Numerical Relativity

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1 Overview

Solutions to Einstein's equations of general relativity describe, among other things, the generation and propagation of gravitational waves. Interest in numerical relativity has been bolstered by the recent construction of gravitational wave observatories such as LIGO (the Laser Interferometric Gravitational-wave Observatory). This NSF-supported project consists of two observatories (near Hanford, Washington and Livingston, Louisiana) that began collecting data in September 2002. Astronomy has been revolutionized in the past by developing the ability to observe electromagnetic radiation in new wavelength regimes (e.g., by X-ray and radio telescopes). Similarly, if successful, the development of gravitational-wave observatories will surely lead to many new surprises. However, gravitational waves reaching earth are incredibly weak and have yet to be directly detected. Being able to predict the gravitational wave signature of various possible events would help interpret any data received and separate the weak signal from noise.

Numerical relativity shares many features with other computational sciences involving systems of partial differential equations, such as computational fluid dynamics, solid mechanics, and optics. Many of the techniques developed to overcome computational difficulties in these areas are relevant to solving the Einstein equations as well. However, numerical relativity has challenges and complexities that often make it difficult to apply standard techniques directly and may require the development of new methodologies in applied and computational mathematics.

The Einstein equations can be written in the deceptively simple tensor form

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

where $G_{\mu\nu}$ is the Einstein tensor (encapsulating geometric information about the structure of space-time) and $T_{\mu\nu}$ is the stress-energy tensor (modeling the mass/energy and momentum of matter that both reacts to and creates the gravitational structure). Most of the discussion at this workshop concerned the vacuum Einstein equations, in which case $T_{\mu\nu} = 0$. The resulting equation $G_{\mu\nu} = 0$ hides a complicated set of geometric equations within the tensor notation. The Einstein equations represent one of the richest and most difficult systems of PDEs describing a viable physical theory. Simulations will play a crucial role in understanding these equations, particularly since experiments cannot be performed. To date, most of what is known about solutions of these equations has been obtained from different approximations that limit the nature of solutions found. Numerical simulations have the potential for unraveling the full consequences of the theory and robust implementations of the equations will certainly give rise to surprises.

This workshop engaged 38 participants in four and a half days of intense discussions of relativity and numerical approaches. Of these participants, roughly half can best be catagorized as physicists, based on their background and academic appointments, and the remainder as mathematicians. The main goal of the workshop was to bring these communities together to share ideas and learn from one another. There was widespread agreement that the workshop was a success in this regard. In addition to the transfer of information in talks and informal discussions, a number of potential new collaborations were initiated.

The speakers did an excellent job of presenting talks aimed educating others about various problems and techniques. The introductory talks the first morning set the stage for more detailed discussions of various aspects of this problem.

Douglas Arnold gave an extremely clear and elegant discussion of the basic differential geometry and tensor calculus that underlies most work in numerical work in relativity. Choptuik followed with a discussion of some of the key issues that distinguishes relativity from cousins in computational science. For example, he used the well known analogue between the Einstein and Maxwell systems as an illustration of how each continuous gauge freedom actually eliminated *two* dynamical degrees of freedom, and he emphasized how the counting of dynamical degrees of freedom was frequently confused in the literature.

The major highlight of the meeting was the talk by Frans Pretorius, who presented very recent numerical results showing the collision of two black holes and the resulting gravitational radiation. Figures 1 and 2 can only capture a small part of the impact of the full animations, and particularly for those who have been following the numerical work on the binary black hole problem, the results are nothing short of stunning.

As will be discussed in more detail Pretorius' code is especially notable (but hardly unique in the history of numerical relativity) in that it uses so little "conventional wisdom" vis a vis past experience in "3-D binary black hole" numerical relativity. For example, the code is based on generalized harmonic coordinates, which, save for some seminal work by Garfinkle [**?**], as well as Winicour and YYY [**?**], have not been previously used in a major numerical relativity effort.

Calculations of the type Pretorius showed have been a goal of many groups working in this area and so there was much interest in the particular set of techniques used to achieve his results. However, we note that the comment made by one of our colleagues (a numerical relativist) at the end of Pretorius' talk—"I believe that this is the beginning of the end of the binary black hole problem"— is symptomatic of an attitude that has plagued numerical relativity since its inception in the early 1970's. Numerical relativists have had a unnerving and demonstrably counterproductive tendency to rush on to the "next problem", typically characterized by substanially higher computational demands, more or less at the earliest sign of success with their current calculation. This has meant that very few problems in numerical relativity have been exhaustively studied, not to mention that progress has been unnecessarily slowed since researchers spend too much time (days to weeks) waiting for calculations to complete.

We would argue that Pretorius' computations rather mark the "end of the beginning" of the the binary black hole era, that his work is an important demonstration of an end-to-end solution of the problem that apparently has workable solutions for all of the key challenges (AMR, black hole excision, treatment of outer boundaries, ...) that were identified long ago, and that most of the exciting work to be done in numerical relativity still lies ahead. Indeed, the fact that the specific calculation showed required 70,000 CPU hours is a sobering fact, even given the capacity of contemporary computers.

Before proceeding to a conference summary kindly provided to us by Carsten Gundlach, the organizers would like to note that we know that we speak on behalf of *all* of the participants in terms of our appreciation for the absolutely brilliant way that the center has been conceived and is operating. We all look forward eagerly to our next trip to BIRS!

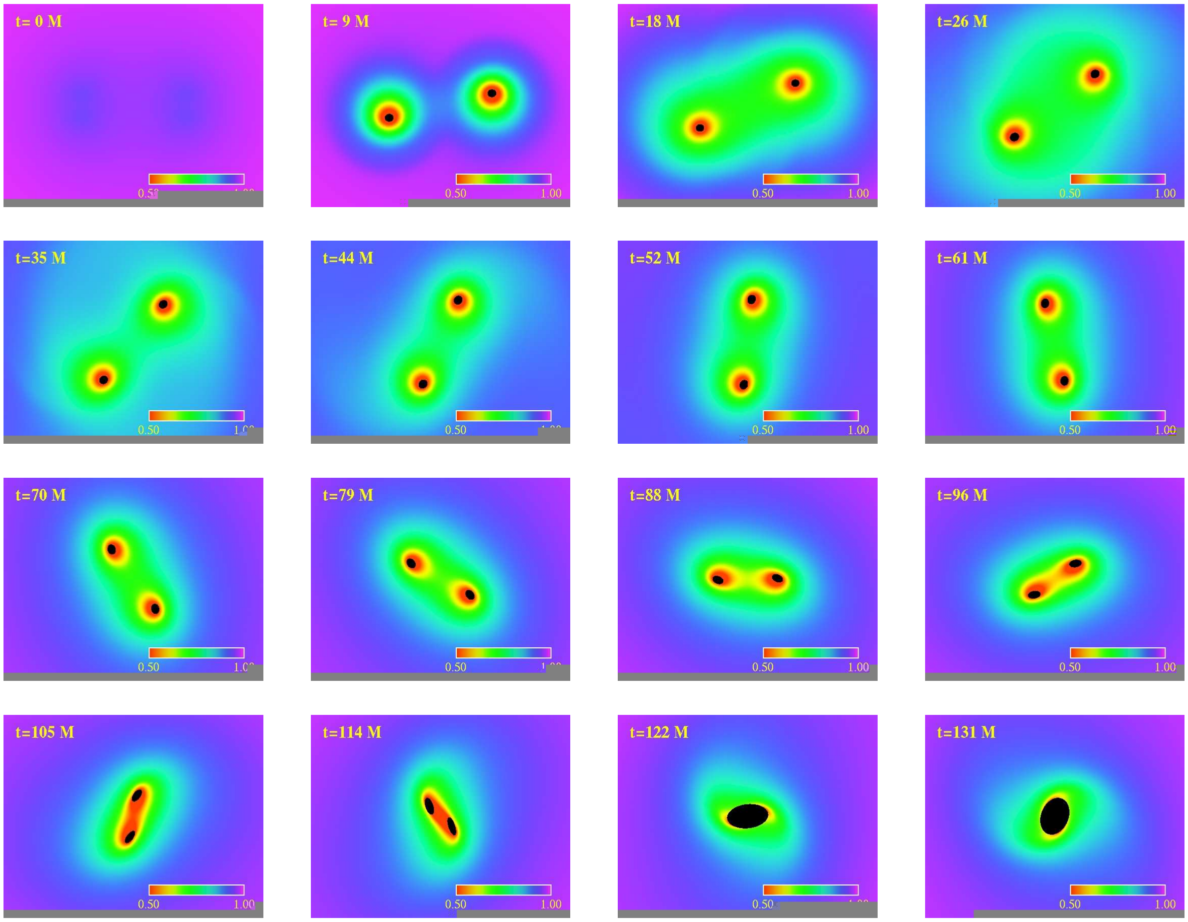

Figure 1: Early time evolution of the lapse function, $\alpha(t, x, y, 0)$, from the "medium-resolution" equal-mass black hole collision described in [**?**]. Time is measured in units of the mass, M, of either of the initial black holes. Values of the lapse function vary from about 0.5 near the black hole horizons to approximately 1.0 at large distances from the holes. See text for further discussion.

2 Conference Summary

Reprinted from *Matters of Gravity* (the newsletter of the Gravitation Topical Group of the American Physical Society), Fall 2005 issue.

