Geometric Analysis and General Relativity

Greg Galloway (University of Miami),
Daniel Pollack (University of Washington),
David Maxwell (University of Alaska Fairbanks),
Richard Schoen (University of California, Irvine)

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1 Overview of the Field

General Relativity is one of the fundamental theories of modern physics. Since its formulation by Einstein in 1915, it has been a cornerstone of our understanding of the universe. Mathematical research on the problems of general relativity brings together many important areas of research in partial differential equations, differential geometry, dynamical systems, analysis and numerical analysis. The fundamental mathematical questions which arise in general relativity have stimulated the development of substantial new tools in analysis and geometry. Conversely, the introduction of modern tools from these fields have recently led to significant developments in general relativity, shedding new light on a number of deep and far reaching conjectures.

The rigorous mathematical study of Einstein's theory began just over sixty years ago, with Yvonne Choquet-Bruhat's proof that Einstein's theory has a well-posed Cauchy problem. This describes the dynamic evolution of space-time from a set of admissible initial data. The Cauchy problem is the cornerstone perspective of most modern mathematical approaches to the subject and has been the setting for many of the great accomplishments in the field. Notable subsequent developments include the Singularity Theorems of Hawking and Penrose during the late 1960's, the proof of the the positive mass theorem in the early 1980's by Schoen and Yau (and independently by Witten, in the case of spin manifolds) and the establishment of the non-linear stability of Minkowski space-time in the early 1990's by Christodoulou and Klainerman.

The Einstein equation shares issues of convergence, collapse and stability with important geometric evolution equations such as the Ricci flow and the mean curvature flow. Asymptotic behavior at singularities as well as questions related to asymptotic geometrization have been and continue to be intensely studied in each of these settings. Elliptic problems and techniques play a central role in studying the initial data sets in the context of the Cauchy problem, and in the study of global inequalities and the role of marginally outer trapped surfaces (apparent horizons).

The analysis of the Riemannian Einstein equation plays a fundamental role in modern Riemannian geometry and geometric analysis. Recent developments in General Relativity have brought many ideas and techniques which have been developed in the Riemannian context to bear on the Lorentzian Einstein equations. One example of this type of development is the realization that marginally outer trapped surfaces (MOTS), which are relevant for our understanding of quasilocal aspects of black holes, have many properties in common with minimal surfaces. Many recent results in this area have been inspired in part by this connection.

2 Objectives of the Workshop

This workshop brought together a wide variety of researchers from around the world whose expertise and interests lie within the realms of geometric analysis and general relativity. Over the past few years there have been remarkable advances on the mathematical problems of general relativity and this workshop built on this by bringing together the primary participants in this recent progress as well as many young researchers, including post-docs and advanced graduate students, working in this area.

Rather than try to equally include the totality of research in mathematical relativity this workshop focused on work which is primarily of geometric and/or elliptic nature from an analytical point of view. This increased the likelihood of fostering significant mathematical interactions and collaborations amongst the participants. Below we outline a number of the themes that were well represented during the workshop.

The subject of marginally outer trapped surfaces (MOTS) and their relationship with solutions of Jang's equation, were initiated in the work of Schoen and Yau in proving the space-time version of the positive mass theorem. Over the last decade, this subject has seen a major resurgence and blossoming in the field. Significant recent mathematical work in this area has been done by Andersson, Eichmair, Galloway, Mars, Metzger, Pollack and Schoen, among others. Some of this work was viewed within the context of quasi-local models for black holes introduced by Hayward and subsequently developed by Ashtekar and Krishnan. MOTS play an important role in the analysis, by Galloway-Schoen and Galloway, of the topology of higher dimensional black holes. More recently the advances in understanding the analysis of MOTS has been applied to questions of topological censorship in the work of Eichmair, Galloway and Pollack.

Global inequalities have played a large role in general relativity, going back at least as far as the proof the positive mass theorem due to Schoen and Yau and more recently via the proofs of the Riemannian Penrose inequality (which is deeply connected to Penrose's cosmic censorship conjecture) by Huisken-Ilmanen and Bray. Attempts to generalize this later work to a Lorentzian space-time setting are topics of current research. The long standing attempt to generalize the notion of mass in relativity to one which is quasi-locally defined has seen new ideas introduced due to the work of Shi and Tam and, more recently, to the extensive work of Wang and Yau. Recognizing the need for low regularity formulations of the fundamental quantities in relativity Huisken has developed formulations of many global inequalities in terms of isoperimetric inequalities which hold in a very rough setting. This perspective has been investigated by Eichmair and Metzger. Fundamental questions of uniqueness and stability of black holes has led to a great deal of important work, by Dain, Chrusciel, Schoen and others, on the angular momentum-mass inequality for axisymmetric black holes.

