context of the variety of abstract mathematical representations, e.g., using graphs or molecular fingerprinting, that were not always compatible and suffered from reconstruction inaccuracies. These unification and communication challenges were noted to arise due to different conventions used in diverse areas of computational chemistry. Enhancing the communication between computational codes required novel and improved algorithms to transcribe and convert among data formats.

Future development

This session was composed of several largely independent topics. All of them were initially suggested by the organizers and later followed by the participants. The session moderator was Alexander E. Kobryn. All the workshop participants actively took place in the debates. In addition, mini presentations on this matter were delivered by A.E. Kobryn and Gabriel Pereira da Costa. The following subsections summarize the most pertinent information and its discussion.

Acceleration of computations with GPUs

Graphics Processing Units (GPUs) are known as programmable processing units independently working from Central Processor Units (CPUs) and originally responsible for graphics manipulation and output. Because of their high performance in data processing, from the beginning of new millennium parallel GPUs started to be actively used for General Purpose Computing on GPU (GPGPU) and later found its way into fields of material science, computational chemistry, and quantum chemistry. Therefore, contemporary High Performance Computing (HPC) clusters often provide, in addition to the so-called regular compute nodes, the GPU nodes where computations can be run on both CPU and GPU cores. At the same time, big scientific software developers started providing the GPU support in their products. This information is easy to trace and can be found, e.g., at the HandWiki list of quantum chemistry and solid-state physics software [1]. In particular, one can identify that such big developers as ADF, GAMESS, Gaussian, MOLCAS, Quantum Espresso, VASP – to name a few – already support GPUs. At the same time, such popular products as DFTB+/++, DMol3, OpenMX, ORCA – also to name a few – do not yet account for the possibility to accelerate computations with the use of GPUs. The workshop participants have agreed and expressed a hope that the future development of the computational modeling software should include the GPUs support, and the architects of the next generation HPC clusters should continue equipping them with a set of GPU nodes. The workshop participants also agreed to circulate this expectation at any other relevant public events and through the interpersonal communications.

Quantum computing

In last decade, the most growing expectation with respect to the increase of the computation speed and complexity was about quantum computers – devices that perform quantum computing, a type of computation that harness collective properties of quantum states, such as superposition, interference, and entanglement [2, 3]. The basic unit of quantum information in quantum computing is quantum bit or qbit – a two-state quantum mechanical system often compared for simplicity with an imaginary spin-up/down system and represented as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A quantum memory may then be found in any quantum superposition $|\psi\rangle$ of the two states $|0\rangle$ and $|1\rangle$, i.e. $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with the coefficients $\alpha, \beta \in \mathbb{C}$, satisfying $|\alpha|^2 + |\beta|^2 = 1$, and called quantum amplitudes. The state of the quantum memory can be manipulated by applying quantum logic gates, in analogy to how classical memory can be manipulated with classical logic gates (AND, OR, XOR, NOT, etc.). With this respect, the most practical type of quantum computers at present seems the quantum circuit model, in which a computation is a sequence of quantum gates and measurements. The great expectations from quantum computing may be explained by the fact that quantum algorithms sometimes offer a polynomial or super-polynomial speed-up over the best known classical algorithms. Figure G.2 and Table G.1 show a schematic chart of a computing cost and a complexity scaling. For material science, computational chemistry, and quantum chemistry problems this factor is the decisive one and

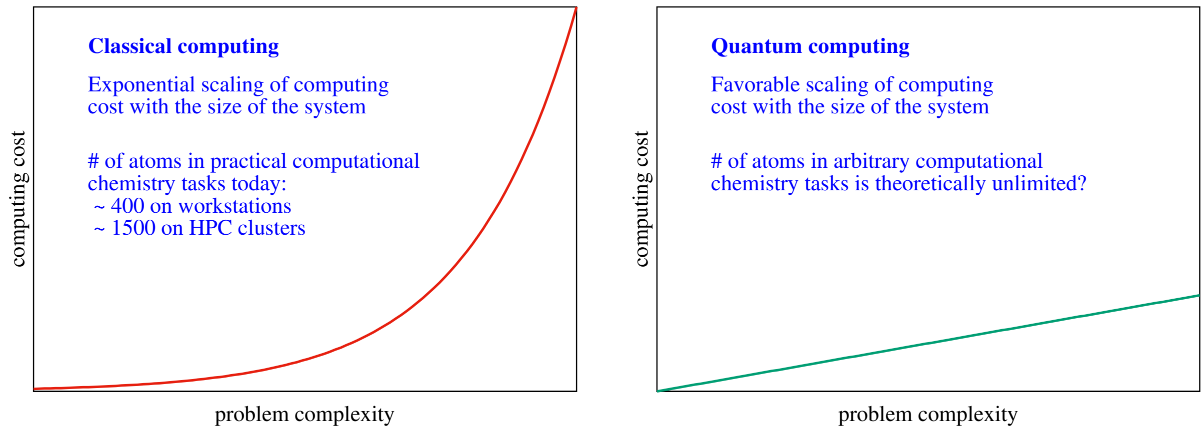

Figure G.2: A schematic chart of scaling of computing cost on classical and quantum computers with the increase of the system size or the problem complexity. On charts, the problem complexity increases from left to right and the computing cost increases from bottom to top.

will determine the technological progress in these fields, provided the engineering task of building a powerful quantum computer is solved. The workshop participants noticed in the discussion that technical challenges of this task include not only the problem of physical scalability to increase the number of qbits, but also building quantum gates that are faster than the decoherence time, and lowering the error rates and bringing them to the level of modern classical computers.

Incremental improvements of computer codes vs rewriting from scratch

In computational science each noticeable progress in the hardware development means that the software should be improved and refactored continuously all the time. Then, the principal question of the scale "to be or not to be" is shall the code improvement be incremental, little-by-little, or comprehensive, with rewriting the whole code from the beginning? A general answer to this question does not exist as every situation is worthy a thoughtful consideration. Before making decision it may be not bad idea to start from the time and cost assessments for the following categories: (i) time and cost of improving the existing code; (ii) time and cost of rewriting from scratch; (iii) time and cost of fixing bugs and adding new features; (iv) time and cost of updating and circulating instructions, manuals, tutorials, etc.; (v) time and cost of team management for each of these cases. A separate assessment should be for the level of impact the changes may have on scientific results and therefore appear for the users to be important and valuable or inessential and unappealing. Because of this, the picture we often observe over the years is that both small and big developers prefer improving the existing codes over rewriting them from scratch. The workshop participants agreed that the future development will rather not alter this picture and that the existing balance between the frequency of the so-called major and minor scientific software updates will be preserved. In both cases, however, one shall be ready to new bugs appearing in the rewritten software. Referring to the Preface of one famous textbook [4]: "In computer science it is generally assumed that any source code over 200 lines contains at least one error". Even so, the story does not end there. Quite oppositely, it marks the beginning of a new cycle in the never-ending line of updates.

How math can drive and speed up the development of computational chemistry

In recent decades, computational methods became major tools of theoretical studies. Accordingly, the mathematical models and numerical analysis that underlie these methods have an increasingly important and direct role to play in the progress of computational and quantum chemistry [5]. No wonder, the number of mathematical challenges in this area remains high. In our discussion we could mention the need for the following:

- Developing of accurate coarse grained models at moderate computational cost;

Table G.1: The performance of classical vs quantum computers for few selected subroutines that are critical for the execution of the entire algorithm. There is also a comparison of error rates and application areas.

Classical computers		Quantum computers	
Subroutine	Complexity	Subroutine	Complexity
Matrix inversion $AX = B \rightarrow X = A^{-1}B$	$O\left(N \log N\right)$	Matrix inversion $\hat{A}\|X\rangle = \|B\rangle \rightarrow \|X\rangle = \hat{A}^{-1}\|B\rangle$	$O\left((\log N)^2\right)$
Eigenvalues and eigenvectors of sparse/low-rank matrices	$O\left(N^2\right)$	Quantum phase estimation (a.k.a. Q-phase)	$O\left((\log N)^2\right)$
Fast Fourier transform	$O\left(N \log N\right)$	Quantum Fourier transform	$O\left((\log N)^2\right)$
Have low error rates (10^{-15}) and can operate at room temperature		Have high error rates (10^{-3}) and need to be kept at ultralow temperatures	
Are best for everyday numerical processing		Well suited for tasks like optimization problems, data analysis, and simulations	

- Developing models that exploit the multiscale nature of computational chemistry problems;
- Developing models that properly reflect and describe quantum stochastic processes;
- Development of appropriate treatment for strongly correlated valence electrons.

The mentioned above problems may have a better chance for a quicker and general solutions if they are tackled by teams composed of experts with complementary professional skills: mathematicians, physicists, chemists, programmers, engineers, etc. The workshop participants willingly shared their personal experience of participation in the past in such multi-expert groups. They also expressed the necessity of including additional advanced mathematical courses on non-mathematical university departments, especially if they are related with the material or computational science. In particular, the courses mentioned include complex calculus, functional analysis, mathematical statistics, differential and integral equations, operator calculus.

Summary

Based on our discussions we conclude that in order to correspond to the modern level of research on new materials, biology, and medicine the computational chemistry needs significant improvements. The most realistic way of success is to combine different approaches, like more efficient numerical methods and new science (e.g. AI/ML) or increase their efficiency on new hardware (e.g. GPUs). Mathematics plays a central role in this development as the whole field is focused on solving systems of integral and differential equations or their appropriate combination and correlation. Also, we believe that the basic mathematical background of students specializing in computational chemistry should be extended to catalyse the application and development of new methods. In addition to the traditionally studied mathematical fields, such as group theory or differential and integral equations, they need to be more familiarized in operators calculus, projection operator techniques, optimization, data analysis, etc.

To verify and expand the ideas discussed we have decided to apply for a 5-days BIRS workshop. In addition, we have expressed the intention to communicate the most interesting details of our discussions by publishing them in a peer-reviewed scientific journal.

Participants

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Takahashi, Ken (Kyoto University)
Veryazov, Valera (Lund University, Sweden)

Bibliography

[1] List of quantum chemistry and solid-state physics software
handwiki.org/wiki/List_of_quantum_chemistry_and_solid-state_physics_software.

[2] J.D. Hidary, Quantum Computing: An Applied Approach, 2^{nd} ed. Springer, Cham, 2021.

[3] M.S. Ramkarthik, P.D. Solanki, Numerical Recipes in Quantum Information Theory and Quantum Computing: An Adventure in FORTRAN 90, CRC Press, Boca Raton, 2022.

[4] D. Frenkel, B. Smith, Understanding Molecular Simulation, 2^{nd} ed. Academic Press, San Diego, 2002.

[5] J. Leszczynski (ed.), Handbook of Quantum Chemistry, 2^{nd} ed. Springer, Cham, 2017.

Appendix H

2022 Math Attack Summer Camp for Girls (22w2001)

July 15 - 17, 2022

Organizer(s): Lauren DeDieu (University of Calgary), Sean Graves (University of Alberta)

Description

The 2022 Math Attack Summer Camp for Girls was an 8-day overnight camp that was held at the University of Calgary and the Banff International Research Station (BIRS) from Sunday, July 10th - Sunday, July 17th. The camp brought 21 grades 6 - 10 students who identify as girls together to engage in fun mathematical activities and build connections. Students stayed in the university residence for the first five nights of the camp and stayed at the Banff Centre for the last two nights.

The camp aimed to encourage girls to pursue their passion for mathematics and make connections with peers who shared similar interests. Throughout the week, students engaged in mathematical sessions that explored topics such as cryptology, data science, probability paradoxes, and actuarial science. They investigated the spread of disease by modelling a zombie outbreak, learned what you can do with a math degree at a Women in Math Panel, and competed in the Amazing (Math) Race. These sessions and panels exposed students to over 20 female role models, including recent high school graduates, undergraduate math students, graduate math students, mathematics faculty, and mathematicians in industry.

During the camp, there was also plenty of time for friendship building and physical activity. Evening activities included sports, swimming, board games, karaoke, and a walk along Bow Falls Trail. On Friday, students also took some time to explore the town of Banff and hiked up Tunnel Mountain.

There was no registration fee for the camp and all meals and accommodations were provided.

Schedule

Time	Sunday July 10th	Monday July 11th	Tuesday July 12th	Wednesday July 13th	Thursday July 14th	Friday July 15th	Saturday July 16th	Sunday July 17th
8 – 9am		Breakfast	Breakfast	Breakfast	Breakfast	Breakfast	Breakfast	Checkout + Breakfast
9 – 10:15am		Visualizing the Pythagorean Theorem (Jenny Lawson) MS 431	The Amazing (Math) Race (Jenny Lawson) MS 431	Modeling Zombies (Ariane Cantin) MS 431	Data Science (Katie Burak) MS 571	Bus to Banff	Mathematical Communication (Lauren DeDieu)	Combinatorics (Dami Wi)
10:15 – 10:30am		Break	Break	Break	Break		Break	Break
10:30 – 11:45am		Probability Paradoxes* (Keira Gunn) ENA 201	The Amazing (Math) Race (Jenny Lawson) MS 431	Modeling Zombies (Ariane Cantin) MS 431	Data Science (Katie Burak) MS 571	Drop-off luggage and explore Banff (*participants purchase their own lunch*)	Undergraduate Student Panel (hosted by Michelle Mo)	Feedback + Closing Ceremony
11:45am – 1pm		Lunch	Lunch	Lunch	Lunch		Group Photo + Lunch	Lunch
1 – 2:15pm		Callysto Hackathon (Rania Mahdi & Jenny Lee) MS 571	Free Time	Women in Math Panel* (hosted by Kristine Bauer) ENA 201	Data Science (Katie Burak) MS 571	Hike (Tunnel Mountain)	Free Time	Bus to Calgary
2:15 – 2:30pm		Break	Break	2:15 – 3pm: Women in Math Meet & Greet MS 457	Break		Break	
2:30 – 3:45pm		Actuarial Science (Ella Charpentier) MS 431	Probability Paradoxes (Keira Gunn) MS 431	3 – 4pm: Free Time	Data Science (Katie Burak) MS 571		Mathematical Card Tricks (Lauren DeDieu)	Departure (Aurora Hall)
3:45 – 4pm		Feedback	Feedback		Feedback		Break	
4 – 5:30pm	Arrival and Registration (Aurora Hall)	Prepare Questions for Panel/ Puzzle Day	Monty Hall* (Vince Chan) ENA 201	Math Contest (Vince Chan) MS 431	Problem of the Day	Check-in and Free Time	Cryptology: Classical Ciphers (Lauren DeDieu)	
5:30 – 6:30pm	Dinner	Dinner	Dinner	Dinner	Dinner	Dinner	Dinner	
6:30 – 9pm	Introduction, Ice Breaker Activities MS 431	Sports	Board Games	Feedback/ T-Shirt Design	Movie	Origami (Dami Wi)	Walk (Bow Falls Trail)	

BIRS Highlights

During the BIRS portion of the camp, the focus was on helping students develop their mathematical logic and communication skills. Sessions began on Saturday with an introduction to mathematical communication. Since many K-12 schools do no emphasize communication, this concept was new to many students. We discussed the importance of communicating results precisely, using correct notation and prose to help the reader navigate. Students then broke into teams and went to the breakout rooms in the basement of BIRS to solve a mathematical logic problems and write their solutions as elegantly as possible; students then ranked the other groups' solutions based on the quality of communication and Dr. Lauren DeDieu ranked them as well and provided feedback. At the end of the session, a winner was announced. In the afternoon, students continued to develop their mathematical proof-writing skills by learning mathematical card tricks and working to explain why the tricks work.

Over the weekend, students also learned about cryptology, combinatorics, and had the opportunity to network virtually with undergraduate and graduate students from Harvard University, MIT, and Caltech; this panel of women shared their experience studying mathematics and answered our participants' many questions.

