Free Probability Theory

Alexandru Nica (Waterloo),
Roland Speicher (Queen's),
Dan Voiculescu (Berkeley)

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1 Introduction

Free probability theory is a line of research which parallels aspects of classical probability, in a non-commutative context where tensor products are replaced by free products, and independent random variables are replaced by free random variables. It grew out from attempts to solve some longstanding problems about von Neumann algebras of free groups. In the twenty years since its creation, free probability has become a subject in its own right, with connections to several other parts of mathematics: operator algebras, the theory of random matrices, classical probability and the theory of large deviations, algebraic combinatorics. Free probability also has connections with some mathematical models in theoretical physics.

The BIRS workshop on free probability brought together a very strong group of mathematicians representing the current directions of development in the area. This continued a sequence of very successful 5-day workshops organized on these lines, like the ones at the Fields Institute in March 1995, at CIRM Luminy in January 1998, and at MSRI in January 2001.

In this report we look in more detail at what are the current directions of development in free probability, with an emphasis on how they were represented in the BIRS workshop.

2 Developments related to operator algebras and random matrices

Free probability has far-reaching connections both with the field of operator algebras (where the subject is originally coming from) and with the one of random matrices. Since the interactions between free probability and these two fields are closely related to each other, we will discuss them together.

2.1 Background and general overview

Let $L(G)$ denote the von Neumann algebra (that is, the weakly closed subalgebra of $B(l^2(G))$) generated by the left regular representation of the discrete group G. The so-called "isomorphism question" for von Neumann algebras of free groups asks: denoting by F and F' the free groups with 2 and respectively 3 generators, is it true or not that $L(F)$ is isomorphic to $L(F')$? This outstanding problem (still open today) was the original motivation for the birth of free probability.

We should mention here that several other questions raised in the 60's about the von Neumann algebras of free groups were still open when free probability was born — e.g. questions about the so-called fundamental groups of these algebras, or the question of whether they admit (a von Neumann algebra version of) Cartan subalgebras.

Substantial progress was brought in these old problems by the development of free probability and of its connection with random matrices. The first such connection appeared with the "random matrix model for freeness" established in [53]; this model very quickly was found to have groundbreaking applications concerning the fundamental groups of the free group von Neumann algebras (see e.g. [42]). Also, random matrix model techniques showed that, with F and F' as above, the von Neumann algebras $L(F)$ and $L(F')$ do become isomorphic when the natural stabilization operation of tensoring with $B(H)$ is applied to them. (This was seeming to suggest that $L(F)$ and $L(F')$ might be, after all, isomorphic to each other!)

On the other hand there was another, very penetrating, line of attack, which came with the appearance of the concept of free entropy for n-tuples of non-commutative random variables. The developments related to free entropy are gathering arguments in the support of the idea that (with F and F' as above) $L(F)$ and $L(F')$ are not isomorphic. At the current stage, free entropy arguments limit the possibilities on how a hypothetical isomorphism between $L(F)$ and $L(F')$ could go; further technical progress in free entropy may completely rule out the existence of such an isomorphism. Free entropy is discussed in the next subsection.

It is worth pointing out that, beyond the drive for solving the celebrated isomorphism question, free probability has built a solid theory of the free group von Neumann algebras, which parallels the one going on in the hyperfinite case. Among von Neumann algebras, the hyperfinite ones are by far the ones with the richest theory, and the free group von Neumann algebras are considered as the "best" among the non-hyperfinite ones. It has become clear that the parallelism between the hyperfinite and and the free case goes deep, and it includes, for example, the development of a type III theory (for infinite von Neumann algebras), see e.g. [22], [46].

Moreover, free probability has also found outstanding aplications in C^*-algebra theory, and in relation to the invariant subspace problem. This was done in work by Haagerup and collaborators ([24],[25], [28]–[31]), and was an important topic of the workshop – cf. the discussion in the subsections 2.3 and 2.4 below.

Finally, the subsections 2.5 and 2.6 of this section discuss two other operator algebra topics that were addressed in the workshop. One is in connection to the Connes embedding problem, and the other is in connection to a q-deformation of the von Neumann algebra of a free group.

2.2 Free entropy

Free entropy is an invariant for n-tuples of non-commutative random variables. There are in fact two versions (both well-motivated) of this concept: the "microstates" free entropy, and the "non-microstates" one.

