Orthogonal Polynomials; Interdisciplinary Aspects

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1 Introduction

Since 1984, the community of researchers in orthogonal polynomials and special functions have held regular international symposia in Europe. Indeed, these meetings, their venues, and the year of the symposium:

1. First International Symposium on Orthogonal Polynomials, Special Functions and Applications, Bar-Le-Duc, France; 1984.

2. Second International Symposium on Orthogonal Polynomials, Special Functions and Applications, Segovia, Spain; 1986.

3. Third International Symposium on Orthogonal Polynomials, Special Functions and Applications, Evian, France; 1992.

4. Fourth International Symposium on Orthogonal Polynomials, Special Functions and Applications, Delft, The Netherlands, 1994.

5. Fifth International Symposium on Orthogonal Polynomials, Special Functions and Applications, Patras, Greece, 1999.

6. Sixth International Symposium on Orthogonal Polynomials, Special Functions and Applications, Rome, Italy, 2001.

7. Seventh International Symposium on Orthogonal Polynomials, Special Functions and Applications, Copenhagen, Denmark, 2003.

The Eighth International Symposium on Orthogonal Polynomials, Special Functions and Applications is currently in the planning stage and will be held in Munich, Germany in July, 2005. In addition, there have been several large Spanish national conferences that have attracted a large international audience, including Laredo (1987, 1992), Gijon (1989), Vigo (1990), Granada (1991), and Sevilla (1997).

In North America, there has been one large NATO-sponsored meeting on orthogonal polynomials (Columbus, Ohio) and several smaller special sessions of national and regional AMS meetings dedicated to orthogonal polynomials and related topics. The organizers of the BIRS Workshop on

Orthogonal Polynomials; Interdisciplinary Aspects felt it was necessary to organize a mid-size conference in this area with a North American venue to further invigorate North American interest as well as to bring the core of researchers in integrable systems closer together to orthogonal polynomials and special functions.

2 Areas of Participation

The following areas of mathematics were represented at the BIRS Workshop on Orthogonal Polynomials; Interdisciplinary Aspects together with the names of the people who gave seminars in these areas; there were 26 talks (each 50 minutes duration) given during the five days of this workshop by participants from 17 different countries.

1. Lie algebras and Darboux transform theory (Luc Vinet, Francisco Marcellan)
2. Integrable Systems (John Harnad, Hans Lundmark, Mark Adler, Marco Bertola)
3. Moment theory (Christian Berg, Andreas Ruffing)
4. Classical theory of OPS's and Special Functions (Richard Askey, Christian Berg, Lance Littlejohn, Mourad Ismail)
5. Asymptotic theory of orthogonal polynomials (Alexei Borodin, Ken McLaughlin)
6. Spectral theory of Differential Operators (Lance Littlejohn)
7. The Bochner-Krall Classification Problem (Dong Won Lee)
8. Riemann-Hilbert Problems (Percy Deift, Arno Kuijlaars, Marteen Vanlessen, Ken McLaughlin, Jeff Geronimo, Peter Miller)
9. Random Matrices (Percy Deift, Alexei Borodin, Arno Kuijlaars, Marco Bertola)
10. Multiple Orthogonality (Walter van Assche, Jorge Arvesu)
11. q-orthogonal polynomials (Andreas Ruffing, Sergei Suslov, Jorge Arvesu, Natig Atakishiyev)
12. Matrix orthogonal polynomials (Alberto Grunbaum)
13. Multivariate Discrete Orthogonal Polynomials (Yuan Xu)
14. Orthogonal Polynomials and Special Functions in Mathematical Physics (Mourad Ismail, Anatol Odzijewicz, Mark Adler)

In addition to the participants listed above, Professor Madan Mehta (Service de Physique Theorique) presented some interesting open problems in the area and there were three graduate students in attendance at the workshop: Keivan Mohajer (University of Saskatchewan), Davut Tuncer (Utah State University), and Tomohiro Takata (Kyoto University).

3 Orthogonal Polynomials: General Overview

The last thirty years has seen a remarkable rebirth of research in the theory of special functions and orthogonal polynomials. Essentially invigorated by Richard Askey's series of NSF lectures [1] at Virginia Tech in 1974, research in various areas of orthogonal polynomials and their applications has kept a steady pace ever since. As a result, key contributions in these subjects during this time period have come from researchers in more than thirty countries.