The analysis of initial data sets and the implications for their spacetime developments has been the subject of a great deal of recent attention. Research in this area has benefited from the introduction of gluing techniques from geometric analysis. Corvino's construction of vacuum spacetimes which are Schwarzschild outside of a compact set, and the extension of this method by Corvino-Schoen and Chruściel-Delay have had a significant impact. The combination of this with conformal gluing methods by Chruściel, Isenberg and Pollack have led to many interesting constructions. Recently, the construction of initial data sets exhibiting gravitational screening by Carlotto and Schoen has made clear that solutions of the constraint equations have a great deal more flexibility than was previously understood. This latter work has led to a number of interesting conjectures.

Aside from gluing, the analysis of the set of solutions of the constraint equations has largely relied on the so-called conformal method. This technique allows for a parameterization of the set of constant-mean curvature solutions of the constraints, but its applicability in the far-from-CMC regime is poorly understood. Lately there have been two major pushes toward understanding the conformal method for arbitrary mean curvatures: the limit equation analysis by Dahl, Gicquaud and Humbert, and select "far-from CMC" existence results due to Holst, Nagy, Tsogtgerel and Maxwell. These results have generated a large amount of recent activity, including results by younger researchers Dilts, Ngô and Sakovich. Despite these efforts, basic questions of the existence and uniqueness for the conformal method remain open, and indeed a very recent paper by Gicquaud and Ngô shows that our current far-from CMC existence results can be seen as merely another variety of near-CMC theorems. In light of these difficulties, Maxwell has reexamined the conformal method by working with special cases and has used this analysis to developed a related "drift" method that offers new perspectives and the potential of fertile ground for future research.

3 Presentation Highlights

The talks for the meeting were solicited in two ways. The organizers asked all of the participants to send us a title and abstract if they wanted to give a talk. In addition we asked specific participants, whose recent work was known to be of widespread interest, to give talks. Talks were scheduled for either 25 or 50 minutes, in large part depending on the preferences of the speakers. In each case the schedule was made to allow sufficient time afterward for questions and discussion.

The schedule was formulated in a manner to group talks by similar subject matter together to some extent. This allowed for a greater sense of continuity and enabled the speakers to easily refer to earlier talks for other relevant contriibutions. Rather than fill every available slot with talks, we made an effort to preserve a good deal of time for informal discussion and collaboration. This was very successful and groups formed spontaneously in the lecture room and lounge as well as in outdoor excursions around BIRS.

The meeting also highlighted the Marsden Memorial Lecture which was given by Richard Schoen on Tuesday afternoon. See below for a description of Rick's talk

A summary of each of the talks given at the meeting, along with some general comments about the subject matter, follows. This summary is organized chronologically in the order with which the talks were presented at the meeting.

3.1 Monday, July 18

Gerhard Huisken found a new formula for the ADM mass involving large isoperimetric hypersurfaces. The advantage of this definition is that it does not depend on derivatives of the metric. The usual definition requires at least two derivatives to decay at an appropriate rate. Since doing the Einstein equations with low differentiability is an important topic, this approach to the ADM mass deserves further study. The first three talks discuss progress in this area.

Yugang Shi: Isoperimetric mass and isoperimetric surfaces in AF manifolds

In his lecture, Y. Shi described his recent joint work with Chodosh, Eichmair, and Yu [1]. He described his recent theorem proving positivity of the isoperimetric mass for all isoperimetric surfaces. He described the proof by Carlotto, Chodosh, and Eichmair showing that there exists isoperimetric surfaces for all enclosed volumes. It is shown that the isoperimetric regions exhaust the manifold as the volume goes to infinity. A further main theorem is that the leaves of the Huisken-Yau CMC foliation near infinity are isoperimetric surfaces. More generally the authors solve a conjecture of Bray showing that any stable CMC surface which lies near infinity is an isoperimetric surface.

Jeff Jauregui: Lower semicontinuity of Huisken's isoperimetric mass I

This talk is also about the Huisken isoperimetric mass and its behavior under convergence of metrics. It is a companion talk to the talk of Dan Lee which follows.

The Bartnik mass requires minimizing the ADM mass over admissible asymptotically flat extensions of a given region with non-negative scalar curvature. The question is whether the ADM mass is lower semicontinuous under the appropriate notion of convergence. It is shown that non-negative scalar curvature and the no-horizon condition are necessary conditions for lower semicontinuity. It is also shown that the ADM mass is not continuous even with non-negative scalar curvature. It is also observed that lower semicontinuity implies the positive mass theorem. The first main theorem shows the lower semicontinuity under the assumption that the convergence is locally C^2. The second main theorem is that the lower semicontinuity holds for pointed intrinsic flat convergence as defined by C. Sormani and S. Wenger. The third main theorem is that lower semicontinuity holds in general with local C^0 convergence in place of C^2 convergence. The limit metric is only required to be C^0 and the ADM mass of the limit is replaced by the isoperimetric mass.

