Report on BIRS workshop 07w5013: Operator spaces and group algebras, Sunday, August 19, 2007 to Friday, August 24, 2007

This workshop was organized by **Eberhard Kaniuth** (Univeristy of Paderborn), **Anthony To-Ming Lau** (University of Alberta), and **Zhong-Jin Ruan** (University of Illinois). The workshop had a total of 42 participants. Our main speaker was Professor Gilles Pisier (Paris VI/Texas A. and M.), who gave two 50-minute talks. The other 22 speakers each gave a 50-minute talk. The following are reports on their talks, problems arising, and impacts following the workshop.

1 The similarity problem

The two lectures of **G. Pisier** at the workshop concerned the similarity problem.

A locally compact group is *unitarizable* if any (continuous) uniformly bounded representation is unitarizable (i.e., if it is similar to a unitary representation). Dixmier asked already in 1950 whether 'unitarizable' implies 'amenable' (the converse was proved by him and Day independently). More precisely, a representation $\pi\colon\ G \to \mathcal{B}(H)$ is *unitarizable* if there is an invertible operator $\xi\colon\ H \to H$ such that $g \mapsto \xi\pi(g)\xi^{-1}$ is a unitary representation on G. In 1955, Ehrenpreis and Mautner showed that $SL_2(\mathbb{R})$ is not unitarizable, from which it follows formally that any discrete free group admitting it as a quotient is also non-unitarizable.

Motivated by this, Kadison [41] formulated the following conjecture: any bounded homomorphism $u\colon\ A \to \mathcal{B}(H)$ from a C^*-algebra into the algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H is similar to a $*$-homomorphism, i.e., there is an invertible operator $\xi\colon\ H \to H$ such that $x \mapsto \xi u(x)\xi^{-1}$ satisfies $\xi u(x^*)\xi^{-1} = (\xi u(x)\xi^{-1})^*$ for all x in A. In this latter case, we say that ξ *unitarizes* u and that u is *unitarizable*. Without loss of generality, one may suppose that A has a unit. Then u is unitarizable if and only if its restriction to the unitary group of A is unitarizable as a group representation.

These conjectures remain unproved, although many partial results are known. In his series of two talks, Pisier surveyed those results, as well as more recent results on the closely related notion of *length* of an operator algebra that he introduced. In particular, he explained why 'length equal to 2' characterizes amenable groups or C^*-algebras. Moreover, he showed that, if one can always force the similarity ξ to be in the von Neumann algebra generated by the range, then the group (or the C^*-algebra) must be amenable. Here are some more precise definitions.

We denote by $\|u\|_{cb}$ the completely bounded (in short, c.b.) norm of a linear mapping between two operator spaces, i.e., two linear subspaces of the space $\mathcal{B}(H)$ of bounded operators on a Hilbert space H. This plays a crucial role in similarity problems because of Haagerup's formula, valid for any (algebra) homomorphism $u\colon\ A \to \mathcal{B}(H)$ defined on a C^*-algebra A: $\|u\|_{cb} = \inf\{\|\xi\|\|\xi^{-1}\|\}$, where the infimum is over all invertibles ξ on H that 'unitarize' u.

The *similarity degree* of a unital operator algebra A is defined (see [58] and more references there) as the smallest $\alpha \geq 0$ for which there is a constant C such that any bounded morphism (= unital homomorphism) $u\colon\ A \to \mathcal{B}(H)$ satisfies $\|u\|_{cb} \leq C\|u\|^\alpha$.

On the other hand, an operator algebra $A \subset \mathcal{B}(H)$ is *of length* $\leq d$ if there is a constant K such that, for any n and any x in $\mathbb{M}_n(A)$, there are an integer $N = N(n,x)$ and scalar matrices $\alpha_0 \in \mathbb{M}_{n,N}(\mathbb{C})$, $\alpha_1 \in \mathbb{M}_N(\mathbb{C}), \dots, \alpha_{d-1} \in \mathbb{M}_N(\mathbb{C})$, $\alpha_d \in \mathbb{M}_{N,n}(\mathbb{C})$ together with diagonal matrices $D_1, \dots, D_d$ in $\mathbb{M}_N(A)$ satisfying

$$x = \alpha_0 D_1 \alpha_1 D_2 \dots D_d \alpha_d \quad \text{with} \quad \prod_0^d \|\alpha_i\| \prod_1^d \|D_i\| \leq K\|x\|\,.$$

We denote by $\ell(A)$ the smallest d for which this holds, and we call it the *length* of A (so that 'A has length $\leq d$' is indeed the same as '$\ell(A) \leq d$'). It is easy to see that, if $\ell(A) \leq d$, then, for any bounded homomorphism $u\colon A \to \mathcal{B}(H)$, we have $\|u\|_{cb} \leq K\|u\|^d$.

