Report on Workshop 03w5074: Perspectives in Differential Geometry

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1 Calibrated geometry

Professor Robert Bryant from Duke University spoke on "Resolving a coassociative cone"

Abstract: I construct some new examples of coassociative submanifolds in R^7 that give resolutions of some homogeneous cones that were already known. (The point is that there is an irreducible $SO(3)$ action that fixes the coassociative calibration and I have been able to figure out (by a rather involved ODE analysis) how to compute the $SO(3)$-invariant coassociative submanifolds. Two of these are cones and were already known. The rest are asymptotic to these cones in one way or another, but the topology of the examples can be quite different from what you might expect. These are interesting partly because they give resolutions of isolated 'orbifold-type' coassociative singularities. However, the main tool is a very delicate ODE analysis rather than any PDE methods. It turns out that the ODE system is completely integrable, though this is far from obvious at first glance.

Professor Alexei Kovalev from Cambridge University gave a progress report on "A special Lagrangian fibration of smooth compact Calabi-Yau 3-fold".

Abstract. We construct a compact smooth Ricci-flat Kähler 3-fold as a carefully chosen 'generalized connected sum' of two asymptotically cylindrical manifolds taken at their cylindrical ends. A special Lagrangian (SL) fibration is obtained on each of the two non-compact pieces with typical fibre a 3-torus (and some singular fibres). By proving a gluing theorem for these fibrations a SL fibration is obtained on the connected sum. This is, to the author's knowledge, the first example of a SL fibration of a compact non-singular simply-connected Calabi–Yau threefold.

SL fibrations are important for the Strominger-Yau-Zaslow (SYZ) conjecture which proposes to explain the 'mirror symmetry' duality between Calabi-Yau (CY) threefolds, so that the mirror Calabi-Yau is obtainable by replacing each non-singular SL torus fibre by the dual torus. It is therefore a rather suggestive further direction to identify the mirror CY for this example of SL fibration, according to the SYZ. Another direction of this project is to generalize the construction and find further examples of SL-fibered irreducible Calabi–Yau threefolds (and verify SYZ for them too).

The abstract of Professor Yann Rollin's, from M.I.T., talk is: We prove a Bennequin inequality for Legendrian knots K in 3-dimensional contact manifolds Y, under the assumption that Y is fillable by a 4-dimensional manifold M with a non zero Seiberg-Witten invariant. The proof requires an excision result for Seiberg-Witten moduli spaces; then, the Bennequin inequality is just a manifestation of the adjunction inequality for surfaces lying inside M. As a corollary we find a new proof of an Eliashberg's theorem without using the theory of pseudo-holomorphic curves, namely, that symplectically fillable contact structures must be tight. (Joint work with Tom Mrowka)

2 Mean curvature flows

Professor Gerhard Huisken from Max Planck Institute for Gravitational Physics, Germany, spoke on "Surgery for geometric evolution equations".

Abstract: Mean curvature flow and Ricci flow are quasilinear parabolic evolution equations describing a deformation of embedded hypersurfaces and Riemannian metrics respectively. They were used in the past to evolve certain initial geometries into known shapes just like a heat equation deforms an initial temperature into an equilibrium. During the last few years and months possible singularities of the flows above have become understood in great detail, creating a chance to extend the flows past singularities and to prove uniformisation results for large classes of manifolds. The lecture discusses recent results of Huisken and Sinestrari on mean curvature flow and their relation to the results of Hamilton and Perelmann on Ricci flow as well as the relation to the geometrisation of 3-manifolds. A particular focus is the surgery for 3-surfaces of positive scalar curvature.

Interesting open problems:

a) Mean curvature flow with surgery for two-dimensional surfaces of positive mean curvature.

b) Convergence of surgery solutions to weak solutions of the flow.

Professor Jiayu Li, from Institute of Mathematics at Academia Sinica, China, discussed some recent developments of mean curvature flows with codimension greater than one. In particular, he considered motions of Lagrangian submanifolds in a Calabi-Yau manifold and symplectic surfaces in a Kähler-Einstein surface. Certain natural convexity assumption is made on the initial submanifolds: in the Lagrangian case one considers an initial Lagrangian submanifold which is almost calibrated by the real part of the holomorphic n-form of the Calabi-Yau n-fold. Mean curvature flow preserves this convexity and the Lagrangian property. In the symplectic case, mean curvature flow preserves the symplectic property. Structure of the tangent cones at the first singular time of these flows was studied. He also discussed the long-time existence and convergence in special cases.