Outcomes of the Meeting

This camp helped inspire our female participants to pursue their passion for mathematics by making connections with female role models and peers who share similar interests. This is reflected in the following quotes from our participants:

- This camp encouraged me to pursue my passion for math because I got to see other people like me and some others to look up to.
- This camp made me realize that having a passion in math could be turned into a very interesting career with applying mathematics to different parts of the world.
- This camp changed the way I viewed mathematics as there are so many things that it is. I used to think that it is plain calculations, but it turns out that there is much more, such as programming, data science, and communication.
- This camp has probably taught me more about the field of stem than any experience i have ever experienced prior.
- This was a once in a lifetime experience, and I really loved it. The supportive attitude of everyone made it feel like home, and in only a week Ive gotten so attached to everyone. It feels like weve been together for so long now. There are so many new concepts and unique ways of thinking that are going to stick with me for the rest of my life, as well as precious school advice from dozens of professionals and current students. If I could go back in time and do it again, I would.

- Thank you so much for opening my eyes up to the world of data science and more! I learnt about so many different careers that I believe I could potentially and realistically pursue. I loved seeing women that were so enthusiastic and confident in their areas, and it was inspiring to take a glimpse of what my future held by seeing the careers of those women.

- It was amazing and I felt really at ease, surrounded by girls like me. I never had to wonder, will they like me? Will they judge my sunburnt Vaseline-covered face? like I would around boys. Often, when Im surrounded by guys, I worry more about my appearance or my presentation or my attitude. But around my peers, I was comfortable with being myself and speaking my mind. Not only was this camp an eye opening experience, but the best part was my friends.

- This camp was such an insightful and inspiring experience. Here, I have not only learned about STEM and womens' contributions, but also about the future of our understanding of our universe, through math, computer science and statistics. The classes were incredibly engaging and allowed me to get exposure to entire new ideas and fields. As well, I have connected with other outstanding and spectacular young women with whom I share similar interests to. I am sure this opportunity will build lasting relationships. This camp is an once in a lifetime experience to learn and explore with direction in STEM fields, while having an astronomical amount of fun.

- I loved being able to talk with the other girls and make connections! I think the people was what made this camp so fun!

Additional Information

Additional photos and information about the camp can be found in the Final Report that is available here: https://science.ucalgary.ca/mathematics-statistics/engagement/educational-outreach/math-attack .

Participants

Aggarson, Jeena (Student)
Andrew, Claire (Student)
Cao, Lisa (Student)
Chen, Michelle (Student)
Chen, Trinity (Student)
DeDieu, Lauren (University of Calgary)
Goyal, Isha (Student)
Hua, Jenny (High School Student)
Ji, Yonghua (Student)
Jiao, Bella (Student)
Kahlon, Karmin (Student)
Li, Sunny (Student)
Li, Ziyu (Student)
Lu, Ashlyn (Student)
Miao, Julia (Student)
Mo, Michelle (High School Student)
Nie, Milly (Student)
Pan, Chenwei (Student)
Pyatalova, Sofia (Student)
Wang, Iris (Student)
Wi, Dami (University of Alberta (Alumni))
Yang, Emily (High School Student)
Yang, Alice (High School)
Zhai, Chloe (Student)
Zhang, Caroline (Student)

Appendix I

Almost Periodicity in Aperiodic Order (22w2232)

September 9 - 11, 2022

Organizer(s): Michael Baake[1] (Bielefeld University), Natalie P. Frank[2] (Vassar College), Nicolae Strungaru[3] (MacEwan University)

Overview of the Field

Objects with long-range order but no translational symmetry have been of longstanding interest, but the field of Aperiodic Order was catalyzed by the discovery of quasicrystals in the early 1980s. An important goal of this field is to gain a better understanding of mathematical diffraction theory, especially for systems with a significant Bragg (or point) diffraction spectrum but crystallographically forbidden symmetry. Aperiodic order connects various different areas of mathematics, such as harmonic and Fourier analysis, spectral theory of dynamical systems, discrete geometry, number theory, and topology, to name just a few. It was the primary goal of this meeting to bring people from various subareas of Aperiodic Order together.

Recent Developments, Open Problems and Presentations

Pure point spectrum

In recent years, the connection between almost periodicity and pure point diffraction has become clearer. This connection has already appeared implicitly in the work of Hof and Solomyak, and explicitly in the work of Lagarias and Baake–Moody. Building and expanding on extensive previous work in this direction, the connection was recently fully characterized by Lenz, Spindeler and Strungaru.

Indeed, a measure ω has pure point diffraction exactly when it is mean almost periodic, while pure point diffraction and the so-called consistent phase property essentially is equivalent to Besicovitch mean almost periodicity. For dynamical systems, pure point spectrum is equivalent to the Besicovitch mean almost periodicity of almost all

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elements, while pure point spectrum, unique ergodicity and continuity of eigenfunctions can be characterized in terms of Weyl mean almost periodicity.

Some of the most interesting examples of highly ordered aperiodic point sets are of number-theoretic origin. Systems like square-free integers (the support of the Möbius function) and the visible points of a lattice fit into the larger class of weak model sets of maximal density, which is a new direction in the field. These systems have pure point (diffraction) spectrum, and the connection to the Besicovitch mean almost periodicity starts to become visible. The full characterisation of model sets and of weak model sets of maximal density via almost periodicity remain two important questions in this direction.

At this conference, systems with pure point spectrum appeared in the talks of Till Hauser, Jan Mazáč and Felipe Garcia-Ramos. Additionally, Jeffrey Lagarias introduced a new aperiodic system based on the floor quotient function, and Lorenzo Sadun explained how the topology of aperiodic tilings affects the mass distributions they admit.

Systems with mixed spectra

As the case of pure point spectrum gets better understood, several people have shifted their interest towards models with mixed spectra. The Eberlein decomposition for weakly almost periodic measures provides a tool into the individual study of the pure point and the continuous spectrum, respectively. The further study of the absolutely continuous and the singular continuous diffraction spectrum becomes more subtle, as the refined Eberlein convolution is so far only established for measures with Meyer set support. Its existence in general remains an important open question in this area.

A recent new direction is the study of substitution tilings on compact alphabets. These systems generalize substitution tilings, and it is expected that many of the resulting objects have mixed (diffraction) spectra. These objects were covered in the talk of Neil Mañibo. Systems with mixed spectrum also occurred in the talk of Reem Yassawi.

Scientific Progress Made

The main goal of the meeting was to introduce some of the new directions in the field to all participants and encourage the collaboration between people from different subfields of aperiodic order. In this direction, the meeting was a success, and there were many discussions between the talks and at the end of the day, which will likely start many new collaborations.

Unfortunately, due to the hybrid format of the meeting, the online participants were often left out of the ongoing discussions, and some potential exchanges of ideas were missed.

Outcome of the Meeting

The area of Aperiodic Order is a new and fast growing area, and conferences like this are essential for its development, especially after the challenging CoVid years. While some international collaborations continued during the CoVid shutdowns, the pandemic issue made it hard for new international collaborations to start. The BIRS meeting created a great opportunity for discussions and new collaborations, which will positively impact the field in the future.

Participants

Baake, Michael (Bielefeld University)
Berthe, Valerie (IRIF)
Bustos-Gajardo, Alvaro (Open University)
Coons, Michael (California State University, Chico)

Cortez, Maria Isabel (Pontificia Universidad Católica de Chile)
Damanik, David (Rice University)
Emilsdottir, Iris (Rice University)
Frank, Natalie (Vassar College)
Gähler, Franz (Universität Bielefeld)
García-Ramos, Felipe (Universidad Autónoma de San Luis Potosi)
Glubokov, Andrey (Purdue University)
Grimm, Jasper (University of York)
Hauser, Till (Max Planck Institute Bonn)
Humeniuk, Adam (MacEwan University)
Kellendonk, Johannes (Université Claude Bernard Lyon 1)
Keller, Gerhard (Universität Erlangen-Nürnberg)
Klick, Anna (Bielefeld University)
Korfanty, Emily Rose (University of Alberta)
Lagarias, Jeffrey (University of Michigan)
Lee, Jeong-Yup (Catholic Kwandong University)
Lenz, Daniel (Friedrich-Schiller-Universität Jena)
Manibo, Neil (University of Bielefeld/Open University)
Mazac, Jan (Bielefeld University)
Miro, Eden Delight (Ateneo de Manila University)
Moody, Robert (University of Victoria)
Pouti, Aisling (MacEwan University)
Richard, Christoph (FAU Erlangen-Nuremberg)
Robinson, E. Arthur (George Washington University)
Rust, Dan (The Open University)
Sadun, Lorenzo (University of Texas at Austin)
Sell, Daniel (Nicolaus Copernicus University)
Sing, Bernd (University of the West Indies)
Spindeler, Timo (University of Bielefeld)
Staynova, Petra (University of Derby)
Strungaru, Nicolae (MacEwan University)
Walton, Jamie (University of Nottingham)
Walton, Jamie (University of Nottingham)
Whittaker, Mike (University of Glasgow)
Yassawi, Reem (Open University)

Appendix J

Recent Progress in Detection and Prediction of Epilepsy (22w2244)

October 14 - 16, 2022

Organizer(s): Elena Braverman (Faculty of Science, University of Calgary), Gordon Campbell Teskey (Cumming School of Medicine, University of Calgary)

Overview of the Field

Human epilepsy is a common neurological disorder that is characterized by abnormal brain activity resulting in seizures. Approximately 0.6% of the Canadian population has epilepsy. The International League Against Epilepsy classified seizure types into 3 major groups, namely generalized onset seizures, focal onset seizures, and unknown onset seizures. Temporal lobe epilepsy (TLE) is the most common form of focal epilepsy. About 6 out of 10 people with focal epilepsy have temporal lobe epilepsy. Seizures in TLE start or involve in one or both temporal lobes in the brain.

Different experimental methods such as electroencephalography (EEG), magnetoencephalography (MEG), functional magnetic resonance imaging (fMRI), and proton emission tomography (PET) and in drug-resistant cases, intracranial EEG (iEEG) are utilized by teams of mathematicians, neurologists, epileptologists, neuroscientists and practicing doctors to diagnose epilepsy. The most commonly used EEG recordings measure electrical brain activity through a grid of electrodes or depth electrodes depending upon if the measurements were made on the scalp or intracranially.

The main purpose of this workshop was to bring together mathematicians, experts in neuroscience, psychology, and practicing medical doctors in order to exchange opinions and report on mathematical analyses of neurological data connected to seizures. The detection and prediction of different types of TLE using the data are two important topics of discussion in this workshop.

The neurological data are intracranial EEG data collected from drug-resistant TLE patients prior to resection surgery. The electric signals measured with iEEG are non-linear. For mathematical analysis of the data, we extensively use complex network theory, signal theory, non-linear time-series analysis, integral transforms, neuronal modelling, statistics and visualization techniques. For example, our recent results on bi-lateral tonic-clonic and impaired awareness seizures using transform methods such as phase coupling and variations of phase coupling methods suggest possible interpretations of the seizures during the pre-ictal, ictal, and post-ictal periods. Non-linear time-series analysis led to investigate the efficacy of estimating the largest Lyapunov exponents at all stages of the above-mentioned seizures. Detection and prediction components are inherent in these methods to seek diagnostics answers. Finally, focal onset seizures, impaired awareness seizure or bi-lateral tonic-clonic seizure, engage different regions of the brain. With this in mind, complex network analysis is performed to understand the community structure of the brain regions at different stages of the seizures.

Presentation Highlights and Scientific Progress

The lectures included a combination of talks focused on a rigorous dynamical systems approach to the brain functioning (Jörn Davidsen and Wilten Nicola) with extensive description of advanced experimental results and comparative analysis of brain functioning (Majid Mohajerani and Asad Beck), either human or implemented with laboratory mice, and machine learning and engineering approaches (Artur Luczak).

Application of Hilbert transform to the processed data for Foothills hospital patients suffering from epileptic seizures combined with the following windowed time-frequency analysis and tedious time frequency correlation analysis was illustrated in the talks of Daniel Girvitz and Yanina Bazhan.

The presentation of Jörn Davidsen focused on the connection of the chimera states with the critical brain hypothesis. One of the pillars of modern physics is the concept of symmetries. Spontaneously breaking such symmetries typically gives rise to non-trivial phenomena and can explain, e.g., why particles have mass. The occurrence of such symmetry-breaking phenomena is not limited to particle physics but occurs across a wide range of physical, chemical and biological systems. Recently discovered examples include chimera states. Chimera states are hybrid states characterized by the coexistence of localized synchronized and unsynchronized dynamics in a given system. Indeed, the name chimera is used here in analogy to the hybrid creature in Greek mythology. Such coexisting behavior can even occur in a homogeneous system, thus breaking the underlying symmetry - something that was long thought to be impossible. While over the last 20 years a significant mathematical understanding of this phenomenon has started to emerge, many challenges are left to be addressed. One particular challenge is the presence and role of chimeras in the context of epilepsy. Preliminary work by us and others has shown not only that chimera states might be play a role in brain dynamics but that they are in particular present at the onset of epileptic seizures. This opens the door for new mathematical approaches to understand and eventually control epileptic seizures and was discussed in detail at the workshop.

The lab of Artur Luczak is using electrophysiological and machine learning methods to study information processing in the brain. One of our main contributions is a development of neuronal packet concept, which describes basic building blocks of neuronal code (Nature Rev Neurosci 2015, Neuron 2013, J Neurosci 2013, Neuron 2009, PNAS 2007). Moreover, they derived a predictive learning algorithm from basic cellular principles, i.e. from maximizing metabolic energy of a neuron, which may offer a step toward a general theory of neuronal learning (Nature Machine Intelligence; 2022). The lab is also studying changes in neuronal activity caused by neurological disorders, especially epilepsy (Brain 2017). To facilitate it, our lab developed Deep Neural Networks for detecting neurological deficits (PLOS Biology 2019).

The work presented by Scott Rich, Taufik Valiante, and Jeremie Lefebvre provided a detailed survey of the collaboration between the Valiante and Lefebvre labs (Dr. Rich is a postdoc co-supervised by Valiante and Lefebvre) applying tools from computational neuroscience and dynamical systems to the study of epilepsy. In mathematical models of epileptogenic neuronal circuits, a common analogue for seizure-onset is the sudden transition of circuit activity from asynchronous firing into hyper-synchronous and hyper-active neuronal oscillations. Mathematical analyses have historically shown that such dynamics are commonly associated with bifurcations, specifically those causing the fixed point of simplified neuronal circuits to shift from stable to an unstable oscillator. The group showcased multiple types of computational circuits in which these behaviors arise, as well as different mechanisms by which these transitions can be disrupted.

One such mechanism is the system's noise: increasing the intrinsic noise in an epileptogenic circuit or adding extrinsic noise (abstractly modeling potential new paradigms for neurostimulation via implantable neuromodulatory devices) stabilize network dynamics and mitigate the sudden transitions echoing seizure onset. Another mechanism is the intrinsic heterogeneity of the model neuron's intrinsic properties, a phenomenon which importantly has strong experimental support from the characterization of human neurons taken from non-epileptogenic and epileptogenic cortical tissue samples. In both cases, these mechanisms prevent sudden transitions into hyper-synchronous and hyper-active dynamics in spiking networks, and suppress multistability and bifurcations in associated stability analyses of systems simplified using mean-field analysis.

The group presented additional work on two fronts: how bistability can explain the tendency for increased interneuronal activity prior to seizure onset, and how neuronal heterogeneity might serve a physiological role in dynamical homeostasis in the brain. All the presented studies showcase the power of interdisciplinary research, particularly collaborations between experimental, computational, and mathematical neuroscientists, in the study of epilepsy. A myriad of pathological changes associated with epilepsy can be recast as decreases in cell and circuit heterogeneity. The group proposed recontextualizing epileptogenesis as a process where reduction in cellular heterogeneity, in part, renders neural circuits less resilient to seizure. By comparing patch clamp recordings from human layer 5 (L5) cortical pyramidal neurons from epileptogenic and non-epileptogenic tissue, we demonstrate significantly decreased biophysical heterogeneity in seizure-generating areas. Implemented computationally, this renders model neural circuits prone to sudden transitions into synchronous states with increased firing activity, paralleling ictogenesis. These findings are extended to the concept of resilience, where a systems dynamics persist despite changes in intrinsic and/or extrinsic control parameters, preserving associated function. To do this we computationally explore how excitability heterogeneity can influence system resilience to ”insults” like increases in network size, connection probability, strength and variability of synaptic weights, and modulatory fluctuations which promote stability transitions. The group found that excitability heterogeneity rendered the network more resilient to these insults. To then understand these computational findings, we used spectral theory for large random systems to reveal that excitability heterogeneity is a generic control mechanism promoting: 1) homeostasis, by tuning the distribution of eigenvalues complex plane in a context-dependent way; and 2) resilience, by anchoring this eigenvalue distribution and gradually making it less dependent on modulatory influences. Taken together, these results provide new vistas on the contribution of a fundamental organising principle of the brain - neural diversity - to brain resilience.

