New Trends in Noncommutative Algebra and Algebraic Geometry

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1 Overview of the Field

Noncommutative algebra is a rich field that has influences rooted in geometry, physics, algebraic combinatorics, and representation theory. Today, the field has many overlapping facets that spill over into other disciplines, but are nevertheless all connected by the common use of algebraic methods in their study. We briefly give an overview of some of the main trends that shape the field today, putting particular emphasis on those trends that were most represented during the workshop.

1. **Quantum Groups and Quantum Cluster Algebras**

   Quantum groups are a class of Hopf algebras that can be regarded, in a natural sense, as noncommutative deformations of classical objects (enveloping algebras of finite-dimensional Lie algebras, coordinate rings of affine algebraic groups, etc.). Recently, there has been a lot of work on Quantum cluster algebras, which were introduced by Berenstein and Zelevinsky [8]. In the non-quantum setting there has been much work showing that many classical objects are in fact cluster algebras and a common theme during the meeting was an investigation into the extent to which quantum analogues of these results hold (see for example, the papers [19, 23, 37]). In addition to this, there are many other directions of study in this area. As an example of some of the other areas of research in quantum groups, we note that these objects often support a rational action by an affine algebraic group, and this action often stratifies the prime spectrum into finitely many disjoint pieces, each of which is homeomorphic to an affine scheme of finite type. This then gives rise to many combinatorial questions: namely, how many strata are there for a given quantum algebra admitting a finite stratification and what are the dimensions of the associated affine schemes. There has been a lot of recent work inspired by these problems [6, 7, 10, 34, 35].

2. **Cherednik algebras**

   Cherednik algebras are, like quantum groups, noncommutative deformations of more classical objects—in this case, skew group rings of polynomial rings by a finite group. In fact, much of the machinery developed in the study of quantum groups applies in this situation. The development of this facet of noncommutative algebra has been very rapid since being introduced by Etingof and Ginzburg [18] in 2002. Many conjectures in algebraic combinatorics have in fact been answered using the Cherednik algebra machinery. See, for example, Gordon [20] to answer a conjecture of Haiman, by Berest, Etingof and Ginzburg [9] to answer questions about rings of quasi-invariants that arise in integrable systems. The immense recent interest in this area and rapid development has led to a large number of open problems in this area.

3. **Noncommutative surfaces** Since curves, having one parameter, are commutative, (or at least finitely generated modules over their centres), as exhibited in [2], [3], it is natural to study noncommutative surfaces. In commutative algebraic geometry there is a classification of surfaces with the most important Theorems being Castelnouvo's and Enriques' which characterize rational and ruled surfaces respectively. Artin's conjecture predicts that noncommutative surfaces are either rational, ruled or module finite over their centres. The study of noncommutative rational surfaces produced seminal works in this area [4], [5], and new techniques and objects such as twisted homogeneous coordinate rings. Chan and Nyman have shown a version of Enriques' Theorem in the noncommutative setting [14].

4. **Hopf Algebra Actions**

   In the noncommutative world, Hopf algebra actions on Artin-Schelter regular algebras forms a natural setting. The classification of finite quantum subgroups of quantum SL_2, namely, so-called quantum binary polyhedral groups, has been worked out recently following the classification of Artin-Schelter regular algebras of dimension 2 [17]. This leads to the study of Kleinian or DuVal singularities of Hopf actions on noncommutative surfaces. Invariant theory of finite group actions (as well as finite dimensional Hopf algebra actions) on Artin-Schelter regular algebras of higher global dimension has been developed, see [28, 29, 30], which includes noncommutative versions of several classical results such as Watanabe theorem (a criterion for the fixed subrings being Gorenstein), Shephard-Todd-Chevalley theorem (a criterion for the fixed subring being regular) and Kac-Watanabe theorerm (a criterion for the fixed subrings being a complete intersection). These studies are closely connected to the representation theory of finite group/Hopf algebras, homological aspects of noncommutative algebra and noncommutative algebraic geometry.

