Current Trends in Representation Theory of Finite Groups

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1 Introduction

It is over a century since Frobenius initiated the study of group representations. One feature of the subject in his day was the interplay between the general theory and the study of the important special groups (for Frobenius, the symmetric groups and $PSL(2,p)$ for example) and this connection has continued to be a central theme of the field ever since, with ideas, questions and motivation flowing two ways.

The conference exhibited this connection in many ways and the program displays the current work on general theories and the study of the special groups, mainly the reflection groups, finite groups of Lie type, and other related groups. This latter work divides into the study of the representation theory for the natural characteristic and the cross characteristic case, and we shall organize the report along these lines.

2 General theory

There are several interesting conjectures, none of which are proved for all finite groups, dealing with modular representations. Most prominent are conjectures of Brauer, Alperin, Alperin-McKay, Broué, and Dade. Broué's conjectures, refined by Rickard, have been studied intensively in the last 10 years. A reference is [8]. Broué and Rickard, at least conjecturally, introduced the theory of derived categories into representation theory of finite groups. In the case that G is a finite reductive group, there is a strong with cohomology of Deligne-Lusztig varieties, as will be discussed in section 4.

Reporting on work with Raphael Rouquier, Joe Chuang spoke on filtered derived equivalences. The authors had proved the conjecture of Rickard establishing derived equivalences between blocks using complexes suggested by Rickard, a very notable recent contribution. These equivalences have remarkable properties giving rise to filtered derived equivalences, a new idea, which was introduced in the talk and then developed during the lecture. This is an example of work on special groups, the symmetric groups, leading to new general results.

Jon Carlson spoke on joint work with Jacques Thévenaz [4,5]. They have proved that relative syzygies of Alperin generate the torsion-free part of the group of endo-trivial modules and give another set of such generators by a homological construction. Group cohomology is used heavily in the arguments. Serge Bouc surveyed his work on biset functors and their connection with endo-permutation modules. In particular, he discussed ideas, which if carried through, would lead to the classification of all such modules. The work of Carlson and Thévenaz, recent work by Mazza and these idea of Bouc give credence to optimism about progress in this area.

Lluis Puig's talk was entitled "The $k*$-localizing functor" and deals with the open question, of interest to homotopy theorists as well as group theorists, whether certain local structure of groups can be "put together" in a construction that leads to a new algebra. Puig lectured on his work and considerable progress. One of the key tools are the endopermutation modules discussed above, in particular to the work of Carlson and Thevenaz.

Radha Kessar spoke on the related topic of fusion systems and whether certain specific ones could arise in the study of non-principal blocks. Using the structure of the representation theory of simple groups, she showed that no such examples could exist, another example of connections between the parts of representation theory being very relevant.

There is a well-known and important character correspondence, due to Isaacs and Glauberman. Watanabe and Horimoto descovered a correspondence of blocks playing a role. Koshitani spoke about his work following up on the work of Watanabe constructing associated Morita equivalnces.

Geoff Robinson reported on his work on the number of irreducible characters in blocks, an old topic with striking results, like the Brauer-Feit theorem and important conjectures. He has developed some striking new estimates.

Gabrielle Nebe spoke on integral representations of finite groups and, in particular, on the radical idealizer chain of symmetric orders. Detailed results were given for the case of cyclic blocks but much remains unknown and intriquing.

3 Equal characteristic: the general linear and symmetric groups

There has been interest for over a hundred years in studying representations of the symmetric group. The irreducible ones at characteristic 0 are well known; forms of these over the integers can be reduced modulo a prime p, and the problem is to decompose these into irreducibles at characteristic p. This problem of finding these decomposition numbers has attracted much attention. Although quite a lot is known, there is no general answer and no conjecture.