Dan Lee: Lower semicontinuity of Huisken's isoperimetric mass II

This is a continuation of the previous talk concerning the lower semicontinuity. The goal is to supply proof outlines for the main theorems.

The proof of the main theorem follows from the condition that the quasi-local isoperimetric mass of a sufficiently large region is uniformly bounded above by the ADM mass plus an arbitrarily small constant depending the size of the region. The mean curvature flow is also used in an interesting comparison to the corresponding flow for the Schwarzschild metric.

Chistopher Nerz: A geometric characterization of asymptotic flatness

The definition of asymptotic flatness traditionally is stated in terms of coordinates defined outside a compact set, and so is inherently non-geometric. There have been various attempts to make the definition purely geometric in nature.

It is proposed to hypothesize a CMC foliation exhausting the manifold with the normalization that the mean curvature of the surface $f = \sigma$ be given by $2/\sigma$. The results presented in Shi's talk give us a unique foliation by isoperimetric surfaces. The main result presented in Nerz's talk is that asymptotic flatness is equivalent to the existence of such a CMC foliation with the conditions that: the limit of the Hawking masses is nonzero, the Ricci curvature decays sufficiently rapidly, and the leaves are stable. The CMC foliation is used to define coordinates. The lapse function of the foliation is used to center the coordinates.

Julien Cortier: On foliations related to the center of mass in General Relativity

The Huisken-Yau definition of the center of mass [4] involves the use of a CMC foliation defined near infinity. The asymptotic conditions required for the construction are somewhat restrictive and work only in the time symmetric case with appropriate asymptotics.

The main theorem described in the talk is a joint work with C. Cederbaum and A. Sakovich. The general constraint equations are assumed with the dominant energy condition and with essentially optimal asymptotic conditions. Under the assumption that the ADM energy-momentum vector is timelike the authors construct a foliation by large spheres satisfying the condition that the Lorentz norm of the spacetime mean curvature vector is constant.

Hugh Bray: Flatly foliated relativity: Gravity without gravitational waves

The talk consisted of two parts. In the first part the speaker described his theory of gravity without dynamics. The requirement is that the spacetime is foliated by flat spacelike hypersurfaces. Examples include the classical cosmological models of Friedman-Lemaitre-Robertson-Walker as well as the Schwarzschild spacetime. In general there are four free functions which determine the metric. The usual action principle is assumed, but variations are required to preserve the flatly foliated condition. The variational conditions are just the constraint equations which become a family of elliptic systems parametrized by the time under appropriate energy conditions and the condition that the cosmological constant be positive. The positive mass theorem follows from the divergence theorem. The foliation seems to avoid the inside of black holes, and so may be nonsingular in general. In some cases the equation reduces to the 2-hessian equation which is an elliptic fully non-linear scalar equation.

In the second part the speaker described the work of his student Henri Roesch on the spacetime Penrose inequality. The idea is to find the quasi-local mass function associated to the Hawking energy. This is monotone under a flow along a null cone under certain convexity assumptions which can be checked for perturbations of the Schwarzschild metric.

3.2 Tuesday, July 19

Marc Mars: Asymptotic behavior of the Hawking energy in null directions and a Penrose-like inequality

Marc Mars reported on joint work with Alberto Soria. The setting here is asymptotically flat initial data sets with one end satisfying the Dominant Energy Condition (DEC) containing a weakly outer trapped surface. The motivation is the Penrose inequality which has been well understood in the Riemannnian case but is widely open in the case of non-vanishing second fundamental form. The weakly outer trapped surface occurs as the minimal enclosure of the trapped region. The aim here is consider situations where considering the minimal enclosure is not necessary.

The Null Penrose inequality gives a lower bound for the Bondi mass on an asymptotically flat null hypersurface satisfying the DEC with a weakly out trapped surface in terms of the area of the weakly outer trapped surface. The main result presented here was a "Shell-Penrose inequality" which eliminates the need to consider the weakly outer trapped surface by solving the characteristic initial value problem on a asymptotically flat null hypersurface and a portion of future null infinity (Scri+). This provides a lower bound on the Bondi mass and an integral of the null expansion (which can be disregarded - because it has the correct sign - in the case of a weakly outer trapped surface).

Note that one open problem here concerns the ambiguity regarding definitions of a "asymptotically flat null hypersurface" N. One needs to define what it means for N to extend to past null infinity - require cross sections of spherical topology and affinely parametrized null geodesics extended to their maximal domain. One also needs to go further and specify decay at infinity for transversal vector fields. The definition of an null hypersurface being asymptotically flat is nontrivial and takes some development . The Bondi energy relies on a choice of observer at infinity (i.es a choice of a geodeic foliation by large spheres at infinity).