5. **Calabi-Yau algebras and skew (or twisted) Calabi-Yau algebras**

   Calabi-Yau algebras were introduced by Ginzburg in 2006 as a noncommutative version of coordinate rings of Calabi-Yau varieties, and Calabi-Yau triangulated categories were introduced by Kontsevich in 1998. Since then, the study of Calabi-Yau algebras/categories has been related to a large number of research areas such as quivers with superpotentials, differential graded algebras, cluster algebras/categories, string theory and conformal field theory, noncommutative crepant resolutions, and so on. In addition to work of Kontsevich and Ginzburg, some foundations of the topic have been established by Keller and Van den Bergh recently. Skew (or twisted) Calabi-Yau algebras are generalization and a companion of Calabi-Yau algebras. For example, all Artin-Schelter regular algebras are skew Calabi-Yau and only a subclass of Artin-Schelter regular algebras are Calabi-Yau. On the other hand, Calabi-Yau algebra can be produced by using skew Calab-Yau algebras. One invariant attached to a skew Calabi-Yau algebra is its Nakayama automorphism. Several identities about the Nakayama automorphism have been proved. These identities helps to understand the questions such as when a smash product of Calabi-Yau algebra with Hopf algebra is Calabi-Yau. Associated to Calabi-Yau algebras there are various combinatroial data (such as super-potential) which might be calculated by computer algorithms.

2 Recent Developments and Open Problems

We summarize some of the main open problems that exist in the area today. Again, we place emphasis on the problems that were viewed as most important by speakers at the conference and we do not claim to give a complete list of all of the most important open problems in the area.

1. *Artin's proposed birational classification of noncommutative projective surfaces*

   Artin [1] conjectured that every division algebra of transcendence degree two that comes from a noncommutative projective surface is either finite-dimensional over its center or is isomorphic to a Skylanin division ring or to a division ring of the form $K(x;\sigma,\delta)$, where K is the function field of a smooth projective curve, σ is an automorphism of K, and δ is a σ-derivation of K. By looking at rank one discrete valuations of these division algebras, it can be shown that they are non-isomorphic. This conjecture is very natural after the landmark work of Artin and Stafford [2, 3] that classified all noncommutative projective curves (or noncommutative graded domains of Gelfand-Kirillov dimension two). It is generally believed that the conjecture is currently beyond available methods, but there is a large amount of work devoted to understanding this conjecture. As some examples, we note the work of Chan, Hacking, Ingalls, Kulkarni [11, 12, 13], the manuscript by Artin and de Jong, and the series of papers by Keeler, Rogalski, Stafford and Sierra [27, 33], some of which look at classifying projective surfaces in a given birational class on Artin's list.

2. *Understanding which quantum groups are quantum cluster algebras*

The notion of quantum cluster algebras was introduced by Berenstein and Zelevinsky [8]. These algebras are noncommutative deformations of cluster algebras and over the past few years there has been a large amount of work devoted to understanding their basic properties. In the classical setting, there is now a wealth of examples of classical objects that are cluster algebras. In the quantum setting, however, there are many questions which ask if quantum analogues of these classical results hold. As a few of the more important examples, we note that Berenstein and Zelevinsky conjectured that the quantized coordinate ring of a double Bruhat cell is in fact a quantum cluster algebra, which was proved by Berenstein, Fomin, and Zelevinsky in the classical setting. Another interesting case is the quantum Grassmannian, which has been shown to be a quantum cluster algebra in only a few cases [23].

3. *Problems on Hopf actions*

One influential program is the quantum McKay correspondence. There are several proposals in the literature, but there is not one that is currently sufficiently broad to include many noncommutative algebras/schemes. This program involves several pieces: (a) Hopf algebra actions on noncommutative algebras/schemes, (b) quotient singularities of noncommutative schemes, (c) representation theory of Hopf algebras and fixed subrings, (d) homological algebra aspects (such that various derived categories are related), (e) combinatorial information coming from algebra, geometry and representation theory, and the most important of all, (f) the connection between these pieces. There is some work in dimension 2 which had been mentioned during the workshop. As the classification of Artin-Schelter regular algebras of dimension 3 is now known, thanks to the work of Artin, Schelter, Tate and Van den Bergh [5], this question could be tractable in dimension 3, and would lead to significant new mathematics.

3 Presentation Highlights

Many participants commented that the level of quality of the lectures at this conference was very high and there were many great presentations during the course of the week.

Many of the talks dealt with the burgeoning area of Hopf algebra actions on algebras. There is a rich history of studying group actions on algebras and the notion of Hopf algebra actions is a natural extension. Daniel Chan, Kenneth Chan, Chelsea Walton and others discussed new developments in this area and gave many open problems that have arisen during the course of their investigations. In particular, Daniel Chan's talk considered the action of Hopf algebras on projective varieties and on twisted homogeneous coordinate rings of projective varieties. Their work also involved results with Ellen Kirkman, Yanhua Wang, and James Zhang [15, 16, 17].