The workshop was organised by Doug Arnold, Matt Choptuik, Luis Lehner, Randy LeVeque and Eitan Tadmor, with the purpose of bringing together researchers in GR working numerically and analytically. 20 invited half-hour talks were given over 4 days, with plenty of time for discussions between talks, over meals, and in the evening.

The BIRS page on the programme can be found on

`http://www.pims.math.ca/birs/`

and Matt Choptuik's page including PDF files of talks is

`http://bh0.physics.ubc.ca/BIRS05/`

To complement this, I shall highlight only a few of the talks.

In the 1990s, some researchers were concentrating on obtaining physics insight from effectively 1+1 dimensional problems: what cosmological spacetimes with two commuting Killing vectors can tell us about the nature of generic singularities (Berger and collaborators), and what we can learn

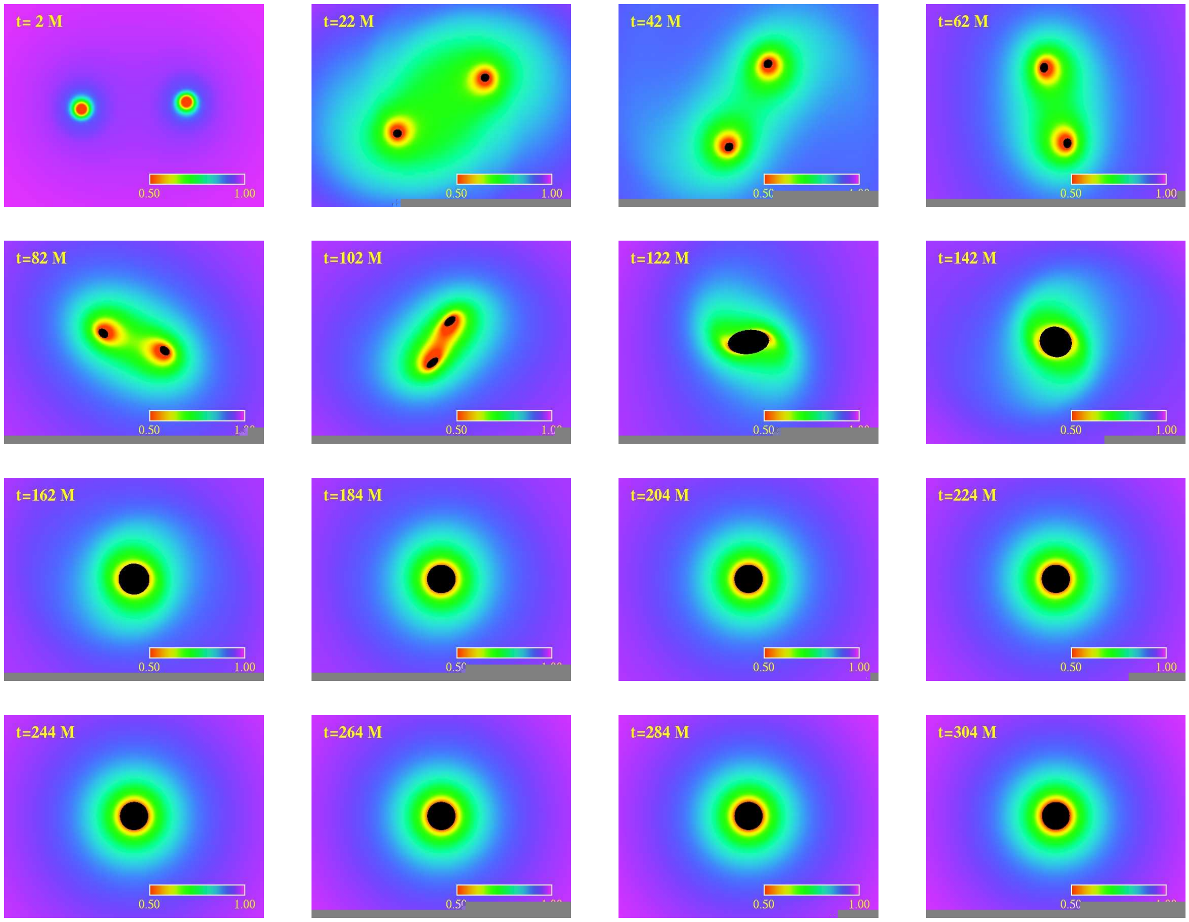

Figure 2: Complete time evolution of lapse function, $\alpha(t, x, y, 0)$, from the "medium-resolution" black hole collision described in [**?**]. The ability of the code to evolve stably for many dynamical times after the merger is evident.

about cosmic censorship from spherical collapse (Choptuik and students). More ambitious, axisymmetric or 3D, work confronted overlapping problems hard to disentangle in the low resolution available in 3D. In particular, instabilities already present in the continuum problem were not clearly distinguished from those added at the discretisation stage. The Banff meeting showed that now at least we have a clearer view of the problems facing us.

3+1 approaches need to start from a well-posed initial-boundary value problem in the continuum, with boundary conditions that are compatible with the constraints. Well-posedness can be proved by energy methods, based on a symmetric hyperbolic form of the field equations. *Olivier Sarbach* drops the energy estimate based on the symmetriser in favour of a "physical" energy plus a constraint energy. The remaining "gauge" energy is estimated separately using elliptic gauge conditions. This intuitively appealing programme has been completed for electromagnetism, although the gauge seems a bit restrictive. Work with Nagy is under way on general relativity. By contrast *Oscar Reula* emphasised that strong hyperbolicity is often enough. He could prove that whenever a first-order system subject to constraints is strongly hyperbolic (eg the BSSN formulation) then so is the associated constraint evolution system. *Heinz Kreiss* surprised some of his disciples in the numerical relativity community by also stressing that energy methods are too limited. In a series of examples, he proposed a general approach based on reducing initial-boundary value problems to half-space problems with frozen coefficients and analysing the dependence of each Fourier mode on its initial and boundary data.

On the numerical methods front, *Manuel Tiglio* reported on collaborative work to discretise systems of first-order strongly hyperbolic equations on multiple touching patches (for example 6 cubes to form a hollow sphere), using summation by parts and penalty methods. Their animations of toy problems looked very impressive, and the whole technology will be available as a general tool through the Cactus infrastructure. *Michael Holst* and *Rick Falk* gave review talks on finite elements for both elliptic and evolution equations. This is promising for nontrivial domains, but has not yet been applied to numerical relativity.

Other talks showed what 3D simulations can do. *David Garfinkle* reported on simulations of cosmological singularities without any symmetries on T^3. The key elements of his approach are the use of inverse mean curvature flow slicing ($\alpha = 1/K$) and a tetrad and connection formulation used successfully by Uggla and coworkers in analytical studies. His results are compatible with the BKL conjecture, although soon the resolution becomes too low to follow the development of ever more decoupled Bianchi IX regions. *Thomas Baumgarte* summarised the state of the art in binary neutron star simulations by himself and others, notably Masaru Shibata. There seems to be no real showstopper for such simulations. Rather what is needed now is more resolution, and the modelling of physical phenomena such as neutrinos, viscosity, and magnetic fields. Interesting results include the formation in binary mergers of a hot neutron star held up only by differential rotation, and expected to collapse later.

The most noted talk of the meeting was that of *Frans Pretorius* giving preliminary results on binary black hole mergers using harmonic coordinates. His simulations no longer seem to be limited by instabilities, but rather by computer power and time, and by unphysical initial data (there is evidence that his initial data are very far from circular inspiral data). The key ingredients seem to be the following: a working 3D AMR code on still massive computers, compactification of the Cartesian spatial coordinates (that is, at i_0) together with damping of outgoing waves, modified harmonic coordinates, and a damping of the harmonic gauge constraint through lower order friction terms (Gundlach). Generalised harmonic gauge (Friedrich) is $(x^\mu)^{;\sigma}{}_{;\sigma} = H^\mu$, where the gauge source functions H^μ are treated as given functions. Pretorius makes H^0 obey a wave equation $\left(H^0\right)^{;\sigma}_{;\sigma} \sim \alpha - 1$, which prevents the lapse from collapsing without affecting the well-posedness. This works less well for critical collapse.

3 Additional Information

3.1 Formulations of the equations

The tensorial nature of the Einstein equations allows a multitude of different formulations as partial differential equations to be discretized and solved numerically. Most numerical work is based on a $3 + 1$ formulation of the equations, in which spacelike slices of the space-time metric are advanced in a timelike direction.

There is considerable freedom in how one chooses the time slicing, as well as the spatial coordinates within the constant-time surfaces, and these choices can have a major impact on the overall efficacy of the solution algorithm. For instance, poor coordinate choices can lead to the development of coordinate shocks or other nonphysical pathologies, or can result in the simulation encountering physical singularities. All of these outcomes must be avoided in practice.

The hyperbolic equations are coupled through highly nonlinear source terms and this has hampered the calculation of energy growth estimates at the continuum level. Related to this is the fact that the stability of the numerical algorithms used is often hard to predict and/or achieve. For instance, simply maintaining a steady state numerically in a stable manner can be a challenge for many existent codes.

