# Interactions between model theory and geometry

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Model theory is in a period of rich activity. Advances in pure model theory are finding immediate applications, in particular to the model theory of fields, while the applications are themselves motivating the abstract developments. Applied model theory is using ideas and methods from other parts of mathematics, ranging from homology theory to complex analytic geometry. These two strands of research were exhibited at the BIRS workshop. The workshop was used as an opportunity to exhibit and elucidate two large pieces of technical work which have been in the process of development for several years. The first of these, "Homology in o-minimal theories" is a prime example of the way in which a mathematical tool can be developed to apply in a wider model-theoretic context. The second tutorial, "Imaginaries in valued fields" illustrates the way in which an applied context (in this case, an algebraically closed valued field) has motivated a theoretical development (the notion of stable domination). The three afternoon sessions presented recent research developments in three different active areas of research in applied model theory.

## 1 A. Berarducci: Tutorial on o-minimality

The principal aim of the tutorial was to describe the "transfer approach" in the study of definable groups and definable manifolds in an o-minimal structure, where by "transfer approach" I essentially mean a reduction to the classical (i.e. locally compact) case. The 5 talks of the tutorial focused in particular on work in collaboration with M. Otero in this direction, as well as on related results and conjectures by other authors, which I describe in the sequel.

The definable sets in any o-minimal structure $M$ admit a cell decomposition analogous to the cylindric cell decomposition for semialgebraic subsets of $\mathbb{R}^n$ with the difference that cells are bounded by graphs of definable continuous functions rather than semialgebraic functions. The analogy with semialgebraic geometry may however be misleading, since in general an o-minimal structures may not admit an easy description of its definable continuous functions (hence of its definable sets). The problem is the lack of quantifier elimination theorems or model completeness theorems.

We are interested in definable groups in an o-minimal structure $M$, namely groups whose underlying set is definable and with a definable group operation. (We will assume in the sequel that $M$ expands a field, although for many results this assumption is not necessary.) Such groups have been studied by many authors and the results so far obtained suggest a close analogy between definable groups and classical Lie groups.

In particular it follows from the results of A. Pillay [26] that if the underlying set of the o-minimal structure is the real line, then every definable group is indeed a Lie group, while Y. Peterzil, A. Pillay and S. Starchenko [24] obtained matrix representation theorems for definable groups which confirm the analogy with Lie groups. Despite these results, many fundamental questions on definable groups remain open. The difficulty is that many tools available in the study of classical Lie groups are not available in the o-minimal context, mainly due to the fact that definable groups and definable manifolds may fail to be locally compact (if the structure $M$ is not locally compact).

This motivates the search for "transfer theorems" which allow, for various problems, a reduction to the case when the underlying set of $M$ is the real line.

A considerable amount of work in this direction was motivated by the following problem. It is known that every compact Lie group has torsion elements (namely elements of finite order) and Peterzil and Steinhorn [25] asked whether the same holds for every "definably compact" group (such groups need not be compact in the classical sense). It was known (Strzebonski [31]) that if a prime $p$ divides the "o-minimal Euler characteristic" $E(G)$ of a definable group $G$, then $G$ has an element of order $p$. (The o-minimal Euler characteristic of a set is defined as the number of its even dimensional cells minus the number of its odd dimensional cells, with respect to any given cell decomposition.) This and other results by the same author establish a beautiful analogy between finite groups and definable groups, with the o-minimal Euler characteristic playing the role of the cardinality. It follows in particular from the above mentioned result that if $E(G) = 0$ then $G$ has elements of any prime order. On the other hand the analogy with Lie groups suggests the conjecture (now verified) that if $G$ is definably compact then $E(G) = 0$.

After the development of o-minimal homology, initiated in the Ph.D. thesis of A. Woerheide [32], a new notion of Euler characteristic became available in the o-minimal category, defined as the alternating sum of the ranks of the o-minimal homology groups. The two notions of Euler characteristic are not equal in general, but they do coincide for closed and bounded sets (any definably compact group admits an embedding as a closed and bounded subset of $M^n$ for some $n$). This opened the way for the homology approach in the study of definable groups.