Some comments on the meeting

When planned in 2020, the meeting was expected to be co-organized with Kris Vasudevan who suddenly passed on August 22, 2022, so a part of the morning meeting was dedicated to his memory and featured a presentation of the department head (Tony Ware) where Kris worked, memories of his colleagues (E. Braverman, G.C. Tesky, J. Davidsen) and his son Alexander Vasudevan.

Due to the original COVID restrictions and other reasons, the workshop was not a large meeting but it included presenters at all the stages of their career: an undergraduate (Daniel Girvitz), a MSc (Yanina Bazhan), a PhD student (Asad Beck), a postdoctoral fellow (Scott Rich), all the others were faculty members from mathematics and statistics, biology departments, schools of medicine, two Brain institutes (Ottawa and Calgary).

Outcomes of the Meeting

The workshop brought in new results and critical comments from the participants, and fostered development of new methods and approaches. Also, it served as an ideal platform for associated postdoctoral fellows and graduate students to report their most recent results and exchange ideas with the experts. The results of the research groups are published in the front-line journals in neuroscience, biology and mathematical modeling, and there are more to come. We believe the useful exchange of opinions and new ideas will lead to new collaborations and stimulate progress in the area of describing, predicting and treatment of epilepsy, as well as our understanding how the brain works.

Participants

Bazhan, Yanina (University of Calgary)
Beck, Asad (University of Washington)
Braverman, Elena (University of Calgary)
Campbell, Sue Ann (University of Waterloo)
Cavers, Michael (University of Toronto Scarborough)

Davidsen, Joern (University of Calgary)
Girvitz, Daniel (University of Calgary - Dept. of Mathematics and Statistics)
Lefebvre, Jeremie (University of Ottawa)
Luczak, Artur (University of Lethbridge)
Mohajerani, Majid (University of Lethbridge)
Nicola, Wilten (University of Calgary)
Patton, Andrew (University of Calgary)
Rai, Juhi (Shri Ramdeobaba College of Engineering and Management)
Rich, Scott (Krembil Brain Institute)
Teskey, Gordon (University of Calgary)
Valiante, Taufik A. (Krembil Brain Institute)
Vasudevan, Alexander (Oxford University)
Ware, Tony (The University of Calgary)

Focused Research Group Reports

Appendix K

Computability and Complexity of Statistical Behavior of Dynamical Systems (22frg253)

March 13 - 20, 2022

Organizer(s): Michael Yampolsky (University of Toronto), Cristóbal Rojas (Universidad Católica de Chile)

For all practical purposes, the world around us is not a deterministic one. Even if a simple physical system can be described deterministically, say by the laws of Newtonian mechanics, the differential equations expressing these laws typically cannot be solved explicitly. Computers are generally not much help either: of course, a system of ODEs can be solved numerically, but the solution will inevitably come with an error due to round-offs of computations and inputs. Commonly, solutions of dynamical systems are very sensitive to such small errors (the phenomenon known as "Chaos"), so the same computation can give wildly different numerical results.

Of course, this difficulty is well known to the practitioners, who analyze chaotic dynamical systems in the language of statistics, based on what is broadly known as *Monte Carlo* technique, pioneered by Ulam and von Neumann in 1946 [6]. Informally speaking, we can throw lots of random darts to select a large number of sets of initial values; run our simulation for the desired duration for each of them; then statistically average the outcomes. We then expect these averages to reflect the true statistics of our system. To set the stage more formally, let us assume that we have a discrete-time (a continuous-time case will require an obvious adjustment) dynamical system

$$f : D \to D, \text{ where } D \text{ is a finite domain in } \mathbb{R}^n$$

that we would like to study. Let $\bar{x}_1, \ldots, \bar{x}_k$ be k points in D randomly chosen, for some $k1$ and consider the probability measure

$$\mu_{k,n} = \frac{1}{kn} \sum_{l=1}^{k} \sum_{m=1}^{n} \delta_{f^{\circ m}(\bar{x}_l)}, \tag{K.0.1}$$

where $\delta_{\bar{x}}$ is the delta-mass at the point $\bar{x} \in \mathbb{R}^n$. The mapping f can either be given by mathematical formulas, or stand for a computer program we wrote to simulate our dynamical system. We then postulate that for $k, n \to \infty$ the probabilities $\mu_{k,n}$ converge to a limiting statistical distribution of our system and thus we can use them to make meaningful long-term *statistical* predictions of its behavior.

Let us say that a measure μ on D is a *physical measure* of f if it is the weak limit of *Birkhoff sums* $\frac{1}{n} \sum_{m=1}^{n} \delta_{f^{\circ m}(\bar{x})}$ for a set of initial values $\bar{x} \in A \subset D$ with positive Lebesgue measure. This means that the limiting statistics of such points will appear in the averages (K.0.1) with a non-zero probability. If there is a unique physical measure in our dynamical system, then one random dart in (K.0.1) will suffice. Of course, there are systems with many physical measures. For instance, Newhouse [4] showed that a polynomial map f in dimension 2 can have infinitely many attracting basins, on each of which the dynamics will converge to a different stable periodic regime. This in itself, however, is not necessarily an obstacle to the Monte-Carlo method, and indeed, the empirical belief is that it still succeeds. Let us say that a map is *non-statistical* if Birkhoff sums do not converge to a well-defined limit on a positive measure set of initial values (D. Ruelle in [5] called such maps "historic").

The empirical belief in Monte Carlo method appears to be unfounded in some cases. C. Rojas and M. Yampolsky have considered the simplest examples of non-linear dynamical systems: quadratic maps of the interval $[-1, 1]$ of the form

$$f_a(x) = ax(1-x),\ a \in (0, 4]$$

and found values of a for which:

1. there exists a *unique* physical measure μ which is the weak limit of

$$\frac{1}{n}\sum_{m=1}^{n}\delta_{f_a^{\circ m}(x)}$$

   for Lebesgue almost all $x \in [0, 1]$;

2. the measure μ is not computable.

Thus, the Monte-Carlo computational approach may fail spectacularly for truly simple maps – not because there are no physical measures, or too many of them, but because the "nice" unique limiting statistics cannot be computed, and thus the averages (K.0.1) will not converge to anything meaningful in practice.

Furthermore, P. Berger [1] introduced the concept of *emergence*, which is the exponential growth rate of the size of the set of finite Birkhoff sums in the space of probability measures. Positive emergence means that the set of numerical observations for a given dynamical system is "all over the place", depending on the number of steps of iteration and the initial condition. With co-authors [2, 3], Berger showed that positive emergence is a common phenomenon in natural spaces of dynamical systems.

The two approaches: emergence and non-computability have been developed by distinct groups of researchers in parallel to address the same question: *how hard is it to describe a typical dynamical system statistically?* Working together at BIRS led to an exciting synthesis of techniques and ideas. We have answered several important open questions and formulated specific directions of future work. As is seen from the following incomplete list, the answer to the above question could be *very hard indeed* for a given map.

Refining the above result of Rojas-Yampolsky, we have shown:

Theorem. *There exist computable parameters $a \in [0, 1]$ for which the map f_a has a unique physical measure μ_a which is not computable.*

Amusingly, using similar techniques we also proved:

Theorem. *For each of the following questions there exist computable parameters $a \in [0, 1]$ such that the answers cannot be obtained in ZFC:*

- *is f_a statistical?*
- *is f_a chaotic?*

Since a has a finite description, namely a computer program which upon input n outputs de n-th decimal digit of a, this is quite striking.

We also asked whether it may be possible to construct examples in which μ_a is non-computable, and yet, absolutely continuous, which is commonly seen as a nice characteristic that natural measures should have. This is unknown at present and would be very surprising, if true. However, we mapped out a conjectural approach to producing such examples.

We showed that our techniques can be combined to show non-computability or unprovability in other settings, such as rational maps of the Riemann sphere and area-preserving maps.

We asked whether positive emergence can be shown in families of dissipative maps in which Newhouse phenomenon occurs generically, in particular, the family of dissipative complex quadratic Hénon maps with semi-Siegel points (based on the work of Yampolsky and Yang).

We cannot yet describe a general model for the dynamics that produces emergence or non-computability (similar to how Smale's horseshoe gives a general model for chaotic dynamics). We feel that such a description may be possible, and would be crucial for understanding the occurence and typicality of these phenomena.

The week of brainstorming in Banff has led us to a synthesis of approaches, which has already resulted in breakthroughs, and will undoubtedly lead to more. We are grateful to BIRS for bringing us together for this meeting and for creating an ideal working environment.

Participants

Berger, Pierre (CNRS - Sorbonne Université)
Gaidashev, Denis (University of Uppsala)
Helfter, Mathieu (Institut Mathématique de Jussieu)
Rojas, Cristóbal (Universidad Católica de Chile)
Yampolsky, Michael (University of Toronto)

Bibliography

[1] P. Berger, Emergence and non-typicality of the finiteness of the attractors in many topologies, Proceeding of the Steklov inst. (2017) Vol. 297, pp. 1 27. Article translated in Russian in Trudy Matematicheskogo Instituta imeni V.A. Steklova (2017), Vol. 297, pp. 7 37.

[2] P. Berger and S. Biebler, Emergence of wandering stable components, Arxiv (2020) 80 p.

[3] P. Berger and J. Bochi, On Emergence and Complexity of Ergodic Decompositions , Advances in Mathematics, Vol. 390, (2021) 52 p.

[4] Newhouse, E. “Diffeomorphisms with infinitely many sinks”. Topology. 13 (1): 9–18, 1974.

[5] D. Ruelle, Historical behaviour in smooth dynamical systems. In *Global Analysis of Dynamical Systems*, Ed. H.W Broer, B Krauskopf, Gert Vegter, CRC Press, 2001.

[6] Ulam, S. and Von Neumann, J. “On combination of stochastic and deterministic processes”. Preliminary report. Fifty-third Summer Meeting of the American Mathematical Society, 1947.

Appendix L

Dynamics of biopolymers across multiple scales (22frg264)

May 22 - 29, 2022

Organizer(s): Brittany Bannish (University of Central Oklahoma), Calina Copos (Northeastern University), Adriana Dawes (The Ohio State University), Kelsey Gasior (University of Ottawa), Minghao W. Rostami (Syracuse University), Rebecca Pinals (MIT)

Overview

The main objective of our Focused Research Group (FRG) was for our pre-established team of collaborators to meet in person for intensive work on extending our current project and completing a manuscript currently in progress. This FRG was originally scheduled to occur in July, 2020, but was postponed to 2022 due to the pandemic. Broadly, we aimed to extend and refine our current working models and devise concrete measurements to holistically capture actin network formation, connecting from a stochastic model describing microscale phenomena at the molecular actin level, to a continuum model describing macroscale behavior at the cellular level. Our group was formed in June 2019 at the Collaborative Workshop for Women in Mathematical Biology, held through the Institute for Pure and Applied Mathematics at the University of California, Los Angeles (IPAM UCLA). This week-long workshop brought together female-identifying mathematicians of varying career levels and backgrounds with the goal of examining an unfamiliar topic from a multifaceted, new perspective. Following this workshop, our group continued to collaborate by meeting remotely each week and we have been highly productive, resulting in one published paper, one paper in preparation for submission, and ideas for at least two more papers. At the time of our initial application in January, 2020, our group consisted of six members in five distinct geographic regions (California, Florida, Ohio, Oklahoma, and New York) at various stages of our academic careers: one graduate student, two postdoctoral researchers, one tenure-track assistant professor, and two associate professors with tenure. At the 2022 FRG meeting, there had been changes in both geography and career stage: we hail from five distinct regions (Ohio, Oklahoma, New York, Boston, and Ottawa) and at various career stages (one postdoctoral researcher, three tenure-track assistant professors, and two full professors). While we have successfully continued our remote collaboration, the 2019 UCLA IPAM workshop revealed how well we function as a collective group and how rapidly we can progress in our research project when gathered in person. The FRG program offered by BIRS was an exceptional opportunity for us to work intensively, in person, on the proposed project. Our time at BIRS was used to catalyze our current research into one complete paper draft, to make substantial progress on the objectives listed below, and to develop a concrete plan for our continuing research.

Objectives and Scientific Progress Made

Objective 1: Develop a spatially- and resource-constrained environment. Our current model, as outlined in our earlier book chapter (Copos et al. 2021 [1]), adopts a simplified approach to actin dynamics in that it assumes an unlimited amount of space and resources for the formation of the actin network. While this setup enabled us to create our initial modeling platform, it could be made more biologically relevant by including the presence of barriers such as the cell membrane. Additionally, the availability of actin monomers necessary for network growth is constrained in both amount and location within a cellular environment, both of which could potentially influence microscopic network dynamics and macroscopic end states. We have recently introduced a spatially constrained environment in which resource availability and dynamics primarily occur at the leading edge of the growing actin network. Our initial results suggest these constraints can produce complex spatial patterns.

Progress toward Objective 1: Some of our time at BIRS was spent collectively analyzing these complex spatial patterns, yielding insights into regulatory mechanisms inside cells.

Objective 2: Expand sensitivity analysis focus and techniques. Copos et al. 2021 [1] connected our stochastic model to our continuum model using a relation between the mean displacement of the actin network and the diffusion coefficient in the continuum model. We then explored the influence of key parameters on this relationship, highlighting different measures by which we should evaluate the network growth and change, such as the spatial spread, density, and fractal dimension of the network, in addition to the distances between actin tips and branches. These measurements will become of particular interest when the environment and resources of the network are constrained. Using more advanced sensitivity techniques, such as the extended Fourier Amplitude Sensitivity Test (eFAST) and Latin Hypercube Sampling (LHS), we will create a more complete picture of the actin network dynamics across both scales. Synthesizing the results of this analysis will be far more efficient and comprehensive when done in person.

Progress toward Objective 2: In the years following our initial FRG submission, our goals temporarily shifted away from sensitivity analysis and toward Machine Learning (see comment below Objective 4). However, we still spent some time at BIRS evaluating the effect of certain parameters (in particular, the capping probability) on resulting actin network architectures as well as the efficacy of a variety of classification techniques. Some techniques, such as shape PCA, provided inconclusive information, while others, such as symmetry quantification using Transformation Information, allowed us to identify groupings of actin network structures with common features. We found that different Machine Learning (ML) techniques have widely varying success in identifying underlying mechanisms that result in a particular network structure. This led us to not only recognize new properties and limitations of ML techniques but also the range of actin network characteristics that can be generated by a straightforward stochastic simulation framework.

Objective 3: Explore alternative continuum models that more closely match the stochastic model. Both Fishers equation and Skellams equation are continuum models frequently used to describe actin dynamics. However, in our previous work, we observed that their fit to the stochastic model was not fully satisfactory. Now, we aim to identify a continuum model that more suitably matches our stochastic model. We have preliminary results showing that a reaction-diffusion equation with a Poisson-type reaction term results in a better match to the stochastic model output.

Progress toward Objective 3: We spent a bit of time at BIRS working collectively (capitalizing on the distinct expertise of all group members) to derive more appropriate partial differential equations from first principles. We now have several avenues we plan to continue to pursue, including: reaction-diffusion PDEs with saturated growth terms, PDEs based on biased or constrained random walks (rather than unbiased), PDEs that can account for the branching of actin networks, and data-driven PDEs whose terms and coefficients are discovered by ML techniques based on experimental or synthetic data.