3.2 Constraint preservation

The hyperbolic system is also coupled with elliptic constraint equations between the dynamical variables: at the continuum level, these constraints are automatically preserved in time (in the boundary-free case) provided that they are satisfied at the initial time. At the discrete numerical level, so called free-evolution schemes [1] in which the constraints are only explicitly solved at the initial time are often susceptible to instabilities that lead to manifest violations of the constraints. A variety of options to address this problem are currently being considered, and no completely satisfactory solution exists yet, although, as Pretorius' work demonstrates, solutions continue to be available on a case by case basis. The situation here is analogous to the case of magnetohydrodynamic equations (MHD), or any other set of equations incorporating Maxwell's equations for the magnetic field, in which the divergence of the magnetic field must converge to zero in the continuum limit, and where it has been found that schemes in which the divergence constraint is treated exactly, or as exactly as possible, have *a priori* superior stability properties. In this case a variety of different approaches have been studied in detail and are still competing. Significant complications here relative to the Maxwell case include the facts that the Einstein constraints

$$G^{0\mu} = 8\pi T^{0\mu} \tag{1}$$

are tensorial and non-linear.

3.3 Black hole excision

In computing waves generated from black hole collisions, the event horizons of the black holes are moving boundaries that must be computed numerically, and where suitable numerical boundary conditions must be imposed. This can be particularly challenging during the merging phase, but again, the empirical evidence suggests that this can now largely be viewed as an "engineering" problem. In particular, Pretorius' results (again, see Figs. XXX and XXX and note that the "blacked out" regions are literally regions that have been excised from the computational domain. Also note that the interior of the computational domain; i.e. the union of the exteriors of the excised regions— is *many* grid points (**FRANS: How many, typically?**) within the apparent horizon. provide a brilliant confirmation of the efficacy of the idea first espoused by Unruh to his graduate students in the early 1980's.

3.4 Outer boundary conditions

[compactification]

3.5 Singularity formation

The fact that gravitational collapse of stars can lead to the formation of black holes follows from theorems of Hawking and Penrose, but these theorems give little information about the structure of the singularities and numerical simulations are being used to fill in some of this knowledge. As a singularity is approached, some terms in the equations blow up and others are negligible by comparison. The BKL conjecture [reference?] states that the time derivatives become more important than spatial derivatives and that at each spatial point the dynamics approach that of a homogeneous solution. David Garfinkle reported on some work on Gowdy spacetimes, a cosmological model in which the spatial derivatives in the metric are multiplied by decaying exponentials. [References, also to Andersson's work?]

Simulations show that spatial derivatives become negligble almost everywhere but that spikes form at some locations and the dynamics become a sequence of epochs and bounces. Most work so far has been on the $1 + 1$ spherically symmetric case using many grid points to capture the spikes. In more dimensions adaptive refinement would be very useful to capture the analogous behavior, which could be co-dimenison 1 surfaces or 1-dimensional spikes. [??].

[1] Here the interested reader is urged to consult the classic paper by Piran [**?**] which describes his detailed study of *cylindrically symmetric* (and thus involving 1+1-D PDEs) systems in general relativity. In this seminal work, Piran not only defines the concepts and nomenclature such as "free evolution" and "constrained evolution" that subsequent generations of numerical relativists would use, he rather exhaustively investigates various implementations of various schemes. Largely on the basis of empirical evidence he draws a conclusion that remains true to this date. Schemes in which the constraints are explicitly satisfied have a higher chance, a priori, of being stable when treated with "sensible" discretizations. (e.g. centred, second-order finite difference schemes with Crank-Nicholson time differencing, than those in which the constraints are explicitly satisfied *only* at $t = 0$.

3.6 Pretorius' binary black hole evolutions

As mentioned in the introduction above, the highlight of the meeting, at least from the numerical relativity vantage point, was Frans Pretorius' talk on his ongoing computation of black hole mergers using his recently developed "generalized harmonic" code. As also mentioned, Pretorius' current effort is quite notable for the number of approaches and techniques that it contains that represent departures from "recommended" or "traditional" practice.

Following Friedrich [**?**] and Garfinkle [**?**], Pretorius chooses coordinates $x^\mu \equiv (t, x^i)$ to satisfy

$$\nabla_\alpha \nabla^\alpha x^\mu = H\left(x^\mu\right) . \tag{2}$$

where ∇_α is the spacetime covariant derivative operator compatible with the metric ($\nabla_\alpha g_{\mu\nu} = 0$) and the H^μ are four *functions*, which are to be (and must be!) specified completely in order to fix the "gauge". They are also to be viewed as *independent* of the fundamental dynamical variables, which in this approach are simply the metric components, $g_{\mu\nu}$ themselves.

The traditional harmonic coordinates, used for example in Choquet-Bruhat's pioneering work in the late 50's on (local) existence and uniqueness of solutions of the Einstein field equations, have also been used in numerical relativity, but have *not* had a major impact. The reason for this is basically that one exhausts *too much* coordiante freedom in demanding

$$\nabla_\alpha \nabla^\alpha x^\mu = 0 . \tag{3}$$

Specifically, in the Cauchy problem context one gets to specify (and thus *must* specify!) precisely 6 numbers per spatial gridpoint at $t = 0$, namely x^i and $\partial_t x^i$, and from then on (3) and (3) determines the coordinatization of the spacetime as it is constructed, constant-time slice by constant-time slice. This means that *anything* that we are to "encode" in x^μ to allow the coordinates, for example, to gracefully cross a horizon with little or no fuss (e.g. in Schwarzschild, ingoing Eddington-Finkelstein coordinates, no fuss; usual Schwarzschild, much fuss), must be "encoded" in the initial slice. (Bear in mind that it will be generally impossible, for example, to tell whether a given intial data set will evolve into a black hole without actually performing the full dynamical evolution).

Thus, at least heuristically, it seems that with harmonic coordinates we exhaust *too much* coordinate freedom at the initial time: we specify x^i and $\partial_t x_i$ on the initial time slice, then hope for the best. This is very much at odds with the *successful* general approaches to coordinate choices for black holes spacetimes which are invariably based on a *local* response to a *local* solution feature, even though the governing PDE for the coordinate function may turn out to elliptic. Examples here include maximal slicing

$$\Delta\alpha = \alpha\left(Q\right) \tag{4}$$

which will, as is well known, drive α *to*0 in the vicinity of a black hole, and ingoing Eddington-Finkelstein (t, r, θ, φ) in spherical symmetry, where r is chosen to be areal (i.e. proper area $=$ $A(r) = 4\pi r^2$), and then t is chosen so that the ingoing tangent vector, $\partial/\partial u$:

$$\frac{\partial}{\partial u} \equiv \frac{\partial}{\partial t} - \frac{\partial}{\partial r} \tag{5}$$

is *ingoing null*

$$g_{\mu\nu}\left(\frac{\partial}{\partial u}\right)^\mu \left(\frac{\partial}{\partial u}\right)^\nu = g_{\mu\nu}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r}\right)^\mu \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r}\right)^\nu = 0 \tag{6}$$

Friedrich's significant insight then—which was then first implemented numerically by Garfinkle as part of his ongoing program of study of the nature of final (cosmological) singularities—was to note that by introducing the source functions H^μ, and then treating the H^μ as *formally* (if not always *operationally*) independent, one recovered the full original *four* degrees of coordinate freedom per spacetime point, *but* that, crucially, the resulting form of the Einstein field equations shared with the "true" harmonic form, the fact that the principal part of the operator acting on the dynamical variables (again, the $g_{\mu\nu}$) is precisely a hyperbolic wave operator. Specifically, we have

$$g^{\gamma\delta} g_{\alpha\beta,\gamma\delta} + 2 g^{\gamma\delta}{}_{,(\alpha} g_{\beta\delta,\gamma} + 2H_{\alpha,\beta} - 2H_\delta \Gamma^\delta{}_{\alpha\beta} + 2\Gamma^\gamma{}_{\delta\beta}\Gamma^\delta{}_{\gamma\alpha} + 8\pi\left(2T_{\alpha\beta} - g_{\alpha\beta} T\right) = 0 . \tag{7}$$

A further crucial insight due to Pretorius came about as a result of his considerable experimentation with various choices for H^μ in *strong field dynamical cases*, most often involving one or more black holes formed through prompt collapse of scalar field packets [**?**]. After trying various strategies that involved setting H^μ directly without much success, Pretorius decided to look at what the generalized harmonic condition had to say about the lapse and shift, $\alpha(t, x^i)$ and $\beta^j(t, x^i)$, even though those "3+1" quantities do *not* explicitly appear in the formulation. Pretorius shows that, writing the metric in the usual 3+1 form ("usual" modulo the use of h for the $3 - metric$ rather than γ or g)

$$ds^s = -\alpha^2 dt^2 + h_{ij} \left(dx^i + \beta^i dt\right) \left(dx^j + \beta^j dt\right) . \tag{8}$$

Then, given (2), we have

$$H \cdot n \equiv H_\mu n^m u = -n^\mu \partial_\mu \ln \alpha - K \tag{9}$$

$$\perp H^i \equiv H_\mu h^{i\mu} = \frac{1}{\alpha} n^\mu \partial_\mu \beta^i + h^{ij} \partial_j \ln \alpha - \Gamma^i{}_{jk} h^{jk} \tag{10}$$

where n^μ is the unit, future-directed normal to the spacelike hypersurfaces, $K \equiv h^{ij} K_{ij}$ is the mean extrinsic curvature, and the $\Gamma^i{}_{jk}$ are the Christoffel symbols computed with respect to **XXX? Frans?**

These last relations can be recast in the form

$$\partial_t \alpha = -\alpha^2 H \cdot n + \ldots \tag{11}$$

$$\partial_t \beta^i = \alpha^2 \perp H^i + \ldots \tag{12}$$

and from these equations, Pretorius has come up with several prescriptions that effectively generate "good" choices for H^μ in the context of "real" calculations such as the collision of two black holes. Specifically, although in his calculations to date spatialy harmonic (or perhaps one should say "morally spatial harmonic", since Pretorius actually uses the conditions $H_i = 0$, rather than $H^i = 0$). In all cases the physical intuition is to stop the lapse from collapsing, as it is prone to do if the temporal coordinate is "too harmonic". If the lapse collapses indefinitely and rapidly (typically exponentially) to 0, then there is a good chance that the solution is enountering a *coordinate* pathology. In addition, even if there ultimately is *no* true pathological behaviour ((such as $\alpha(t, r_p) \to 0$, Pretorius has observed that values of α that are "too low" are apparently correlated with numerical instability; so, again, the indicated rule-of-thumb is "keep your lapse up" (clearly, there is a great deal of mathematics to be done here!).

