Aperiodic Order: Dynamical Systems, Combinatorics, and Operators

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The field of Aperiodic Order is concerned with the structure and properties of point sets that display long-range orientational order, and of all structures that can be derived from such point sets. The latter include tilings, discrete structures in general, measures, operators etc. Although there is no standard monograph on the subject yet, several review volumes on key topics are available by now [48, 53, 8, 56]. This indicates the activity of the field, see also [3] for a guide to further literature.

It was the aim of this workshop to bring people from the various mathematical disciplines together and to exchange the state of the art as well as to communicate open problems.

1 Model sets and diffraction theory

An important class of ordered Delone sets are Meyer sets and, among them, model sets, compare [41] for details. Model sets are also known as cut and project sets, and admit a rather general formulation in the setting of locally compact Abelian groups, compare [47, 63]. Though they made their appearance in the context of algebraic number theory already in the seventies [47], their importance was only recognized after the discovery of quasicrystals whose spatial structure can be described by model sets, see [69] for a recent review from an experimental perspective.

Mathematically, model sets are defined on the basis of a *cut and project scheme*. The latter is a triple $(G, H, \widetilde{L})$ consisting of locally compact Abelian groups G and H, of which G is also σ-compact, and a lattice $\widetilde{L}$ in $G \times H$ (i.e., a co-compact discrete subgroup)

$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H & \xrightarrow{\pi_2} & H \\ & & \cup & & \\ & & \tilde{L} & & \end{array} \tag{1}$$

such that the natural projections $\pi_1 : G \times H \longrightarrow G$, $(t, h) \mapsto t$ and $\pi_2 : G \times H \longrightarrow H$, $(t, h) \mapsto h$ satisfy the following properties:

- The restriction $\pi_1|_{\tilde{L}}$ of π_1 to $\tilde{L}$ is injective.
- The image $\pi_2(\tilde{L})$ is dense in H.

Let $L := \pi_1(\tilde{L})$ and $^\star : L \longrightarrow H$ be the mapping $\pi_2 \circ (\pi_1|_{\tilde{L}})^{-1}$. Note that * is indeed well defined.

Given a cut and project scheme (1) and a compact $W \subset H$, we define $\curlywedge(W)$ by

$$\curlywedge(W) := \{x \in L : x^\star \in W\}.$$

A *model set*, associated with the cut and project scheme (1), is a non-empty subset $\varLambda$ of G of the form

$$\varLambda = x + \curlywedge(y + W),$$

where $x \in G$, $y \in H$, and $W \subset H$ is compact with $W = \overline{W^\circ}$. A model set $\varLambda = x + \curlywedge(y + W)$ is called *regular* if the Haar measure of the boundary ∂W of W is zero. A regular model set is called *generic* if $\partial W \cap L^\star = \varnothing$. Any model set is a Delone set. Namely, it is uniformly discrete (as W is compact) and relatively dense (as W has nonempty interior). In fact, they are even Meyer sets, because $\varLambda - \varLambda \subset \curlywedge(W - W)$ and $W - W$ is compact, so that $\varLambda - \varLambda \subset \varLambda + F$ with F a finite set. Moreover, a regular model set has uniform patch frequencies (i.e., the associated dynamical system is uniquely ergodic) and a generic model set is repetitive, see [49] for a review of the properties of model sets.

A prominent feature of regular model sets is their pure point diffraction. If we start from the Dirac comb $\omega = \delta_\varLambda = \sum_{x\in\varLambda} \delta_x$, with δ_x the normalized point measure at x, there exists a natural *autocorrelation*

$$\gamma_\omega = \lim_{r\to\infty} \frac{1}{\mathrm{vol}(B_r)} \tilde{\omega} * \omega|_r$$

where B_r is the ball of radius r around 0 for $G = \mathbb{R}^d$, or a suitable generalization of this concept for general G, see [64] for details. Moreover, $\omega|_r$ is the resriction of ω to B_r and $\tilde{\omega} = \delta_{-\varLambda}$ is the origin inverted variant of ω. This autocorrelation is unique (w.r.t. averaging sequences of van Hove type) which reflects the unique ergodicity of the corresponding dynamical system (see below). What is more, it is always a positive definite and translation bounded measure on G. Consequently, it is transformable, and its Fourier transform, $\hat{\gamma}_\omega$, is a positve measure, called the *diffraction measure* of ω. It describes the outcome of standard diffraction experiments, compare [37, 24].

