Report for BIRS Workshop 16rit691: New applications of Menger Curvature to Complex Analysis

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1 Introduction

This serves as a preliminary report of our activities during the Research in Teams Workshop on *New applications of Menger Curvature to Complex Analysis* (BIRS Workshop 16rit691) that took place August 10 - 17, 2016. We have made substantial progress and are in the process of writing a research article for submission in a peer-reviewed journal in which we will acknowledge BIRS for providing critical support to this project (we anticipate completion by December 2016).

For the background and history of this project we refer to the workshop proposal (submitted in January 2016): this report will focus solely on the progress made during our Research in Teams Workshop.

2 New results in complex dimension 1

We say that a singular integral operator T with Schwartz kernel $K(z,w)$ has *Property A relative to a measure* μ if the three-point symmetrization of $K(z,w)$ for any three distinct points $\{z_1,z_2,z_3\}$ that lie in the support of μ is a positive function of $\{z_1,z_2,z_3\}$ which factors as $F(z_1,z_2,z_3)\,c^2(z_1,z_2,z_3)$ for some (positive) function $F(z_1,z_2,z_3)$, and with $c(z_1,z_2,z_3)$ denoting the Menger curvature for the triple $\{z_1,z_2,z_3\}$.

Mattila, Melnikov and Verdera [10] proved that a certain version of the classical Cauchy integral has *Property A* relative to a class of measures supported on a Lipschitz curve, in fact with $F(z_1,z_2,z_3)\equiv 1$, and this led them to a new proof of the the $L^2(\mu)$-regularity of the Cauchy integral that relies on a "simplified version" [13] of the earlier $T(1)$-theorem [1] which had been originally manufactured to study the Cauchy Integral for Lipschitz curves in $\mathbb{C}$.

During our Research in Teams Workshop we have proved that if T satisfies *Property A* under the more general condition that if $F(z_1,z_2,z_3)\in L^\infty(\mu)$ then T is bounded on $L^2(\mu)$ via a "simplified version" of the classical $T(1)$-theorem that extends Tolsa's original treatment [13] (which dealt with the special case when T is the Cauchy integral and $F(z_1,z_2,z_3)\equiv 1$).

In earlier work pre-dating our BIRS Workshop, we had proved that a number of operators related to the single- and double-layer potentials (which are inherently "real", yet arise from a modified Cauchy kernel) satisfy *Property A* with non-constant factor $F(z_1, z_2, z_3)$, thus opening the road to a new proof (in progress) of $L^2(\mu)$-regularity for these operators, that will rely on a "simplified version" of $T(1)$.

3 New results in complex dimension $n \geq 2$

Property A is of course meaningful also in higher dimension (for measures supported on, say, hypersurfaces in $\mathbb{R}^n$). During our Research in Team Workshop we have proved that none of the classical operators of Complex and Harmonic Analysis has *Property A*. Specifically, we have proved that none among: the Cauchy-Leray integral; the essential part of the Cauchy-Leray integral; the Henkin-Ramirez integral; the Double-Layer potential; the Szegő projection, has *Property A* – with failure occurring already in the simplest possible setting of measures supported in the unit ball of $\mathbb{C}^2$ (or, for the double layer potential, the unit ball of $\mathbb{R}^4$).

One can also state a version of *Property A* that relies on the $(2p+1)$-point symmetrization of the Schwartz kernel for some suitable choice of $p \neq 1$ related to, say, the dimension of the support of the measure, and this would lead to the study of $L^{2p}(\mu)$-regularity of T (as opposed to $L^2(\mu)$-regularity, corresponding to $p = 1$): alas, during our Research in Teams Workshop, we have proved that the double layer potential turns out to fail this more flexible version of *Property A*, and we expect that all other operators (mentioned above) will also fail it.

4 Conclusion

In complex dimension one *Property A* is valid for an extended class operators that includes the variant of the Cauchy Integral studied by Mattila, Melnikov and Verdera, and this leads to a new proof of $L^2(\mu)$-regularity for all these operators that relies on a simplified version of $T(1)$.

On the other other hand, in higher dimension none of the classical operators of complex function theory (nor of potential theory) satisfies *Property A* (nor more flexible versions of it); thus to study these operators' regularity in $L^p(\mu)$, one can solely rely on the previously versions of $T(1)$ (as opposed to new, "simplified versions").

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