Objective 4: Improve the computational efficiency of the stochastic simulations. In our previous work, we found that simulating the stochastic model becomes quite challenging in terms of both run time and memory as the filamentous actin network becomes increasingly dense. This occurred at long simulation times and high probabilities of branching and polymerization, and prohibited us from exploring certain parameter regimes crucial in connecting our model to higher-order phenomena observed on a full cell level. We plan to improve various aspects of our stochastic model implementation, including parallelization.

Progress toward Objective 4: Prior to arrival at BIRS, we had synchronized our efforts and ensured uniformity of microscale stochastic code across all group members; we also wrote and tested a parallel version of the code which allows us to leverage high-performance computers to efficiently generate large amounts of data for the ML algorithms (see the next objective). At BIRS, we were able to confidently run new simulations as necessary, share results, and collectively talk through the implications of those results.

New Objective: In the few years following the submission of our proposal to BIRS, our goals have expanded due to the groups growing interests and expertise in ML. While we have made good progress towards Objectives 1, 3, and 4 in the proposal, what we decided to focus on in our next paper and during our stay at Banff is using ML techniques to uncover the dominant mechanisms underlying an actin network.

Progress towards the New Objective: Before arriving at Banff, we had developed the necessary software and written a draft of the paper. By meeting as a group in Banff, we were able to work intensively on the paper and finish the bulk of the project.

Outcome of the Meeting

Indeed, our FRG was highly productive from a scientific perspective. As a group, we examined, debated, and validated the ML techniques used to obtain the preliminary results. Since we are all beginners of ML, having these discussions in person was necessary and educational. They were also invigorating as we have diverse research expertise, and each of us offered unique perspectives and asked refreshing, thought-provoking questions. We realized that while our pre-Banff work used supervised ML algorithms only, unsupervised ML algorithms can be used in conjunction with supervised ML algorithms to produce better results. This is a brilliant realization and strengthens our paper.

Our FRG was also highly productive with respect to our manuscript writing. Before coming to Banff, despite having obtained promising results on multiple fronts, we were unsure about the theme, structure, and presentation of the next paper. It felt like we had many interesting pieces but did not know how they fit together to tell a coherent story. Utilizing the scenic meeting rooms at Banff and with the support of the extremely helpful staff of BIRS, we had many brainstorming sessions and breakout sessions to finalize the content and structure of our next paper. (It turns out the old-fashioned outlining on the blackboard, printing out our draft, and taping the relevant pages next to the outline are still highly effective.) This allowed us to form working groups” that tackled different parts of the paper. Upon leaving Banff, we had a considerably improved draft that we anticipate will be submitted in the near future.

Participants

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Bibliography

[1] C. Copos, B. Bannish, K. Gasior, R. Pinals, M. Rostami, A. Dawes, Connecting actin polymer dynamics across multiple scales, in: R. Segal, B. Shtylla, S. Sindi (Eds.), Using Mathematics to Understand Biological Complexity, Vol. 22, Springer, Cham, 2021, pp. 733. doi:10.1007/978-3-030-57129-0_2.

Appendix M

Geometric Interpretation and Visualization of Multi-Parameter Persistent Homology (22frg267)

June 5 - 19, 2022

Organizer(s): Claudia Landi (University of Modena and Reggio Emilia), Robyn Brooks (Boston College), Celia Hacker (EPFL), Barbara Ilse Mahler (KTH)

Overview of the Field

Digital data are being produced at a constantly increasing pace, and their availability is changing the approach to science and technology. The fundamental hypothesis of Topological Data Analysis (TDA) is that data come as samples taken from an underlying shape, and unveiling such shape is important to understanding the studied phenomenon. Topological shape analysis amounts to determining non-trivial topological holes in any dimension. Computational Topology provides tools to derive specific signatures – topological invariants – which depend only on topological features of the shape of data and are robust to local noise. Among them, persistent homology [10] stands out as the most useful. A useful generalization of persistent homology is multiparameter persistent homology which, contrarily to persistent homology, allows us to consider multiple aspects of the data simultaneously in order to compute topological summaries of the data.

The first step in the persistence pipeline is to build a family of nested simplicial complexes that model the data at various scales varying one or more parameters. The second step focuses on the maps induced in homology by the simplicial inclusions, to extract invariants such as persistence modules and their rank invarant. The third step is to use persistence invariants as a source of feature vectors in machine learning contexts. As final goal is to use the acquired topological information to improve the understanding of the underlying data, an important feature of this pipeline is its robustness with respect to noise in the input data [10]. This is guaranteed by appropriate metrics, such as the matching distance between persistence modules, which gives a measure of dissimilarity of the underlying data sets.

Single parameter persistent homology [17] has proven to be useful in many applications [2, 3, 11, 14, 16], yielding a summary of the data through a one-dimensional filtration. However, some data requires to be filtered along multiple parameters to fully capture its information. This is the role of multiparameter persistent homology [7, 8], the topic of interest in this report.

Recent Developments and Open Problems

Unfortunately, understanding, visualizing and computing invariants in multiparameter persistent homology remains a difficult task theoretically and computationally. This difficulty holds as well when it comes to computing distances between such invariants. In the one-dimensional case there are several ways to compare persistent homology modules, such as the bottleneck distance and Wasserstein distances, which exhibit some stability property with respect to variations in the input [10].

For more than one parameter, there are also various definitions of distances between persistence modules [15, 5]. Amongst them, the matching distance [4] is attracting the attention of multi-parameter persistence practitioners. Using the fact that by restricting an n-parameter filtration to any line of positive slope through the parameter space one gets a 1-parameter filtration, one can use knowledge of the 1-dimensional case. Indeed, following this idea, the matching distance is defined as a supremum of the one-dimensional bottleneck distance, over the collection of all lines of positive slope in the parameter space, i.e.,

$$d_{match}(M,N) := \sup_{L:\ u=s\vec{m}+b} \hat{m}^L \cdot d_B(\operatorname{dgm} M^L, \operatorname{dgm} N^L),$$

where $\hat{m}^L$ is a weight necessary for this distance to yield 1-Lipschitzianity of the persistent homology transform. However, computing exactly this distance is not an easy task given the nature of its definition. As a first step towards an exact computation, several approximations of this distance have been provided [9, 13].

The exact computation of the matching distance is currently only possible for 2-dimensional modules [12], with recent computational improvements in terms of time complexity [6]. These methods exploit the duality of points and lines in the plane, which means that they are difficult to generalize to persistence modules with more than two parameters. Moreover, the geometric interpretation of the optimal lines achieving the matching distance is not clarified.

Workshop summary

Our research group, i.e. the meeting organizers together with participants Asilata Bapat (ANU) and Elizabeth Stephenson (IST-Austria), focused on exploring a method to compute the matching distance based on a refinement of the framework developed in [1]. In that work, we propose a step towards the interpretation and visualization of the rank invariant for persistence modules for any given number of parameters. We show how discrete Morse theory may be used to compute the rank invariant, proving that it is completely determined by its values at points whose coordinates are critical with respect to a discrete Morse gradient vector field. These critical values partition the set of all lines of positive slope in the parameter space into equivalence classes, such that the rank invariant along lines in the same class are also equivalent. We show that we can deduce all persistence diagrams of the restrictions to the lines in a given class from the persistence diagram of the restriction to a representative in that class.

The critical values (closed under least upper bound) described in [1] capture all the changes in homology occurring throughout the multifiltration and fully determine the rank invariant, which is equivalent to barcodes in 1-dimensional persistence modules. Based on this intuition, we formulate the idea that the critical values must be relevant to the choice of lines for the computation of the matching distance, which is the question we focused on during the workshop.

Dealing with the matching distance from this perspective allows us to reduce the number of lines necessary to compute it to a finite set, thus reducing the computation to a maximum rather than a supremum without exploiting the point-line duality used in [12].

During our stay at BIRS, we built on this framework to derive a new method of computing the matching distance. We worked through both the theoretical and computational aspects of this question, proving the theoretical completeness of our method of computation, as well as initializing and developing an implementable algorithm for computation in Python.

Although at first we have focused in two dimensions, the advantages of exploiting this framework is that there is the potential to extend it to more than two parameters. The method we propose aims at producing algorithms with comparable time complexity to [12], however since we do not exploit the point-line duality we may achieve a reduction of the space complexity.

Scientific Progress Made

Through some examples we have shown that considering only lines passing through pairs of points in the closure of critical values C_M and C_N of 2-parameter persistence modules M and N, with respect to the least upper bound, is not sufficient. Indeed, the definition of matching distance uses the bottleneck distance of the restrictions along lines. However, lines in the same equivalence class might not have the bottleneck distance always given by the same pairing even though there is a bijection between their persistence diagrams.

To overcome this problem, we have analyzed where switches might happen in the matching giving the bottleneck distance, identifying a set Ω of points in the projective completion of the parameter space, called switch points. This set allows us to refine our equivalence relation on the set of positive lines by considering the set of points $\overline{C}_M \cup \overline{C}_N \cup \Omega$. Using this set of points it is possible to identify all the lines at which the matching distance can potentially be realised, reducing the computation of a supremum to a that of a maximum over a finite set of lines through the parameter space. We have found a detailed explanation of how to compute these points and shown that the matching distance is attained either at a line through a pair of points in $\overline{C}_M \cup \overline{C}_N \cup \Omega$ or a line of diagonal slope through exactly one of the points.

In contrast to other methods such as [12, 5], we thus provide a geometric understanding of different lines, horizontal, vertical, diagonal, as well as passing through critical points, and their contribution to the matching distance.

In conclusion, the progress achieved by this Focused Research Group has been to advance the state of the art, although still restricted to two dimensions, in two ways: computing the matching distance in a way which is both geometrically interpretable and implementable.

Outcome of the Meeting

As a result of the meeting, this Focused research Group has achieved enough theoretical results to explain important lines for the matching distance computation in 2-parameter persistence. We aim at posting these results on arXiv within the next few weeks. Moreover, all our proofs are constructive and such construction will lead to algorithms.

Our next goal is the implementation of such algorithms in order to perform numerical tests to ascertain the performances of the method in terms of speed, memory consumption, scalability, and, possibly, parallelizability.

Participants

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Bibliography

[1] A. Bapat, R. Brooks, C. Hacker, C. Landi, and B. I. Mahler, Morse-Based Fibering of the Persistence Rank Invariant, Association for Women in Mathematics Series **30**, 27–62 (2022), Springer International Publishing.

[2] P. Bendich, J. S. Marron, E. Miller, A. Pieloch, and S. Skwerer, Persistent homology analysis of brain artery trees, Ann. Appl. Stat., **10** (2016), 198–218.

[3] S. Bhattacharya, R. Ghrist, and V. Kumar, Persistent homology for path planning in uncertain environments, IEEE Transactions on Robotics, **31** (2015), 78–590.

[4] S. Biasotti, A. Cerri, P. Frosini, D. Giorgi, and C. Landi, Multidimensional size functions for shape comparison, Journal of Mathematical Imaging and Vision, **32** (2008), 161–179.

[5] H. Bjerkevik, On the Stability of Interval Decomposable Persistence Modules, Discrete Comput. Geom., **66** (2021), 92–121.

[6] H. B. Bjerkevik and M. Kerber, Asymptotic improvements on the exact matching distance for 2-parameter persistence, arXiv:2111.10303 (2021).

[7] G. Carlsson and A. Zomorodian, The theory of multidimensional persistence, Discrete Comput. Geom., **42** (2009), 71–93.

[8] G. Carlsson, G. Singh, and A. Zomorodian, Computing multidimensional persistence. In Algorithms and computation, Lect. Notes Comput. Sci. **5878**, 730–739, Springer, Berlin, 2009.

[9] A. Cerri and P. Frosini, A new approximation algorithm for the matching distance in multidimensional persistence, Journal of Computational Mathematics, **38** (2020), 291–309.

[10] D. Cohen-Steiner, H. Edelsbrunner, and J. Harer, Stability of persistence diagrams, Discrete Comput. Geom., **37** (2007), 103–120.

[11] S. B. Green, A. Mintz, X. Xu, and J. Cisewski-Kehe, Topology of our cosmology with persistent homology, CHANCE, **32** (2019), 6–13.

[12] M. Kerber, M. Lesnick, and S. Oudot, Exact computation of the matching distance on 2-parameter persistence modules. In SoCG 2019, **129**, LIPIcs, 46:1–46:15 (2019).

[13] M. Kerber and A. Nigmetov, Efficient approximation of the matching distance for 2-parameter persistence. In 36th International Symposium on Computational Geometry (S. Cabello and D. Z. Chen eds.), SoCG 2020, LIPIcs.

[14] Y. Lee, S. D. Barthel, P. Dłotko, S. M. Moosavi, K. Hess, and B. Smit, Quantifying similarity of pore-geometry in nanoporous material, Nature Communications, **8** (2017).

[15] M. Lesnick, The theory of the interleaving distance on multidimensional persistence modules, Found. Comput. Math., **15** (2015), 613–650.

[16] M. Sinhuber and N. T. Ouellette, Phase coexistence in insect swarms, Phys. Rev. Lett., **119**:178003 (2017).

[17] A. Zomorodian and G. Carlsson, Computing persistent homology, Discrete Comput. Geom., **33** (2004), 249–274.

Appendix N

Cohomogeneity Two Manifolds of Positive Sectional Curvature (22frg800)

June 13 - 24, 2022

Organizer(s): Catherine Searle (Wichita State University)

Our group split our time at BIRS between two projects: the first was the cohomogeneity two project we described in our proposal, and the second was on estimating the lengths of geodesics.

Project 1: Cohomogeneity Two

Setup

A manifold admitting a group action with k-dimensional orbit space is a cohomogeneity k manifold. Manifolds of positive curvature admitting isometric group actions of cohomogeneity 0, that is homogeneous spaces, have been classified. (see Berger [3], Bérard-Bergery [2], Wallach [20], Aloff and Wallach [1], Wilking [21], and Wilking and Ziller [23]). A classification of cohomogeneity one manifolds of positive curvature was achieved in all dimensions except for 7, where a list of candidates have been given (see Searle [12], Verdiani [18, 19], and Grove, Wilking, and Ziller [10]). It is then natural to consider the problem of classifying simply-connected, closed, cohomogeneity two manifolds of positive curvature.

Let G be a compact, connected Lie group. Recall that an isometric G-action on M^n, a complete n-manifold, is called polar if it admits a section, that is, an immersed submanifold, Σ, of dimension equal to $\dim(M/G)$, that meets every orbit orthogonally. By work of Fang, Grove, and Thorbergsson [6] a closed, simply-connected, positively curved manifold admitting a polar cohomogeneity k action with $k \geq 2$ is equivariantly diffeomorphic to a compact rank one symmetric space (CROSS) with the corresponding polar action. Thus, in order to classify simply-connected, closed, cohomogeneity two manifolds of positive curvature, it remains to understand the case of non-polar actions.

Goals

The broad goal here is to classify such cohomogeneity two actions: that is, to find all possible M, up to diffeomorphism, and G, and to describe the action of G on M up to (equivariant) diffeomorphism. Coming into our stay at BIRS, our goal was to prove a classification theorem in low dimensions for closed, simply-connected, positively curved manifolds admitting an isometric, non-polar, cohomogeneity two action.

A useful tool for such actions is that of the G-manifold reduction, see Grove and Searle [9] and Skjelbred and Straume [16]. The idea is to reduce the G-action on M to the case of a core group, ${}_cG$, acting on a core manifold,

${}_cM$. The core group ${}_cG$ is defined to be $N_G(H)/H$, where $N_G(H)$ denotes the normalizer of H in G, the set of g such that $gHg^{-1} = H$. The core manifold ${}_cM$ is defined to be the closure of the set M^H of points in M fixed by (a particular copy of) H. The quotient ${}_cM/{}_cG$ is isometric to M/G, and the principal isotropy of the action of ${}_cG$ on ${}_cM$ is the identity only. Thus, the only properties of the original G-action that are not preserved in the reduction are that ${}_cM$ might not be simply connected and ${}_cG$ might not be connected. Note that the original action is polar if and only if the core group ${}_cG$ is finite. Using this technique, we have been able to prove the following proposition.