In a paper with M. Otero [1] we obtain some "transfer results" establishing natural isomorphisms which connect the o-minimal versions of the homology groups and the fundamental group with the corresponding classical notions, and we apply the result to show that every definable manifold of dimension not equal to 4 corresponds in a natural way to a classical topological manifold. (We could not eliminate the dimension assumption, but it turned out to be harmless in the intended applications to groups.) Using these results we then obtain [2]) a proof that for every definable compact group $G$ we have $E(G) = 0$ (hence $G$ has torsion). A different solution to the same problem had been obtained by M. Edmundo (our work is independent). Other results by Edmundo allow to count the number of torsion elements of a given order and give information on the o-minimal co-homology rings.

Recently Anand Pillay ([27] stated an insightful conjecture which, if solved positively, would greatly clarify the relationship between Lie groups and definable groups. The conjecture states that one can associate in a canonical way to every definable group $G$ a classical Lie group $G/G^{00}$. Here $G^{00}$ is the smallest "type definable subgroup of bounded index" (whose existence is part of the conjecture) and $G/G^{00}$ has a suitable "logic topology" studied by Lascar and Pillay. The conjecture states in addition that if $G$ is definably compact then $G/G^{00}$ resembles $G$ very closely. In particular the o-minimal dimension of $G$ coincides with the dimension of $G/G^{00}$ as a Lie group and these two groups have the same torsion. The results in[27] establish the conjecture for every definable groups of dimension one and for every definably simple group. Other partial results were obtained in our recent preprint with M. Otero [4]. Thanks to these results and the exchange of ideas and collaborations made possible during the Banff meeting, we now know that the type definable subgroups of bounded index satisfy the DCC (in analogy with the closed subgroups of a Lie group) and therefore $G^{00}$ exists and $G/G^{00}$ is a Lie groups. What remains open is the computation of the dimension of $G/G^{00}$.

The analogy between definable groups and Lie groups suggests the possibility of defining an analogue of the Haar measure in any definably compact group. With this aim in mind in the preprint [3] we define a finitely additive real valued measure in every o-minimal expansion $M$ of a field and we show that the definable subsets of $M^n$ contained in the finite part of $M^n$ are measurable (if the underlying set of $M$ is the real line one obtains the Jordan measure). This entails that we can indeed define the analogue of the Haar measure for certain definably compact groups, but the general problem remains open.

# 2 D. Macpherson: Tutorial on imaginaries in valued fields

A tutorial was presented on recent work on elimination of imaginaries in algebraically closed valued fields, together with extensions and applications. This consisted of 5 lectures, two by Dugald Macpherson, two by Deirdre Haskell, and one by David Lippel. Material from [15], [16] and the survey [17] was presented, together with work from a manuscript in preparation of E. Hrushovski and B. Martin.

A complete first order theory $T$ has *elimination of imaginaries* (e.i.) if, for every $M \models T$, every $n \geq 1$, and every $\emptyset$-definable equivalence relation $E$ on $M^n$, there are $m \geq 1$ and an $\emptyset$-definable function $f = f_E : M^n \to M^m$ such that, for $x, y \in M^n$, $xEy$ holds if and only if $f(x) = f(y)$. The $E$-classes are called *imaginaries.* Close model-theoretic study of a structure usually requires one to understand the imaginaries, and e.i. ensures that each imaginary is coded by a tuple in the structure. Elimination of imaginaries can always be guaranteed by adjoining a new sort $M^n/E$ for each $n, E$ as above, to obtain $M^{\mathrm{eq}}$; in the process, however, one can lose sight of definability. An intermediate step is to adjoin to $M$ certain specific, well-understood, sorts from $M^{\mathrm{eq}}$, and prove elimination of imaginaries with these sorts.

