Representations of Kac-Moody Algebras and Combinatorics

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The particular focus of this workshop was on the combinatorial aspects of representation theory. It brought together senior mathematicians working in the representation theory of Kac-Moody algebras with students and postdoctoral fellows who are in the initial stages of their career in this field. The participants represented the field quite well, in subjects ranging from the algebraic aspects of the representation theory of infinite–dimensional algebras, the combinatorial aspects of the crystal base theory and the path model, the geometric aspects of quiver varieties and the mathematical physics aspects of the Bethe Ansatz. Towards the end of the conference a good picture emerged of the development and the interplay between the different aspects of the subject.

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We outline the main developments which were presented and discussed in the workshop.

2.1 Algebraic Aspects

The study of Kac–Moody Lie algebras began in the 1970's and were a natural generalization of the theory of semisimple Lie algebras. A Kac–Moody Lie algebra, [14] of rank n is defined by an $n \times n$ integer valued matrix $A = (a_{ij})$ (called the generalized Cartan matrix) satisfying the conditions: $a_{ii} = 2$ and for $i \neq j$ $a_{ij} = 0 \iff a_{ji} = 0$. Such matrices were classified by Vinberg and were shown to be satisfy one of the following mutually exclusive conditions: (a) the matrix is positive definite, (b) the matrix is positive semidefinite and and every proper principal minor is positive definite, (c) there exists a vector v of positive integers such that Av is a negative vector. In case (a) the associated Lie algebra is a semisimple finite dimensional Lie algebra while in the other cases the Lie algebra is infinite dimensional. If the matrix A satisfies condition (b) or (c) it is said to be of affine or indefinite type respectively.

The affine Lie algebras and the representation theory of these algebras is widely studied and is motivated by important applications in physics. One such application comes from the underlying symmetry of two dimensional conformal field theories which also led to the study of vertex algebras. Another application is that of the quantized universal enveloping algebras in the theory of quantum integrable systems. Both these applications in turn, brought important ideas to the study of the representation theory of these algebras. The talks in the conference that focussed on the algebraic side were given by Bakalov, Brundan, Greenstein, Hernandez, Loktev and de Moura. The talks dealt with representations of affine Lie algebras and its applications to physics. The representation theory or indeed the structure of an arbitrary Kac–Moody algebra is in general poorly understood. However in the last two years some progress has been made in understanding the representation theory of certain infinite families of Kac–Moody Lie algebras and S. Viswanath reported on this new development.

All the talks discussed so far focussed on the integrable representations of the Kac–Moody algebra which are the analogs of the finite–dimensional representations of a semisimple Lie algebra. A closely related category is the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. C. Stroppel presented some new results for the category $\mathcal{O}$ associated to a Kac-Moody algebra and indicated a very intriguing new connection with knot invariants.

2.1.1 Vertex operator algebras

These were introduced by R. Borcherds as an algebraic tool to study the underlying the operator product expansion operation in conformal field theory. They were instrumental in the proof of monster moonshine conjecture. Chiral algebras which are a generalization of vertex algebras were introduced later and play an important role in the geometric Langlands program. An algebraic structure that emerged from the study of chiral algebras in conformal field theory are the Lie conformal algebras and its higher dimensional analogs the Lie Pseudoalgebras, [3]. B. Bakalov discussed the classification of Lie Pseudoalgebras and the relation to solutions of the classical Yang–Baxter equations.

2.1.2 Representations of indefinite type Kac–Moody algebras

For many years there was limited progress in the representation theory of Kac-Moody algebras associated to a generalized Cartan matrix of indefinite type. The best studied amongst these were the algebras of hyperbolic type and even there, results are hard to come by. More recently, M. Kleber and S. Viswanath identified infinite families of algebras of indefinite type whose representation theory parallels and in fact generalizes that of sl_n. Roughly speaking, the algebras they consider are obtained by "extending" the Dynkin diagram of a Kac–Moody algebra by a tail which is the Dynkin diagram of sl_n. Clearly the finite–dimensional algebras of classical type belong to this picture. But now, one can also allow the infinite series coming from the simple laced exceptional algebras, this includes the hyperbolic Lie algebra E_{10} which has been studied by mathematicians and physicists. In his lecture, S. Viswanath discussed the representation ring of these algebras and showed that the tensor product of the integrable representations decomposed in a stable way: namely as the length of the tail went to infinity, the multiplicity of the isotypical components remained the same. Using this, he explained how to define a stable product on a suitable vector space, analogous to the ring of symmetric functions coming from the representation theory of sl_n.