Specifically then, Pretorius finds that the following choices for the evolution of the temporal "gauge function" (source function), H_t, help maintain $\min \alpha(t, r) \sim 0.5$, even in the vicinity of trapped surfaces, during rather generic black hole collisions.

$$\tag{13}$$

Again, what is especially notable about this development is that without thinking about things from a "3+1" perspective it clearly would have taken Pretorius *much* longer to come up with "good" coordinates for the binary problem.

Another continuum-level technique incorporated in Pretorius' new code is a successful realization of an idea that many researchers have been expounding and investigating in recent years. The core idea, again, at least at the heuristic level, is to add "constraint damping" terms to the evolution equations so that one starts with initial data, that satisfy (some discrete version of) the constraints, and then subsequently develop that data to the future, *without* reimposing the constraints (i.e. if we perform *free evolution*), then any modes in the system that drive the solution away from constraint-satisfaction are highly damped. In this instance it is not the Hamiltonian and momentum constraints on which one focuses (although they are intimately related to the constraints which *do* consider), but the generalized harmonic conditions themselves.

Specifically, defining, C^μ, via

$$C^\mu \equiv H^\mu - \nabla^\alpha \nabla_\alpha x^\mu \tag{14}$$

the constraints are simply

$$c^\mu = 0\,. \tag{15}$$

Another small miracle that occurs in Pretorius' implementation of this idea in his "generalized harmonic code" is that the technique, as worked out most recently by workshop participant Gundlach and his collaborators, following early work by XXX and YYY, apparently works even better in the generic 3-D collision of two black holes that in has in any of the model problems in strong field gravity to which it has been applied.

The constraint damping modification of the equations of motion is quite straightforward. Defining the covector field, n_μ, via

$$n_\mu \equiv -\alpha \nabla_\mu t \tag{16}$$

the new equations of motion are given by (see (**??**) for the terms denoted ...)

$$\tag{17}$$

where the adjustable constant, κ controls the strength of the damping and can be determined empirically. Pretorius also supplies an argument that gives an appropriate order of magnitude for κ based on dimensional grounds. The final continuum-level approach that appears cruicial to Pretorius recent success is his use of a compactified spatial domain, both at the continuum *and* discrete levels. In the discrete case, this compactification, combined with Pretorius use of $O(h^2)$ finite difference approximations to the equations of motion, and, *cruically*, his use of the now familiar "Kreiss-Oliger" type dissipation, which acts as a low-pass filter and effectively quenches any and all outgoing radiation at large distances, while minimizing "finite resolution" backscatter. The use of spatial compactification avoids a whole host of problems that arise when one attempts to model a scenario such as a black hole collision as a mixed initial-boundary value problem. Although participants in this workshop, most notably Helmut Friedrich, have now worked out necessary and sufficient conditions for the mixed problem to be well posed, it is safe to say that the translation of these conditions into operational prescriptions for numerical relativists remains far from being adequately worked out.

As important as the various continuum level techniques described above are to the success of Pretorius' calculations, the importance of his use of algorithms with near-optimal scaling properties can not be underestimated. While researchers with unigrid codes can only worry about the basic factor of 16 in computational cost that accompanies a factor of 2 change in resolution, and use "optimization" techniques to get their operation counts per space-time gridpoint and thus the computational cost of the calculation down by a few, at best, AMR provides literally (many) orders of magnitude in computational efficiency. Since there is never a "magical" resolution at which some physical phenomena is "well resolved" using any specific finite difference scheme, once one has AMR, traditional optimizations are largely irrelevant, since a change in overall resolution by a factor of 2, which will *never* impact our ability to extract physics, will cost a factor of 16, which is entirely likely to mask any constant factor speed ups due to traditional speed-ups (including, one should note, such techniques as only using two iterations in iterated Crank Nicholson, rather than iterating to some fixed convergence criteria, as common-sense and defensive numerical analysis would have us do.)

Almost as important is the fact that Pretorius code is built on top of what one might call a "second generation" parallel-ready software infrastructure for finite difference calculations on block structured AMR meshes. The term second-generation is appropriate, since Pretorius studied carefully the extant implementations that were designed to provide MPI-based parallel support for adaptive finite difference computations. Specifically, he looked in detail at Parashar and Browne's DAGH (now called GRaCE), a parallel infrastructure that was specifically commissioned by, and coded for, the US numerical relativity community during the BBH project (workshop particpants: BBH PIs Choptuik, Laguna, Winicour; BBH Collaborators/PDFs/Grad Students: Arnold, Holst, Friedrich, Baumgarte, Lehner, Gundlach, Hirschmann). Pretorius reanalyzed some of the motivating factors for certain design decisions (most notably the use of space-filling curves (Peano-, Hilbert) to "linearize" the storage on a Berger & Oliger-style mesh hierarchy, the rationale being to keep grid values on distinct levels of discretization, but defined at the same physical space-time point, close to one another in storage (see any of Parashar's presentations from that era; they will contain the exhortation "Locality, locality, locality".)) Pretorius major contribution, which is a new implementation of middleware to isolate the application programmer from the burden of MPI programming, started with a detailed investigation of a mock-up of the refinement pattern likely to result from the use of Berger & Oliger AMR, where he discovered that the locality afforded by the use of space-filling curves, which is only an issue for interlevel transfers (and which, without loss of generality, we can assume to be between two levels with discretizations h_ℓ and $h_{\ell+1} = h_\ell/2$), would typically be completely swamped by the basic interior computation costs. In other words, due to the extreme "computational intensity" (quantified, for example, in terms of the number of floating point ops per space-time grid-point-update: this measure can exceed 10^5 for certain discrete implementations of the Einstein equations) expected of the general calculations, *all* reasonable amounts of interlevel communication work are small (the old volume *vs* area argument), and thus, at least for calculations such as the BH-BH collision, where the features that exist and/or develop on a wide variety of scales nonetheless *remain volume filling on their own scale* (so that there is no need for the rotated refinments discussed in the late 70's by Brandt, and implemented by Berger & Oliger in their 1984 JCP paper), there was no need to fuss with the, elegant and interesting, but rather complicated-to-implement and debug approach used by Parashar and Browne. Rather, in the coding of his new library (`C`-language application program interface (API)), while still retaining much of the formal, axiomatic coding style for objects such as bounding boxes found in DAGH and various antecedents such as LPARX `libpamr.a`, Pretorius distributes storage across processors on a SIMD distributed machine *disretization level by discretization level*. Note that this means that the data at different levels of discretization, but at a common physical space-time point *will* live on distinct processors in general, in contrast to the DAGH scheme where *locality, locality, locality* was a mantra. Again, any small overhead/loss of efficiency inherent in the level-by-level approach (and it is not even clear that, theoretically, we *should* expect a loss of efficiency for the problems that we are considering) *vs* the space-filling-curves approach is completely incidental since the code is *anything* but communication-bound.

In practice, Pretorius has seen near-perfect scaling and optimal processor usage on all machines with "fast" interconnects (Myrinet, Quadrix, scaling here to several dozen processors), and very good scaling and reasonable processor usage on GigE machines (scaling here to 200 or more processors). It is crucial to note that these near-ideal speed-ups are seen in the context of *fully adaptive* (in both space and time) finite-difference solutions in 3 spatial dimensions. Other parallel-enabling infrastructure in wide use in the numerical relativity community, most notably Cactus, and the various "mesh refinement" (FMR, box-in-box, etc) extensions thereof, does not exhibit good scaling to our knowledge.

4 Critical phenomena

Although the prime focus of the numerical relativity community continues to be the solution of the compact binary problem (collisions of black holes and/or neutron stars), there is also substantial continuing interest in black hole critical phenomena.

Black holes from from gravitational collapse, whereby, in very loose terms, the gravitational field within the collapsing object (in astronomical contexts, typically a core of "nuclear ashes" too large to be supported by neutron-degeneracy pressure), becomes strong enough to "trap light". Due to the universally attractive nature of the gravitational field (for matter sources with positive energy), and the fact that the gravitational field acts as its own source in general relativity, it is reasonably intuitive that the collapse process per se is unstable or irreversible in the sense that, once it gets going it keeps going. Thus gravitational collapse is an inherently dynamical process, and, further, one in which the gravitational fields are, by definition, just about as strong as they can be without being hidden from external view.

