Number Theorists Weekend

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This short workshop focused on developments in Number Theory at the interface of Computational Methods with Analytic Number Theory and Diophantine Approximation. These represent two particular strengths of the Canadian mathematical community in general, and of the PIMS region in particular. The workshop was designed as a bridge between the Computational workshop preceding it and that in Analytic Number Theory and Diophantine Approximation which followed. An emphasis was placed on expository talks with graduate student and postdoctoral fellow involvement highlighted.

Computational tools for Diophantine problems include implementations of the Lenstra-Lenstra-Lovascz lattice basis reduction algorithm (typically DeWeger's integer-arithmetic version), Wildanger's algorithm for unit equations in number fields, various techniques for computing information about the unit groups of number fields (typically fundamental units, with or without assuming the Generalized Riemann hypothesis), related algorithms for solving Thue equations, algorithms related to algebraic curves, genus calculations, ranks of elliptic curves and Jacobians (critical for applications of Chabauty-type techniques for bounding rational points on higher genus curves). Modular forms computations à la William Stein (i.e. modular symbols) for computations of Hecke eigenvalues of Galois conjugacy classes of, e.g. weight 2 cuspidal newforms of level N also are critical in modern Diophantine analysis, as are computations of zeros of Dirichlet L-functions (see e.g. Rubinstein) and related zero-free regions (Khadiri). These latter results are involved in producing explicit Chebyshev-type bounds for primes in short intervals in arithmetic progressions, which figure in estimating nonarchimedean contributions in the hypergeometric method of Thue and Siegel.

Multiplicative Number Theory also utilizes a variety of computational tools, both in order to make asymptotic estimates explicit, and also to inspire or provide evidence for conjectures. Some of the major tools of analytic number theory involve the theory of meromorphic functions (which was in large part commenced by the study of the Riemann zeta function in connection with the distribution of prime numbers), the evaluation and estimation of exponential sums, sieve methods, and many techniques from the fields of harmonic analysis, probability, and random matrix theory. In many of these areas, computations inform and suggest directions for future research.

The primary problems to which these computational tools are applied include, on the analytic side, the distribution of prime numbers and of the prime factors of integers, special values of zeta functions (including multiple zeta values) and L-functions, and uniform distribution of arithmetic sequences; and on the Diophantine side, determining the transcendentality of natural constants and of values of modular functions, irrationality measures for these values and for algebraic numbers, and applications to rational points on algebraic varieties and solutions of Diophantine equations (see e.g. [2], [4], [5], [6]).

The workshop kicked off with an expository exploration of open problems by Richard Guy (Calgary) in prime number theory, arithmetic function theory and the arithmetic of elliptic curves. Subsequent talks focused on pseudoprimes (with applications to primality testing), and inequalities for arithmetic functions, before the direction of the workshop changed on the second day to more algebraic computation. Along these lines, the workshop featured talks on Hilbert modular form computations, with related work on ternary Diophantine equations, via modularity of Q-curves.

References

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