Ben Webster, a graduate student at Berkeley and one of the participants of the conference noticed that this stabilization feature can be explained clearly using quiver realizations of representations of Kac–Moody algebras. His preprint is now available on the archive, [29].

2.1.3 Representations of affine Kac–Moody algebras

Affine Kac-Moody algebras are one of the most important and well studied class of Kac-Moody algebras. The main reason for this is that they can be realized as the universal (one–dimensional) central extension of the Lie algebra of Laurent maps from to a semisimple Lie algebra. The representation theory of the affine algebras are "controlled" by the center, and there is striking difference between the representations where the center acts by a positive integer (positive level representations) and those where the center acts trivially (level zero representations). One outcome of the workshop was a very good understanding and formalizing of the connection between these two families of representations.

2.1.4

The irreducible finite–dimensional representations of quantized affine algebras play a key role in the theory of quantum integrable systems. The structure of these representations is quite complicated and there are a number of approaches to studying them, [1], [4], [5], [7], [9], [10], [27]. Two of these approaches have had significant success recently and were discussed in the conference. One is the approach of q–characters, an idea that was introduced by Frenkel and Reshetikhin and further studied by Frenkel and Mukhin. D. Hernandez discussed his work on q–characters and showed how his methods could be used to solve a conjecture on the structure of a particular family of modules, the so-called Kirillov–Reshetikhin modules. This also allowed him to establish that these characters solved a system of equations called the Q–system arising in the study of integrable systems. A. deMoura (joint work with V. Chari) presented an alternate approach to defining the q–characters which leads to a parametrization of the blocks (generalizing results of [8]) in the category of finite–dimensional representations of quantized affine algebras.

A second approach is to consider the $q = 1$ limit of representations of quantum affine algebras which led to the idea of Weyl modules in this context. These modules were defined and initially studied by Chari and Pressley who also had a conjecture on the structure of these modules. About the same time, B. Feigin and S. Loktev defined the notion of a fusion product of finite–dimensional representations of a simple Lie algebra and showed that these could be regarded as modules for the polynomial valued subalgebra of the affine algebra. S. Loktev discussed (joint work with various coauthors) in his lecture the relationship between the Weyl modules and fusion products and also generalizations of Weyl modules to other algebras. J. Greenstein (joint work with Chari) discussed extensions in the category of finite dimensional representations of affine Lie algebras and a new realization of current algebras.

2.1.5

A new connection between the theory of Yangians [6] and W-algebras was also presented during the conference. W-algebras are endomorphism algebras of certain induced modules for a finite dimensional reductive complex Lie algebra. There is a natural way to associate to nilpotent element such an algebra (Slodowy slice). In fact, the coordinate ring of the Slodowy slice is isomorphic to the an appropriate associated graded version of the W-algebra. In special cases it has been observed before that in this way one gets the Yangian of level ℓ. J. Brundan reported on his joint work with A. Kleschev, where they consider arbitrary nilpotent matrices. They describe a presentation of these algebras which leads to a generalization of the of Yangians, the so-called shifted Yangians. Because of the Schur-Weyl duality or rather it's quantized version, they obtain also a close connection with the degenerate cyclotomic Hecke algebra and representations of the Lie algebra gl_∞

2.1.6

Projective functors in the Bernstein–Bernstein–Gelfand category $\mathcal{O}$ [2] are the functors obtained as direct summands of the functors given by tensoring with finite dimensional representations. Such functors have been classified by Gelfand and Bernstein.

A different approach to this problem was presented during the meeting by C. Stroppel. The advantage of the approach is that it not only recovers the known results but also can be easily generalized to the Kac-Moody case. Further, the approach by deformation theory also opens a new and very interesting connection to knot and tangle invariants

2.2 Combinatorial Representation theory

2.2.1

Macdonald polynomials play an important role in representation theory and govern in many cases the combinatorial aspects of a theory. In his talk M. Haiman explained the latest developments in this field. In particular, he explained a new combinatorial formula for Macdonald polynomials. The advantage of this formula is that fact that it gives deep insights into the structure of these polynomials and provides a new approach to understanding the charge formula of Lascoux and Schützenberger. M. Shimozono gave a talk on finding a Schubert calculus on affine Grassmanians and explained the importance of this in enumerative algebraic geometry. Roughly speaking the idea is to find a pairing between the Schubert bases of the cohomology and homology of the affine Grassmanian associated to sl_n. Using this he and his collaborators hope to find the structure constants of the homology. This should give the decomposition of the fusion product of positive level representations generalizing the the Littlewood Richardson rule for the tensor product of finite dimensional representations of sl_n.

