Entropies, the Geometry of Nonlinear Flows, and their Applications (18w5069)

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1 A Brief Introduction to the Field

Many natural processes involving the interaction of a very large number of particles, such as conduction of heat, fluid flows and chemical reactions, possess an *entropy*, a quantity that increases during the evolution. A powerful strategy for quantitatively understanding the properties of such systems is to establish mathematical relations between entropies and other quantities characterizing the state of the system. The investigation of these relations has been extremely successful in explaining and predicting the properties of the dynamics of such large and complicated systems. In order to draw the precise conclusions it is important to establish the optimal relations between relevant quantities.

There is also an important geometric aspect to the evolutions of such systems. Mathematically, many nonlinear evolution equations can be interpreted as gradient flows of entropy/energy functionals, *i.e.*, steepest ascent (or descent) over the entropy/energy landscape with respect to an appropriate notion of distance. This interpretation is useful not only to understand the abstract geometric framework of the nonlinear equations, but also to deal, for instance, with particle approximations and handle mass conservative models in connection with mass transport theory.

A large scientific community has been involved in this area since the first meeting held in Banff in 2006. This meeting played an important role in the development of the topic, and entropy methods have now reached a certain maturity through the geometric interpretation of nonlinear flows. The area is now more vibrant than ever involving a growing network of interactions between branches of mathematics, physics, biology and social sciences. This meeting has been intended to consolidate this progress and set the stage for new advances.

2 Recent Developments and Open Problems

2.1 State of the Art

Entropy estimates for nonlinear evolution equations have proved to be the basis of extremely successful methods for the investigation of many quantitative issues such as rates of convergence and sharp constants in functional inequalities. These methods are well-suited to marry with multi-scale analysis and useful to understand physical systems that are described by systems of evolution equations on different scales, for instance a rarefied gas on the microscopic individual particle scale, the mesoscopic kinetic many-particle scale and the continuum scale of hydrodynamics. The existence of entropies at all these levels is a crucial property underpinning the physical/biological derivation of systems of equations, allowing for the understanding of qualitative properties as diverse as stability of steady states, asymptotic behavior and convergence of numerical schemes.

A major break-through in the theory was the observation that many nonlinear evolution equations constitute gradient flows of entropy functionals with respect to the Wasserstein or related transport metrics. Understanding the connection between nonlinear flows and the geometry of optimal transportation has paved the road towards a broad variety of profound results on large-time asymptotics of diffusion, towards novel proofs of sharp functional inequalities, and towards structure preserving discretizations of evolution equations and manifolds, just to name a few. Generalizing the ideas of optimal mass transport to coupled systems of evolution equations is not straightforward at all. Finding sharp conditions under which, *e.g.*, a given reaction-diffusion system has a good entropy or gradient flow structure is still an open problem, although some recent advances have been done.

Another very novel direction is the application to the "discrete world". There has been a surge of activity around using entropy like ideas in Markov chains considered by themselves or as numerical schemes for some of the standard Fokker-Planck equations related to linear/nonlinear diffusion equations, as well as the extension of these methods to the quantum setting. Understanding the different notions of steepest descent in these methods, their interconnections and their potential in other applications such as clustering in machine-learning applications is another topic of current interest.

Another important theme to emerge from the work discussed at the previous meetings is the role of stability theorems for sharp functional inequalities in the study of nonlinear evolution equations. The use of nonlinear flows for proving functional inequalities has developed into a powerful method for establishing new remainder terms for classical inequalities – which have then been applied to study other nonlinear laws. It will certainly be fruitful to continue this development in the context of further problems arising in mathematical biology and physics. Interesting variational arguments involving the use of entropies have been proposed in order to show symmetry of optimizers in the Caffarelli-Kohn-Nirenberg inequalities or in stationary states of nonlinear diffusion variants of the Keller-Segel system. Finding an analytical framework to attack the symmetry breaking in some problems of this type would be an excellent tool in various problems such as questions related to interaction potential functionals.

The extension of gradient flow methods into the domain of non-convex functionals has been another source of recent progresses. Non-convex functionals appear naturally in many physical/biological problems related to phase transitions. These free energies have several stationary states and finding good mathematical approaches to understand the basin of attraction for these cases is an interesting topic. Isotropy or not in the long-time asymptotics is observed or not depending on the presence and intensity of noise/diffusion. This has been realized only in few cases in one spatial dimension, and therefore its generalization to more physical dimensions would be a large push to the theory. Hypocoercivity tools might be a good track for this question.

Nonlocal diffusion equations have been so far elusive to the use of entropy methods except when the interaction potential is defined as the inverse nonlocal Laplacian of the density or in one dimensional settings. In these cases the connection to obstacle problems recently found is an excellent tool bridging to the world of nonlocal elliptic equations. The potential use of these ideas for nonlocal versions of nonlinear diffusion equations is to be explored.

And finally, since these entropy methods are now used not only as tools to solve problems of pure functional analysis but also as techniques in various applied and even computational areas, it was important to bring together a group of researchers who can profitably interact, but who might not meet at other meetings.

2.2 Topics of Current Interest

Here are some emerging topics in entropy methods in PDEs and the study of their associated flows which will probably play an important role in the development of the field in the near future:

- Reaction-Diffusion equations & systems as gradient flows.
- Advances on gradient flows of non-geodesically convex functionals.
- Symmetries of critical points which are not necessarily global minimizers of entropies.
- Stability inequalities for remainder terms involving entropies and rates of decay.
- Hypocoercivity & multiplicity of stationary states.
- Discrete Entropies: Markov chains & clustering in machine learning.
- Non-equilibrium steady states.
- Non-locality and non-isotropy in diffusions and mean-field interactions.

2.3 Identification of Relevant Topics for the Workshop

The workshop has been in part intended as a follow-up to the workshops "Nonlinear diffusions: entropies, asymptotic behavior and applications", "Nonlinear Diffusions and Entropy Dissipation: From Geometry to Biology" and "Entropy Methods, PDEs, Functional Inequalities, and Applications", held at BIRS in April 2006, in May 2010 and in June 2014, respectively. Ideas emanating from stimulating exchanges at these events have led to important advances in optimality issues in the theory of functional inequalities, in problems of mathematical biology and geometric flows, with seemingly far-apart applications such as quantum Markov semigroups and collective behavior models. All in all there is an outstanding track record of this series of workshops as effective catalyzers for far-reaching consequences outside the comfort zone of mathematical applications of these tools. The organizers took care to bring together different researchers of the "core communities" involved: PDE analysts, differential geometers and probabilists; and potential users of these techniques in pure and applied mathematics.

