A t-Pieri rule for Hall-Littlewood P-functions and $QS(t)$-functions

Soojin Cho (Ajou University)
Sarah Mason (Wake Forest University)
Jim Haglund (University of Pennsylvania)
Vasu Tewari (University of British Columbia)
Steph van Willigenburg (University of British Columbia)
Meesue Yoo (Korea Institute for Advanced Study)

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1 Overview of the Field

One of the central areas of research in algebraic combinatorics is the study of Macdonald polynomials. These were introduced in the late 1980s by Macdonald as symmetric polynomials with two parameters q, t, which reduce to certain well-known polynomials such as Hall-Littlewood P-functions, Jack polynomials and zonal polynomials when q and t are set to certain values. Despite their relatively straightforward definition in terms of a scalar product, they were notoriously difficult to manipulate until a combinatorial formula for them was discovered [2].

Another vibrant area of research in algebraic combinatorics is that of quasisymmetric functions. The Hopf algebra of quasisymmetric functions was introduced by Gessel in the 1980s as a source of generating functions for P-partitions. However, since then, like Macdonald polynomials, they have arisen in other areas of mathematics such as the representation theory of the 0-Hecke algebra, encoding the flag f-vector of posets in discrete geometry, and the study of riffle shuffles in probability.

At the natural intersection of these two areas are the symmetric functions known as Schur functions, which are refined by quasisymmetric functions and are specializations of Macdonald polynomials at $q = t = 0$. Schur functions are themselves a hub of investigation due to their appearance as the irreducible characters of the symmetric group in representation theory, connections to Schubert classes in algebraic geometry and as generating functions for tableaux in enumerative combinatorics.

2 Recent Developments and Open Problems

In [4] a new basis for quasisymmetric functions, termed quasisymemtric Schur functions, was discovered. These functions were founded on the combinatorics of Macdonald polynomials, were further developed in [1, 3], and furthermore showed that many of the beautiful properties of Schur functions can be refined to the Hopf algebra of quasisymmetric functions. Amongst the many properties that refine from Schur functions to quasisymmetric Schur functions are the expansion in terms of fundamental quasisymmetric functions, Kostka numbers, two formulations of the Littlewood-Richardson rule, and the Pieri rules. The Pieri rules for Schur functions are rules for expressing the product of a Schur function indexed by a diagram of row or column shape with a generic Schur function indexed by a diagram of partition shape as a sum of Schur functions whose indices are determined by operations on the diagram of partition shape. Similarly, the Pieri rules for quasisymmetric Schur functions are rules for expressing the product of a Schur function indexed by a diagram of row or column shape with a generic quasisymmetric Schur function indexed by a diagram of composition shape as a sum of quasisymmetric Schur functions whose indices are determined by operations on the diagram of composition shape. In all of these cases classical proofs for the Schur function scenario were able to be adapted in order to yield proofs for the quasisymmetric Schur function case.

In [4] it was shown that some of the combinatorial structure of the quasisymmetric Schur functions can be extended to include an extra parameter t. In particular they showed that these $QS(t)$-functions refine a well-known Hall-Littlewood symmetric function basis in the same way that quasisymmetric Schur functions refine Schur functions, that is, that the sum over all compositions, whose parts rearrange the parts of a given partition, of the $QS(t)$-function corresponding to the composition, equals the Hall-Littlewood P-function corresponding to the partition. Therefore, what other properties of Hall-Littlewood P-functions refine to $QS(t)$-functions?

Recently Haglund noted empirically using Maple that the expansion of the product of a Schur function and a Hall-Littlewood P-function, when expanded in terms of the P-function basis, has coefficients which are polynomials in t with nonnegative integral coefficients. In the special case when we multiply by a Schur function indexed by a diagram of row shape, Yoo refined this conjecture and obtained an elegant combinatorial description of these coefficients, called a t-Pieri rule. When $t = 0$ this reduces to the classical Pieri rule for Schur functions. Additionally, Yoo developed an elegant combinatorial conjecture for expressing the product of a Schur function indexed by a diagram of row or column shape with a $QS(t)$-function indexed by a diagram of composition shape, as a sum of $QS(t)$-functions whose indices are determined by operations on the diagram of composition shape that also generate powers of t and $(1-t)$.

Therefore, our major goal for the FRG was to prove these latter two conjecures as a means to understanding the t-Pieri rule conjecture of Yoo and eventually the conjecture of Haglund.

3 Scientific Progress Made

A key step in proving both the Pieri rules for Schur functions and the Pieri rules for quasisymmetric Schur functions is to prove the special case of a product with a Schur function indexed by the diagram consisting of one cell, called Monk's rule. Therefore our first step was to prove the conjecture for expressing the product of a Schur function indexed by the diagram consisting of one cell, s_1, with a generic $QS(t)$-function indexed by a diagram of composition shape α, QH_α.

Although we were not able to complete the proof of $QH_\alpha s_1$ during our FRG we were successful in developing a number of avenues necessary to adapt classical proofs for this case. More precisely, we

- reformulated Yoo's conjecture into a combinatorial identity equivalent to the original conjecture;
- extended the insertion algorithm of Mason, which, when fully verified, would be a fundamental ingredient to proving the above combinatorial identity bijectively;
- established that this new algorithm is reversible;
- conjectured that the change in power of t associated to a tableau before and after insertion is equal to the number of skipped steps that occur during insertion;
- extended all these facets in terms of superfillings, which generate an equivalent definition for QH_α and hence give a second approach to proving the conjecture;
- produced a plethora of data to support the algorithm.

Additionally, Yoo's conjecture was verified for the cases $QH_{(n)}s_1$, $QH_{(1^n)}s_1$, $QH_{(m1^{n-m})}s_1$ and when $t = 0$. Lastly, during an investigation of the dual Hopf algebra basis to $QS(t)$-functions, we conjectured that the image under the forgetful map is a modified Hall-Littlewood function, and that those indexed by a diagram of a partition or reverse partition shape when expressed as a sum of Schur functions have coefficients which are polynomials in t with nonnegative integral coefficients that can be explicitly computed combinatorially.

4 Outcome of the Meeting

Further to making substantial progress on our project from which a journal article will result, we also learned a range of techniques and relationships between our areas of expertise, through informal lectures we gave to each other.

We would like to thank BIRS for this invaluable Focussed Research Group opportunity.

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