In the classical theory of orthogonal polynomials in the real variable x, a sequence $\{p_n\}_{n=0}^{\infty}$ of polynomials, where the degree of each p_n is exactly n, is said to be an orthogonal polynomial sequence with respect to the Borel measure μ (possibly signed) if

$$\int_{-\infty}^{\infty} p_n p_m d\mu = k_n \delta_{n,m} \quad (k_n \neq 0; n, m \in \mathbb{N}_0).$$

If the measure μ is positive, this is the positive-definite case, then each $k_n > 0$; it is particularly this case that several classical textbooks have been written, most notably by Szegö [20] and Chihara [6]. Standard topics in these books include a detailed and fundamental general theory of orthogonal polynomials with several detailed examples ranging from discrete measures (e.g. Charlier, Krawtchuck), continuous measures (e.g. the classical orthogonal polynomials of Jacobi, Laguerre, Hermite), and signed measures (Bessel polynomials). Furthermore, topics in these books include an indepth discussion of the three-term recurrence relation for orthogonal polynomials, connections to the theory of continued fractions and the theory of moments, asymptotic properties of the classical orthogonal polynomials and their zeros, inequalities associated with the classical orthogonal polynomials, various characterizations of the classical orthogonal polynomials, approximation and expansion properties, and various classical applications of orthogonal polynomials to numerical analysis and mechanical quadrature methods as well as to the theory of electostatics. Moreover, complex orthogonal polynomials are discussed in [19].

The past thirty years may, arguably, be called the golden years of orthogonal polynomials. Indeed, the topics listed above have been greatly enriched and generalized and, furthermore, the subject has witnessed the openings of several new avenues of research. Several key results and open problems have fallen during this time period. Indeed, among many results, we mention that a key result leading to the solution of the Bieberbach conjecture involved a new inequality involving certain Jacobi polynomials. Major developments in asymptotic theory have been made during this period, including the solution of the Freud conjecture. The long standing Bessel moment problem was also elegantly solved during this period. The electrostatic interpretation of the roots of a large class of orthogonal polynomials has also been achieved in recent years.

This time period also saw the important development of the theory of q-orthogonal polynomials and basis hypergeometric series leading to the development of the Askey tableau in orthogonal polynomials and a new meaning for 'classical' orthogonal polynomials. These q-orthogonal polynomials and new special functions have proved useful in both theory and applications to mathematical physics and statistical mechanics.

Interpreting certain problems in terms of Riemann-Hilbert problems has led to some impressive new results, as well as some powerful new tools, in the asymptotic theory of orthogonal polynomials; further remarks on this and connections to the theory of random matrices are described below in the next section.

Sobolev orthogonality, that is orthogonality with respect to bilinear forms of the type

$$(p, q)_M := \sum_{k=0}^{M} \int_{\mathbb{R}} p^{(k)} \overline{q}^{(k)} d\mu_k,$$

made significant strides during this period as has the theory of matrix orthogonal polynomials. Key advances have also been made in the past fifteen years to the characterization of orthogonal polynomial sequences to differential equations, in particular to the so-called $BKS(N, M)$ classification problems, and to the spectral theory of the associated operators. A sequence of polynomials $\{p_n\}$ belongs to the $BKS(N, M)$ class if $\{p_n\}$ is orthogonal with respect to a bilinear form of the type $(\cdot, \cdot)_M$ above and each p_n is an eigenfunction of a fixed differential equation of order N. Left-definite spectral analysis, whose roots can be traced to fundamental work of Hermann Weyl, has proved promising in establishing new orthogonality results for polynomials in the $BKS(N, M)$ sets as well as helped to obtain new characterization results for this class of polynomials. Development of the theory of Darboux transforms has also shed considerable light on the $BKS(N, M)$ classification problem and given us a powerful tool for constructing new orthogonal polynomials in these classes.

During the past few years, significant advances in the theory of multiple orthogonal polynomials have been made with some very interesting, and important applications including a new proof of the irrationality of $\zeta(3)$; there is much hope that further classical results and new applications to number theory will be found using the tools that have recently been developed in this area. The theory of multivariable orthogonal polynomials or orthogonal polynomials in several variables, with respect to discrete and continuous measures, is a subject of much recent interest and significant developments.