The organizers had given the participants complete freedom in the choice of topic they want to present, as long as it fits with the general scope of the conference. The portfolio of proposed talks is therefore a good indicator for general directions and specific problems that are currently topical in the various branches of the community. In comparison to the previous edition of this workshop in 2014, there was a stronger focus on the further development of various aspects of the theory, while the adaptation of existing theory to study specific PDEs played a less prominent role. The new theoretical developments that have been presented were rich and profound.

Already in the past, we could observe that gradient flow dynamics on metric spaces, specifically flows in the Wasserstein distance, played a role of continuously increasing significance. The current event has been no exception to that trend: two morning sessions have been devoted exclusively to applications of optimal transport in the context of entropy methods, and many further talks featured Wasserstein gradient flows as an essential element. A recurrent theme in that context was that of mean field games, a class of equations and models with surprising, but yet not fully understood relations to smoothed optimal transport.

For the more classical aspects of the theory, the organizers could observe a shift of interest from fully dissipating to hypocoercive systems. The first general characteristic of the latter is that there is a part of phase space in which entropy is dissipated at exponential rates, while in other parts, the flow is almost entropy conserving. However, the second key feature of hypocoercivity is the existence of a global, possibly slow dynamics that forces any solution to pass through the rapidly dissipating region at a quantifiable frequency, and thus leads to a continuous decay of the entropy in the end.

3 Presentation Highlights

The organizers were extremely pleased both with the wide range of topics that had been proposed by the participants and the quality of each individual presentation. The speakers took great care to make their results understandable to the audience.

3.1 Nonlinear diffusion

Nonlinear diffusion equations of the form $\partial_t u = \Delta u^m$ constitute one of the classical playgrounds for the entropy method, and possibly the field where the method had its most significant achievements in the past twenty years. The field is still remarkably lively.

One of the highlights of the conference was the impressive talk by **Esteban** in which she gave a precise characterization of the symmetry breaking in the Caffarelli-Kohn-Nirenberg inequality [28]. It has been relatively easy to conjecture the line of transition between radial symmetry and non-symmetry of minimizers by a linear stability analysis of the Euler-Lagrange equations and then to identify a region where symmetry breaking must occur. On the contrary, the rigorous proof for radial symmetry in the complement of that region is highly demanding both on the conceptual and the technical level. The application of entropy method to nonlinear diffusion equations on particular manifolds plays a key role in the proof.

Figalli gave the first results ever on qualitative properties of the equation $\partial_t u + (-\Delta)^s u^m = 0$ on a *bounded* domain involving the fractional Laplacian, in the porous medium case $m > 1$. One of the first difficulties lies in the appropriate definition of the fractional Laplacian in that context. Three alternative definitions were given, and for two of them, uniform relative convergence to a self-similar profile could be proven. The key key tool is a comparison principle. These results are explained in the works [5, 4].

Not far from that is the singular diffusion equation considered by **Iacobelli**, which is of the form $\partial_t u = -(u^p(u/u_\infty)_x)_x$, where the target density u_∞ is given, and p is a *negative* power. Under positivity conditions on u_∞ and the initial datum, it has been shown that solutions converge exponentially fast to u_∞. As before, the key element of the proof is a comparison principle. The main objective of her talk was to apply these ideas to the quantization of measures problem [7, 36].

The very general class of coupled systems of reaction-diffusion equations presented by **Fellner** bears a problem that is similar to hypocoercivity. He applies the entropy method to the sum of the physical entropies of the individual components and uses it as Lyapunov functional. There is no uniform exponential rate of decay for that quantity: the dissipation vanishes if one of the components is depleted, and becomes arbitrarily slow close to such solutions.

3.2 Nonlinear aggregation/diffusion

More recent than the studies of pure nonlinear diffusion is the combination of diffusion with binary particle interactions moderated by a (typically singular) interaction kernel K. This leads aggregation-diffusion equations of the type $\partial_t u = \Delta u^m + \nabla \cdot (u \nabla K * u)$. Several speakers presented novel results on the long time asymptotics of solutions. Their general line of approach was to exploit the equation's gradient flow structure in the Wasserstein distance.

Craig introduced a method to approximate optimizers for the generalizations of Poincaré 's problem, that is, one wants to find a set of given measure on which the integral of a given kernel K is minimized. By means of Γ-convergence, she could show that the associated aggregation-diffusion equation with exponent $m > 1$ converges as $m \to \infty$ to a flow on the space of functions with $0 \leq u \leq 1$, and that it approximates in the long time limit a solution to a relaxation of the Poincaré problem. The results are related to several recent works [24, 22, 23, 13] and a work in preparation with Topaloglu.

Yao considered a model for the reproduction of coral, which leads to a aggregation-diffusion equation with an additional harvesting term. The main question, namely if the combination of diffusion and interaction leads to a faster rate of convergence to the trivial equilibrium than diffusion alone, boils down to understanding the long time asymptotics in a linear Fokker-Planck equation with an attractive, but slowly growing potential. The general answer to the question is quite subtle, as she demonstrated on a particular example where the interactions even leads to a slow-down. The results are based on a work in preparation.

Volzone gave an overview over new conditions for existence, uniqueness and radial symmetry of the stationary solutions to the Keller-Segel equation in the diffusion dominated regime, *i.e.*, $m > 2 - 1/n$. The dynamical stability of these states was also discussed. He showed an impressive result of radial symmetry of stationary states in the degenerate case that leads to uniqueness of steady states in the Newtonian kernel in two dimensions [12, 16, 17]. A result sought for a long time. Convergence results were obtained without rate of convergence in the two dimensional Keller-Segel model with nonlinear diffusion $m > 1$.

The first goal of **Patacchini**'s presentation was essentially opposite: for a general class of aggregation/diffusion equations, he derived conditions for the *non*-existence of stationary solutions. He also derived conditions in order to have ground states which are sharp for the case of linear diffusion [14].

Schlichting talked about phase transitions in McKean-Vlasov equations (aggregation equation with diffusion) with periodic boundary conditions. Together with Carrillo, Gvalani and Pavliotis in [15], he has shown that in the H-stable regime the relative entropy to the minimizer is decaying exponentially fast, while for H-unstable systems the asymptotic dynamics can be substantially richer. In particular they consider the bifurcation of nontrivial steady states as the parameter controlling the strength of the nonlocal term (aggregation) increases. Continous and discontinuous phase transtions can exist depending on assumptions on the Fourier coefficients of the interaction potential.

3.3 Linear kinetic equations

Several talks have been devoted to novel estimates on the rate of equilibration in linear BGK models, or even more general linear kinetic equations. The main difficulty that was present in each talk was the hypocoercive nature of the dynamics, which is highly entropy dissipative but only with respect to the velocity variable. To measure global convergence rates, one has to understand how the transport operator transfers the relaxation to the spatial variable.