For the Penrose inequality. the key object is a functional on spacelike surfaces. It has the physical dimensions of length (energy) but is not truly quasi-local (since there are null vector fields). One needs to understand its monotonicity properties and asymptotic behavior. This is shown to be monotonic under DEC if the surface has non-zero genus. It is not monotonic in the spherical case but there only one bad term which can be controlled.

By defining "geodesic asymptotically Bondi (GAB) foliations for past asymptotically Null hypersurfaces" Mars exhibits a setting where they can establish as Penrose-like inequality. This gives an actually null Penrose inequality in the case where the foliation approaches large spheres. Two conditions arerequired - one asymptotical condition on the null expansion (the shear free vacuum case satisfies this - strengthening a prior result by Sauter) and a curvature condition.

Pengzi Miao: Total mean curvature, scalar curvature and a variational analog of the Brown-York mass

Miao presented joint work with Christos Mantoloulidis. in the area of work going back to to the 2002 result by Shi-Tam [5]. One considers compact connected, Riemannian 3-manifolds with non-negative scalar curvature and boundary with positive Gauss and mean curvature and establishes a comparison with the total mean curvature compared to the isometric embedding in R^3, with equality iff there is only one boundary component and the domain is isometric to a convex domain in R^3. Uses Bartnik quasi-spherical and PMT.

The main question they address is what happens when you drop the positive Gauss curvature assumption on the boundary. To start with, assuming the boundary is a topological 2-sphere then there is a constant Λ only depending on the intrinsic geometric on the boundary which bounds the total mean curvature. (this was independently derived by Lu in work on isometric embeddings). The proof uses key ideas of Wang-Yau and Shi-Tam on boundary behavior of compact manifolds with a negative lower bound on scalar curvature in the context of the quasi-local mass problem in general relativity.

The Shi-Tam result leads one to study the Brown-York mass of the boundary. Mantoulidis and Miao use their invariant of the induced metric to define a variant of the Brown-York mass in the case when K is not necessarily positive (which, by Shi-Tam, coincides with the Brown-York mass in the case of positive Gauss curvature). This variant is nonnegative and vanishes only if the domain is flat (note: it is open to determine whether it is zero for all flat domains i.e. whether the mass is achieved). One could also allow extra boundary components which are minimal surfaces. The supremum over this larger class still agrees with the previous one.

Naqing Xie: Brown-York mass and the trapped surface conjecture

A well know folk belief in general relativity states “A large mass concentrated into a small region leads to gravitational collapse”. This was made more precise by Kip Thorne’s hoop conjecture. In 1979 Seifert restated this idea in terms of a of the existence of a universal constant such that if the mass of of domain is bounded below by this constant times a measurement of its size than the domain must contain a trapped surface. This is the trapped surface conjecture.

Xie’s talk, reporting on joint work with E. Malec, explores, sin spherical symmetry, what happens if the quasi-local mass concentrates for a small region. If the Brown-York mass is bounded below by the square root of the area of the boundary divided by π that the boundary is a trapped surface. The biig question that remains is what happens if we drop spherical symmetry? Xie believes that the Quasi-local mass does not behave well if the surface is too oblate. Their main result is in the context of conformally flat spaces with equipotential foliations, if $M_{BY}(\Sigma, g) > I + II + II$ then Sigma is a trapped surface. where I, II and II measures size in terms of conformal geometric and analysis data - which they interpret physically.

Xin Zhou: Progress on the min-max theory of minimal surfaces

In this talk, Zhou introduced the min-max theory for minimal surfaces with free boundary. A minimal submanifold with free boundary is a natural critical point of the area functional among all smooth submanifolds with boundary properly embedded in a compact ambient manifold with boundary, where one requires the boundary of the submanifold to be contained on the boundary of the ambient manifold. The fundamental question in this subject asks that given a compact smooth domain in the Euclidean space, can we always find an embedded minimal hypersurface with free boundary inside this given domain. This problem has been studied in many sub-cases by many manthematicians, e.g. Meeks-Yau, A. Fraser, Gruter-Jost, Jost, M. Li, etc. Recently, Fraser-Schoen found deep relations between minimal surfaces with free boundary in the standard ball with the extremal eigenvalue problems. In a joint work with M. Li, Zhou totally solves this problem using the min-max theory for minimal surfaces. They prove that for an arbitrary compact smooth domain in the Euclidean space, there always exists an embedded minimal hypersurface inside this domain with free boundary. The strategy is to develop a general Morse theory for the area functional among all hypersurfaces with boundary, called the min-max theory. The resulting minimal hypersurface is called the min-max hypersurface with free boundary. This result finishes the program initiated by Almgren, which was largely advanced by Pitts and Schoen-Simon in the case of closed ambient manifolds. At the end of the talk, Zhou discussed a fundamental issue in this theory, i.e. the issue of properness of the min-max hypersurface with free boundary and introduced two conjectures related to this issue.