2.2.2

The theory of crystal bases developed independently by Kashiwara [15], [16] and Lusztig [23] has become a very important tool in many aspects of representation theory. The associated graph reflects in many ways important properties of the representation. Different aspects of the theory of crystals were discussed at the meeting: in the case of finite dimensional representations of affine Kac-Moody algebras for example, it is only conjectured [17] that crystal bases exist in the general case. Proofs for the existence in special cases need case by case considerations. The other important point is that of constructing combinatorial models of these graphs, [20], [21], [26]. Of course, different models of the same graph may be particularly adapted to different properties, so a third important point is to understand the relationship between different existing models

2.2.3

K. Misra gave a report on further development of the second problem mentioned above. He presented results on a joint work with Kashiwara, Okado and Yamada. They construct perfect crystals for the integrable highest weight $D_4^{(3)}$-modules of level $k > 0$. These perfect crystals are finite graphs, but the graphs for the infinite dimensional integrable highest weight modules can then be constructed as semi-infinite tensor products of these graphs.

2.2.4

The crystal graph can be also very helpful in constructing bases of the representation spaces. One case was reported on the conference by A. Premat. The aim was to construct a monomial basis for Demazure modules. Of course, there is the global / canonical basis by Kashiwara and Lusztig, but which is in general not always easy to compute in an explicit way. Using a combinatorial model for the crystal basis by Young diagrams, she reported that the transition matrix between the monomial basis constructed by her and global bases are upper triangular with ones in the diagonal.

2.2.5

An important step in developing a crystal graph theory for finite dimensional representations of untwisted affine Kac-Moody algebras was presented by D. Sagaki and S. Naito. The set of Lakshmibai-Seshadri paths makes sense for affine Kac-Moody algebras even in the case where λ is not a weight in the Tits cone. Suppose λ is of level zero and dominant integral for the underlying finite dimensional Lie algebra. They show that after the projection on the space modulo the imaginary root one can endow this set with the structure of a crystal graph. In fact, this set has a tensor product decomposition, it is the product of the corresponding sets for the fundamental weights. Since these are combinatorial models for the crystal graph of quantum Weyl modules, it follows that in this way they provide a uniform way to get a combinatorial way for the quantum Weyl modules of all untwisted affine Kac-Moody algebras.

2.2.6

Another successful tool to obtain combinatorial models for crystal bases / crystal graphs for irreducible highest weight crystals of quantum (affine) algebras.

The Young walls consist of colored blocks with various shapes that are built on a given ground-state wall and can be viewed as generalizations of Young diagrams. The rules for building Young walls and the action of Kashiwara operators are given explicitly in terms of combinatorics of Young walls. The crystal graph of a basic representation is characterized as the set of all reduced proper Young walls. The character of a basic representation can be computed easily by counting the number of colored blocks that have been added to the ground-state wall.

This theory has been developed by Seok-Jin Kang, J. H. Kwon, J.-A. Kim, H. Lee, D.-U. Shin and others. A report on the present state of the theory was given and a possible connection between modular representation theory and crystal bases.

2.2.7

A first step in the understanding of the connection between the Kyoto path model for representations of affine quantum algebras and the path model by Littelmann was presented by P. Magyar. In the case of the basic level-one representation, he derives a direct connection between the two path models by generalizing the path model to a class of semi-infinite concatenations of paths, called skeins.

2.2.8

Let $\mathfrak{g}$ be a simple complex Lie algebra and denote by $\widehat{\mathfrak{g}}$ the affine Kac–Moody algebra associated to the extended Dynkin diagram. It is a natural approach to understand the infinite dimensional highest weight representations of $\widehat{\mathfrak{g}}$ by first studying them as $\mathfrak{g}$–modules. To do so, one needs restriction formulas. A natural filtration by finite dimensional subspaces of such a representation is given by its $\mathfrak{g}$–stable Demazure modules.