Arnold started by showing a seemingly easy way to prove exponential decay of a suitable norm for general linear systems $\dot{x} = Cx$, where the matrix C is positive definite but not symmetric. The applicability of his strategy to linear BGK equations, however, requires to a find a suitable adapted norm, with a choice that is independent of the Fourier mode under consideration, and this is a challenging task, that could only be undertaken by sophisticated trial and error so far. Beyond the systematic method he developed with his collaborators, a significant achievement compared to standard approaches based on hypo-elliptic methods is that norms can be explicitly constructed and rates are optimal. The results presented were based on [1, 2]. **Evans** in [32] was also using a relatively algebraic approach, but took a different angle of attack: she studied the dissipation of Φ-entropies and related functional inequalities. Φ-entropies are a natural interpolation between the logarithmic Boltzmann entropy functional and the quadratic norms considered by Arnold.

A different approach to linear BGK models has been presented by **Canizo**. His ideas originated from Harris' ergodic theorem, which states that if a markovian semi-group admits a uniform lower bound after a transition time $t_0 > 0$, then solutions converge exponentially fast in total variation. Canizo showed how to massage the technique to be applicable to BGK equations, using geometric and probabilistic arguments. In contrast to the aforementioned speakers, **Guillin** did not study the BGK approximation, but a more general linear kinetic equation with diffusion in the velocity variable and a non-quadratic external potential, see [18]. Previous approaches like those by Villani were relying on log-Sobolev inequalities and thus induced demanding convexity assumptions on the potential. Guillin showed a strategy how to make the proof work under essentially weaker hypotheses.

3.4 Kac master equation

In two related talks, **Carvalho** and **Loss** discussed new results on the old problem of propagation of chaos in the Boltzmann equation [8]. Specifically, they were looking at the Kac walk on the sphere and gave novel estimates on the rate of convergence to equilibrium. For the Kac equation (for infinitely many particles), Cercignani's conjecture states that the entropy production $D(f)$ dominates the entropy $H(f)$ in a linear way, yielding exponential convergence to the steady state. It was shown by Villani that this conjecture is wrong in general, but one can at least expect $D(f) \geq c_\epsilon H(f)^{1+\epsilon}$ for every $\epsilon > 0$. Carvalho showed that a similar inequality is still true for the Kac walk, with a c_ϵ independent of the number of particles. Her proof, however, uses a strong assumption on the disorder of the distribution, for which propagation along solutions is unclear.

Loss instead verifies the linear dependence $D(f) \geq cH(f)$ originally conjectured by Cercignani, however for a variant of the Kac model, where the particles are coupled to a huge (but still finite) heat bath, and in a regime which is away from equilibrium.

3.5 Entropy methods on discrete structures

One of the goals of the entropy method is to provide guidelines for the definition of structure preserving discretizations. Currently, the main interest is to build a consistent discrete theory that reproduces key elements of the original continuous one, but many of the concepts that have been developed so far are directly applicable in the design of numerical methods that preserve features like entropy dissipation.

One of the forerunners in the field of discretizing the theory of optimal transport is **Maas**. He had already developed a successful strategy to define analogues of the Wasserstein metric, associated gradient flows, and curvature properties on graphs [31, 30, 35]. In his presentation, he showed a surprising limitation of his concepts for the passage the continuum limit: in general, one cannot expect Gromov-Hausdorff convergence of his transport distance on the dual graph of a finite-element decomposition of a domain unless rather restrictive geometric conditions are met. A remarkable fact is that the associated heat equation converges under much more general hypotheses. On the basis of Maas' discretized transport distance, **Erbar** has developed a notion of super-Ricci flow on graphs. The main quantities of interest are lengths associated to the edges of the graph, and these change according to an evolution inequality. Moreover, at certain instances of time, the graph itself changes as well, when either an edge collapses because its length becomes zero, or a new edge is born out of a point. Erbar justifies his *ansatz* by proving discrete analogues of various equivalent characterizations of the continuous super-Ricci flow, like estimates on the heat kernel or the modulus of convexity of the entropy, see [29].

A complementary approach was taken in the discretization of a model for traffic flow considered by **Di Francesco**, who did not use a graph structure but instead a moving mesh, define by the Lagrangian map. The underlying evolution equation is of aggregation type, but with a nonlinear mobility function that saturates, corresponding to a maximal admissible density of cars on the road. Albeit the PDE has been introduced in arbitrary dimensions, the traffic flow model is naturally one-dimensional, and that is where most of the analysis has been carried out. The main result is the convergence of the discrete solutions to an entropy solution, in total variation, see [27]. Finally, **Plazotta** discussed a spatially continuous but time discrete approximation to Wasserstein gradient flows. The scheme, which is formally of second order in time, mimicks the JKO method, but the variational functional contains the difference of two distances (instead of one distance). He has shown that discrete solutions converge strongly in L^p to a weak solution in the continuous limit [38]. However, second order convergence could not be verified due to the roughness of the Wasserstein space.

3.6 Derivation of models

Principles like monotonicity of the entropy have always been of great importance in the derivation of PDEs for models in physics, and — more recently and possibly surprisingly — biology. This was illustrated in the talks of Degond, Filbet and Raoul, all of which studied micro-to-macro-limits of biological models.

Degond considered swelling materials (like gels and tumors) and explained why Darcy's law alone is not sufficient to derive a closed system of evolution equations [26]. He henceforth proposes as an alternative first principle simple rules on the microscopic scale: essentially maximal packing density and minimal motion. These indeed lead to a consistent macroscopic model. **Filbet** started from an spatially extended version of the FitzHugh-Nagumo model, with forced local interactions. The resulting equation is of kinetic type. The dynamics of appropriate "macroscopic" quantities, *i.e.*, suitable moments of the kinetic solution, converges to the classical non-local reaction-diffusion model in the hydrodynamic limit, see [25]. For **Raoul**, who considered a model for sexual reproduction, the initial equation was of collisional Boltzmann type, and his goal was to obtain the macroscopic Kirkpatrick-Barton model from population genetics, which is of reaction-diffusion system, as a limit in suitable parameter ranges. The key tool in his derivation were Tanaka-like contraction estimates in the Wasserstein distance. The derivation of such models is of crucial importance for the understanding of ecological problems like the ones induced by climate change, see [39].

A very different set of problems was considered by **Kinderlehrer**. He studied structure of grain-boundary networks in polycrystaline materials, as well as their evolution. He discussed the observation that during the evolution the grain boundary network quickly achieves a stable distribution of grain boundaries with the average length of boundaries of a certain energy characteristics. This distribution is known as the *grain boundary character*. The talk focused on ode-based models to explain and predict the grain boundary character distribution. The models were derived following ideas of entropy and gradient flows in appropriate metrics. At this point it is worth to be mentioned that Kinderlehrer's ideas have been extraordinarily influential in the mathematical community targeted by the workshop and once more he has challenged the participants by proposing a new area of investigation in [3].