Alessandro Carlotto: Effective index estimates via Euclidean isometric embeddings

Carlotto’s talk focused on the class of minimal hypersurfaces possibly with suitable boundary conditions in a compact Riemannian manifold. The aim is to establish what he called universal comparison theorems. An example of this is given by the 2010 result of A. Savo providing a lower bound on the Index of a non-equitorial minimal surface of genus g by $g/2 + 4$. This a remarkable improvement of the result from 1991 by F. Urbano which was used by Marques and Neves in their proof of the Willmore conjecture [3].

Carlotto presented two joint works, one with Lucas Ambrozio and the other with Benjamin Sharp, with a focus on the second - concerning free boundary minimal hypersurfaces in Euclidean domains. The main result presented establishes a lower bound for the index of a free boundary minimal surface which is linear both with respect to the genus and the number of boundary components. Applications to compactness theorems, to the explicit analysis of known examples (due to Fraser-Schoen and to Folha-Pecard-Zolotareva) and to novel classification theorems were also mentioned. One corollary of these results lead to the existence of free boundary minimal surfaces with arbitrarily high index in the unit 3-ball.

Richard Schoen: Marsden Memorial Lecture: the constraint manifold of general relativity

Schoen's talk focused on the Einstein constraint equations in the context of the Cauchy problem. The global study of the space of solutions of the Einstein constraint equations goes back to work of Jerry Marsden and co-workers in the early 1970's. As Schoen pointed out there are a number of global settings for this study, from the analysis point of view these can be viewed as boundary conditions. One case is the compact without boundary case, known in the Relativity literature as the "cosmological" setting. The other is the case of asymptotically flat spacetimes which is used to model isolated systems and has the Minkowski and Schwarzschild spacetimes as basic examples. The third setting is that of asymptotically hyperbolic manifolds.

Marsden's work in this area focused on the notion of *linearization stability*. A solution of the Einstein equations is said to be linearization stable if every infinitesimal deformation is tangent to a family of deformations. Schoen reviewed prior work in this area and then presented ways that subject has evolved in recent years. A primary focus was the issue of the localization of supports and the subsequent localization of solutions of the constraint equations. It has been possible to localize the deformation theory in certain cases to deform solutions inside a chosen region without changing them outside. This work, joint with A. Carlotto [2], relies on a careful analysis of the asymptotic behavior of solutions to the constraint equations. Schoen also addresses second issue which arises in the deformation theory, namely the derivative loss problem which occurs when one attempts to place a manifold structure on the constraint manifold of solutions with a finite degree of differentiability.

3.3 Wednesday, July 20

Justin Corvino: Deformation and gluing for the Einstein constraints with the dominant energy condition

Corvino described work involving localized deformation of initial data sets. The talk focused on a result showing that generic solutions of the constraint equations satisfying the dominant energy condition (DEC) can be modified on a compact set such that the DEC is strictly satisfied in a subregion. The ability to perform such deformations is a useful tool in applications where the extra room provided by the strict inequality is useful.

Corvino described several such applications. The Bartnik conjecture states that a minimal mass initial data set satisfying an interior boundary condition must be static. Corvino has shown this to be true in the time-symmetric setting where the DEC corresponds to non-negative scalar curvature. If a metric is not static, the scalar curvature can be increased to be strictly positive in a compact subregion, and this can be parlayed to obtain another extension that lowers the mass. In the non-time symmetric setting, a previous gluing result of P. Chrusciel, J. Isenberg, and D. Pollack requires that the strict dominant energy condition hold strictly at the point of gluing; using Corvino's method one can take data sets that satisfy the DEC, perhaps not strictly, and modify them locally near the two gluing points to obtain strict satisfaction of the DEC. Finally, the promotion of the DEC to the strict DEC allows for an application to interpolating two data sets across an interior dividing region.

Paul Allen: Weakly asymptotically hyperbolic solutions to the Einstein constraint equations

Allen described the construction of initial data in the asymptotically hyperbolic setting. Previous work in this area by L. Andersson, P. Chursciel and H. Friedrich brought to the fore the importance of the so-called shear-free condition for initial data, which is a necessary condition for the data to evolve into a spacetime admitting a conformal infinity, as well as delicate issues regarding regularity at conformal infinity.