Proposition N.0.1 *Let G be a compact, connected Lie group acting isometrically, effectively and by cohomogeneity two on M^n a closed, simply-connected manifold of positive curvature. Suppose further that the action is non-polar and that there is a G-manifold reduction of the G-action on M with core group of rank ≥ 2. Then M decomposes as a G-invariant union of disk bundles.*

The proof of this theorem relies on understanding the orbit space of the group action, M/G. In particular, M/G is homeomorphic to S^2 or D^2. If $M/G = S^2$, then there can be at most 2 isolated exceptional orbits, and if $M/G = D^2$, there are 4 possibilities: M/G has 0, 1, 2, or 3 vertices. In all but the case where $M/G = D^2$ and has 3 vertices, it is well-understood that M admits a G-invariant disk bundle decomposition. In the last case, in order to show that M admits a G-invariant disk bundle decomposition, it suffices to prove that two of the vertices of M/G are right angles.

This still leaves us to better understand non-polar G-actions on M with ${}_cG_0$, the connected component of ${}_cG$, is isomorphic to S^1. This case is well-understood when $M = S^n$, see Straume [14] and [15], but remains to be explored for all other such manifolds. While we have not encountered any examples of such manifolds that do <u>not</u> admit a G-invariant disk bundle decomposition, we also cannot rule out the possibility that such examples do exist.

We were able to make significant progress on the classification question. Going into our stay, we were not yet able to rule out the possibility that the Wu manifold, $SU(3)/SO(3)$, could admit such an action with a G-invariant metric of positive curvature in dimension 5, nor had we completed the classification in dimension 6. At the end of our stay, we were able to prove the following theorem.

Theorem N.0.2 *Let G be a compact, connected Lie group acting isometrically, effectively and by cohomogeneity two on M^n a closed, simply-connected n-manifold of positive curvature. Suppose further that the action is non-polar, $n \leq 6$, and for $n = 5$, the action cannot be almost free. Then M^n is (equivariantly) diffeomorphic to S^n, $\mathbb{C}\mathrm{P}^{n/2}$, or $V^6 = SU(3)/T^2$.*

Observation 1 *The theorem is rigid: we can show that all such manifolds admit an isometric, non-polar, cohomogeneity two action.*

Observation 2 *The case where $n = 5$ and the action is free has recently been resolved by work of Cavenaghi, Grama, and Sperança [5], who have shown that there is no such action. Using the Connectedness Lemma of Wilking [22], one sees that if the action is almost free and the exceptional orbits are not isolated, that there is no such action. However, the cases where the action is almost free and has isolated exceptional orbits have yet to be understood. They have been studied by Simas [13] who showed that the only candidates are diffeomorphic to the two S^3 bundles over S^2.*

Project 2: Lengths of Geodesics

We note that this second project also includes Isabel Beach as a collaborator and will form part of the focus of our team at the Women in Geometry 3 workshop to be held in November 2023 at BIRS.

Setup

Let $\mathcal{M}_{k,v}^D(n)$ denote the set of n-dimensional closed Riemannian manifolds M, with sectional curvature bounded below by k, volume bounded below by v, and diameter bounded above by D. The property that distinguishes this class of manifolds is their uniform local contractibility: a result of Grove and Petersen in [8], states that there exist r and R, both depending on k, v, D, n, such that every ball of radius r is null-homotopic in the concentric ball of radius R.

A theorem of Serre states that in a closed Riemannian manifold M, given any two points p and q there are infinitely many geodesics from p to q. The goal of our project is to estimate the growth of the lengths of these geodesics, as M ranges over $\mathcal{M}_{k,v}^{D}(n)$. In particular, we have the following conjecture:

Conjecture 3 *Let $M \in \mathcal{M}_{k,v}^{D}(n)$ and assume M is simply connected. Then there is a constant $C(k, v, D, n)$, such that for every $\ell \in \mathbb{N}$, there are at least ℓ geodesics from p to q of length at most $C(k, v, D, n) \cdot \ell$.*

Existing results

Our project is based on results of Nabutovsky and Rotman [11]. In [11], they paper prove that for any points p and q in closed manifold M of diameter D, there are at least ℓ geodesics from p to q of length at most $4nD \cdot \ell^2$. In particular, the sequence of lengths in order grows at most quadratically with ℓ. The hope is that by introducing the bounds on curvature and volume, we can improve the bound so that the lengths grow linearly with ℓ.

Immediate goal

We have sketched a proof of our conjecture with an additional hypothesis, namely, we suppose there is a constant c such that all loops of length at most $2D$ are null-homotopic through loops of length at most cD. At BIRS we began the process of writing up this result. With this additional constraint, instead of a obtaining a constant $C(k, v, D, n)$, we get a constant $C(k, v, D, n, c)$ that depends on c as well. One special case is when there are no loops of length at most $2D$ that are local minima of length. In this case, every loop of length at most $2D$ is null-homotopic through a path of loops that never increases in length, so we have $c = 2$, and our constant depends only on k, v, D, n. We hope to prove the conjecture without this additional hypothesis, but so far do not have good techniques for eliminating it.

Next goal

As we worked on writing up our result, we also explored an extension of the problem, where instead of geodesics between two points we consider periodic geodesics. While a periodic geodesic is a closed loop that is geodesic all along the loop, a geodesic loop is a geodesic from a point p to itself, but the incoming and outgoing directions at p might have some angle between them. The work of Gromoll and Meyer [7] gives topological conditions on a closed manifold M that guarantee infinitely many periodic geodesics, and Sullivan and Vigué-Poirrier [17] show that these conditions are satisfied if the rational cohomology of M cannot be generated by a single generator. Under these hypotheses, it makes sense to ask about the growth of the length of these periodic geodesics, and we can hope that the conclusion of our theorem for geodesics is also true for periodic geodesics. When we count periodic geodesics, each periodic geodesic can be iterated and/or shifted to produce infinitely many other periodic geodesics; thus, we say that periodic geodesics are distinct only if their images are distinct loops in M.

What we hope to prove here is that if $M \in \mathcal{M}_{k,v}^{D}(n)$ and the rational cohomology of M cannot be generated by a single class, and all loops of length at most $2D$ are null-homotopic through loops of length at most cD, then for all $\ell \in \mathbb{N}$, there are at least ℓ geometrically distinct periodic geodesics of length at most $C(k, v, D, n, c) \cdot \ell$. We spent some of our time at BIRS trying to understand the proof of Gromoll and Meyer, which is based on careful analysis of index and nullity of a periodic geodesic and its iterates based on paper [4] of Bott, to see how easily it could be made quantitative in this way.

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Bibliography

[1] Aloff, S., Wallach, N. L., “An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures”, Bull Amer. Math. Soc., v. 81, pp. 93–93 (1975).

[2] Bérard Bergery, L., “Les Variétés Riemannienes homogenes simplement connexes de dimension impair á courbure strictement positive”, J. Math. Pures Apl., v. 55, pp. 47–68 (1976).

[3] Berger, M., “Les Variétés Riemannienes homogenes normales simplement connexes a courbure strictement positive”, Ann. Scuola Norm. Pisa, v. 15, pp. 179–246 (1961).

[4] Bott, R., “On the iteration of closed geodesics and the Sturm intersection theory”, Comm. Pure Appl. Math. 9, 171-206,(1956).

[5] F. Cavenaghi, L. Grama, and Ll. Sperança, The Petersen–Wilhelm conjecture on principal bundles, arXiv preprint:2207.10749 (2022).

[6] F. Fang, K. Grove, and G. Thorbergsson, Tits geometry and positive curvature, Acta Math., **218**, no. 1 (2017), 1-53.

[7] Gromoll, D., Meyer, W., “Periodic geodesics on compact Riemannian manifolds”, Journal of Diff. Geom., v. 3, pp. 493-510 (1969).

[8] Grove, K., Peterson, P. V, “Bounding homotopy types by geometry”, Annals of Mathematics, v. 128, (1988), 195-206.

[9] K. Grove and C. Searle, Global G-Manifold Resolutions and Reductions, Annals of Global Analysis and Geometry, **18** (2000), 437–446.

[10] K. Grove, B. Wilking, and W. Ziller, Positively curved cohomogeneity one manifolds and 3-Sasakian geometry, J. Differential Geom., **78** no. 1 (2008), 33–111.

[11] Nabutovsky, A., Rotman, R., “Length of geodesics and quantitative Morse theory on loop spaces”, Geometric and Functional Analysis, v. 23 (2013), 367-414.

[12] C. Searle Cohomogeneity and Positive Curvature in Low Dimensions, Math. Zeitschrift **214** no.3 (1993) 491-498, *Corrigendum*, Math. Z., **226** (1997) 165–167.

[13] F. Simas, Non-negatively curved 5-manifolds with non-abelian symmetry, Geom. Dedicata **181** (2016), 61-82.

[14] Straume, E., “Compact connected Lie transformation groups on spheres with low cohomogeneity. I”. Mem. Amer. Math. Soc. **119** (1996), no. 569, pp. vi–93.

[15] Straume, E., “Compact connected Lie transformation groups on spheres with low cohomogeneity. II”. Mem. Amer. Math. Soc. **125** (1997), no. 595, pp. viii–76.

[16] T. Skjelbred and E. Straume, On the reduction principle for compact transformation groups, preprint (1993).

[17] Sullivan, D., Vigué-Poirrier, M., "Thehomology theory of the closed geodesic problem", Journal of Diff. Geom., v. 11, no. 4 (1976), 633-644,

[18] L. Verdiani Cohomogeneity one Riemannian manifolds of even dimension with strictly positive sectional curvature, I Math. Z. **241** (2002) 329–229.

[19] L. Verdiani Cohomogeneity one manifolds of even dimension with strictly positive sectional curvature J. Diff. Geom. **68** no. 1 (2004) 31–72.

[20] Wallach, N. L., "Compact homogeneous Riemannian manifolds with strictly positive curvature", Ann. Math., **96** (1972), 277–295.

[21] B. Wilking, The normal homogeneous space $SU(3) \times SO(3)/U^\bullet(2)$ has positive sectional curvature, Proceedings of the AMS, **127** no. 4(1999), 1191–1194.

[22] B. Wilking, Torus actions on manifolds of positive sectional curvature, Acta Math. **191**, no. 2 (2003), 259–297.

[23] B. Wilking and W. Ziller, Revisiting homogeneous spaces with positive curvature, J. Reine Ange. Mathematik, **2018** no. 738 (2018), 313–328.

Appendix O

Studying PDE Dynamics Via Optimization With Integral Inequality Constraints (22frg243)

July 10 -17, 2022

Organizer(s): David Goluskin (University of Victoria), Dávid Papp (North Carolina State University), Ian Tobasco (University of Illinois Chicago)

Recent Developments and Open Problems

A sizeable fraction of the discussion during the FRG centered on the pointwise dual relaxation (PDR) method described in [1, 3] and references therein. This is a method for relaxing integral variational problems into sum-of-squares (SOS) polynomial optimization problems. In general it is an open problem to characterize when this method introduces a relaxation gap and so cannot converge to the global optimum of the original integral variational problem. Partial results that guarantee sharpness under fairly strong assumptions, and that give counterexamples to sharpness in other cases, have appeared this year in preprints by FRG participants [2, 4]. As for computational implementation of this relaxation strategy for particular variational problems, the examples published so far [1, 3] are simple enough that they could have been solved by more standard computational strategies. It has remained a practical challenge to produce computational results using the PDR method for examples that are intractable by existing methods. **[David: What about various examples in Valmorbida et al's work? There are some PDE dynamics problems that maybe are hard to solve otherwise.]**

Scientific Progress Made

Parts of each day were devoted to informal presentations from various participants on recent progress. Fantuzzi and Tobasco presented their results [2] on sharpness conditions for the PDR method, and Korda presented his results [4] on the same topic. The results are complementary with neither being contained in the other. Korda's coauthor joined some conversations via zoom, and the discussions led to an updated version of their main counterexample, reflected in the third version of [4] on the arXiv. Tobasco presented topics from variational analysis, including quasiconvexity and polyconvexity, and possible applications of SOS optimization were explored. Examples include verifying that a function is or is not quasiconvex, or constructing the largest possible supporting functions that are polyconvex.

Fantuzzi and Fuentes shared unpublished work on a computational approach to integral variational problems that is a possible alternative or complement to the PDR approach. Their method relies on finite element discretizations of the variational problem and SOS computations for the discretized problems. Under reasonable assumptions this is guaranteed to converge to the optimum of the original problem with increasing finite element resolution and polynomial degree, and it even provides global optimizers to the discretized problems. Computational cost makes it hard to reach convergence for 2D problems, and, unlike the PDR method as polynomial degrees are raised, the convergence is not monotone. Nonetheless, some promising computational examples were shown, and an effort was begun to compute results for the same examples using the PDR method so that a one-to-one comparison can be made.

Outcome of the Meeting

Collaborations were initiated on many different projects and involving many different subsets of participants. Here we describe only a few representative topics.

Questions concerning the combination of finite elements and SOS

Fantuzzi and Fuentes studied integral variational problems problems of the form

$$f^* := \min_{u \in W_0^{1,p}(\Omega)} \int_\Omega f(x, u, Du)\,\mathrm{d}x, \tag{O.0.1}$$

where $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$ is a quasiconvex polynomial function, and $W_0^{1,p}(\Omega)$ is the usual Sobolev space of weakly differentiable p-integrable functions vanishing on the domain boundary. Their approach approximates optimal u using a "discretize & relax" approach consisting of two steps. First, the variational problem is discretized on a finite element mesh whose elements have size h. This transforms problem (O.0.1) into a polynomial optimization problem (POP) where the number of variables is inversely proportional to h, but which has a high degree of structure. Specifically, variables are coupled directly only if they are used to represent the function u on a single mesh element. The second step is to use sum-of-squares polynomials of degree $\omega \in \mathbb{N}$ to relax this POP into a convex semidefinite program (SDP), whose optimal value $\lambda(\omega, h)$ bounds that of the POP from below. Under suitable technical conditions, one can show that

$$\lim_{h\to 0} \lim_{\omega\to\infty} \lambda(\omega, h) = f^*. \tag{O.0.2}$$

Moreover, the solutions of the SDP relaxations converge to the optimal u when the latter is unique.

Despite these initial results, more should be done to better understand the theoretical properties of this approach and, consequently, make it more useful in practice. This FRG identified three areas where progress is desirable.

Order of limits Can the order of limits in (O.0.2) can be reversed? If so, what is the $h \to 0$ limit of the SDPs obtained with fixed relaxation order ω? Such questions are not just a mathematical curiosity: a full understanding of the $h \to 0$ limit for fixed ω could lead to a new type of convex relaxations for (O.0.1) which, contrary to existing convexification approaches, produce arbitrarily accurate lower bound on the global minimum f^*. Some progress was made by considering the setup of periodic domains and optimizing over periodic functions, which leads to an optimization over polynomials with infinitely many variables, but the lack of a positivity representation theorem of such mathematical objects, even for a compact domain, is an obstacle.

Removing technical assumptions The known proof of (O.0.2) requires two restrictive technical assumptions. The first is the so-called running intersection property (RIP), which requires the coupling between variables of each finite-element discretization to be described by a chordal graph. This condition is never satisfied for variational problems over two- or higher-dimensional domains Ω, unless one introduces "fictitious" variable couplings that increase computational costs to prohibitive levels. On the other hand, practical experience suggests that the RIP may often be unnecessary. This FRG discussed the possibility of dropping the RIP requirement by exploiting a connection with the PDR method. Specifically, some of the participants sketched an argument showing that the RIP is unnecessary if the PDR method is sharp. What remains to be done is to rigorously confirm this and to prove more the sharpness of the PDR method for more interesting classes of problems than those studied by the participants in [2] and [4].