Now let G be a discrete group, and let $A = C^*(G)$. We wish to restrict the above factorization to the case when the entries of the diagonal matrices $D_1, \ldots, D_d$ sit in G itself (viewed as a subset of $C^*(G)$ in the usual way). We denote by $\ell(G)$ the smallest d as above, but for these restricted factorizations. Analogously, we denote by $d(G)$ the smallest α such that, for some C, all uniformly bounded representations π satisfy $\|u_\pi\|_{cb} \leq C \sup_{g\in G} \|\pi(g)\|^\alpha$, where $u_\pi\colon [G] \to \mathcal{B}(H)$ is the homomorphism linearly extending π.

To summarize the basic known results, we state the following from [55, 57]; the last assertion is due to Erik Christensen [12].

Theorem 1.1 (i) *For any discrete unitarizable group G, we have $d(G) = \ell(G)$.*

(ii) *Any G containing a non-abelian free group has infinite length (i.e. is not unitarizable).*

(iii) *The Dixmier question whether unitarizable groups are amenable is equivalent to the assertion that there are no groups G with $2 < \ell(G) < \infty$.*

(iv) *For an infinite discrete group, G is amenable if and only if $\ell(G) = 2$.*

Recall that a C^*-algebra A is *nuclear* (equivalently *amenable* in B. E. Johnson's sense) if, for all C^*-algebra B, there is a unique C^*-norm on $A \otimes B$.

Theorem 1.2 (i) *For any unital operator algebra A, we have $d(A) = \ell(A)$.*

(ii) *The Kadison conjecture that all C^*-algebras are unitarizable is equivalent to the assertion that there is a fixed d such that any C^*-algebra A has length $\ell(A) \leq d$.*

(iii) *For an infinite-dimensional C^*-algebra A, we have $d(A) = 2 \Leftrightarrow \ell(A) = 2 \Leftrightarrow A$* is nuclear.

(iv) *There are examples (e.g., $A = \mathcal{B}(\ell_2)$) for which $\ell(A) = 3$.*

The Banff meeting was followed in October 07 by a conference at the American Institute of Mathematics (AIM) in Palo Alto on the dichotomy between amenable and non-amenable groups. The Dixmier problem was one of the main problems discussed in that workshop, but the participants were from a different background from those at Banff, with many from geometric (infinite) group theory or random walks on groups. Pisier again gave a series of two talks, concentrating on the group case rather than on operator algebras, at that meeting.

2 Operator algebras

A number of talks during the week circled around the interplay between groups and operator algebras, e.g., von Neumann II_1-factors arising either from ergodic group actions on probability spaces (through the Murray–von Neumann group-measure space construction) or as the II_1-factor $L(G)$ associated with the regular representation of a discrete group with infinite conjugacy classes. It must be observed that, if a countable group G acts ergodically on a probability space (X, μ), then the group-measure space factor $M = L^\infty(X, \mu) \rtimes G$ depends only on the equivalence relation induced by G on X; in particular, amenability properties of the equivalence relation translate into amenability properties of M.

At the workshop, **C. Anantharaman-Delaroche** gave a lucid survey talk on various possible definitions of amenability for equivalence relations, group actions, and more general groupoids. She put the emphasis on asymptotic properties of random walks on groupoids. A basic open question about II_1-factors M is: Up to conjugacy, how many Cartan subalgebras are there in M?