The last few years has seen a new group of researchers, namely people in integrable systems, enter the orthogonal polynomial scene. As further described below, several integrable systems can be solved using orthogonal polynomials and moment theory. One of the central reasons for organizing this BIRS meeting was to bring this group together with a core of researchers in orthogonal polynomials for an exchange of ideas and for further collaboration. To this end, this meeting was hugely successful!

4 Riemann Hilbert Problems and Orthogonal Polynomials

The introduction of random matrices to theoretical physics dates back to Wigner in the 1950's, and was motivated by the attempt to explain resonances in the scattering of slow neutrons off heavy nuclei. Physical observations made it clear that a statistical theory was needed to explain the events. A particularly salient feature of the data was the "repulsion of energies" – on average, the resonance energies stayed "far" apart. In earlier work with von Neumann in 1927, Wigner had shown that the degeneracy of the eigenvalues of a Hermitian matrix was a "rare" event (of co-dimension 2), and this led Wigner to suggest the eigenvalues of matrices distributed according to some probabilistic law as an appropriate statistical model for the resonance energies. It was believed that the phenomenon of resonance scattering was universal, subject only to some general symmetry restrictions, and so the statistics should be largely independent of the details of the theory. Early work in the theoretical physics community on the theory of random matrices was due to Wigner, Dyson, as well as Mehta and Gaudin.

The theory of random matrices is concerned with the distribution of the eigenvalues of ensembles of matrices distributed according to some probability measure. The Unitary Ensembles (UE's) consist of $N \times N$ Hermitian matrices with the measure

$$\frac{1}{Z_N} e^{-N \operatorname{Tr} V(M)} dM, \tag{1}$$

where M is an $N \times N$ Hermitian matrix; dM denotes the Lebesgue product measure on the (algebraically independent) entries of M, and V is a real-valued function growing sufficiently rapidly at infinity. The term "Unitary Ensembles" refers to the fact that the measure in (1) is invariant under unitary conjugation: $M \to UMU^*$. In the particular case $V(x) = x^2$ the ensemble is called the Gaussian Unitary Ensemble (GUE). The function Z_N is the partition function for the ensemble:

$$Z_N = \int e^{-N \operatorname{Tr} V(M)} dM.$$

The principal goal of random matrix theory is to calculate the basic statistical quantities for the eigenvalues of matrices distributed according to a given probability distribution and to evaluate these quantities in the limit as $N \to \infty$. A statement of the universality of the theory for large N is as follows: Note first that for any $a < b$,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Exp} \sharp \ \{ \text{ eigenvalues in } (a,b)\}$$

exists and equals $\int_a^b \rho(x)dx$ for some $\rho(x) \geq 0$, the so-called density of states. We consider a point $x = x_0$ of positive density, $\rho(x_0) > 0$, and then rescale the matrices so that the expected number of eigenvalues per unit length at x_0 is equal to 1. Then for $\theta > 0$ and for "reasonable" V's (polynomial V's are certainly included!),

$$\lim_{N\to\infty} \Pr\left\{\text{no eigenvalues in } \left(x_0 - \frac{\theta}{N\rho(x_0)}, x_0 + \frac{\theta}{N\rho(x_0)}\right)\right\} = \det(I - S_\theta), \tag{2}$$

where S_θ is the Fredholm integral operator on $L^2(-\theta, \theta)$ with kernel

$$S_\theta(x, y) = \frac{\sin \pi(x - y)}{\pi(x - y)},$$

and $\det(I - S_\theta)$ denotes the Fredholm determinant. In other words, under appropriate scaling, the probability that there are no eigenvalues in an appropriately scaled interval is universal, and in particular is independent of the choice of V.

