# REPORT ON THE 5-DAYS WORKSHOP "EXTREMAL KÄHLER METRICS" BANFF, JUNE 28TH-JULY 3RD 2009

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The workshop *Extremal Kähler metrics*, held at BIRS from June 28 to July 3 2009, was the occasion for a broad overview of the most recent advances and active interactions in a very central topic of current research, situated at the crossroad of seemingly separate disciplines. The problem itself, first proposed by Calabi in the 1980s, asks whether one can find canonical Kähler metrics, called *extremal*, in a given Kähler cohomology class of a compact complex variety. In rough terms, the search of extremal Kähler metrics is a natural generalisation of the existence of metrics of constant Gauss curvature on 2-dimensional surfaces.

The key point is to decide what one means by *canonical* metric among all the metrics (which form an infinite dimensional space). Inspired by the case of two-dimensional surfaces, a natural choice of "best" metric would be the one with has curvature as close to constant as possible. In dimensions great than two, various notions of curvature exist forming a natural hierarchy of sectional, Ricci and scalar curvatures.

The problem of characterising complex Kähler manifolds which admit a metric with constant holomorphic sectional curvature was completely solved in the first part of the twentieth century; it turns out that very few differentiable manifolds admit such metrics.

Great attention has since been given to the case of constant Ricci curvature. This natural geometric condition is equivalent to a Riemannian version of the Einstein equations, and the corresponding system of PDEs describing Einstein metrics stimulated a huge amount of research both from the geometrical and analytical viewpoints. In the context of complex geometry, this system was reduced to a single complex Monge-Ampère equation. For compact Kähler manifolds, work of Aubin and Yau in the 1970s gave a complete solution to the existence of Kähler metrics of constant non-positive Ricci curvature.

For Kähler manifolds whose first Chern class is not a multiple of a Kähler form, there can exist no Kähler-Einstein metrics. This leads one to consider the natural generalisation of Kähler metrics of constant scalar curvature.
In the 1950s Matsushima found an obstruction to the existence of such metrics in terms of the structure of the group of biholomorphisms. In the 1980s Futaki found a more subtle obstruction using the same group. This has been further generalised by Yau, Tian and Donaldson to obtain a much deeper obstruction related to stability of the Kodaira embeddings of a given polarised projective variety. It is conjectured that this so-called *K-stability* is the full obstruction to the existence of a Kähler metric on constant scalar curvature on a polarized projective variety. This indicates a fundamental difference between the complex and real settings—in the real case the existence of constant scalar curvature is unobstrucuted by Aubin–Schoen's solution of the Yamabe problem.
It was at the beginning of the 1980s that Calabi identified a natural and weaker condition on the curvature of a Kähler manifold which he called *"extremal Kähler"*. While again, as discovered few years later by Levine, not every manifold admits an extremal metric, it has been explicitly conjectured by Tian that such an existence result is true up to a finite number of degenerations of the complex structure.
The fundamental question is then to identify which manifolds admit any of the metrics described above, and possibly give an explicit description of the metric itself.
When seeking any of the above special Kähler metrics, it is always convenient to translate the system of PDE into a single PDE on one smooth function—the Kähler potential: given one representative $\omega$ of the cohomology class, one can deform it by a function $\omega + i\partial\bar{\partial} f$ to any other. This approach leads to a number of natural questions:

(1) Describe the geometry of the space of Kähler potentials for a fixed cohomology class on a compact complex manifold. Stated differently, one would like to know is there a preferred metric on this infinite dimensional space of functions and what are the geometrical properties describing the extremal Kähler metrics ?
(2) Which is the best way to deform a metric to the *best* representative of its class? What happens to the deformation of the metric (e.g. in terms of Gromov–Hausdorff convergence) when the manifold does not admit an extremal metric? Does the method still identify an inetersting geometric

object (which will not be a metric and might be singular under these assumptions) ?

The workshop put together about 35 international experts in this field, varying from algebraic geometers through differential geometers to experts in PDE. They presented very recent results on all of the above mentioned general problems, some before the results were available in written form. We now review in more detail the 17 one-hour talks that were given during the workshop.

Thanks to the work of Mabuchi–Semmes–Donaldson, the space $\mathcal{H}$ of all Kähler potentials on a compact Kähler manifold $X$ is equipped with a structure of infinite dimensional Frechét Kähler manifold: the corresponding metric is essentially the $L^2$-metric on $\mathcal{H}$ and has the structure of an infinite dimensional symmetric space of negative curvature. On the other hand, in the case when $X$ is a polarized variety, $\mathcal{H}$ naturally contains finite dimensional submanifolds of the Kähler potentials corresponding to the pull-backs of the Fubini-Study metrics induced by some embedding into a projective space. A fundamental result of Tian and shows that these "projective potentials" are dense in $\mathcal{H}$. It is then of crucial importance to understand how all geometrics notions (such as geodesics, energy functionals and geometric flows) on $\mathcal{H}$ can be approximated by the corresponding quantities on these submanifolds. This was the central theme in the talks of Fine, Sun and Berndtsson.