The investigation of black hole critical phenomena takes consideration of gravitational collapse a step further and asks "What happens precisely at the threshold of black hole formation?". Such a threshold can be itself precisely defined by considering parametrized families of solutions representing collapse scenarios, so that the family p, controls the overall strength of the gravitational field as the collapse proceeds. In particular, in critical collapse, the family is chosen so that as one tunes p from low values (say $p \to 0$) to high (say $p \to \infty$) one goes from situations where no black holes form to those holes do form. For any such family, there is then a *critical* parameter value, $p = p^\star$ at which black hole formation sets in, and solutions that appear precisely at criticality are known as critical solutions.

The phenomenology that is observed near-criticality has rather precise analogues in the critical phenomena familar from the statistical mechanical study of phase transitions, and, indeed, viewing the black hole mass as an order parameter, we see the analogues of both first and second order phase transitions in these studies. The second order case, in which black hole formation is conjectured to"turn on" at infinitesimal mass, has attracted the most interest, not least since the critical solutions represent violations of certain weak forms of cosmic censorship.

At our meeting, an informal discussion group was held to debate whether or not one really *can*, in principle, make arbitrarily small black holes using, for example, a spherically-symmetric, massless scalar field as a matter source. The conventional answer to this question, is "yes", as first conjectured by Choptuik in the early 90's. However, recent caculations by participant Aichelburg and collaborators, using an approach that, in contrast to Choptuik's early calculations, has a computational domain that includes future null infinity, which, at least according to a large group of "die-hards" in the relativity community, is the best place to talk about the amount of mass in a system that gets radiated away as massless radiation. Aichelburg's calculations suggest that for any given family there will, in fact, be a *minimum* mass black hole that can be formed, apparently due to radiation backscatter effect. However, as became very clear in our debate, Aichelburg's claim apparently leads to other assertions about the asymptotic behaviour of the solution that don't seem to make physical sense, and it is probably fair to say that there were no converts one way or the other re whether or not Aichelburg was seeing a "real" rather than "numerical" effect.

5 Research interests of participants

Each participant in the workshop was invited to submit a brief summary of their research interests in advance. These were made available on the workshop webpage and in the lounge during the workshop.

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Fine structure phenomena in critical collapse [?] New numerical studies of critical phenomena in black hole formation for the spherically symmetric SU(2)-sigma model (wave maps) coupled to gravity. An interesting feature of this model is that the nature of the critical behavior depends on the coupling constant α characterizing the strength of gravitational interaction: for small α the critical solution is continuously self-similar (CSS), for large α the critical solution is discretely self-similar (DSS), while in-between there is a region of so called episodic self-similarity where a competition between the CSS and DSS behaviors is observed. In order to understand better the character of the CSS/DSS transition we performed high precision computations near the bifurcation point and found an unexpected fine structure. In this region, given an interpolating family of initial data parametrized by p, there is an infinite series of critical values $p_n \to p^*$ such that for each p_n one has the CSS critical behavior with n episodes, while for p^* one has the DSS critical behavior. This is also reflected in the corresponding black hole mass scaling. An explanation of this phenomenon in the language of dynamical systems theory is conjectured.

News from Critical Collapse: Bondi Mass, Tails and Quasinormal Modes [?] This research tries to answer the question how critical collapse "looks from far away". What are the characteristic features to be observed? We analyze global aspects of the best studied model: the self-gravitating massless scalar field in spherical symmetry, however in a compactified context. Our evolution system is based on Bondi coordinates, the mass function is used as an evolution variable. Radiation quantities like the Bondi mass and news function are calculated and we find that they reflect the DSS behavior. Surprisingly, the period of radiation at null infinity is related to the formal result for the leading quasi-normal mode of a black hole with rapidly decreasing mass. Furthermore, our investigations shed some light on global versus local issues in critical collapse, and the validity and usefulness of the concept of null infinity when predicting detector signals.

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Construction and validation of constraint-preserving boundary conditions for hyperbolic formulations of Einstein's equations, well-posedness of differential boundary conditions, long term stability estimates for Einstein's system, hyperbolic formulations, energy estimates for systems first order in time second order in space.

Differential constraint-preserving boundary conditions. Consider a vector wave equation in a bounded domain Ω subject to a differential constraint (indices are raised with the flat metric)

$$\partial_t u_i = \partial^l \partial_l u_i, \qquad \partial^i u_i = 0. \tag{18}$$

Dirichlet data for system (18) can not be set arbitrary, since not every solution to the wave equation is automatically divergence-free. Even the "visually divergence free" Dirichlet data, $u_i|_{\partial\Omega} = (\operatorname{curl}\varphi)_i|_{\partial\Omega}$, where φ_j is a vector defined in the neighborhood of the boundary, is constraint-compatible only in special cases.

Instead, let n_j be the unit normal to the (flat) boundary, vectors m_j and l_j complement n_j to an orthonormal triple. Consider a vector function g_j defined on the boundary, with a property $g \cdot n = g^j n_j = 0$, the proposed boundary conditions are ($(n \times u)_i = \varepsilon_i^{pq} n_p u_q$ is the usual cross product)

$$n \times u|_{\partial\Omega} = g, \quad \frac{\partial}{\partial n} u \cdot n|_{\partial\Omega} = \frac{\partial}{\partial l} g \cdot m - \frac{\partial}{\partial m} g \cdot l. \tag{19}$$

These boundary conditions are constraint-compatible and enforce the constraint through the second expression in (19).

Long term stability for Einstein's equations. The first BSSN equation reads (k is the extrinsic curvature, indices are raised with the inverse spatial metric)

$$\partial_0 k_i^i = a k^{li} k_{li} - D^l D_l a. \tag{20}$$

Unless the lapse function a is chosen with care, equation (20) is expected to be unstable. For example, for a spatially independent lapse and zero shift vector, equation (20) yields an estimate $\partial_t k \geq \frac{1}{3} a k^2$ which implies that $k \geq [(1/3)\int_0^t a(\tau)\mathrm{d}\tau + 1/k(0)]^{-1}$, or that the solution k is unbounded in a finite time, which is a well-known example of a coordinate singularity.

Energy estimates for systems first order in time second order in space. The following is the energy identity for the linearized BSSN system with the densitized lapse ($\|\ \|$ is the L_2-norm)

$$\begin{aligned}
&\partial_t \Big[\|\kappa\|^2 + 36\|\partial_l \varphi\|^2 + \|\Gamma_j - 8\partial_j \varphi\|^2 + \|A\|^2 + \|\frac{1}{2}\partial_l \tilde{\gamma}_{ji} - (\Gamma_{(i} - 8\partial_{(i}\varphi)\delta_{j)l}\|^2 \Big] \\
&= \ -6\int_{\partial\Omega} (\frac{\partial}{\partial n}\varphi)\kappa - \int_{\partial\Omega} (\frac{\partial}{\partial n}\tilde{\gamma}_{ij}) A^{ij} + 2\int_{\partial\Omega} (\Gamma_i - 8\partial_i \varphi) n_j A^{ij}.
\end{aligned}$$

Partial references. A.Alekseenko [**?**], A.Alekseenko and D.Arnold [**?**], L.Lindblom *et al* [**?**], C.Gundlach and J.M.Martin-Garcia [**?**, **?**], N.Tarfulea [**?**], G.Calabrese *et al* [**?**], G.Nagy *et al* [**?**].

JAMES BARDEEN
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My focus has been on exploring the potential of tetrad-based formulations for numerical relativity, with particular emphasis on hypersurface-orthogonal gauges, in which tetrad congruence is kept orthogonal to the constant-t hypersurfaces. The acceleration of the tetrad congruence is then equal to the gradient of the lapse, and the extrinsic curvature tensor of the orthogonal three-space is symmetric. The lapse can be made dynamic, satisfying an equation of the Bona-Masso type, which induces a dynamical equation for the tetrad acceleration. I argue that the angular velocity of the spatial triad relative to Fermi-Walker transport should also be made dynamic, rather than being set to zero. In spacetimes such as Kerr black holes in which there is large differential frame dragging, triads propagating by Fermi-Walker transport may become highly twisted near the horizon. A simple dynamic gauge condition for the angular velocity is to assume the angular velocity is the gradient of a scalar potential. An evolution equation for this scalar potential, and a corresponding dynamical equation for the triad angular velocity, is proposed which allows twists in the spatial triad to relax by propagation at light speed away from where they are generated.

Evolution equations based on these gauge conditions can be made symmetric hyperbolic with a few simple substitutions of constraint equations. Both "Einstein-Ricci" systems, in which the primary variables are the Ricci rotation coefficients, and "Einstein-Bianchi" systems, in which the components of the Weyl tensor are promoted to independent dynamical variables evolved by the Bianchi identities, have been considered. All symmetries of the Ricci rotation coefficients and the Weyl tensor are enforced explicitly, which leads to propagation different from light speed for most of the "constraint" and "gauge" modes. This may be in important ways an advantage for numerical relativity, particularly in spacetimes with horizons.

So far these systems have been tested on vacuum 1-D spacetimes, Schwarzschild and non-linear colliding plane waves. In the Schwarzschild case, excellent accuracy and long-term stability can be obtained with the Einstein-Bianchi system and lapse gauge conditions which produce asymptotically hyperbolic spacelike hypersufaces. Such hypersurfaces have normals pointing outward relative to the coordinate grid at the outer boundary as well as at the inner excision boundary. This minimizes the need for incoming mode boundary conditions, and makes enforcing the constraints at the boundaries trivial. Stability is most impressive when the lapse gauge conditions give highly superluminal propagation in the vicinity of the event horizon for modes coupling the radial acceleration and longitudinal extrinsic curvature.