Great progress has been made by Gordon James and Alexander Kleshchev (two of our speakers.) The Iwahori-Hecke algebra $\mathcal{H}_q$ of type A is a deformation of the group algebra of the symmetric group S_n; when the parameter q is a p-th root of unity where p is prime, the decomposition numbers for the representations of $\mathcal{H}_q$ are related, but not equal, to the decomposition numbers for representations of the symmetric group at characteristic p. In recent years a remarkable conjecture, partly inpsired by Kleshchev's work, was made by A. Lascoux, Bernard Leclerc (one of our speakers), and R. Thibon [7], relating decomposition numbers for $\mathcal{H}_q$, where q is a p-th root of unity, to canonical bases of certain Fock spaces, which are representations of quantized enveloping algebras of affine Kac-Moody Lie algebras of type $A^{(1)}_{p-1}$. This conjecture was proved by Ariki [1] and Grojnowski (both participants in this workshop.)

The representation theory of the symmetric groups has long been known to be related to the representation theory of general linear groups $GL(n, K)$. At characteristic 0 this was discovered by Frobenius and Schur in the early 1900's; there is still much interest at characteristic p. There are several ways of looking at this relation. The general linear group $GL(V)$ acts on the r-fold tensor power $\otimes^r V$, and the symmetric group S_r also acts by permuting the ordering of the the vectors in a tensor. It is known that these actions are mutual centralizers of each other. This is called Schur-Weyl duality. There is also a finite dimensional algebra, called the Schur algbera $S(n, r)$ whose representation theory is a portion of the theory for $GL(n, K)$, and also is related to representations of S_r. There is also a quantized version of this, the q-Schur algebra, closely related to the Hecke algbra $\mathcal{H}_q$.

Gordon James has a conjecture that in certain cases, the decomposition numbers for representations of the symmetric group at characteristic p are equal to the decomposition numbers for $\mathcal{H}_q$ at a p-th root of unity, which are now known by Lascoux-Leclerc-Thibon-Ariki-Grojnowski. James spoke at our meeting on an interesting new conjecture which equates some decomposition numbers for the symmetric group at different primes. He also gave some intriguing consequences of this conjecture. In the smallest case where his conjecture is not known to hold, if it were false then his earlier conjecture on decomposition numbers for H_q would also fail. He also discussed a way of calculating some of the decomposition numbers of certain blocks of the symmetric group.

Anne Henke spoke on comparting the representation theory of Schur algebras $S(n,r)$ for different values of n and r. A method was given involving the use of exterior powers of the natural representation V of $GL(V)$ to compare theories for different Schur algberas.

Jon Brundan spoke on Kazhdan-Lusztig theories in type A. The original Kazhdan-Lusztig conjecture for $gl_n(C)$ describes the composition multiplicities of Verma modujles in terms of certain Kazhdan-Lusztig polynomials arising from the Hecke algebra of type A. Thanks to Schur-Weyl duality these polynomials can be defined in a completely different way – replacing the Hecke algebra with the quantized enveloping algebra $U_q(gl_\infty)$ and defining a canonical basis in its natural tensor space using Lusztig's quasi-R-matrix. This leads to a reformulation of the Kazhdan-Lusztig conjecture peculiar to type A which admits several remarkable generalizations – for example, the well known Lascoux-Leclerc-Thibon conjecture proved by Ariki and Grojnowski is one such. Brundan spoke about the finite dimensional representation theory of the supergroups $GL(m|n)$ and $Q(n)$ which fit neatly into this philosophy.

A. Kleshchev spoke on joint work with Brundan. To every nilpotent class e in a semisimple complex Lie algebra, Premet, in 2002, associated a certain associative algebra W. Roughly speaking, W is the endomorphism algebra of a generalized Gelfand-Graev representation corresponding to e. Little is known about the structure and representation theory of W. He explained the importance of W.

Hyohe Miyachi spoke on some results on canonical bases of Fock space; he elaborated on some of his joint work with Bernard Leclerc. The interest stems from the Lascoux-Leclerc-Thibon conjecture. There are also connections with conjectures of Broué. In general the canonical bases are not explicitly computed, but in certain cases one can give explicit closed formulas for them.