J. Fine (Brussels University) focused on the problem of approximating the Calabi flow on $\mathcal{H}$ via algebraic flows (called *balancing* flows) on the finite dimensional approximations. This gives a dynamical interpretation of Donaldson's theorem stating that if a constant scalar Kähler metric exists then it is the limit of special points (called balanced metrics) in the spaces of projective potentials.

S. Sun (University of Wisconsin) reported on his joint work with X.-X. Chen with a similar theme, giving a proof of the $C^0$-convergence of the smooth geodesics in the finite dimensional symmetric spaces to the Chen geodesic in $\mathcal{H}$. This was used to show that the geodesic distance of Mabuchi–Semmes–Donaldson's metric on $\mathcal{H}$ is the limit (with a controlled rate) of the distances in the approximating finite dimensional symmetric spaces. This leads to a proof of the fact that $\mathcal{H}$ has non-positive curvature in the sense of Alexandrov, as well as of some key K-energy inequalities previously obtained by X.-X. Chen.

B. Berndtsson (University of Gothenberg) presented an alternative approach to results on the geometry of $\mathcal{H}$. One aspect of this new approach is a theorem on positivity of vector bundles that appear as direct images of adjoint bundles of semi-positive line bundles over a Kähler fibration. Secondly he obtains improved convergence of the geodesics in finite dimensional symmetric spaces to Chen's geodesic, and, in joint work with J. Berman, a new proof (using a weaker regularity of geodesics) of the Bando–Mabuchi uniqueness theorem for Kähler–Einstein metrics.

The idea of approximating geometric quantities on $\mathcal{H}$ with corresponding quantities on the finite dimensional symmetric spaces of projective potentials is at the heart of the Yau-Tian-Donaldson conjecture. Roughly speaking, this conjecture predicts that the existence of a constant scalar curvature Kähler metric is governed by algebraic properties of all possible algebraic degenerations of complex structure of the original manifold induced by the actions of $\mathbb{C}^*$-subgroups on the projective embeddings of the variety. A version of this conjecture has also been proposed in the case of extremal metrics by Székelyhidi using a relative version of stability. Such algebraic degenerations have been called *special degenerations* by Ding-Tian or *test configurations* by Donaldson, and the corresponding key algebraic property required is called *K-stability*. Its precise relation to other GIT stability notions is still under scrutiny. The solution to the above Conjecture is the Holy Grail of the subject and most of the activity of the Workshop focused on recent progress on this problem. Thanks to the results of Tian-Donaldson-Mabuchi and Stoppa half of the Conjecture has been proved: the existence of the *best metric* implies *K-stability* for smooth algebraic manifolds. Moreover, Donaldson proved that *K-stability* implies asymptotic Chow stability (a more classical algebraic GIT notion) when the projective manifold has no continuous families of automorphisms, while in the general case Mabuchi has found an obstruction for the same result to hold.

R. Thomas (Imperial College) reported on his joint work with J. Ross on an extension of the above conjecture to the case of orbifolds with cyclic singularities, where the main theory and known results for manifolds can be proved to hold. In particular, this provides an alternative understanding of an obstruction to the existence of Einstein–Sasaki metrics (in one real dimension higher) found by Martelli-Sparks-Yau.

Y. Sano (Kyushu University) presented an important example of a manifold with constant scalar curvature Kähler metric which is therefore K-stable but *not* asymptotically Chow (semi)-stable. This shows that Mabuchi's obstruction is nontrivial.

Still on this algebraic side of the problem, A. Della Vedova (University of Parma) presented recent joint results with Arezzo and LaNave on the minimal singularities necessary to K-destabilize a manifold. Remarkably, it turns out that while degenerations of complex structure can produce extremely singular schemes, only reduced simple normal crossing varieties are enough to detect *K-stability*, hence putting analytic methods back in center-stage.

By looking at the original motivation for the definition of *K-stability*, S. Paul (University of Wisconsin) has introduced a whole new machinery, based on the asymptotic analysis of the K-energy, which lead him to a new definition of stability. The relationship between these notions, which he can show to give very promising results, and the old ones will clearly be of great importance in the near future.

Turning back to the original PDE approach to the existence of extremal metrics, it is natural to look at the successful cases for the Einstein problem, due to Aubin (negative scalar curvature), Yau (negative or zero scalar curvature) and Tian (partial answers in the positive case). One should try to understand the link between these methods and the K-stability obstruction in order to get a complete picture covering the remaining open positive Kähler–Einstein case, as well as the constant scalar curvature and extremal cases.
Calabi himself proposed a way to attack the existence problem by deforming a metric through a special curve of Kähler metrics which would converge to the sought-after special metric at time 1 (this is known as the *continuity method*).