Luisa Buchman and Olivier Sarbach at Caltech are working on an implementation of the 3-D Einstein-Bianchi system within the framework of the Cornell-Caltech pseudo-spectral code.

ROBERT BARTNIK
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- Energy in general relativity:
  - quasi-local measures, geometric properties and numerical evaluation.
  - Hamiltonian phase space structure
  - Positive energy theorems

  Numerical relativity in characteristic coordinates:
  - structure of the hypersurface equations,
  - optimal metric and connection parameters,
  - asymptotic structure at null infinity,
  - interaction of gravitational waves with a black hole

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Over the past years I have worked on many different aspects of numerical relativity, including both the initial value problem and dynamical evolution calculations. I have been particularly interested in compact binaries [?], and have constructed initial data for both binary neutron stars [?] and binary black holes [?] in quasicircular orbit. I have worked on reformulations of the ADM equations for the dynamical evolution of the gravitational fields that have dramatically improved the stability of their numerical implementations [?, ?]. I have also worked relativistic hydrodynamics [?] for the dynamical evolution of binary neutron stars. More recently I have studied mixed black hole-neutron star binaries, and have constructed initial data describing such binaries in quasicircular orbit [?].

LUISA BUCHMAN

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My research interests are focussed on how to best formulate the Einstein equations, in order to (eventually) yield accurate, efficient, and long-term stable numerical evolutions of coalescing binary black holes for projects such as LIGO and LISA. In particular, I have been working on orthonormal frame based approaches. I have performed numerical experiments in 1D with a formulation which evolves the 24 Ricci rotation coefficients and the tetrad vector components, for an arbitrary orientation of the timelike congruence generated by the tetrad fields with respect to the constant-t spacelike hypersurfaces[1]. Currently, I am implementing an orthonormal frame based Einstein Bianchi system[2,3] in a 3D pseudo-spectral Caltech-Cornell code written by L. Kidder, M. Scheel, and H. Pfeiffer. In this system, the timelike congruence is explicitly orthogonal to the constant-t hypersurfaces. The work I am doing is in collaboration with James Bardeen, Frank Estabrook, and Olivier Sarbach.

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I'm interested in structure-preserving discretizations of partial differential equations. My background is in finite element discretizations of boundary integral equations for Maxwell's equations. More recently I have tried to relate Regge calculus to finite element techniques (especially Whitney forms) [?], I have developed tools for analyzing the convergence of some non-linear quantities under div-curl control in the Galerkin setting [?], and I have investigated constraint preservation in discretizations of the Yang-Mills equations [?].

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Full time numerical relativist since 1980. M.Sc. work (UBC 1982) concerned the initial value problem, corrected a minor error in numerical work by York and Piran, but, more importantly, was accomplished via an O(N) multi-grid solution with adaptive mesh refinement in one of the two coordinate directions, and was thus was the first published use in numerical relativity of both multi-grid and AMR (Choptuik 1986).

PhD (UBC 1986) concerned gravitational collapse in spherical symmetry, a topic which is still being studied by our research group, although calculations are increasingly often performed in axial symmetry (2D) or no symmetry (3D).

Once sufficient computer resources and visualization/analysis capacity were available, discovery of critical collapse in the early 1990's (Choptuik 1993) was the immediate and direct result of previous work to implement full Berger & Oliger AMR (1984) for spherical collapse in 1987-1988 in a vain attempt to use AMR to ameliorate coordinate problems encountered at late times in collapse ("lapse collapse", for example).

Starting in the mid-90's, and largely due to work with (and mostly by!) Steve Liebling and Eric Hirschmann, our group's focus switched from spherical (1D) to axial (2D) symmetry; Pretorius joined this effort in 2000, and, for his PhD thesis (UBC 2002) demonstrated fully critical collapse of a massless scalar field *without* restriction to spherical symmetry; follow-up work included non-spherically symmetric, but axially-symmetric critical solutions.

Work in general relativistic hydrodynamics started with the UT Austin PhD of David Neilsen (1999), followed by Scott Noble (UT Austin PhD, 2003) and Ignacio (Inaki) Olabarrieta (UBC PhD 2004) and Martin Snajdr (UBC PDF 2005-present) and primarily has focused on the rich phenomenology of critical collapse with a perfect fluid. In the near term, our group will be concentrating a great deal on 3D calculations with perfect fluids, with an aim to work on various aspects of neutron star physics and astrophysics, including inspiraling collisions and supernovae explosions.

Other pertinent interests include the design and implementation of high level software tools for our own use as well as the community's, to the extent that others want to use them. This inlcudes the special purpose language RNPL (Robert Marsa, UT Austin PhD 1995) as well as visualization/analysis tools `xvs` (Choptuik) and `DV` (Pretorius & Choptuik).

In general, as physicists, group members are most interested in *obtaining new solutions* of the Einstein equations (as well as those for whatever matter fields are present), and then understanding, so far as possible, the "physics" that underlies the results. Our work is characterized by an exclusive use of finite differencing (minimum second order if at all possible), as well as, following LF Richardson (Richardson 1910), an agressive use of convergence testing so that one ultimately can ultimately tackle the most general and difficult problems where calibration with a known (closed form) solution is either impossible or, more likely, relatively meaningless.

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My main interest in general relativity is concerned with the structure of gravitational fields in the large, with their conformal and their asymptotic structure, and with concepts of gravitational radiation from isolated systems. On the technical side this amounts to

– studies of the structure of the field equations under various conditions,
– the analysis of different representations of the field equations,
– the understanding of the 'conformal structure' of the field equations,
– the analysis of the constraints and the structure of initial data,
– the analysis of gauge conditions and their long time behavior,
– the reduction of initial and initial-boundary value problems for the Einstein and the conformal Einstein equations,
– the analysis of the interrelations of the systems involved in the reduction of initial or of initial-boundary value problems for Einstein's field equations,
– local, semi-global, and global existence proofs for solutions of Einstein's field equations under various assumptions.

May main goals are to control the evolution of gravitational fields in the large analytically and to reduce the problem of calculating entire space-times, including their asymptotic structure and their radiation content, to the calculation of solutions to finite Cauchy problems solely from the Cauchy data. Parts of this work are discussed in the survey articles [**?**, **?**]

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My research interest in numerical relativity involves trying to use simpler model problems with some of the same features to understand why standard numerical methods for the Einstein equations fail and to help design methods that overcome these problems. One particular issue that I am considering is in what sense to satisfy the constraint equations as time evolves. Although the constraint equations are automatically satisfied by the exact solution if they are satisfied initially, these equations are complicated, so in general will not be satisfied by a numerical approximation. One might hope, however, to design numerical schemes which preserve appropriate approximations to these identities. Such issues arise in a much simpler context in the solution of Maxwell's equation, where solutions which are initially divergence free remain so without explicitly enforcing this condition as an additional constraint. Good numerical methods for Maxwell's equations preserve a discrete divergence free condition on the approximate solution again without explicitly enforcing it. A related question is what effects the constraints have on the overall stability of the problem.

I am also interested in studying the use of the Regge calculus, a discrete version of general relativity, that has previously been used as a numerical approximation to the Einstein equations. The Regge calculus may be viewed as a type of finite element method and I am interested in exploring this connection with the aim of additional understanding and improvement of this approach to numerical relativity.

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I am interested in numerical implementation of Friedrich's conformal field equations. We are currently working on a 3D implementation to study the incidence of gravitational waves on a black hole. Recent investigations also include the analysis of constraint divergence in the Bianchi system. Furthermore, we are developing and testing methods to use discrete differential forms or even more general notions of discrete geometry in GR. We are currently investigating spherically symmetric and plane wave space-times.

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Much of my recent research in numerical relativity has been on (i) properties of singularities, (ii) critical gravitational collapse and (iii) cosmic censorship.

Singularities occur at the centers of black holes and at the big bang. Due to the singularity theorems, we know that singularities form in a wide range of circumstances; but these theorems give us very little information about the nature of singularities. To find the nature of singularities, I have performed numerical simulations of the formation of singularities. These simulations were first done in the case of symmetry; but recently I have done simulations of the case with no symmetry. The results support the so called BKL conjecture that the approach to the singularity is locally homogeneous and oscillatory.

Critical collapse, as first found by Choptuik, is the scaling properties of gravitational collapse at the threshold of black hole formation. I have investigated several aspects of this phenomenon including (i) scaling of tidal force for systems that just barely fail to form a black hole, (ii) critical gravitational collapse in spacetime dimensions other than 4, (iii) closed form solutions describing critical gravitational collapse (iv) critical gravitational collapse of a massive vector field and (v) an analog of critical gravitational collapse in Ricci flow.

Cosmic censorship is the question of whether the singularities that form in gravitational collapse are hidden inside black holes. Recently, a system of a scalar field with negative potential energy had been proposed as a counterexample to cosmic censorship. However, I performed numerical simulations of this system and showed that it is not a counterexample.

In addition to these studies, I have also (i) investigated the use of harmonic coordinates as a method for the numerical simulation of Einstein's equation and (ii) investigated (along with Pretorius and Lehner) the properties of the horizon of an unstable black string.