Open problems:

a) what is the size of the singular set of the symplectic or the Lagrangian mean curvature flows at the first singular time?

b) how does one run the flows in the above two cases after the first singular time?

3 Ricci flow

Professor Huai-dong Cao, from Texas A&M University, spoke on some recent work on Kähler Ricci flow. The abstract of his talk is: An important open problem in the study of the Ricci flow on compact Kahler manifolds is whether a solution to the normalized flow is nonsingular. In this talk I will show how to combine the recent local injectivity radius estimate of Perelman with my Li-Yau-Hamilton estimate for the Kaehler-Ricci flow to derive uniform curvature estimate for solutions to the Ricci flow on compact Kähler manifolds with positive bisectional curvature, i.e., such solutions must be nonsingular. This is a joint work with B. L. Chen and X.P. Zhu.

4 Harmonic mappings and harmonic functions

Professor Chikako Mese of Connecticut College spoke on "Harmonic maps from a 2-complex to a R-tree".

For a finitely generated group acting an a R-tree T, the equivariant harmonic map from the universal cover of a flat admissible 2-complex X to T defines a holomophic quadratic differential on each face of X. This in turn defines a measured foliation on each face of X which piece together to give a foliated 2-complex. This foliation gives rise to a R-tree T'. We say the action on T is geometric if T' is isometric to T. This generalizes the notion of geometric action of a surface group.

Open questions and research directions:

a) What is the proper definition of a holomorphic quadratic differential on a 2-complex?

b) What properties are needed so the a holomorphic quadratic differential defines a measured foliation on a 2-complex?

The title of Professor Peter Topping from Mathematics Institute at University of Warwick is: Harmonic map flow update.

He discussed some recent developments in the study of the two dimensional harmonic map heat flow. These included the discovery and partial understanding of the 'reverse bubbling' phenomenon, and a negative resolution of the problem of whether the flow $u(T)$ at a singular time T need be continuous. We also discussed some new hypotheses on flows which guarantee that $u(T)$ must be continuous.

Professor Lei Ni from University of California at San Diego discussed a new monotonicity formula for the plurisuhbarmonic functions on complete Kähler manifolds with nonnegative bisectional curvature. As applications we derive the sharp estimates for the dimension of the spaces of holomorphic functions (sections) with polynomial growth, which in particular, partially solve a conjecture of Yau. The finite generation of the ring of polynomial growth holomorphic functions is still open.

5 Fully nonlinear equations and conformal geometry

Professor Pengfei Guan of McMaster University discussed certain class of locally conformally flat manifolds with positive σ_k scalar curvature, relies a fundamental result of Schoen-Yau on developing maps. For $k \geq 1$, we say (M^n, g) has σ_k-positive scalar curvature if $\sigma_j(S_g) > 0$ for all $j \leq k$, where S_g the Schouten tensor. If (M, g) has positive σ_k scalar curvature for some $k \geq \frac{n}{2}$, then it must be conformally equivalent to a spherical space form. We address the question: *for $k < \frac{n}{2}$, given a locally conformally flat manifold (M, g_0), when there is a k-admissible metric, i.e., $g \in [g_0]$ such that (M, g) has σ_k-positive scalar curvature?*

First, as in the scalar curvature case, there are some topological obstructions. We prove that $H_p(M) =$ for an optimal range of p depending on k. On the other hand, we obtained a sufficient condition for the existence of k-admissible metrics according to certain Yamabe type functionals. Recall the Yamabe constant of $[g]$ can be defined as

$$Y_1([g]) = \inf_{g \in [g]} (vol(g))^{-\frac{n-2}{n}} \int_M \sigma_1(g) dvol(g).$$