The microstates free entropy was introduced in [54]. The key idea for this concept is to look at sets of n-tuples of matrices of large size (the "matricial microstates") which approximate in distribution a given n-tuple of non-commutative random variables. Very soon after being introduced, the microstates free entropy found powerful applications to free group von Neumann algebras. One such application was the proof (in [55]) of the fact that these von Neumann algebras do not have Cartan subalgebras. Another application was the so-called primality of the free group von Neumann algebras (in the sense that they cannot be decomposed as tensor products in a non-trivial way), see [26].

The non-microstates free entropy was introduced in [56], and uses an approach where one first considers the free analogue for the concept of Fisher information. There are a number of particular cases when the two approaches to free entropy (via matricial microstates and via the free Fisher information) are known to coincide; it is in fact believed that this should be true in general. Finding bridges between the two approaches to free entropy is at present one of the central problems in free probability. A notable progress on this line was made in [10]; by using large deviation techniques for n-tuples of matrices, it was shown there that one always has an inequality between the two free entropies (the free entropy defined with microstates can never exceed the one defined via free Fisher information).

An invariant related to free entropy is the free entropy dimension. In the microstate approach this is, very roughly, a normalized dimension of Minkowski type for the sets of matricial microstates. There now exist several versions of the the free entropy dimension; in particular we should mention that a surprising facet of this invariant has recently occurred in a von Neumann homological context, in the work of Connes and Shlyakhtenko [21].

For a survey on free entropy and free entropy dimension, see [58].

Free entropy was a central topic at the BIRS workshop. In the opening talk of the workshop, Dima Shlyakhtenko presented his work on estimates for free entropy dimension (cf. [21], [47]). The talk by Kenley Jung addressed the issue of free entropy dimension inequalities for subfactors (cf. [35]). The talk by Fumio Hiai was devoted to presenting his work in [28] on the free analogue of pressure, a concept which is dual via a Legendre transform to a free entropy type of invariant. Related to that, the talk by Denes Petz presented the work in [34] on free transportation cost inequalities via random matrix approximation.

2.3 The invariant subspace problem

Let $A \subset B(H)$ be a von Neumann factor (i.e. a von Neumann algebra such that the centre of A is reduced to $\mathbb{C}1_A$). The invariant subspace problem for A asks if every element $a \in A$ has an invariant subspace of the form $\mathrm{Ran}(p)$ with $p = p^* = p^2 \in A$. In the case when $A = B(H)$, this is the "classical" invariant subspace problem; but the problem is also very interesting (and still open) for other von Neumann factors – in particular for those of type II_1 (a class of von Neumann factors which includes the von Neumann algebras of free groups).

Significant progress in the invariant subspace problem in the type II_1 case was made in the last few years, by using the concept of Brown measure for an element of a II_1 factor; this was introduced by L. Brown [16], and is a generalization (defined via subharmonic function theory) for the notion of spectral distribution for normal elements. The very recent groundbreaking work of Haagerup and Schultz [29] shows that an element of a II_1 factor has invariant subspaces whenever the support of its Brown measure is not reduced to a single point. This is the the culmination of intense work building in this direction, by Haagerup [28] (proving the same result under a restriction related to the Connes embedding problem mentioned in Section 2.5), and by several authors ([24], [25], [49]) who studied Brown measures and invariant subspaces in examples occurring naturally in free probability.

The recent progress in the invariant subspace problem was one of the highlights of the BIRS workshop, and was reflected in the combined talks given by Uffe Haagerup and Hanne Schultz.

2.4 Applications to C*-algebra theory

Free probability considerations can also be made in a C^*-algebra (rather than von Neumann algebra) framework. In the C^*-algebra universe, the counterpart of $L(G)$ is $C^*_r(G)$, the reduced C*-algebra of the group G.

In the recent paper [31], Haagerup and Thorbjornsen obtain a sweeping generalization to several matrices for a number of results concerning the largest eigenvalue of a Gaussian random matrix. This has strong consequences for the reduced C^*-algebras of free groups. In particular, they obtain that, for G a free group on 2 or more generators, the Ext semigroup of $C^*_r(G)$ is not a group (this has been one of the most popular open questions in C^*-algebra theory since the late 70's).