An important result is that the diffraction measure for the Dirac comb of a regular model set is a pure point measure, or, in other words, that regular model sets are pure point diffractive. In this generality, it was proved by M. Schlottmann in [64], though it has several predecessors [37, 68]. These aspects, and in particular their various relations to dynamical systems, were summarized in the opening lecture by Robert Moody, and reappeared in many other talks throughout the meeting.

The cornerstone of most proofs of this result is the connection to pure point spectra of dynamical systems, to which we will come back below. One alternative proof is known [9] that relates pure point diffraction spectra directly to strong almost periodicity of the autocorrelation measure, the latter being a consequence of a Weyl type result on uniform distribution in model sets [63, 50]. This is also related to recent results of J.-B. Gouéré [34]. Using this approach, one can see in general that a complex translation bounded measure ω on G is pure point diffractive (w.r.t. an averaging van Hove sequence $\mathcal{A} = \{a_n \mid n \in \mathbb{N}\}$, compare [64, 9]) if and only if its autocorrelation γ_ω (obtained w.r.t. $\mathcal{A}$) is strongly almost periodic, compare also [31] for background.

Various generalizations of model sets are studied, such as multi-component model sets (e.g., in the talk by J.-Y. Lee) or deformed model sets (in the talk by D. Lenz), where the latter go considerably beyond the setting of Delone sets of finite local complexity. Of considerable interest is the question for the diffractive properties of Meyer sets. Though it is known [47] that they are subsets of model sets, their diffraction is much more involved. In particular, one can have mixed spectrum (i.e., both pure point and continuous components), and Meyer sets can have positive entropy density (e.g., the union of $2\mathbb{Z}$ with an arbitrary subset of $2\mathbb{Z}+1$ is Meyer). First general results on the systematic study of Meyer set diffraction were presented by N. Strungaru, building on the more detailed theory of almost periodic measures (strong versus weak, see [31] for details).

Still of interest is the systematic investigation of symmetry, and the development of efficient methods to determine the symmetry of a given Delone set, ranging from translation over point to inflation symmetries. Well-known heuristics from the physics literature, see [46] and references given there, are now reformulated in the context of the cohomology of groups, and the survey talk of B. Fisher showed the present state of affairs, compare [28].

Finally, one common theme of many talks on aperiodic order was the apparent similarity of model sets to lattices. Sometimes, it requires a slight reformulation of the classic concepts, but very often a new and simplifying point of view emerges. The talk by U. Grimm might serve as an illustration, where combinatorial problems of crystallography [5] were reformulated and solved in a unified fashion for lattices and model sets, see [4, 6] for more.

2 Aperiodic order and dynamical systems

The dynamical systems approach to the theory of aperiodic order has gained prominence in recent years. Let us explain how dynamical systems appear in the simple setting of Delone sets in $\mathbb{R}^d$. Given a Delone set Λ, which can be viewed as a model of an atomic configurations, we can consider its *hull* X_Λ. It can be defined as the closure of the set $\{\Lambda - x : \ x \in \mathbb{R}^d\}$ of all translates of Λ in the natural (local) topology. In this topology, two sets are close if they almost agree on a large ball around the origin. The group $\mathbb{R}^d$ acts continuously on X_Λ by translations, which is our topological dynamical system. Given an invariant probability measure on X_Λ (it always exists and is often unique; then the system is said to be *uniquely ergodic*), we get a measure-preserving system $(X_\Lambda, \mathbb{R}^d, \mu)$, which may be studied using the tools of ergodic theory. In particular, we can consider the *dynamical spectrum*, that is, the projection spectral measure, or a family of scalar spectral measures, associated with the group of unitary operators on $L^2(X_\Lambda, \mu)$ given by $(U_x f)(\xi) = f(\xi - x)$ for $x \in \mathbb{R}^d$. The spectral type of the dynamical system may be pure point (pure discrete), pure absolutely continuous, pure singular continuous, or a mixture. A key observation made by Dworkin [25] in 1993 is that pure point dynamical spectrum implies pure point diffraction spectrum, which has been widely viewed as the key feature of an ordered (crystalline or quasi-crystalline) structure.