Suppose that $K$ is an algebraically closed field equipped with a non-trivial valuation $v : K \to \Gamma$, where $\Gamma$ is an ordered group. Let $R := \{x \in K : v(x) \geq 0\}$ be the valuation ring of $M$, $\mathcal{M}$ its maximal ideal, and $k := R/\mathcal{M}$ be the residue field. The model theory of $(K, v, \Gamma)$ was investigated by Abraham Robinson in [28]. From his work a quantifier elimination can also be obtained, and it can be shown that $\Gamma$ is o-minimal, $k$ is strongly minimal, and both are stably embedded. However, imaginaries are rather complicated. The main theorem of [15] is that one obtains e.i. after adding, for each $n \geq 1$, two sorts $S_n$ and $T_n$. Here, $S_n$ is the set of all $R$-lattices in $K^n$, that is, free rank $n$ $R$-submodules of $K^n$. If $A \in S_n$ then $\mathcal{M}A$ is an $R$-submodule of $A$, and $T_n$ is the union, over all $A \in S_n$, of the $k$-vector spaces $\mathrm{red}(A) := A/\mathcal{M}A$; in particular, there is a map $\pi_n : T_n \to S_n$ whose fibres are $n$-dimensional $k$-vector spaces. It is also possible to identify $S_n$ with the coset space $\mathrm{GL}_n(K)/\mathrm{GL}_n(R)$, and to treat $T_n$ similarly. It is known that if all but finitely many of the $T_n$ are omitted, then e.i. no longer holds.

The paper [16] consists of a close model-theoretic study of ACVF, the theory of algebraically closed valued fields with the sorts $S_n$ and $T_n$ (in addition to the 'home sort' $K$). This theory is certainly not stable (it interprets $\Gamma$). However, many ideas from stability theory are applicable. If $C$ is a parameter set, there is a many-sorted structure $\mathrm{Int}_{k,C}$ with a sort for $\mathrm{red}(A)$ for each $C$-definable lattice $A$, with all the induced $C$-definable structure. This structure is $\omega$-stable with elimination of imaginaries, and the elements of any $C$-definable stable and stably embedded set are coded by tuples from $\mathrm{Int}_{k,C}$.

Certain types over $C$ are 'stable dominated', meaning that their independent extensions are somehow determined by their 'trace' in $\mathrm{Int}_{k,C}$. Some model theory of stable domination is developed in [16] in complete generality, i.e. not specifically for ACVF. In ACVF, it is shown that a type is stably dominated precisely if it is 'orthogonal' to $\Gamma$. Given sufficient orthogonality to $\Gamma$, all reasonable notions of independence (of which there are several) coincide.

In a recent preprint, Hrushovski and Martin have shown that the $p$-adic field also has elimination of imaginaries, again with the sorts $S_n$ (but the $T_n$ are not required). The proof uses both the elimination of imaginaries for ACVF, and ideas from its proof, but the result is formulated in greater generality, with many potential applications to further structures. As a consequence, they show that certain power series associated with finitely generated nilpotent groups are rational. If $G$ is such a group, the $n$-dimensional complex characters $\chi_1, \chi_2$ of $G$ are said to be *twist-equivalent* if there is a 1-dimensional complex character $\phi$ of $G$ such that $\chi_2 = \chi_1 \circ \phi$. By work of Lubotzky and Magid it is known that the number $a_n(G)$ of twist-equivalence classes of $n$-dimensional irreducible complex characters of $G$ is finite. Hrushovski and Martin show that if $p$ is a prime, the power series

$\Sigma_{n\geq 0} a_{p^n} T^n$ is rational. The proof uses some of the methods developed in [14] and extended by du Sautoy, but appears to require an understanding of imaginaries in $\mathbf{Q}_p$.

Many issues arise from the above work: elimination of imaginaries in other valued fields, and their expansions by subanalytic functions or by generic automorphisms or derivations; the structure of definable/interpretable groups and fields; uniformity in $p$ for the $p$-adic e.i., and uniformity in $p$ for group-theoretic rationality results; further development of the model theory of stable domination. There has been progress on some of these, for example in work by Hrushovski on groups with a stably dominated type, and work by Mellor on imaginaries in real closed valued fields.