3.7 Optimal transportation

Optimal transport has played a key role in understanding nonlinear flows. Primarily by being the metric on the space of configurations with respect to which many nonlinear flows are gradient flows of an appropriate free energy. Studying the geometry of optimal transportation, its properties and role in functional inequalities has led to remarkable progress. Several speakers have focused on new variants of optimal transportation and their connections to dissipative systems.

Carlier gave a talk on entropy regularized optimal transport and connections to mean-field games. Entropic regularization of optimal transport was independently introduced by Galichon and Cuturi. In particular the associated Sinkhorn numerical algorithm by Cuturi has become one of the most popular approaches to computational optimal transport. Optimal transport is recovered as the regularization parameter $\varepsilon \to 0$. In contrast, Carlier focused on fixed entropic regularization, $\varepsilon = 1$, and developed the connections with variational descriptions of mean-field games. In particular he discussed the connections between mean-filed games of Lasry and Lions [37] with the dynamic description of the entropy regularized optimal transport studied by Chen, Georgiu, and Pavon, [19], and by Gentil, Léonard, and Ripani. [33]. Carlier also introduces the time discretized dynamical version as a new and useful tool.

Savaré spoke about dynamic formulation for the entropic regularization of the optimal transportation and connections to mean-field games. In particular he studied the dual formulation of the dynamic description of optimal transportation, due to Benamou and Brenier. The dual formulation leads to a Hamilton-Jacobi equation and connections to mean-field games. Savaré carefully discussed the notions of solutions, hence building the setup that would allow to study the limit of viscous regularizations of the equations and consequently the limits of the entropic regularizations of optimal transport.

Ghoussoub gave a presentation on the *Theory of Transfers* that he recently introduced with Bowles [6]. Namely using notions of convex duality they introduced the notion of (forward and backward) linear transfer which encompasses all of the classical optimal transportation, martingale optimal transport, entropy regularized optimal transport (Schrödinger bridges), as well as weak mass transports. For the classical optimal transport the notion is closely related to the dual formulation of Kantorovich. Ghoussoub also introduced the notion of convex transfers with particular focus on its subset of entropic transfers. In particular he showed that Donsker-Varadhan entropy can be described as an entropic transfer.

Palmer talked about a generalization of the dynamic description of optimal transportation where the end time when a mass reaches its destination is not fixed but can vary over the mass being transferred. Furthermore he considered general Lagrangian terms in the dynamical description and introduced *density processes* to be able to present an eulerian description of the problem. Using Hamilton-Jacobi-Bellman variational inequalities he gave a characterization of the new transportation distance [34]. Moreover Palmer talked about the extension of the transportation distances with free end times to a stochastic setting. These have important applications in mathematical finance and have strong connections to stochastic optimal control. Again he was able to obtain an eulerian characterization of the optimal stochastic transport with free end times.

3.8 Other

There were several talks that did not fall in any of the categories above. In his talk **McCann** explored possible geometric explanations for the second law of thermodynamics. Namely just as Einstein's theory of general relativity provides a geometric explanation for gravity, one is wondering if geometric explanations can be given for entropy in statistical mechanics. As a step in that direction McCann investigated the relation between entropy and curvature on Lorentzian manifolds. **Gangbo** spoke about stochastic processes on the space of probability measures. Developing a theory of stochastic processes in the infinite dimensional space of probability measures is a topic of substantial current interest which has applications in mean-field games. Chow and Gangbo [20] introduce the notion of *partial Laplacian* which allows for only a certain family of stochastic paths. Gangbo explained its well posedness, basic properties and relevance to the field.

Fathi discussed a new approach to proving rates in Central Limit Theorems (CLT) by proving regularity estimates for the Monge-Ampère equation. Namely, in order to compare probability measures one often uses the Stein method, by proving the existence of a Stein kernel. In their very interesting work, Fathi and collaborators use a type of Monge-Ampère equation studied by Cordero-Erausquin and Klartag, [21] to show the existence of Stein Kernels. Fathi also discussed the consequences of the existence of the new Stein Kernels to CLT rates for the Wasserstein metric.

4 Outcome of the Meeting

This 5-day workshop at BIRS had been a perfect venue for an event like this. The group size of around 40 people has been large enough for bringing together a representative cross section of the different involved communities, in the whole range from the established leaders like Robert McCann (differential geometry), Wilfried Gangbo (optimal transport), Michael Loss (kinetic equations), and Guiseppe Savaré (gradient flows), to young faculty like Katy Craig (nonlinear diffusions, aggregation), Mikaela Iacobelli (nonlinear diffusions, kinetic equations), and Yao Yao (aggregation models), to PhD students and Post-Docs, like Josephine Evans (kinetic equations), Aaron Palmer (optimal transport), Francesco Patacchini (diffusion, aggregation models), and Simon Plazotta (gradient flows). And on the other hand, the group was compact enough for the exchange of ideas in a relaxed and friendly atmosphere. According to the feedback that we received from young researchers, they enjoyed the workshop, got a broad overview of the field, and benefitted from interactions with prominent researchers in the field.

Although we gave essentially everybody the opportunity to present his or her results, we had a strict time limit of thirty minutes for most of the talks, including questions. The longer slots of forty-five minute were reserved for a few selected speakers presenting overview talks. This format gave the participants enough freedom for individual discussions and collaboration in small groups.

Overall there was a sense that the range of applications of entropy methods and related functional inequalities has significantly broadened over the last five years and that there are subfields and connections with other fields that are advancing at rapid pace. Among these we noted analysis on discrete spaces, applications to statistics, numerical computation, highly non-trivial mathematical modeling with applications in particular in mathematical biology, connections with new forms of optimal transport (martingale, unbalanced, etc.). In addition substantial progress on some classical problems, as highlighted by the talks of Esteban, Volzone and Loss, has been made.

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Appendix A. Abstracts of the lectures

• LARGE-TIME BEHAVIOR IN HYPOCOERCIVE BGK-MODELS. **Arnold, Anton**. BGK equations are kinetic transport equations with a relaxation operator that drives the phase space distribution towards the spatially local equilibrium, a Gaussian with the same macroscopic parameters. Due to the absence of dissipation w.r.t. the spatial direction, convergence to the global equilibrium is only possible thanks to the transport term that mixes various positions. Hence, such models are hypocoercive.

We shall prove exponential convergence towards the equilibrium with explicit rates for several linear, space periodic BGK-models in dimension 1 and 2. Their BGK-operators differ by the number of conserved macroscopic quantities (like mass, momentum, energy), and hence their hypocoercivity index. Our discussion includes also discrete velocity models, and the local exponential stability of a nonlinear BGK-model.

The proof is based, first, on a Fourier decomposition in space and Hermite function decomposition in velocity. Then, the crucial step is to construct a problem adapted Lyapunov functional, by introducing equivalent norms for each mode.