If M has no Cartan subalgebra, then M cannot come from the group-measure space construction; if M has at least two Cartan subalgebras, then M comes from at least two genuinely different actions (i.e., non orbit equivalent). The fact that the factor $L(\mathbb{F}_r)$ of the free group $\mathbb{F}_r$ has no Cartan subalgebra is a success of Voiculescu's free probability theory. The existence of a II_1-factor with two non-conjugate Cartan subalgebras is a joint result by two Fields medallists, A. Connes and V. F. R. Jones in 1982 [14].

N. Ozawa lectured about his joint work with Popa on II_1-factors with zero or one Cartan subalgebra. He sketched the proof [53] that, if M is a II_1-factor with the complete metric approximation property (CMAP), then $M \otimes L(\mathbb{F}_r)$ has no Cartan subalgebra (a far-reaching generalization of Voiculescu's result), and that, if the probability space (X, μ) carries a profinite ergodic action of $\mathbb{F}_r$, then the group-measure factor $L^\infty(X, \mu) \rtimes \mathbb{F}_r$ has a unique Cartan subalgebra (namely $L^\infty(X, \mu)$).

A purely group-theoretical result of the Ozawa–Popa study is the fact that, if a wreath product $H \wr G$ has CMAP, then the acting group G is amenable. Since the workshop, Ozawa and Popa, generalizing further [54], have shown that, if Γ is a non-amenable group with a strong form of the Haagerup property (also called a-T-menability), which moreover has CMAP, then $L(\Gamma)$ has no Cartan subalgebra and $L^\infty(X, \mu) \rtimes \Gamma$ has a unique Cartan subalgebra when (X, μ) carries a profinite ergodic action of Γ. These results raise the questions of having more examples of groups satisfying CMAP and/or the Haagerup property.

During the workshop, two further talks addressed these questions. **E. Guentner** explained his result with Higson [35] about groups acting properly and isometrically on finite-dimensional $CAT(0)$. Also **A. Valette** explained his result with Y. de Cornulier and Y. Stalder [15] that the Haagerup property is preserved under wreath products. Together with the Ozawa–Popa result mentioned above, this disproves a conjecture by M. Cowling in 1996 that the class of CMAP groups coincides with the class of Haagerup groups; see [9]. Another by-product is that the finite-dimensionality assumption cannot be removed from the Guentner–Higson result. These three sets of results give rise to obvious questions about these classes. What are their permanence properties? In particular, which kinds of semi-direct products preserve these classes?

In a slightly different direction, **B. Bekka** explained his remarkable super-rigidity result [4] for $SL_n(\mathbb{Z})$, where $n \geq 3$, in a von Neumann-algebra setting. Let $f : SL_n(\mathbb{Z}) \to U(M)$ be a homomorphism to the unitary group of a finite factor M. Then either M is finite-dimensional (and f factors through a congruence subgroup), or there exists a finite index subgroup Γ of $SL_n(\mathbb{Z})$ such that f extends to a normal homomorphism $L(\Gamma) \to M$. A corollary is that the full C^*-algebra of $SL_4(\mathbb{Z})$ has no faithful trace, which answers negatively a question of Kirchberg [42] (a positive answer would have proved the famous Connes' embedding conjecture, also mentioned in **K. Dykema**'s talk during the week: Every countable group embeds into the unitary group of the ultra-power of the hyperfinite II_1-factor!) An interesting open question is: What happens if $SL_n(\mathbb{Z})$ is replaced by some other higher rank lattice?

Coming from theoretical physics, the Bessis–Moussa–Villani conjecture, going back to 1975, states that the function $t \mapsto \mathrm{Tr}(\exp(A + itB))$ is positive-definite on the real line for any two self-adjoint matrices A and B of the same size. Of course this conjecture can be generalized to any C^*-algebra with a faithful trace, and **M. Bozejko** explained the proof [6] of this generalized form when A and B are q-Gaussian random variables, with $q \in [-1, 1]$; the case where $q = 0$ corresponds to free probability. Interesting connections were made with completely bounded maps on Coxeter groups.