Ellen Kirkman and Ken Brown gave closely related talks to those of Daniel Chan, Kenneth Chan, and Chelsea Walton. Ken Brown looked at the problem of determining the Hopf algebras over a field of characteristic zero that can be viewed as deformations of the commutative polynomial algebra in n variables over k. This is a problem that generated some interest at the conference and Ken Brown initiated related work with other participants on this problem while at the meeting.

Ellen Kirkman spoke on a closely related topic of the ring of invariants A of a graded ring R under the action of a finite group G of graded automorphisms. Such algebras have been extensively studied in the commutative case and in particular one has a beautiful characterization, due to Gulliksen [24] of when such algebras are complete intersections. This holds when any of the following equivalent conditions holds:

(a) the Ext-algebra $E(A) := \bigoplus_{n=0}^{\infty} \mathrm{Ext}_A^n(k,k)$ of A has finite GK-dimension;

(b) the Ext-algebra $E(A)$ is noetherian.

This naturally leads one to the notion of a noncommutative complete intersection. Kirkman described her joint work with James Kuzmanovich and James Zhang [30] on the relationship between these two notions in the noncommutative case—specifically when the algebra A is of the form R^G for some Artin-Schelter regular algebra R and some finite group G.

Birge Huisgen-Zimmerman gave an excellent survey on the so-called Finitistic Dimension Conjectures for finite-dimensional algebras, which are usually attributed to Bass. Although the conjectures themselves are roughly fifty years old, a large amount of progress has been made in the past ten years. In particular, the lecture examined the representation theory and homology of biserial algebras, a class of algebras that encode the representation theory of the Lorentz group.

Some of the presentations dealt with quantum groups and the connections between noncommutative algebra and algebraic combinatorics and geometry.

Ken Goodearl described his recent work with Milen Yakimov [19]. This work extends many classical results on unique factorization in coordinate rings. Due to the pioneering work of Fomin and Zelevinsky [25, 26], there has been considerable interest in the study of cluster algebras. In particular, many coordinate rings of affine varieties have been shown to be cluster algebras and, moreover, many of these families of cluster algebras have also been shown to be UFDs. The noncommutative analogue of being a unique factorization domain involves a condition on height one prime ideals being principal and generated by a normal element—this definition is equivalent to being a UFD in the commutative case. The notion of a quantum cluster algebra is a relatively recent concept and not as much work has been done regarding whether certain quantized coordinate rings are quantum cluster algebras and whether they are noncommutative UFDs. Ken Goodearl's presentation discussed how for a large class of quantum algebras one has that they are both quantum cluster algebras and noncommutative UFDs.

Stéphane Launois discussed minimal conditions on positivity of minors in a matrix needed to verify that it is either totally positive or totally nonnegative [31]. This is a problem of some interest within the combinatorial community. Launois showed how one could give algorithms inspired by techniques from the study of quantum groups. In particular, Cauchon's deleting derivations algorithm can be used to give an algorithm that requires $\mathrm{O}(n^4)$ operations to determine if a matrix is a totally nonnegative matrix in a given totally nonnegative cell. (The complexity of the "naïve" algorithm grows exponentially with n.)

Some of the lectures gave overviews of recent solutions and counter-examples to previously open problems from the literature. David Saltman spoke on joint work with Louis Rowen [32], in which they give a surprising counter-example to the question of whether $D_1 \otimes_F D_2$ is necessarily a domain when D_1 and D_2 are division algebras and F is an algebraically closed field that is contained in the centres of both D_1 and D_2. It is well-known that when D_1 or D_2 is a field then the tensor product does not have zero divisors, but the general problem had been open for many years. By using a combination of geometric techniques and results from the theory of Brauer groups, Saltman was able to produce explicit examples of tensor products that meet the conditions described above and which fail to be domains.

Milen Yakimov's presentation gave an overview of his recent proof of two conjectures in the field of quantum groups [36]. Namely, he gave proofs of both the Andruskiewitsch-Dumas and Launois-Lenagan conjectures, which give a conjectural classification of the automorphism groups of certain families of noncommutative associative algebras. It should be pointed out that the method of his proof in fact applies to a very large class of noncommutative algebras that contains the families examined by both Andruskiewitsch-Dumas and Launois-Lenagan. In particular, he gave a general rigidity theorem that can be applied in many situations to find the automorphism group of an associative algebra.