Allen and his collaborators J. Lee and I. Stavrov have designed novel function spaces that are well-adapted to this setting and have the key property that they are strong enough to permit a satisfactory elliptic theory, but are sufficiently weak to allow for the low regularity at conformal infinity that naturally arises. In particular, they have found a function space for metrics that is naturally closed for the Yamabe problem.

I.e., if one starts with a metric in these function spaces and solves the Yamabe problem, the resulting metric stays in the same space. Additionally, they have discovered a new conformally-invariant tensor that can be used to enforce the shear-free condition. Using these tools, one arrives at a successful adaptation of the conformal method for constructing constant-mean curvature, asymptotically hyperbolic initial data satisfying the shear-free condition.

James Dilts: The conformal method gives a poor parameterization of initial data

The conformal method is a key tool for the construction of initial data in relativity. Indeed, the previous talk by Allen used the method to construct certain constant-mean curvature (CMC) initial data sets. The method is most successful in the construction of CMC data sets in various settings, and its good properties can often be extended to the near-CMC regime as well. Nevertheless, recent work by Maxwell and by Nguyen has given evidence that, at least in some very limited circumstances, the conformal method encounters difficulties in the far-from-CMC setting where it breaks down as a parameterization. That is, for certain seed conformal data, the conformal method variously generates zero or multiple solutions of the Einstein constraint equations. Dilts described a numerical approach using the ODE bifurcation analysis tool AUTO to explore the parameter space more broadly than is currently possible using analytic methods. He presented a variety of families of conformal seed data that suggest that bifurcations seem to be pervasive in the far-from-CMC setting and that past expectations that the good properties of the conformal method would extend generally for non-CMC initial data are not likely to be fulfilled.

The-Cang Nguyen: Improving the recent results for the Vacuum Einstein conformal constraint equation by using the half-continuity method

Modern existence theory for the conformal method of generating initial data sets relies on fixed-point theorems, typically variations of the Schauder Fixed-Point Theorem. In this talk Nguyen showed how a rather technical fixed-point tool, the half-continuity method, can be applied to obtain stronger generalizations of current results.

One application concerns bifurcation behavior for the conformal method for Yamabe-positive metrics on compact manifolds. In previous work Nguyen had shown that for very particular seed data for the conformal method, parameterized by a real number a, if a is sufficiently large there is no corresponding initial data set, and that there is a sequence $a_k \to 0$ such that for each a_k the seed data generates at least two solutions of the Einstein constraint equations. Using the half-continuity method Nguyen has shown that, in fact, there are multiple solutions for all a sufficiently small.

A second, very interesting application of the half-continuity method concerns the so-called limit equation of R. Gicquaud, E. Humbert and M. Dahl. Those authors showed that if the conformal method does not generate a solution, then there is a solution of a blowup profile equation, the limit equation. Their work was restricted, however, to mean curvatures that do not change sign. Nguyen has used the half-continuity method to demonstrate that there is a generalized limit equation that also applies for arbitrary mean curvatures.

3.4 Thursday, July 21

Anna Sakovich: On the positive mass conjecture in the asymptotically hyperbolic setting

In her talk, Anna Sakovich discussed an approach to proving the positivity of mass of asymptotically hyperbolic and more generally, asymptotically hyperboloidal initial data sets. While adequate proofs exist in 3-dimensions, without spin assumption only partial results are known in higher dimensions. The approach taken in her talk (first considered by Schoen and Yau) is to use Jang's equation, a certain quasi-linear elliptic equation, to reduce the problem to a setting where on can apply the well-known positive mass theorem of Schoen and Yau. While in her talk she restricted to 3 dimensions, the method should in principle extend to higher dimensions. The method requires obtaining a solution of Jang's equation with special asymptotics. A density theorem of Dahl and Sakovich is important to the overall argument.