R. Archbold described some recent results [1, 2] that he had obtained with Kaniuth on the *stable rank* $sr(C^*(G))$ and *real rank* of group C^*-algebras $C^*(G)$ and of compact transformation group C^*-algebras $C_0(X) \rtimes G$. In the case of an almost connected, nilpotent, locally compact group, we have

$$sr(C^*(G)) = 1 + \left[\frac{1}{2}\text{rank}(G/[G,G])\right],$$

which generalizesg the result of Sudo and Takai for simply connected, nilpotent Lie groups. On the other hand, an unresolved dichotomy was described for $RR(C^*(G))$ in the case where $\text{rank}(G/[G,G])$ is even. For a second countable transformation group (G, X) with G compact, detailed formulae were described for $sr(C_0(X) \rtimes G)$ and $RR(C_0(X) \rtimes G)$, subject to the proviso that the space X is locally of finite G-orbit type. The meeting stimulated further work on these problems, using some very recent results [7] of Brown on the real rank of CCR C^*-algebras. This has resulted in substantial progress by Archbold and Kanuith on the dichotomy for $RR(C^*(G))$, and also on the removal of the somewhat restrictive assumption of 'finite G-orbit type' in the case where G is a compact Lie group acting on a second countable space X.

A C^*-algebra A is *exact* if the minimal tensor product by A preserves short exact sequences of C^*-algebras. These algebras were the topic of the talk of **M. Dadarlat**.

A fundamental result of Kirchberg [43] asserts that the separable exact C^*-algebras are precisely those C^*-algebras which embed in the Cuntz algebra $\mathcal{O}_2$. A C^*-algebra which can be represented as an inductive limit of finite-dimensional C^*-algebras is called an *AF algebra*. A major open problem in the structure theory of C^*-algebras is to characterize the C^*-algebras which embed in separable AF-algebras. One has the following conjecture.

Conjecture: A separable C^*-algebra is AF-embeddable if and only if it is exact and quasidiagonal.

Recall that a separable C^*-algebra A is *quasidiagonal* if there is a sequence $\varphi_n : A \to \mathbb{M}_n(\mathbb{C})$ of completely positive contractions such that $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b))\| \to 0$ as $n \to \infty$ for all $a, b \in A$. Voiculescu proved that quasidiagonality is a homotopy invariant in the class of separable C^*-algebras. While the above conjecture is very much open, there are some promising results towards its validity. N. Ozawa proved in [52] that AF-embeddability is a homotopy invariant in the class of separable exact C*-algebras. In particular the cone, and hence the suspension, of any separable exact C^*-algebra is AF-embeddable. Dadarlat [16] has verified the conjecture for the class of separable, residually finite-dimensional algebras that are equivalent in KK-theory to commutative algebras. Using a result of J. L. Tu, one then concludes that the C^*-algebra of a countable amenable residually finite group is AF-embeddable. This brings us to another very interesting open problem inspired by work of Vershik and Rosenberg.

Problem: Is the C^*-algebra of a countable amenable discrete group AF-embeddable or at least quasidiagonal?

Dadarlat has shown that this is the case for central extensions of amenable, residually finite groups by $\mathbb{Z}^n$. Nevertheless this problem seems very difficult even for the class of elementary amenable groups. It is expected that the methods required to solve this problem would inspire powerful generalizations of the Berg technique, and would lead to significant progress concerning the structure of C^*-algebras associated to amenable groups.

K. Dykema discussed Horn inequalities. Indeed, given a Hermitian $n \times n$ matrix A, let $\lambda_A(1) \geq \lambda_A(2) \geq \cdots \geq \lambda_A(n)$ be its eigenvalues listed according to multiplicity. The classical *Horn inequalities*, for Hermitian $n \times n$ matrices A, B, and C such that $A + B + C = 0$, are the inequalities

$$\sum_{i \in I} \lambda_A(i) + \sum_{j \in J} \lambda_B(j) + \sum_{k \in K} \lambda_C(k) \leq 0$$

for certain triples (I, J, K) of subsets of $\{1, \ldots, n\}$ with $|I| = |J| = |K|$, known as Horn triples.

It was proved about ten years ago, due to work of due to work of Klyachko [45] and Knutson and Tao [46] that the Horn inequalities, together with the trace equality

$$\sum_{i=1}^{n} \lambda_A(i) + \sum_{j=1}^{n} \lambda_B(j) + \sum_{k=1}^{n} \lambda_C(k) = 0,$$

exactly chacterize the set of possible eigenvalues of such A, B, and C.

