Report on BIRS workshop 06rit072
"Generalized Harish-Chandra modules of gl(∞)"
August 26 - Sept. 4

Due to the absence of Professor Gregg Zuckerman and his replacement by Elizabeth Dan-Cohen, we modified slightly the main topic of our workshop. More precisely, we decided to concentrate on the following two subtopics of the general topic "Generalized Harish-Chandra modules of gl(∞)":

(i) highest weight gl(∞)-modules, existence of maximal submodules in Verma modules induced from general locally finite subalgebras of gl(∞), and the problem of describing the conjugacy classes of maximal locally finite subalgebras of gl(∞).

(ii) a geometric realization of highest weight modules of diagonal direct limit Lie algebras as higher cohomology groups of line bundles on flag ind-spaces.

As a work-mode we chose "working sessions" (we did not hold formal talks due to the small size of our group). Topic (i) was a joint topic for the three of us, while topic (ii) was a working topic for Ivan Dimitrov and Ivan Penkov only, as it was part of a joint paper in progress.

Review of our work on topic (i). One of the purposes of the workshop was to introduce our junior participant, Elizabeth Dan-Cohen to some problems which have arisen in the joint work of the two senior participants. The first and most important topic was the existence of unique maximal submodule of Verma module induced from a general locally solvable subalgebra of gl(∞). The three of us shared all the information which is available on the subject. Roughly speaking, the status quo is that Dimitrov and Penkov have a sufficient condition for the existence of such a submodule, based on results of [DP1], and [BB]; however it is not known whether such a unique submodule exists if the condition is not satisfied. Our general feeling is that there should exist a counterexample to the statement for sufficiently general maximal locally solvable subalgebras, and this was suggested to Dan-Cohen as an excellent thesis problem.

We also discussed in detail the conjugacy problem for maximal locally solvable subalgebras of gl(∞). Here Dan-Cohen was in leading position because of her experience in the solution of the analogous problem for Cartan subalgebras, [DPS]. We were able to obtain a first partial result: in the special case of a maximal closed generalized flag with exactly one predecessor-successor pair of codimension 1 which is not strongly closed, [DP2], we were able to show that the stabilizers of all such generalized flags are conjugate. This is an encouraging first result in a difficult problem.

Finally, another problem we discussed was the existence of opposite maximal locally solvable subalgebras. Opposite maximal locally solvable subalgebras would be useful in studying Verma modules, as in the case of finite dimensional Lie algebras. Based on our discussion, we modified our conjectured definition for one maximal locally solvable subalgebra to be opposite to another. Oddly

enough, the definition is not symmetric, as it uses the notion of a spanning toral subalgebra inside one of the maximal locally solvable subalgebras. While there are many nonobvious examples of maximal locally solvable subalgebras, the existence of an opposite maximal locally solvable subalgebra is not known in general.

Review of our work on topic (ii). Dimitrov and Penkov had been working for several years on the Bott-Borel-Weil theorem for diagonal ind-groups. The main challenge has been to prove that any higher cohomology group $H^i(G/B, \mathcal{L})$ is isomorphic to the algebraic dual of a B-highest weight module. Here $G = \varinjlim G_n$ is a locally simple diagonal ind-group, $B = \varinjlim B_n$ is the direct limit of Borel subgroups with $B_n \cap G_{n-1} = B_{n-1}$, and $\mathcal{L}$ is a G-equivariant line bundle on G/B. In 2005 there had been a significant advance in the problem: through a geometric construction the problem was reduced to a combinatorial problem which was expected to be solvable by the method of [DPW]. Earlier in 2006, after the attempt to make a first draft of the paper, it was discovered that the desired reduction to [DPW] can be carried out only in a particular case, and that the general combinatorial question is therefore open. During this meeting, we were able to achieve a decisive breakthrough in this problem. Our continued work (after the workshop) has showed that indeed the problem is solved and that we have a Bott-Borel-Weil theorem for locally simple diagonal ind-groups. The joint paper of Dimitrov and Penkov is currently in its final stages of preparation.

In summary, we have had a much needed workshop which has stimulated the development of the subject in several different directions. We would like to thank the entire staff at BIRS for their outstanding support.

References

[BB] Yu. Bahturin, G. Benkart, Highest weight modules for locally finite Lie algebras, Studies in Advanced Mathematics, 4, AMS/IP (1997), p. 1-32.

[DP1] I. Dimitrov, I. Penkov, Weight modules of direct limit Lie algebras, IMRN 1999, no. 5, p. 223-249.

[DP2] I. Dimitrov, I. Penkov, Borel subalgebras of $gl(\infty)$, Resenhas IME-USP 6 (2004), p.111-119.

[DPS] E. Dan-Cohen, I. Penkov, N. Snyder, Cartan subalgebras of root-reductive Lie algebras, Journ. of Algebra, to appear.

[DPW] I. Dimitrov, I. Penkov, J. A. Wolf, A Bott-Borel-Weil theory for direct limits of algebraic groups, American Journ. of Math. 124 (2002), p. 955-998.