Geometric Inequalities

Marcus Khuri: Penrose Inequalities with Angular Momentum

Aghil Alaee: Mass-angular momentum-charge inequality in higher dimensions

Mara Eugenia Gabach Clement: A mass-size-angular momentum inequality for objects

Geometric inequalities play a deep and fundamental role in General Relativity. By an elegant physical argument, Penrose was led to conjecture an inequality which gives a lower bound on the total mass of an isolated gravitating system in terms of the surface area of all black holes present. Violations of the Penrose inequality would spell trouble for a long-conjectured, widely-accepted scenario for gravitational collapse. The conjecture was established by Huisken-Illmanen (for a single black hole) and Bray (for multiple black holes) in the special case of time-symmetric asymptotlcaly flat (AF) initial data sets. Based on new insights, Sergio Dain was able to establish a Penrose-type inquality for AF axisymmmetric initial data sets, whereby the mass is bounded below by the angular momentum. Various extensions and improvements to this result were made by Dain and his collaborators and others. Several talks at the workshop presented contributions in this direction. Markus Khuri presented his work with Gilbert Weinstein etablishing a Penrose type inequality involvng mass, angular momentum and rod parameter, which yields a variational characterization of the Kerr solution, and implies, in particular, that the inequality obtain is saturated uniquely by the Kerr solution. Khuri discussed the charged case, as well. The technique (pioneered by Dain) involves bounding from below the ADM mass by a 'mass functional', and exploiting a harmonic map structure contained within. Khuri also discussed the charged case. There has been recent progress extending these techniques to higher dimensional black holes. Aghil Alaee described his work with Khuri and Kundari, establishing a mass-angular momenta inequality for 4 dimensional AF bi-axisymmetric initial data sets, where in this case equality is achieved uniquely by an extreme Myers-Perry black hole. Alaee also discussed the charged case. As emphasized by Dain, one may also consider geometric inequalities for regular material bodies. At the workshop Maria Gabach Clement discussed work in progress on establishing mass-area-angular momentum inequalities for material bodies. Subject to various assumptions, she described how inverse mean curvature flow can be used to obtain an estimate for the size of the body in terms of the angular momentum and the mass.

Higher Dimensional Black holes

Sumio Yamada: Bi-axisymmetric stationary solutions to the vacuum Einstein equation with non-spherical horizons

Mattias Dahl: Constructions of outermost apparent horizons with non-trivial topology

String theory and related developments, such as the AdS/CFT correspondence, have generated much interest in higher dimensional gravity and, in particular, higher dimensional black holes. The discovery of Emparan and Reall of the so-called black ring, a 4+1 dimensional stationary vacuum black hole spacetime with horizon topology $S^2 \times S^1$ caused a great surge of activity in the study of higher dimensional black holes. The black hole topology theorem of Galloway and Schoen, which requires horizons to be of positive Yamabe type, has provided additional impetus for constructing black holes with other topologies compatible with this condition. In his talk, Sumio Yamada discussed work in progress with Khuri and Weinstein for constructing a 4+1 dimensional stationary, bi-axsymmertic black hole spacetime satisfying the vacuum Einstein equations, with horizon having lens space topology. Using Weyl-Papapetrou coordinates reduces the problem to establishing the existence of a certain harmonic map, which they have succeeded in doing. However to ensure global regularity, as a last step they must prove the absence of 'strut' type singularities. Mattias Dahl, in his talk, discussed work with his student Eric Larsson concerning the topology of outermost apparent horizons (minimal surfaces) in AF time-symmetric initial data sets of nonnegative scalar curvature. They construct a one parameter family of AF metrics, conformal to the Euclidean metric, of zero scalar curvature, which, when the parameter is small enough produces an outer trapped surface. Fundamental existence and regularity results for apparent horizons in dimensons $n \leq 7$ then yields a smooth outermost apparent horizon. By the nature of their procedure, the resulting horizon, for sufficiently small parameter value, has the topology of the unit bundle to any compact $p-1$-dimensional submanifold sitting in an R^p factor in R^7, (p between 1 and 5). Such manifolds are known to carry metrics of positive scalar curvature, which provided some motivation for the construction. Their result significantly improves earlier work of Fernando Schwartz.

Cosmic Censorship, etc

Katharina Radermacher: Strong Cosmic Censorship in cosmological Bianchi class B perfect fluids and vacuum

Philippe LeFloch: The global nonlinear stability of Minkowski space for the Einstein-massive field system and the $f(R)$-theory of modified gravity

The initial data formulation of the strong cosmic censorship conjecture may be briefly stated as follows: For generic initial data, the maximal globally hyperbolic development is inextendible. There has been some progress in proving this conjecture under certain restrictive assumptions, specifically under certain symmetry assumptions, which make the Einstein equations more tractable. In her talk, Katharina Radermacher considers this conjecture in the context of spatially homogeneous cosmological models. Here one assumes that the spacetime metric is invariant under the action of a 3-dimensional Lie group that acts transitively on compact spacelike orbits. Spatially homogeneous cosmologies are classified according to nine Bianchi types, which fall into two classes, Type A and Type B. Radermacher establishes strong cosmic censorship for the class of Type B Bianchi models which satisfy either the vacuum or perfect fluid (with linear equation of state) Einstein equations. (Type A models have previously been handled by Ringstrom and others.) Using a dynamical systems approach, together with an analysis of the field equations, Radermacher shows in the vacuum case that the Kretshcmann scalar (square of the curvature tensor) blows up along incomplete causal geodesics, implying C^2 inextendibility. Certain exceptions to this involve more symmetries and hence are nongeneric. In the fluid case she proves that the square of the Ricci tensor blows up along incomplete causal geodesics. Given the exteme complexity of the Einstein equations, one is not always able to understand the dynamics of all relevant solutions to these equations. To simplify, one instead, can consider the problem of studying the dynamical evolution of small perturbations of the initial data from that of important known solutions, such as Minkowski space, or the Kerr solution. An especially important result of this kind is the theorem on the global nonlinear stability of Minkowski space due to Christodoulou and Klainerman. In the workshop, Phillip Lefloch presented his work, with Y. Ma, on the global nonlinear stability of Minkowski space for the Einstein-massive field system and the $f(R)$-theory of modified gravity. Their proof makes use of the hyperboloidal foliation method introduced by them to deal with coupled systems of wave-Klein-Gordon equations posed on a curved space. They also prove that the Cauchy developments of modified gravity converge to those associated with the Einstein equations when the function $f(R)$ approaches R.

