Residually finite groups, graph limits and dynamics

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1 Overview of the Field

The Focussed Research Group brought together seven active researchers in the following fields: asymptotic group theory (Abert, Jaikin-Zapirain and Nikolov), ergodic theory (Bowen), discrete mathematics (Szegedy) and probability theory (Lyons and Virag).

The common object of interest is residually finite groups, that each field investigates from a different angle. An infinite group Γ is called residually finite, if the intersection of its subgroups of finite index is trivial. This means that finite images approximate the group structure of Γ. Important examples are lattices in linear Lie groups. More generally, finitely generated linear groups, and specifically, arithmetic groups.

From the abstract group theoretical point of view, residual finiteness is a natural condition that allows one to analyze such groups using finite group theory and discrete mathematics. A natural generalization of residual finiteness is soficity. The definition comes from Gromov and means that the group can be approximated by finite structures in a strong sense. The notion is wide enough to put amenable groups in the net – in fact, no finitely generated non-sofic groups are known. On the other hand, it is strong enough to prove general results. For instance, every sofic group satisfies the Kaplansky direct finiteness conjecture.

The topics addressed in the meeting included covering towers, graph limits, weak containment, entropy, groups acting on rooted trees, spectral measure, free spanning forests, percolation, L2 Betti numbers, unimodular random networks, cost, rank gradient, property (τ) and expander graphs.

2 Recent Developments and Open Problems

The field can be described as something lying at the crossroads of graph theory, group theory, ergodic theory and percolation theory. The field is only half-existing in the sense that while there are already many exciting results and even more questions, some of the researchers active in the area have not assimilated each other's point of view and major directions of research are waiting to be explored. The Focussed Research Group aimed to address these problems.

We now quote some important recent results and problems in the field; most connect to the work of one of the participants.

By the work of Abert and Nikolov, the growth of rank (called rank gradient) equals the cost of the corresponding boundary action of Γ. It is not known whether the rank gradient depends on the chain; both possible answers would solve a distinguished problem, one in 3-manifold theory and the other in topological dynamics. One way to solve this is to decide whether the cost-1 is multiplicative for arbitrary free actions.

Dense graphs have been investigated successfully with analytic methods. For graphs of bounded degree, there are strong hints of the existence of such analysis, but it has not yet been born. A crucial challenge is to understand the shape of random d-regular graphs in some sense. A major test problem here is to show that the independence ratio converges on a random graph sequence.

The residually finite group Γ acts by automorphisms on the corresponding coset tree (a locally finite rooted tree). The action extends to a measurable action on the boundary of the tree. One can connect the dynamics of this boundary action to asymptotic properties of the chain. In particular, the boundary action gives us a graphing (a measurable graph) that is the limit of the graphs coming from the actions on the levels of the coset tree. These special kind of graphings (profinite graphs) need to be investigated in depth. When taking a random point of the boundary, the rooted graph starting there gives us a unimodular random network.

Under mild conditions, the spectral measure of the Markov operator on finite quotients converges to the spectral measure of the Markov operator on Γ. In some cases, this allows one to compute the spectral measure. The core of the Lück approximation theorem is that the spectral measures converge even in a stronger sense. There is a lot of math waiting to be explored here. By Lück approximation, the growth of the first $\mathbb{Q}$-homology equals the first L^2-Betti number of Γ. In particular, the homology growth does not depend on the chain. When taking mod p homology, it is not even known whether the limit always exists.

Lattices in $SL_2(C)$ deserve a special attention among residually finite groups. For instance, the growth of the Heegaard genus on a covering tower of 3-manifolds can be analyzed using spectral properties of the corresponding chain. These topological investigations have already lead to new, exciting pure group theoretical results and more is expected in this direction.

Free spanning forests of $\mathbb{Z}^d$ are widely investigated in probability theory, because of their connection to random walks and percolation. Maybe the most direct way to introduce the first L^2 Betti number of a group is from the expected degree of a free spanning forest on a Cayley graph of it. There are also higher dimensional analogues. A good direction is to exploit this connection and prove new results on L^2 Betti numbers using percolation theory. The cost is also involved in this game, as it gives strong general estimates between the critical values of percolation. There are various beautiful natural problems in this area: for instance, show that $p_c(G) = 1$ implies that G has two ends.

3 Presentation Highlights

We had numerous three hour long presentations, typically in the mornings. These presentations were very enjoyable, with a lot of questions and dialogue. Speakers usually provided a general picture on the subject and then went into proofs, detailed as the audience requested. The list of three hour presentations included:

- Lyons on percolation and factors of i.i.d.;
- Szegedy on graph limits;
- Abert on rank gradient, weak containment and cost;
- Bowen on entropy in the non-amenable setting;
- Jaikin-Zapirain on Luck approximation and Lackenby's results;
- Lyons on L2 Betti numbers;

The afternoons were typically discussion sessions. People digested each others questions – sometimes answered them, sometimes found new ones.