• ON A NEW PROOF OF THE HARRIS ERGODIC THEOREM AND RELATED SUB-EXPONENTIAL CONVERGENCE RESULTS. **Cañizo, José A.** We revisit a result in probability known as the Harris theorem and give a simple proof which is well-suited for some applications in PDE. The proof is not far from the ideas of Hairer & Mattingly (2011) but avoids the use of mass transport metrics and can be readily extended to cases where there is no spectral gap and exponential relaxation to equilibrium does not hold. We will also discuss some contexts where this result can be useful, particularly in a model for neuron populations structured by the elapsed time since the last discharge. This talk is based on joint works with Stéphane Mischler and Havva Yolda.

• ENTROPIC REGULARIZATION OF OPTIMAL TRANSPORT AND APPLICATIONS. **Carlier, Guillaume**. Entropic regularization of optimal transport is appealing both from a numerical and theoretical perspective. In this talk we will discuss two applications, one from incompressible fluid dynamics and the other from mean-field games theory.

• ENTROPY PRODUCTION INEQUALITIES FOR THE KAC WALK. **Carvalho, Maria C.** We investigate new functional inequalities for the well-known Kac's Walk, and largely resolve the 'Almost' Cercignani Conjecture on the sphere. A new notion of chaoticity plays an essential role. The results we obtain validate Kac's suggestion that functional inequalities for the Kac walk could be used to quantify the rate of approach to equilibrium for the Kac-Boltzmann equation. This is joint work with E. Carlen and A. Einav.

• FROM SLOW DIFFUSION TO A HARD HEIGHT CONSTRAINT: A SINGULAR LIMIT OF KELLER-SEGEL. **Craig, Katy**. For a range of physical and biological processes from dynamics of granular media to biological swarming the evolution of a large number of interacting agents is modeled according to the competing effects of pairwise attraction and (possibly degenerate) diffusion. We prove that, in the slow diffusion limit, the degenerate diffusion becomes a hard height constraint on the density of the population, as arises in models of pedestrian crown motion. We then apply this to develop numerical insight for open conjectures in geometric optimization.

• A NEW CONTINUUM THEORY FOR INCOMPRESSIBLE SWELLING MATERIALS. **Degond, Pierre**. Swelling media (e.g. gels, tumors) are usually described by mechanical constitutive laws (e.g. Hooke or Darcy laws). However, constitutive relations of real swelling media are not well-known. Here, we take an opposite route and consider a simple heuristics relying on the following rule: (i) particles are at packing density; (ii) any two particles cannot swap their position; (iii) motion should be as slow as possible. These heuristics determine the medium velocity uniquely. In general, this velocity cannot be retrieved by a simple Darcy law.

• DETERMINISTIC PARTICLE APPROXIMATIONS OF LOCAL AND NONLOCAL TRANSPORT EQUATIONS. **Di Francesco, Marco**. Nonlinear convection and nonlocal aggregation equations are known to feature a "formal" gradient flow structure in presence of a "nonlinear mobility", in terms of the generalized Wasserstein distance "à la" Dolbeault-Nazaret-Savaré. Such a structure is inherited by the discrete Lagrangian approximations of those equations in a quite natural way in one space dimension, and this simple remark allows to formulate a discrete-to-continuum "many particle" approximation. I will describe some recent results in this direction, which include the discrete (deterministic) particle approximation for scalar conservation laws and (more recently) a large class of nonlocal aggregation equations as main examples. The results are in collaboration with M. D. Rosini (Ferrara), S. Fagioli and E. Radici (L'Aquila).

• SUPER RICCI FLOWS FOR MARKOV CHAINS. **Erbar, Matthias**. I will present a discrete notion of super Ricci flow that applies to time dependent Markov chains or weighted graphs. This notion can be characterized equivalently in terms of a discrete time-dependent Bochner inequality, gradient estimates for the heat propagator on the evolving graph, contraction estimates in discrete transport distances, or dynamic convexity of the entropy. I will also discuss several examples. This is joint work with Eva Kopfer.

• WHY IN SOME CASES THE ASYMPTOTIC LINEARIZED PROBLEM YIELDS OPTIMAL RESULTS FOR A NONLINEAR VERSION OF THE *carré du champ*. **Esteban, Maria J.** Using a nonlinear parabolic flow, in this talk I will explain why the optimal regions of symmetry and symmetry breaking for the extremals of critical and subcritical Caffarelli-Kohn-Nirenberg inequalities are related to the spectral gap of the linearized problem around the asymptotic Barenblatt solutions. This is a surprising result since it means that a global test yields a global result. The use of the parabolic flow also allows to get improved inequalities with explicit remainder terms.

• HYPOCOERCIVITY IN PHI-ENTROPY FOR LINEAR RELAXATION BOLTZMANN EQUATION. **Evans, Josephine**. As well as results in Hilbert spaces, Villani's memoire on hypocoercivity contains convergence to equilibrium results measured in relative entropy. Since then hypocoercivity in more general Phi entropies has been studied by several authors. These works have mainly been for diffusion equations which can be put in a "Hörmander sum of squares form". The linear relaxation Boltzmann equation is a simple equation not of this form for which hypocoercivity in Phi entropies can still be shown but with extra terms added to the functional which would not be needed for a diffusion.

• STEIN KERNELS, OPTIMAL TRANSPORT AND THE CLT. **Fathi, Max**. Stein kernels are a way of measuring distance between probability measures, defined via integration by parts formulas. I will present a connection between these kernels and optimal transport. The main result is a way of deriving rates of convergence in the classical central limit theorem using regularity estimates for a variant of the Monge-Ampère PDE. As an application, we obtain new rates of convergence for the multi-dimensional CLT, with explicit dependence on the dimension.

• EQUILIBRATION OF RENORMALISED SOLUTIONS TO NONLINEAR CHEMICAL REACTION-DIFFUSION SYSTEMS. **Fellner, Klemens**. We prove exponential convergence to equilibrium for renormalised solutions to general complex balanced reaction-diffusion systems without boundary equilibria and even for systems with boundary equilibria provided a finite dimensional inequality holds along solutions trajectories. Our proofs are based on the entropy method and represent the most general results on the convergence to equilibrium for complex balanced RD systems currently available. (joint works with Bao Quoc Tang)

• GLOBAL ESTIMATES FOR LOCAL AND NONLOCAL POROUS MEDIUM TYPE EQUATIONS ON BOUNDED DOMAINS. **Figalli, Alessio**. The behavior of solutions to the classical porous medium equation is by now well understood: the support of the solution expands at finite speed, and for large times it behaves as the separate-variable solution. When the Laplacian is replaced by a nonlocal diffusion, completely new and surprising phenomena arise depending on the power of the nonlinearity and the one of the diffusion. The aim of the talk is to give an overview of this theory.