Further discussion revolved around a second restrictive technical assumption required to prove (O.0.2): the uniqueness of the (global) minimizer of (O.0.1). In general, the optimal solutions of the SDP relaxations converges to the sequence of moments of a probability measure supported on the set of global minimizers of (O.0.1). If multiple minimizers exist, individual ones can be recovered only if the SDP relaxation for a given finite ω and h is exact (meaning that it gives a sharp bound on the finite-element discretization for mesh size h) and its matrix variables satisfy a rank condition called "flatness". These finite convergence and flatness conditions often hold in practice, but there is currently no proof that they hold generically for SDP relaxations that exploit the POP structure. (They do when the structure is ignored, but this is clearly not desirable in practice.) An alternative to bypass this problem would be to find extreme optimal solutions to the SDP relaxations, which are moment sequences of probability measures supported on individual minimizers. To the best of our knowledge, however, no SDP solvers have the ability to reliably produce extreme solutions.

Efficient solution of large-scale SDP relaxations In addition to theoretical questions, the "discretize & relax" of Fantuzzi & Fuentes poses practical challenges. Indeed, the SDP relaxations one must solve for small mesh size h and/or large relaxation order ω are beyond the reach of available general-purpose SDP solvers. This is especially true when tackling integral variational problems in two or more spatial dimensions. One possible way forward is to exploit the particular structure of the SDP relaxations. It seems possible to combine a variation of recently developed decomposition strategies for SDPs with "first-order" algorithms, such that each iteration of the algorithm consists of many small independent subproblems that can be solved efficiently and—crucially—in a distributed manner. While the formulation of such algorithms is relatively straightforward, there are many possible variations and it is unclear which one will offer the best compromise between iteration simplicity, parallelization, and speed of convergence. Answering this question will require efficient software prototypes as well as further theoretical convergence analysis.

Application areas for the PDR method

An application area that lies outside existing sharpness guarantees for the PDR, but where improvement on existing analytical results seems possible, is the estimation of optimal constants for a linear Korn's inequality, which allows one to control the full gradient of a map $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ in terms of its symmetric gradient. We work in dimensions $d \geq 2$ and define $e_{ij}(u) = \partial_i u_j + \partial_j u_i$ for $i, j = 1, \ldots, d$. Without boundary conditions, the inequality guarantees the existence of C such that

$$C \int_\Omega |e(u)|^2 \geq \min_{W \in \text{Skew}} \int_\Omega |\nabla u - W|^2$$

for all $u(x)$. With periodic boundary conditions, that

$$C \int_{\mathbb{T}^d} |e(u)|^2 \geq \int_{\mathbb{T}^d} |\nabla u|^2$$

for periodic $u : \mathbb{T}^d \to \mathbb{R}^d$. These are classical inequalities in the theory of elasticity, but the optimal constants C are unknown except in very simple domains. We wish to compute convergent approximations to the optimal constants, and to find the $u(x)$ that saturate these optimal constants. The PDR method is directly applicable in the case without boundary conditions, and ways to implement periodic boundary conditions were discussed.

A second application concerns the question of how much energy is required to carry out elastic crumpling. We seek sharp lower bounds on a model problem:

$$\min_{(u,w):\mathbb{T}^2 \to \mathbb{R}^2 \times \mathbb{R}} \int_{\mathbb{T}^2} |e(u) + \frac{1}{2}\nabla w \otimes \nabla w - I|^2 + h^2 |\nabla\nabla w|^2 \, dx.$$

Letting $E_h(u, w)$ denote the above energy functional, the minimum is known to obey $\min\, E_h \lesssim h^{5/3}$ by a "minimal-ridge" construction that suitably smooths an origami crease pattern. It is also not hard to prove that $\min\, E_h \gtrsim h^2$, and a more subtle argument shows the better asymptotic bound $\min\, E_h \gg h^2$. It is widely believed that the upper bound is sharp, meaning there are positive constants C and C' such that

$$Ch^{5/3} \leq \min\, E_h \leq C' h^{5/3}$$

for $h \leq 1$. Thus we aim to apply the PDR method to find a lower bound strictly better than $E_h \gg h^2$, and perhaps scaling like $h^{5/3}$.

Participants

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Bibliography

[1] A. Chernyavsky, J. J. Bramburger, G. Fantuzzi, and D. Goluskin, Convex relaxations of integral variational problems: pointwise dual relaxation and sum-of-squares optimization, arXiv:1210.03079v1 (2021).

[2] G. Fantuzzi and I. Tobasco, Sharpness and non-sharpness of occupation measure bounds for integral variational problems, arXiv:2207.13570v2 (2022).

[3] M. Korda, D. Henrion and J. B. Lasserre, Moments and convex optimization for analysis and control of non-linear PDEs. In Numerical Control: Part A (E. Trlat and E. Zuazua, ed.), Handbook of Numerical Analysis, **23**, 339, Elsevier, 2022.

[4] M. Korda and R. Ríos-Zertuche, Sharpness and non-sharpness of occupation measure bounds for integral variational problems, arXiv:2205.14132v2 (2022).

[5] D. Papp and S. Yildiz, Sum-of-squares optimization without semidefinite programming, SIAM J. Optim. **29** (2019) 822–851.

Research
in
Teams

Appendix P

Shape From Echoes (22rit002)

March 20 - 27, 2022

Organizer(s): Mireille Boutin (Purdue University), Gregor Kemper (Technical University of Munich)

Problem Statement

Assume we have a room, i.e., an arrangement of planar walls, which may include ceilings, floors, and sloping walls. An omnidirectional loudspeaker and some omnidirectional microphones are in the room. The loudspeaker, modeled as a point source, emits a short duration sound at a frequency high enough so that the ray acoustics approximation is valid. The microphones receive several delayed responses corresponding to the sound bouncing back from each wall. These are the first-order echoes. This project centers around the problem of reconstructing the shape of the room from the measured delay times of the first-order echoes. The difficulty of this lies in the fact that in order to determine the wall positions, echoes received by different microphones but coming from the same wall need to be matched. This has to be accomplished even though the delay times detected by each microphone come as an unlabeled set. So the problem is to figure out under which circumstances, and how, one can find out the correct echo-wall matching and reconstruct the wall positions.

Overview of the Field

For a long time, researchers have been trying to understand the relation between the geometry of a room its acoustic. For examples, a considerable amount of work was done in the later part of last century to model the source-to-receiver acoustic impulse response so to improve the design of concert halls. For example, the two-point impulse response within rectangular rooms was analyzed in [1]. The results were later extended to arbitrarily polyhedral rooms in [3]. In the last 20 years, there has been a strong interest in developing methods to reconstruct the geometry of a room using various arrangements of microphones and sound sources. Some reconstruction methods assume that the room is two-dimensional (e.g., [7],[2]], [[9]]) while others consider three-dimensional, but still constrained, geometry (e.g., [14], [8]). In some cases, the wall points are reconstructed directly (e.g., [6], [11], [19], [9], [14]) while in other cases, the reflection of a loudspeaker with respect to a wall plane, called a "mirror point," is reconstructed (e.g., [22],[13],[8],[18], [10]). In some setups, the clocks of the microphones are synchronized (e.g., [22], [8], [12], [17]); in other setups, they are not (e.g., [20], [16], [2]). The microphones are sometimes placed on a vehicle carrying the microphones (e.g. [15], [4]), for example a cell phones (e.g, [21], [23]).

By and large, previous work has been mostly numerical and experimental. A theoretical standpoint was recently taken by Dokmanić et al. [8], who proved the following result:

Theorem P.0.1 *(Dokmanić et al. [8]). Consider a room with a loudspeaker and at least four microphones placed at random positions. Then with probability 1, the first-order echoes of a sound from the loudspeaker uniquely specify the room.*

Our research project started when we got interested in this result and tried to verify it by using methods from computational commutative algebra. Using the modeling done in [5], this turned out to be not very hard, so we started wondering if Theorem P.0.1 can be strengthened. In the theorem, all four (or more) microphones are placed independently at random positions, so the microphone placement requires a configuration space of dimension (at least) 12. For applications, it would be useful if four microphones could be mounted on the body of some vehicle, say a drone, so their relative positions are fixed, but the vehicle is moving in the room. For a vehicle such as a drone, this means that the microphone placement requires only a 6-dimensional configuration space. For many applications it would be even more useful if the loudspeaker, too, could be mounted on the vehicle. Again using methods from computational commutative algebra and computer algebra, we were able to prove the following results:

Theorem P.0.2 *(Boutin and Kemper [4]). Consider a given arrangement of walls and a loudspeaker. Also consider a drone that carries four non-coplanar microphones at fixed locations on its body. Within the 6-dimensional configuration space of possible drone positions, those where an incorrect detection of a wall may occur lie in a subspace of dimension 5.*

Theorem P.0.3 *(Boutin and Kemper [4]). The same is true if the loudspeaker is carried by the drone.*

We also gave an algorithm for the wall detection. In this, the Cayley-Menger determinants are used for solving the problem of matching echoes, heard by different microphones, that are coming from the same wall. It is easy to see that this algorithm detects every wall from which an echo is heard by all four microphones. So in the above theorems, incorrect detection means that walls are detected which do not actually exist (ghost walls). There can be no method that completely rules this out, so finding that the occurrence of ghost walls is unlikely is the best possible result. Theorem P.0.3 is much harder to prove than Theorem P.0.2. In fact, the proof required a lot of experimenting with different computational methods and some preconditioning of the problem by choosing suitable coordinate systems.

Scientific Progress Made

During the workshop at Banff we continued the above line of investigation by considering a situation with even fewer degrees of freedom: we considered the case where the microphones are mounted on a ground-based vehicle. It may be counter-intuitive that fewer degrees of freedom make the problem harder, but this is because less freedom of movement make it harder to get out of a position where an incorrect wall detection occurs (a **bad position**). And indeed we found wall configurations in a three-dimensional space where a ground-based vehicle cannot get out of a bad position by an infinitesimal movement. So fewer degrees of freedom do make a difference. In fact, we managed to classify the wall arrangements where such a phenomenon occurs, and characterized them by the catchphrase of "an unlucky stack of mirror points." For the exact definition, we refer to our preprint [5]. Perhaps surprisingly, the very same exceptional wall configurations hold for the situation of a "hovering drone", which has four degrees of freedom.

In summary, we have proved the following result:

Theorem P.0.4 *(Boutin and Kemper [5]) Consider a given arrangement of walls and a loudspeaker, and assume that this does not have an unlucky stack of mirror points. Also consider a ground-based vehicle or a hovering drone that carries four non-coplanar microphones at fixed locations on its body. Within the configuration space of possible positions, those where an incorrect detection of a wall may occur lie in a subspace of lower dimension.*

As mentioned above, a result as general as Theorem 2 cannot be obtained in this situation, and Theorem 4 is n this sense the best possible. The theorem considers the physically (and technically) relevant situation of a vehicle that is restricted to two dimensions in a 3-dimensional world. We also considered the truly 2-dimensional case, which is also relevant in some potential applications, and obtained the following smoother result.

Theorem P.0.5 *(Boutin and Kemper [5]) Consider a given two-dimensional scene with finitely many walls. Also consider a vehicle in the scene which carries three microphones that do not lie on a common line. A loudspeaker is either placed at a fixed location or mounted on the vehicle. Then within the configuration space of possible vehicle positions, those where an incorrect detection of a wall may occur lie in a subspace of lower dimension.*

Outcome of the Meeting

A direct outcome of the meeting is the new preprint [5], which has already been submitted. But a week at BIRS also provided an optimal forum for holding brainstorming sessions and thinking about new research projects. During the research stay, we initiated the following projects:

- How can our methods take care of inaccurate measurements?
- Can a vehicle work out its own position from the echoes of a sound event?
- Is echo matching also possible if the time of sound emission is unknown?

We are optimistic that tangible results will come out of these projects.

Participants

Boutin, Mireille (Purdue University)
Kemper, Gregor (Technical University of Munich)

Bibliography

[1] JB Alien and DA Berkley. Image method for efficiently simulating small-room acoustics. The Journal of the Acoustical Society of America, 60(S1):S9–S9, 1976.

[2] Fabio Antonacci, Jason Filos, Mark RP Thomas, Emanuël AP Habets, Augusto Sarti, Patrick A Naylor, and Stefano Tubaro. Inference of room geometry from acoustic impulse responses. IEEE Transactions on Audio, Speech, and Language Processing, 20(10):2683–2695, 2012.

[3] Jeffrey Borish. Extension of the image model to arbitrary polyhedra. The Journal of the Acoustical Society of America, 75(6):1827–1836, 1984.

[4] Mireille Boutin and Gregor Kemper. A drone can hear the shape of a room. SIAM J. Appl. Algebra Geometry, 4:123–140, 2020.

[5] Mireille Boutin and Gregor Kemper. Can a ground-based vehicle hear the shape of a room? ArXiv preprint `https://arxiv.org/abs/2204.00244`, 2022.

[6] Antonio Canclini, Fabio Antonacci, Mark RP Thomas, Jason Filos, Augusto Sarti, Patrick A Naylor, and Stefano Tubaro. Exact localization of acoustic reflectors from quadratic constraints. In 2011 IEEE Workshop on Applications of Signal Processing to Audio and Acoustics (WASPAA), pages 17–20. IEEE, 2011.

[7] Ivan Dokmanić, Yue M Lu, and Martin Vetterli. Can one hear the shape of a room: The 2-d polygonal case. In 2011 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pages 321–324. IEEE, 2011.

[8] Ivan Dokmanić, Reza Parhizkar, Andreas Walther, Yue M. Lu, and Martin Vetterli. Acoustic echoes reveal room shape. Proceedings of the National Academy of Sciences, 110, 2013.

[9] Youssef El Baba, Andreas Walther, and Emanuël AP Habets. Reflector localization based on multiple reflection points. In 2016 24th European Signal Processing Conference (EUSIPCO), pages 1458–1462. IEEE, 2016.

[10] Youssef El Baba, Andreas Walther, and Emanuël AP Habets. 3d room geometry inference based on room impulse response stacks. IEEE/ACM Transactions on Audio, Speech, and Language Processing, 26(5):857–872, 2017.

[11] Jason Filos, Antonio Canclini, Mark RP Thomas, Fabio Antonacci, Augusto Sarti, and Patrick A Naylor. Robust inference of room geometry from acoustic measurements using the hough transform. In 2011 19th European Signal Processing Conference, pages 161–165. IEEE, 2011.

[12] Ingmar Jager, Richard Heusdens, and Nikolay D Gaubitch. Room geometry estimation from acoustic echoes using graph-based echo labeling. In 2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pages 1–5. IEEE, 2016.

[13] Edwin Mabande, Konrad Kowalczyk, Haohai Sun, and Walter Kellermann. Room geometry inference based on spherical microphone array eigenbeam processing. The Journal of the Acoustical Society of America, 134(4):2773–2789, 2013.

[14] Sooyeon Park and Jung-Woo Choi. Iterative echo labeling algorithm with convex hull expansion for room geometry estimation. IEEE/ACM Transactions on Audio, Speech, and Language Processing, 29:1463–1478, 2021.

[15] Fangrong Peng, Tiexing Wang, and Biao Chen. Room shape reconstruction with a single mobile acoustic sensor. In 2015 IEEE Global Conference on Signal and Information Processing (GlobalSIP), pages 1116–1120. IEEE, 2015.

[16] Marc Pollefeys and David Nister. Direct computation of sound and microphone locations from time-difference-of-arrival data. In 2008 IEEE International Conference on Acoustics, Speech and Signal Processing, pages 2445–2448. IEEE, 2008.

[17] Tilak Rajapaksha, Xiaojun Qiu, Eva Cheng, and Ian Burnett. Geometrical room geometry estimation from room impulse responses. In 2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pages 331–335. IEEE, 2016.

[18] Luca Remaggi, Philip JB Jackson, Philip Coleman, and Wenwu Wang. Acoustic reflector localization: Novel image source reversion and direct localization methods. IEEE/ACM Transactions on Audio, Speech, and Language Processing, 25(2):296–309, 2016.