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I am trying to find ADM-like formulations of the Einstein equations, together with gauge choices and boundary conditions that make the initial-boundary value problem well-posed. Mostly I have been using energy methods to investigate well-posedness, and so I am looking for symmetric hyperbolic systems, in particular second-order in space and first-order in time ones. I am also trying to understand better how to define hyperbolicity for such systems.

More recently, I have become interested in finite-differencing methods for second-order in space systems, using mostly discrete energy methods. I am interested in boundary conditions in the presence of a shift vector. It should not be necessary to impose more or fewer boundary conditions, depending on the normal component of the shift. I am investigating these issues in the wave equation and electromagnetic toy models for gravity.

With Ian Hawke, I am just starting to look at gravitational waves from neutron star mergers and core collapse, with the twin aims of going beyond barotropic equations of state, and of improving gauge choices and boundary conditions.

I have worked a lot on critical phenomena in gravitational collapse, mostly by constructing critical solution using a self-similar ansatz, and then using perturbation theory to calculate critical exponents. Critical collapse in spherical symmetry is well-understood now, but we don't know much in less symmetry. In particular, interesting things will happen when one looks for the black hole threshold for rotating initial configurations. I am not doing rotating collapse simulations, but hope someone else will soon!

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My primary interest are the numerical simulation of the gravitational waves emitted by collapsing stars. Towards this end I am primarily interested in wave extraction ([**?**]), particularly Cauchy-characteristic extraction, simulating hydrodynamics, particularly with high resolution shock capturing (HRSC) methods ([**?**]), formulations of Einstein's equations for numerical relativity ([**?**]), and technical improvements to computational code including mesh refinement ([**?**]) and multiple-patch evolutions ([**?**]).

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Discrete differential forms. Discrete differential forms can be regarded as finite elements for differential forms: they provide piecewise polynomial p-forms with respect to a piecewise linear triangulation of the underlying manifold. Their simplest representatives on simplicial complexes are known as Whitney forms. In this case, they can be regarded as a way to extend p-cochains to proper p-forms.

For any polynomial degree a judicious construction of discrete differential forms can ensure that the exterior derivative d can be restricted to the discrete forms. The resulting discrete co-homology becomes isomorphic to the deRham co-homology.

Discrete differential forms allow the straightforward discretization of physical laws that can be expressed by means of differential forms, e.g. Maxwell's equations. The discrete model preserves algebraic identities arising from $d \circ d = 0$ as well as degrees of freedom arising from the topology of the triangulated manifold.

Discrete Hodge operators. Metric-dependent constitutive laws defy a canonical treatment in the context of discrete differential forms, because their formulation relies on the so-called Hodge operator, which lacks a clear discrete counterpart. Yet, a few fundamental algebraic requirements can be stated that have to be satisfied by meaningful discrete Hodge operators. It turns out that many discretization schemes ranging from primal and dual finite elements to finite volume methods and Galerkin approaches fit this framework and can be regarded as particular realizations of discrete Hodge-operators.

Discrete Lie derivatives. Quasistatic models for magnetic diffusion in conducting fluids lead to initial boundary value problems of the form

$$d \star d\omega = \pm \frac{d}{dt}(\star\omega) + L_{\mathbf{v}}(\star\omega) \ ,$$

where ω is a time-dependent p-form and $L_{\mathbf{v}}$ is the Lie derivative with respect to a given vector field $\mathbf{v}$. The issue is how to incorporate the Lie derivative into the calculus of discrete differential forms. How to do this on a semi-discrete level and in a fully discrete Lagrangian framework is research in progress.

Einstein-Dirac equations in spherical symmetry. The Einstein-Dirac equations for a 2-fermion system in the spherical symmetric case lead to a linear wave equation that propagates non-linear constraints. These arise from the conservation of total charge and ADM-mass. Current research aims at developing a stable numerical scheme that is charge conserving and prevents a drift of the mass.

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There is a tendency to write the equations of general relativity as a first order symmetric system of time dependent partial differential equations. However, for numerical reasons, it might be advantageous to use a second order formulation like one obtained from the ADM equations.

I have been involved in the study of the mathematical properties of the ADM equations and the study of the numerical methods for second order hyperbolic systems of partial differential equations.

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Numerical simulations and gravitational wave emission of excised black holes, extreme-mass-ratio binaries and tidal stellar disruptions by a massive black hole. Applications of the isolated and dynamical horizon framework to numerical relativity.

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Analysis both at numerical and analytical levels of Einstein equations (in Cauchy and characteristic forms) and their actual implementation to study strongly gravitating systems.
At the analytical level the understanding how different formulations of the the equations behave and the interplay of constraints and boundary conditions in the future evolution of a given system. In particular to learn how to exploit the freedom in the theory to single out re-formulations which might be better behaved at the implementation level and the coordinate freedom to simplify tracking the system. At the numerical level the search for a robust implementation of the equations which requires the use of carefully developed algorithms, non-trivial coordinate charts and the use of adaptive gridding techniques. At the application level my interests range from astrophysical systems –to understand their process and possible gravitational wave output– to investigations of higher dimensional scenarios –which arise naturally in string theory–. I collaborate on different aspects of the mentioned problems with M. Choptuik, D. Garfinkle, S. Liebling, F. Pretorius, O. Reula and M. Tiglio.

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I work on finite volume methods for solving hyperbolic systems of equations, software implementing these methods, and various applications where these problems arise. An on-going project is the further development of the CLAWPACK (Conservation LAW PACKage) software for solving hyperbolic systems [?]. This has recently been extended to a version for two dimensional manifolds specified in terms of a general metric, in work with James Rossmanith and Derek Bale [?]. These high-resolution methods are based on solving Riemann problems orthogonal to each face of a grid cell in a local coordinate system and using the resulting waves to update the solution in nearby grid cells. They include slope limiters that are particularly useful for problems with discontinuous solutions, such as shock waves, but that can also improve accuracy of smooth solutions by reducing dispersive effects. Outflow boundary conditions are often particularly easy to implement with these methods. Adaptive mesh refinement is also incorporated. More information can be found at
http://www.amath.washington.edu/∼claw/clawman.html

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Physics of compact objects: A significant portion of my present research effort is dedicated toward studying binary black hole systems and black hole/boson star interactions. The corresponding 3D/2D-axisymmetric code uses an evolution scheme based on generalized harmonic coordinates, a coordinate domain compactified to spatial infinity to facilitate the imposition of physically correct outer boundary conditions, and (dynamical) excision to deal with the singularities inside of black holes [?]. The numerical solution method is finite difference, with Berger and Oliger style (parallel) adaptive mesh refinement. Current matter sources are real and complex scalar fields, though a long term goal is to include fluid and electromagnetic fields.

Horizons in higher-dimensional spacetimes: The uniqueness theorems in 4D Einstein gravity state that the only stationary (vacuum) black hole solutions are the Kerr family; furthermore, these black holes are stable under dynamical perturbations. The situation in higher dimensional (string-theory motivated) spacetimes is much richer, with larger classes of black "objects" including black holes, strings, branes and rings, and interestingly many of these objects are not stable. A long-standing project of collaborators and myself is to discover the end-state of the evolution of a dynamically perturbed unstable black string. Our initial effort [?] hinted that the string my "fracture" into a sequence of spherical black holes, thought the code failed due (we think) to bad coordinate conditions at late times before we could ascertain the final state. A new effort is underway to include gauge conditions that are better able to follow the late-time dynamics of the spacetime, as well as use a better-behaved formulation of the field equations.

Critical phenomena in gravitational collapse: I am involved in an ongoing effort to explore the critical behavior that occurs at the threshold of gravitational collapse, in particular that of scalar field and pure gravitational wave sources. The threshold solutions, being discretely self-similarity in nature, are challenging to solve for numerically due to the larger range of relevant spatial and temporal scales that must be resolved during an evolution. Thus, aside from the fascinating physical aspects of the problem, critical collapse is an excellent test bed for numerical solution methods for the Einstein field equations.

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Numerical and analytic questions about evolution of hyperbolic systems. In particular: a) systems with constraints which are not very well preserved along evolution. b) problems with boundary conditions which are physically interesting, but nevertheless not of the maximally dissipative type. c) Evolution on manifolds with non-trivial topology and therefore where several coordinate charts, and therefore grids, are needed.

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An important goal in astrophysics is to model diverse phenomena such as the accretion of matter onto black holes and the interaction of colliding black holes. Although the metric is dynamically important in the black hole collision problem, while remaining relatively fixed in the accretion problem, a key piece in both of these examples is the evolution of the fluid variables. Under some simplifying assumptions, the fluid variables evolve according to the relativistic hydrodynamic equations, which form a system of hyperbolic balance laws that are strongly nonlinear and generically exhibit shock formation. In order to accurately simulate the evolution of such fluids, several types of numerical methods have been proposed, including finite difference, pseudo-spectral, smoothed-particle hydrodynamic, and various high-resolution shock-capturing methods. Although these methods each have their own advantages, perhaps the class of methods that has received the most attention in recent years is that of high-resolution shock-capturing schemes such as those based on Godunov, ENO (essentially non-oscillatory), and central schemes (see Font [**?**] for a detailed review of these approaches).

In our research, we are exploring an alternative to the high-resolution schemes currently in use in the astrophysics community. We are considering a class of methods known as residual distribution schemes, which are based on a truly multi-dimensional extension of the scalar upwind scheme (see Abgrall [**?**] for a review). The method is naturally formulated on triangular grids, making it an ideal approach for flows in complex geometries such as those encountered in astrophysical flows near compact objects. Residual distribution (RD) schemes are conservative and shock-capturing, and high-order can be achieved entirely through the use of compact stencils.