One direction of research in the recent years has been to reverse the implication, that is, to deduce pure point *dynamical* spectrum from pure point *diffraction* spectrum. This was done in a restricted setting of Delone sets of finite local complexity in [42], and more recently, in much greater generality, by J.-B. Gouéré [34] and M. Baake & D. Lenz [7]. In fact, in [7], instead of Delone dynamical systems on $\mathbb{R}^n$, dynamical systems on translation bounded measures on rather general locally compact Abelian groups are considered. This approach via measures is both more general and well-suited for applications. The talks by D. Lenz and J.-B. Gouéré described these achievements, among other things.

Another class of dynamical systems related to aperiodic order is that of tiling dynamical systems. They are defined similarly to Delone dynamical systems, starting with a tiling of the Euclidean space, and considering the translation action on the hull. In recent years, topological methods have been increasingly used to study such systems. Some of these developments were described in the talk on "Tilings, tiling spaces, and topology" by L. Sadun. One of the key questions is: what are the possible perturbations of a tiling and what happens to the dynamical systems under these perturbations? It turns out that the tiling spaces may be homeomorphic (e.g., if the two tiling systems have identical combinatorics), but that their dynamical properties can still differ. In the recent work by A. Clark and L. Sadun [22], the Čech cohomology H^1 of the tiling space is used to determine when the perturbation yields a topologically conjugate system, and when it yields a mutually locally derivable system. The latter notion corresponds to the the existence of a "local code"; unlike in symbolic dynamics, for tiling dynamical systems not every conjugacy is given by a local code.

Symbolic substitution systems form a rich and interesting class of examples, studied in dynamical systems and ergodic theory for several decades. More recently, generalizations to higher dimensions, including substitution tiling systems and substitution Delone sets, largely motivated by the theory of aperiodic order, were introduced and investigated. They provide examples with various spectral types: pure discrete, pure singular, and partially absolutely continuous. But even in the classical, one-dimensional symbolic setting, there remain many open questions. We describe one of them in detail.

Let $\mathcal{A} = \{1, \ldots, d\}$ be a finite alphabet, with $d \geq 2$, and denote by $\mathcal{A}^* = \bigcup_{i=0}^{\infty} \mathcal{A}^i$ the set of finite words. A *substitution* is a map $\zeta : \ \mathcal{A} \to \mathcal{A}^*$; it is extended to a map $\mathcal{A}^* \to \mathcal{A}^*$ by concatenation. The $d \times d$ matrix associated with the substitution ζ is defined by $M_\zeta(i, j) = \ell_i(\zeta(j))$, where $\ell_i(w)$

denotes the number of occurrences of the letter i in the word w. The substitution is *primitive* if there exists a k such that all entries of M_ζ^k are strictly positive. A primitive substitution gives rise to a uniquely ergodic dynamical system: the space is the set of all sequences all blocks of which occur in $\zeta^k(i)$ for some k and i, and the dynamics is given by the shift. One of the outstanding problems is to resolve the *Pisot discrete spectrum conjecture* which is that every Pisot type substitution system has pure point spectrum. The substitution is of *Pisot type* if all the eigenvalues of the matrix M_ζ, except the largest eigenvalue, are strictly between 0 and 1 in absolute value. It is still open, although recently the case of two symbols was settled affirmatively, see [11, 38]. The important special case of unimodular Pisot substitutions, where it is also assumed that $\det(M_\zeta) = 1$, is open, too. There is a related *coincidence conjecture* which we do not describe here. At the workshop, we were fortunate to have several groups present from around the world who are working in this field, and there were many lively discussions of various approaches to the problems. Among the participants were Sh. Akiyama [1], M. Baake and B. Sing [10], and V. Sirvent and Y. Wang [66] who made contributions to this area. J. Kwapisz, in his talk entitled "Geometric coincidence conjecture and pure discrete spectrum for unimodular tiling spaces," described his work in progress, jointly with M. Barge and B. Diamond. Their approach uses the space of "strands", which was first introduced in [11], to represent the dynamical system. One of the new results is that, for unimodular Pisot substitutions, pure point spectrum is equivalent to the model set representation. A geometric approach to the Pisot Substitution Conjecture was pioneered by G. Rauzy [59]; it uses what is now known as the "Rauzy tiling" of the substitution. At the workshop, A. Siegel represented this direction; she described some combinatorial conditions for pure discrete spectrum in her talk, based on [19, 65].