In the case where $V = V(\ell\Lambda_0)$ corresponds to a multiple of the highest weight of the vacuum representation (and some more general cases), G. Fourier (joint work with P. Littelmann) presented a very effective approach. In this case the Demazure modules are indexed by dominant coweights, and it was explained that the Demazure module decomposes as $\mathfrak{g}$-module into a tensor product of Demzure modules corresponding to fundamental coweights. This decomposition can be viewed as the natural generalization and uniform formulation of many partial results know before.

For these "smallest modules" an explicit decomposition is given in the classical case (and in many nonclassical cases). In fact, it turns out that as $\mathfrak{g}$-module they are isomorphic to some Kirillov–Reshetikhin-modules.

As a consequence one can give a description of the $\mathfrak{g}$-module structure of $V(\Lambda)$ for an arbitrary dominant weight as a semi-infinite tensor product of finite dimensional $\mathfrak{g}$–modules.

2.3 The Bethe Ansatz

The Bethe Ansatz is a method to obtain eigenvectors for a certain set of operators. The corresponding Bethe vectors correspond then in the general case to certain parameters satisfying the Bethe equations. There are two methods to obtain these vectors, one coming from the crystal base theory and another method to obtain to obtain eigenvectors comes from representation theory of affine Lie algebras.

2.3.1

E. Mukhin showed that the Bethe equation for the nonhomogenous Gaudin model could be solved by certain orthogonal polynomials. He also addressed a similar problem for other models, namely the trignometric model and the XXZ model. All these involve looking at suitable finite dimensional representations of affine Lie algebras.

2.3.2

Another problem on this subject was addressed by Anne Schilling. In the case of a given spin model, the Bethe vectors are indexed by certain rigged configurations, whereas the solutions obtained by representation theory are indexed by elements of a crystal graph. So it is natural to ask for the relationship between these to methods.

A. Kirillov and N. Reshetikhin provided a combinatorial bijection between certain restricted rigged configurations and highest weights in crystal. This bijection was generalized later by A. Kirillov, A. Schilling and M. Shimozono. A. Schilling gave a report on this subject and presented the latest development: An extension of the bijection above to a bijection between the rigged configurations parameterizing the Bethe vectors and the crystals parameterizing the eigenvectors obtained by representation theoretic methods. This result was obtained by defining a crystal graph structure on the set of rigged configurations.

2.3.3

The tensor product of evaluation representations of affine Kac-Moody algebras lifts to the fusion product of integrable modules. The fusion tensor product induces the grading of the multiplicity spaces for the decomposition of tensor product of irreducible modules over the underlying simple Lie algebra. Poincaré polynomials for graded multiplicity spaces can be regarded as generalizations Kostka-Foulkes polynomials. The structure of these polynomials is closely related to the structure of irreducible characters of corresponding affine Kac-Moody algebras. The latest progress in this direction was reported by R. Kedem.

3 Talks

Speaker: Bojko Bakalov

Title: Lie Pseudoalgebras

Abstract: One of the algebraic structures that has emerged recently in the study of the operator product expansions of chiral fields in conformal field theory is that of a Lie conformal algebra. A Lie pseudoalgebra is a "higher-dimensional" generalization of the notion of a Lie conformal algebra. On the other hand, Lie pseudoalgebras can be viewed as Lie algebras in certain pseudo-tensor categories.

I will review the classification of finite simple Lie pseudoalgebras, and I will discuss their relationship to solutions of the classical Yang-Baxter equation and to linear Poisson brackets. I will also describe the irreducible representations of the Lie pseudoalgebra $W(\mathfrak{d})$, which is closely related to the Lie-Cartan algebra W_N of vector fields, where $N = \dim \mathfrak{d}$. (Based on a joint work with A. D'Andrea and V. G. Kac.)

Speaker: Jon Brundan

Yangians, Whittaker modules and cyclotomic Hecke algebras.

There has recently been some progress in understanding some algebras introduced originally by Kostant in 1978. These algebras can be viewed as quantizations of the Slodowy slice associated to a nilpotent orbit in a semisimple Lie algebra. In type A, it turns out that these quantizations of the Slodowy slice are closely related to the Yangian of the Lie algebra gl_n. Actually, they are generalizations of the Yangians which we call shifted Yangians.

In recent work with A. Kleshchev, we have worked out the combinatorics of the finite dimensional representations of shifted Yangians. The approach uses in an essential way a theorem of Skryabin relating representations of these algebras to certain categories of generalized Whittaker modules. In particular, we are able to reprove and generalize the known results about representations of Yangians, all as a direct application of the Kazhdan-Lusztig conjecture.