Bernard Leclerc spoke on Lusztig's semicanonical bases. In 2000, Lusztig introduced a new basis of the positive part of the enveloping algebra of a Kac-Moody algebra : the semicanonical basis. He spoke on a comparison between the canonical and the semicanonical basis and multiplicative properties of the dual semicanonical basis

Christine Bessenrodt spoke on invariants such as the determinant or the Smith normal form of certain integral matrices which come from the character tables of the symmetric groups and its double covers. As a consequenc0e, she has a new proof of a strengthened version of a conjecture of Mathas on the determinant of the Cartan matrix of a Hecke algebra $\mathcal{H}_q$ of type A, at a primitive p-th root of unity.

A natural question is whether there is a version of Schur-Weyl duality for groups other than $GL(V)$, for instance for symplectic or orthogonal groups. Richard Dipper spoke on this. Tensor space V^r is replaced by a mixed tensor space, defined as r-fold tensor product of V tensored with the s-fold tensor product of the dual space of V. There is an action of the Brauer algebra on tensor space and it is known by a theorem of Brauer, that Schur-Weyl duality holds, if K has characteristic zero. In case of mixed tensor space, the same holds, replacing tzhe Brauer algebra by a certain subalgebra, called walled Brauer algebra, by a theorem of Bankart et.al Dipper considered the case that K has finite characteristic.

Alison Parker spoke on determining all the homomorphisms between Weyl modules for $SL_3(K)$ where K is an algebraically closed field of characteristic at least five. As a corollary of this result she obtained all the homomorphisms between Specht modules for the symmetric group when the partitions have at most three parts and the prime is at least five. She also found that the Hom spaces are always at moat one dimensional in both cases.

Karin Erdmann spoke on certain modules for the Schur algbebra $S(n,r)$, called tilting modules. for Schur algebras $S(3,r)$ over characteristic two The Schur algebras $S(n,r)$ are quasi-hereditary, where the standard modules are the usual Weyl modules, and the costandard modules are the duals of the Weyl modules, The 'tilting modules' which are the modules which have a filtration by Weyl modules and also a filtration by dual Weyl modules, play an important role for the understanding of decomposition numbers for general linear groups, and also for symmetric groups. When $p = 2$

and $n = 3$, the only tilting modules missing to get an induction going are the ones labelled by the partition with one part. We have a new parametrization of these tilting modules (and also of some classes of Weyl modules), in terms of their restriction to the finite groups $GL(3, 2)$. This uses finite-group technology and alsio Auslander-Reiten theory.

Andrew Mathas spoke on elementary divisors of Specht modules. Let $\mathcal{H}_q(\mathfrak{S}_n)$ be the Iwahori–Hecke algebra of the symmetric group. This algebra is semisimple over the rational function field $\mathbb{Q}(q)$, where q is an indeterminate, and its irreducible representations over this field are q-analogues $S_q(\lambda)$ of the Specht modules of the symmetric groups. The q-Specht modules have an "integral form" which is defined over the Laurent polynomial ring $\mathbb{Z}[q, q^{-1}]$ and they come equipped with a natural bilinear form with values in this ring. Now $\mathbb{Z}[q, q^{-1}]$ is not a principal ideal domain. Nonetheless, one can try to compute the elementary divisors of the Gram matrix of the bilinear form on $S_q(\lambda)$. When they are defined, one gets a precise relationship between the elementary divisors of the Specht module $S_q(\lambda)$ and $S_q(\lambda')$, where λ' is the conjugate partition. Also, the elementary divisors can be computed when λ is a hook partition. There are examples to show that in general the elementary divisors do not exist.

4 Representations of finite reductive groups, cross characteristic

There is a large amount of work on representations of finite groups of Lie type, such as $SL(n, k)$, where k is a field of characteristic p, where the groups act on vector spaces over a field K of characteristic 0 or characteristic $\ell \neq p$. Enormous progress was made by Deligne and Lusztig, who found representations coming from ℓ-adic cohomology of certain algebraic varieties.