Toby Stafford spoke on joint work with Iain Gordon [21, 22] on equidimensionality of charactersitic varieties over Cherednik algebras and $\mathbb{Z}$-algebras. Gordon and Stafford have developed a great deal of machinery during the course of their study of Cherednik algebras and it has resulted in many successes, including Gordon's proof of Haiman's deep $n!$-Theorem. Stafford's lecture showed that as an application of the homological machinery he developed with Gordon, one can generalize work of Gabber's on unitary algebras, thus providing an answer to another open problem.

There was also some work that dealt with topics that are of some interest in physics, in particular in string theory: Calabi-Yau algebras and Mirror symmetry. In particular, Raf Bocklandt discussed connections between noncommutative projective geometry and Artin-Schelter regular algebras and mirror symmetry for Riemann surfaces. These connections have not been fully explored at this point, but they will undoubtedly lead to new results in noncommutative geometry.

Daniel Rogalski spoke on twisted Calabi-Yau algebras—his talk concerned forthcoming joint work with James Zhang and Manuel Reyes. These are algebras with some especially nice homological properties. In the connected graded setting, they are essentially just the Artin-Schelter regular algebras, which have a dualizing complex equal to a complex shift of the algebra, possibly twisted on one side by an automorphism (the Nakayama automorphism). He discussed some nontrivial formulas for what happens to the Nakayama automorphism when one takes a smash product with a Hopf algebra action or does a graded twist. Our methods rely on graded local cohomology and the formulas involve the notion of the homological determinant of an automorphism. We also note that in wide generality, the homological determinant of the Nakayama automorphism is equal to 1.

Quanshui Wu also spoke on Calabi-Yau algebras and focused on the problem of how to compute the Hochschild and cyclic cohomology for three-dimensional graded Calabi-Yau algebras via the Poisson cohomology for Poisson algebras.

There were some talks related to noncommutative projective geometry and to representations of Clifford algebras. Representations of a Clifford algebras of homogeneous forms correspond to the class of Ulrich bundles on the associated hypersurface. Rajesh Kulkarni spoke about how the connection can be used to study the vector bundles on cubic and quartic surfaces, and in particular can be used to show that a smooth quartic surface is a linear Pfaffian. He also described how one can construct stable Ulrich bundles on a class of three-folds by using the so-called twisted tensor product construction in Clifford algebras.

Susan Sierra spoke on forthcoming joint work with Daniel Rogalski and Toby Stafford on maximal orders in the Sklyanin algebra. Her talk dealt with understanding the maximal orders in the three-Veronese subalgebra of a generic Skylanin algebra, and showed that each subalgebra could be interpreted, in a natural way, as a blow-up of the ambient algebra at a divisor on an associated elliptic curve E. As a result, they are able to classify all subalgebras in the three-Veronese subalgebra of a generic Sklyanin algebra that are maximal orders. This provides an complete study of noncommutative Del Pezzo surfaces and represents a large first step in trying to understand all connected graded algebras of GK-dimension three that are birationally isomorphic to a Sklyanin algebra.

Finally, Louis Rowen gave a survey of the polynomial identities of algebras with involution, with a particular focus on Specht's problem.

4 Scientific Progress Made

As a result of this meeting, many people looking at different facets of noncommutative algebra were able to have discussions and start projects with people working on related areas. In fact, there are many examples of projects that were begun at this meeting (we give a few testimonials to this effect in the following section). We note that this meeting had many younger researchers, postdocs, and graduate students and this meeting was particularly useful in allowing them to network and create joint projects with more senior researchers. One of the often repeated remarks made by different participants was an interest in Hopf algebra actions—some of these participants had not previously been introduced to this thriving area and many expressed interest in working on some of the problems posed by younger researchers during the meeting.

5 Outcome of the Meeting

This workshop brought together 38 researchers, which included a mix of Ph.D. students, several postdocs, and researchers from Europe, Australia, China, and North and South America, working in different areas of noncommutative algebra and geometry, including ring theory, noncommutative algebraic geometry, representation theory, the study of Hopf algebra actions, and quantum groups.

Many of the participants privately remarked that this was one of the best conferences to which they had been. There were many great talks and several of the participants gave testimonials about research programs that they started at Banff—we include a few examples below.