A question asked in the talk of Dykema was whether the analogues of the Horn inequalities hold for self-adjoint elements in all II_1-factors. This is related to Connes' embedding problem, a deep question that has many equivalent formulations. The main new result that was given in the talk appeared about a year after the BIRS workshop [5]: it is that all Horn inequalities do hold in all II_1-factors. The proof of this result is actually a construction: given arbitrary flags $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ in a II_1-factor and a triple (I, J, K) whose corresponding Littlewood–Richardson coefficient is equal to 1, there is a projection in the intersection $S(\mathcal{E}, I) \cap S(\mathcal{F}, J) \cap S(\mathcal{G}, K)$ of the corresponding Schubert varieties. This construction seems to be new even in the case of matrices. A more intricate eigenvalue question, analogous to Horn's question, but with 'matrix coefficients'; by [13], this is equivalent to Connes' embedding problem.

R. Smith discussed masas in von Neumann algebras. For an inclusion $B \subseteq M$ of finite von Neumann algebras, a unitary operator $u \in M$ *normalizes* B if $uBu^* = B$. The group of normalizing unitaries is denoted by $\mathcal{N}(B)$, while $\mathcal{N}(B)''$ denotes the von Neumann algebra that it generates inside M. Interest in these operators goes back to Dixmier in the 1950's, who used $\mathcal{N}(B)''$ to classify various types of maximal abelian self-adjoint subalgebras (masas) in factors. It is always the case that $B \subseteq \mathcal{N}(B)''$, and B is *singular* if equality holds. A natural question is how singularity relates to tensor products and, as part of a larger study with Sinclair, White, and Wiggins [61], it was shown that the tensor product of singular masas is again singular. Subsequently this was generalized by Chifan [10] to the formula

$$\mathcal{N}(B_1 \mathbin{\overline{\otimes}} B_2)'' = \mathcal{N}(B_1)'' \mathbin{\overline{\otimes}} \mathcal{N}(B_2)''$$

for arbitrary masas in finite factors.

Since then Smith has investigated normalizers for irreducible inclusions of factors (with White and Wiggins [62]), where it is actually possible to determine them explicitly, unlike the masa case. This has led recently to the resolution of the case where the subalgebra satisfies $B' \cap M \subseteq B$, which includes the two special cases already mentioned. For tensor products one has to replace normalizers by a wider class of operators called *groupoid normalizers*, but then the analogous result holds true [25]. This has already had a slightly suprising application to the theory of maximal injective von Neumann subalgebras [8].

Various examples show that $B' \cap M \subseteq B$ is a natural boundary for such theorems, but all of this discussion holds true only for finite von Neumann algebras, and all such questions are open for the other type of von Neumann algebras. In the finite case one has available the basic construction algebra $\langle M, e_B \rangle$ of Jones. The main current difficulty is to find a suitable modification of the techniques to make progress on the other types of von Neumann algebras.

3 Fourier algebras

The Fourier algebra $A(G)$ of a locally compact group G has been a very fruitful object for interactions between operator algebras and harmonic analysis since the 1960s. Though the Banach algebra structure is commutative, the spatial structure is not, and the theory of operator spaces, which recognizes this non-commutativity, interacts surprisingly well with various aspects of $A(G)$. This has been known since the 1980s, with the pioneering work of Haagerup *et al.* on multipliers. In 1995, Ruan showed that questions of amenability, a Banach algebra property, can be resolved by tweaking that property in a way that takes operator space structures into account, which leads to the notion of *operator amenability*; indeed, he showed that $A(G)$ is operator amenable precisely when G is amenable [59]. Ruan's work led to more intense research into the operator space structure of $A(G)$, for example, see the work of Forrest, Kanuith, Lau and Spronk [27] on the complemented ideal problem, and work of Ilie and Spronk [39] on the structure of homomorphisms.