It is impossible to discuss all the recent developments related to substitutions here. There is an excellent recent book on this topic [56] with the chapter on spectral theory written by A. Siegel.

Substitution tilings in $\mathbb{R}^d$ represent a far-reaching generalization of substitution sequences. The role of the alphabet is played by a finite set of *prototiles*. The substitution map replaces a tile by a "patch" of tiles in a consistent manner. We do not go into the details of the definition here. A crucial point comes in deciding how the tiles of the tiling are obtained from the prototiles: (a) using translations only, or (b) using arbitrary Euclidean motions. We have a much better understanding of the class (a), in large part due to the commutativity of the translation group. The class (b), however, is gradually being investigated as well. Its best known example is the *pinwheel tiling* of the plane [58]. The dynamical and diffraction spectrum of this tiling are still poorly understood: it is known that it has no non-trivial discrete spectral component, but we do not know whether the spectrum is singular or it has an absolutely continuous component. At the Problem Session, N. Strungaru described his recent result with R. V. Moody and D. Postnikov [51] which says that the diffraction spectrum is rotation-invariant under the action of S^1.

An important feature of many substitution systems is *unique decomposition*. It can be defined by saying that the substitution map defines a homeomorphism of the tiling space. It was proved in [67] in the translationally-finite setting that the unique decomposition property is equivalent to the tiling being non-periodic (i.e., it should have no translation symmetries). C. Holton reported on the recent progress for non-translationally finite tilings. In joint work with C. Radin and L. Sadun (in preparation), the unique decomposition property was verified under some assumptions, the main one being that the set of relative orientations of a tile in the tiling leaves no subspace of $\mathbb{R}^d$ invariant. On the other hand, a counter-example in $\mathbb{R}^3$ was constructed to demonstrate that the latter condition cannot be dropped.

3 Combinatorics on words

A popular way to measure complexity is via subword or pattern complexity. Aperiodic order then manifests itself in low complexity with respect to this measure. In one dimension, investigations in this direction date back at least to the 1930's and have evolved into an independent mathematical subdiscipline, often called "Combinatorics on Words." Currently, this field is particularly active in France, and the French school has put the theory on a firm footing and made it more accessible to a broad audience by the publication of a series of textbooks, [43, 44, 45] (see also [56]). We were happy to have Valérie Berthé and Julien Cassaigne participate in the workshop and report on recent progress of combinatorics on words in dimensions greater than one—a subject that is still in its early stages.

Before summarizing the key results and open problems in higher dimensions, let us briefly discuss the one-dimensional case. Here, one studies words over a finite alphabet. That is, if $\mathcal{A}$ is a finite set, one considers the sets $\mathcal{A}^*, \mathcal{A}^{\mathbb{Z}_+}, \mathcal{A}^{\mathbb{Z}}$ of finite, one-sided infinite, and two-sided infinite words over $\mathcal{A}$, respectively. Given a one-sided or two-sided infinite word w, define its complexity function $p_w : \mathbb{Z}_+ \to \mathbb{Z}_+$ by

$$p_w(n) = \#\{\text{subwords of } w \text{ having length } n\}.$$

It is obvious that p_w is a bounded function if w is (eventually) periodic. A surprising, albeit elementary, result of Hedlund and Morse [36] states that the converse is true and, moreover, there is some minimum growth of the complexity function when w is not eventually periodic. For $w \in \mathcal{A}^{\mathbb{Z}_+}$, the following are equivalent,

(i) w is ultimately periodic, that is, there are $n_0, q \in \mathbb{Z}_+$ such that $w_{n+q} = w_n$ for $n \geq n_0$.

(ii) p_w is bounded.

(iii) There exists $n_1 \in \mathbb{Z}_+$ such that $p_w(n_1) \leq n_1$.