3.5 Friday, July 22

Christina Sormani: Almost Rigidity of the Positive Mass Theorem

The rigidity statement of the Positive Mass Theorem states that an asymptotically Euclidean metric with non-negative scalar curvature and zero ADM mass is isometric to Euclidean space. Sormani considered the implications of small ADM mass and, in particular, if given a sequence of such manifolds with ADM mass converging to zero whether the manifolds converge, in some sense, to Euclidean space. Using the notion of *intrinsic flat distance* defined by Sormani and S. Wenger, she presented theorems that show this is indeed the case in several settings. In particular, almost rigidity holds in the spherically-symmetric case (joint work with D. Lee), for data sets that are graphs over Euclidean space (joint work with L.-H. Huang and D. Lee), and for geometrostatic data sets (joint work with I. Stavrov, who described more results on geometrostatic data sets in the following talk).

Iva Stavrov: A continuous matter distribution arising as an intrinsic flat limit of point particle configurations

Stavrov's presentation continued with the theme of applications of intrinsic flat distance in the context of geometrostatic solutions of the Einstein constraint equations. Such data sets describe a kind of discrete point-particle configuration generalizing static slices of Schwarzschild. After providing more context on the almost-rigidity theorem discussed in Sormani's talk, she described a pair of results, also joint work with Sormani, connecting discrete and continuous dust matter distributions. First, appropriate sequences of geometrostatic data converge in the intrinsic flat sense to dust solutions of the constraint equations. Conversely, every time-symmetric asymptotically flat dust solution of the constraint equations can be obtained as the limit of such a sequence.

Jan Sbierski: The C^0 inextendibility of the Schwarzschild spacetime

The maximal analytic extension of Schwarzschild is easily seen to be C^2 inextendable because the Kretschmann scalar blows up along inextendable geodesics. Questions surrounding the Strong Cosmic Censorship conjecture motivate determining if the Schwarzschild spacetime can nevertheless be extended in a less regular sense. The Strong Cosmic Censorship conjecture states that generic maximal globally hyperbolic solutions of Einstein's equations are inextendable as (suitably regular) Lorentzian metrics. Finding the correct notion of *generic* and *suitably regular* is part of the challenge of resolving the conjecture. Recent work by M. Dafermos and J. Luk has shown that perturbations of Kerr remain C^0 close to Kerr and admit C^0 extensions. Hence C^0 is too weak a notion of extensibility for the Strong Cosmic Censorship conjecture. Nevertheless, C^2 is too strong because recent low-regularity existence theorems for the Cauchy problem for Einstein's equations (S. Kleinerman, I. Rodnianski, J. Szeftel, J. Luk) allow for metrics with L^2 curvatures.

Standard tools for detecting inextendibility, such as curvature invariants, are limited to C^2 regularity. Sbierski's talk introduced a new geometric quantity, *spacelike diameter*, a companion concept to the notion of timelike diameter. Using these tools, he presented an elegant argument using causality theory to show that Schwarzschild does not admit even a C^0 extension.

References

[1] Otis Chodosh, Michael Eichmair, Yuguang Shi, and Haobin Yu, Isoperimetry, scalar curvature, and mass in asymptotically flat Riemannian 3-manifolds, *a*rXiv:1606.04626 [math.DG]

[2] Alessandro Carlotto and Richard Schoen, Localizing solutions of the Einstein constraint equations, *a*rXiv:1407.4766 [math.DG]

[3] Fernando C. Marques and André Neves, Min-max theory and the Willmore conjecture. *Ann. of Math. (2)* **179** (2014), no. 2, 683?782.

[4] Gerhard Huisken and Shing-Tung Yau, Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature, Invent. Math. **124** (1996), no. 1-3, 281?311.

[5] Yuguang Shi and Luen-Fai Tam, Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature, *J. Differential Geom.* **62** (2002) 79–125.