M. Neufang spoke on his impressive work with Junge and Ruan [40] on completely bounded multipliers of locally compact quantum groups. This extends theory pioneered by Haagerup in the $A(G)$ case, and Størmer and Ghahramani in the 'dual' group/measure algebra case. The present work unifies the aforementioned cases, and provides new insight into cases with less commutativity. This work has promoted further work between Hu, Neufang, and Ruan, who now have several papers (e.g., [38]) on various Banach algebras related to these multipliers, and on multipliers in a more classical setting. Following progress made at this meeting, Neufang was able to conduct work with his student Kalantar on defining and characterizing in various ways an intrinsic group for a locally compact quantum group, providing a beautiful analogue of the intrinsic group found in the theory of algebraic quantum groups. They also describe an invariant which is a certain subgroup of the torus, and in fact coincides with the latter in the case of Woronowicz's $SU_q(2)$ group. Promising insights into classification and structure of general locally compact quantum groups will stem from these efforts.

While the characterization of operator amenability of $A(G)$ was established by Ruan in 1995, the characterizations of other forms lagged by several years. Spronk in 2002 [63] showed that $A(G)$ is always *operator weak amenability*, and Forrest and Runde [26] characterized the amenability of $A(G)$ in 2004.

E. Samei presented on his work [28] with Forrest and Spronk, addressing such questions on certain Fourier algebras of symmetric spaces. Since this meeting there has been exciting progress. Critically using work presented at the meeting, Forrest, Samei, and Spronk [29] have characterized weak amenability of $A(G)$ for compact groups G. It remains an open question as to whether $A(G)$ is weakly amenable for various groups lacking a non-abelian connected compact subgroup; for example $G = SL_2(\mathbb{R})$, the $ax + b$ group, or any of the Heisenberg groups.

M. Monfared reported on his work [37] with Hu and Traynor on the *character amenability* of Banach algebras, extending work on a concept introduced and studied recently by Kanuith, Lau, and Pym and by Monfared. Based on consultations at the meeting, Runde and Spronk were able to improve one of their main results on the Fourier–Stieltjes algebra $B(G)$; the paper [60] considers the operator amenability of $B(G)$. However, it is an open question to classify the groups G for which $B(G)$ is operator amenable. When does $B(G)$ admit a point derivation?

The *Herz–Figá-Talamanca algebras* $A_p(G)$ can be defined for any $1 < p < \infty$. This class includes the Fourier algebras $A(G)$ at $p = 2$. Without the theory of von Neumann algebras associated to them, the algebras $A_p(G)$, in general, have a much more subtle theory than does $A(G)$. **V. Runde** reported on his work with Lambert and Neufang [47] establishing an operator space structure on $A_p(G)$ which allows a generalization of Ruan's amenability theorem. In new work, conducted in part at this meeting, Neufang and Runde have gained further insights into the fine structures of $A_p(G)$. Open questions remain as to whether the algebras $A_p(G)$, for amenable G, admit a nice homomorphism theorem, such as was proved for $A(G)$ by Cohen for abelian G and Ilie and Spronk for general amenable G.

N. Spronk reported on his generalization [64] of Feichtinger's Segal algebras from abelian groups to the general locally compact case, presented in a Fourier-algebra context. Through the meeting, he was invited to France to meet with Ludwig, and they found a minimality condition characterizing this algebra in the dual group algebra setting. This suggests that a definition for Feichtinger's Segal algebra for locally compact quantum groups may be possible.

4 Banach algebras and other topics

The lecture of **G. Dales** involved the topological centres of some Banach algebras.

Let A be a Banach algebra, and regard A as a closed subspace of its second dual A''. Then there are two natural products on A''; they are called the *first* and *second Arens products*, and are denoted by $\Box$ and $\Diamond$, respectively. We briefly recall the definitions. As usual, A' and A'' are Banach A-bimodules. For $\lambda \in A'$ and $\Phi \in A''$, define $\langle a,\, \lambda \cdot \Phi\rangle = \langle \Phi,\, a \cdot \lambda\rangle$ and $\langle a,\, \Phi \cdot \lambda\rangle = \langle \Phi,\, \lambda \cdot a\rangle$ for $a \in A$, and, for $\Phi, \Psi \in A''$, define

$$\langle \Phi \Box \Psi,\, \lambda\rangle = \langle \Phi,\, \Psi \cdot \lambda\rangle\,, \quad \langle \Phi \Diamond \Psi,\, \lambda\rangle = \langle \Psi,\, \lambda \cdot \Phi\rangle \quad (\lambda \in A')\,.$$

Then $(A'', \Box)$ and $(A'',\, \Diamond\,)$ are both Banach algebras containing A as a closed subalgebra. The *left topological centres* of A'' is defined by

$$\mathfrak{Z}_t^{(\ell)}(A'') \quad = \quad \{\Phi \in A'' : \Phi \Box \Psi = \Phi \Diamond \Psi \ (\Psi \in A'')\}\,,$$

and similarly for the *right topological centre* $\mathfrak{Z}_t^{(r)}(A'')$. See [18, 20, 48, 49, 50, 51] for extensive discussions of these centres.