Sarah Witherspoon said: "The workshop was great, and I am very happy that I went. The lectures were largely very good, with several on topics very close to my own research, and others further away but on topics that will benefit my research program as I learn more. I particularly liked the variety of speakers, including young mathematicians working on very current problems. The workshop also gave me a very valuable opportunity to meet with collaborators who were there: I am starting to work on a project with Chelsea Walton, and it was helpful to touch base with her in Banff. I have been discussing with Nicolas Andruskiewitsch a plan to (at least partially) prove a conjecture on cohomology of Hopf algebras, and it was important that we were able to talk in person in Banff over a period of several days."

Ulrich Kraehmer said: "Many talks raised more questions than giving answers and equipped the audience with some ideas that the speaker had. This really is the most inspiring way to hold workshops, and together with another participant I have started thinking seriously about one of the questions that have been asked."

Ken Brown said: "Discussions with James Zhang and his student Guangbin Zhuang following my seminar led to significant developments in our ongoing research programme on connected Hopf algebras, which will certainly greatly improve the paper I am currently preparing on this, and may well lead to James and Guangbin becoming co-authors."

Tomasz Brzenski said: "My own research is in differential aspects of Noncommutative Geometry. This means that I am studying algebras of the kind similar to those discussed during the workshop, but from a different point of view. Most of the lectures were very useful for me, allowing me to become more familiar with both mathematical concepts and literature as well as to learn new techniques and approaches. I am convinced that I will use these while working on my current project that involves studying of quantum orbifolds obtained by actions (of groups and Hopf algebras) on quantum spaces. Definitely, the meeting in Banff was one of the most exciting meetings I attended in the past few years."

Stéphane Launois wrote: "This meeting has been a great opportunity for me to present my latest work, and to hear about the latest developments in the field. It allowed me to meet my collaborators from America. In particular, I had the chance to discuss with Karel

Casteels his recent work (posted on the arXiv the week before the workshop took place), so he could explain me the general strategy of his proof of the Goodearl-Lenagan Conjecture. This is especially appreciated as his paper is more than 50 pages long, and so having the insights of the author is crucial in order to fully understand the details of his proof. Finally, the workshop certainly allowed me identify new directions of research which I am now planning to pursue with my PhD students. Overall this has been a great workshop which will certainly influence my future research!"

Martin Lorenz wrote: "The conference was outstanding; the lectures presented a comprehensive panorama of a large portion of current noncommutative algebra. I also very much appreciated the thorough coverage that quantum invariant theory received during the conference. This will be most useful to my own work and that of my students."

Milen Yakimov said the following: "The BIRS workshop was a great opportunity for me to learn about the current developments in the area of noncommutative algebraic geometry and its relations to Hopf algebras and representation theory. It will have a great influence on my future research, both in terms of working on problems related to the talks and using methods described in lectures. The workshop was also very stimulating in relation to working on ongoing research projects with other participants."

Following this workshop, there will be a closely related half-year program at MSRI that will be held from January to May, 2013. Many participants have pointed out that the BIRS workshop has really provided a focus to their research and will have some role in shaping the research conducted at MSRI.

References

[1] M. Artin, Geometry of quantum planes, *Azumaya algebras, actions, and modules* (Bloomington, IN, 1990), 1–15, Contemp. Math., 124, AMS, Providence, RI, 1992.

[2] M. Artin and J. T. Stafford, Noncommutative graded domains with quadratic growth, *Invent. Math.,* **122** (1995) 231-276.

[3] M. Artin and J. T. Stafford, Semiprime graded algebras of dimension two, *J. Algebra* **227** (2000), no. 1, 68–123.

[4] M. Artin, J. Tate, and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, *The Grothendieck Festschrift,* Vol. I, 33–85, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990.

[5] M. Artin, J. Tate, and M. Van den Bergh, Modules over regular algebras of dimension 3, *Invent. Math.* **106** (1991), no. 2, 335–388.

[6] J. Bell, K. Casteels, S. Launois, Enumeration of torus-invariant strata with respect to dimension in the big cell of the quantum minuscule Grassmannian of type Bn. New trends in noncommutative algebra, 27–40, *Contemp. Math.,* **562**, Amer. Math. Soc., Providence, RI, 2012.

[7] J. Bell, K. Casteels, S. Launois, Enumeration of H-strata in quantum matrices with respect to dimension. *J. Combin. Theory Ser. A* **119** (2012), no. 1, 83–98.

[8] A. Berenstein and A. Zelevinsky, Quantum cluster algebras, Adv. Math. **195** (2005), no. 2, 405–455.