In the tutorial, an overview of ACVF was presented. Then aspects of the proof of e.i. were described, with a slightly different treatment due to Lippel. The main ideas of stable domination were sketched, along with some of the independence theory. Finally, it was shown how $p$-adic e.i. can yield rationality results for finitely generated nilpotent groups.

# 3 L. Lipshitz: session on Rigid Analytic Geometry and $p$–adic and Motivic Integration

**1. Rigid subanalytic sets and rigid analytic quantifier elimination** — Leonard Lipshitz

This talk surveyed the current state of knowledge on the subjects in the title. Let $K$ be a complete, algebraically closed valued field. $K^0 = \{x \in K : |x| \leq 1\}$. Let $\mathcal{C}$ be a class of analytic function $(K^0)^n \to K$, $n \in \mathbb{N}$. $D\colon K^2 \to K$ is restricted division. $\mathcal{L}_{an}(\mathcal{C})$ is the language of $K$ enriched with symbols for the functions in $\mathcal{C}$ and $\mathcal{L}_{an}^D(\mathcal{C})$ is $\mathcal{L}_{an}(\mathcal{C})$ with $D$ adjoined.

The corresponding (global) semi–analytic (resp. $D$–semianalytic) subsets of $(K^\circ)^m$ are those defined by quantifier free $\mathcal{L}_{an}(\mathcal{C})$-(resp. $\mathcal{L}_{an}^D(\mathcal{C})$-) formulas. The corresponding subanalytic subsets of $(K^\circ)^m$ are the projections of semi–analytic subsets of $(K^\circ)^{m+n}$.

If $K$ has quantifier elimination in $\mathcal{L}_{an}^D(\mathcal{C})$ (or quantifier simplification (model completeness) in $\mathcal{L}_{an}(\mathcal{C})$) then one obtains a natural theory of subanalytic sets in close analogy to the real and $p$–adic cases.

Natural classes to consider for $\mathcal{C}$ are

1. $\mathcal{J}$ the strictly convergent power series
2. $\mathcal{O}$ the overconvergent elements of $\mathcal{J}$.
3. $\mathcal{S}$ the separated power series.

• $K$ has $QE$ in $\mathcal{L}_{an}^D(\mathcal{S})$, giving a natural theory of rigid subanalytic sets, including dimension theory and Lojasiewicz inequalities. [18]

• $K$ has quantifier simplification in $\mathcal{L}_{an}(\mathcal{J})$ giving the theorem on the complement for afinoid (i.e. based on $\mathcal{J}$) subanalytic sets. [19]

• $K$ has quantifier elimination in $\mathcal{L}_{an}^D(\mathcal{O})$ giving a natural theory of "strongly" subanalytic sets, which are the images of analytic sets under affinoid proper maps. [29]

• In [12], [13] and [30] it is claimed that $K$ has $QE$ in $\mathcal{L}\mathcal{J}_{an}^D(\mathcal{J})$. A key step (the global flattening theorem) in the proof is incorrect. A counter example to global flattening is given in [21].

**2. Henselian Fields with Analytic Structure, Denef-Pas Cell Decomposition, and its Extension to the Analytic Category**— Zachary Robinson– (joint work with Leonard Lipshitz and Raf Cluckers)

Algebraic cell decomposition for Henselian valued fields has been used by Denef [6] in developing $p$-adic integration techniques, and more recently by Denef and Loeser [8], and others, in developing the theory of motivic integration.

This talk has two parts. The first is an exposition of the algebraic cell decomposition techniques of P. J. Cohen [5], J. Denef [7] and J. Pas [23]. The second part treats the work in progress to extend these methods to handle analytic functions. Here, one must first define carefully the notion of analytic structure on a valued field that may not be complete in a rank one valuation, e.g., a non-standard model of a field that is complete (cf. [9] and [22].) In this setting, Weierstrass Preparation techniques and the completeness of the coefficient ring compensate for the lack of completeness of the domain to yield a suitably rich and general analytic function theory (cf. [20].) One then uses this theory to obtain analogues of classical results of Mittag-Leffler on functions analytic in an annulus over the complex numbers (cf. [10].) This permits a reduction of the problem of analytic cell decomposition to the algebraic case.