4 Scientific Progress Made

Most importantly, people in the group learned each other's angles on problems that were interesting to everyone involved. The group found an array of new questions – in fact, the final collection of questions counts around forty. Some were solved on site, but many remained unanswered. Naturally, it is hard to judge now how hard these will prove to be. We quote some of the new questions, either found at this workshop and some that already existed but have not been publicized widely.

1. Can every d-regular graphing be properly edge colored by $d+1$ colors (measurable Vizing theorem).
2. For a nonamenable Cayley graph G let $U(G)$ denote the set of factors of i.i.d. on G with a unique infinite cluster. Is the density bounded below on $U(G)$?
3. Let G be a non-amenable Cayley graph with one end. Does there exist a $d' < d$ such that if ω is a factor of i.i.d. with expected degree at least d', then ω has a unique infinite cluster?
4. Are factors of i.i.d. on the 3-regular tree closed in the weak topology? In particular, look at the weak limit of majority functions on n-balls. Is that a factor of i.i.d.?
5. Is it true that $\mathrm{cost}(\Gamma, X) = \mathrm{cost}(\Gamma, X^2)$ for free actions?
6. Suppose G and H are Cayley expanders on n vertices and you can almost match them. Is it true that they are isomorphic?
7. Can you show $\beta_1^2(\Gamma) \leq \mathrm{cost}(\Gamma) - 1$ by combinatorial means (say, free spanning forest)?
8. Let Γ be amenable, acting ergodically and essentially faithfully on X (every nontrivial element moves a set of positive measure). Is the cost of the action 1? If Γ is finitely presented, is it true for any infinite ergodic action?
9. Let G be a Cayley graph and G_n be a sofic approximation of G. Can you label G_n so that it soficly approximates the Cayley diagram?
10. Let Γ be a Property (T) group and let G_n be a sofic approximation to G, that is, a sequence of finite graphs that converges to a Cayley graph of G. Surely, G_n does not have to be an expander family. But can we modify the sequence by an asymptotically vanishing amount (in the edit distance) to a sequence G'_n such that any subsequence of connected components of G'_n is an expander family? To put it another way, can one 'pullback' the ergodic decomposition of the invariant measure on the ultraproduct space?
11. Let Γ be a nonamenable group. Does $\{0,1\}^\Gamma$ factor onto $\{0,1,2\}^\Gamma$?
12. Let S be a finite set and let $\Gamma = (F_S, R)$ be a presentation. For $r > 0, \varepsilon > 0$ and n let $\mathrm{Sof}(r, \varepsilon, n)$ denote the set of sofic approximations of Γ with degree n up to radius r with error at most ε. That is, a function $f : S \to \mathrm{Sym}(n)$ belongs to $\mathrm{Sof}(r, \varepsilon, n)$, if for all $w \in F_S$ with $|w| \leq r$ we have $\mathrm{fix}(w(f)) < \varepsilon$ if $w \notin R$ and $\mathrm{fix}(w(f)) > 1 - \varepsilon$ if $w \in R$. Here $\mathrm{fix}(\sigma)$ denotes the fixed point ratio of σ. Now define the *sofic dimension* of Γ as
$$\sigma(\Gamma) = \inf_r \inf_\varepsilon \lim_{n\to\infty} \frac{\log|\mathrm{Sof}(r, \varepsilon, n)|}{\log|\mathrm{Sym}(n)|}$$
This is a finitary version of the free entropy dimension. Relate it to the first L^2 Betti number of Γ. Can one define the sofic dimension of a m.p. action accordingly?
13. If an invariant percolation on a Cayley graph dominates the FSF and is finitely dependent, must it be connected a.s.?
14. Take an action of a free group with positive f-invariant (in terms of Bowen). Does it factor on an i.i.d.?
15. Let G be a strongly ergodic, bounded degree graphing that weakly contains a finite graph H. Does G factor on H?
16. Are periodic measures dense among all invariant measures for one relator groups?

5 Outcome of the Meeting

The meeting was a clear success in every aspect. The participants learned a lot of exciting new math. It was nice to observe as people in somewhat distant subjects, like Bowen, Szegedy, Lyons and Abert, had very similar questions and were speaking each other's mathematical language naturally. A lot of new questions (and some answers) have been found. As a result of the meeting, we expect further interaction and maybe collaboration among the participants.