Orthogonal polynomials play a significant role in the theory of Unitary Ensembles. By classical theory, there is a sequence of polynomials $p_{j,N} = \gamma_{jN} x^j + \dots$, $\gamma_{jN} > 0$, $j = 0, 1, 2, \dots$, which are orthonormal with respect to the measure $e^{-NV(x)}dx$, that is,

$$\int_{-\infty}^{\infty} p_{j,N} p_{k,N} e^{-NV(x)} dx = \delta_{jk}, \quad j, k \geq 0,$$

and it turns out that in the basic contributions [17], [16], Gaudin and Mehta computed the probability distribution of the eigenvalues of matrices in UE's directly in terms of these orthogonal polynomials. Set

$$\phi_{j,N}(x) = e^{-\frac{N}{2}V(x)} p_{j,N}$$

and define

$$K_N(x, y) = \sum_{j=0}^{N} \phi_{j,N}(x)\phi_{j,N}(y).$$

By the Christoffel-Darboux identity,

$$K_N(x, y) = e^{-\frac{N}{2}(V(x)+V(y))} \frac{\gamma_{N-1,N}}{\gamma_{N,N}} \frac{p_{N,N}(x)p_{N-1,N}(y) - p_{N,N}(y)p_{N-1,N}(x)}{x - y}, \tag{3}$$

where $\gamma_{j,N}$ is the leading coefficient of $p_{j,N}$ as above. The basic statistical quantities for the eigenvalues in the Unitary Ensemble can be computed in terms of $K_N(x, y)$. For example, the m-point correlation function $R_{m,N}$ is given by

$$R_{m,N}(x_1, \dots x_m) = \det ||K_N(x_j, x_k)||_{1 \leq j,k \leq m}; \tag{4}$$

and the gap probability for any $a < b$ by,

$$\Pr\{\text{no eigenvalues in } (a, b)\} = \sum_{m=0}^{N} \frac{(-1)^m}{m!} \int_a^b \cdots \int_a^b R_{m,N}(x_1, \dots, x_m) dx_1 \cdots dx_m. \tag{5}$$

Also, for the density of states we have

$$\rho(x) = \lim_{N\to\infty} \frac{1}{N} K_N(x, x).$$

Universality for the (scaled) kernel K_N takes the form (for $\rho(x_0) > 0$),

$$\lim_{N\to\infty} \frac{1}{N\rho(x_0)} K_N\left(x_0 + \frac{u}{N\rho(x_0)}, x_0 + \frac{v}{N\rho(x_0)}\right) = \frac{\sin \pi(u - v)}{\pi(u - v)},$$

which in turn leads to the V-independent limit (2) above, via (5).

¿From (3), (4), and (5) it is clear that the computation of the limiting eigenvalue statistics as $N \to \infty$ requires precise knowledge of the asymptotic behavior of the associated orthogonal polynomials $p_{j,N}$ as $N \to \infty$. Universality for random matrix theory is thus a direct consequence of the asymptotic properties of the polynomials.

The discovery by Fokas, Its, and Kitaev [14] that the orthogonal polynomials can be obtained as the solution of a Riemann-Hilbert problem, was a significant step toward this goal. The authors in [9], [12] have shown the steepest descent method for Riemann-Hilbert problems to be a powerful tool in obtaining the asymptotic behavior of orthogonal polynomials, and hence in proving universality (see also [5]). The steepest descent method for Riemann Hilbert problems was introduced by Deift and Zhou in 1993 [11], and further developed together with Venakides [13] to include fully nonlinear oscillations. Inverse scattering problems in the theory of completely integrable systems are often formulated as Riemann-Hilbert problems, and the asymptotic behavior of solutions to integrable equations has been obtained by the application of the steepest descent method to the associated Riemann-Hilbert problems. (cf. the bibliography in [10].)

The situation for unitary ensembles is now in a fairly satisfactory state, but the proof of universality remains open for many other types of ensembles.

5 Orthogonal Polynomials and Integrable Systems

A number of discrete integrable systems, such as the Toda flows, [18], [3], multipeakon flows [2], relativistic Toda lattice, [7], [8], can be solved in terms of orthogonal polynomials or orthogonal rational functions. See the bibliography in [4].