[19] Luca Remaggi, Philip JB Jackson, Wenwu Wang, and Jonathon A Chambers. A 3d model for room boundary estimation. In 2015 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pages 514–518. IEEE, 2015.

[20] Jan Scheuing and Bin Yang. Disambiguation of tdoa estimation for multiple sources in reverberant environments. IEEE Transactions on Audio, Speech, and Language Processing, 16(8):1479–1489, 2008.

[21] Oliver Shih and Anthony Rowe. Can a phone hear the shape of a room? In 2019 18th ACM/IEEE International Conference on Information Processing in Sensor Networks (IPSN), pages 277–288. IEEE, 2019.

[22] Sakari Tervo and Timo Tossavainen. 3d room geometry estimation from measured impulse responses. In 2012 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pages 513–516. IEEE, 2012.

[23] Bing Zhou, Mohammed Elbadry, Ruipeng Gao, and Fan Ye. Batmapper: Acoustic sensing based indoor floor plan construction using smartphones. In Proceedings of the 15th Annual International Conference on Mobile Systems, Applications, and Services, pages 42–55, 2017.

Appendix Q

Random Motions in Markov and Semi-Markov Environments and Their Applications (22rit234)

June 19 - 26, 2022

Organizer(s): Anatoliy Swishchuk (University of Calgary, Calgary, AB, Canada), Anatoliy Pogoruy (Zhytomyr Ivan Franko State University, Zhytomyr, Ukraine), Ramon M. Rodriquez-Dagnino (Tecnologico de Monterrey, Monterrey, Mexico)

The theory of dynamical systems operating under the influence of random factors is one of the fields of modern mathematics that is under intensive study. The theory of stochastic processes is the basic mathematical tool to study these systems, and a good representative of them is the theory of random evolutions or random motion. Random evolution in a random environment means that the system depends on the state of the environment, and it occurs in many real systems in nature. Similarly, if the evolution of the system does not affect the random environment, but the environment is described by a random process, say a Markov renewal processes, then such systems are called stochastic. Some stochastic system change their states abruptly, that is, in every state the system is spending a random holding time and then immediately is transferred to another state. A better modelling strategy for such systems is the notion of semi-Markov (Markov) evolution, which is given by two processes: the switching Markov renewal process describing the random environment, and the switched process that describes the evolution of the system (see [20, 21, 22, 23]).

Overview of the Field

In the study of the asymptotic distribution of probability of reaching a ”hard-to-reach domain” by semi-Markov processes, the theory of perturbation for linear operators is systematically used, as well as potential operators and generators theory. These theories were used for the asymptotic analysis (large deviations) of semi-Markov processes. A phase merging scheme or state lumping procedure for Markov chains and for the investigation of the probability distribution of reaching ”hard-to-reach domain” in semi-Markov processes have been studied.. This state lumping scheme was developed and introduced in seminal book by Korolyuk V. and Turbin A. in 1993 (see [19]), and it is also called asymptotic average scheme [6]. Transport stochastic processes with reflecting boundaries is a good model for multiphase supplying system with feedback which are also studied in this book. For instance, the estimation of effectiveness of a supplying system with feedback is reduced to the calculations of the stationary distribution of a switched process that models the system. The superposition of processes is also used to model aggregates in combination with reservoirs.

S. Goldstein in 1951 (see [14]) and M. Kac in 1974 (see [17]) studied the movement of a particle on a line with a speed that changes its sign driven by the Poisson process. Subsequently, this process was called the telegraph process or the Goldstein- Kac process. Further developments of this theory have been presented in the works of R. Griego and R. Hersh in 1969 (see [15]), who gave a definition of stochastic evolutions in a general setting. R. Hersh wrote a nice survey paper in 1974 (see [16]) on the results and problems in random evolutions and their applications. Several models of random evolutions in Markov and semi-Markov media, which generalize the Goldstein-Kac telegraph process, are considered and distributions of such evolutions. Papers [25, 26, 27, 28] investigated fading evolutions, where the velocity of a particle tends to zero as the number of switches growths at infinite.
The papers [25, 26, 27, 28] also studied a generalization of the telegraph process to the case of Erlang inter-arrivals between successive switches of particle velocities. For such processes a differential equation for the pdf of the particles position on the line was obtained. In addition, a method for solving such equations by using monogenic functions was developed. A recursive expression for the conditional characteristic functions of random walk with Erlang switching considering a non-Markov switching process was also obtained. Papers [4, 25] generalized the result obtained by A. D. Kolesnik [18] (an integral equation for the characteristic function of multidimensional walk in the case of Poisson switching process) for the semi-Markov case, and investigated the case of Erlang distributed stay of the switching process in the states. The paper [28] extended these results to multidimensional random motion at random velocities.

Recent Developments and Open Problems

In the recent two-volume book [9, 10] both discrete systems for which the model are Markov and semi-Markov processes, and continuous systems which are simulated by random evolutions have been studied. More precisely, this two-volume book [9, 10] is devoted to the description of different homogeneous and inhomogeneous one- and multi-dimensional random motions in Markov and semi-Markov random environment and their applications, including financial ones. The latter application contains, e.g., modelling of financial markets with Markov and semi-Markov volatilities and pricing of covariance and correlation swaps. This book incorporated the approach based on martingales and also considers applications such as telegraph process in finance to model stock price and an analogue of Black-Scholes formula, a generalization of Black-76 formula in commodity markets by regarding Markov or semi-Markov modulated volatility in the forward pricing of energy products. We apply there the method of inhomogeneous random evolutions introduced in [24]. The set of particles with interaction, where each particle moves on a line according to a telegraph process (with Markov and semi-Markov switching) up to collision with another particle, are also studied in this book.
Recent paper [8] considered various transformations of classical telegraph process. We also gave three applications of transform telegraph process in finance: 1) application of classical telegraph process in the case of balance, 2) application of classical telegraph process in the case of dis-balance, and 3) application of asymmetric telegraph process in finance. For these three cases, we presented European call and put option prices. The novelty of the paper consists of new results for transformed classical telegraph process, new models for stock prices and new applications of these models to option pricing.
One of many important directions in the study and applications of telegraph process is collision of particles under different motion conditions, see [1, 2, 3]. And one of the open problems here was of how to asymptotically estimate two telegraph particle collisions and to find an application of this result. We solved this problem in [11], see Sec. 4.
Other problems associated with telegraph process are: telegraph random motion on an ellipse and telegraph Cox-based process. That's because they have important applications in random harmonic oscillators theory and in finance, respectively. We started to consider and to solve these problems during our workshop [12, 13]. See Sec. 4.

Presentation Highlights

During our workshop we made several presentations devoted to reviewing many papers on the topic including [29, 30, 31] and the following book [18].
We also discussed, reviewed and corrected our paper [11] that finally was sent to Mathematics journal and published during our stay at BIRS!

Scientific Progress Made

During our workshop we reviewed, discussed and successfully published the following paper "Asymptotic Estimation of Two Telegraph Particle Collisions and Spread Options Valuations", Mathematics, 2022, 10, 2201. See [11]. In this paper [11] we studied collisions of two telegraph particles on a line that are described by telegraph processes between collisions. We obtained an asymptotic estimation of the number of collisions under Kac's condition for the cases where the direction-switching processes have the same parameters and different parameters. We also considered the application of these results to evaluate Margrabe's spread option (see [7]) for two assets of spot prices modelled by two telegraph processes.
Regarding the problem associated with the telegraph process on an ellipse [12], we made a significant progress in solving this problem: we proposed the model, stated main results and highlighted main methodologies and approaches in proving those results. Application to random harmonic oscillator on an ellipse was considered as well.
Regarding the problem associated with the telegraph Cox-based (see [5]) process [13], we also made a significant progress in solving this problem by proposing the model, by creating main results with highlighted approaches and methodologies, and by proposing an application in finance, in particular, how to price option for a stock modelled via telegraph Cox-based process.

Outcome of the Meeting

The main outcomes of the workshop are two working papers we started to develop and write with prospective to publish them:

1) [12] -"Telegraph process on an ellipse": we introduced and described a new model for telegraph process on an ellipse, highlighted main approaches and application of this process in random harmonic oscillator theory. The novelty of the paper consists in a new model for telegraph process on an ellipse and in a new application: before the telegraph process was considered only on a circle [29];

2) [13]-"Telegraph Cox-based process": we introduced and described a new model for telegraph Cox-based process, highlighted main approached and application of this process in finance. The novelty of the paper consists of a new model for telegraph Cox-based process, namely, the governing process was replaced by Cox process comparing with previous models with Poisson process. Also, we consider a new application of this process in finance associated with frequency of treading described by Cox process.

Acknowledgement

All three members of the workshop (22rit234) would like to thank BIRS very much for their hospitality and friendly environment that resulted in a very productive week. Also, Anatoliy Swishchuk thanks NSERC for continuing support.

Participants

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Bibliography

[1] Harris, T.E. Diffusion with "Collisions" between Particles. J. Appl. Probab. **1965**, 2, 323–338.

[2] Spitzer, F. Uniform Motion with Elastic Collision of an Infinite Particle System. J. Math. Mech. **1969**, 18, 973–989.

[3] Major, P.; Domokos, S. On the Effect of Collisions on the Motion of an Atom in R^1. Ann. Probab. **1980**, 8, 1068–1078.

[4] Pogorui, A.A.; Rodríguez-Dagnino, R.M. One-dimensional semi-Markov evolutions with general Erlang sojourn times. Random Oper. Stoch. Equ. **2005**, 13, 399–405.

[5] Cox D.R. Renewal Theory, Published by Methuen; John Wiley & Sons: London, UK; New York, NY, USA, 1962.

[6] Korolyuk, V.S.; Limnios, N. Average and diffusion approximation for evolutionary systems in an asymptotic split phase space. Ann. Appl. Probab. **2004**, 14, 489–516.

[7] Margrabe, W. The value to exchange one asset for another. J. Financ. **1978**, 33, 177–186.

[8] Pogorui, A.; Swishchuk, A.; Rodríguez-Dagnino, R.M. Transformations of Telegraph Processes and Their Financial Applications. *Risks* **2021**, 9, 147. https://doi.org/10.3390/risks9080147

[9] Pogorui, A.; Swishchuk, A.; Rodríguez-Dagnino, R.M. *Random Motion in Markov and Semi-Markov Random Environment 1: Homogeneous and Inhomogeneous Random Motions*; ISTE Ltd. & Wiley: London, UK, 2021; Volume 1.

[10] Pogorui, A.; Swishchuk, A.; Rodríguez-Dagnino, R.M. *Random Motion in Markov and Semi-Markov Random Environment 2: High-Dimensional Random Motions and Financial Applications*; ISTE Ltd. & Wiley: London, UK, 2021; Volume 2.

[11] Pogorui, A.; Swishchuk, A.; Rodríguez-Dagnino, R.M. Asymptotic estimation of two telegraph particle collisions and spread options valuations. *Mathematics* 2022, 10, 2022. 14 pages.

[12] Swishchuk, A.; Pogorui, A.; Rodríguez-Dagnino, R.M. Telegraph Process on Ellipse. *Working paper*. Created during BIRS RiT workshop (22rit234), June 19-26, 2022.

[13] Swishchuk, A.; Pogorui, A.; Rodríguez-Dagnino, R.M. Telegraph Cox-based Process. *Working paper*. Created during BIRS RiT workshop (22rit234), June 19-26, 2022.

[14] Goldstein, S.: On diffusion by discontinuous movements and on the telegraph equation. *Q. J. Mech. Appl. Math*. 4, 129-156 (1951).

[15] Griego, R. and Hersh, R.: Random evolutions, Markov chains, and system of partial differential equations. *Proc. Nat. Acad. Sci. USA*, 62(2), 305-308 (1969).

[16] Hersh, R.: Random evolutions: A survey of results and problems. *Rocky Mountain J. Math.* 4, no. 3, 443–478 (1974).

[17] Kac, M.: A stochastic model related to the telegrapher's equation. *Rocky Mt. J. Math*. 4, 497-509 (1974).

[18] Kolesnik, A.D.; Ratanov, N. Telegraph Processes and Option Pricing; Springer: Heidelberg, Germany, 2013; Volume 204.

[19] Korolyuk, V. S. and Turbin, A.: *Mathematical foundations of the state lumping of large systems*. Kluwer AP, 1993.

[20] Korolyuk, V.S., Swishchuk, A.V. *Evolution of Systems in Random Media*. CRC Press, Boca Raton, U.S.A., 1995.

[21] Korolyuk V.S. and Swishchuk A.V.: *Semi-Markov Random Evolutions*. Kluwer Academic Publishers, Dordrecht, The Netherlands. 310 p. 1995.

[22] Swishchuk, A.: *Random Evolutions and their Applications*, Kluwer Academic Publishers, 1997.

[23] Swishchuk, A.V.: *Random Evolutions and Their Applications. New Trends*. Kluwer AP, Dordrecht, 2000.

[24] Swishchuk, A.: *Inhomogeneous Random Evolutions*. CRC Press, Boca Raton, USA, 2019.

[25] Pogorui, A. A., and Rodríguez-Dagnino, R. M. (2011). Isotropic random motion at finite speed with K-Erlang distributed direction alternations. *Journal of Statistical Physics*, 145(1), 102.

[26] Rodríguez-Said, R. D., Pogorui, A. A., and Rodríguez-Dagnino, R. M. (2008). Stationary Effectiveness of an Information Server with a Single Buffer and Bursty Demands of Two Different Customers. *Stochastic Models*, 24(sup1), 246-269.

[27] Pogorui, A. A., and Rodríguez-Dagnino, R. M. (2012). Random motion with uniformly distributed directions and random velocity. *Journal of Statistical Physics*, 147(6), 1216-1225.

[28] Pogorui, A. A., and Rodríguez-Dagnino, R. M. (2013). Random motion with gamma steps in higher dimensions. *Statistics and Probability Letters*, 83(7), 1638-1643.

[29] De Gregorio, A. and Iafrate, F. Telegraph random evolutions on a circle. *Stoch. Proc. Appl.*, 141 (2021), 79-108.

[30] Marchione, M. and Orsingher, E. Hitting distribution of a correlated planar Brownian motion in a disk. *Mathematics*, 2022, 10, 536. See also *arXive:2019.00144v3*, 20 Mar 2022.

[31] Cinque, F. and Orsingher, E. Random motions in R^3 with orthogonal directions. *arXive:2205v1*, 18 May 2022.

Appendix R

Functor Calculus, Cartesian Differential Categories, and Operads (22rit268)

June 26 - July 3, 2022

Organizer(s): Kristine Bauer (University of Calgary), Brenda Johnson (Union College), Sarah Yeakel (University of California at Riverside)

Overview of the Field

A basic goal of algebraic topology is to find algebraic invariants that classify topological spaces up to various notions of equivalence. Computing such invariants can be extremely difficult, yet can lead to spectacular outcomes. Some of the most acclaimed results in algebraic topology and K-theory in recent years, including Hill, Hopkins and Ravenel's solution to the Kervaire Invariant One problem and Voevodsky's use of motivic homotopy theory in resolving longstanding conjectures in K-theory, are founded on such computations.

The calculus of homotopy functors, first introduced by Goodwillie in the 1980's, is a framework that makes it possible to understand and gain information about algebraic invariants even when they cannot be computed explicitly. Within this framework, one replaces an algebraic invariant with a related functor of topological objects and then approximates this functor by a tower of "polynomial" functors analogous to the Taylor series for a real-valued function. These polynomial functor approximations have properties that often make them easier to analyze than the original invariants. This approach has led to significant breakthroughs in the understanding of periodic homotopy theory, algebraic K-theory, and embeddings of manifolds. (See [AM99, DGM13, Wei99].)