G. Székelyhidi (Columbia University) reported on his recent work dealing with the existence of positive Kähler–Einstein metrics. He presented a method to quantitatively understand the maximal time of existence of solutions along the continuity method. Remarkably, he found a connection between the maximum value of time for which the deformation exists and the K-stability of the manifold. Besides the fact that this number is explicitly computable in many cases, this result is likely to have applications for future research.

N. Pali (University of Paris XI) and V. Tosatti (Univerity of Columbia) have analysed the continuity method in two different situations. Pali focussed on the case when the cohomology class does not support a genuine Kähler metric, but instead lies at the boundary of the Kähler cone. In joint work with J.-P.-Demailly they found a special subset of the boundary of the cone where existence and uniqueness of singular Kähler-Einstein metrics hold and gave a general theorem to detect the blow-up locus of the solutions of the equation in terms of purely algebraic data. This result clearly has great scope for further study.

Tosatti focussed on the case of zero Ricci curvature Kähler manifolds (called Calabi-Yau manifolds) and their degenerations to the boundary of the Kähler cone. This is a special case of the Demailly–Pali theorem which allows a geometrically complete description of the blow-up locus and indicates some connections of this problem to the classical birational geometry of Calabi-Yau manifolds.

Singular Kähler–Einstein metrics also played a central role in Tsuji's lecture in which he surveyed his extraordinary program relating singular Kähler–Einstein currents to the canonical model programme of Mori. In particular he explained how to link Kähler-Einstein manifolds and canonical models, families thereof, and their moduli spaces, by using metrised canonical models. He obtained canonical Kähler–Einstein currents on log canonical varieties of log general type.

Following Perelman's spectacular breakthrough in the geometrisation of three-dimensional manifolds, a number of similar methods using various geometric flows have been proposed to solve the existence problem for Kähler–Einstein and constant scalar curvature Kähler metrics. Regarding the Kähler–Einstein case, one can use the Kähler version of the Ricci flow, firstly introduced by Cao in the 1980s, which has been proved by Perelman to detect the existence as the infinite time limit of a Kähler-Einstein metric. A similar result has been proved by Tian–Zhu for solitons.

X. Zhu (Beijing University) lectured on his results with Tian establishing convergence (in the Cheeger–Gromov sense) of the normalised Kähler–Ricci flow starting from a metric which is $C^{2,\alpha}$ close to a Kähler–Ricci soliton.

B. Wang (Princeton University) showed how one can prove Tian's classification of Kähler–Einstein rational complex surfaces by using the normalized Kähler–Ricci flow. This approach has the advantage of producing Kähler–Ricci solitons in the obstructed case.

G. La Nave (Yeshiva University) reported on his joint work with Tian, in which they study singularities of the Kähler–Ricci flow in the case when the manifold is not Fano, and the link to Mori's program on classification of complex varieties. A special case of interest is when the manifold has an $S^1$ principal symmetry; then a very interesting reformulation of the problem on the stable quotient was presented that removes many of the problems with singularities by working on the total space of the flow.

Most of the results and techniques described so far focus on general algebro-geometric and analytic problems which are usually very hard to see explicitly at work. On the other hand, when the manifolds have some special structure or symmetry it is likely that variations on the general arguments can be simplified to get optimal and explicit results. An excellent example of this is provided by Donaldson's recent resolution of the main conjecture on the existence of constant scalar curvature metrics on complex surfaces with toric symmetry. It is of fundamental importance to have a good source of examples as a testing ground for the general conjectures, and also because in physical applications the manifold has symmetries and an explicit expression for the metric is crucial.

In this vein, A. Futaki (Tokyo) described Kähler–Ricci soliton metrics on the total spaces of powers of the anti-canonical bundle over a Fano manifold with a toric symmetry. The construction relies on a general existence result for toric Sasaki–Einstein metrics that he obtained in collaboration with Ono and Wang, and gives new examples of such solitons with a singularity at the zero section.

C. Boyer (University of New Mexico) introduced analogues of the notions of Kähler cone and extremal Kähler metrics on a Sasaki manifold. He discussed these notions through the use of examples, in particular toric ones, and gave a complete discription of the extremal Sasaki metrics compatible with the standard CR sphere.

C. Tonnesen-Friedman (Union College) lectured on results obtained in collaboration with V. Apostolov, D. Calderbank and P. Gauduchon, showing that the existence of an extremal Kähler metric in certain classes on the the total space of a projective bundle over a complex curve is equivalent to decomposability of the underlying vector bundle as a direct sum of stable bundles. She further specialised to the case of projective plane bundles over a curve and introduced explicit methods for settling the existence problem for extremal metrics in this case.

Thus, in 5 intensive days, the program succeeded in tying together most of the new results in the subject, and a variety of new projects and collaborations were born as a result of the many informal discussions during this week.

We take this opportunity to thank the staff of the BIRS for their highly professional work and the excellent environment that they provided for all the participants during the workshop.