We illustrate this approach with a recent example from integrable systems. The Camassa-Holm equation,

$$u_t - \frac{1}{4}u_{xxt} + \frac{3}{2}(u^2)_x - \frac{1}{8}(u_x^2)_x - \frac{1}{4}(uu_{xx})_x = 0,$$

is formally an isospectral deformation of the differential operator $D^2 - zm - 1$, where $2m = (4 - D^2)u$. By a Liouville transformation the eigenvalue problem for this differential equation on the real line is carried into the acoustic spectral problem [2]

$$D^2\varphi = zg(y)\varphi(y,z), \quad -1 \le y \le 1, \quad \varphi(\pm 1, z) = 0. \tag{6}$$

The special case of multi-peakons (non-smooth solitons) is obtained when g is a sum of Dirac delta functions:

$$g(y) = \sum_{j=1}^{n} g_j \delta(y - y_j), \qquad -1 < y_1 < \cdots < y_n < 1.$$

The discrete spectral problem associated with (6) was studied by Krein [15]. The inverse problem recovers the masses and spacings between the mass points from the spectral data, and is realized succinctly as a direct application of Stieltjes' solution of the classical moment problem [2]. Stieltjes original solution applies to positive masses, but the solution extends immediately to both positive and negative masses.

The solution of the inverse problem is directly related to the construction of a system of orthogonal polynomials relative to a positive measure: Let $\varphi(y,z)$ be a solution of (6). The Weyl function is defined by

$$W(z) = \frac{\varphi_y(1,z)}{\varphi(1,z)}.$$

The Weyl function can be constructed knowing the spectral data, that is the eigenvalues and coupling coefficients of the left and right wave functions of (6). These give respectively the location of the poles and the associated residues of the Weyl function. The Weyl function can thus be represented as the Stieltjes transform

$$\frac{W(z)}{z} = \int_{-\infty}^{\infty} \frac{1}{z - \lambda} d\mu(\lambda),$$

where $d\mu$ is a positive discrete measure. This positive measure has a sequence of orthogonal polynomials

$$\int_{-\infty}^{\infty} P_j(\lambda,t)P_k(\lambda,t)e^{-2t/\lambda}d\mu(\lambda) = \delta_{jk}.$$

For fixed t the $P_j(\lambda,t)$ satisfy a second order recursion relation

$$\lambda P_j(\lambda,t) = b_j P_{j+1}(\lambda,t) + d_j P_j(\lambda,t) + b_{j-1}P_{j-1}(\lambda,t), \quad 1 \le j \le n-1,$$

where the coefficients b_j are given. The multipeakon solutions may be expressed in terms of the orthogonal polynomials $P_j(0,t)$ as follows:

$$g_j = -\frac{1}{b_{n-j}P_{n-j+1}(0)\,P_{n-j}(0)}$$

$$y_j = 1 + b_{n-j}\Big(P'_{n-j}(0)P_{n-j+1}(0) - P_{n-j}(0)P'_{n-j+1}(0)\Big),$$

where the primes denote differentiation with respect to λ. The impossibility of triple collisions follows directly from the classical Christoffel-Darboux formula in orthogonal polynomials [2].

A bijective map from a discrete string problem to Jacobi matrices [3] gives a unified picture of the Toda, Jacobi, and multipeakon flows, and leads to explicit solutions of the Jacobi flows in terms of orthogonal polynomials.

6 Exit comments of participants

M. Adler (Brandeis University)

The work I discussed involved understanding the Dyson process, Airy and Sine processes and deriving p.d.e.'e for correlation probabilities. Implicitly it involved deriving equations for the coupled 2-matrix modeling, which tied in with work of Harnad and Bertola, discussed at the conference. The Dyson process is an Ornstein -Uhlenbeck process on Hermitian matrices arranged so as to be stationary, by taking the equilibrium measure at infinity for initial data, and correlating probabilities at two distinct times, which precisely corresponds to computing a coupled two-matrix integral and then deriving pde's in the coupling constant and the endpoints of the 2 spectral intervals in question. Scaling according to the edge or the bulk in the classical Hermite case then leads to pde's for the Airy and Sine processes. Now the coupled two-matrix integral is the focal point for Harnad and Bertola's work, where now they integrate over spectral sets which may even be curves in the complex plane, which of course leaves the realm of Hermitiam matrices. They however are interested in isomonodromy properties of the two-matrix intregral, so it is entirely permissable to make such deformtions and study their properties. It turns out that all of the isomonodromic data can be computed via residue calculations made on various differential forms on an associated spectral curve, which is basically very good news, as that means all of the data can be effectively computed in any example.