RD methods can be applied to balance laws of the form:

$$\frac{\partial}{\partial t} q + \vec{\nabla} \cdot \vec{F} = \psi(q, \vec{x}), \tag{21}$$

where $\vec{F} = [f^1, f^2, f^3]$ is the flux tensor and q, f^i, $\psi \in \mathbb{R}^m$ are the vectors denoting the conserved variables, flux functions, and source term, respectively. Space is discretized with a mesh of triangles (2D) or tetrahedrals (3D) that consists of elements, $\mathcal{T}_i$, and nodes $\vec{x}_j$. On each node at each time level, we define an approximate solution: $Q_j^n \approx q(\vec{x}_j, t^n)$. The approximate total residual in element $\mathcal{T}_i$ is given by

$$\phi^{\mathcal{T}_i} \approx \int\int\int_{\mathcal{T}_i} \left[\vec{\nabla} \cdot \vec{F} - \psi(q, \vec{x}) \right] dV. \tag{22}$$

This residual is then distributed to the nodes of element $\mathcal{T}_i$ through a multi-dimensional upwinding procedure. The approximate solution is updated by summing up all the contributions to node j:

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{V_j} \sum_{\mathcal{T}: j \in \mathcal{T}} \phi_j^{\mathcal{T}}, \tag{23}$$

where V_j is the volume of the median dual and $\phi_j^{\mathcal{T}}$ is the part of the total residual in element $\mathcal{T}$ that is distributed to node j. Nonlinear limiters are employed to guarantee that the methods are (essentially) non-oscillatory near shocks and high-order in smooth regions.

Further development of RD schemes is still an active area of research. Recently, Abgrall and Roe [?] have developed third and fourth-order RD schemes for scalar problems in the steady-state. Time accuracy is still under research; perhaps the most promising strategy is that proposed by Abgrall and Mezine [?]. The goal of this research is to develop additional improvements to the current RD schemes and to apply them to problems in astrophysical flows. Some preliminary results in this direction are presented in [?].

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I am currently working on the numerical implementation of a curvature-based formulation of General Relativity. In this formulation the geometry of spacetime is described by an orthonormal tetrad field, connection coefficients and the tetrad components of the Weyl tensor. When augmented with suitable gauge conditions the system gives rise to a first order symmetric hyperbolic evolution. Furthermore, using the work by Friedrich and Nagy [?], it is possible to obtain a well posed initial-boundary value formulation via maximally dissipative boundary conditions. This offers the possibility of constructing highly accurate discretization schemes for which numerical stability can be guaranteed in the linearized case. I have started looking at pseudo-spectral discretization methods which should be well-suited for the vacuum equations where the solutions are expected to be smooth. This work is in collaboration with Gabriel Nagy, Jim Bardeen and Luisa Buchman.

I am also studying the evolution of homogeneous, spherically symmetric bubble spacetimes in five-dimensional Kaluza-Klein theories. These spacetimes, which are everywhere regular and asymptotically approach the Kaluza-Klein vacuum, have attracted some interest in the string community since their mass is not necessarily positive. Furthermore, unlike four dimensional vacuum spacetimes, where there exists no non-static spherically symmetric configurations, bubble spacetimes admit non-trivial dynamical degrees of freedom. Indeed, we have found [?, ?] rich dynamical scenarios including the formation of black strings and critical phenomena. The bubble configurations provide a nice toy model problem of physical interest where several aspects of the binary black hole problem, like outer boundary conditions, gauge conditions, extraction of invariant quantities, the formation of apparent horizons, etc., can be analyzed. This is work in collaboration with Luis Lehner.

I am also working on the initial-boundary value formulation of Einstein's field equations. In numerical relativity one usually solves the field equations on a spatially truncated domain with artificial boundaries. Boundary conditions have to be specified in such a way that they ensure constraint propagation, incorporate some physics (for example, in many application it is desirable that the boundary conditions minimize the reflection of gravitational radiation) and are stable in the sense that they yield a well posed initial-boundary value problem. Besides the formulation in [?] which is based on a tetrad representation of the field equations, no boundary conditions satisfying all the above properties have been found for the more commonly used metric based formulations. The major difficulty lies in the fact that for first order hyperbolic systems the conditions requiring constraint propagation typically yield boundary conditions in the form of differential equations instead of the more familiar algebraic conditions for which much is known in the literature. In order to understand the structure of these boundary conditions Oscar Reula and I recently analyzed a related model problem in electromagnetism and derived [?] a well posedness result using methods from semigroup theory. Currently, I am working on the generalization of our result to the case of the Einstein equations.

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Analysis of time-dependent problems governed by linear and nonlinear PDEs; the development of novel high-resolution algorithms for the approximate solution of these problems, including finite-difference, finite volume and spectral methods, and the interplay between the theory and computational aspects of such approximate methods.

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My general research interests include the areas of partial differential equations, numerical analysis, and general relativity. In recent years, most of my effort has been directed toward hyperbolic formulations of Einstein's equations, constraint preserving boundary conditions for such formulations, and the initial data problem. My work on this subject has been motivated by the necessity to derive consistent boundary conditions and initial data for various hyperbolic formulations of Einstein's equations that lead to long term stable and accurate three dimensional simulations in numerical relativity. Most of my results are contained in my Ph.D. thesis ("Constraint Preserving Boundary Conditions for Hyperbolic Formulations of Einstein's Equations," University of Minnesota, 2004) and M.A. paper ("On the Hamiltonian Constraint Equation in General Relativity," Penn State University, 2001), both under the advisership of Prof. Douglas N. Arnold. My current research program is a natural continuation of the work done in my Ph.D. thesis. In particular, I am interested in extending some results on constraint preserving boundary conditions obtained in the case of polyhedral domains for the linearized (about the Minkowski spacetime) Einstein equations to the more general case of curvilinear domains. I am also interested in studying the boundary conditions problem for Einstein's equations linearized around other interesting solutions, such as Schwarzschild spacetime. Numerical implementations of these results is another important direction of current and future work.

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Numerical evolutions of Einstein's equations, mostly involving black holes and neutron stars, and associated mathematical, numerical and scientific computing issues. High order finite difference methods, multi-block simulations, boundary conditions for Einstein's equations, gauge conditions, methods to supress constraint violations.

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My work in numerical relativity began with development of a code based upon the characteristic initial value problem[1], in which spacetime is foliated by light cones (retarded time) rather than by Cauchy hypersurfaces (which define a standard time coordinate). This allows a description of infinity as a compactified boundary (Penrose boundary) where the emitted gravitational radiation can be unambiguously computed. This code has been successful treating a generic single black hole spacetime and for the outer region of a binary black hole. But the characteristic approach fails in the inner region of a two black hole spacetime because of the appearance of caustics in the light cones. However, an exterior characteristic evolution can (in principle) be matched to an interior Cauchy evolution to supply the outer Cauchy boundary condition and and compute the emitted waveform. This matching has be successful in many test cases but not in a general 3-dimensional black hole spacetime. The difficulty seem to arise from the lack of a good computational/analytical treatment of the outer boundary condition for the Cauchy codes. This has turned my recent interest to the Cauchy initial-boundary value problem (IBVP) in general relativity.

We have developed a version of this IBVP in harmonic coordinates for which the IBVP is well-posed for homogeneous or small boundary data. A code based upon this harmonic formulation has been implemented and successfully tested on model problems[2]. Initial progress has been made on the analytic/computational issues necessary to make the code applicable to black holes[3,4].

[1] "Characteristic Evolution and Matching", J. Winicour, *Living Reviews* 2001.

[2] B. Szilágyi and J. Winicour, *Phys. Rev.*, **D68**, 041501 (2003).

[3] "Some mathematical problems in numerical relativity", M. Babiuc, B. Szilágyi and J. Winicour, gr-qc/0404092.

[4] "Modeling the Black Hole Excision Problem", B. Szilágyi, H-O. Kreiss and J. Winicour, gr-qc/0412101

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Most of my research during the last years has been focused on numerical methods for partial differential equations, in particular theory for finite element methods. I have also worked on theory and numerical schemes for nonlinear hyperbolic conservation laws. Central themes of my activity on finite element methods have been mixed methods, saddle point systems, preconditioning and iterative schemes.

Two of my recent projects appear to be related to numerical relativity. In a joint paper with Snorre Christiansen we have performed numerical studies of a version of the Yang-Mills system. In particular, we investigated how the nonlinear constraints, preserved exactly by the equation itself, behaves for different numerical schemes. The conclusion was that unless special constraint preserving discretizations are used, a substantial drift of the constraints will take place. This strongly suggests that also for numerical solution of the Einstein equation, formulated as an evolution system, constraint preserving schemes should be used.

In a joint effort with Douglas Arnold and Richard Falk we have used ideas from differential forms and discrete differential complexes to construct stable finite element schemes, derived from the Hellinger–Reissner variational principle, for linear elasticity. A key tool in our proach is the study of discrete analogs of the so called "elasticity complex." This complex involves a second order differential operator, characterizing divergence free symmetric matrix fields, and this operator is closely related to a linearizion of the spatial operator occurring in the evolution system derived from the Einstein equation. Therefore, it may be possible that there is link between mixed finite element methods for elasticity and numerical schemes for the Einstein system.