There has been great interest in block theory for groups of Lie type at cross characteristic. This started with important work of Fong, Srinivasan, and Broué, all of whom were at this workshop. Complex reflection groups are increasingly playing an important role in this theory.

Broué's conjectures predict that the ℓ-adic cohomology complex of a Deligne-Lusztig variety is a tilting complex for a derived equivalence between the principal ℓ-block of G and that of the normalizer of the ℓ-Sylow subgroup of G.

Ongoing work on block theory at cross-characteristic by Broué, Fong, and Srinivasan was described by Bhama Srinivasan. They have a conjecture giving explicit bijections of characters which would imply the Dade's ordinary conjecture, including the Isaacs-Navarro conjecture, for unipotent blocks of finite reductive groups. The Isaacs-Navarro conjecture [6] is a very interesting refinement of conjectures of McKay and Alperin on correspondences between numbers of characters of certain degrees in a group, or a block, and those of a Sylow subgroup, or a defect group. In addition, the conjecture extends the work of Broué, Malle, and Michel [2] to the case of nonabelian defect group. The bijection conjecture has been checked for $GL(n, q)$.

Character values of finite reductive groups G can be calculated, at least in principle, using Lusztig's geometric theory of character sheaves on G. Toshiaki Shoji described what still has to be done to obtain all the character values. First, one must establish a conjecture of Lusztig that certain known class functions are given in terms of so-called almost characters. Secondly, there are cetain constants which turn up in these theories which must be explicitly calculate. Finally, one must know the scalars involved in generalized Green functions. Shoji's work shows that Lusztig's conjecture is true, for certain primes p, if $G(\overline{k})$ has connected center, where k is the algebraic closure of the finite field k. The case of $SL(n, k)$ has long been complicated, in part because $SL(n, \overline{k})$ it has disconnected center. Shoji explained how all the character values of $SL(n, k)$ can be computed, if the characteristic p of k is large enough. Before Shoji's lecture there was a long discussion, on Tunnel mountain, among Broué, Cliff, Michel, and Shoji, about whether or not one can actually write down all the irreducible character values of $SL(n, k)$, using Lusztig's character sheaves. It was decided that, given Shoji's work, this perhaps would be a suitable project for a Ph.D. student.

Gunter Malle spoke on joint work with Raphael Rouquier (also attending) on the determination of families of irreducibles characters of certain finite complex reflection groups. These families generalize the notion of two-sided (Kazhdan-Lusztig) cells of finite Weyl groups, and they share many of the properties of these two-sided cells. Also, as for Weyl groups, they have a connection to the hypothetical representation theory of spetses associated to spetsial complex reflection groups. This completes earlier work by Broué and Kim for imprimitive complex reflection groups.

Gerhard Hiss spoke on unitary designs, representations of finite unitary Groups, and the function fields of the Fermat curves. Fermat curves over for special degrees and fields yield unitary designs with high symmetry (containing the 3-dimensional projective unitary group.) An analogous construction gives a family of unitary designs, called Ree unitals, with the Ree groups as automorphism groups. Here, the Fermat curves are replaced by 1-dimensional Deligne-Lusztig varieties. Again, the representation theory of the Ree groups and properties of the function fields of these Deligne-Lusztig varieties yield information about the elementary divisors of the incidence matrices of the Ree unitals.

There has been much recent work on Schur indices of irreducible characters of finite groups of Lie type. There is a conjecture that the Schur index of an irreducible character of any finite quasi-simple group G is one or two. Due to the classification of finite simple groups, one is led to case that that G is a finite reductive group. Through the work of Gow, Lusztig, Ohmori and Geck, the Schur indices of all unipotent characters of finite groups of Lie type are now known. Geck spoke on the last open case which was only settled recently: the cuspidal unipotent characters of $E_7(q)$ have Schur index 2, but only if q is an even power of a prime congruent to 1 modulo 4 (a rare case of non-generic behaviour !)

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