C. Berg (Copenhagen) My talk on 'Orthogonal Polynomials associated to positive definite matrices' focused on several new characterizations of indeterminate moment proble ms in terms of Hankel matrices. This raised the question of how this was related to Riemann-Hilbert problems and brought me in contact with Ken McLaughlin. He had treated asymptotic questions for weigts of the form $w(x) = exp(-|x|^a)$, which for $0 < a < 1$ corresponds to indeterminate moment problems. He did not know about the problems of determining the order of the entire functions in the Nevanlinna matrices for these problems, and one can hope that this can be done by Riemann-Hilbert methods.

In the talk by Natig Atakishiev was presented some duality results for certain systems of q-orthogonal polynomials. To several people in the audience it was felt that one needed a precise notion of duality in this area, and after several hours of discussions and thoughts I think that he and I have reached such a notion and this could lead to a joint paper.

Jeff Geronimo, whom I did not meet since the Columbus meeting 1989, is now interested in orthogonal polynomials in two variables, a subject I had touched upon years back, so we gave fruitful exchange of information.

I had opportunity to discuss my ongoing project with Mourad Ismail on Kibble-Slepian kind of formulas - we came further ahead.

M. Ismail (University of Central Florida)

1. The presence of a mix of mathematicians from integrable systems and orthogonal polynomials was a great idea. I hope in future people like Charles Dunkl, Dennis Stanton, Erik Koolink, Tom Koornwinder, H. Rosengren, who do special functions and orthogonal, will be invited.

2. I did get mathematical ideas from the meeting which will be part of the paper I am writing on the tau function. It was nice to listen and talk to others.

3. The BIRS environment is ideal for these meetings and workshops and the limitation on the size makes it easy to interact with others.

4. I was impressed that two of the organizers did not give talks to provide space for others. This is a nice touch.

J. Harnad (Concordia University)

1) Summary of my presentation: Biorthogonal Polynomials and 2-Matrix Models

A survey was given of results relating the spectral theory of coupled pairs of random matrices and the theory of biorthogonal polynomials. This was largely based on joint work with Marco Bertola and Bertrand Eynard, and included the following topics: 1) In the case of measures involving exponentials of polynomial potentials, the associated Christroffel Darboux kernels, which are projectors onto spaces spanned by the first N biorthogonal polynomials, determine the spectral properties of coupled unitarily dialgonalizable random matrices, with spectra supported on curves in the complex plain. All correlation functions are given by d eterminantal formulae involving these kernels, and the gap probabilities are given by Fredholm determinants of the associated integral operator, supported on the complement of the region considered. 2) The associated systems of differential-deformation-difference equations (obtained by "folding" the first order differential deformation equations satisfied by the biorthogonal polynomial sequences through use of the recursion relations) satisfied on dual "windows" of consecutive biorthogonal polynomials of sizes equal to the degrees of the potentials are compatible as overdetermined systems. These therefore admit joint fundamental systems, integral representations of which were given, as well as their large x and y asymptotics. 3) By virtue of the compatibiility of these sequences of differential-deformation equations, the generalized monodromies of each of the dual sequences of differential equations with respect to the dual variables are invariant under deformations in the polynomial potentials determining the measures, and independent of the integer determining the positions of the "window" . 4) A theorem of "spectral duality" was given, stating that the spectral curves defined by the characteristic polynomials of the corresponding Lax matrices for the dual systems, as well as those satisfied by the sequences of Fourier Laplace transforms of the biorthogonal polynomials, were all identical. Due to lack of time, a new, simplified proof and generlization of a theorem of Soshnikov concerning Janossy densities and their relation to gap probabilities in multi-matrix models was not presented in the seminar, but was discussed in private sessions with other participants.

2) Comments on presentations by other participants.