• RIGOROUS DERIVATION OF THE NONLOCAL REACTION-DIFFUSION FITZHUGH-NAGUMO SYSTEM. **Filbet, Francis**. We introduce a spatially extended transport kinetic FitzHugh-Nagumo model with forced local interactions and prove that its hydrodynamic limit converges towards the classical nonlocal reaction-diffusion FitzHugh-Nagumo system. Our approach is based on a relative entropy method, where the macroscopic quantities of the kinetic model are compared with the solution to the nonlocal reaction-diffusion system. This approach allows to make the rigourous link between kinetic and reaction-diffusion models

• A PARTIAL LAPLACIAN AS AN INFINITESIMAL GENERATOR ON THE WASSERSTEIN SPACE. **Gangbo, Wilfrid**. We study stochastic processes on the Wasserstein space, together with their infinitesimal generators. One of these processes plays a central role in our work. Its infinitesimal generator defines a partial Laplacian on the space of Borel probability measures, and we use it to define heat flow on the Wasserstein space. We verify a distinctive smoothing effect of this flow for a particular class of initial conditions. To this end, we will develop a theory of Fourier analysis and conic surfaces in metric spaces. We note that the use of the infinitesimal generators has been instrumental in proving various theorems for Mean Field Games, and we anticipate they will play a key role in future studies of viscosity solutions of PDEs in the Wasserstein space (Joint work with Y. T. Chow).

• WHEN OTTO MEETS NEWTON AND SCHRÖDINGER, AN HEURISTIC POINT OF VIEW. **Gentil, Ivan**. We propose a generalization of the Schrödinger problem by replacing the usual entropy with a functional $\mathcal{F}$ which approaches the Wasserstein distance along the gradient of $\mathcal{F}$. From an heuristic point of view by using Otto calculus, we show that interpolations satisfy a Newton equation, extending the recent result of Giovani Conforti. Various inequalities as Evolutional-Variational-inequalities are also established from a heuristic point of view. As a rigorous result we prove a new and general contraction inequality for the usual Schrödinger problem under Ricci bound on a smooth and compact Riemannian manifold. This is a joint work with L. Ripani and C. Léonard.

• A THEORY OF TRANSFERS. **Ghoussoub, Nassif**. I introduce and study the class of "linear transfers" between probability measures. This class contains all cost minimizing mass transports, including "equivariant mass transports" and "martingale mass transports". It also contain the "Schrödinger bridge" associated to a reversible Markov process, and the "weak mass transports" of Talagrand, Marton, Gozlan and others. However, what motivated us to develop the concept are the stochastic mass transports in their various forms. We also introduce the cone of "convex transfers," which in addition to linear transfers, include any p-power of a linear transfer, but also the logarithmic entropy, other energy functionals, as well as the Donsker-Varadhan information. The ultimate goal: Stochastic Weak KAM theory.

• LONG TIME BEHAVIOR OF KINETIC LANGEVIN EQUATION. **Guillin, Arnaud**. We will present here two different approaches to study the long time behavior of the kinetic Langevin equation : 1) hypocoercivity technique for entropic convergence via a new weighted logarithmic Sobolev inequality ; 2) Wasserstein convergence via a particular reflection coupling.

• EQUILIBRIA IN THE DIFFUSION-DOMINATED REGIME AND RELATED FUNCTIONAL INEQUALITIES. **Hoffmann, Franca**. We study interacting particles behaving according to a reaction-diffusion equation with nonlinear diffusion and nonlocal attractive interaction. This class of partial differential equations has a very nice gradient flow structure that allows us to make links to different families of functional inequalities. Depending on the nonlinearity of the diffusion, the choice of interaction potential and the space dimensionality, we obtain different regimes. Understanding the asymptotic profiles of this model is related to the minimization problem of the corresponding energy functional. We will provide an overview of recent advances. These are joint works with José A. Carrillo, Vincent Calvez, Jean Dolbeault, Rupert Frank, Edoardo Mainini and Bruno Volzone.

• ASYMPTOTICAL ANALYSIS OF A WEIGHTED VERY FAST DIFFUSION EQUATION ARISING IN QUANTIZATION OF MEASURES VIA THE JKO SCHEME. **Iacobelli, Mikaela**. In this talk I would like to present some recent results on the asymptotic behavior of a very fast diffusion PDE with periodic boundary conditions. This equation is motivated by the gradient flow approach to the problem of quantization of measures. I prove exponential convergence to equilibrium under minimal assumptions on the data, and I also provide sufficient conditions for W2-stability of solutions. Moreover, I will present a work in progress with Filippo Santambrogio and Francesco Saverio Patacchini where we use the JKO scheme to relax the hypotheses of my previous convergence result.

• TOWARDS A GRADIENT FLOW FOR MICROSTRUCTURE. **Kinderlehrer, David**. A central problem of microstructure is to develop technologies capable of producing an arrangement, or ordering, of the material, in terms of mesoscopic parameters like geometry and crystallography, appropriate for a given application. Is there such an order in the first place? We describe very briefly the emergence of the grain boundary character distribution (GBCD), a statistic that details texture evolution, and illustrate why it should be considered a material property. Its identification as a gradient flow by our method is tantamount to exhibiting the harvested statistic as the iterates in a mass transport JKO implicit scheme, which we found astonishing. Consequently the GBCD is the solution, in some sense, of a Fokker-Planck Equation. The development exposes the question of how to understand the circumstances under which a harvested empirical statistic is a property of the underlying process (joint work with P. Bardsley, K. Barmak, E. Eggeling, M. Emelianenko, Y. Epshteyn, X.-Y. Lu and S. Ta'asan)

• ENTROPY DECAY FOR THE KAC MASTER EQUATION. **Loss, Michael**. The Kac master equation models the behavior of a large number of randomly colliding particles. Due to its simplicity it allows, without too much pain, to investigate a number of issues. E.g., Mark Kac, who invented this model in 1956, used it to give a simple derivation of the spatially inhomogeneous Boltzmann equation. One important issue is the rate of approach to equilibrium, which can be analyzed in various ways, using, e.g., the gap or the entropy. Explicit entropy estimates will be discussed for a Kac type master equation modeling the interaction of a finite system with a large but finite reservoir. This is joint work with Federico Bonetto, Alissa Geisinger and Tobias Ried.

• GROMOV-HAUSDORFF CONVERGENCE OF DISCRETE OPTIMAL TRANSPORT. **Maas, Jan**. For a natural class of discretisations of a convex domain in R^n, we consider the dynamical optimal transport metric for probability measures on the discrete mesh. Although the associated discrete heat flow converges to the continuous heat flow, we show that the transport metric may fail to converge to the 2-Kantorovich metric. Under an additional symmetry condition on the mesh, we show that Gromov-Hausdorff convergence to the 2-Kantorovich metric holds. This is joint work with Peter Gladbach and Eva Kopfer.