That is, words displaying aperiodic order should have a complexity function that is bounded from below by $n+1$ but does not grow much faster than that. One is then interested in consequences of low complexity. The case of minimal complexity, $p_w(n) = n+1$, has been completely analyzed; see [15, 36, 44]. Words w with this complexity function are called *Sturmian* and they have a large number of equivalent descriptions. Aside from the combinatorial description in terms of their complexity function above, they can also be characterized geometrically, in terms of certain balance properties, their palindromic subwords; to mention just a few. Other classes of words, for which strong general structure results are known, are those satisfying $p_w(n) = n + k$ for some k and large enough n, the so-called quasi-Sturmian words (see, e.g., [23, 54]), or $p_w(n) = O(n)$, the words having linearly bounded complexity (cf. [27]).

The Hedlund-Morse result for two-sided infinite words looks slightly more elegant in that ultimate periodicity can be replaced by periodicity. For $w \in \mathcal{A}^{\mathbb{Z}}$, the following are equivalent,

(i) w is periodic, that is, there is $q \in \mathbb{Z}_+$ such that $w_{n+q} = w_n$ for every $n \in \mathbb{Z}$.

(ii) p_w is bounded.

(iii) There exists $n_1 \in \mathbb{Z}_+$ such that $p_w(n_1) \leq n_1$.

In dimensions greater than one, the most basic open problems concern suitable analogues of the results described above, that is, a suitable version of the Hedlund-Morse theorem and a characterization of a suitable class of low-complexity objects. Of course, one has to define a notion of complexity first. A natural way to do this is the following. Given $w \in \mathcal{A}^{\mathbb{Z}^d}$ and $n_1, \dots, n_d \in \mathbb{Z}_+$, one defines

$$p_w(n_1, \dots, n_d) = \#\{ \text{ subwords of } w \text{ having "shape" } n_1 \times \cdots \times n_d\}.$$

The function p_w on box shapes could be called the box complexity function or, in the case $d = 2$, the rectangle complexity function. In search of an analogue of the Hedlund-Morse theorem, a naive guess could be that if there is some shape $(n_1, \dots, n_d)$ such that $p_w(n_1, \dots, n_d) \leq n_1 \times \cdots \times n_d$, then w has a periodicity vector. A simple example found by Sander and Tijdeman [62] shows such a statement cannot hold when $d \geq 3$. The question in $d = 2$ is open, but the answer is conjectured to be affirmative.

Nivat's Conjecture [52]. Let $w \in \mathcal{A}^{\mathbb{Z}^2}$. If there exist $n_1, n_2 \in \mathbb{Z}_+$ such that $p_w(n_1, n_2) \leq n_1 n_2$, then w has a periodicity vector.

However, unlike in the one-dimensional case, Nivat's conjecture is not an equivalence. In fact, there exists a word w, possessing a periodicity vector, for which $p_w(n_1, n_2) > n_1 n_2$ for all pairs (n_1, n_2) [17].

There are partial results saying that if $p_w(n_1, n_2) \leq c\, n_1 n_2$ for some $n_1, n_2 \in \mathbb{Z}_+$ and $c = 1/144$ [26] or $c = 1/16$ [57], then w has a periodicity vector. There are surveys of results and questions centered around Nivat's conjecture by Cassaigne [21] and Tijdeman [70]. Cassaigne's talk at our workshop dealt with these and related issues.

Two-dimensional words of low complexity, and in particular analogues of one-dimensional Sturmian words, have been studied in a number of papers (e.g., [16, 17, 18, 20]). Since Sturmian words in one dimension admit various equivalent descriptions, there are multiple ways to approach such a generalization. It turns out that they lead to different classes of two-dimensional words and hence the situation is more complicated than in one dimension. Valérie Berthé presented an overview of these results at the workshop, together with possible directions for future research.

4 Topological aspects of aperiodic order

Beginning from a tiling or Delone set in $\mathbb{R}^d$, denoted Λ, one may consider the set of all its translates and endow this with a natural metric. Under the hyptothesis of finite local complexity, this metric space is pre-compact. That is, its completion, denoted X_Λ, is a compact metric space. Moreover, the translation action of $\mathbb{R}^d$ extends continuously. In the case that the periodic vectors for the tiling form a spanning set for $\mathbb{R}^d$, the space is just a torus of dimension d.

