Analytic Methods for Diophantine Equations

Michael Bennett (University of British Columbia),
Chantal David (CRM),
William Duke (University of California, Los Angeles),
Andrew Granville (University de Montreal),
Yuri Tschinkel (Courant Institute NYU and University of Goettingen)

May 13 – 19

Some of the oldest questions in mathematics stem from the desire to find integer solutions to equations. From the equation in Pythagoras' theorem, toFermat's last theorem, Waring's problem, the abc-conjecture and Manin's conjecture, professional and amateur mathematicians alikeare thrilled in trying to prove that there are no solutions, or to determine solutions, or to count solutions. With such a venerable topic it is not surprising that there are many competing approaches to such questions, somewhose time has already come, some that are very hot methods right now, and some whose time is yet to come. At this meeting at BIRS therewere participants from many of the different schools of thought in this fascinatingsubject; it was an interesting opportunity for them to come togetherand find common ground.

During the last academic year two of the world's major research institutes, the Centre de Recherche en Mathematiques in Montreal, and the Mathematical Sciences Research Institute in Berkeley, have hosted semester long programs on different aspects on these questions. It was decidedto get together at the end of the academic year for a joint meeting to discuss issues that arise at the thematic programs at each institute. Thus the participants were primarily people who had been at one special year or the other, though perhaps a third were other researchers who are expert in Diophantine equations.

Perhaps the most consistent theme of this meeting was the topic of counting points on higher dimensional varieties, particularly Manin's conjecture. We heard a highly motivating survey by Yuri Tschinkel (Gauss chair at Goettingen), exciting new research from a geometric perspective by Par Salberger (Chalmers, Sweden), from a perspective of automorphic forms by Ramin Takloo-Bighash (Princeton) and from a perspective closer to Diophantine approximations by Jeff Thunder (Northern Illinois U).

There were exciting and controversial new perspectives on Manin's conjecture on K3 surfaces from Arthur Baragar (Nevada) and Ronald von Luijk (who was a CRM and MSRI postdoc this year, and will be a PIMS postdoc next year).

To understand Manin's conjecture on del Pezzo surfaces we heard an explanation of a basic example by Michael Joyce (Tulane) and saw a representation theoretic approach to universal torsors by Alexei Skorobogatov (Imperial College), and a direct approach to these torsors by Ulrich Derenthal (Gottingen).

Among new results was one announced by de la Breteche (Orsay) who showed that a specific height zeta functions (for a toric cubic surface) cannot be analytically continued to the whole complex plane (it has a natural boundary), so that the "Riemann Hypothesis" is not, in general, even a sensible question.

To count points on higher dimensional varieties one can also proceed by the classical circle method. Roger Heath-Brown (Oxford) told us about his recent major breakthrough on counting points on cubic hypersurfaces (reducing the number of variables in Davenport's famous result), the extension to quartic varieties was discussed by Tim Browning (Bristol). Trevor Wooley (Michigan) explained his idea to prove that the local-global principle works almost always and discussed what he has shown to date.

Noam Elkies (Harvard) showed how root numbers in families of elliptic curves, in combination with heuristics, could be used to predict surprising behavior regarding uniform boundedness of ranks of elliptic curves over number fields, and to contradict a well-known conjecture on the topology of rational points. Andrew Granville (Montreal) explained his new conjectures on the distribution of rational and integral points on curves and specifically how they impact in a provocative way on the question of ranks of elliptic curves. Aaron Levin (MSRI/Brown) developed techniques of Vojta to bound the number of rational points on curves of genus 1 over fields of bounded degree; and Jordan Ellenberg (Wisconsin) gave impressive new upper bounds, from his work with Akshay Venkatesh, on the heights of points of curves of genus 1, breaking through what had seemed to be a difficult barrier from the work of Heath-Brown.

There were also several talks on related questions: Noriko Hirata-Kohno (Nihon) improved Evertse's theorem giving good bounds on the total number of solutions to certain Fermat-type Diophantine equations. Preda Mihailescu (Gottingen) showed that techniques in the theory of cyclotomic fields could be used to bound solutions to certain Ljunggren-Nagell type equations. Valentin Blomer (Toronto) improved the error term in the known approximations for representations by ternary quadratic forms using his recent work on convexity-breaking. Pietro Corvaja (Udine) explained how to show that there are large prime factors of any Markov pair, Patrick Ingham (UBC) showed that multiples of integral points on elliptic curves cannot themselves be integral, except in certain obvious cases. Jean-Louis Colliott-Thelene (Paris Sud) presented an extension of the Brauer-Manin obstruction to integral points (instead of rational points), and showed how it explained recent results on integral quadratic forms. Hershy Kisilevsky (Concordia) showed how points on cubic twists give rise to points on certain K3 surfaces; combining this with work of the Dokshitzers one discovers surprising families of surfaces which must contain rational points. Finally Harald Helfgott (Montreal) conjectured that the only extreme examples in the large sieve are the images of points from a finite set of curves, and indicated how he proved this, with Akshay Venkatesh, in two dimensions.

All participants seemed to have greatly enjoyed the meeting. It was an interesting "coming together" of different approaches to important questions, and most speakers tried to be accessible, so a lot was learned. There were several new collaborations formed during the meeting, and even some results proved, while in Banff.

The meeting was well situated. The lecture hall and the rooms were appropriate, the local BIRS staff was excellent (particularly Brenda Shakotko), as well as of the Banff center. The weather could not have been better and everyone went home having enjoyed the mathematics and re-invigorated by the mountain surroundings.