We define a sequence of conformal invariants for $2 \leq l \leq n/2$ by letting

$$Y_l = \begin{cases} \inf\limits_{g \in \mathcal{C}_{l-1}} (vol(g))^{-\frac{n-2l}{n}} \int_M \sigma_l(g) dvol(g), & \text{if } \mathcal{C}_{l-1} \neq \emptyset, \\ -\infty, \quad \text{if } \mathcal{C}_{l-1} = \emptyset, \end{cases}$$

where $\mathcal{C}_k = \{\hat{g} \in [g] \,|\, l(S_{\hat{g}})(x) \in \Gamma_k^+, \forall x \in M.\}$, $\quad \Gamma_k^+ = \{\lambda \in \mathbf{R}^n \,|\, \sigma_j(\lambda) > 0, \forall 1 \leq j \leq k\}$. Our main result is: if $Y_k([g]) > 0$, then $\mathcal{C}_k \neq \emptyset$, i.e., there is a k-admissible $g \in \mathcal{C}_k$. A direct consequence is: if $dimM = 2m$ and $\mathcal{C}_{m-1} \neq \emptyset$ and the Euler characteristic of M is positive, then (M, g_0) is conformally equivalent to S^{2m}.

We prove that, like in the case of the Yamabe problem, one can deform the metric to an "extremal" metric with $\sigma_k(S_g) = constant$ which minimizes the corresponding curvature functional

$$\mathcal{F}_k(g) = vol(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g)\, dg$$

if there is a k-admissible metric.

Similar to a result of Schoen-Yau on scalar curvature case, we show that if M_1, M_2 are two locally conformally flat manifolds of same dimension with σ_k ($k < \frac{n}{2}$) positive scalar curvature, one may assign a metric on the connected sum $M_1 \# M_2$ such that it is locally conformally flat with positive σ_k scalar curvature. This leads to conjecture that: for $k = [\frac{n-1}{2}]$, if (M, g) is a locally conformally flat manifold with positive σ_k scalar curvature, then M is of the following form: $L_1 \# \cdots \# L_i \# H_1 \# \cdots \# H_j$, where $L_i' s$ and H_j' are quotients of the standard sphere S^n and $S^{n-1} \times S^1$ respectively. When $n = 3, 4$, it is true following the results of Schoen-Yau and Izeki.

Professor Jeff Viaclovsky from M.I.T. spoke on "Fully nonlinear equations in conformal geometry". The abstract of the talk is: We present a conformal deformation involving a fully nonlinear equation in dimension 4, starting with positive scalar curvature. Assuming a certain conformal invariant is positive, one may deform from positive scalar curvature to a stronger condition involving the Ricci tensor. A special case of this deformation gives a more direct proof of the result of Chang, Gursky and Yang. We also give a new conformally invariant condition for positivity of the Paneitz operator, which allows us to give many new examples of manifolds admitting metrics with constant Q-curvature. Another problem is to find a conforml deformation to make the k-th elementary symmetric function of the eigenvalues of the Schouten tensor equal to a constant. In dimension 3, this is solved if the manifold is not simply connected, and in dimension 4 the problem is completely solved. This is joint work with Matt Gursky.

Professor Jose Escobar from Cornell University talked about uniqueness and non-uniqueness of metrics with prescribed scalar curvature and mean curvature on compact manifolds with boundary. He also discussed a joint work with Garcia on the prescribed zero curvature and mean curvature problem on the n-dimensional Euclidean ball for $n \geq 2$. They considered the limits of solutions of the regularization obtained by decreasing the critical exponent and characterize those subcritical solutions which blow-up at the least possible energy level, determining the points at which they can concentrate and their Morse indices. When $n = 3$ this is the only blow-up which can occur for solutions. When $n \geq 4$ conditions are given to guarantee there is only one simple blow-up point.

6 General relativity and Einstein metrics

Professor Hugh Bray of Columbia University talked about "Quasi-Local Mass in General Relativity". Abstract: we discuss quasi-local mass funcitonals in general relativity, with an emphasis on explicit functionals such as the Hawking mass and the Brown-York mass. We note how it is possible to generalize the Hawking mass to compute generalized Hawking masses, each of which is monotone with respect to certain flows which are themselves generalizations of inverse mean curvature flow. We will comment how the constructions of Shi-Tam have allowed for upper bounds for the Bartnik mass of a region, and discuss the conclusion that the Hawking mass of an outer-minimizing region is less than or equal to the Brown-York mass. Finally we will discuss "inverse mean curvature vector" flows of surfaces in a spacetime and how it relates to observations made by Frauendiener, Mars, Malec, Simon, and Hayward.

Professor Rafe Mazzeo from Stanford University demonstrated how to use the gluing technique to produce Poncaré-Einstein metrics.