The quality and level of interest of presentations by other participants was very high. The background and specialized interests of the various participants included several different orientations. There were those whose primary interest was in the classical theory of orthogonal polynomials, with measures that are both continuous and discrete, and their placement within the general framework of special functions - particularly, of hypergeometric and q-hypergeomtric type, as well the classical moment problem. There was also considerable interest in generalizations, such as biorthogonal systems or multiple orthogonal polynomials, as well as the applications of orthogonal polynomials, e.g.

to random matrices and other probabliistic and combinatorial systems, Thiere were those particalu- arly intersted in the study of large N asymptotics, either in random matrices, or in the representation theory of Lie groups, in relation to asymptotics of orthogonal polynomials. There were also many interested in applications to integrable systems, or the use of methods developed in that area, such as the matrix Riemann-Hilbert (inverse spectral) and dbar methods. The interactions between these various perspectives was particularly useful and stimulating. The talks having the greatest imme- diate interest in relation to my own work were those on multiple orthogonal polynomials and their applications (by Van Assche and Kuijlaars), those on large N asymptotics and universality (by Deift, McLaughlin and Vanlessen) and on the dbar steepest descent method for Riemann-Hilbert problems (Miller). Of course, the talk by Marco Bertola, which was based on joint work that we had done on the isomonodromic deformation problems arising in the study of semi-classical orthogonal poly- nomials, and the relation of the isomonodromic tau function, the spectral curve and the partition function for the corresponding matrix models was of immediate interest. Other talks of considerable interest, where I felt that I learned something new, included those on the relation between the ran- dom matrix recursion relations and various Markov processes (Askey and Grunbaum), the analysis of the indeterminancy problem in reconstruction of measures from moments (Berg), the problem of Darboux and Christoffel tranformations in terms of LU factorization of the recursion matrices, (Marcellan), the relation between matrix model processes and Brownian motion (Adler)and the large N asymptotics for representations of the unitary group (Borodin).

3) Interaction with other participants.

Perhaps the most important aspect of this meeting was the opportunity it provided for interacting with others in the area - some of whom I had known before, but would not have otherwise had an opportunity to interact with or discuss recent developments with at this time, and others, whom I had only known through their published works, or indirectly, e.g. from lectures given by others. The most fruitful interactions were with Arne Kuijlaars, whose work on applications of multiple orthogonal polynomials, both to random matrices with external coupling (jointly with Pavel Bleher) and to 2-matrix models and the Riemann-Hilbert problem for biorthogonal polynomials (joint work with Ken McLaughlin) relates closely to my own work. Our group (Bertola, Eynard, Harnad) had previously developed a version of the Riemann-Hilbart problem for biorthogonal polynomials based on the notion of duality described above, but recently Kuijlaars and McLaughlin found another one, based on the relation to multi-orthogonal polynomials. Through discussions initiated at this meeting, we seem to have clarified the relation between these two approaches. This will very likely lead to new work developing this link. There was also another possible joint project initiated, based on using the interpretation of KP tau function as a determinant over an infinite Grassmannian, to obtain new tau functions from the deformation theory of multiple orthohogonal polynomials. The ongoing open question of large N asymptotics for biorthogonal polynomials remains the most important objective, and this meeting gave a chance for the two groups most actively involved in resolving this problem to interact, and develop new ideas for its eventual solution. This included very helpful discussions with Ken McLaughlin, who is perhaps the most experienced in application of Riemann-Hilbert methods to the asymptotic analysis of orthogonal polynomials arising in matrix models, through his joint work with Deift, Krichebaur, Venakides and Zhou, one of the main pioneering works in this direction. There were also useful discussions with other workshop participants - in particular, with Alexei Borodin, who has also done some work on multi-level determinantal ensembles (of which coupled chains of random matrices are an example) and the computation of Janossy densities and gap distributions through the use of multi-orthogonal functions. Useful references and information on other developments relating more broadly to the theory of isomonodromic deformations, which we are also working on, followed from these discussions. There were also useful discussions with Mourad Ismail, on the relation of our work on recursion relations, isomonodromic deformations, and tau functions to the general framework of orthogonal polynomials, as well as many smaller discussions with other workshop participants, in which useful information was exchanged.

A. Kuijlaars (Leuven)

It was an excellent workshop and I thank the organizers for bringing together a wonderful group of people, coming from a variety of different areas. What brought us together was our interest in orthogonal polynomials and special functions, which in one way or another played a crucial role in our research.

The interaction between people with different backgrounds was therefore an important goal of the workshop and it was achieved wonderfully well.