Goodwillie's original formulation of functor calculus has been generalized and applied to a variety of settings, all sharing the common theme of approximating functors (invariants) with easier to control "polynomial" functors. The resulting tower of approximations can be analyzed through study of the fibers, which can often be classified by "derivatives." Some versions of functor calculus include

- homotopy calculus for Quillen model categories and ∞-categories, generalizes Goodwillie's homotopy calculus [Ku07, Lur17],
- abelian calculus, developed by Johnson and McCarthy [JM04] and applied to algebraic invariants like Hochschild homology (by Kantorovicz and McCarthy, see [KM02]), and
- orthogonal and manifold calculus (see [Wei95, GW99]) , invented by Weiss and Goodwillie-Weiss and applied to questions in embedding and surgery theory.

One major thread of current activity in functor calculus is unifying these approaches categorically. For example, comparisons between various functor calculi have been studied by Bauer, Johnson & McCarthy [BJM15] and Barnes & Eldred [BE15]. Work of Johnson and Hess seeks a categorical context uniting manifold, homotopy, and abelian calculi that can also be used to generate new functor calculi. The WIT II project by Bauer, Johnson, Osborne, Riehl and Tebbe [BJO+18] and ongoing work of Bauer, Burke and Ching tie abelian and homotopy calculus with the notion of cartesian differential categories of Blute-Cockett-Seely [BCS09].

Recent Developments and Open Problems

Broadly speaking, we seek to further these comparisons by

1. finding analogues of theorems from homotopy calculus in abelian calculus,
2. identifying how such theorems are a result of the differential category structure in abelian calculus, and
3. generalizing this relationship to create and compare results in various versions of functor calculus.

In the long term, we expect the flow of information to yield a new framework for dealing with unreduced functors in homotopy calculus, a topic with few results and important applications.

Our first goal is to find an operad structure for derivatives in abelian calculus. Ching showed that the derivatives of the identity functor in homotopy calculus form an operad ([Chi05, Chi]). Synthesizing results from [BJO+18] and methods in [Yea19], we will show that the derivatives of certain functors (including the identity functor) in abelian calculus form a functor-operad, which recovers an operad upon evaluation at particular objects. As part of this process, we will show that the operad structures naturally arise as a consequence of a particular lax monoidal functor built using the differential category structure identified in [BJO+18].

The next goal will be to compare classifications of polynomial functors given by Arone-Ching in homotopy calculus [AC15] and Johnson-McCarthy in abelian calculus [JM03a, JM03b] and determine how these classifications are tied to differential categories. Ching's operad was instrumental in the classification of functors obtained by Arone and Ching [AC11], and we plan to use our operad in a similar way to obtain classifications in abelian calculus that can be compared to those observed by Johnson and McCarthy.

Scientific Progress Made

Prior to the BIRS RIT program, we had established that our desired operad structure for the derivatives in abelian calculus could be obtained by finding a bicategory homomorphism from $AbCat$, the bicategory of abelian categories (suitably defined) and another bicategory which we will call Faà$(AbCat)$. The latter category is a bicategorical version of the Faà category originally defined by Cockett and Seely [CS11], and may be of independent interest. During the BIRS RIT, we focused on constructing this bicategory homomorphism, ∇.

Our goal of extracting specific operad structures from this framework dictates what the source and target of this bicategory homomorphism should be while the definition of bicategory homomorphism requires that two technical conditions, the hexagon axiom and the unit axiom, are satisfied by the homomorphism. In particular, to successfully construct the homomorphism, we needed a very concrete version of a chain rule for abelian functor calculus – we needed to find a concrete natural weak equivalence

$$D_1 F \odot D_1 G \to D_1(F \odot G),$$

where D_1 denotes the degree 1 homogeneous approximation of a functor, and $\odot$ denotes the horizontal composition in the bicategory $AbCat$. Abstractly, such an equivalence is known to exist by work of Bauer, Johnson, Osborne, Riehl, and Tebbe [BJO+18], but verifying the hexagon and unit axioms entails building an explicit model for this homomorphism and showing that it is a natural weak equivalence. Part of the challenge in doing so arises from the manner in which horizontal composition in $AbCat$ is defined – the definition relies on the Dold-Kan correspondence, a well-known equivalence between categories of chain complexes and simplicial objects in abelian categories.

During the first half of our RIT program, we constructed a candidate for this chain rule map. We showed that it provided the desired natural weak equivalence between $D_1F \odot D_1G$ and $D_1(F \odot G)$, and we were able to prove that the hexagon axiom holds for that candidate. In the process, we proved several technical lemmas that should prove useful as we continue our work on this project.

During the second half of the week, we attempted to verify the unit axiom - but here we ran into trouble. This will be resolved in ongoing collaboration.

Outcome of the Meeting

The Research in Teams program provided us with the opportunity to focus on a highly technical aspect of our project. We made far more progress on the construction of the bicategory homomorphism in this one week than we had in many previous months of long-distance collaboration. We are very grateful to Banff International Research Station for making this possible. In addition to the results we obtained while in Banff, we now have several new tools and ideas that we can use in tackling the remaining steps in this problem. Once these steps have been completed, we will have a paper that

- proves a new chain rule for abelian functor calculus,
- provides a bicategorical version of Cockett and Seely's Faà category,
- demonstrates how operad structures in functor calculus arise from these categorical constructions, at least in the case of the abelian functor calculus.

These results will pave the way for future work in two significant directions. First, as outlined in Section 2, identification of the operad structures is the first step in a program to obtain a classification of polynomial functors in abelian functor calculus in a manner similar to that done for the calculus of homotopy functors by Arone and Ching [AC15], and compare that with the classification obtained by Johnson and McCarthy [JM03a, JM03b]. The second direction would explore the new Faà bicategory and the extent to which Cockett and Seely's characterization of cartesian differential categories in terms of the Faà comonad can be extended to a bicategorical setting, and the consequences of such an extension.

Participants

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Bibliography

[AC11] Greg Arone and Michael Ching, Operads and chain rules for the calculus of functors, Astérisque (2011), no. 338, vi+158. MR2840569 (2012i:55012)

[AC15] Greg Arone and Michael Ching, A classification of Taylor towers for functors of spaces and spectra, Adv. in Math. **272** (2015), 471-552. MR3303239

[AM99] Greg Arone and Mark Mahowald, The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres, Invent. Math. **135** (1999), no. 3, 743–788. MR1669268 (2000e:55012)

[BCS09] R. F. Blute, J. R. B. Cockett, and R. A. G. Seely, Cartesian differential categories, Theory Appl. Categ. **22** (2009), 622–672. MR2591951

[BE15] David Barnes and Rosona Eldred, Comparing the orthogonal and homotopy functor calculi, Journal of Pure and Applied Algebra **220** (2015). MR3506473

[BJM15] Kristine Bauer, Brenda Johnson, and Randy McCarthy, Cross effects and calculus in an unbased setting, Trans. Amer. Math. Soc. **367** (2015), no. 9, 6671–6718, With an appendix by Rosona Eldred. MR3356951

[BJO+18] Kristine Bauer, Brenda Johnson, Christina Osborne, Emily Riehl, and Amelia Tebbe, Directional derivatives and higher order chain rules for abelian functor calculus, Topology Appl. **235** (2018), 375–427. MR3760211

[Chi] Michael Ching, Infinity-operads and Day convolution in Goodwillie calculus, J. Lond. Math. Soc. (2) **104** (2021), no. 3, 12041249. MR4332476

[Chi05] Michael Ching, Bar constructions for topological operads and the Goodwillie derivatives of the identity, Geom. Topol. **9** (2005), 833–933 (electronic). MR2140994 (2006a:55012)

[CS11] J. R. B. Cockett and R. A. G. Seely. The Faà di Bruno construction. Theory Appl. Categ., 25:No. 15, 394–425, 2011. MR2861113

[DGM13] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy, The local structure of algebraic K-theory, Algebra and Applications, vol. 18, Springer-Verlag London, Ltd., London, 2013. MR3013261

[GW99] Thomas G. Goodwillie and Michael Weiss, Embeddings from the point of view of immersion theory. II, Geom. Topol. **3** (1999), 103–118. MR1694808

[JM03a] B. Johnson and R. McCarthy, A classification of degree n functors. I, Cah. Topol. Géom. Différ. Catég. **44** (2003), no. 1, 2–38. MR1961524 (2004b:18016)

[JM03b] B. Johnson and R. McCarthy, A classification of degree n functors. II, Cah. Topol. Géom. Différ. Catég. **44** (2003), no. 3, 163–216. MR2003579 (2004e:18015)

[JM04] B. Johnson and R. McCarthy, Deriving calculus with cotriples, Trans. Amer. Math. Soc. **356** (2004), no. 2, 757–803 (electronic). MR2022719 (2005m:18016)

[KM02] Miriam Ruth Kantorovitz and Randy McCarthy, The Taylor towers for rational algebraic K-theory and Hochschild homology, Homology Homotopy Appl. **4** (2002), no. 1, 191–212. MR1983016

[Ku07] Nicholas J. Kuhn, Goodwillie towers and chromatic homotopy: an overview Proceedings of the Nishida Fest (Kinosaki 2003), 245-279, Geom. Topol. Monogr., 10, Geom. Topol. Publ., Conventry, 2007. MR2402789

[Lur17] Jacob Lurie, Higher Algebra, available at `https://www.math.ias.edu/~lurie/papers/HA.pdf`, version dated 18 September 2017.

[Wei95] Michael Weiss, Orthogonal calculus, Trans. Amer. Math. Soc. **347** (1995), no. 10, 3743–3796. MR1321590

[Wei99] Michael Weiss, Embeddings from the point of view of immersion theory. I, Geom. Topol. **3** (1999), 67–101 (electronic). MR1694812 (2000c:57055a)

[Yea19] Sarah Yeakel, A lax monoidal model for multilinearization, Homology Homotopy Appl. **22** (2020), no. 1, 319331. MR4042795

Appendix S

Algebraically Integrable Domains (22rit259)

July 24 - July 31, 2022

Organizer(s): M. Agranovsky (Bar-Ilan University), A. Koldobsky (University of Missouri), D. Ryabogin (Kent State University), V. Yaskin (University of Alberta).

The research efforts of our team were dedicated to algebraically integrable domains. The first study of such domains goes back to Newton and is connected to Kepler's laws of planetary motion. Let K be an infinitely smooth bounded domain in $\mathbb{R}^n$. For a non-zero vector ξ in $\mathbb{R}^n$ and a real number t, consider the cut-off functions of K':

$$V_K^+(\xi,t) = \mathrm{vol}_n(K \cap \{x \in \mathbb{R}^n : \langle x,\xi\rangle \le t\}),$$
$$V_K^-(\xi,t) = \mathrm{vol}_n(K \cap \{x \in \mathbb{R}^n : \langle x,\xi\rangle \ge t\}).$$

The domain K is called algebraically integrable if the two-valued function $V_K^{\pm}$ is an algebraic function, which means that there is a polynomial P of $n+2$ variables such that

$$P(\xi_1,\xi_2,\cdots,\xi_n,t,V_K^{\pm}(\xi,t)) = 0$$

for all ξ and t such that $\langle x,\xi\rangle = t$ intersects K.

In Lemma XXVIII of his Principia [7], Newton proved that there are no algebraically integrable ovals in $\mathbb{R}^2$. Three centuries later, Arnold asked for extensions of Newton's result to other dimensions and non-convex domains; see problems 1987-14, 1988-13, and 1990-27 in his book [5]. Arnold's problem in even dimensions was solved by Vassiliev [8], who showed that there are no algebraically integrable bounded domains with infinitely smooth boundaries in $\mathbb{R}^{2n}$. It is still an open problem whether in odd dimensions the only algebraically integrable smooth domains are ellipsoids.

In order to attack this problem Agranovsky [1] introduced a related concept of polynomially integrable domains. Let K be a bounded domain in $\mathbb{R}^n$. The parallel section function of K in the direction $\xi \in S^{n-1}$ is defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap \{\langle x,\xi\rangle = t\}), \qquad t \in \mathbb{R}.$$

We say that K is polynomially integrable if there is an integer N such that

$$A_{K,\xi}(t) = \sum_{m=0}^{N} a_m(\xi) t^m \tag{S.0.1}$$

for all ξ and t such that $\langle x,\xi\rangle = t$ intersects K.

It is not difficult to see that ellipsoids are polynomially integrable in odd dimensions (but not in even). Agranovsky asked whether these are the only polynomially integrable domains in Euclidean spaces. He also gave a partial answer to this question: there are no polynomially integrable bounded domains with smooth boundaries in even dimensions. In odd dimensions, Agranovsky proved that such domains must be convex and obtained some results towards the affirmative answer. Koldobsky, Merkurjev, and Yaskin [6] showed that the only infinitely smooth polynomially integrable convex bodies in odd dimensions are ellipsoids, thus completing the solution of the problem. In [2] Agranovsky partially solved Arnold's problem (for the so-called domains free of real singularities) by reducing it to the polynomially integrable case.

As we mentioned above, the parallel section function of ellipsoids is not polynomial in even dimensions. However, the square of this function is polynomial for ellipsoids in all dimensions. Thus it is natural to ask if ellipsoids are the only bounded domains in $\mathbb{R}^n$ such that

$$(A_{K,\xi}(t))^2 = \sum_{m=0}^{N} a_m(\xi)t^m$$

for all ξ and t such that $\langle x, \xi \rangle = t$ intersects K.

In $\mathbb{R}^2$ this question has been answered affirmatively by Agranovsky [4] for domains with algebraic boundaries. In higher dimensions the problem is open. During our time at BIRS we worked on this problem by identifying possible approaches via the Fourier transform and connections to floating bodies.

In addition to the question described above we also worked on closely related problems. For example, Agranovsky in his paper [3] suggested to study unbounded surfaces that are polynomially integrable near their boundaries. He conjectured that only quadrics in odd dimensions can have this property.

Another question was raised by Yaskin in [9]. Let K be a convex body in $\mathbb{R}^n$. Consider the following generalization of the parallel section function.

$$A_{K,m,\xi}(t) = V_m(K \cap \{\langle x, \xi \rangle = t\}), \qquad t \in \mathbb{R},$$

where V_m is the m-th intrinsic volume. If $m = n - 1$, this definition coincides with the one given above. If $m = n - 2$, this function gives the surface area of the relative boundary of $K \cap \{\langle x, \xi \rangle = t\}$.

It can be checked that for ellipsoids $A_{K,m,\xi}(t)$ is a polynomial in t precisely when m is even. However, it is an open problem whether ellipsoids are the only bodies with this property when m is even.

As we mentioned above, there are no polynomially integrable domains in even dimensions. It is natural to ask what conditions (similar to (S.0.1)) we should imposed on the parallel section function to obtain ellipsoids in even dimensions. We were able to identify such a condition, and based on the results obtained in Banff, we are preparing a manuscript tentatively titled "Polynomially integrable domains in even-dimensional spaces".

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Bibliography

[1] M. AGRANOVSKY, On polynomially integrable domains in Euclidean spaces, Complex analysis and dynamical systems, 1–21, Trends Math., Birkhäuser/Springer, Cham, 2018.

[2] M. AGRANOVSKY, On algebraically integrable bodies, Functional analysis and geometry: Selim Grigorievich Krein centennial, 33–44, Contemp. Math., 733, Amer. Math. Soc., Providence, RI, 2019.

[3] M. AGRANOVSKY, Locally polynomially integrable surfaces and finite stationary phase expansions, J. Anal. Math. **141** (2020), no. 1, 23–47.

[4] M. AGRANOVSKY, Domains with algebraic X-ray transform, Anal. Math. Phys. **12** (2022), no. 2, Paper No. 60, 17 pp.

[5] V. I. ARNOLD, Arnold's problems, 2nd edition, Springer-Verlag, Berlin (2004).

[6] A. KOLDOBSKY, A. MERKURJEV, V. YASKIN, On polynomially integrable convex bodies Adv. Math. **320** (2017), 876–886.

[7] I. NEWTON, Philosophiae Naturalis Principia Mathematica, London, 1687.

[8] V. A. VASSILIEV, Newton's lemma XXVIII on integrable ovals in higher dimensions and reflection groups, Bull. Lond. Math. Soc. **47** (2015), 290–300.

[9] V. YASKIN, An extension of polynomial integrability to dual quermassintegrals, International Mathematics Research Notices **13** (2020), 3978–3990.