The topology of the space X_Λ has been the focus of much research. It was observed very early that, for aperiodic tilings, the space, locally, is the product of an open ball in $\mathbb{R}^d$ with a totally disconnected space. A global extension of this result was obtained by Sadun and Williams [61] who showed that the space was a fibre bundle over a torus with totally disconnected fibres.

Anderson and Putnam [2] considered the case of substitution tilings and showed that the space could be written as an inverse limit of spaces which are quite tractable. They are branched oriented manifolds and, if the tiles are polygons meeting edge to edge and vertex to vertex, they are also finite cell complexes. In fact, the inverse limit is stationary in the sense that each space is the same and each map is the same. This result (with non-stationary system) was extended to the general situation in two distinct ways; first by Bellissard, Benedetti and Gambaudo [12] and secondly by Gähler [30] and Sadun [60], the latter paper being based on a talk given by Gähler on a previous meeting. Jean-Marc Gambaudo gave a presentation at the workshop on the former and subsequent generalizations (with Benedetti [14]) to other situations, including tilings of non-Euclidean spaces.

A result of Sadun and Williams [61] states that, under the hypothesis of finite local complexity, the space X_Λ is homeomorphic to one obtained by performing a d-fold suspension of a free minimal action of $\mathbb{Z}^d$ on a Cantor set X. This then connects the subject with the very active area of $\mathbb{Z}^d$ dynamical systems. The construction of Bellissard, Benedetti and Gambaudo may be used in this context to give approximating finite subequivalence relations of the orbit relation for such an action.

The presentation by Anderson and Putnam of the space as an inverse limit made it possible to compute its K-theory and Čech cohomology. (The methods in the general situation also work in principle, but for practical computations they seem unwieldy.) The computation of cohomology and K-theory for projection method tilings can be done using methods of Forrest, Hunton and Kellendonk [29]. The method uses some advanced techniques from algebraic topology: finding resolutions of certain modules and spectral sequences. In his lecture at the workshop, John Hunton sketched the basic ideas. He also discussed the Euler characteristic for these spaces. This can be obtained from the cohomology, of course, but its computation is much simpler. Moreover, examples suggest that there are some interesting questions regarding its sign.

Franz Gähler reported on his work in actually carrying out these cohomology computations (with the title "Examples and counter-examples ..."). In particular, he has developed software to implement the Anderson-Putnam method, as well as to do some calculations using his own technique. This resulting evidence was quite interesting. In particular, he found an example where the cohomology has a torsion component. This contradicted results of Forrest, Hunton and Kellendonk. By the end of the meeting, the problem seemed to have been resolved and a correct version of the Forrest-Hunton-Kellendonk result found, compare [30] for more.

Jean Bellissard showed how one may construct a C^*-algebra from an aperiodic tiling [13]. A different version is due to Kellendonk [39]. By present knowledge, this is – up to strong Morita equivalence – the crossed product construction by the action of $\mathbb{R}^d$ on X_Λ, see [40]. Let us denote this C^*-algebra by A_Λ. It is important from a physical viewpoint because Schrödinger operators (in the tight binding approximation) associated with an electron's movement in an aperiodic material are in this C^*-algebra.

A reasonable amount of information is now known about these C^*-algebras. First of all, their K-theory is computable. By a result of Connes, it is isomorphic to the K-theory of the space X_Λ. The internal structure of the algebras has been investigated by Giordano, Herman, Putnam and Skau [35, 32] in dimension one and by N.C. Phillips [55] in higher dimensions, where the analysis becomes much more difficult. The technique is to make use of the finite approximations of the Bellissard-Benedetti-Gambaudo inverse limit to construct approximating subalgebras which are themselves finite dimensional. Many nice properties of finite dimensional C^*-algebras may then be transferred to the larger algebra. Phillips gave a presentation summarizing these methods.

An alternate approach to this problem is in the work of Giordano, Putnam and Skau to classify minimal Cantor set dynamics up to orbit equivalence. Skau gave a summary of the past work in the program (in dimension one) and its relations with the structure of the C^*-algebras. Giordano gave a presentation of the current work in dimension two [33]. Here, better finite approximations of the orbit space are obtained by using cocycles for the action, the drawback being that little is currently known about the existence of such cocycles. This problem is equivalent to a more complete understanding of the cohomology of the space X_Λ.

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