There is also a close connection between shifted Yangians and the degenerate cyclotomic Hecke algebras, thanks to a Schur-Weyl duality which interpolates between the classical Schur-Weyl duality and Drinfeld's affine analogue of it. This leads to a natural representation theoretic construction of some higher level Fock spaces for the Lie algebra gl_∞, complete with their dual canonical bases.

Speaker: Jacob Greenstein

An application of free Lie algebras to current algebras

We realize the current algebra of a Kac-Moody algebra as a quotient of a semi-direct product of the Kac-Moody Lie algebra and the free Lie algebra of the Kac-Moody algebra. We use this realization to study the representations of the current algebra. In particular we see that every ad-invariant ideal in the symmetric algebra of the Kac-Moody algebra gives rise in a canonical way to a representation of the current algebra. These representations include certain well-known families of representations of the current algebra of a simple Lie algebra. Another family of examples, which are the classical limits of the Kirillov-Reshetikhin modules, are also obtained explicitly by using a construction of Kostant. Finally we study extensions in the category of finite dimensional modules of the current algebra of a simple Lie algebra.

Speaker: Mark Haiman

Title: A combinatorial formula for Macdonald polynomials

Abstract: I'll explain recent joint work with Jim Haglund and Nick Loehr, in which we prove a combinatorial formula for the Macdonald polynomial $\tilde{H}_\mu(x;q,t)$ which had been conjectured by Haglund. Such a combinatorial formula had been sought ever since Macdonald introduced his polynomials in 1988.

The new formula has various pleasant consequences, including the expansion of Macdonald polynomials in terms of LLT polynomials, a new proof of the charge formula of Lascoux and Schutzenberger for Hall-Littlewood polynomials, and a new proof (and more general version) of Knop and Sahi's combinatorial formula for Jack polynomials.

In general, our formula doesn't yet give a new proof of the positivity theorem for Macdonald polynomials, because it expresses them in terms of monomials, rather than Schur functions. However, it does yield a new combinatorial expression for the Schur function expansion when the partition μ has parts ≤ 2, and there is hope to extend this result.

Speaker: David Hernandez

Title:The Kirillov-Reshetikhin conjecture and solutions of T-systems.

In this talk we present a proof of the Kirillov-Reshetikhin conjecture for all untwisted quantum affine algebras : we prove that the characters of Kirillov-Reshetikhin modules solve the Q-system, and so we get explicit formulas for the characters of their tensor products. Moreover we establish exact sequences involving tensor products of Kirillov-Reshetikhin modules and prove that their q-characters solve the T-system. For simply-laced cases these results were first obtained by Nakajima with geometric arguments which are not available in general. The proof we present is different and purely algebraic, and so can be extended uniformly to non simply-laced cases.

Speaker: Seok-Jin Kang

Title: Combinatorics of Young walls and crystal bases.

We will discuss the construction of irreducible highest weight crystals using Young walls. We will also discuss the possible connection between modular representation theory and crystal bases.

Speaker: Rinat Kedem

Title: Constructions of affine Lie algebra modules via graded tensor products via generalized Kostka polynomials.

The graded tensor product is a tensor product of finite-dimensional g-modules, endowed with a g-equivariant grading. This grading is related to the action of the loop algebra on the "fusion product" of representations of conformal field theory, and was originally defined by Feigin and Loktev. A conjecture, which has been proven in some special cases is that the graded multiplicity of an irreducible g-module in the graded tensor product is related to the Kostka polynomial or one of its generalized or level-restricted versions.

I will discuss how this graded tensor product allows us to construct integrable modules in two very different ways. One is in terms of the inductive limit of the graded tensor product of an infinite number of g-modules. The other is a generalization of the semi-infinite construction of Feigin and Stoyanovksy, which allows us to compute the characters of arbitrary highest weight integrable modules. This last requires use of the inverse of the matrix of generalized Kostka polynomials, and hence gives an interesting alternating sum expression for characters corresponding to non-rectangular highest weights in terms of rectangular ones.

Speaker: Sergei Loktev

Title: Weyl modules over sl_r-valued currents

Abstract: We discuss Weyl modules over sl_r-valued currents in one and two variable.

For one–dimensional currents a construction of basis, proposed by V.Chari and the speaker, will be described. If there will be enough time, the relation to Demazure modules and fusion modules will be discussed.