[9] Y. Berest, P. Etingof, V. Ginzburg, Cherednik algebras and differential operators on quasi-invariants. *Duke Math. J* **.118** (2003), no. 2, 279–337..

[10] K. Casteels, Quantum Matrices by Paths, arXiv:1210.6399.

[11] D. Chan, P. Hacking, C. Ingalls, Canonical singularities of orders over surfaces, *Proc. Lond. Math. Soc.* (3) **98** (2009), no. 1, 83–115.

[12] D. Chan and C. Ingalls, Noncommutative coordinate rings and stacks, *Proc. London Math. Soc.* (3) **88** (2004), no. 1, 63–88.

[13] D. Chan, R. Kulkarni, Numerically Calabi-Yau orders on surfaces, *J. London Math. Soc.* (2) **72** (2005), no. 3, 571–584.

[14] D. Chan, A. Nyman, Noncommutative Mori Contractions and P1 Bundles, arXiv:0904.1717.

[15] K. Chan, C. Walton, J. J. Zhang, Hopf actions and Nakayama automorphisms, arXiv:1210.6432.

[16] K. Chan, C. Walton, Y.-H. Wang, J. J. Zhang, Hopf actions on filtered regular algebras, arXiv:1211.6513.

[17] K. Chan, E. Kirkman, C. Walton, J. J. Zhang, Quantum binary polyhedral groups and their actions on quantum planes, in preparation.

[18] P. Etingof, V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. *Invent. Math.* **147** (2002), no. 2, 243–348.

[19] K. R. Goodearl, M. T. Yakimov, From quantum Ore extensions to quantum tori via noncommutative UFDs, arXiv:1208.6267.

[20] I. Gordon, On the quotient ring by diagonal invariants. *Invent. Math.* **153** (2003), no. 3, 503–518.

[21] I. Gordon, J. T. Stafford, Rational Cherednik algebras and Hilbert schemes. II. Representations and sheaves, *Duke Math. J.* **132** (2006), no. 1, 73–135.

[22] I. Gordon, J. T. Stafford, Rational Cherednik algebras and Hilbert schemes, *Adv. Math.* **198** (2005), no. 1, 222–274.

[23] J. Grabowski and S. Launois, Quantum cluster algebra structures on quantum Grassmannians and their quantum Schubert cells: the finite-type cases, *Int. Math. Res. Not.* (2011), no. 10, 2230–2262.

[24] T. Gulliksen, A homological characterization of local complete intersections. *Compositio Math.* **23** (1971), 251–255.

[25] S. Fomin and A. Zelevinsky, Cluster algebras I. Foundations, *J. Amer. Math. Soc.* **15** (2) (2002) 497–529.

[26] S. Fomin and A. Zelevinsky, Cluster algebras. II. Finite type classification, *Invent. Math.* **154** (2003), no. 1, 63–121.

[27] D. Keeler, D. Rogalski, J. T. Stafford, Naïve noncommutative blowing up. *Duke Math. J.* **126** (2005), no. 3, 491–546.

[28] E. Kirkman, J. Kuzmanovich, J.J. Zhang, Shephard-Todd-Chevalley Theorem for skew polynomial rings, *Algebr. Represent. Theory*, **13** (2010), no. 2, 127–158.

[29] E. Kirkman, J. Kuzmanovich, J.J. Zhang, Gorenstein subrings of invariants under Hopf algebra actions, *J. Algebra* **322** (2009), no. 10, 3640C3669.

[30] E. Kirkman, J. Kuzmanovich, J.J. Zhang, Noncommutative complete intersections, in preparation.

[31] S. Launois, T. H. Lenagan, Efficient recognition of totally nonnegative matrix cells, arXiv:1207.3613.

[32] L. Rowen, D. Saltman, Tensor Products of Division Algebras and Fields, arXiv:1106.5517.

[33] S. Sierra, Classifying birationally commutative projective surfaces, *Proc. Lond. Math. Soc.* (3) **103** (2011), no. 1, 139–196.

[34] M. Yakimov, On the spectra of quantum groups, arXiv:1106.3821.

[35] M. Yakimov, Strata of prime ideals of De Concini-Kac-Procesi algebras and Poisson geometry, arXiv:1103.3451.

[36] M. Yakimov, The Launois-Lenagan conjecture, arXiv:1204.5440.

[37] A. Zelevinsky, Cluster algebras: origins, results and conjectures, *Advances in algebra towards millennium problems*, 85–105, SAS Int. Publ., Delhi, 2005.