In the BIRS workshop, Steen Thorbjornsen presented another application of random matrices to reduced C^*-algebras of free groups – a new proof for the fact that these C^*-algebras have no non-trivial projections (cf. [30]).

2.5 Connes embedding problem

The Connes embedding problem [20] asks whether every type II_1 factor can be embedded into an ultraproduct of the hyperfinite II_1 factor. The problem has several equivalent reformulations, one of them being that every n-tuple of elements of a II_1 factor has matricial microstates.

The BIRS workshop had a talk on this topic, given by Florin Radulescu. He presented his work in [43] showing that the Connes embedding problem is equivalent to a statement on matrix trace inequalities which is, in a certain sense, an analytic version of Hilbert's 17th problem (formulated in a von Neumann algebra framework).

2.6 q-deformations of the free group factors

A basic way of producing examples of free families of random variables goes by using creation and annihilation operators on the full Fock space. The most important such example is the "free semicircular system" of Voiculescu, which is the analogue in free probability for a family of independent Gaussian random variables. The von Neumann algebra generated by a semicircular system with n elements is isomorphic to $L(F_n)$, where F_n is the free group on n generators.

In [14], Bozejko and Speicher put into evidence a deformation, depending on a parameter $q \in (-1, 1)$, for the creation and annihilation operators on the full Fock space. The corresponding q-deformation of a free semicircular system is called "q-Gaussian family". In the case when $q = 0$ one has the full Fock space and the semicircular families, while in the limit $q \to 1$ one gets back to the classical objects, independent families of Gaussians. The von Neumann algebra generated by a q-Gaussian family was given quite a bit of attention in the last few years, but remains fairly mysterious. In fact even the basic question of whether this algebra has trivial center was only partly solved until the recent work of Ricard [45]. In his talk at the workshop, Eric Ricard presented this work, showing the factoriality of the von Neumann algebra generated by a q-Gaussian family with n elements, for all $n \geq 2$ and all $q \in (-1, 1)$.

3 Relations to probability theory

An important insight brought by free probability is that the concept of freeness for a family of non-commuting random variables in a von Neumann algebra should be treated as an analogue of the notion of independence from classical probability. Acting on this line, one is prompted to start a programme of developing free counterparts for fundamental theorems of classical probability.

It is remarkable how far this programme can go. There exist now quite a few deep theorems about the internal structure of free probability, inspired (though certainly not following from!) developments in classical probability.

For example, there exists a notion of free convolution of distributions, and a well-developed analytic machinery (replacing the classical machinery of the Fourier transform) for dealing effectively with this new type of convolution. We have characterizations for freely infinitely divisible distributions, also for freely stable distributions and their domains of attraction [8]. On the other hand there exists a well-developed theory of stochastic integration and of stochastic analysis for the free Brownian motion [12]. Quite a few talks at the BIRS workshop addressed free analogues of the classical situation.

3.1 Analytic properties of free convolution

Free convolution is a binary operation on probability measures on the real line which corresponds to the sum of two free random variables (in the same way as the usual convolution corresponds to the sum of two independent random variables). There exists also a multiplicative version of the free convolution, which goes with the product of free random variables. Many theorems about classical convolution have free counterparts, see e.g. [57], [59]. The analytic treatment of such questions relies on a good understanding of the Cauchy transforms of the convolved probability measures, and of a couple of other transforms specifically used by free probability, the R-transform and the S-transform. Thus on one hand many questions about free convolution result in interesting new statements about analytic functions, and on the other hand complex analysis is an important tool for investigating properties of the free convolution. It should be pointed out that although free convolution and classical convolution have many properties in common, there are also differences –

in particular, free convolution has better regularity properties than classical convolution. Whereas the basic analytic theory of free convolution is by now a well-established theory, there are many questions in this context which are still open, giving rise to interesting investigations.

The BIRS workshop had two talks about recent developments in this direction: by Hari Bercovici, on Hincin's theorem for multiplicative free convolution and by Serban Belinschi, on the regularity properties of the free convolution. It is a remarkable fact (first noticed in [40]) that any probability measure μ on the real line belongs to a partial semigroup μ_t $(t \geq 1)$ relative to the free additive convolution. In their work, Belinschi and Bercovici proved similar results for free multiplicative convolution of measures supported on the positive half-line and (in a slightly less general context) on the unit circle. They also investigated regularity properties of measures in these semigroups, and connections with additive and multiplicative Boolean convolutions of probability measures. See [5]–[7].