For two–dimensional currents relation to the space of diagonal coinvariants and parking functions, observed by B.Feigin and the speaker, will be explained.

Speaker: Kailash Misra

Title: Perfect crystal for $D_4^{(3)}$

Abstract: The crystal base theory developed by Kashiwara and independently by Lusztig provides an important combinatorial tool to study the representations of symmetrizable Kac-Moody algebras. It is known that the crystal base for affine Kac-Moody Lie algebras can be concretely realized as a subset of the semi-infinite tensor products of perfect crystals. In this talk we will present a perfect crystal for the integrable highest weight $D_4^{(3)}$-module of level $k > 0$. This is a joint work with Kashiwara, Okado and Yamada.

Speaker: Adriano A Moura

Title: Blocks of Finite Dimensional Representations of Classical and Quantum Affine Algebras.

Abstract: It is well known that the category of finite dimensional representations of classical or quantum affine algebras is not semisimple. To understand its block decomposition in the quantum case, P. Etingof and the speaker introduced the notion of Elliptic Characters. However, the original definition using analytic properties of the R-matrix imposed some un-natural restrictions to the problem (—q— should be different from 1). In particular, it was unclear how to compute the classical limit of the block decomposition. In this talk based on joint work with V. Chari we present a definition of Elliptic Characters from the point of view of the Braid Group action and the theory of q-Characters. This allow us to obtain the block decomposition for generic q as well as for q=1.

SPEAKER: Evgeny Mukhin

TITLE: Multiple orthogonal polynomials in Bethe Ansatz.

ABSTRACT: We show that the Bethe Ansatz equation for the non-homogeneous sl_n Gaudin model and two finite dimensional representations one of which is a symmetric power of vector representation, is solved in term of zeroes of multiple orthogonal Jacobi-Piñeiro polynomials. Equivalently, the spaces of polynomials with two finite ramification points with special exponents at one of the points have a basis explicitly given via multiple orthogonal Jacobi-Piñeiro polynomials. In a similar way, multiple orthogonal Laguerre polynomials appear in the Bethe Ansatz related to the trigonometric Gaudin model and multiple orthogonal little q-Jacobi polynomials in the Bethe Ansatz related to the XXZ model.

This is a joint work with A. Varchenko.

Speaker: Alejandra Premat

Monomial Bases for Demazure Modules

Abstract: We will discuss certain monomial bases of quantum Demazure modules for the algebra Uq(affine-sln) and show how to compute them using a description of the crystal graphs by Young diagrams. We will also see that the transition matrices from these bases to the Global bases are upper triangular with ones in the diagonal.

Speaker: D. Sagaki - S. Naito

Crystal of Lakshmibai-Seshadri paths associated to a level-zero integral weight for an affine Lie algebra

Let $\lambda = \sum_{i \in I_0} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$, be an integral weight of level zero that is a sum of level-zero fundamental weights ϖ_i, $i \in I_0$, for an affine Lie algebra $\mathfrak{g}$. We study a certain crystal $\mathbb{B}(\lambda)_{cl}$, which is (modulo the null root of $\mathfrak{g}$) the crystal of all Lakshmibai-Seshadri paths of shape λ, and prove that the $\mathbb{B}(\lambda)_{cl}$ is isomorphic as a crystal to the tensor product $\bigotimes_{i \in I_0} \mathbb{B}(\varpi_i)_{cl}^{\otimes m_i}$ of the crystals $\mathbb{B}(\varpi_i)_{cl}$, $i \in I_0$. Here we note that for each $i \in I_0$, the $\mathbb{B}(\varpi_i)_{cl}$ turns out to be isomorphic as a crystal to the crystal base of the level-zero fundamental module $W(\varpi_i)$ over the quantum affine algebra $U'_q(\mathfrak{g})$.

Speaker: Anne Schilling

Title: Crystal structure on rigged configurations

Abstract: Rigged configurations label the Bethe vectors of a given spin model. According to a bijection by Kirillov and Reshetikhin (generalized by Kirillov, S., Shimozono) rigged configurations correspond to highest weight crystal paths. The natural question arises whether there exist ”unrestricted” rigged configurations corresponding to any crystal path, not necessarily highest weight. In this talk we define unrestricted rigged configurations and describe the crystal structure on this set.