Let A be a Banach algebra. Then A is *Arens regular* if $\mathfrak{Z}_t^{(\ell)}(A'') = \mathfrak{Z}_t^{(r)}(A'') = A''$; *left strongly Arens irregular* if $\mathfrak{Z}_t^{(\ell)}(A'') = A$, *right strongly Arens irregular* if $\mathfrak{Z}_t^{(r)}(A'') = A$, and *strongly Arens irregular* if A is both left and right strongly Arens irregular. A subset V of A'' is *determining for the left topological centre* of A'' if $\Phi \in A$ whenever $\Phi \in A''$ and $\Phi \Box \Psi = \Phi \Diamond \Psi$ $(\Psi \in V)$.

For example all C^*-algebras are Arens regular, but each group algebra $L^1(G)$ is strongly Arens irregular. There has been recent interest in improving the latter result by finding 'small' sets that are determining for the left topological centre of $L^1(G)''$.

Let S be a cancellative semigroup. Then it is shown in [20] that certain subsets of βS of cardinality 2 are determining for the left topological centre of $\ell^1(S)''$. The lecture, based on [17], discussed analogous results for various weighted convolution algebras of the form $\ell^1(S,\omega)$; see also [51]. There are several open questions in [20] and [17]; here is one from [17]. Is there a weight ω on $\mathbb{R}^+$ such that $\ell^1(\mathbb{R}^+,\omega)$ is Arens regular?

Current research is given in [21], where many related results are obtained. For example it is shown that, for each locally compact group G, the spectrum Φ of $L^\infty(G)$ is determining for the left topological centre of $L^1(G)''$. Here are two questions that are so far unresolved in [21]. (1) Is there a finite subset of Φ that is determining for the left topological centre of $L^1(G)''$? (2) Is the related measure algebra $M(G)$ strongly Arens regular for each locally compact group G? This is shown for non-compact groups G (of non-measurable cardinality) in [50], but it is open for the case where $G = \mathbb{T}$.

The lecture of **V. Paulsen** involved the projectivity and injectivity of C^*-algebras and G-maps.

There is a well-known contra-variant functor that connects compact spaces and abelian, unital C^*-algebras. Thus many results on the injectivity of C^*-algebras correspond to results about the projectivity of compact spaces. Gleason's classical theorem is central here. Paulsen gave an attractive, simple, and complete expositon of these notions, based on [34].

In the second part of the lecture, the above notions were generalized to a dynamical situation. Let G be a discrete group. An *action of G on a topological space X* is a homomorphism of G into the group of homeomorphisms of X that sends e_G to the identity map. Now X is a G-*space*. The notions of G-*cover* and G-*projective*, etc., are defined by analogy with the standard definitions. The definition of a *G-projective cover* requires care because the 'obvious' definition does not work.

The aim of the authors was to prove that every G-space has a G-projective cover, seeking to duplicate the theory of the first paragraph. This is an open question, and it leads to several interesting questions discussed in [34].

The authors do show that certain 'minimal' G-spaces have G-projective covers, and they derive various properties of G-projectivity that are related to topics in topological dynamics. For this, they use some results on the algebra of βG; these results were taken from the monograph [36], which was also refered to in some other talks at the meeting. Some of the questions raised in [34] concern amenable groups, another favourite topic of the meeting.

The lecture of **L. Turowska** was on operator multipliers.