3.2 Free extreme values, free transport inequalities, and free de Finetti's theorem

As mentioned above there are quite a few parts of classical probability theory which have a free counterpart – well established are by now, e.g., the theory of free convolution (as addressed in 3.1) or the basic theory of free stochastic processes and free stochastic integration. The BIRS workshop had several talks about new analogues of classical theories.

• Gerard Ben-Arous talked about free extreme values; in recent joint work [4] with Dan Voiculescu, they obtained free probability analogues of the basics of extreme value theory, based on Ando's spectral order. This includes classification of freely max-stable laws and their domains of attraction, using free extremal convolutions on the distributions.

• A recent development in free probability theory is to look for analogues of cost of transportation inequalities in the free setting. A free Wasserstein distance was introduced and investigated by Biane and Voiculescu [13]. In particular, they proved a free version of Talagrand inequality. In her talk titled "About optimal transport for non-commutative variables", Alice Guionnet reported on her ongoing joint work with Cedric Villani on how to develop this program even further.

• Franz Lehner reported on his free version of de Finetti's theorem, characterising amalgamated free products as noncrossing exchangeability systems which satisfy a so-called weak singleton condition, see [36].

3.3 Free probabilistic aspects of random matrices

Whereas the above developments show that free probability really deals with a beautiful and rich structure, it happens at the same time that the framework created by free probability may indicate new conceptual approaches to problems from other fields. This was in fact the main theme in the section 2 above, where the other field was the one of operator algebras. Another outstanding illustration of how this can happen is provided by work of Ben-Arous, Guionnet, and Cabanal-Duvillard (see e.g. [3], [17], [57]), who showed that free entropy is useful in the study of the rate functions of large deviation principles.

In the BIRS workshop, James Mingo talked about fluctuations of random matrices and second order freeness. He reported his joint work with Speicher [37] and with Sniady and Speicher [38], where they extend the relation between random matrices and free probability theory from the level of expectations to the level of fluctuations by introducing the new concept of second order freeness.

Another connection between special random matrix ensembles and concepts from free probability was made in the talk of Benoit Collins. He considered the product of two independently randomly rotated projectors, the square of whose radial part turns out to be distributed as a Jacobi ensemble, and studied its global and local properties in the large dimension scaling relevant to free probability theory, see [19].

3.4 Other versions of non-commutative probability theory

Free probability theory is a prominent example of a non-commutative probability theory, where one tries to extend notions and ideas from classical probability theory to a non-commutative setting (by replacing the commutative algebra of random variables by a non-commutative algebra). Free probability theory is that part of non-commutative probability theory where the notion of freeness (as a replacement for independence) plays a crucial role. However, there are also some other possibilities for non-commutative versions of independence; in particular, we have the theories of conditional freeness (which is a generalization of freeness), of boolean independence, and of monotonic independence.

At the BIRS workshop, Marek Bozejko talked about a relation between conditional freeness and some classes of free Levy processes. In joint work with W. Bryc, they have shown a relation with free pairs of random variables which have linear regressions and quadratic conditional variances when conditioned with respect to their sum, see [15].

There exist also more general axiomatic notions for non-commutative independence. The investigation of processes with independent increments in such a general frame was presented in the talk of Claus Koestler. In joint work with J. Hellmich and B. Kümmerer [32], they introduced a non-commutative extension of Tsirelson-Vershik noises, which they call continuous Bernoulli shifts, and they established a bijective correspondence between additive and unital shift cocycles in this setting.

4 Combinatorial aspects of free probability

4.1 Cumulants and R-transforms

Even when restricted to a purely algebraic framework, free random variables have an interesting combinatorics, which stems from their non-commuting character. The combinatorics of free probability turns out to be governed by the theory of Moebius inversion in lattices (as developed by Rota and collaborators), applied to lattices of non-crossing partitions. The specific way how the lattices of non-crossing partitions show up in this framework is via the non-crossing cumulant functionals associated to a non-commutative probability space – see e.g. the survey paper [51].