Speaker: Mark Shimozono

Title: Schubert calculus on the affine Grassmannian

Abstract: We present a generalization of the Robinson-Schensted-Knuth correspondence which conjecturally realizes the Cauchy identity that gives the perfect pairing between the Schubert bases of cohomology and homology of the affine Grassmannian of type $A_{n-1}^{(1)}/A_{n-1}$. This involves two kinds of tableaux that are defined using respectively the weak and strong Bruhat orders on the affine Weyl group. When n goes to infinity the bijection converges to the usual RSK map. We state a Pieri rule for the multiplication in cohomology, which uniquely determines the basis.

We are also investigating the properties of a jeu de taquin algorithm on weak order tableaux which may lead to a rule for the structure constants for homology. These constants generalize the fusion Littlewood-Richardson coefficients that come from the tensor product of representations at a given level.

This is ongoing joint work with Thomas Lam, Luc Lapointe, and Jennifer Morse.

Speaker: Catharina Stroppel

Title: The classification of projective functors for Kac-Moody Lie algebras

We consider the Bernstein-Gelfand–Gelfand category O attached to a semisimple complex Lie algebra. Projective functors are the direct summands of the functors given by tensoring with finite dimensional representations. These functors were classified by Bernstein and Gelfand. We want to give an alternative approach to this classification using deformation theory. We will explain how this alternative proof can be generalized to the Kac Moody situation giving rise to a classification of projective functors. As an explanation we briefly mention the connection to knot and tangle invariants.

Speaker: S. Viswanath

Dynkin diagram sequences and tensor product stabilization

In this talk, we will consider sequences of Dynkin diagrams Z_k of the form $X - o - o - o - \cdots - o - o - Y$ where X and Y are two fixed Dynkin diagrams and k is the number of intermediate nodes. The classical series A_k, B_k, C_k, D_k are all of this form and we can construct many more such series of indefinite Kac-Moody algebras as well (e.g $E_n, G_n, (E-E)_n, \cdots$).

Our goal will be to show that for the Z_k, multiplicities of irreducible representations in tensor product decompositions exhibit a stabilization behavior as $k \to \infty$. This parallels the situation for the series A_k where this result is implied directly by the Littlewood-Richardson rule. We'll use Littelmann's path model to do this.

The stable values of these multiplicities can be used as structure constants to define a "stable tensor product" operation on a space $\mathcal{R}(X|Y)$ that could be called the "stable representation ring". We'll show that this multiplication operation is indeed associative, making $\mathcal{R}(X|Y)$ a bonafide C algebra that captures tensor products in the limit $k \to \infty$.

Speaker: Milen Yakimov

General finiteness of the fusion tensor product

Kazhdan and Lusztig proved a finiteness result for the fusion tensor product for smooth modules over an affine Kac-Moody algebra which can be viewed as an analog of the fact that the product of finite dimensional modules over a simple Lie algebra is finite dimensional. In the classical situation Kostant's theorem from the late 70's provides a much more general finiteness: for any subalgebra k of a complex simple Lie algebra g which is reductive in g, the category of finite length, admissible (g,k)-modules is stable under tensoring with finite dimensional g-modules (with applications to category O, Harish-Chandra modules, etc.). We will describe a proof of an analog of this theorem for the fusion tensor product of smooth affine modules, based on an approach different from the one of Kazhdan and Lusztig.

4 Conclusion

The important ideas which emerged from the workshop were the relation between the Demazure modules, the level zero representations of affine Lie algebras , the Weyl modules and the path model for these representations. It is hoped that these relations should help in solving a conjecture of Kashiwara which predicts that the Kirillov–Reshetikhin modules for the quantum affine algebras admit a crystal basis. Also, it now appears very likely that the specialization to $q = 1$ of the tensor product of representations of the quantum affine algebra should be the fusion product of the representations of the classical affine algebras. While much of the work reported was on the untwisted affine algebras, it also became clear that the corresponding problems for the twisted affine algebras were also important.

The average age of participants was younger then usual and women were well represented among the speakers and participants. We consider this a success. The workshop has already stimulated research activity amongst its participants. S. Viswanath [28] and Ben Webster [29] have already posted articles following up on results presented at the conference. Several other collaborations between the participants, Hernandez and Greenstein, Hernandez and deMoura are ongoing and preprints should be available soon. On the whole we believe that the workshop was very useful and provided a good venue for interaction between the various directions of research in representation theory.

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