**3 and 4. A New Framework for Motivic Integration** — Raf Cluckers and Francois Loeser

In two talks (by R. Cluckers and F. Loeser) a new framework for motivic integration is presented. This is joint work by Cluckers and Loeser.

In this new framework, several new concepts, new arguments, new results, and generalised results are introduced. A class of motivic constructible functions is defined, as well as the notion of positive motivic constructible functions, by analogy to a class of p-adic functions one would like to interpolate motivically. For these classes of functions a motivic integral is defined. Essential is that now the integrals may depend on parameters, in other words, given a constructible function, one can integrate some variables out, and end up with a constructible function in the other variables. In the definition of the integrals, no completion process is needed, i.e., no approximation process is used to define the integrals. Instead, a parametrisation process is used where the parameters run over the residue field and the value group (the integers) of the valued field of equicharacteristic 0. Hence, a notion of measure is needed not only on the valued field, but also on the value group (the counting measure), and on the residue field. On the residue field, one takes as measure just formally the class of the set under isomorphisms. When infinite sums occur, we show that these can always be written as geometric power series and hence, their sum exists in certain localisations without needing any completion.

The notion of positivity is based on the observation that one can work with semigroups instead of groups, and every element of a semigroup is understood to be positive. We let the motivic measure and the positive constructible functions take values in the semi-Grothendieck group G of isomorphism classes of a certain kind of subsets of vector spaces over the residue field. After inverting additively any element in these semigroups, one can go to Kontsevich's notion of motivic integrals by applying a "forgetfull"-morphism from G to (a certain localisation of) the Grothendieck ring of varieties and by completing this. One can also specialise to Denef-Loeser's notion of arithmetic motivic integral by applying the Denef-Loeser map from G to (a certain localisation of) the Grothendieck ring of Chow motives (tensored with the rational numbers) and by completing this. Finally, one can interpolate $p$-adic integrals for $p$ big enough, i.e., our notion of motivic integrals gives a geometric understanding of $p$-adic integrals for $p$ big enough.

In this framework, a completely general change of variables is obtained. In this generality, a direct image formalism is developed.

In the first talk by Cluckers, a general introduction of the new concepts and new framework is given. The notion of positivity, as well as the measures on the valued field, residue field and value group are explained. The proof of change of variables is explained.

In Loeser's talk, more exact definitions are given than in the general introduction by Cluckers, the proof of Fubini's theorem is given, and the direct image formalism is explained.

The work presented here is available in resume form at the arxiv, and a paper containing all the proofs will be available soon.

# 4 T. Scanlon: session on jets

The portion of the program on jets was organized around the jet space construction and its applications to problems in the model theory of difference and differential fields. Talks in this session were presented by Alexandru Buium, Zoé Chatzidakis, Rahim Moosa, Anand Pillay and Thomas Scanlon.

Alexandru Buium spoke about $p$-jet spaces of modular curves. A general construction attaches to any smooth scheme $X$ over a $p$-adic ring a tower of formal schemes called the $p$-jet spaces of X. When this construction is applied to modular curves a theory emmerges that generalizes the theory of $p$-adic modular forms. Structure theorems can be obtained for the resulting rings of "$p$-differential" modular forms.

Zoé Chatzidakis spoke about two results related to the jet space methods. First, she reported on work of Bustamante showing that the methods and results of Pillay-Ziegler extend to finite dimensional sets defined in difference-differential fields. Secondly, the jet space methods show that (in appropriate theories) if $\mathrm{SU}(a/c) < \omega$ and $c = \mathrm{Cb}(a/c)$, then $\mathrm{tp}(c/a)$ is internal to the non-locally modular minimal sets. Possible extensions of ths result to general supersimple theories were discussed.

Rahim Moosa discussed jet spaces from the point of view developed by Grothendieck in the 1950s. He then discussed how the Campana-Fujiki theorems on complex analytic spaces, which served as precursors to the Pillay-Ziegler theorem, could be understood in this geometric language.