The combinatorial study of freeness had an important impact in the development of the R-transform, which is the free probabilistic counterpart of the Fourier transform from classical probability. The R-transform for one bounded selfadjoint random variable has both an analytic and a combinatorial incarnation, and each of these two incarnations generated its own direction of research. A success of the combinatorial approach to the R-transform was that it could be extended (by using non-crossing partitions) to the case when joint R-transforms for several non-commuting random variables are considered. Moreover, the pursuit of the multivariable R-transform lead to a whole collection of combinatorial tools, which can often streamline and make transparent complicated algebraic computations involving freeness. A nice illustration of the power of the combinatorial method is provided for instance by the solution to the problem of computing the distribution of a free commutator, see [41].

The basic theory of non-crossing cumulants has inspired a number of variations. For instance: by starting from the fact that the lattices of non-crossing partitions have analogues of type B (cf [44]), one can work on developping an analogue of type B for the theory of non-crossing cumulants – see [11]. Another very interesting variation was presented at the workshop by Mireille Capitaine; in a very recent joint work with Muriel Casalis [18], they have introduced a concept of "cumulants for $N \times N$ random matrices", where letting $N \to \infty$ gives back the non-crossing cumulants used in free probability.

Last but not least let us mention that a substantial part of the theory of the non-crossing cumulants and R-transforms can be extended to an "operator-valued" version, where we talk about free independence with amalgamation over a subalgebra B (rather than over $\mathbb{C}$). For a survey of non-crossing cumulants in this more general framework, see the memoir [52]. At the BIRS workshop, the operator-valued R-transform was the main tool used by Ken Dykema in his talk on nearest neighbor random walks on amalgamated free product groups (cf [23]).

4.2 Asymptotic representation theory for symmetric groups

The combinatorial machinery of free probability has found powerful applications in the asymptotic representation theory of symmetric groups. The study of limit shapes of Young diagrmas goes back to the 70's, and was well-developped in a sequence of papers by S. Kerov in the 90's. The connection with free probability came with the paper [9], where P. Biane showed how the non-crossing cumulants of the transition measure of a limit diagram can be used to study asymptotics for characters and for natural operations with representations. Research on this topic was presented at the BIRS workshop in a survey talk given by Aikihito Hora, and in a talk by Piotr Sniady, presenting his work in [48].

4.3 Other combinatorial connections

There were several other talks at the BIRS workshop which touched on combinatorial aspects pertinent to developments in free probability.

• Michael Anshelevich presented his work [1] where he puts the basis for a theory of orthogonal polynomials in several non-commuting indeterminates, and where some non-trivial examples are obtained by looking at free analogues for the classical Meixner systems.

• A new possible direction of development was addressed in the talk by Ed Effros, on the relations between non-crossing cumulants and combinatorial Hopf algebra theory.

• The talk by Ian Goulden discussed the well-known formula of Harer and Zagier (which is directly related to the moments of a GUE random matrix), and presented a direct combinatorial way of deriving this formula, via tree enumeration (cf. [27]).

• The talk by Michael Neagu discussed another occurrence of free independence in connection to the symmetric groups – random permutation matrices are asymptotically free from GUE or Wishart matrices (cf. [39]).

5 Reaching out to new directions

The organizers of the BIRS workshop found it important to invite also some participants who do not work in free probability, but are interested in it and whose areas of interest indicate possible new connection points and new directions of development for free probability. The talks of those participants did not report on genuine free probability results, but gave some kind of introduction to questions and results in their respective areas.

• Robert Bauer gave a survey talk on conformal invariance and stochastic Loewner equations. In [2], he showed, using concepts of noncommutative probability, that Loewner's evolution equation can be viewed as providing a map from paths of measures to paths of probability measures, whose fixed point is the convolution semigroup of the semicircle law.

• Michael Pimsner gave a talk about graded groups in KK-theory, with emphasis on an application to computing the K-theory groups for a crossed-product by an action of a symmetric group.

• Alexander Soshnikov talked about Poisson statistics for the largest eigenvalues of Wigner and Wishart random matrices with heavy tails. He considered large Wigner random matrices in the case when the marginal distributions of matrix entries have heavy tails and proved that the largest eigenvalues of such matrices have Poisson statistics, see [50].

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