Having a background in approximation theory, I had inspiring discussions with people like Mark Adler, Marco Bertola and John Harnad, who work in integrable systems. With Marco and John I talked at length about a problem from random matrix theory that we are all interested in, but that we approach from different backgrounds. The interaction with them was very fruitful.

There were excellent talks at the workshop. Personally I enjoyed the talks by Percy Deift and Alexei Borodin very much, since they showed how orthogonal polynomials can play a role in seemingly unrelated areas. There were many other great talks as well.

It was a good idea of the organizers to invite a number of graduate students and recent Ph.D.'s as well. I am sure the workshop has been a great experience for them and will stimulate them in their future research into orthogonal polynomials and their applications.

M. Vanlessen, (Leuven)

At the workshop I presented a talk about universality questions for eigenvalue correlations at the origin of the spectrum, which was joint work with Arno Kuijlaars. In this talk I focussed on unitary ensembles with a singularity at the origin of the spectrum. This singularity appears through a factor $|detM|^{2\alpha}$ in the ensemble. The aim of the talk was to show how the Riemann-Hilbert approach could be used to get a universality class (in terms of Bessel functions) at the origin of the spectrum for this ensemble.

The Riemann-Hilbert approach was first applied by Deift, Kriecherbauer, McLaughlin, Venakides and Zhou. Therefore, for my point of interest, it was useful that Percy Deift and Ken McLaughlin were present at the workshop. Especially the talk of Percy was very interesting since he talked about universality questions for orthogonal and symplectic ensembles, which was one of my proposed projects for my postdoc application.

The presence of Alexei Borodin was also useful because of his knowledge about discrete orthogonal polynomial ensembles. This is because Ken McLaughlin and I are currently very interested in the Gamma kernel (of Alexei) and we had some useful conversations with Alexei.

K. McLaughlin, University of North Carolina

A number of presentations pertained to the asymptotic analysis of Riemann-Hilbert problems, and applications to a variety of different areas of mathematics.

The asymptotic analysis of orthogonal polynomials is directly connected to the statistical behavior of eigenvalues of random Hermitian matrices. In order to describe the limiting statistical behavior of the eigenvalues when the size of the matrices grow to infinity, it is required to understand the so-called "semi-classical" asymptotic properties of orthogonal polynomials. In the late 90s, the problem of determining such global asymptotic behavior of orthogonal polynomials was re-cast in terms of Riemann-Hilbert problems [12], [9], and several of the presentations concerned new results for the asymptotic analysis of Riemann-Hilbert problems. These results, in turn, yield new results for random matrices.

Here are three examples: (1) Maarten Vanlessen described a new universality class for the local statistical behavior of eigenvalues of random Hermitian matrices (work with A. Kuijlaars). (2) Peter Miller described an extension of the above results to the asymptotic analysis of Riemann-Hilbert ($\bar{\partial}$) problems, which establish asymptotic properties of orthogonal polynomials under much weaker assumptions on the orthogonality weights (work with K. McLaughlin). (3) Arno Kuijlaars and Walter Van Assche gave presentations describing the asymptotic analysis of some 3×3 Riemann-Hilbert problems, multiple orthogonal polynomials, and applications to random matrices with external sources.

In other directions, new universality results were stated (for the first time) concerning the limiting statistical behavior of random symmetric matrices (rather than Hermitian) by P. Deift (with D. Gioev). Ken McLaughlin presented joint work with V. Pierce and N. Ercolani in which continuum limits of the Toda lattice are used to obtain explicit formulae for coefficients in a recently established expansion for the partition function of random matrices. Alexei Borodin spoke about the connection between discrete orthogonal polynomials and asymptotic representation theory.

What links this subset of presentations is the interplay between the asymptotic behavior of orthogonal polynomials, and applications. Here are a few examples of the cross-fertilization that occurred during this meeting:

(1) Researchers investigating the asymptotic behavior of Riemann-Hilbert problems associated to "multiple orthogonal polynomials" got together with researchers investigating biorthogonal polynomial asymptotics, and discovered new connections between seemingly disparate Riemann-Hilbert problems appearing in their respective work.

(2) A new Gamma kernel, identified in asymptotic representation theory, is being investigated from the point of view of universal behavior of discrete orthogonal polynomial ensembles, using recent results from Riemann-Hilbert analysis and asymptotic representation theory.

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