Geometry and deformation theory of hyperbolic 3-manifolds

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July 3, 2004 – July 8, 2004

Our small group convened to discuss, informally, current and new directions for research in Kleinian groups, in view of the tremendous progress that has occurred in recent years. Major old problems have been solved, and some powerful techniques have been introduced to the field (drilling theorems, model manifolds and curve complexes, and more recently end reductions), which provide new opportunities to obtain a deeper understanding of an intricate subject.

Topology of the deformation space

The recent solution of the Ending Lamination Conjecture by Brock-Canary-Minsky [16, 4] gives a complete classification of (finitely-generated) Kleinian groups, but it does not give a *topological* description of the deformation space of a group because the invariants involved in the classification do not vary continuously, as discovered and explored by Anderson-Canary [1], Brock [3], and others.

Bromberg [5] more recently showed that even some remaining optimistic conjectures about this topological structure were false, when he proved that the deformation space of punctured-torus groups is not locally-connected at its boundary.

We discussed at length some approaches to proving similar results for *Bers slices*: these are special slices of the representation space where most of the discontinuity phenomena do not occur, and so there was some room for hope that the topology of these slices is tamer than the general case. However it is possible that Bromberg's approach can also prove non-local-connectivity in this case. It may also be that a computer-driven search can produce and verify examples of this.

On the positive side, we discussed an ongoing project to encode the continuity properties of the ending invariants in a complete way. The machinery in Masur-Minsky [11] and Brock-Canary-Minsky [4] gives some tools for attempting this, but there is so far no single coherent description.

Another promising direction is the question of *bumping* and *self-bumping* (of components of the deformation space) in the case of manifolds with compressible boundary. In the incompressible case this phenomenon has been well-studied by Anderson-Canary-McCullough [2], Holt [9], McMullen [13], Bromberg-Holt [6], Ito [10] and others. It is known for example that only finitely many components can meet at one point, but conversely arbitrarily complex finite bumping has been shown to occur. Generalizing this to the compressible case remains a challenge. The discontinuity of topological type of quotient manifold at bumping points can be more intricate: handles can switch sides in a compression body, for example. However we still expect that only finitely many components can meet at one point.

Uniform models and quantitative bounds

An important theme we explored was that of improving our quantitiative understanding of hyperbolic 3-manifolds. A number of foundational theorems in the field are proved using "soft" methods, such as compactness of parameter spaces, and hence the bounds obtained are not even in principle computable. Here are some sample questions:

Thurston's bounded image theorem: A self-map of Teichmüller space $\mathcal{T}(\partial M)$ associated to a 3-manifold M plays a central role in Thurston's geometrization theorem and in subsequent developments [17, 12]. The image of this self-map, in certain cases (M acylindrical and ∂M incompressible) is a bounded set. Nothing is known about the diameter of this image. It is open, in particular, whether a bound exists depending only on the genus of ∂M. We discussed on the one hand the possible construction of examples with arbitrarily large image diameter, and on the other hand we searched for a proof yielding constructive upper bounds. We constructed examples where the complexity of M goes to infinity (with fixed ∂M) while the image diameter remains bounded.

Uniform models: The solution of the Ending Lamination Conjecture includes a construction of a *bilipschitz model* for a given hyperbolic manifold, depending only on its topology and its ending invariants. The bilipschitz constants obtained in the proof are non-constructive, but at least in the surface-bundle case they depend only on the topology of the manifold. There is a fairly clear line of argument that we expect will yield similar topologically-dependent bounds in the incompressible-boundary case. However, the case of manifolds with compressible boundary (notably handlebodies) is considerably harder.

Brock-Souto have made some progress on obtaining uniform models in the compressible case, as has Namazi (a student of Minsky). However in general this area is wide open and promises to involve some delicate topological questions.

Miscellaneous topics

Infinitely-generated Kleinian groups: This area remains quite open, except for various constructions of examples. What is a good general theory of such groups? Is there a reasonable setting in which one can establish global rigidity results? McMullen's rigidity theorem controls quasiconformal deformations given an upper bound on injectivity radius. Other infinitely-generated examples, suggested by Bromberg, exhibit *no* deformations of any sort.

Projective structures and cone manifolds: Complex projective structures on surfaces are closely related to hyperbolic cone manifold structures on 3-manifolds with boundaries. This relationship figures heavily in Bromberg's work on Bers' density conjecture. Is every complex projective structure induced from some cone manifold? A positive answer would perhaps give new geometric tools for understanding general (not just discrete) representations of surface groups in $PSL(2, \mathbb{C})$, in view of the theorem (Gallo-Kapovich-Marden [8]) that every non-elementary representation of a surface group into $SL(2, \mathbb{C})$ is the monodromy of some complex projective structure.

Cannon-Thurston maps and local connectivity of limit sets: The limit set of a finitely-generated Kleinian group is conjectured to be locally connected. This was proved for pseudo-Anosov fibre groups by Cannon-Thurston [7], for bounded-geometry surface groups by Minsky [15], and for punctured-torus groups by McMullen [14] (and there are additional generalizations). The general problem remains open, although it seems that the main tool, which worked in the previous cases, is now available – namely the model manifolds from the solution of the Ending Lamination Conjecture. We discussed some plausible approaches to carrying through a proof; some considerable difficulties remain, but this is a very interesting direction to pursue.

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