The study of Schur multipliers has its origins in the work of Schur in the early $20^{\text{th.}}$ century. These objects have a simple definition: a bounded function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is a *Schur multiplier* if, whenever a matrix $(a_{ij})_{i,j\in\mathbb{N}}$ gives rise to a (bounded) transformation of the space l_2, the matrix $(\varphi(i,j)a_{ij})_{i,j\in\mathbb{N}}$ does so as well. A characterization of Schur multipliers was given by Grothendieck in his *Résumé*: Schur multipliers are precisely the functions φ of the form $\varphi(i,j) = \sum_{k=1}^{\infty} a_k(i)b_k(j)$, where $a_k, b_k : \mathbb{N} \to \mathbb{C}$ are such that $\sup_i \sum_{k=1}^{\infty} |a_k(i)|^2 < \infty$ and $\sup_j \sum_{k=1}^{\infty} |b_k(j)|^2 < \infty$. Schur multipliers have had many important applications in analysis, see e.g. [3], [23], and [56]. One of the forms of the celebrated Grothendieck inequality can be given in terms of these objects [56].

The lecture described generalization of an approach of Birman and Solomyak to the multi-dimensional setting, so extending many results known for classical Schur multipliers to ones about operator multipliers.

First Turowska introduced multi-dimensional Schur multipliers imposing some metric conditions as for the usual ('continuous') Schur multipliers and characterized them as elements of the *extended Haagerup tensor product*, generalizing results by Grothendieck and Peller. The result may be useful in connection with the theory of multi-dimensional operator integrals and their applications to the differentiation theory of operator functions and in the theory of perturbation.

Among other results she established a non-commutative and multi-dimensional version of the characterisation of Grothendieck and Peller which shows that the universal multipliers (i.e., multipliers with respect to any pair of representations) can be obtained as a certain weak limit of elements of the algebraic tensor product of the corresponding C^*-algebras with uniformly bounded Haagerup tensor norm. This was formulated as an open problem in [44], and is a generalization of the Grothendieck theorem to non-commutative multipliers.

A project discussed at the workshop concerns the study of compactness properties of operator multipliers. Schur multipliers φ whose associated linear operator $S_\varphi((a_{ij})) = (\varphi(i,j)a_{ij})$ is compact were studied by Hladnik. The notion of complete compactness is an operator space version of compactness which was defined and studied by Saar and Webster. A classification of completely compact universal operator was obtained.

The relations between completely compact and compact multipliers and between completely compact maps and compact maps are not fully understand so far. It was proved that the inclusion of completely compact multipliers in the set of compact ones is strict in general. To formulate necessary and sufficient conditions about automatic complete compactness of compact multipliers will be challenging. Some other interesting questions about multipliers which have to be investigated are: the connection between the space of the Fourier transforms of n-measures, completely bounded multipliers of the multidimensional Fourier algebra, and the space of multidimensional Schur multipliers; the property of closability of multipliers, important in connection with mathematical physics; factorisation of bounded multipliers and its connection with the study of means (geometric, algebraic, harmonic, and others) of Hilbert space operators.

The lecture of **Y. Zhang** concerned the approximate amenability of Banach algebras.

Let A be a Banach algebra A. Then: a continuous derivation $D : A \to X$ is *approximately inner* if there exists a net $(\xi_\nu) \subset X$ such that, for each $a \in A$, $D(a) = \lim_\nu (a \cdot \xi_\nu - \xi_\nu \cdot a)$; A is *approximately amenable* if, for each Banach A-bimodule X, every continuous derivation $D : A \to X'$ is *approximately inner*.

This definition and many variants were introduced by Ghahramani, Loy, and Zhang [30, 33]; see also [11, 31], for example. Many results about these properties are now known; for example, the Banach sequence algebras ℓ^p are not approximately amenable [22].

The lecture discussed Segal algebras. Let G be a locally compact group. A *Segal algebra* S on G is a dense left ideal of the group algebra $L^1(G)$ such that $(S, \|\cdot\|)$ is a Banach algebra for a norm $\|\cdot\|$, where $\|L_x f\| = \|f\| \geq \|f\|_1$ for $f \in S$ and $x \in G$. Of course, $L^1(G)$ itself is amenable, and hence approximately amenable. The conjecture is that any proper Segal algebra fails to be approximately amenable. This is proved in some cases in [32]; see also [19] for some further recent results, where the authors concentrate on the case where $G = \mathbb{T}$.

Related to this problem, it is shown in [11] that a Segal algebra on a SIN group is always approximately permanently weakly amenable.

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