• ENTROPIC CONCAVITY AND POSITIVE ENERGY. **McCann, Robert**. On a Riemannian manifold, lower Ricci curvature bounds are known to be characterized by geodesic convexity properties of various entropies with respect to the Kantorovich-Rubinstein-Wasserstein square distance from optimal transportation. These notions also make sense in a (nonsmooth) metric measure setting, where they have found powerful applications. In this talk I describe the development of an analogous theory for lower Ricci curvature bounds in time-like directions on a Lorentzian manifold. In particular, by lifting fractional powers of the Lorentz distance (a.k.a. time separation function) to probability measures on spacetime, I show the strong energy condition of Penrose is equivalent to geodesic concavity of the Boltzmann-Shannon entropy there.

• GRADIENT FLOWS IN ABSTRACT METRIC SPACES: EVOLUTION VARIATIONAL INEQUALITIES AND STABILITY. **Muratori, Matteo**. We study the main consequences of the existence of a Gradient Flow (GF for short), in the form of Evolution Variational Inequalities (EVI), in the very general framework of an abstract metric space. In particular, no volume measure is needed. The hypotheses on the functional associated with the GF are also very mild: we shall require at most completeness of the sublevels (no compactness assumption is made) and, for some convergence and stability results, approximate λ-convexity. The main results include: quantitative regularization properties of the flow (in terms e.g. of slope estimates and energy identities), discrete-approximation estimates of a minimizing-movement scheme and a stability theorem for the GF under suitable gamma-convergence-type hypotheses on a sequence of functionals approaching the limit functional. Existence of the GF itself is a quite delicate issue which requires some concavity-type assumptions on the metric, and will be addressed in a future project. This is a joint work with G. Savaré.

• OPTIMAL TRANSPORTATION WITH FREE END-TIMES. **Palmer, Aaron**. We explore a dynamic formulation of the optimal transportation problem with the additional freedom to choose the end-time of each trajectory. The dual problem is then posed with a Hamilton-Jacobi variational inequality, which we analyze with the method of viscosity solutions. We find properties that imply the optimal stopping-time is the hitting-time of the free boundary to the variational inequality. Joint work with N. Ghoussoub and Y.H. Kim.

• EXISTENCE OF GROUND STATES FOR AGGREGATION-DIFFUSION EQUATIONS. **Patacchini, Francesco Saverio**. We analyze free energy functionals for macroscopic models of multi-agent systems interacting via pairwise attractive forces and localized repulsion. The repulsion at the level of the continuous description is modeled by pressure-related terms in the functional making it energetically favorable to spread, while the attraction is modeled through nonlocal forces. We give conditions on general entropies and interaction potentials for which neither ground states nor local minimizers exist. We show that these results are sharp for homogeneous functionals with entropies leading to degenerate diffusions while they are not sharp for fast diffusions. The particular relevant case of linear diffusion is totally clarified giving a sharp condition on the interaction potential under which the corresponding free energy functional has ground states or not. This is joint work with J. A. Carrillo and M. G. Delgadino.

• A BDF2-APPROACH FOR THE NON-LINEAR FOKKER-PLANCK EQUATION. **Plazotta, Simon**. In this talk I will discuss the construction of approximate solutions for the Non-linear Fokker-Planck equation. We utilize the L^2-Wasserstein gradient flow structure of this PDEs to perform a semi discretization in time by means of the variational BDF2 method. Our approach can be considered as the natural second order analogue of the Minimizing Movement or JKO scheme. In comparison to our own recent work on constructing solutions to λ-contractive gradient flows in abstract metric spaces, the technique presented here exploits the differential structure of the underlying L^2-Wasserstein space. We directly prove that the obtained limit curve is a weak solution of the non-linear Fokker-Planck equation without using the abstract theory of curves of maximal slope. Additionally, we provide strong L^m convergence instead of merely weak convergence in the L^2-Wasserstein topology of the time-discrete approximations.

• WASSERSTEIN ESTIMATES AND MACROSCOPIC LIMITS IN A MODEL FROM ECOLOGY. **Raoul, Gaël**. We are interested in evolutionary biology models for sexual populations. The sexual reproductions are modeled through the so-called Infinitesimal Model, which is similar to an inelastic Boltzmann operator. This kinetic operator is then combined to selection and spatial dispersion operators. In this talk, we will show how the Wasserstein estimates that appear naturally for the kinetic operator can be combined to estimates on the other operators to study the qualitative properties of the solutions. In particular, this approach allows us to recover a well-known (in populations genetics) macroscopic model.

• ENTROPIC OPTIMAL TRANSPORT AND NONLINEAR PDE'S. **Savaré, Giuseppe**. We discuss two examples of "dynamical optimal transport problems", whose formulations involve a relative entropy functional. The first case is related to the Hellinger-Kantorovich distance and induces an interesting geometric structure on the space of positive measures with finite (but possibly different) mass. In particular, contraction estimates of nonlinear flows are strongly related to geodesic convexity of the generating entropy functionals.

In the second example an entropy functional penalizes the density of the connecting measures with respect to a given reference measure (typically the Lebesgue one) and leads to a first order "mean field planning" problem, which is classically formulated by a continuity equation and a Hamilton Jacobi equation with a nonlinear coupling. In this case, the variational approach and the displacement convexity of the entropy functionals (in the usual sense of optimal transport) provide crucial tools to give a precise meaning to the PDE system and to prove the existence of a solution.

• PHASE TRANSITIONS FOR THE MCKEAN-VLASOV EQUATION ON THE TORUS. **Schlichting, André**. In the talk, the McKean-Vlasov equation on the flat torus is studied. The model is obtained as the mean field limit of a system of interacting diffusion processes enclosed in a periodic box. The system acts as a model for several real-world phenomena from statistical physics, opinion dynamics, collective behavior, and stellar dynamics.

This work provides a systematic approach to the qualitative and quantitative analysis of the McKean-Vlasov equation. We comment on the longtime behavior and convergence to equilibrium, for which we introduce a notion of H-stability.

The main part of the talk considers the stationary problem. We show that the system exhibits multiple equilibria which arise from the uniform state through continuous bifurcations, under certain assumptions on the interaction potential. Finally, criteria for the classification of continuous and discontinuous transitions of this system are provided. This classification is based on a fine analysis of the free energy.

The results are illustrated by proving and extending results for a wide range of models, including the noisy Kuramoto model, Hegselmann-Krause model, and Keller-Segel model (joint work with José Carrillo, Rishabh Gvalani, and Greg Pavliotis).

• PATTERN FORMATION DRIVEN BY TRANSPORT, DRIFT, AND LOCALIZED INTERACTIONS. **Stevens, Angela**. An exemplary drift-reaction system with mass conservation is studied w.r.t. pattern formation. The occurrence of rippling patterns in this system relates to an aggregation equation, whose qualitative behavior will also be discussed.