Anand Pillay discussed the category of algebraic $D$-varieties and algebraic $D$-groups. Jet space results were used (in joint work with Kowalski) to show that the category of algebraic $D$-groups has quantifier-elimination. On the other hand, for trivial reasons the category of algebraic $D$-varieties does not have quantifier elimination. He discussed the issue of finding new "complete" objects among algebraic $D$-varieties. Some positive answers were given within the context of groups.

Thomas Scanlon spoke about joint work with Moosa and Pillay in which arc spaces are developed for possibly infinite dimensional partial differential varieties to prove a dichotomy theorem for regular types in partial differential fields. Namely, every regular type in a partial differential field is nonorthogonal to some regular generic type of a definable additive group.

# 5 C. Steinhorn: session on o-minimality

O-minimality has been one of the central areas of research in model theory for about twenty years. The wealth of mathematically important examples of o-minimal structures now known combined with the powerful model-theoretic tools that have been developed have led to applications in fields as diverse as representation theory and statistics. The talks in this session were selected to represent a range of topics in and around o-minimality. Starchenko's contribution continues to advance the theme that abelian groups definable in o-minimal structures resemble real Lie groups; Aschenbrenner's and Miller's talks concern new model-theoretic contexts beyond o-minimality in which the definable sets of an ordered structure are still what might be called "tame." The abstracts follow.

Matthias Aschenbrenner, University of Illinois at Chicago. Title: Gaps in $H$-fields

Abstract: The class of $H$-fields is a common algebraic abstraction of Hardy fields and of (certain) fields of transseries. In a joint project, Lou van den Dries, Joris van der Hoeven and myself are trying to obtain a model-theoretic understanding of this class. In my talk I will focus on a particularly troublesome phenomenon (gaps) connected with the presence of transexponential elements in Hardy fields.

Chris Miller, Ohio State University Title. Expansions of o-minimal structures by trajectories of definable planar vector fields.

Abstract. An expansion of the real field is said to be o-minimal if every definable set has finitely many connected components. Such structures are a natural setting for studying "tame" objects of real-analytic geometry such as non-oscillatory trajectories of real-analytic planar vector fields. It turns out that even some infinitely spiralling trajectories of such vector fields have a reasonably well-behaved model theory; this motivates the notion of d-minimality, a generalization of o-minimality that allows for some definable sets to have infinitely many connected components. The following trichotomy illustrates why we are interested in this notion. Let $U \subseteq \mathbb{R}^2$ be open and $F\colon U \to \mathbb{R}^2$ be real analytic such that the origin 0 is an elementary singularity of $F$ (i.e., $F^{-1}(0) = \{0\}$ and the Jacobian of $F$ at 0 has a nonzero eigenvalue). Let $g\colon (0, b) \to \mathbb{R}^2$ be a solution to $y' = F(y)$ such that $g(t) \to 0$ as $t \to 0^+$. Then, after possibly shrinking $b$, the expansion of the real field by the curve $g((0, b))$ either is o-minimal, is d-minimal and not o-minimal, or defines $\mathbb{Z}$.

Sergei Starchenko, University of Notre Dame Title: On torsion free groups definable in o-minimal structures.

Abstract: (Joint work with Y. Peterzil) We consider groups definable in the structure $\mathbb{R}_{an}$ and certain o-minimal expansions of it. We prove: If $\mathbb{G} = \langle G, * \rangle$ is a definable abelian torsion-free group then $\mathbb{G}$ is definably isomorphic to a direct sum of $\langle \mathbb{R}, + \rangle^k$ and $\langle \mathbb{R}^{>0}, \cdot \rangle^m$, for some $k, m \geq 0$. Futhermore, this isomorphism is definable in the structure $\langle \mathbb{R}, +, \cdot, \mathbb{G} \rangle$. In particular, if such $\mathbb{G}$ is semialgebraic, then the isomorphism is semialgebraic. We show how to use the above result to give an "o-minimal proof" to the classical Chevalley theorem for abelian algebraic groups over algebraically closed fields of characteristic zero.

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