If time permits, aggregation equations with local interactions will be presented, respectively Chemotaxis-models with a non-diffusive chemical.

All models have in common that their qualitative features are more of "hyperbolic type". Thus pattern formation and the analysis of these systems is different from the one of their "parabolic counterparts".

• RECENT RESULTS ON NONLINEAR AGGREGATION-DIFFUSION EQUATIONS: RADIAL SYMMETRY AND LONG TIME ASYMPTOTICS. **Volzone, Bruno**. One of the archetypical aggregation-diffusion models is the so-called classical parabolic-elliptic Patlak-Keller-Segel (PKS for short) model. This model was classically introduced as the simplest description for chemotatic bacteria movement in which linear diffusion tendency to spread fights the attraction due to the logarithmic kernel interaction in two dimensions. For this model there is a well-defined critical mass. In fact, here a clear dichotomy arises: if the total mass of the system is less than the critical mass, then the long time asymptotics are described by a self-similar solution, while for a mass larger than the critical one, there is finite time blow-up. In this talk we will show some recent results concerning the symmetry of the stationary states for a nonlinear variant of the PKS model, of the form

$$\partial_t \rho = \Delta \rho^m + \nabla \cdot (\rho \nabla (W * \rho)), \tag{1}$$

being $W \in C^1(\mathbb{R}^d \setminus \{0\})$, $d \geq 2$, a suitable aggregation kernel, in the assumptions of dominated diffusion, i.e. when $m > 2 - 2/d$. In particular, if W represents the classical logarithmic kernel in the bidimensional case, we will show that there exists a unique stationary state for the model (1) and it coincides, according to one of the main results in the work [9], with the global minimizer of the free energy functional associated to (1). In the case $d = 2$ we will also show how such steady state coincides with the aymptotic profile of (1). Finally, we will also discuss some recent results concerning the model (1) with a Riesz potential aggregation, namely when $W(x) = c_{d,s}|x|^{2s-d}$ for $s \in (0, d/2)$, again in the diffusion dominated regime, *i.e.* for $m > 2 - (2s)/d$. In particular, all stationary states of the model are shown to be radially symmetric decreasing and that global minimizers of the associated free energy are compactly supported, uniformly bounded, Hölder regular, and smooth inside their support. These results are objects of the joint works [10], [11].

• ENHANCEMENT OF BIOLOGICAL REACTION BY CHEMOTAXIS. **Yao, Yao**. In this talk, we consider a system of equations arising from reproduction processes in biology, where two densities evolve under diffusion, absorbing reaction and chemotaxis. We prove that chemotaxis plays a crucial role to ensure the efficiency of reaction: Namely, the reaction between the two densities is very slow in the pure diffusion case, while adding a chemotaxis term greatly enhances reaction. While proving our main results we also obtain a weighted Poincare's inequality for the Fokker-Planck equation, which might be of independent interest.

Appendix B. Schedule

• *Monday, April 9*

09:00 - 09:30 : David Kinderlehrer: Towards a gradient flow for microstructure
09:30 - 10:00 : Gaël Raoul: Wasserstein estimates and macroscopic limits in a model from ecology
10:30 - 11:00 : Pierre Degond: A new continuum theory for incompressible swelling materials
11:00 - 11:30 : Francis Filbet: Rigorous derivation of the nonlocal reaction-diffusion FitzHugh-Nagumo system

14:20 - 15:05 Angela Stevens: Pattern formation driven by transport, drift, and localized interactions
15:30 - 16:00 : Yao Yao: Enhancement of biological reaction by chemotaxis
16:00 - 16:30 : Katy Craig: From slow diffusion to a hard height constraint: a singular limit of Keller-Segel
16:30 - 17:15 : Nassif Ghoussoub: A Theory of Transfers

• *Tuesday, April 10*
08:45 - 09:15 : Guillaume Carlier: Entropic regularization of optimal transport and applications
09:15 - 10:00 : Robert McCann: Entropic concavity and positive energy
10:30 - 11:15 : Wilfrid Gangbo: A partial Laplacian as an infinitesimal generator on the Wasserstein space
11:15 - 12:00 : Aaron Palmer: Optimal transportation with free end-times
14:00 - 14:30 : Anton Arnold: Large-time behavior in hypocoercive BGK-models
14:30 - 15:00 : Arnaud Guillin: Long time behavior of kinetic Langevin equation
15:30 - 16:15 : Michael Loss: Entropy decay for the Kac master equation
16:15 - 16:45 : Maria C Carvalho: Entropy production inequalities for the Kac Walk
16:45 - 17:30 : Josephine Evans: Hypocoercivity in Phi-entropy for linear relaxation Boltzmann equation

• *Wednesday, April 11*
08:45 - 09:15 : Klemens Fellner: Equilibration of renormalised solutions to nonlinear chemical reaction-diffusion systems
09:15 - 10:00 : Alessio Figalli: Global estimates for local and nonlocal porous medium type equations on bounded domains
10:30 - 11:00 : Maria J. Esteban: Why in some cases the asymptotic linearized problem yields optimal results for a nonlinear version of the *carré du champ*
11:00 - 11:30 : Mikaela Iacobelli: Asymptotical analysis of a weighted very fast diffusion equation arising in quantization of measures via the JKO scheme

• *Thursday, April 12*
08:45 - 09:15 : Simon Plazotta: A BDF2-Approach for the Nonlinear Fokker-Planck Equation
09:15 - 10:00 : Giuseppe Savaré: Entropic optimal transport and nonlinear PDE's
10:30 - 11:00 : Jan Maas: Gradient flows and quantum entropy inequalities via matrix optimal transport
11:00 - 11:30 Matteo Muratori: Gradient flows in abstract metric spaces: evolution variational inequalities and stability
14:00 - 14:30 : Marco Di Francesco: Deterministic particle approximations of local and nonlocal transport equations
14:30 - 15:00 : Bruno Volzone: Recent results on nonlinear aggregation-diffusion equations: radial symmetry and long time asymptotics
15:30 - 16:00 : Francesco Patacchini: Existence of ground states for aggregation-diffusion equations
16:00 - 16:30 Ivan Gentil: When Otto meets Newton and Schrödinger, an heuristic point of view
16:30 - 17:15 : Max Fathi: Stein kernels, optimal transport and the CLT

• *Friday, April 13*
08:45 - 09:15 : Matthias Erbar: Super Ricci flows for Markov chains
09:15 - 10:00 : José Alfredo Cañizo: On a new proof of the Harris ergodic theorem and related sub-exponential convergence results
10:30 - 11:00 : André Schlichting: Phase transitions for the McKean-Vlasov equation on the torus